18.821 PROJECT 2: ASYMMETRIC PROCESSES

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Abstract. We examine asymmetric processes on two different types of graph: the closed loop and the open loop. We show that these processes can be likened to discrete Markov chains, and evaluate them as such. For the closed loop, we derive the stationary distribution and the relaxation time. For the open loop, we prove existence of the stationary distribution and provide conditions for uniqueness and uniformity arbitrary for arbitrary length lines. Finally, we conjecture a relation between equiprobable groups in the stationary distribution.

1. Introduction

In this paper, we investigate properties of *asymmetric processes*, processes in which particles can move left or right with a preference (which we call the asymmetry parameter q) for one of the two directions. In particular, we consider two versions of asymmetric processes we call the *closed* and *open* case: The closed case is a closed loop of k particles and N positions, and the open case is a variable number of particles and N positions (see Sections 2 and 3 for a detailed description of both cases respectively).

We model both the closed and the open case as a *discrete-time Markov chain*, which is a sequence of random variables $X_1, X_2, X_3, ...$ with the *Markov property*; i.e., the probability of moving to the next state depends only on the present state and not on the previous states. Formally,

$$\mathbb{P}[X_{n+1} = x | X_1 = x_1, X_2 = x_2 ... X_n = x_n] = \mathbb{P}[X_{n+1} = x | X_n = x_n]$$

if the conditional probabilities are well defined; i.e,

$$\mathbb{P}[X_1 = x_1, \dots, X_n = x_n] > 0.$$

The possible values of X_i is a countable set S called the *state space*.

Markov chains can be described by a sequence of directed graphs, where the edges of graph n are labeled by the probabilities of going from one state at time n to the other states at time $n+1$ (see Figure 1). The same information is represented by the transition matrix from time n to time $n+1$ (see Figure 2).

Let π be a probability distribution on the state space S. If for such π,

$$\pi = \pi \cdot P$$

we say that π is the *stationary distribution*, or *equilibrium distribution* denoted as π^*.

In this paper, we investigate the stationary distribution π^*, for the open and closed case, and additionally the time of convergence to π^* for the closed case. Namely, in Section 2, we prove that for $k = 1$, the stationary distribution π^* for the closed

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case is the uniform distribution on N states, and generalize the result to k particles by showing that the stationary distribution is the uniform distribution on $\binom{N}{k}$ states (Subsection 2.1). Also in Section 2, we show that for $k = 1$ and even N, the asymmetric process is periodic, and for odd N or $k \neq 1$ the process converges to the stationary distribution (Subsection 2.2). We also provide empirical results on how quickly the closed case converges to periodicity or to a stationary distribution (Subsection 2.3).

For the open case, we show that in Section 3 there exists a unique stationary distribution π^*, and provide an explicit expression for $N = 3$. Moreover, we provide a value of the asymmetry parameter q for which such a unique stationary distribution is uniform. We conclude by conjecturing that in a stationary distribution, the number of groups of states with equal probability is asymptotically bounded by the number of states regardless of the asymmetry parameter q.

2. Closed case

In the closed case, the k particles are on a closed loop with N positions. Each position can be occupied by at most one particle, so there are $\binom{N}{k}$ states of the system. Given an *asymmetry parameter* q, we perform the following procedure indefinitely:

(1) Choose one of the k particles uniformly at random.
(2) Flip a biased coin with probability of heads equal to q.
(3) If heads, move the chosen particle one position clockwise (if that position is vacant). If tails, move the chosen particle one position counterclockwise (if that position is vacant).

Figure 1 illustrates the asymmetric process on a closed loop for $k = 1$ particles and $N = 6$ positions.

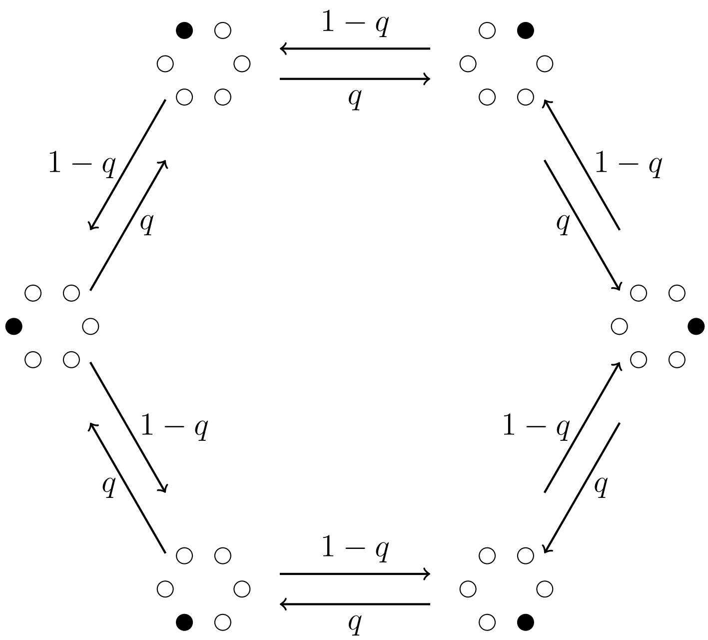


FIGURE 1. Illustration of the asymmetric process on a closed loop for $k = 1$ particles and $N = 6$ positions. The filled circles indicate the location of the particle in each of the 6 states.

2.1. **Stationary distributions on the closed loop.** Firstly, we consider the case where $k = 1$, so there is one particle and hence N states.

Theorem 2.1. *For a closed loop with N positions and $k = 1$ particle, the stationary distribution is the uniform distribution on the N states, i.e. each state is equally likely with probability $\frac{1}{N}$.*

Proof. Firstly, $\pi = (\pi_1, \ldots, \pi_N)$ is a stationary distribution if and only if $\sum_i \pi_i = 1$ and all the following system of equations hold:

$$\begin{aligned}(1-q)\pi_N + q\pi_2 &= \pi_1\\ (1-q)\pi_1 + q\pi_3 &= \pi_2\\ &\cdots\\ (1-q)\pi_{N-1} + q\pi_1 &= \pi_N\end{aligned}$$

Call this system $(*)$. Note that the uniform distribution $\pi^* = (\frac{1}{N}, \ldots, \frac{1}{N})$ indeed satisfies this system. If $q \in \{0, 1\}$, it is clear that π^* is the only solution. Hence assume $q \neq 0, 1$. We claim that this is still the only solution. Squaring every equation in $(*)$ yields the system

$$\begin{aligned}(1-q)^2\pi_N^2 + q^2\pi_2^2 + 2q(1-q)\pi_2\pi_N &= \pi_1^2\\ (1-q)^2\pi_1^2 + q^2\pi_3^2 + 2q(1-q)\pi_3\pi_1 &= \pi_2^2\\ &\cdots\\ (1-q)^2\pi_{N-1}^2 + q^2\pi_1^2 + 2q(1-q)\pi_1\pi_{N-1} &= \pi_N^2\end{aligned}$$

