Week 3 Review Session

Gov January Linear Algebra Review
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Review of weeks 1 and 2

$$\begin{array}{r c}
 & \begin{array}{ccc} \overbrace{\quad}^{\text{free trade } (\boldsymbol{c}_1)} & \overbrace{\quad}^{\text{abortion access } (\boldsymbol{c}_2)} & \overbrace{\quad}^{\text{gun control } (\boldsymbol{c}_3)} \end{array} \\
\begin{array}{r} \text{Maria } (\boldsymbol{r}_1)\{ \\ \text{Dev } (\boldsymbol{r}_2)\{ \\ \text{Jinyang } (\boldsymbol{r}_3)\{ \end{array} & \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} = \mathbf{A}
\end{array}$$

Some important techniques and concepts we have learned so far:

- Transformation: $\mathbf{A}x = b$
  - ex: transforming vector of observed liberal party positions x by $\mathbf{A} \rightsquigarrow$ liberal-conservative summary of voters b.
- Elimination: solving for unknown x in $\mathbf{A}x = b$ by manipulating rows of $\mathbf{A}$
  - ex: solving for unobserved liberal party positions x, given observed voters positions $\mathbf{A}$ and summaries b such that $\mathbf{A}x = b$.
- Inversion: solving for $\mathbf{A}$'s inverse to solve for x in $\mathbf{A}x = b$ as $x = \mathbf{A}^{-1}b$.
  - ex: inverting voters' positions by performing 'double elimination' on augmented matrix $\begin{bmatrix} \mathbf{A} & | & \mathbf{I} \end{bmatrix}$.

Review of weeks 1 and 2

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Some important techniques and concepts we have learned so far:

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent if $\sum_{j=1}^{k} \boldsymbol{v}_j a_j = 0$ only if $a_1 \ldots a_k = 0$ (read: no redundant vectors).
  - ex: $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ not multiples/shifts of each other (unique voters).
  - ex: $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ not multiples/shifts of each other (unique issues).
- A subspace is a subset of a space that can be defined by the span (all possible linear combinations) of some minimal basis of independent vectors.
  - ex: column space of **A** is all linear combos of issues $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$
  - ex: row space of **A** is all linear combos of voters $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$
  - ex: null space of **A** is all party positions x such that $\mathbf{A}x = 0$.

Review of weeks 1 and 2

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- The rank of **A** is (1) the max number of linearly independent columns that can be found (2) dimensions of column space, $\mathbf{C}(\mathbf{A})$.
- **A** is full rank if it achieves the greatest possible number of linearly independent rows and columns.
  - **A** is square and full rank $\rightsquigarrow$ all rows & columns independent.
  - **A** is wide and full rank $\rightsquigarrow$ all rows independent.
  - **A** is long and full rank $\rightsquigarrow$ all columns independent.

Useful: If **A** is square, we <u>know</u> it's invertible if it's full rank.

How to linear algebra in R

Three convenient ways of making our running matrix **A**:

```
A <- matrix(c(1,2,4,-2,3,1,-4,1,2),
            nrow=3, byrow=TRUE)
```

```
A <- rbind(c(1,2,4),
           c(-2,3,1),
           c(-4,1,2))
```

```
A <- cbind(c(1,-2,-4),
           c(2,3,1),
           c(4,1,2))
```

How to linear algebra in R

Example matrix operations:

```r
t(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1   -2   -4
## [2,]    2    3    1
## [3,]    4    1    2
```

```r
rowMeans(A)
```

```
## [1]  2.3333333  0.6666667 -0.3333333
```

```r
colMeans(A)
```

```
## [1] -1.666667  2.000000  2.333333
```

How to linear algebra in R

Matrix multiplication:

```
x <- c(1,3,4)
A %*% x
```

```
##      [,1]
## [1,]   23
## [2,]   11
## [3,]    7
```

Solving for a system of equations:

```
b <- c(23,11,7)
solve(A, b)
```

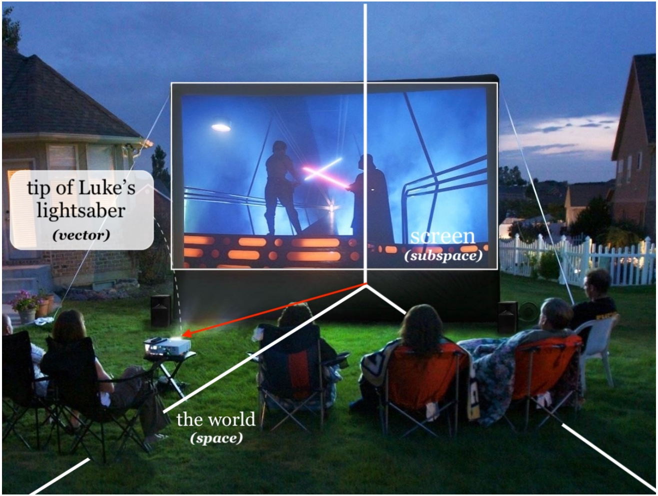
```
## [1] 1 3 4
```

Quick `R` exercise

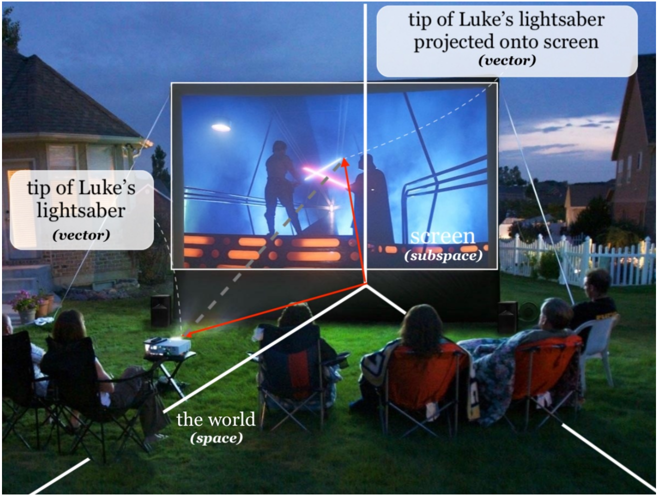
Find the solutions to these exercises from previous weeks using `R`:

1. Exercise 1 in Problem Set 1 (`1.2.1`): dot products
2. Exercise 8 in Problem Set 1 (`2.2.1-2`): elimination
3. Exercise 5 in Problem Set 2 (`3.3.19`): rank
   - hint: the function to find rank is highly google-able!

