The Optimal Roulette Betting Strategy

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Abstract

In this paper we explore the optimal betting strategy in the popular casino game of Roulette. Roulette is a betting game where a small ball is rolled into a spinning wheel with numbered slots, and players place bets on where they think the ball will land when the wheel is finished spinning. There are many different types of bets, and this paper discusses the optimal strategy of Kuhn-Tucker gambling applied to Roulette.

1 Introduction

Similar to the horse-race result[1], one might think the optimal betting strategy for Roulette is proportional gambling (i.e. Kelly gambling). However, the caveat to Roulette and most other gambling games is that they have subfair odds (explained in Section 2.3). In this case, the gambler is no longer able to make a dutch book bet which guarantees they will make a profit on a bet [1]. In the case of subfair odds, the optimal strategy is the one which loses money at the slowest rate. Thus, the optimal strategy would be to not bet any money at all and therefore not lose any, but this is not in the spirit of the game. Since we are interested in the best betting strategy when actually making bets, we will show that there is a new optimal betting strategy in the case of subfair odds, Kuhn-Tucker gambling, which takes the place of the Kelly gambling as the optimal strategy for Roulette.

Roulette is a popular casino game where a wheel with numbered and colored pockets on it is spun and a ball is rolled into it. Before the ball has landed in a pocket, players place bets on where the ball will land and get paid if they guess correctly. In the following paper, we will discuss the structure of the game (Section 2: Preliminaries), the optimal gambling strategy (Section 3: Gambling Strategy), and look at how our strategies perform in the long run (Section 4: Experimental Analysis).

2 Preliminaries

2.1 The Game of Roulette

A popular game in any casino across the globe is Roulette. Roulette involves the spinning of wheel (Figure 1) into which a small ball is rolled. While the wheel is spinning, players make bets on which

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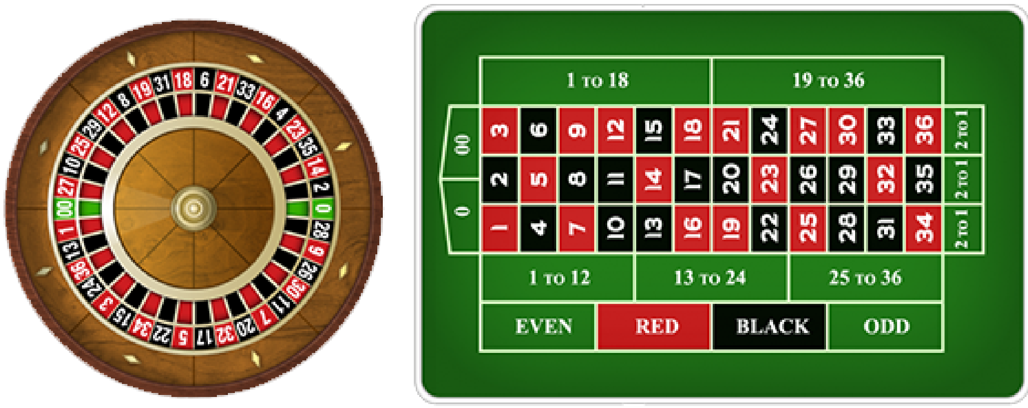

Figure 1: American Roulette Table[3] [3]

pocket of the wheel the ball will stop in.

In this paper, we will focus on the *American* version of the wheel (which, in a capitalistic display, is constructed to favor the casino more than its European counterpart). The American wheel is constructed as follows:

- Pockets numbered $1 - 36$, plus the additional 0 and 00 pockets.
- 18 of the pockets number $1 - 36$ are colored Red, and the remaining 18 are colored Black. The 0 and 00 are colored Green.
- The 0 and 00 are opposite each other on the wheel, and there should be no consecutive numbers or colors.
- Generally, there should be a uniform distribution of high and low, even and odd, red and black pockets across the wheel. However, this is violated several times in the American wheel.

If the ball lands in a pocket that the player bet on, then the player wins a payout specific to that particular bet. If not, then the player loses their money to the casino.

2.2 Types of Bets

Within Roulette, there are 11 different types of bets a player can make. These types of bets, their payouts, and odds [2] are listed in Table 1. For each bet, there is a payout listed in the format $x : 1$, where if you bet \$1 and win, you make a profit of $\$x$ (and now have $\$(x + 1)$). Additionally, the odds of winning each bet are derived from the probability of a bet winning based on the physical structure of the Roulette wheel. The implied odds are derived from the payout, and are what the odds of winning would be in a fair game where payout is proportional to win probability (derived by $1/(payout + 1)$). The rightmost column represents the difference between implied odds and actual odds. Each of the possible bets is described as follows:

Single Number: placing a bet on a single value in the set of pockets $\{00, 0, 1, 2, ..., 36\}$.