Summing all of them, we get

$$\begin{aligned}(1-q)^2\left(\sum_i \pi_i^2\right) + q^2\left(\sum_i \pi_i^2\right) + 2q(1-q)\left(\sum_i \pi_i\pi_{i-2}\right) &= \left(\sum_i \pi_i^2\right)\\ \iff 2q(1-q)\left(\sum_i \pi_i^2\right) &= 2q(1-q)\left(\sum_i \pi_i\pi_{i-2}\right)\\ \iff \sum_i \pi_i^2 &= \sum_i \pi_i\pi_{i-2}\end{aligned}$$

where indices are taken modulo N. However,

$$\sum_i (\pi_i - \pi_{i-2})^2 = 2\left(\sum_i \pi_i^2 - \sum_i \pi_i\pi_{i-2}\right) = 0$$

It follows that $\pi_i = \pi_{i-2}$ for all i. This implies that $\pi_N = \pi_2$, so that $\pi_1 = (1-q)\pi_N + q\pi_2 = \pi_2$. This means that $\pi_1 = \pi_2 = \cdots = \pi_N$. Since they sum to 1, $\pi_i = \frac{1}{N}$ for all i, and we're done. $\square$

More generally, the statement of Theorem 2.1 holds for any number of particles k. Consider the $\binom{N}{k} \times \binom{N}{k}$ transition matrix $\mathbf{P}$ for this asymmetric process, where p_{ij} is the probability of transitioning from state i to state j. Figure 2 shows $\mathbf{P}$ for the case where $k = 1$ and $N = 6$. Clearly, the rows of $\mathbf{P}$ sum to 1. We prove that $\mathbf{P}$ is *doubly stochastic*: its columns also sum to 1.

Lemma 2.2. *Let $\mathbf{P}$ be the $\binom{N}{k} \times \binom{N}{k}$ transition matrix for the asymmetric process on k particles and N positions on a closed loop. Then the columns of $\mathbf{P}$ sum to 1.*

$$\mathbf{P} = \begin{bmatrix} 0 & q & 0 & 0 & 0 & 1-q \\ 1-q & 0 & q & 0 & 0 & 0 \\ 0 & 1-q & 0 & q & 0 & 0 \\ 0 & 0 & 1-q & 0 & q & 0 \\ 0 & 0 & 0 & 1-q & 0 & q \\ q & 0 & 0 & 0 & 1-q & 0 \end{bmatrix}$$

FIGURE 2. Transition matrix for the asymmetric process with $k = 1$ and $N = 6$.

Proof. Consider some state U. Our goal is to prove that the sum of all the transition probabilities from other states to U is equal to 1. The key insight is to group these transition probabilities into k pairs, one for each particle, and where each pair sums to $\frac{1}{k}$.

Consider any one of the k particles in U, say u. There are two possible states U_1, U_2 that can transition to U by moving (or attempting to move) a particle adjacent to u. Specifically, let v_1 be the position one step counter-clockwise of u, and let v_2 be the position one step clockwise of u. Then define U_1, U_2 as:

- U_1: The state prior to an attempt to move a particle from v_1 to u.
- U_2: The state prior to an attempt to move a particle from v_2 to u.

Figure 3 illustrates an example of U, U_1 and U_2. Note that U_1, U_2 do not have to be different from U. For instance, $U_2 \equiv U$ in this case because both u and v_2 were occupied, so the transition $v_2 \to u$ does not change the state.

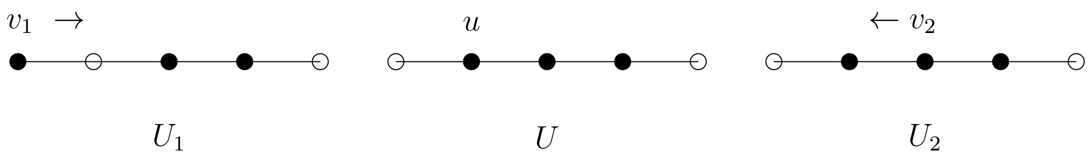


FIGURE 3. Illustration of U, U_1, and U_2. Note that $U \equiv U_2$.

Notice that $\mathbb{P}(\text{transition to } U|U_1) = q$, $\mathbb{P}(\text{transition to } U|U_2) = 1 - q$. Since we pick u with probability $\frac{1}{k}$ in each case,

$$\mathbb{P}(\text{transition to } U \text{ by moving } u) = \frac{1}{k} \cdot q + \frac{1}{k} \cdot (1-q) = \frac{1}{k}$$

Summing over all u, i.e. over all k particles, we find that

$$\sum_i a_{iU} = \sum_i \mathbb{P}(\text{transition to } U|i) = \sum_{u \in U} \mathbb{P}(\text{transition to } U \text{ by moving } u_i) = 1$$

which proves the Lemma. □

Theorem 2.3. *For a closed loop with N positions and k particles, the stationary distribution is the uniform distribution on the $\binom{N}{k}$ states.*

Proof. As before, let $\mathbf{P}$ be the transition matrix for this asymmetric process. Recall that a vector π is a stationary distribution iff $\pi_i \geq 0$ for all i, $\sum_i \pi_i = 1$, and $\pi\mathbf{P} = \pi$. Let v be the state with the greatest probability. π must satisfy the system:

$$\sum_i p_{iu}\pi_i = \pi_u \text{ for all states } u \tag{2.1}$$

By our Lemma, $\sum_i a_{iv} = 1$. Therefore,

$$\pi_v = \left(\sum_i p_{iv}\right)\pi_v \geq \sum_i p_{iv}\pi_i = \pi_v$$

It follows that each π_i is equal to π_v, i.e. π is uniform. It remains to note that the uniform distribution π clearly satisfies Equation 2.1, since

$$\sum_i p_{iu}\binom{N}{k}^{-1} = \binom{N}{k}^{-1}$$

which concludes our proof. $\square$

2.2. **Long-term behavior on the closed loop.** Let $S = \binom{N}{k}$ be the number of states. We now know that the stationary distribution of the asymmetric process on a closed loop is the uniform distribution π^*. If $p = (p_1, \ldots, p_S)$ is our starting distribution, then the state distribution after t iterations of the process is $p\mathbf{P}^t$. We're interested in the *time to equilibrium* of the process: how quickly does p coverge to π^*, if at all?

