Linear dependence :

A sequence of vectors $\{v_1 \dots v_k\}$ is said to be linearly dependent if there exists scalars $\{a_1 \dots a_k\}$, not all 0, such that :

$$\boxed{a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0}$$

$\longrightarrow$ E.g Suppose $a_1 \neq 0$, this implies that

$$\boxed{v_1 = \frac{-a_2}{a_1} v_2 + \dots + \frac{-a_k}{a_1} v_k}$$

i.e v_1 is a linear combination of the remaining columns

Conversely, a sequence of vectors is linearly independent if the equation :

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$$

can be satisfied only by $a_i = 0$ for $i = 1 \dots k$

Column space

- Suppose we have a 3×3 matrix, A

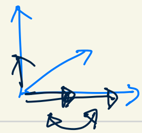
$$A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}$$

- Think of applying the linear transformation, A, to some vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. i.e consider:

$$Av = v_1 \begin{bmatrix} a_1 \end{bmatrix} + v_2 \begin{bmatrix} a_2 \end{bmatrix} + v_3 \begin{bmatrix} a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e A is a function that transforms $v \longrightarrow b$.

- Now consider all possible vectors v. Which vectors b can we get to? This gives the column space of A, $C(A)$



Rank

- The dimensions of the column space is called the 'rank' of a matrix.

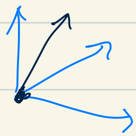
  ↳ It is the maximum number of linearly independent columns that can be chosen from the matrix.

- 'Full rank': when rank = largest possible for matrix of same dimensions

Null space

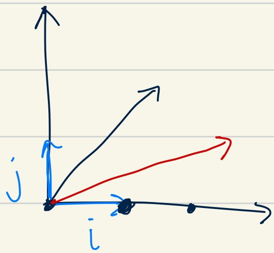
- Now consider the equation $\boxed{Av = 0}$, this has a solution $v \neq 0$ when the columns, are linearly dependent

- The null space, $N(A)$, consists of all vectors v for which $Av = 0$.



Question 1

$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



C_j

C_i

Question 2

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & b \\ | & | & | & | \end{bmatrix}$$

dimensions of $C(A)$ $=$ rank $=$ max # of independent columns

$$\left\{ Av = \underline{b} \right.$$

Question 3

$$AB = 0 \quad \Big| \quad \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

2×2

$$Ab_1 = 0$$

$$A = \begin{array}{ccc} & 2c_1 & 4c_1 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$$

<u>5</u>

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underset{(3\times 2)\ (2\times 3)}{A^T A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = \begin{array}{ccc} & & 1c_1 + 4c_2 \end{array}$$
$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 6 & 5 & 26 \end{bmatrix}$$

$$\underset{2\times 3\ \ 3\times 2}{A A^T}$$

$$\text{Rank}(A) = \text{Rank}(A^T A) = \text{rank}(A A^T)$$

$v_1 \ , \ v_2 \ , \ v_3$

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$$

$$-v_1 + v_2 + 0 v_3 - v_4 = 0$$

Question 7

$$v_1 - v_2 + v_3 = 0 = (w_2 - w_3) - (w_1 - w_3) + (w_1 - w_2)$$

$a_1 = 1 \quad a_2 = -1 \quad a_3 = 1$