Week 1 Exercise Solutions

Gov January Linear Algebra Review

2021-01-04

1. (Strang 1.1.4) From the vectors, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

   $3v + w = (7, 5)$ and $cv + dw = (2c + d, c + 2d)$.

2. (Strang 1.2.1) Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   $u \cdot v = -2.4 + 2.4 = 0$,
   $u \cdot w = -.6 + 1.6 = 1$,
   $u \cdot (v + w) = u \cdot v + u \cdot w = 0 + 1$,
   $w \cdot v = 4 - 6 = -2 = v \cdot w$.

3. (Strang 1.2.2) Compute the lengths $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, and $\|\boldsymbol{w}\|$ of the vectors in the last problem. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$.

   $||u|| = 1$ and $||v|| = 5$ and $||w|| = \sqrt{5}$. Then $|u \cdot v| = 0 < (1)(5)$ and $|v \cdot w| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

4. (Strang 1.2.19 and 1.2.21) There are two equivalent ways to write the triangle inequality:

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \qquad \Longleftrightarrow \qquad \|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$$

   Use the Schwarz inequality to prove the first of these two inequalities. Hint: use these facts about dot products: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

   First, recall that $||a||^2 = (a \cdot a)^2$.
   Let's look at $||v + w||^2 = ((v + w) \cdot (v + w))$. Note that the following equality holds if we use the given rules in the right way (1.2.19):

$$\begin{aligned}
(v + w) \cdot (v + w) &= (v + w) \cdot v + (v + w) \cdot w && \text{(fact 2)} \\
&= v \cdot (v + w) + w \cdot (v + w) && \text{(fact 1)} \\
&= v \cdot v + v \cdot w + w \cdot v + w \cdot w && \text{(fact 1)} \\
&= v \cdot v + 2v \cdot w + w \cdot w && \text{(fact 2)} \\
&= ||v||^2 + 2v \cdot w + ||w||^2 && \text{(definition of length)}
\end{aligned}$$

Now, let's take for given the Triangle inequality which says that $2v \cdot w \leq 2||v|| \cdot ||w||$ (technically there's an absolute value on the left, but if it's negative, the inequality *definitely* holds true!). We can apply this to the middle term in the right hand side of the above equality to get the answer:

$$\begin{aligned} ||v+w||^2 &= (v+w)\cdot(v+w) \\ &= ||v||^2 + 2v\cdot w + ||w||^2 \\ &\leq ||v||^2 + 2(||v||\cdot||w||) + ||w||^2 && \text{(Triangle Ineq.)} \\ &= (||v|| + ||w||)^2 && \text{(quadratic factorization)} \end{aligned}$$

5. (Strang 1.2.29) If $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$, what are the smallest and largest possible values of $\|\boldsymbol{v} - \boldsymbol{w}\|$? What are the smallest and largest possible values of $\boldsymbol{v} \cdot \boldsymbol{w}$?

For a specific example, pick $v = (1, 2, -3)$ and then $w = (-3, 1, 2)$. In this example, using the **cosine of angle** rule on p. 18, $\cos\theta = v \cdot w/(||v|| \cdot ||w||) = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x + y + z = 0$:

$$v \cdot w = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $v \cdot w = 0 - \frac{1}{2}||v|| \cdot ||w||$. Then $\cos\theta = \frac{1}{2}$.

6. (Strang 2.1.9) Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:

$$\text{(a)} \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) $Ax = (18, 5, 0)$

(b) $Ax = (3, 4, 5, 5)$

7. (Strang 2.1.10) Compute each $A\boldsymbol{x}$ from the previous question as a combination of the columns of A.

Multiplying as linear combinations of the columns gives the same $Ax = (18, 5, 0)$ and $(3, 4, 5, 5)$. By rows or by columns: 9 separate multiplications when A is 3 by 3.

8. (Strang 2.2.1-2) Use elimination to solve the following system of linear equations:

$$2x + 3y = 1,$$
$$10x + 9y = 11.$$

If the right side changed to $(4, 44)$ what would the answer be?

Multiply row 1 by 5 and subtract from row 2 to get row 1 of the augmented matrix as $[1 \mid 6]$. The pivots to circle are 2 and -6. And the solution is $(2, -1)$.

Multiplying the RHS by 4 would multiply the solution also by 4 to produce $(8, -4)$.

9. (Strang 2.3.24) Apply elimination to the 2 by 3 augmented matrix $[A \quad \boldsymbol{b}]$. What is the triangular system $U\boldsymbol{x} = \boldsymbol{c}$? What is the solution $\boldsymbol{x}$?

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

The upper triangular matrix is

$$\begin{bmatrix} 2 & 3 & | & 1 \\ 0 & -5 & | & 15 \end{bmatrix}$$

We can either further reduce or perform back substition to get $x_1 = 5$ and $x_2 = -3$.

10. (Strang 2.4.1) Let A be 3 by 5, B be 5 by 3, C be 5 by 1, and D be 3 by 1. All entries of all of these matrices are 1. Which of the following matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DC \qquad A(B+C)$$

BA is 5×5. AB is 3×3. ABD is 3×1. DC and $A(B + C)$ are not defined.

11. (Strang 2.5.12) If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

$A^{-1} = BC^{-1}$ (see review slides for steps).

12. (Strang 2.5.13) If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

$M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply the left by C and the right by A. We get $B^{-1} = CM^{-1}A$.

13. (Strang 2.7.39, Challenge) Suppose Q^T equals Q^{-1} (transpose equals inverse, so $Q^TQ = I$).

(a) Show that the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are the unit vectors: $\|\boldsymbol{q}_i\|^2 = 1$.
(b) Show that every two different columns of Q are perpendicular: $\boldsymbol{q}_1^T \boldsymbol{q}_2 = 0$.

Start from $Q^TQ = I$ as in $\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} \cdot \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) The diagonal entries give $q_1^Tq_2 = 1$ and $q_2^Tq_2 = 1$ (unit vectors).

(b) The off-diagonal entry is $q_1^T q_2 = 0$ (and in general $q_i^T q_j = 0$)

(c) The leading example for Q is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.