Suppose $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_S|$ are the eigenvalues of $\mathbf{P}$, corresponding to eigenvectors $v_1, \ldots, v_S$. Note that we're working with magnitudes because the eigenvalues and eigenvectors can be complex. The following result shows that 1 is the eigenvalue with largest magnitude.

Theorem 2.4. *If λ is an eigenvalue of $\mathbf{P}$, then $|\lambda| \leq 1$. Furthermore, if $|\lambda| = 1$, then $\lambda \in \{-1, 1\}$, and*

- *The only eigenvector with $\lambda = 1$ is the uniform distribution,*
- *$\lambda = -1$ is an eigenvalue of $\mathbf{P}$ if and only if N is even and $k = 1$, and the only eigenvector with this eigenvalue is $(1, -1, \ldots, -1)$, where the signs alternate.*

Proof. We use a similar argument as in Theorem 2.3, but work in the complex numbers instead. If v is an eigenvector of $\mathbf{P}$ with eigenvalue λ, and m is the index of the greatest $|v_m|$, then

$$|v_m| = \left(\sum_i p_{im}\right)|v_m| \geq \sum_i p_{im}|v_i| \tag{2.2}$$

$$\geq \left|\sum_i p_{im}v_i\right| = |\lambda v_m| \tag{2.3}$$

This implies that $|\lambda| \leq 1$, proving the first part of the theorem. Next, assume that $|\lambda| = 1$. Then equality must hold in Eq. 2.2, so in fact $|v_m| = |v_i|$ for all states i. Equality must also hold in Eq. 2.3, which is the triangle inequality, which implies that all the v_i are collinear in the complex plane, i.e. $v_i = rv_1$ for some $r \in \mathbb{R}$ for all states i. But since $|v_i| = |v_1|$, we must have $v_i = s_iv_1$ for all i, where $s_i \in \{-1, 1\}$. Now take any state u and note that

$$\sum_i p_{iu}v_i = \lambda v_u \Rightarrow \sum_i p_{iu}s_iv_1 = \lambda s_uv_1$$

$$\Rightarrow \sum_i p_{iu}s_i = \lambda s_u \tag{2.4}$$

But since the p_{iu} and s_i are real, λ must also be real. Since $|\lambda| = 1$, $\lambda \in \{-1, 1\}$, as claimed.

Next, we consider each case separately:

(1) $\lambda = 1$. When we showed that the uniform distribution is the only stationary distribution of $\mathbf{P}$ in Theorem 2.3, note that we did not use the condition that $\pi_i \geq 0$. Thus what we proved is actually that the unique solution to $\pi\mathbf{P} = \mathbf{P}$ is the uniform distribution, i.e. the only eigenvector with eigenvalue $\lambda = 1$ is the uniform distribution.

(2) $\lambda = -1$. Then Eq. 2.4 becomes

$$\sum_i p_{iu} s_i = -s_u \Rightarrow \left|\sum_i p_{iu} s_i\right| = |s_u| = 1 \tag{2.5}$$

for any state u. But notice that $\sum_i p_{iu} = 1$. Considering only the terms where $p_{iu} > 0$, we have $\sum_{i, p_{iu}>0} p_{iu} = 1$. Since $s_i \in \{-1, 1\}$, by triangle inequality we have

$$1 = \left|\sum_{i, p_{iu}>0} p_{iu} s_i\right| \leq \sum_{i, p_{iu}>0} |p_{iu} s_i| = \sum_{i, p_{iu}>0} p_{iu} = 1$$

Since equality holds and all terms here are nonzero real numbers, each $p_{iu} s_i$ for which $p_{iu} > 0$ must have the same sign. Furthermore, from Eq. 2.5, s_u must have the opposite sign. Therefore

$$p_{iu} > 0 \Rightarrow s_i = -s_u.$$

In other words, if the direct transition $i \to u$ is possible, then s_i and s_u have opposite signs.

Suppose there is only one particle, $k = 1$. For state i corresponding to the case where position i is occupied, the only possible transitions to i are from states $i-1$ and $i+1$. Thus $s_i = -s_{i+1}$, i.e. the s_i are alternating. In particular, this implies that N is even: if N were odd, then $s_1 = -s_2 = \cdots = s_N$, but then s_N and s_1 have the same sign, which is impossible because state N can transition to state 1.

Now suppose $k > 1$, and consider a state C where all k particles are next to each other, for instance, the state where positions $1, \ldots, k$ are occupied. Then there is a nonzero probability of transitioning from C to itself, because we can choose particle 1 with probability $\frac{1}{k}$, move clockwise with probability q, and fail because position 2 is occupied. Thus the transition probability $C \to C$ is at least $\frac{q}{k} > 0$. But then $s_C = -s_C$, which is impossible.

Finally, if $k = 1$ and N is even, it is easy to check that the following system holds:

$$\begin{aligned}
(1-q)(-1) + q(-1) &= -(1) \\
(1-q)(1) + q(1) &= -(-1) \\
&\cdots \\
(1-q)(1) + q(1) &= -(-1)
\end{aligned}$$

It follows that $(1, -1, \ldots, 1, -1)$ is indeed an eigenvector of $\mathbf{P}$ with eigenvalue -1, and we're done.

$\square$

Definition 2.5. The *Jordan normal form* of a matrix A is a block diagonal matrix J:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_p \end{bmatrix}$$

where each block J_i is a square matrix

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \dots & \\ & & \dots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

such that A is similar to J, i.e. there exists an invertible matrix Q such that $Q^{-1}AQ = J$. The diagonal entries of J are the eigenvalues of A, counting multiplicities.

Let J be the Jordan normal form of our transition matrix $\mathbf{P}$, and let J_i be its Jordan blocks. Note that powers of Jordan blocks are easy to compute: if m is the dimension of a block J_i, and $k \geq m-1$, then a well-known equation is

$$J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \dots & \binom{k}{m-1}\lambda_i^{k-m+1} \\ & \lambda_i^k & \dots & \binom{k}{m-2}\lambda_i^{k-m+2} \\ & & \dots & \dots \\ & & \dots & \binom{k}{1}\lambda_i^{k-1} \\ & & & \lambda_i^k \end{bmatrix}.$$

In particular, if $|\lambda_i| < 1$, note that all terms in J_i^k go to zero as k goes to infinity. Because $\binom{a}{b} \leq \frac{a^b}{b!}$, for fixed $0 \leq r \leq m-1$,

$$\binom{k}{r}\lambda_i^{k-m+r} \leq \frac{k^r}{r!}\lambda_i^{k-m+r} = O(k^r \cdot \lambda_i^k)$$

which clearly goes to zero.