The **most** important concept in week 3



The **most** important concept in week 3



The **most** important concept in week 3

The **projection** p (e.g. lightsaber on screen) of a vector b (e.g. lightsaber in projector) onto a subspace $\boldsymbol{V}$ (screen) gives us the closest 'location' on $\boldsymbol{V}$ to b.

(1) Here, $\boldsymbol{V}$ is the horizontal plane in $\mathbb{R}^3$, so $p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b$

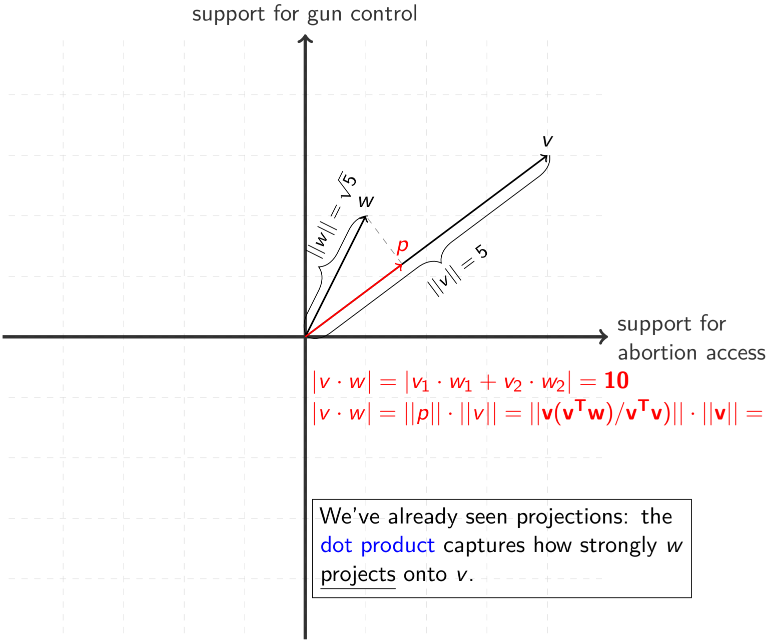
(2) If $\boldsymbol{V}$ is the line formed by vector $\boldsymbol{a}$ in $\mathbb{R}^2$, $p = a\frac{a^T b}{a^T a}$.

- ex: a and b are two voters in a 2-d ideological space.

(3) If $\boldsymbol{V}$ is the column space of matrix $\mathbf{A}$, $p = \underbrace{\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T}_{\text{projection matrix}} b$

- ex: $\mathbf{A}$ is a stack of voters' positions on 3 issues, b is their single-dimensional ideology.

Projecting onto a line in $\mathbb{R}^2$ (example from week 1)



$|v \cdot w| = |v_1 \cdot w_1 + v_2 \cdot w_2| = \mathbf{10}$

$|v \cdot w| = ||p|| \cdot ||v|| = ||\mathbf{v}(\mathbf{v}^\mathbf{T}\mathbf{w})/\mathbf{v}^\mathbf{T}\mathbf{v})|| \cdot ||\mathbf{v}|| = \mathbf{10}$

We've already seen projections: the dot product captures how strongly w projects onto v.

Problem 3 (4.2.13)

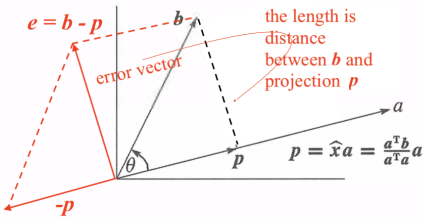
Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Let's find the projection matrix and try out the projection in `R`.

Hint: to create an identity matrix, look up `diag`.

Problem 6 (4.2.23)

For projection $\boldsymbol{p}$ of $\boldsymbol{b}$ on $\boldsymbol{a}$ where $\boldsymbol{p} = \widehat{\boldsymbol{x}}\boldsymbol{a} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a}$, the error vector is $\boldsymbol{e} = \boldsymbol{p} - \boldsymbol{b}$.



Which of the following is the error vector $\boldsymbol{e}$ orthogonal (producing a dot product of 0 with) to?

- $\boldsymbol{b}$
- $\boldsymbol{p}$
- $\boldsymbol{e}$
- $\widehat{\boldsymbol{x}}$

Looking ahead to the semester

- Know the important terms from JLAR review sessions, how they conceptually **relate** to each other, and what they <u>tell</u> us about data.
- When learning methods, always helpful to use meaningful but simple examples in interpretable dimensions.
- A preview of why projection is extremely useful:
  - With real world data (IVs as columns of $\mathbf{A}$ and a DV b) there is often not one solution for $\mathbf{A}x = b$.
  - In other words, b is not in the column space of $\mathbf{A}$! .
  - This is fine since we can <u>always</u> project b onto column space of $\mathbf{A} \rightsquigarrow$ finds the combination $p = \mathbf{A}\hat{x}$ <u>closest</u> to a given vector b.