Types of Roulette Bets				
Bet	Payout	Odds	Implied Odds	Delta Odds
Single Number	35 : 1	2.6%	2.8%	0.2%
2 Number Combination	17 : 1	5.3%	5.6%	0.3%
3 Number Combination	11 : 1	7.9%	8.3%	0.4%
4 Number Combination	8 : 1	10.5%	11.1%	0.6%
5 Number Combination	6 : 1	13.2%	14.3%	1.1%
6 Number Combination	5 : 1	15.8%	16.7%	0.9%
Column	2 : 1	31.6%	33.3%	1.7%
Dozen	2 : 1	31.6%	33.3%	1.7%
Even/Odd	1 : 1	47.4%	50.0%	2.6%
Red/Black	1 : 1	47.4%	50.0%	2.6%
Low/High	1 : 1	47.4%	50.0%	2.6%

Table 1: Roulette Bets

X **Number Combination**: placing a bet on any X number of pockets in $\{00, 0, 1, 2, ..., 36\}$. For example, this could be splitting a \$100 bet evenly among pockets 1 ,21, and 36 to make a 3 number combination.

Column: placing a bet on $\frac{1}{3}$ of the board (excluding 0 and 00), equating to 12 pockets. This refers to a column on the betting table.

Dozen: similar to the Column bet, but instead betting on one of the 12 pocket intervals $\{[1-12], [13-24], [25-36]\}$. This is taking 3 rows of the betting table opposed to the single column of a Column bet.

Even/Odd: placing a bet on whether the winning pocket is an even or odd number.

Red/Black: placing a bet on whether the winning pocket is red or black.

Low/High: placing a bet on whether the winning pocket is numbered $[1 - 18]$ or $[19 - 36]$.

Additionally, there are other "called" bets that a player can make when playing the game of Roulette (e.g. Neighbors of Zero, The Orphans, etc.). However, we will not discuss these special bets as they are typically exclusive to the European version of the game.

2.3 Subfair Odds

As previously stated, Roulette has subfair odds. This is described in Definition 2.3.

Definition 1 *Let o_i be the odds of bet i winning. A game is said to have subfair odds when the following condition is true:*

$$\sum_i \frac{1}{o_i} > 1 \tag{1}$$

Additionally, a game is said to have fair odds when $\sum_i \frac{1}{o_i} = 1$ and super fair odds when $\sum_i \frac{1}{o_i} <$ 1.

Here, Kelly gambling is the best way to bet all of your money on each spin of the Roulette wheel and lose money at the slowest rate. In Section 3.2, we discuss the optimal strategy to lose money at the slowest rate if we are instead able to only bet a fraction of our money on each spin.

3 Gambling Strategy

3.1 Kelly Gambling

In our casino of interest, let X_k be a random variable representing the winning pocket on the k^{th} spin of the wheel. For ease of discussion, we define the following variables:

$$\begin{aligned} o(i) = o_i &:= \text{odds for a bet } i \\ p(i) = p_i &:= \text{probability of bet } i \text{ winning} \\ b(i) = b_i &:= \text{fraction of the player's bankroll bet on bet } i \end{aligned}$$

By definition, it is implied that $\sum b_i = 1$ and $b_i \geq 0 \; \forall i$. For a player making bets, they are interested in maximizing their wealth. But, in order to measure how well this player is doing, we will first need the definition below.

Definition 2 *Wealth relative is defined as how much wealth a player won or lost on a single bet in relation to their entire bankroll:*

$$S(X_k) = o(X_k)b(X_k) \tag{2}$$

After n spins of the wheel, this equates to:

$$\begin{aligned} S_n &= \prod_{k=1}^{n} S(X_k) \\ &= \prod_{k=1}^{n} o(X_k)b(X_k) \end{aligned} \tag{3}$$

Using Definition 2, we are able to define our true metric of interest to the gambler, the *doubling rate*:

Definition 3 *The doubling rate is the length of time it takes for the player to double their wealth.*

$$\begin{aligned} W(b,p) &= \mathbb{E}[\log_2 S(X)] \\ &= \sum_{i=1}^{m} p_i \log_2(b_i o_i) \end{aligned} \tag{4}$$

Where $m :=$ the size of the set of possible bets.