For the remainder of this section, we shift our notation so that p is now a column vector. Then a transition under $\mathbf{P}$ is now $\mathbf{P}p$. Given a starting distribution p, the distribution over states after n time steps is given by

$$\mathbf{P}^n p = QJ^nQ^{-1}p$$

Recall that in Theorem 2.4 we showed that the eigenvalues of $\mathbf{P}$ satisfy $|\lambda_i| < 1$ except for $\lambda_i = 1, -1$. Since all Jordan blocks with $|\lambda_i| < 1$ go to zero, only the Jordan blocks corresponding to $\lambda_i = 1, -1$ affect the long-term behavior of $\mathbf{P}^n p$.

Definition 2.6. A vector v is a **generalized eigenvector of rank** m of a matrix A with eigenvalue λ if $(A - \lambda I)^m v = 0$ but $(A - \lambda I)^{m-1} v \neq 0$.

It is another well-known fact that the columns of Q are exactly the generalized eigenvectors of $\mathbf{P}$. In other words, the generalized eigenvectors of $\mathbf{P}$ form a basis, so that we can write each starting distribution p as a linear combination of the columns q_i of Q, $p = \sum_{i=1}^{S} c_i q_i$. It follows that

$$\mathbf{P}^n p = \sum_{i=1}^{S} c_i \mathbf{P}^n q_i = C\mathbf{P}^n Q = CQJ^n, \tag{2.6}$$

where C is the diagonal matrix with $c_1, \dots, c_N$. This brings us to the following results:

Theorem 2.7. *If $k = 1$ and N is even, then the asymmetric process tends to a periodic sequence.*

Proof. From Theorem 2.4, we know that $\mathbf{P}$ has exactly two eigenvalues and eigenvectors with $|\lambda| = 1$. Therefore, the Jordan blocks J_1, J_2 have dimension 1, and Q has columns $q_1 = (1, \dots, 1)$ corresponding to $\lambda = 1$, and $q_2 = (1, -1, \dots, -1)$ corresponding to $\lambda = -1$. From Eq. 2.6, we have

$$\mathbf{P}^n p = CQJ^n \approx c_1 q_1 (1)^n + c_2 q_2 (-1)^n$$

and therefore $\mathbf{P}^n p$ tends to a periodic sequence. $\square$

Theorem 2.8. *If N is odd or $k \neq 1$, then the asymmetric process converges to the stationary distribution.*

Proof. As above, from Theorem 2.4, we know that $\mathbf{P}$ has exactly one eigenvalue and eigenvector with $|\lambda| = 1$. Hence from Eq. 2.6, we have

$$\mathbf{P}^n p = CQJ^n \approx c_1 q_1$$

so $\mathbf{P}^n p$ tends to the stationary distribution $c_1 q_1$. $\square$

Example 2.9. To illustrate Theorem 2.7, consider the case where $N = 4$, $k = 1$, and $q = 0.4$. We can check (e.g. from Wolfram Alpha) that in this case the eigenvalues are $1, -1, \frac{i}{5}, -\frac{i}{5}$, corresponding to the eigenvectors $(1, 1, 1, 1), (1, -1, 1, -1), (-i, -1, i, 1)$, and $(i, -1, -i, 1)$. If our starting distribution is $(1, 0, 0, 0)$, we can write it as

$$\begin{aligned}(1, 0, 0, 0) = &\frac{1}{4} \cdot (1, 1, 1, 1) + \frac{1}{4} \cdot (1, -1, 1, -1) + \\ &\frac{i}{4} \cdot (-i, -1, i, 1) - \frac{i}{4} \cdot (i, -1, -i, 1)\end{aligned}$$

It follows that the asymmetric process tends to

$$\frac{1}{4} \cdot (1, 1, 1, 1) + \frac{1}{4} \cdot (1, -1, 1, -1)(-1)^n,$$

which means that it oscillates between the following two distributions:

$$(.5, 0, .5, 0), \;\; (0, .5, 0, .5).$$

2.3. **Time to equilibrium on the closed loop.** In the previous section, we established the long-term behavior of the asymmetric process on closed loops. Here, we look into the question of *how quickly* the process converges, either to a periodic sequence (for $k = 1$ and N even), or the stationary distribution (for all other cases).

Recall Eq. 2.6, which says that $\mathbf{P}^n p = CQJ^n$. As in the previous section, every eigenvalue λ_i with absolute value less than 1 disappears, taking with it its Jordan block. It follows that the rate at which J^n converges depends on the largest eigenvalue such that $|\lambda_i| < 1$: the smaller it is, the quicker $\mathbf{P}^n p$ converges.

Example 2.10. Figures 4 and 5 illustrate the largest magnitude of the eigenvalues of the transition matrix such that $|\lambda| < 1$, for the simple case with $k = 1$ particle. Call this eigenvalue λ_{crit}. Note the following:

- For fixed q, $|\lambda_{crit}|$ increases as N increases. As the number of positions N increases, it takes longer for the asymmetric process to converge to the uniform distribution. Furthermore, the pattern with which $|\lambda_{crit}|$ increases seems to depend on the parity of N. The blue and orange lines in Figure 4 illustrate this.

- For fixed N, the smallest $|\lambda_{crit}|$ occurs when $q = 0.5$, which is when clockwise and counter-clockwise movements are equally likely. When q is biased in one direction, $|\lambda_{crit}|$ is higher.

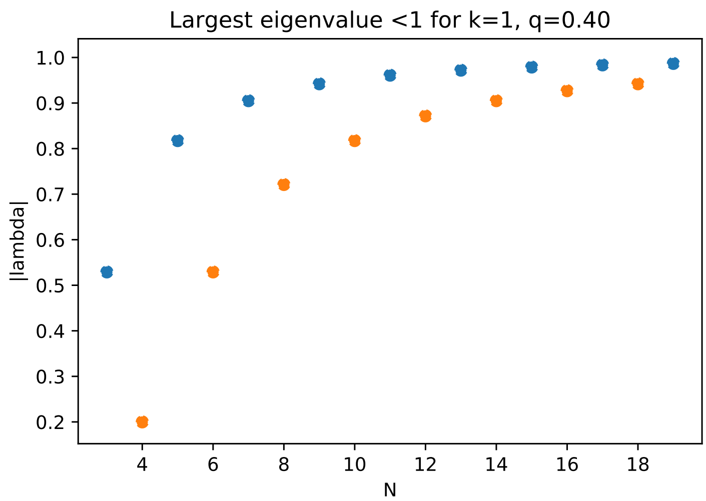


FIGURE 4. Magnitude of the largest eigenvector less than 1 as N varies, for the case with $k = 1$ particle and asymmetric parameter $q = 0.4$. The blue dots are for odd N, and the orange dots are for even N.

