Week 1 Review Session

Gov January Linear Algebra Review
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Harvard University

Jan 11, 2021

Broad Overview

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Why should we care? *Every* statistical method in social science explicitly or implicitly relies on this accounting system.

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- Summarising and combining vectors
  - Length (Schwarz inequality, Triangle inequality)
  - Dot products
- Gaussian elimination
- Gauss-Jordan elimination
- Matrix inversion

Plan

Cover four problems $\rightsquigarrow$ discuss intuition and connect them to a real application in political science.

- Problem 3 (1.2.2): summarising, combining vectors
- Problem 6 (2.1.9): transforming matrices
- Problem 9 (2.2.1-2): solving unknowns between matrices
- Problem 11 (2.5.12): inverting matrices

Don't feel self-conscious about interrupting if you're confused or have questions!

Problem 3 (1.2.2): summarising, combining vectors

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The simplest linear algebraic unit is a **vector** in some space (e.g. $\mathbb{R}^2$), where each dimension usually "means something":

$$u = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow \text{support for gun control} \\ \leftarrow \text{support for abortion access} \end{matrix}$$

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$$||u|| = \sqrt{(-0.6)^2 + (0.8)^2} = 1, \quad ||v|| = 5, \quad ||w|| = \sqrt{5}$$

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But, how does this dot product relate to their individual lengths?

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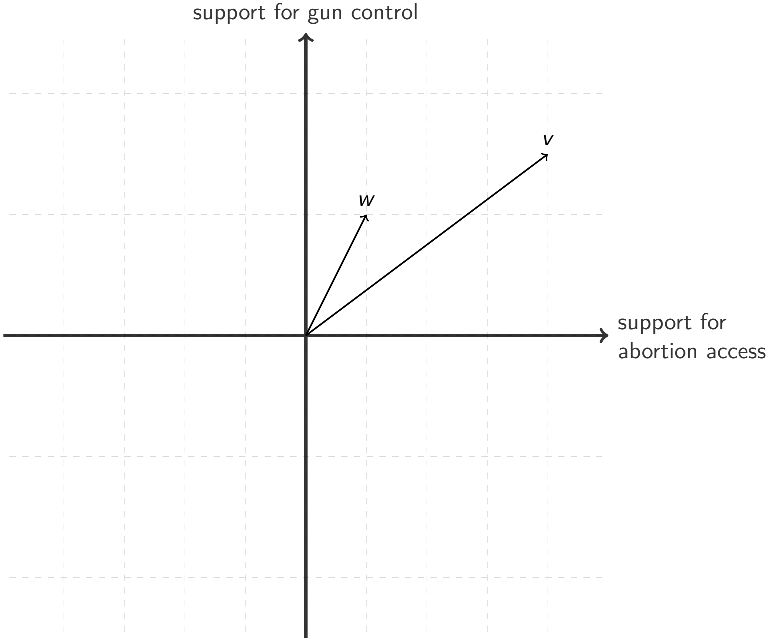
Let's visualize this in our example space.