We know that in order for the gambler to make the most money, they will want to maximize their doubling rate $W(b,p)$. To get a sense of how doubling rate affects wealth growth, we look to Theorem 1.

Theorem 1 *Wealth outcomes grow exponentially at rate[1]:*

$$S_n \doteq 2^{W(b,p)} \tag{5}$$

Proof: *Note that each X_i is independent and each $\log_2 S(X_i)$ is independent. As $n \to \infty$ and applying Law of Large Numbers,*

$$\frac{1}{n} S_n = \frac{1}{n} \sum_{k=1} n \log_2 S(X_k) \tag{6}$$

$$\xrightarrow{n\to\infty} \mathbb{E}[\log_2 S(X)] \tag{7}$$

Rewritten using Definition 3, this says:

$$\frac{1}{n} S_n \to W(b,p)$$

$$S_n \to 2^{nW(b,p)} \tag{8}$$

QED

Continuing with the thought that the gambler wants to maximize their earnings, we attempt to maximize the doubling rate. This is done by betting cleverly as follows to get the optimal doubling rate:

$$W^*(p) = \max_b W(b,p)$$

$$= \max_b \sum_{k=1}^{m} p_k \log_2(b_k o_k) \tag{9}$$

Using a Lagrange Multiplier, we have:

$$J(p) = \sum_{i=1}^{m} p_i \ln(b_i o_i) + \lambda \sum_{i=1}^{m} b_i \tag{10}$$

$$\frac{\partial J}{\partial b_i} = \frac{p_i}{b_i} + \lambda \tag{11}$$

Setting this equal to zero to solve for the maximum, we get:

$$b_i = -\frac{p_i}{\lambda} \tag{12}$$

From the definition of b_i, we know that $\sum b_i = 1$. Substituting this constraint in yields that $\lambda = -1$ and $b_i = p_i$ resulting in the following definition, Definition 4.

Definition 4 *The optimum doubling rate is the maximum doubling rate over all choices of the portfolio b[1]. This is achieved when $b = p$.*

Above, we showed that $b = p$ is a stationary point of the function $J(b)$, meaning the optimum doubling rate is a result of betting proportionally to the probability that a specific bet will win. This is called proportional Gambling or Kelly gambling.

To prove that this is a maximum and not a minimum we use the guess and verify method, which is left as an exercise to the reader.

In fact, Kelly gambling is a log-optimal strategy [1].

Theorem 2 *Kelly gambling is a log-optimal betting strategy. The optimum doubling rate is given by:*

$$W^*(p) = \sum p_i \log_2 o_i - H(p) \tag{13}$$

Where $H(p)$ is the entropy of p. The optimum doubling rate is achieved when $b^* = p$ as in Kelly gambling.

Proof: We rewrite the doubling rate $W(b,p)$ in a form where the maximum is obvious:

$$\begin{aligned} W(b,p) &= \sum p_i \log_2 b_i o_i && (14) \\ &= \sum p_i \log_2 \frac{b_i}{p_i} p_i o_i && (15) \\ &= \sum p_i \log_2 o_i - H(p) - D(p||b) && (16) \\ &\leq \sum p_i \log_2 o_i - H(p) && (17) \end{aligned}$$

Where $D(p||b)$ is the k-l divergence of p and b. To maximize, we get the equality statement in Equation 17 by setting $p = b$ to get $D(p||b) = 0$.

QED

3.2 Kuhn-Tucker Gambling

Kuhn-Tucker gambling is a method derived from the idea that one shouldn't bet their entire bankroll on each spin of the wheel. To do this, we will redefine our portfolio b of bet proportions. Let:

$$\begin{aligned} b_0 &= \text{proportion of bankroll kept "in pocket"} \\ (b_1, ..., b_m) &= \text{proportion of bankroll bet on bets } 1 \to m\text{, i.e. the normal betting practice as before} \end{aligned}$$

Using these new variable definitions, we can redefine wealth relative as the following:

Definition 5 *Wealth relative for Kuhn-Tucker gambling is derived from Definition 2 and is now the following:*

$$S(X_k) = b_0 + o(X_k) b(X_k) \qquad \forall k \geq 1 \tag{18}$$

As our definition of wealth relative has been updated, the doubling rate is also updated:

Definition 6 *Wealth Relative for Kuhn-Tucker gambling is derived from Definition 3 as the following:*

$$\begin{aligned} W(b,p) &= \mathbb{E}[\log_2 S] \\ &= \sum_{i \geq 1} p_i \log_2 (b_0 + b_i o_i) && (19) \end{aligned}$$

As in Kelly gambling, the goal is again to maximize the doubling rate. Through the following algebraic rewriting, we are able to get the doubling rate into a form that is easy to maximize:

$$
\begin{aligned}
W(b,p) &= \sum_{i\geq 1} p_i \log_2(o_i(b_0/o_i + b_i)) \\
&= \sum_{i\geq 1} p_i \log_2(p_i o_i(\frac{b_0/o_i + b_i}{p_i})) \\
&= \sum_{i\geq 1} p_i \log_2 p_i o_i + \log_2 K - D(p||r) \qquad (20)
\end{aligned}
$$

Where $K = b_0(\sum_{i\geq 1} 1/o_i - 1) + 1$, $r = \frac{b_0/o_i + b_i}{K}$, and $D(p||r)$ represents the k-l divergence between p and r. In Equation 20, the way to maximize is by maximizing the value of K and minimizing the value of $D(p||r)$. This analysis ends up breaking down because $b_0^* = 1$ and we are unable to minimize $D(p||r)$. Additionally, this would mean not actually betting which goes against the spirit of this paper.

To account for this, we define the Kuhn-Tucker gambling strategy using the following algorithm:

1. Let the proportion of bankroll bet on $b = (b_1, ..., b_m)$ sum to 1. This means there is no money kept in pocket ($b_0 = 0$) at the start.
2. Sort b_i $\forall i > 0$ in order of decreasing $b_i o_i$. Thus, the m^{th} entry of b should be the smallest $b_i o_i$.
3. Define a new portfolio using the following system:

$$b'_i = b_i - \frac{b_m o_m}{o_i} \qquad \forall i > 0 \qquad (21)$$

   Since $b_i o_i \geq b_m o_m$, we know that $b'_i \geq 0$.
4. Set b_o to the following:

$$
\begin{aligned}
b_0 &= 1 - \sum_{i=1}^{m} b'_i \\
&= \sum_{i=1}^{m} \frac{b_m o_m}{o_i} \qquad (22)
\end{aligned}
$$

As a result of this, our theorem arises:

Theorem 3 *Kuhn-Tucker gambling is the optimal betting strategy for subfair odds games such as Roulette, given we are able to have $b_0 > 0$.*

Proof: Expected return on a bet i in the old portfolio is $b_i o_i$. With the new portfolio b', expected return is:

$$
\begin{aligned}
b_i' o_i =& (b_i - \frac{b_m o_m}{o_i}) o_i & \forall i > 0 \quad (23) \\
=& b_i o_i + b_m o_m (\sum_{i=1}^{m} 1/o_i - 1) & (24) \\
>& b_i o_i & (25)
\end{aligned}
$$

Equation 25 follows by the definition of subfair odds in Equation 1, because we know that the last term in Equation 24 is greater than zero. Thus, the new portfolio b' outperforms the old portfolio b. In the case where portfolio b was set up according to Kelly gambling, our new Kuhn-Tucker portfolio b' will perform better than Kelly Gambling. In Theorem 2, we proved that Kelly gambling is a log-optimal strategy. Thus, it follows that Kuhn-Tucker gambling is also an optimal strategy for subfair odds becuase it strictly improves upon the performance of Kelly gambling.