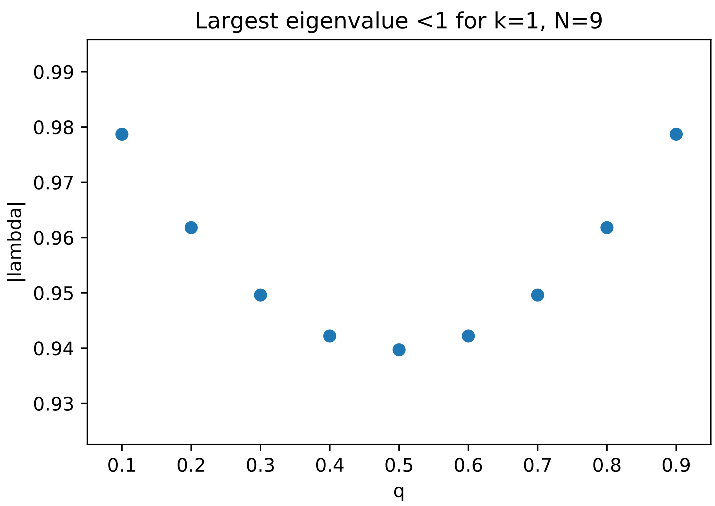


FIGURE 5. Magnitude of the largest eigenvector less than 1 as q varies, for the case with $k = 1$ particle and $N = 10$.

Example 2.11. Here we compute $|\lambda_{crit}|$ for $k = 1$ particles on $N = 3$ positions. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0 & q & 1-q \\ 1-q & 0 & q \\ q & 1-q & 0 \end{bmatrix}$$

The eigenvalues of $\mathbf{P}$ are the roots of the polynomial

$$\lambda^3 - q^3 - (1-q)^3 - 3\lambda q(1-q) = 0 \iff \lambda^3 - 1 = (\lambda - 1) \cdot 3q(1-q)$$

$$\iff \lambda^2 + \lambda + 1 = 3q(1-q) \text{ if } \lambda \neq 1$$

The roots of this quadratic equation are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}(2q-1)i$, so here $|\lambda_2|^2 = \frac{1}{4}(1 + 3(2q-1)^2) < 1$. The minimum possible $|\lambda_{crit}|$ is attained at $q = \frac{1}{2}$, where $|\lambda_2| = \frac{1}{2}$.

3. Open Case

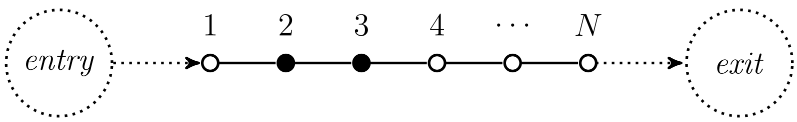


FIGURE 6. Open Markov chain with N nodes, and particles currently on positions 2 and 3. New particles are added to position 1 using the *entry* and particles are removed from position N using the *exit*.

In the open case, we consider the chain of N positions on a line as shown in Figure 6. Particles can enter on the left and exit on the right, meaning the number of particles on the line is variable and there are 2^N possible states of the Markov chain. As in the closed case, we choose an *asymmetry parameter* $q \in [0, 1]$ but now additionally choose *entry* and *exit* parameters $\alpha, \beta \in [0, 1]$. The process behaves as follows:

(1) Choose with uniform probability $\frac{1}{N+2}$ one of the N positions on the line, *entry*, or *exit*.
   - If one of the N positions on the line (call these "inner nodes") is chosen and there is a particle in that position, move the particle left with probability q and to the right with probability $1-q$, if there is a position there and it is vacant.
     Note: *entry* and *exit* are not positions and only serve as a representation of adding and removing particles from the line.
   - If *entry* is chosen, add a particle to the leftmost position with probability α if it is vacant.
   - If *exit* is chosen, remove the particle at the rightmost position with probability β if it is occupied.

(2) Repeat.

We examine the stationary distribution for this process in relation to q, α, and β, and observe that there exists a stationary distribution for the open Markov chain, it is unique when $q < 1$ and $\alpha, \beta > 0$, and is uniform when $q = \frac{1-(\alpha+\beta)}{2}$ for $\alpha = \beta$. Finally, we conjecture a relationship between the number of states in the stationary distribution with equal probabilities and the values of q, α, β, and N.

3.1. **Stationary Distribution on Open Loop.** To begin, we will examine the case where $N = 3$ and expand from there. In this case, there are 8 possible states:

State 1: ○○○

States 2, 3, 4: ●○○ ○●○ ○○●

States 5, 6, 7: ●●○ ●○● ○●●

State 8: ●●●

Following the rules outlined above we know that State 1 can only go to State 2 or itself, State 2 can go to State 3 or itself, State 3 can go to States 2, 4, 5, or itself, and so on. This is shown explicitly in the state diagram in Figure 7. Subsequently, we derive the transition matrix $\mathbf{P}$ for this outlined Markov chain.

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{Vmatrix} 1-\frac{\alpha}{N+2} & \frac{\alpha}{N+2} & & & & & & \\ & 1-\frac{1-q}{N+2} & \frac{1-q}{N+2} & & & & & \\ & \frac{q}{N+2} & 1-\frac{1+\alpha}{N+2} & \frac{1-q}{N+2} & \frac{\alpha}{N+2} & & & \\ \frac{\beta}{N+2} & & \frac{q}{N+2} & 1-\frac{q+\alpha+\beta}{N+2} & & \frac{\alpha}{N+2} & & \\ & & & & 1-\frac{1-q}{N+2} & \frac{1-q}{N+2} & & \\ & \frac{\beta}{N+2} & & & \frac{q}{N+2} & 1-\frac{1+\beta}{N+2} & \frac{1-q}{N+2} & \\ & & \frac{\beta}{N+2} & & & \frac{q}{N+2} & 1-\frac{q+\alpha+\beta}{N+2} & \frac{\alpha}{N+2} \\ & & & & \frac{\beta}{N+2} & & & 1-\frac{\beta}{N+2} \end{Vmatrix} \end{array}$$

Using the transition matrix we are able to get the probability distribution across states π_k for which state the chain will be in at any time step k.

$$\pi_k = \pi_0 \mathbf{P}^t$$

Where π_0 is the starting distribution and π_k is the distribution at time step k. For example, if we start in State 2, then $\pi_0 = [0, 1, 0, 0, 0, 0, 0, 0]$. After some number of iterations, the distribution will reach an equilibrium if at least on state is *positive recurrent*. This distribution is called the stationary distribution π^*. If all the Markov chain's states are positive recurrent, then the Markov chain is *irreducible* and then there exists a *unique* stationary distribution. Intuitively this means if each state can eventually be reached then there is a unique equilibrium distribution independent of the starting state of the process.

