Two-dimensional motion in celestial mechanics

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Outline

Introduction

Ideal model of projectile motion

The ideal model of projectile motion assumes that

1. the acceleration due to gravity is constant and is pointing downwards and
2. there is no air resistance.

Ideal model of projectile motion

Let $\mathbf{v}(0) \equiv \mathbf{v}_0$ be the initial velocity of an object pointed at an angle θ. According to Galileo, the horizontal and the vertical motion of the object is independent and can be expressed as the sum of both motions

$$\mathbf{v}_0 = v_{0x}\hat{\mathbf{i}} + v_{0y}\hat{\mathbf{j}} \tag{1}$$

where

$$v_{0x} = |\mathbf{v}_0| \cos\theta, \tag{2}$$
$$v_{0y} = |\mathbf{v}_0| \sin\theta. \tag{3}$$

Ideal model of projectile motion

Since the acceleration due to gravity is constant and only applies to the vertical direction, the velocity in the horizontal direction remains constant. The components of acceleration are:

$$a_x = 0, \quad (4)$$

$$a_y = -g \quad (5)$$

where g is the acceleration due to gravity.

Concepts

Kepler's laws of planetary motion and the Kepler problem

Johannes Kepler first formulated the laws that describe planetary motion:

1. Each planet moves in an ellipse with the sun at one focus.[1]
2. The sector from the sun to a planet sweeps out equal area in equal time.
3. The period of revolution T of a planet about the sun is related to the semi-major axis a of the ellipse by $T^2 = ka^3$ where k is the same for all planets.

[1] We will only be able to show this.

Newton's law of gravitation

The gravitational force between two objects is directly proportional to the product of their masses and inversely proportional to the square of the distance between them, i.e.

$$F = G\frac{m_1 m_2}{r^2} \tag{6}$$

where m_1, m_2 are the masses of two objects, r is the distance between the two objects, and $G = 6.6743 \times 10^{-11}\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ is the gravitational constant.

Celestial mechanics

Two-body problem

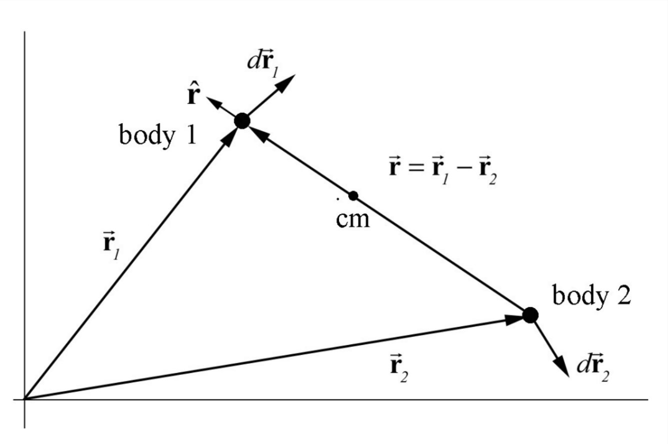

Figure 1: Coordinate system for the two-body problem.

Two-body problem

System of two bodies and their gravitational effect toward each other:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{7}$$

in accordance with Newton's third law of motion.

Two-body problem

Two-body problem

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position of bodies 1 and 2. The motion of the bodies is modeled by the equations

$$\mathbf{F}_{12} = m_1 \mathbf{r}_1'' \tag{8}$$

$$\mathbf{F}_{21} = m_2 \mathbf{r}_2'' \tag{9}$$

Reducing the two-body problem into a one-body problem

The motion of two interacting bodies is equivalent to the motion of a single body acted on by an external central gravitational force where the mass of the single body is

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \implies \mu = \frac{m_1 m_2}{m_1 + m_2}{}^{2}. \tag{10}$$

[2] This central mass is located in between the two bodies, and the ratio of the distance between the a body and the central mass is equal to the inverse ratio of their masses.

Reducing the two-body problem into a one-body problem

The force on body 1 can be described by

$$\mathbf{F}_{21} = -F_{21}\frac{\mathbf{r}}{r} = -F_{21}\hat{\mathbf{r}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \tag{11}$$

where $F_{21} = |\mathbf{F}_{21}|$, $r = |\mathbf{r}|$, and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$.