support for gun control
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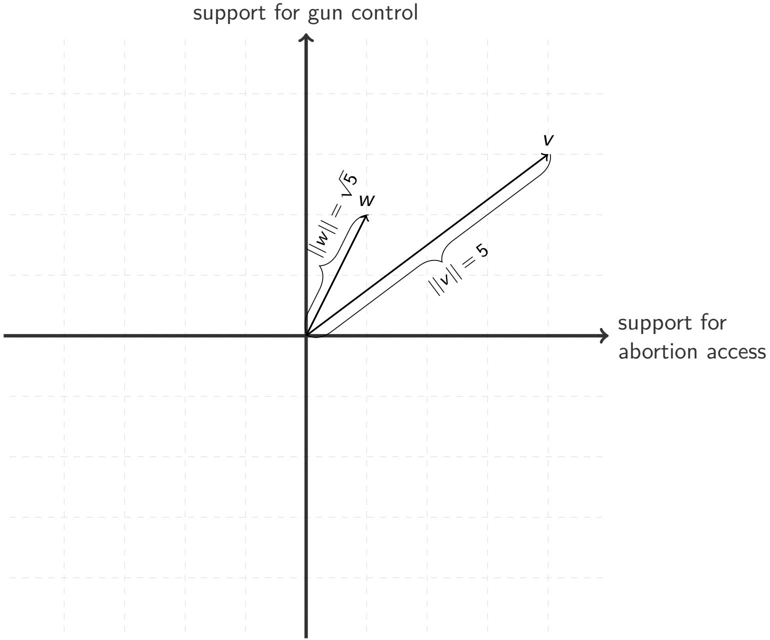
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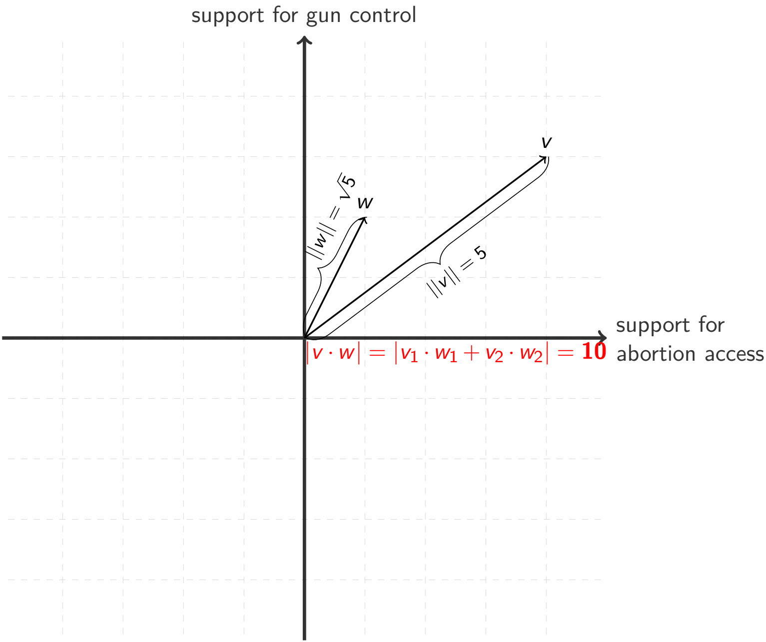
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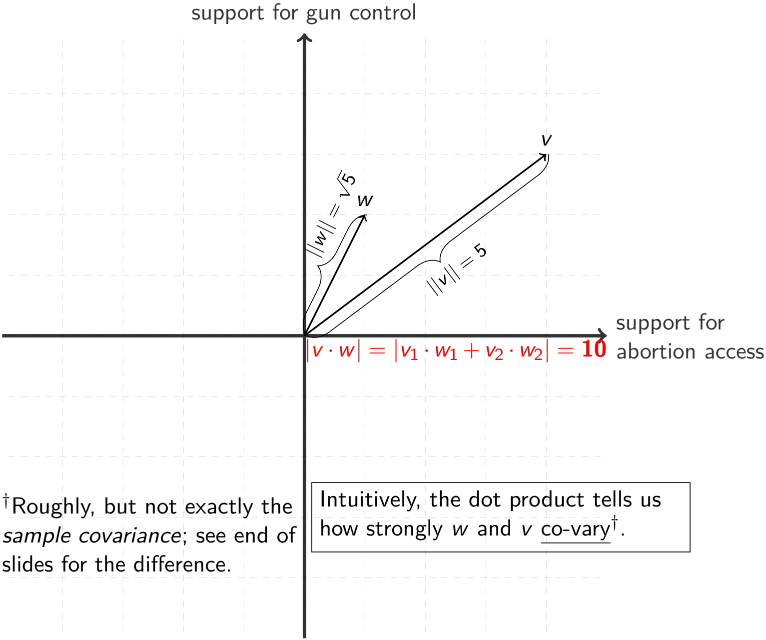
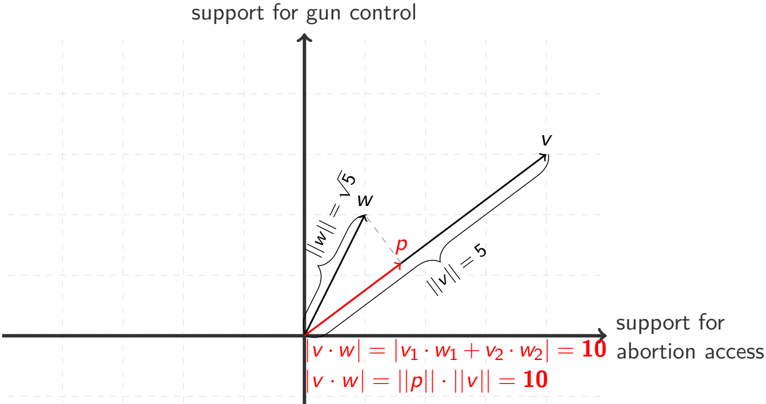


Intuitively, the dot product tells us how strongly w and v co-vary†.