Definition 3.1. A state u is *positive recurrent* if the expected time to return to u is finite. For example if a Markov chain starts in state u, as long as the process returns to s it is positive recurrent regardless of the number of intermediate states the process visits in between.

Definition 3.2. A Markov chain is *irreducible* if from each state u in the set of possible states S, it is possible to transition to all states $v \in S$ including u through a

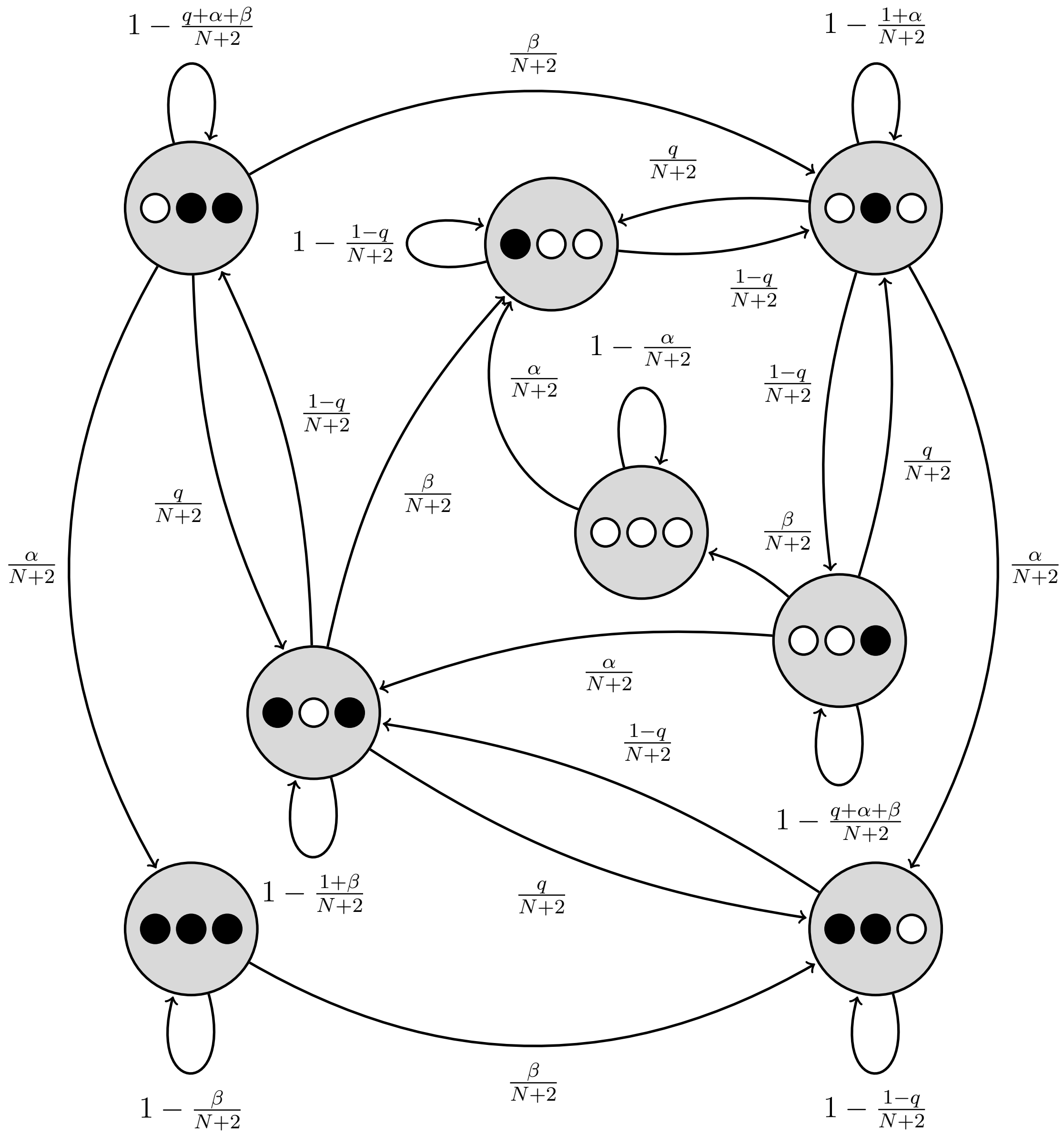


FIGURE 7. State diagram for the open Markov chain with $N = 3$. The state diagram shows each state of the Markov chain as a node, and has directed edges between nodes representing the probability of transitioning from one state to another.

finite number of transitions (i.e. each state is reachable from each state). Explicitly,

$$\mathbb{P}[u \to v] > 0 \qquad\qquad \forall u, v \in S$$

Regardless of the number of states visited between u and v. This definition follows from Definition 3.1: if all states of a Markov chain are positive recurrent, then the Markov chain is irreducible.

Irreducibility can be thought of intuitively as the inability to "shrink" the state space of the Markov chain. For the $N = 3$ example in this section, if the Markov chain only stays in state 1 then the state space can be reduced to just state 1.

The structure of the open Markov chain means that positive reccurrence and irreducibility are satisfied for certain values of q, α, β, leading to Theorems 3.3 and 3.5.

Theorem 3.3. *There exists a stationary distribution π^* for the open Markov chain for all values of q, α, β, and N.*

Proof. Looking at the state diagram in Figure 7 its clear regardless of the values of q, α, and β that there is always at least one state that is positive recurrent because all states have self-loops. Looking at the structure of the Markov chain in Figure 6, there are self-loops on each state for any N. Thus, a stationary distribution exists $\forall N$ which concludes the proof. □

Claim 3.4. The stationary distribution $\pi*$ for $N = 3$ is

$$\pi \propto \begin{bmatrix} \beta^3(q-1)^2 \\ \alpha\beta^2(q^2+2q\alpha+\alpha^2+\alpha\beta+\beta) \\ -\alpha\beta^2(q-1)(q+\alpha+\beta) \\ \alpha\beta^2(q-1)^2 \\ \alpha^2\beta(q^2+2q\beta+\alpha\beta+\alpha+\beta^2) \\ -\alpha^2\beta(q-1)(q+\alpha+\beta) \\ \alpha^2\beta(q-1)^2 \\ \alpha^3(q-1)^2 \end{bmatrix}$$

Proof. A stationary distribution is an eigenvector of the transition matrix. Specifically, it is the eigenvector corresponding to the $\lambda = 1$ eigenvalue of $\mathbf{P}$. □

Theorem 3.5. *There exists a unique stationary distribution $\pi*$ for the Open Markov chain if $\alpha, \beta > 0$ and $q < 1$, for any value of N.*

Proof. For $q = 1$, particles will always move to their leftmost possible position in the Markov chain. Once this configuration is reached there will be no particle movement, thus states in the chain may fail the condition for positive recurrence and the chain as a whole is now reducible.

