Week 1 Review Session

Gov January Linear Algebra Review
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Harvard University

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Broad Overview

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Why should we care? *Every* statistical method in social science explicitly or implicitly relies on this accounting system.

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- Summarising and combining vectors
  - Length (Schwarz inequality, Triangle inequality)
  - Dot products
- Gaussian elimination
- Gauss-Jordan elimination
- Matrix inversion

Plan

Cover four problems $\rightsquigarrow$ discuss intuition and connect them to a real application in political science.

- Problem 3 (1.2.2): summarising, combining vectors
- Problem 6 (2.1.9): transforming matrices
- Problem 9 (2.2.1-2): solving unknowns between matrices
- Problem 11 (2.5.12): inverting matrices

Don't feel self-conscious about interrupting if you're confused or have questions!

Problem 3 (1.2.2): summarising, combining vectors

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The simplest linear algebraic unit is a **vector** in some space (e.g. $\mathbb{R}^2$), where each dimension usually "means something":

$$u = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow \text{support for gun control} \\ \leftarrow \text{support for abortion access} \end{array}$$

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$$||u|| = \sqrt{(-0.6)^2 + (0.8)^2} = 1, \quad ||v|| = 5, \quad ||w|| = \sqrt{5}$$

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But, how does this dot product relate to their individual lengths?

Problem 3 (1.2.2): summarising, combining vectors

According to the **Schwarz inequality**,

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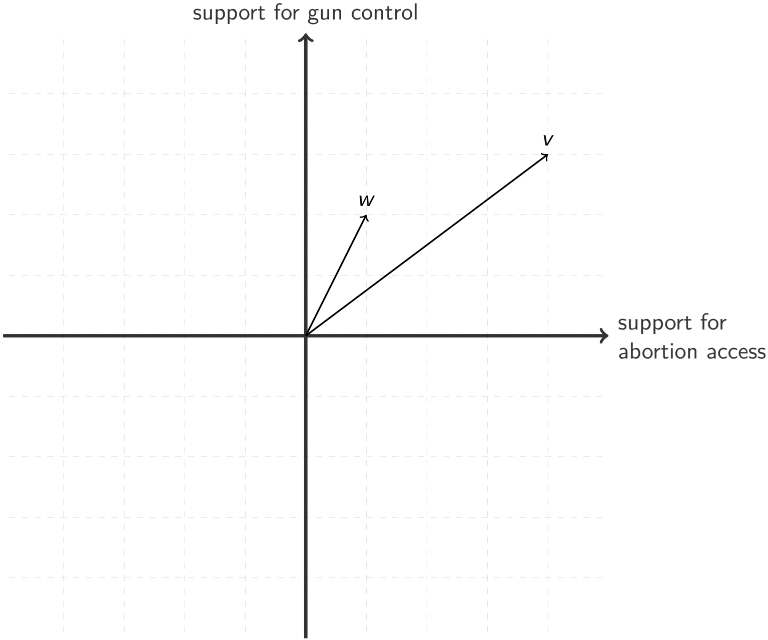
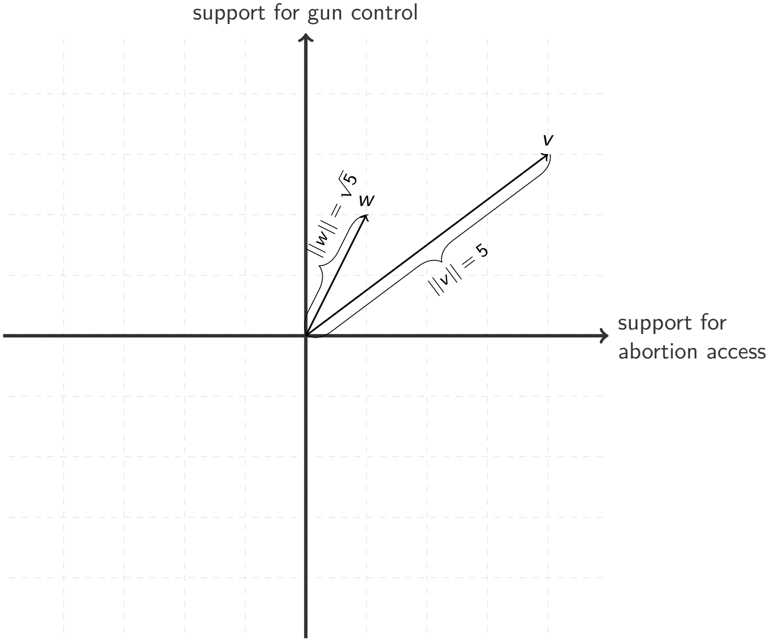
Let's visualize this in our example space.