Reducing the two-body problem into a one-body problem

Dividing through by the mass in both Equations 8 and 9,

$$\frac{\mathbf{F}_{12}}{m_1} = \mathbf{r}_1'' \tag{12}$$

$$\frac{\mathbf{F}_{21}}{m_2} = \mathbf{r}_2'' \tag{13}$$

Reducing the two-body problem into a one-body problem

Subtracting Equation 13 from Equation 12, we get

$$\frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2} = \mathbf{r}_1'' - \mathbf{r}_2'' = \mathbf{r}''. \tag{14}$$

Using the initial result on Equation 7, we get

$$\frac{\mathbf{F}_{12}}{m_1} - \frac{-\mathbf{F}_{12}}{m_2} = \mathbf{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \mathbf{F}_{12}\frac{1}{\mu} = \mathbf{r}''. \tag{15}$$

Reducing the two-body problem into a one-body problem

And so, we reduce the system of two equations to

$$\mathbf{F}_{12} = \mu \mathbf{r}''. \tag{16}$$

The orbit equation

Deriving the orbit equation

The (gravitational) potential energy with the choice of zero reference point $U(\infty) = 0$ is

$$U(r) = -\frac{Gm_1 m_2}{r^2} r = -\frac{Gm_1 m_2}{r} \tag{17}$$

The total energy of the system is constant, so

$$E = \text{KE} + \text{PE} = \frac{1}{2}\mu v^2 - \frac{Gm_1 m_2}{r} \tag{18}$$

where v is the scalar of the relative speed of two bodies – i.e. $v = |\mathbf{v}|$.

Deriving the orbit equation

Choose polar coordinates such as

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta} \qquad v = |\mathbf{v}| = \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \left(r\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2} \tag{19}$$

where $v_r = r'$ and $v_\theta = \theta'$.

Equation 18 becomes

$$E = \frac{1}{2}\mu\left[(r')^2 + (r\theta')^2\right] - \frac{Gm_1m_2}{r^2} \tag{20}$$

Deriving the orbit equation

The angular momentum with respect to the origin is

$$\mathbf{L}_O = \mathbf{r}_O \times \mu\mathbf{v} = r\hat{\mathbf{r}} \times \mu(v_r\hat{\mathbf{r}} + v_\theta\hat{\theta}) = \mu r v_\theta\hat{\mathbf{k}} = \mu r\theta'\hat{\mathbf{k}} =: L\hat{\mathbf{k}} \tag{21}$$

where $L = \mu r v_\theta = \mu r\theta'$[3].

[3] L denotes the angular momentum of the body.

Deriving the orbit equation

We then remove the θ dependence from Equation 20 using the result:

$$\theta' = \frac{L}{\mu r^2} \tag{22}$$

We get

$$E = \frac{1}{2}\mu(r')^2 + \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{Gm_1m_2}{r^2} \tag{23}$$

Differential equation

We can rearrange Equation 23 to get a first-order differential equation:

The orbit equation

$$r' = \sqrt{\frac{2}{\mu}} \left(E - \frac{1}{2} \frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r^2} \right)^{\frac{1}{2}} \tag{24}$$

Differential equation

Instead of solving r w.r.t. t, we shall find a geometric description of the orbit by finding $r(\theta)$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\theta'}{r'} = \frac{L}{\mu r^2}\left[\sqrt{\frac{2}{\mu}}\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r^2}\right)^{\frac{1}{2}}\right]^{-1} \tag{25}$$

$$= \frac{L}{\sqrt{2\mu}} \frac{\frac{1}{r^2}}{\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r^2}\right)^{\frac{1}{2}}} \tag{26}$$

Differential equation

We solve the differential equation in (26):

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{L}{\sqrt{2\mu}} \frac{\frac{1}{r^2}}{\left(E - \frac{1}{2}\frac{L}{\mu r^2} + \frac{Gm_1m_2}{r^2}\right)^{\frac{1}{2}}}, \tag{27}$$

$$\mathrm{d}\theta = \frac{L}{\sqrt{2\mu}} \frac{\frac{1}{r^2}}{\left(E - \frac{1}{2}\frac{L}{\mu r^2} + \frac{Gm_1m_2}{r^2}\right)^{\frac{1}{2}}}\mathrm{d}r. \tag{28}$$

Differential equation

We substitute $u = \frac{1}{r}$ and $\mathrm{d}u = -\frac{1}{r^2}\mathrm{d}r$ to get

$$\mathrm{d}\theta = -\frac{L}{\sqrt{2\mu}} \frac{\mathrm{d}u}{\left(E - \frac{L^2}{2\mu}u^2 + Gm_1m_2u\right)^{\frac{1}{2}}}, \tag{29}$$

$$\mathrm{d}\theta = -\frac{\mathrm{d}u}{\left[\frac{2\mu E}{L^2} - u^2 + 2\left(\mu\frac{Gm_1m_2}{L^2}\right)u\right]^{\frac{1}{2}}}. \tag{30}$$