For $\alpha, \beta = 0$, there is a similar argument. If $\alpha = 0$ then no new particles will enter the chain and eventually all will exit. If $\beta = 0$ then eventually the chain will be completely populated with particles. In either case, the chain fails irreducibility and positive recurrence.

It follows that the converse must be true for the Markov chain to satisfy irreducibility and positive recurrence and these facts are independent the value of N, which concludes the proof. □

Taking it further, we are able to determine the conditions that make this stationary distribution uniform for any length open Markov chain.

Lemma 3.6. *The open loop is analytically equivalent to the closed loop from Section 2 with the same value of q when:*

$$q = \frac{1-(\alpha+\beta)}{2}$$

Subject to the probability constraints $q \in [0, 1)$, $\alpha = \beta \in (0, 1]$ for a unique stationary distribution from Theorem 3.5.

Proof. Recall on the open loop that a particle in position 1 cannot move left and a particle in position N cannot move right. However, a particle in position N can exit the line through the exit node with probability β. That exiting particle can "reenter" the line again through the entry node with probability α. When $\alpha = \beta$ the number of expected number of particles entering and exiting the line at any given time step is equal, mimicking the constant number of particles in the closed loop.

Consider a particle on some node of a line. If $q = \frac{1}{2}$ (i.e. random movement) the particle has equal probability of moving one node to the right or left. However if $q \neq \frac{1}{2}$, then the probability that the particle moves right is not equal to the probability of moving left. Comparing probability of movement right to movement left we get a directional scaling factor to represent the flow of the particle along the line. Call this factor δ:

$$\begin{aligned}\delta_{right} &= (1-q) - q = 1 - 2q\\ \delta_{left} &= q - (1-q) = 2q - 1\\ \delta_{right} &= -\delta_{left}\end{aligned}$$

Intuitively, δ represents the relative, directional proportion of times the particle moves to the right or left. For example if $\delta_{right} > 0$ then flow tends rightwards and if $\delta_{right} < 0$ then flow tends leftwards ($\delta_{right} = 0$ means random movement). The magnitude of δ represents difference between the probabilities of moving in either direction (e.g. $\delta_{right} = 0.5$ means rightward movement is 0.5 more probable than leftward movement).

To make the open loop analogous to the closed loop, we want the connection between the exit and entry nodes in the open loop to have the same δ_{right} as any other edge between inner nodes. Therefore, we define the δ of the exit-entry edge:

$$\begin{aligned}\delta_{right}(exit \to entry) &= \mathbb{P}[\text{particle moves right from node } N \to 1]\\ &\qquad - \mathbb{P}[\text{particle moves left from node } 1 \to N]\\ &= (\beta + \alpha) - 0\end{aligned}$$

And set $\delta_{right}(exit \to entry) = \delta_{right}$:

$$\begin{aligned}\delta_{right}(exit \to entry) &= \delta_{right}\\ \alpha + \beta &= 1 - 2q\end{aligned}$$

Thus the open loop is analogous to a (clockwise-flowing) closed loop with the same value of q when the open loop has:

$$q = \frac{1 - (\alpha + \beta)}{2}$$

□

Theorem 3.7. *The open loop stationary distribution is uniform when:*

$$q = \frac{1 - (\alpha + \beta)}{2}$$

Subject to the probability constraints $q \in [0, 1)$, $\alpha = \beta \in (0, 1]$ *for a unique stationary distribution from Theorem 3.5.*

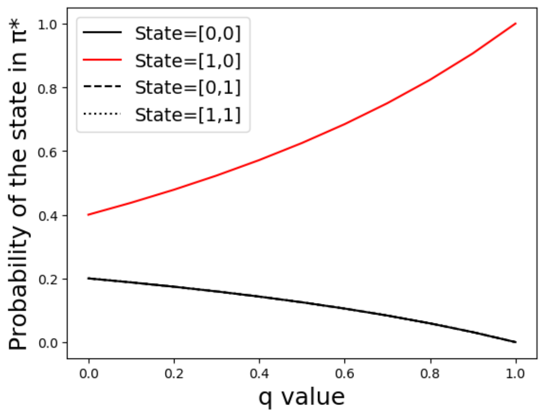
Proof. Following from Lemma 3.6, the open loop can be evaluated as a closed loop. Thus, Theorem 2.3 holds and the stationary distribution is uniform. This can be seen empirically in Figures 8(B) and 9(A). □

Additionally, we can examine how varying q over $[0, 1)$ affects the unique stationary distribution.

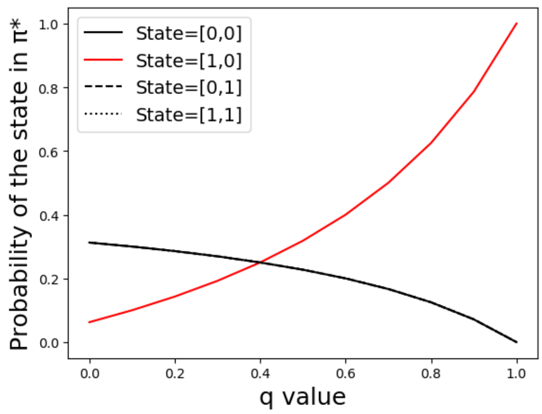
Example 3.8. Consider the cases

$$N = 2 \qquad\qquad N = 3$$

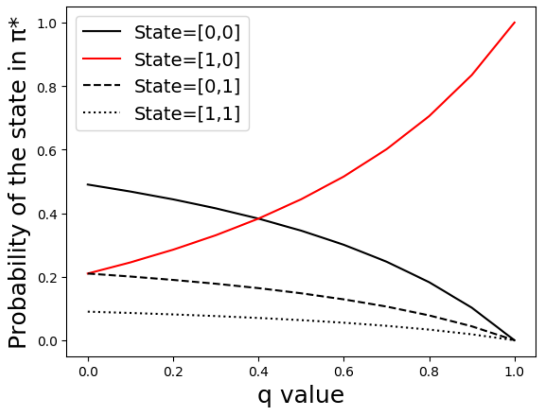
Varying q over $[0, 1)$, we obtain the graphs in Figures 8 & 9 for different relationships between α and β.