support for gun control
support for abortion access
w
v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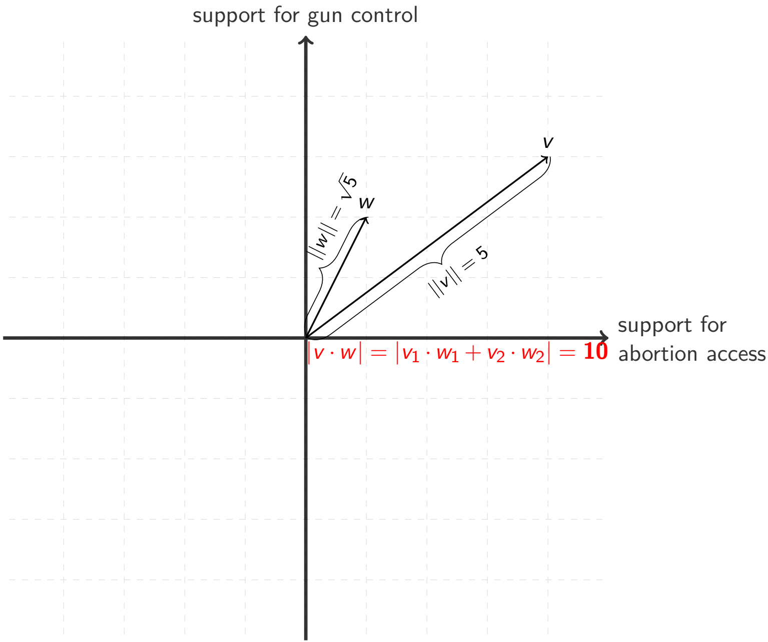
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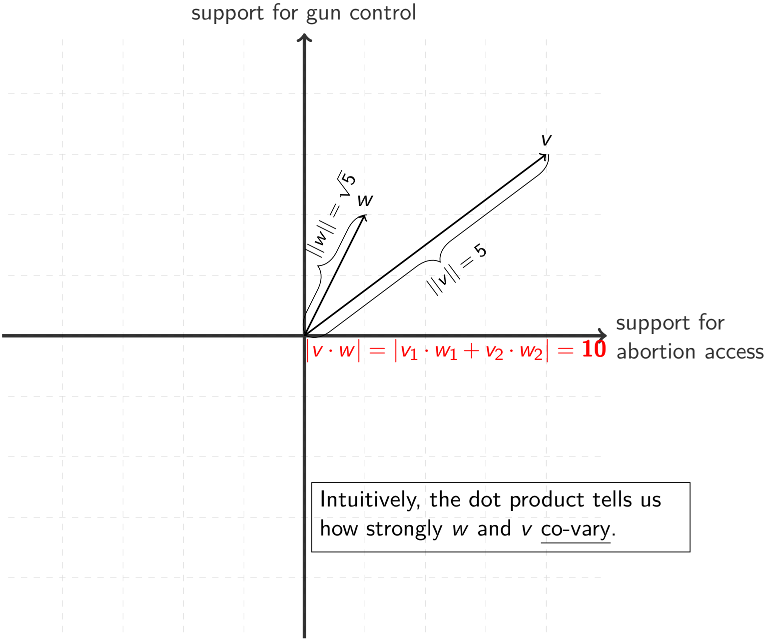
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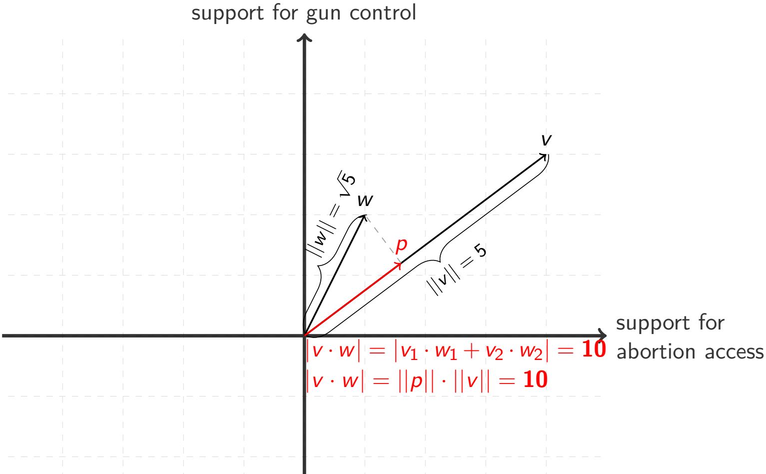

