Week 3 Review Session

Gov January Linear Algebra Review
Soubhik Barari
Harvard University

Jan 25, 2021

Plan

- Review of weeks 1 and 2
- Useful matrix utilities in `R`
- The **most** important concept in week 3
  - Problem 3 (4.2.13)
  - Problem 6 (4.2.23)
- Looking ahead to the semester

Review of weeks 1 and 2

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- Inversion: solving for $\mathbf{A}$'s inverse to solve for x in $\mathbf{A}x = b$ as $x = \mathbf{A}^{-1}b$.
  - ex: inverting voters' positions by performing 'double elimination' on augmented matrix $\begin{bmatrix}\mathbf{A} & | & \mathbf{I}\end{bmatrix}$.

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- $\mathbf{A}$ is full rank if it achieves the greatest possible number of linearly independent rows and columns.
  - $\mathbf{A}$ is square and full rank $\rightsquigarrow$ all rows & columns independent.
  - $\mathbf{A}$ is wide and full rank $\rightsquigarrow$ all rows independent.
  - $\mathbf{A}$ is long and full rank $\rightsquigarrow$ all columns independent.

Useful: If $\mathbf{A}$ is square, we <u>know</u> it's invertible if it's full rank.

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Three convenient ways of making our running matrix **A**:

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How to linear algebra in `R`

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t(A)
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```
##      [,1] [,2] [,3]
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colMeans(A)
```

```
## [1] -1.666667  2.000000  2.333333
```

How to linear algebra in R

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```r
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```
##      [,1]
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## [3,]    7
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Solving for a system of equations:

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Solving for a system of equations:

```
b <- c(23,11,7)
solve(A, b)
```

```
## [1] 1 3 4
```

Quick `R` exercise

Find the solutions to these exercises from previous weeks using `R`:

1. Exercise 1 in Problem Set 1 (`1.2.1`): dot products
2. Exercise 8 in Problem Set 1 (`2.2.1-2`): elimination
3. Exercise 5 in Problem Set 2 (`3.3.19`): rank
   - hint: the function to find rank is highly google-able!

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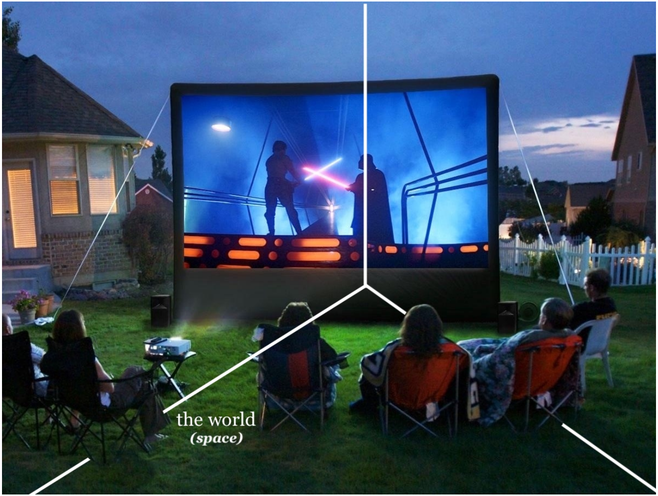

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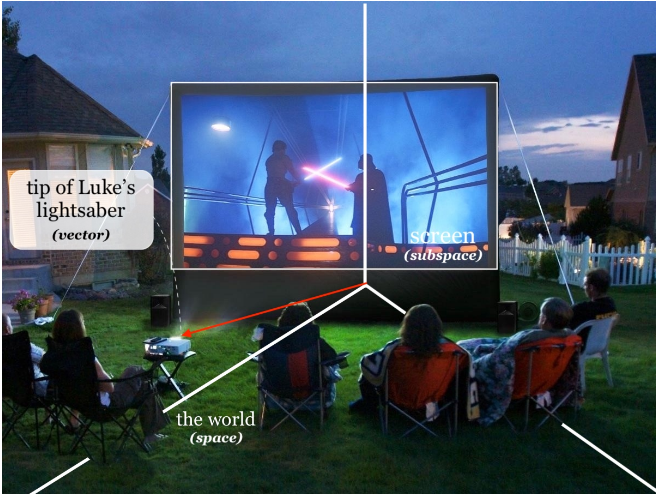

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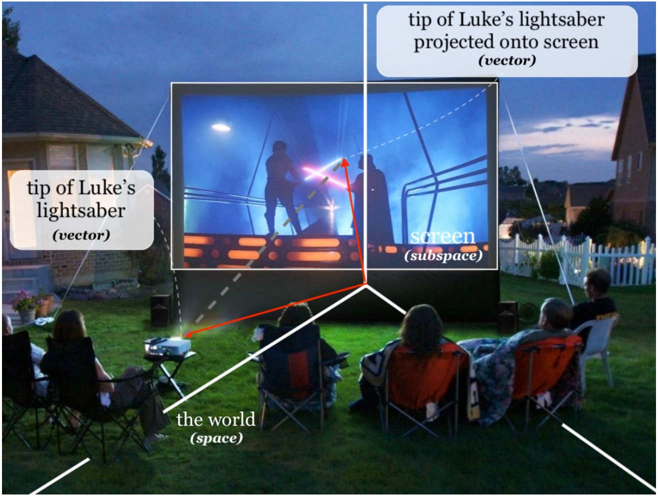

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(1) Here, $\boldsymbol{V}$ is the horizontal plane in $\mathbb{R}^3$, so $p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b$

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- ex: a and b are two voters in a 2-d ideological space.

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(1) Here, $\boldsymbol{V}$ is the horizontal plane in $\mathbb{R}^3$, so $p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b$

(2) If $\boldsymbol{V}$ is the line formed by vector $\boldsymbol{a}$ in $\mathbb{R}^2$, $p = a\frac{a^T b}{a^T a}$.

- ex: a and b are two voters in a 2-d ideological space.

(3) If $\boldsymbol{V}$ is the column space of matrix $\mathbf{A}$, $p = \underbrace{\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T}_{\text{projection matrix}} b$

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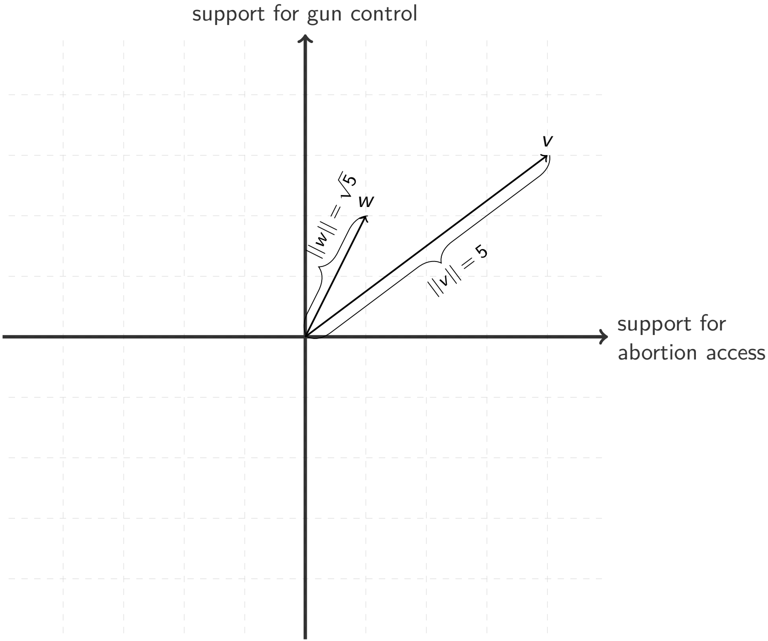
