Week 2 Solutions

Gov January Linear Algebra Review

2021-01-11

1. (Strang 3.1.19) Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

The column space of A is the x-axis or all vectors $(x, 0, 0)$ *or* a line, since the second column is dependent on the first (there's only one truly informative column).

The column space of B is the xy-plane, the possible linear combinations of the two independent column vectors we can produce from row reduction: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The column space of C is the line of vectors $(x, 2x, 0)$. This is apparent if you look at the system of equations generated by $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the possible solutions according to Gauss-Jordan elimination.

2. (Strang 3.1.23) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless _______________. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{x} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $[A \;\; \boldsymbol{b}]$.

The extra column b enlarges the column space unless b is **already in the column space**, or when the column space *doesn't* get larger—it is the same for A and $[A \;\; \boldsymbol{b}]$.

It is solvable because if the column space doesn't get larger, it means that b *is reachable via linear combinations of the column vectors of A.* – ergo, there is a unique solution!

3. (Strang 3.2.22) If $AB = 0$ then the column space of B is contained in the _________ of A. Why?

If A times every column of B is zero – i.e., $A \cdot B_1 = 0, \ldots, A \cdot B_n = 0$ – then the column space of B – $\mathbf{C}(B)$ – is contained in the **nullspace** of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\mathbf{C}(B)$ equals $\mathbf{N}(A)$.

4. (Strang 3.2.39) Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & 9 & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \end{bmatrix}$$

Things that are true if these matrices have rank 1:

- There are no independent columns *or* rows.
- There is only one pivot after Gauss-Jordan elimination.

How do you make a matrix with rank 1? You could place variables, e.g. a, b, in the blank spaces and then perform Gauss-Jordan elimination and then determine what values would make the matrix only have one pivot. Or you could "eyeball" it.

For A (1) the first column is the same as the first row and (2) the second and third entries are multiple of the first, so we can make each row/column the same multiple of the first row/column.

For B, the third row is complete – if we just fill in the second row to be a multiple of this third row and then be careful to make sure our fill-ins for the first row *do not* produce multiples of the other rows and columns, we're golden.

For M, we could turn the second row in a c/a multiple of the first row – the first entry in the second row is already a c/a multiple of a, and $\mathbf{bc/a}$ makes the second entry a multiple.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{2} & 9 & \mathbf{-3} \\ 1 & \mathbf{3} & \mathbf{-3/2} \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \mathbf{bc/a} \end{bmatrix}$$

5. (Strang 3.3.19) Find the rank of A and also the rank of $A^T A$ and also the rank of AA^T:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Both matrices A have rank 2: an easy way to tell is there are 2 pivots in each (see Strang page 155 for a nice summary).

$A^T A$ and AA^T will always have the same rank as A. Intuitively, this is because we can't "create more unique data" by just multiplying a matrix by itself.

Technically, we can see this by writing out the individual columns of the result of $A^T A$ for any matrix A:

$$A^TA = \begin{bmatrix} A^T \underbrace{a_1'}_{\text{first column of A}} & \cdots & A^T a_n' \end{bmatrix}$$

Each column is of the form $A^T a_i'$, which by definition must lie in the column space of A^T. Therefore, all columns of A^TA must be in the column space of A^T; since they share the same column space, they share the same rank. The same argument applies to AA^T.

6. (Strang 3.4.2) Find the largest possible number of independent vectors among:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \boldsymbol{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \boldsymbol{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

v_1, v_2, v_3 are independent. The -1's are in different positions, so they each present unique but "incomplete" ways to navigate R^4, only **spanning** it when we consider them all together (i.e. they form a **basis**). The rest can all be created using different combinations of the first three rows.

7. (Strang 3.4.7) If $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are independent vectors, show that the differences $\boldsymbol{v}_1 = \boldsymbol{w}_2 - \boldsymbol{w}_3$ and $\boldsymbol{v}_2 = \boldsymbol{w}_1 - \boldsymbol{w}_3$ and $\boldsymbol{v}_3 = \boldsymbol{w}_1 - \boldsymbol{w}_2$ are *dependent*. Find a combination of the $\boldsymbol{v}$'s that gives zero.

After some intense eyeballing, you can garner that the sum $v_1 - v_2 + v_3 = 0$. This means that there is a linear combination of these vectors that sums up to 0 where the weights are *not* all zero – recalling one of our definitions of linear independence, they are dependent!