QED

4 Experimental Analysis

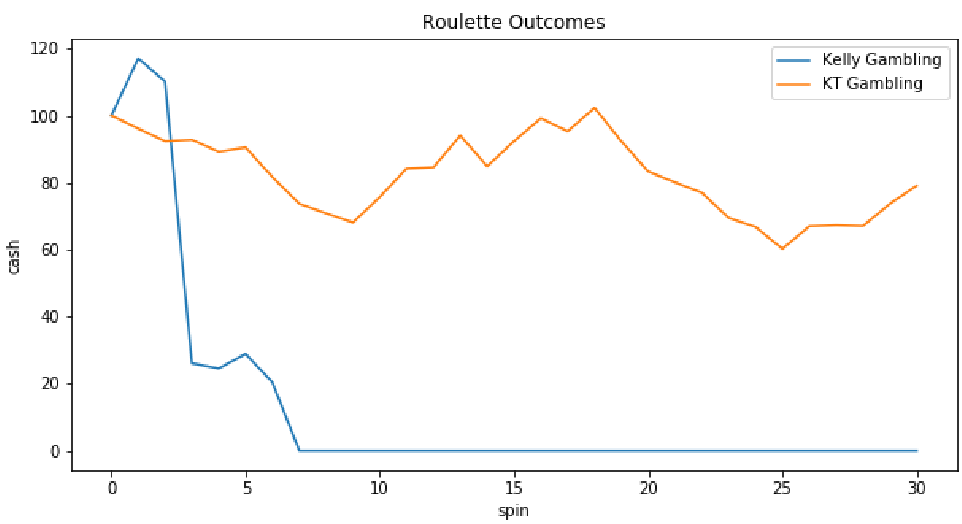

Figure 2: Single game results of employing Kelly gambling versus Kuhn-Tucker gambling

In Section 3, we discussed Kelly gambling and Kuhn-Tucker gambling, and proved that Kuhn-Tucker outperforms Kelly gambling in the case of subfair odds such as in Roulette. In this section, we illustrate how profound these effects are. In Figure 2, we see a dramatic example of how Kuhn-Tucker gambling has a much lower rate of decay. This is further illustrated in Table 2.

Gambling Strategy	Avg. Proportion of Starting Cash	Avg. Rate of Decay
Kelly	0.602	0.034
Delta - Kelly Based	0.624	0.031
Kuhn-Tucker (KT)	0.959	0.004
Delta - KT Based	0.935	0.007

Table 2: Effects on bankroll after $10,000$ games of Roulette, 20 spins of the wheel each

4.1 Delta Gambling

Note: This method is not proven.

When looking at Table 1, one may notice that the delta odds grow as actual and implied odds grow. This is illustrated in Figure 3, where it shows the increase of deltas grows while odds grow. This spurred the idea of accounting for these deltas when constructing a portfolio. The idea is to take our list of delta odds, normalize these to determine which bets are closest to their true odds (call this δ) and then scale b_i's proportional to their δ.

The red bar in Figure 3 is the motivating bet: it stands out in that its delta odds aren't monotonically increasing like the rest of the bets. Thus, it has lower delta odds but higher actual odds, and may be a way to exploit the payout scheme of Roulette.

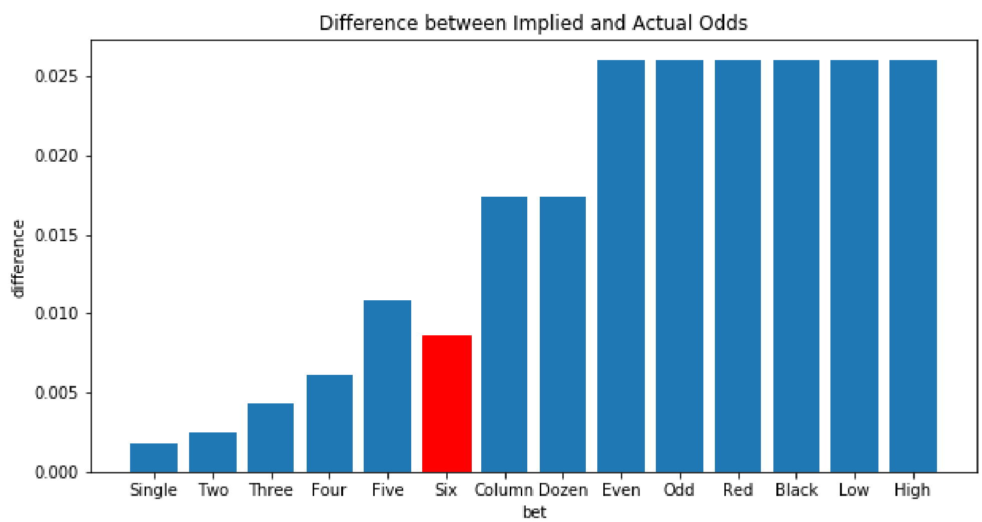

Figure 3: Graph of the deltas between implied odds and actual odds.

The results of the Delta gambling experiment are shown in Table 2. Here, it is shown that it slightly outperforms Kelly gambling when basing it off of a Kelly gambling portfolio, but slightly underperforms relative to Kuhn-Tucker gambling when based off of a Kuhn-Tucker portfolio.

References

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