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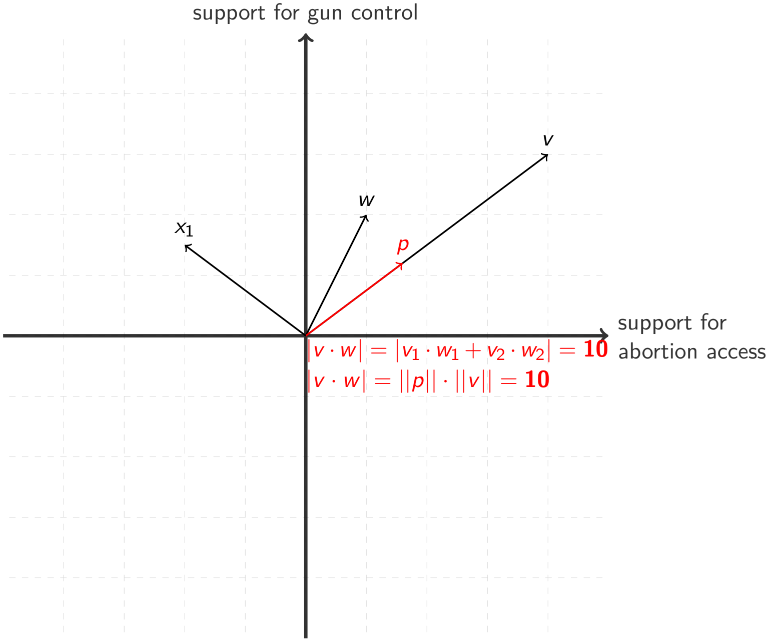
Geometrically, the dot product can be defined as how strongly w projects onto v.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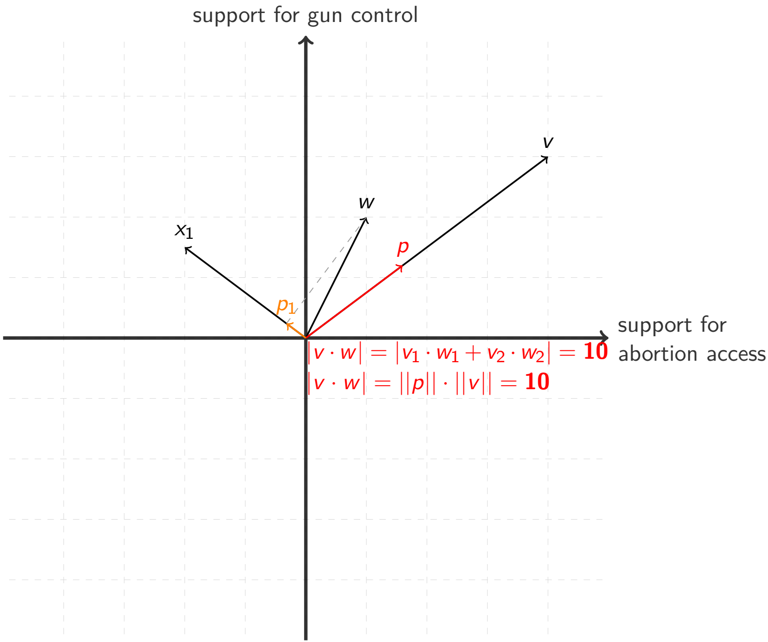
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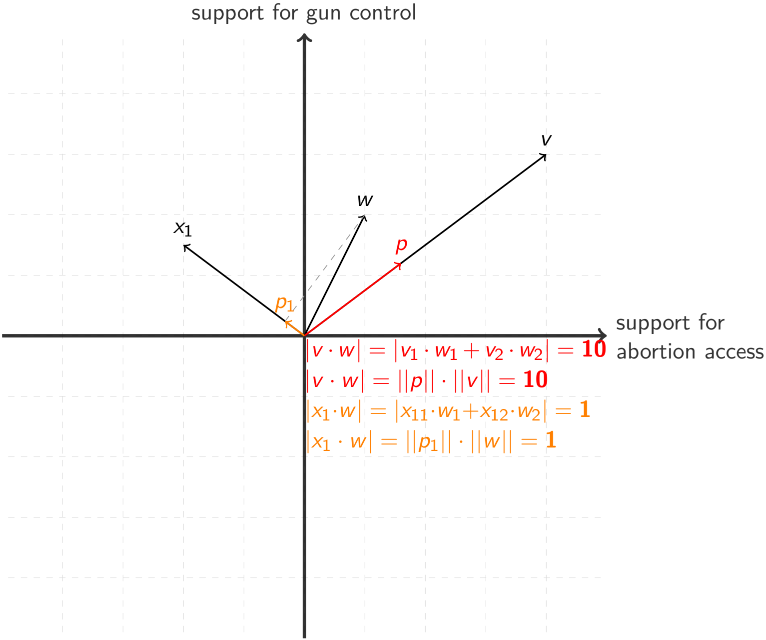
