The University of Western Australia
SCHOOL OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2014 Problems

The Competition begins Tuesday, 27 May and ends Friday, 26 September 2014.

You may use any source of information except other people. In particular, you may use computer packages (though contributed solutions are expected to have complete mathematical proofs). Extensions and generalisations of any problem are invited and are taken into account when assessing solutions. Also, elegance of solution is particularly favoured and desired.

Solutions are to be mailed or given to Greg Gamble, School of Mathematics and Statistics, The University of Western Australia, Crawley, 6009 **before 4 pm on Friday, 26 September.**

Remember, you don't have to solve all the problems to win prizes!

Include a cover page with your name, address, e-mail address, University, and the number of years you have been attending any tertiary institution. Please submit a hardcopy of your entry, to the above address. However, it is recommended that you also submit a PDF-scanned copy to `greg.gamble@uwa.edu.au`.

Start each problem on a new page, and write your name on every page.

---

1. Polyhistoric accuracy

Does there exist a polynomial $p(x)$ with integer coefficients such that $p(2014) = 1915$ and $p(1788) = 1901$?

---

2. Diagonal revolution

Consider the solid of revolution obtained by rotating a rectangle with side lengths ℓ and b, $\ell > b$, along one of its diagonals.

What is the volume of this solid?

---

3. Fractionally sequential

Let O_n (resp. E_n) be the number obtained by writing (in base 10) the first n odd (resp. even) numbers one after the other.

Does the sequence of rational numbers O_n/E_n have a limit as n goes to infinity? (The first few terms of the sequence are: $1/2$, $13/24$, $135/246$, $1357/2468$, $13579/246810, \dots$)

---

4. Imperfect square

Show that no number of the form $(2n-1)(6n-1)$, where n is a positive integer, can be a perfect square.

---

5. Trigonometrically rational

What are the real solutions of the equation

$$\sin x \cdot \cos^2 x \cdot \tan^3 x \cdot \cot^4 x \cdot \sec^5 x \cdot \operatorname{cosec}^6 x = \frac{256}{27}?$$

---

6. Powerfully whole

For what positive integers n is $(2013 + \frac{1}{2})^n + (2014 + \frac{1}{2})^n$ also an integer?

---

7. Piling matches

Starting with a finite number of matches divided into a finite number of piles, at each step, we perform the following procedure:

> Choose two piles and remove from the bigger one as many matches as there are in the smaller pile and add these to the smaller pile. In the case that the two chosen piles have the same size, they are merged into one, reducing the overall number of piles by one.

For example, if we start with three piles with 3, 12, and 33 matches, respectively, we can do:

$$[3, 12, 33] \to [6, 12, 30] \to [12, 12, 24] \to [24, 24] \to [48]$$

so that we get one pile after 4 steps.

For which of the following five starting situations, is it possible to obtain only one pile after a finite number of steps?

(i) $[1, 2, 3]$

(ii) $[1, 2, 2, 3, 3, 4, 5, 12, 17, 15]$

(iii) $[16, 4, 5, 5, 8]$

(iv) $[51, 54, 75, 75, 81, 48]$

(v) $[1, 2, 3, 4, 5, 6, 7]$

(vi) $[1, 1, 2, 2, 2, 3, 3, 3, 3, 5]$

---

8. Tangentially circumcircled

The tangents at B and C to the circumcircle of acute-angled triangle ABC intersect in X. Let M be the midpoint of BC.

Prove that $AM/AX = \cos \angle CAB$.

---

9. Tetrahedral triangle

Prove that every tetrahedron has at least one vertex such that the three edges incident with this vertex have lengths equal to the sides of some triangle.

---

10. Recursively limited?

Let the sequence $a_1, a_2, \ldots$ be defined by

$$a_1 = \sqrt{2},$$
$$a_{n+1} = (\sqrt{2})^{a_n}, \text{ for } n \geq 1.$$

Does this sequence converge? If so, what is its limit?