†Roughly, but not exactly the *sample covariance*; see end of slides for the difference.



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Geometrically, the dot product can be defined as how strongly w projects onto v.

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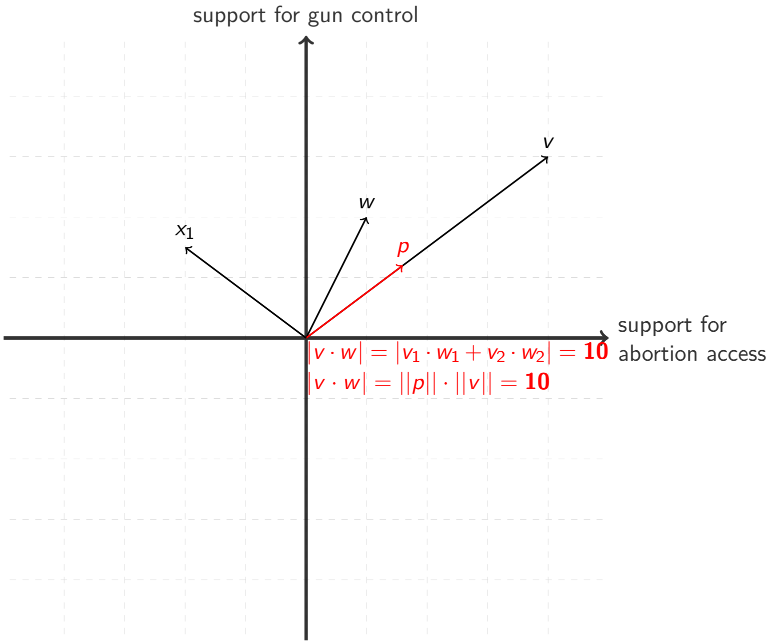