(A) $\boldsymbol{\alpha, \beta = 1}$. States $[0, 0]$, $[0, 1]$, and $[1, 1]$ all overlap in the black line.



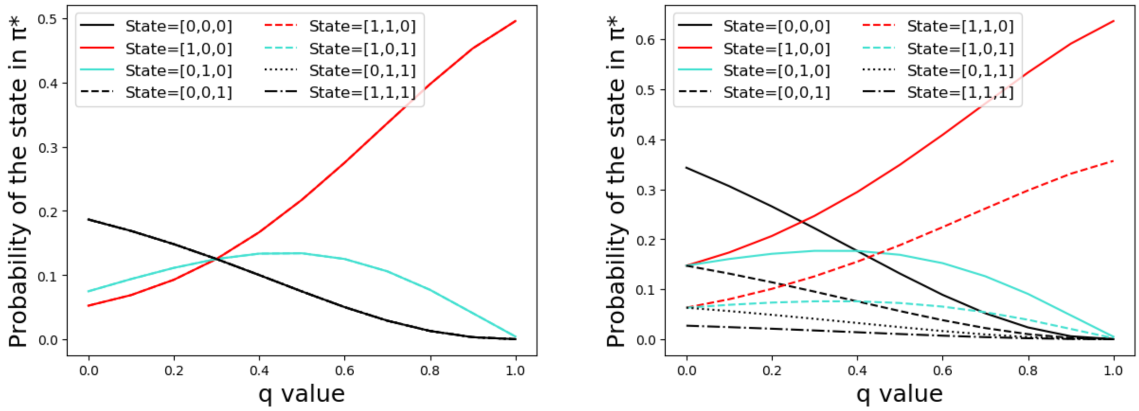
(B) $\boldsymbol{\alpha, \beta = 0.1}$. States $[0, 0]$, $[0, 1]$, and $[1, 1]$ all overlap in the black line.



(C) $\boldsymbol{\alpha = 0.3}$, $\boldsymbol{\beta = 0.7}$. Line colors correspond to the overlapping line in which they appear in charts 8(A) and 8(B).

FIGURE 8. Stationary distribution for N=2 as a function of q. The notation "State $= [1, 0]$" represents the state where position one is occupied and position two is empty, and so on for other states.

From these experimental results, it is clear to see that when $\alpha = \beta$, there is significant overlap between the distribution curves.



(A) $\boldsymbol{\alpha, \beta = 0.2}$. States $[0,0,0]$, $[0,0,1]$, $[0,1,1]$, $[1,1,1]$ overlap on the black line, states $[0,1,0]$, $[1,0,1]$ on the blue line, and states $[1,0,0]$, $[1,1,0]$ on the red line.

(B) $\boldsymbol{\alpha = 0.3}$, $\boldsymbol{\beta = 0.7}$. Line colors correspond to the overlapping line in which they appear in figure 9(A).

FIGURE 9. Stationary distribution for N=3 as a function of q. The notation "State $= [1,0,1]$" represents the state where position one and three are occupied and position two is empty, and so on for other states.

Notice the number of visible lines Figures 8 and 9 when $\alpha = \beta$. We see the number of independent probability groupings is proportional to the value of N, leading to Conjecture 3.10.

3.2. **Equiprobable Groups in the Stationary Distribution.** Within the stationary distribution, there are interesting relationships between the states with the same probability. In this section we conjecture the number of such groups of states called *equiprobable groups* and some of the underlying relationships that causes this. We see that the number of equiprobable groups is on the order of N.

Definition 3.9. A *equiprobable group* Γ is a grouping of states that all have equal probability of occurring. For example, consider some random events A, B, and C.

$$\mathbb{P}[A] = \mathbb{P}[B] = \frac{1}{4}$$
$$\mathbb{P}[C] = \frac{1}{2}$$

In this case A and B are in an equiprobable group, and C is in another.

$$\Gamma_1 = \{A, B\}$$
$$\Gamma_2 = \{C\}$$

Additionally, let the total number of equiprobable groups in a distribution equal γ.

Conjecture 3.10. For the open Markov chain with $\alpha = \beta$ the number of equiprobable groups γ in the stationary distribution π is asymptotically bounded by N, regardless of the value of q.

$$\gamma_{\alpha=\beta} = O(N)$$

If follows that that γ is bounded below by N and above by 2^N for any combination of q, α, β.

$$N \leq \gamma \leq 2^N$$

Proof. Several patterns emerge from empirical analysis that guide our thinking. An interesting observed pattern is that, regardless of N, when we imagine the line "filling" from the right as below in Figure 10 then each of these states are always in the same equiprobable group. This is also always the least probable group for any valid choice of q and $\alpha = \beta$. For this reason, we will call this equiprobable group Γ_0.

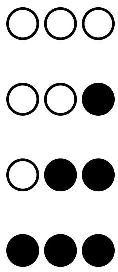


FIGURE 10. Demonstration of "filling" from the right for $N = 3$.

Because Γ_0 has $N + 1$ states, it decreases the maximum number of equiprobable groups from 2^N to $\left[2^N - (N+1)\right] + 1 = 2^N - N$. Finding more relationships like this will further reduce the max number of equiprobable states. Another interesting relationship is that the most probable state is always the state where the first $\lceil \frac{N}{2} \rceil$ nodes are occupied for any valid choice of q and $\alpha = \beta$. Thus, we know there exist relationships between states in equiprobable groups such as in Γ_0, which get more complicated for larger N. While we analytically calculate the stationary distribution for $N = 1, 2, 3$, this proves to be tedious for larger N. Thus, through simulation we get the approximate $\gamma_{\alpha=\beta}$ for different values of N below:

N	$\boldsymbol{\gamma_{\alpha=\beta}}$
1	1
2	2
3	3
4	~ 5
5	~ 6
6	~ 8
7	~ 11

From this, we conjecture that the relationship between N and γ is linear though we do not explicitly prove it. □

There is more work that can be done on proving the number of equiprobable groups but the limiting factor is that the number of states scales exponentially with N. Thus, it is difficult to simulate the process for large N due to computer memory limitations. It is also hard to differentiate equiprobable groups as N gets larger because there are many states and therefore the smallest probabilities in the distribution are on the order of 10^{-N} where error begins to play a larger role. In this case, determining where to distinguish groups becomes a guessing game. Regardless of these limitations, we believe there exists a nearly linear if not linear relationship between γ and N.

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