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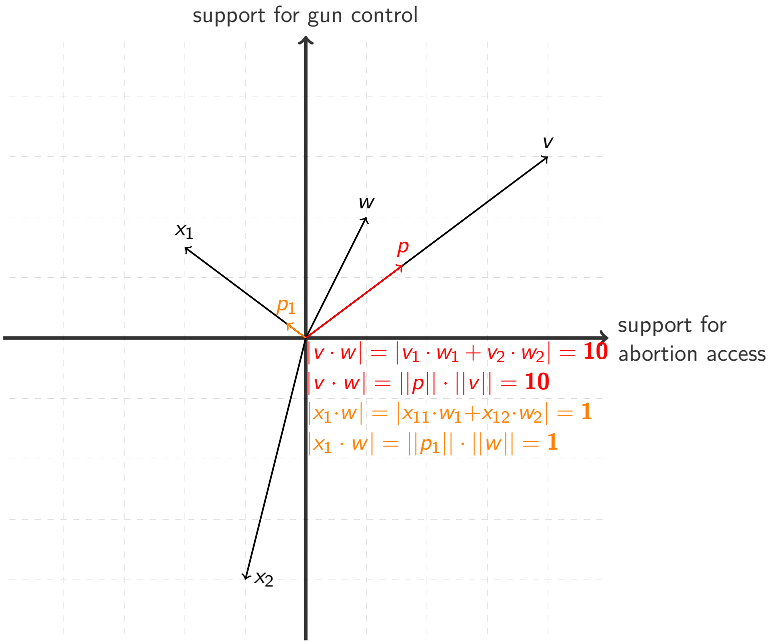
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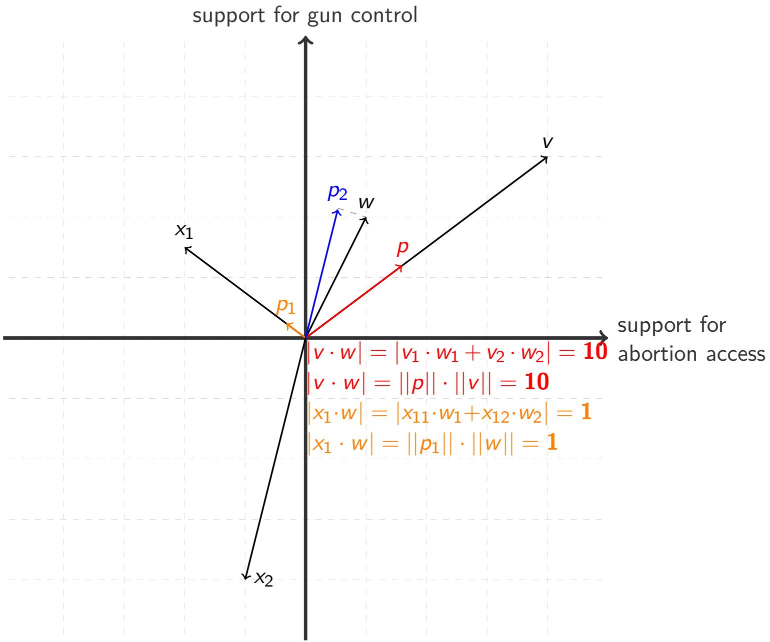
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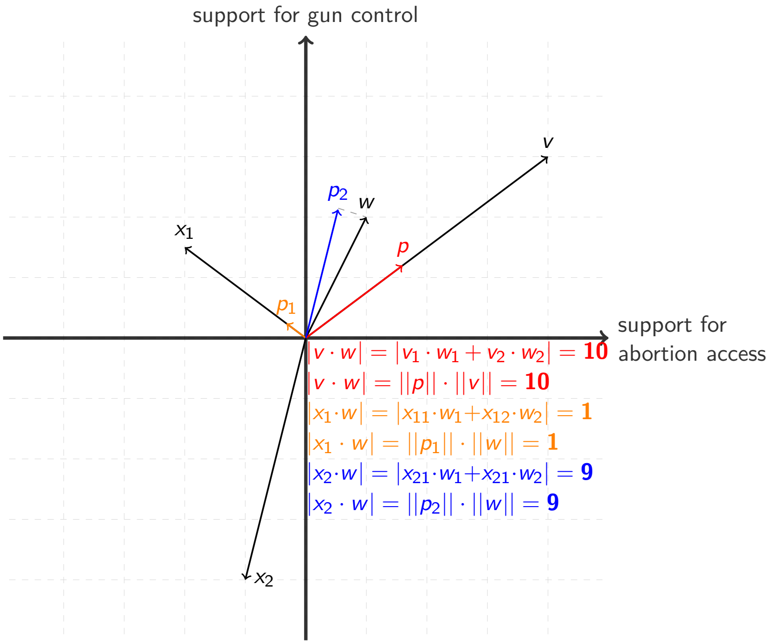

