Week 1 Exercises

Gov January Linear Algebra Review

2021-01-04

1. (Strang 1.1.4) From the vectors, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

2. (Strang 1.2.1) Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:
$$\boldsymbol{u} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3. (Strang 1.2.2) Compute the lengths $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, and $\|\boldsymbol{w}\|$ of the vectors in the last problem. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$.

4. (Strang 1.2.19 and 1.2.21) There are two equivalent ways to write the triangle inequality:
$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2 \qquad \Longleftrightarrow \qquad \|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$$
Use the Schwarz inequality to prove the first of these two inequalities. Hint: use these facts about dot products: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

5. (Strang 1.2.29) If $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$, what are the smallest and largest possible values of $\|\boldsymbol{v} - \boldsymbol{w}\|$? What are the smallest and largest possible values of $\boldsymbol{v} \cdot \boldsymbol{w}$?

6. (Strang 2.1.9) Compute each $A\boldsymbol{x}$ by dot products of the rows with the column vector:
$$\text{(a)} \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

7. (Strang 2.1.10) Compute each $A\boldsymbol{x}$ from the previous question as a combination of the columns of A.

8. (Strang 2.2.1-2) Use elimination to solve the following system of linear equations:
$$2x + 3y = 1,$$
$$10x + 9y = 11.$$
If the right side changed to $(4, 44)$ what would the answer be?

9. (Strang 2.3.24) Apply elimination to the 2 by 3 augmented matrix $[A \quad \boldsymbol{b}]$. What is the triangular system $U\boldsymbol{x} = \boldsymbol{c}$? What is the solution $\boldsymbol{x}$?

$$A\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

10. (Strang 2.4.1) Let A be 3 by 5, B be 5 by 3, C be 5 by 1, and D be 3 by 1. All entries of all of these matrices are 1. Which of the following matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DC \qquad A(B+C)$$

11. (Strang 2.5.12) If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B.

12. (Strang 2.5.13) If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C.) Find a formula for B^{-1} that involves M^{-1} and A and C.

13. (Strang 2.7.39, Challenge) Suppose Q^T equals Q^{-1} (transpose equals inverse, so $Q^T Q = I$).

(a) Show that the columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are the unit vectors: $\|\boldsymbol{q}_i\|^2 = 1$.
(b) Show that every two different columns of Q are perpendicular: $\boldsymbol{q}_1^T \boldsymbol{q}_2 = 0$.