Differential equation

We define $r_0 := \frac{L^2}{\mu G m_1 m_2}$[4] which simplifies our differential equation to

$$\mathrm{d}\theta = -\frac{\mathrm{d}u}{\left[\frac{2\mu E}{L^2} - u^2 + \frac{2u}{r_0}\right]^{\frac{1}{2}}} \tag{31}$$

[4] r_0 is called the *semilatus rectum*

Differential equation

We now rewrite the denominator in order to express eccentricity.

$$\mathrm{d}\theta = -\frac{\mathrm{d}u}{\left[\frac{2\mu E}{L^2} + \frac{1}{r_0^2} - u^2 + \frac{2u}{r_0} - \frac{1}{r_0^2}\right]^{\frac{1}{2}}} \tag{32}$$

$$= -\frac{\mathrm{d}u}{\left[\frac{2\mu E}{L^2} + \frac{1}{r_0^2} - \left(u - \frac{1}{r_0}\right)^2\right]^{\frac{1}{2}}} \tag{33}$$

$$= -\frac{r_0 \mathrm{d}u}{\left[\frac{2\mu E r_0^2}{L^2} + 1 - (r_0 u - 1)^2\right]^{\frac{1}{2}}} \tag{34}$$

Differential equation

We define $\varepsilon^2 := \frac{2\mu E r_0^2}{L^2} + 1$[5]. We are left with

$$\mathrm{d}\theta = \frac{r_0 \mathrm{d}u}{\left[\varepsilon^2 - (r_0 u - 1)^2\right]^{\frac{1}{2}}} \tag{35}$$

Substituting $r_0 u - 1 = \varepsilon \cos\alpha$ and $r_0 \mathrm{d}u = -\varepsilon \sin\alpha \mathrm{d}\alpha$, we get our final result that

$$\mathrm{d}\theta = \frac{-\varepsilon \sin\alpha \mathrm{d}\alpha}{\left(\varepsilon^2 - \varepsilon^2 \cos^2\alpha\right)^{\frac{1}{2}}} = \mathrm{d}\alpha \tag{36}$$

[5] ε denotes eccentricity.

Closed-form solution

We get the result

$$\theta = \alpha + C, C \in \mathbb{R}. \tag{37}$$

Let $C = \pi$ which gives us $\cos\alpha = \cos(\theta - \pi) = -\cos\theta$. We then get

Solution to the orbit equation

$$r(\theta) = \frac{1}{u} = \frac{r_0}{1 - \varepsilon\cos\theta} \tag{38}$$

Model constants

Angular momentum

$$L = (\mu G m_1 m_2 r_0)^{\frac{1}{2}} \tag{39}$$

Total energy

$$E = \frac{G m_1 m_2 (\varepsilon^2 - 1)}{2 r_0} \tag{40}$$

Cases

Circular orbits

We obtain this orbit when $\varepsilon = 0$ which gives us

$$r = \frac{r_0}{1 - (0)\cos\theta} = r_0 \tag{41}$$

– i.e the distance is constant for all θ.

Circular orbits

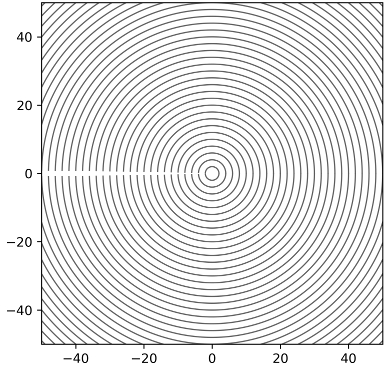

Figure 2: Plot of possible circular orbits with the origin as the center.

Elliptical orbits

We obtain this orbit when $\varepsilon \in (0, 1)$ which gives us

$$r = \frac{r_0}{1 - \varepsilon \cos\theta} > 0 \tag{42}$$

for all θ. This means that r remains finite since the denominator will never be 0.

Elliptical orbits

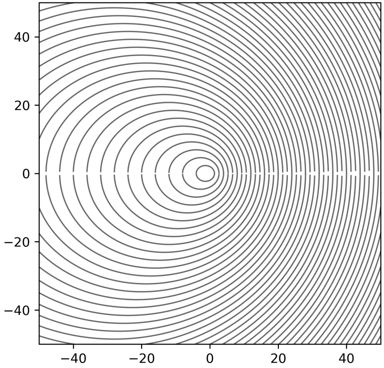

Figure 3: Plot of possible elliptical orbits. Note that the origin is one of the foci.

Parabolic orbits

We obtain this orbit when $\varepsilon = 1$ which gives us

$$r = \frac{r_0}{1 - \cos\theta}. \tag{43}$$

Note that as $\theta \to \pi/2$, $1 - \cos\theta \to 0$, and it follows that $r = \frac{r_0}{1-\cos\theta} \to \infty$. This is the first case where the body reaches escape velocity.