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We can also combine vectors by **addition**:

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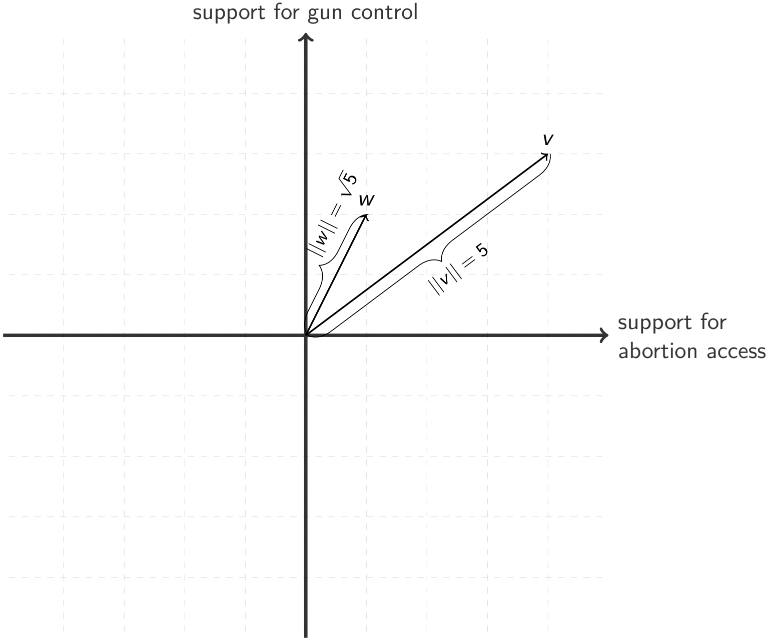
In our case:

$$||v + w|| = 5\sqrt{2} \leq 5 + \sqrt{5} = ||v|| + ||w||$$



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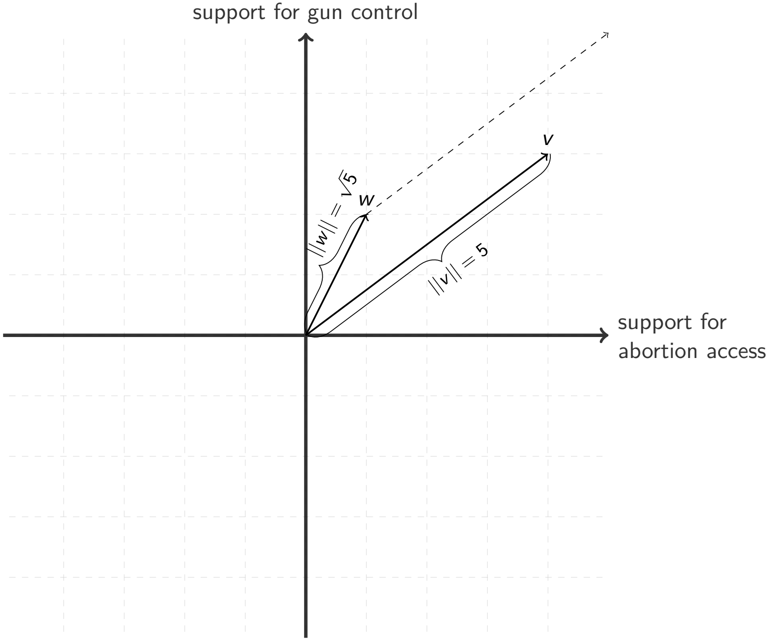
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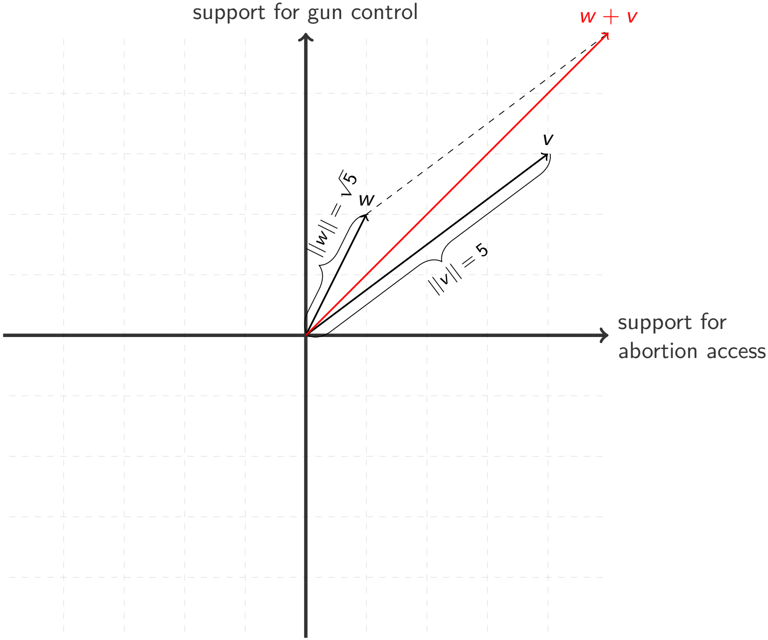