- ex: $\mathbf{A}$ is a stack of voters' positions on 3 issues, b is their single-dimensional ideology.

Projecting onto a line in $\mathbb{R}^2$ (example from week 1)



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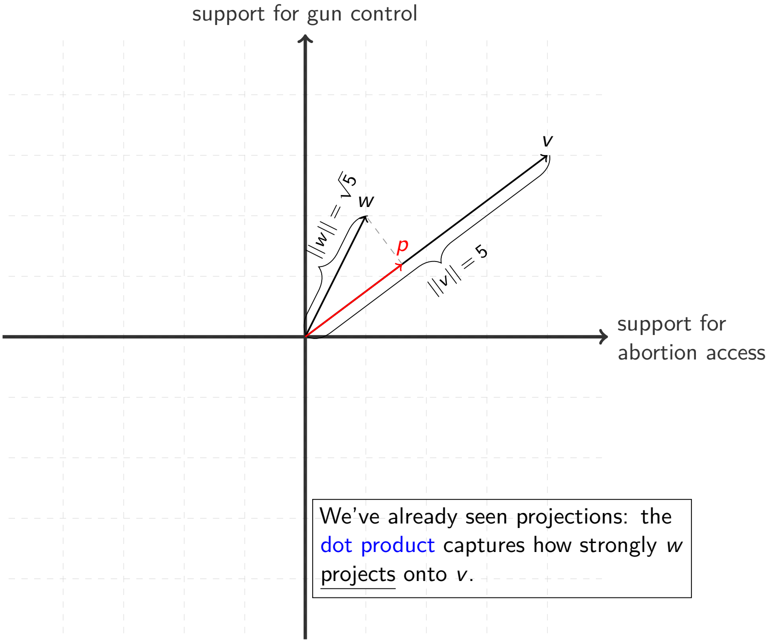

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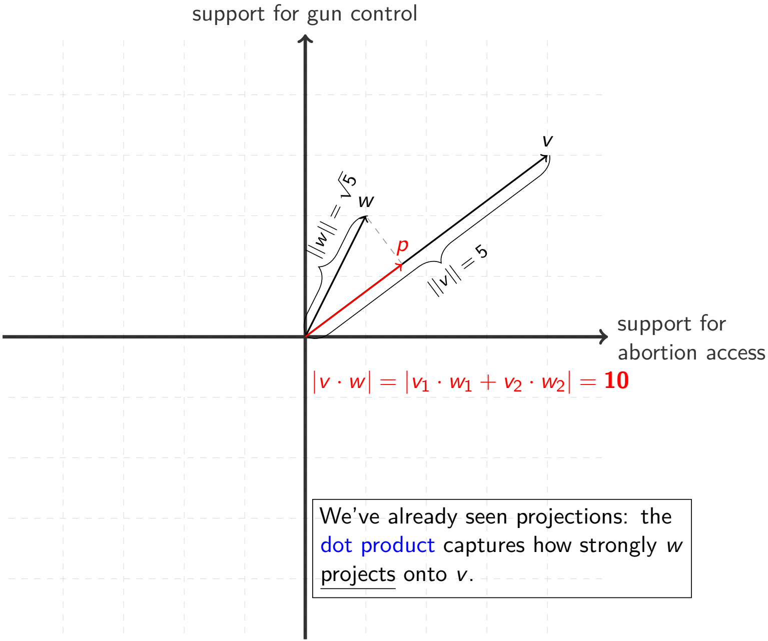

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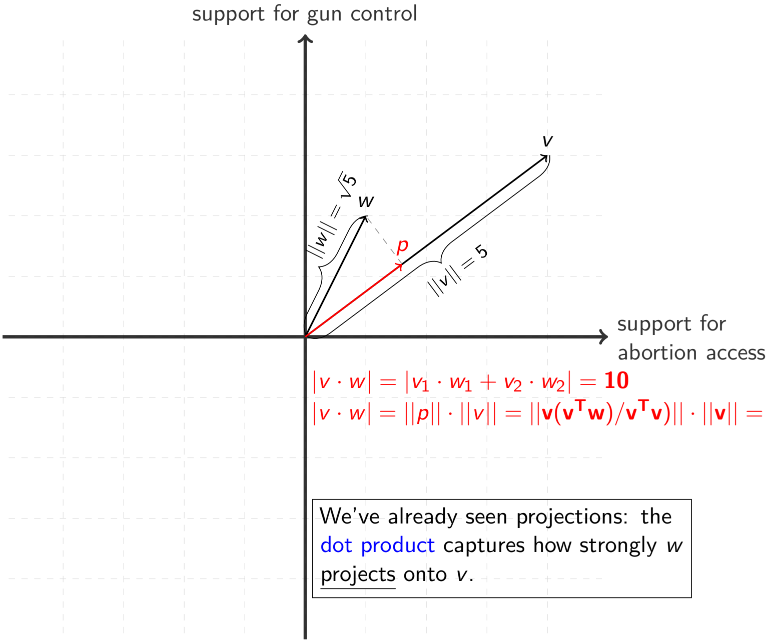

$|v \cdot w| = |v_1 \cdot w_1 + v_2 \cdot w_2| = \mathbf{10}$

$|v \cdot w| = ||p|| \cdot ||v|| = ||\mathbf{v}(\mathbf{v}^\mathbf{T}\mathbf{w})/\mathbf{v}^\mathbf{T}\mathbf{v})|| \cdot ||\mathbf{v}|| = \mathbf{10}$

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Let's find the projection matrix and try out the projection in `R`.

Hint: to create an identity matrix, look up `diag`.

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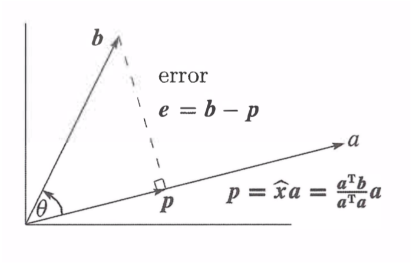
For projection $\boldsymbol{p}$ of $\boldsymbol{b}$ on $\boldsymbol{a}$ where $\boldsymbol{p} = \widehat{\boldsymbol{x}}\boldsymbol{a} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a}$, the error vector is $\boldsymbol{e} = \boldsymbol{p} - \boldsymbol{b}$.



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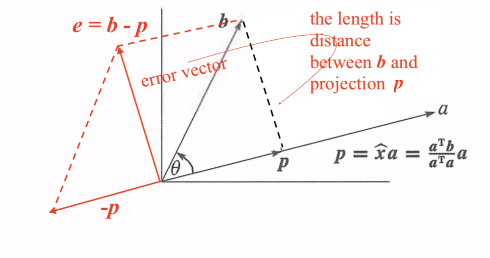

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  - This is fine since we can <u>always</u> project b onto column space of $\mathbf{A} \rightsquigarrow$ finds the combination $p = \mathbf{A}\widehat{x}$ <u>closest</u> to a given vector b.