Parabolic orbits

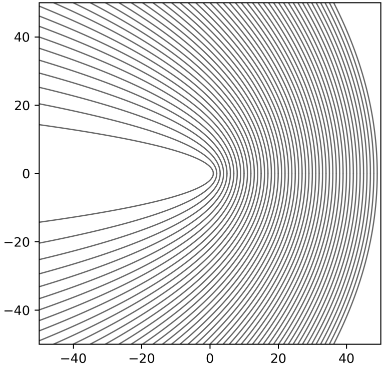

Figure 4: Plot of possible elliptical orbits. Note that the origin is the focus.

Hyperbolic orbits

We obtain this orbit when $\varepsilon > 1$ which gives us

$$r = \frac{r_0}{1 - \varepsilon \cos\theta} \tag{44}$$

which can go negative.

Hyperbolic orbits

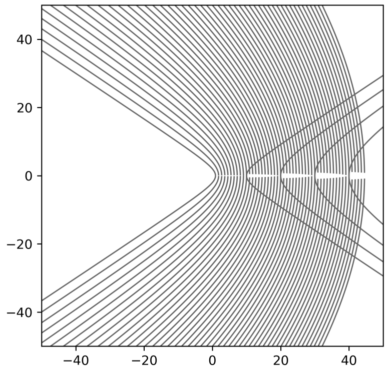

Figure 5: Plot of possible hyperbolic orbits.

Applications

Escape velocity

Note that orbits that escape the gravitational pull of a body has $\varepsilon > 1$. From Equation 40,

$$E = \frac{Gm_1m_2(\varepsilon^2 - 1)}{2r_0} = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{r} \tag{45}$$

Hence we can isolate v and obtain

$$v = \left[\frac{Gm_1m_2(\varepsilon^2 - 1)}{\mu r_0} + \frac{2Gm_1m_2}{\mu r}\right]^{\frac{1}{2}} \geq \left(\frac{2Gm_1m_2}{\mu r}\right)^{\frac{1}{2}} \tag{46}$$

Escape velocity

Escape velocity

In order for two objects to escape their gravitational interaction, their relative speed should be v such that

$$v \geq \left(\frac{2Gm_1m_2}{\mu r}\right)^{\frac{1}{2}} \tag{47}$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.

Special case

Suppose the system consists of a small body and a big body. Without loss of generality, let m_2 be the mass of the large object. Hence

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \approx \frac{1}{m_1} \implies \mu = m_1 \tag{48}$$

Special case

Escape velocity from a large body

In order for a small body to escape the gravitational influence of a larger body, the smaller body should be travelling at a velocity v_{small} such that

$$v_{\text{small}} \geq \left(\frac{2Gm_2}{r}\right)^{\frac{1}{2}} \tag{49}$$

where m_2 is the mass of the larger body.

Solving for Earth’s escape velocity

Given $m_2 = 5.972 \times 10^{24}$kg and $r = 6.3781 \times 10^6$m and letting $m_1 = 1000.0$kg,

$$v \approx 11180\text{m} \cdot \text{s}^{-1} \qquad v_{\text{small}} \approx 11180\text{m} \cdot \text{s}^{-1} \tag{50}$$

Geostationary orbit

A geostationary orbit is a circular geosynchronous orbit on Earth with an orbital period equal to Earth's rotational period.

Given $\varepsilon = 0$,

$$E = -\frac{Gm_1m_2}{2r} = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{r} \tag{51}$$

General and special models for geostationary orbits

$$v = \left(\frac{Gm_1m_2}{\mu r}\right)^{\frac{1}{2}} \approx \left(\frac{Gm_2}{r}\right)^{\frac{1}{2}} \tag{52}$$

Solving for the geostationary orbit altitude

Let $v = \frac{2\pi r}{T}$ where T is the orbital period and r is the radius of the orbit.

Given $T = 86164\text{s}$, $m_1 = 1000.0\text{kg}$, and $m_2 = 5.972 \times 10^{24}\text{kg}$,

$$r = \left(\frac{Gm_1m_2T^2}{4\mu\pi^2}\right)^{\frac{1}{3}} \approx 4.216 \times 10^7\text{m} \approx 42160\text{km}. \tag{53}$$

Given $r_{\text{earth}} = 6.3781 \times 10^6\text{m}$, the altitude of the body - we denote by h - is

$$h \approx 4.216 \times 10^7 m - 6.3781 \times 10^6\text{m} \approx 35790\text{km}. \tag{54}$$

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