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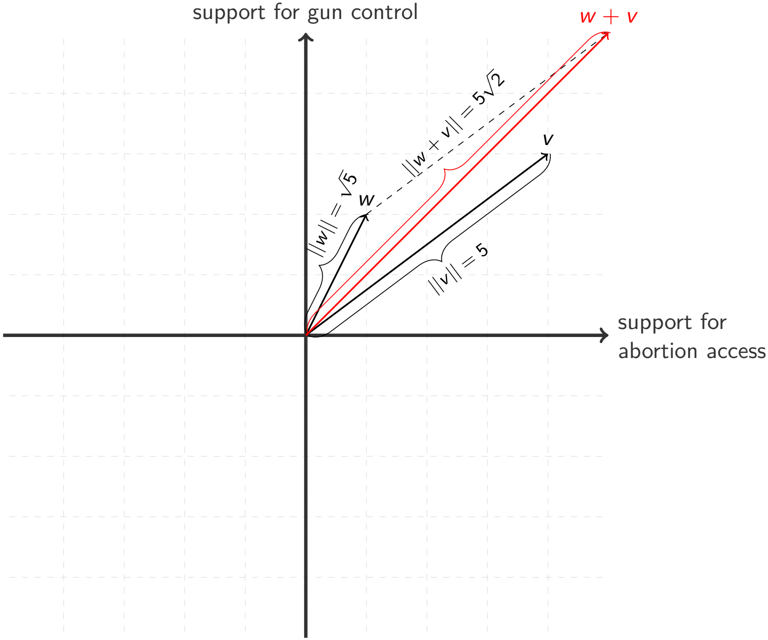

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$$\begin{array}{r c} & \begin{array}{ccc} \overbrace{}^{\text{free trade}} & \overbrace{}^{\text{abortion access}} & \overbrace{}^{\text{gun control}} \end{array} \\ \begin{array}{r} \text{Maria}\{ \\ \text{Dev}\{ \\ \text{Jinyang}\{ \end{array} & \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \end{array} = \mathbf{A}$$

Suppose we want to scale down each person's views to a single ideology measure in $\mathbb{R}$ and collect it in a vector b. How would we do this?

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Let's say we have used some algorithms to compute a vector of the U.S. Democratic party platform's intensity of support/opposition of these three issues $x = [1, 3, 4]^T$.

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One method for ideologically scaling our voters is treating their positions as a **linear transformation** of the party's positions $\rightsquigarrow \mathbf{A}x$.

$$\mathbf{A}x = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$$

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There are two ways to conduct a linear transformation.

Problem 6 (2.1.9): transforming matrices

1. As a dot product of row vectors with x:

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In either case, the results b of the linear transformation of the party positions (x) by our voters' issue positions ($\mathbf{A}$) reveals Maria to be the "strongest Democrat".

Problem 9 (2.2.1-2): solving unknowns between matrices

Now, suppose that we've collected two voters' issue positions, $\mathbf{A}$, *and* we already asked them to numerically scale the strength of their Republican-Democrat affiliation (in $\mathbb{R}$), collected as vector b.

Problem 9 (2.2.1-2): solving unknowns between matrices

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$$\begin{array}{r c} & \begin{array}{cc} \overbrace{}^{\text{public healthcare}} & \overbrace{}^{\text{military spending}} \end{array} \\ \text{Rashida}\{ & \multirow{2}{*}{} \\ \text{Jeb}\{ & \end{array} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \mathbf{A}, \quad \begin{array}{r} \text{Rashida}\{ \\ \text{Jeb}\{ \end{array} \begin{pmatrix} 1 \\ 17 \end{pmatrix} = b$$

Problem 9 (2.2.1-2): solving unknowns between matrices

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This time, we *don't* have measures of x, the "Democratic-ness" of support for each issue. Can we solve for this, though?

$$\mathbf{A}x = b \quad \sim \quad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

Problem 9 (2.2.1-2): solving unknowns between matrices

Gaussian elimination: A generalized “linear algebraic” way of solving systems of equations.

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$$\left[\begin{array}{cc|c} U_{11} & U_{12} & b_1' \\ 0 & U_{22} & b_2' \end{array}\right]$$

via three row operations: (1) swapping rows, (2) multiplying a row by a real number, (3) adding multiple of one row to another.

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{aligned} 2x_1 + \quad 3x_2 &= 1 \\ {\color{red}-5x_2} &{\color{red}= 15} \end{aligned}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rcl} 2x_1 + & 3 \cdot -3 & = 1 \\ & x_2 & = -3 \end{array}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rrl} x_1 & & = 5 \\ & x_2 & = -3 \end{array}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{lcl} x_1 & & = \mathbf{5} \\ & x_2 & = \mathbf{-3} \end{array}$$

Option b: Perform more row operations until the identity matrix appears on left-hand side (**Gauss-Jordan elimination**).

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In either case, the solution x tells us that:

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In either case, the solution x tells us that:

- support for public healthcare is a Democratic position $\rightsquigarrow$ increased support = increases Democratic affiliation by 5
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In many cases, there are not enough "independent" columns or rows in $\mathbf{A}$ $\rightsquigarrow$ we fail Step 2 and cannot find exactly one solution (more this week).

Problem 11 (2.5.12): inverting matrices

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Another method for solving $\mathbf{A}x = b$ would be eliminating $\mathbf{A}$ from the left-hand side of the equation completely via its **inverse**:

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Noting that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$, we can do this sort of elimination to create helpful formulas such as in Problem 11:

$$\mathbf{C} = \mathbf{AB}$$

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$$
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$$

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$$\begin{aligned} \mathbf{C} &= \mathbf{AB} \\ \implies \mathbf{A}^{-1}\mathbf{C} &= \mathbf{A}^{-1}\mathbf{AB} \\ \implies \mathbf{A}^{-1}\mathbf{C}\mathbf{C}^{\mathbf{-1}} &= \mathbf{B}\mathbf{C}^{-1} \\ \implies \mathbf{A}^{-1} &= \mathbf{B}\mathbf{C}^{-1} \end{aligned}$$

Problem 11 (2.5.12): inverting matrices

If $\mathbf{A}^{-1}$ does not exist, $\mathbf{A}$ is called **singular**. How would we know $\mathbf{A}^{-1}$ singular?

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- if there aren't enough independent rows or columns.
- if the "volume" $\mathbf{A}$ takes up in space (determinant) is zero.
- if there is more than one solution to the system $\mathbf{A}x = \mathbf{0}$.

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Ok, so how exactly do you invert a non-singular matrix?

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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We can split these into two systems and solve each via Gauss-Jordan elmination:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4-4 & 1-6 & 0-2 & 1 \end{array}\right] +(-2 \cdot \text{row } 1)$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] \cdot -1/5$$

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can split these into two systems and solve each via Gauss-Jordan elmination:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3{\color{red}-3} & 1{\color{red}-6/5} & 0{\color{red}+3/5} \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] {\color{red}+(-3 \cdot \text{row } 2)}$$

Problem 11 (2.5.12): inverting matrices

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 0 & -1/5 & 3/5 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right]$$

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} \color{red}{1} & 0 & \color{red}{-1/10} & \color{red}{3/10} \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] \color{red}{\cdot 1/2}$$

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 1 & 0 & \mathbf{-0.1} & \mathbf{0.3} \\ 0 & 1 & \mathbf{0.4} & \mathbf{-0.2} \end{array}\right]$$

Problem 11 (2.5.12): inverting matrices

We can verify this in R:

```r
A <- matrix(c(2,4,3,1), ncol=2, nrow=2)
solve(A)
```

```
##      [,1] [,2]
## [1,] -0.1  0.3
## [2,]  0.4 -0.2
```

This Week

Suggested concepts to focus on:

- Space and subspace
  - column space
  - row space
  - nullspace
- Linear independence
  - rank
  - basis

Questions?