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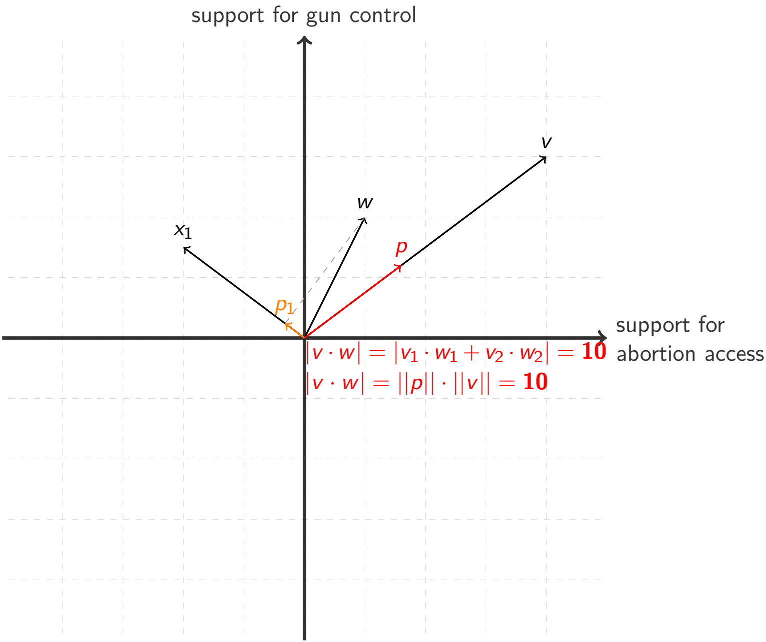
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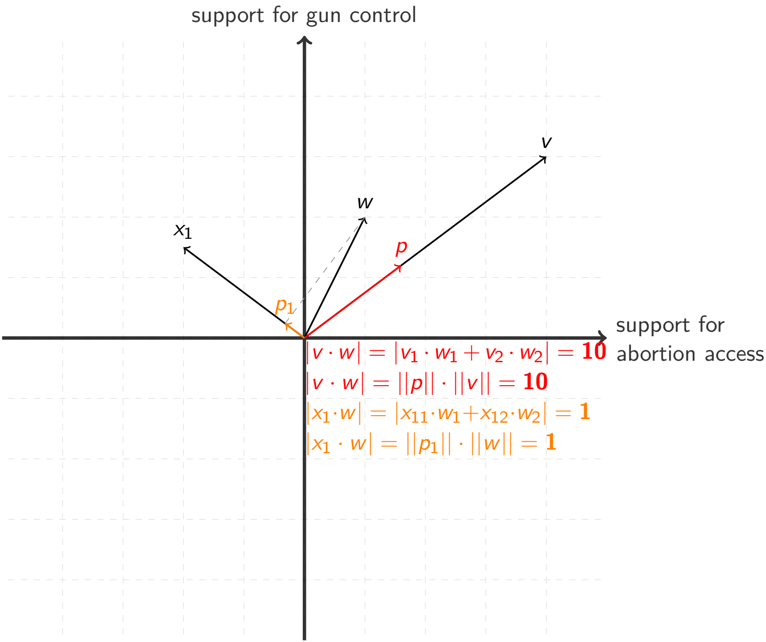
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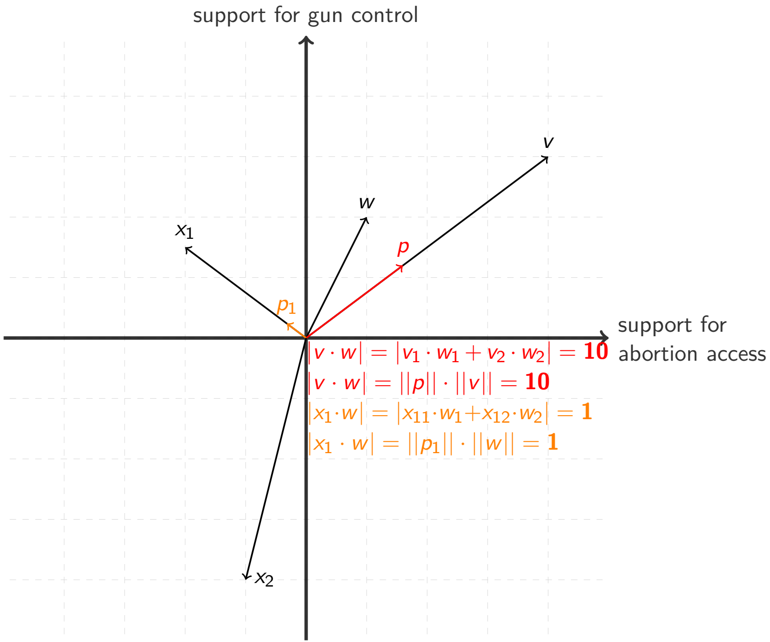
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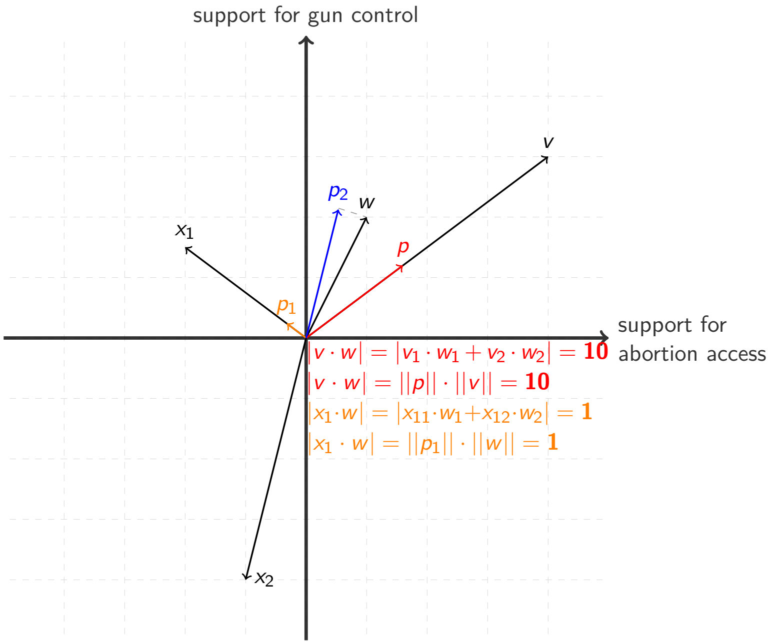

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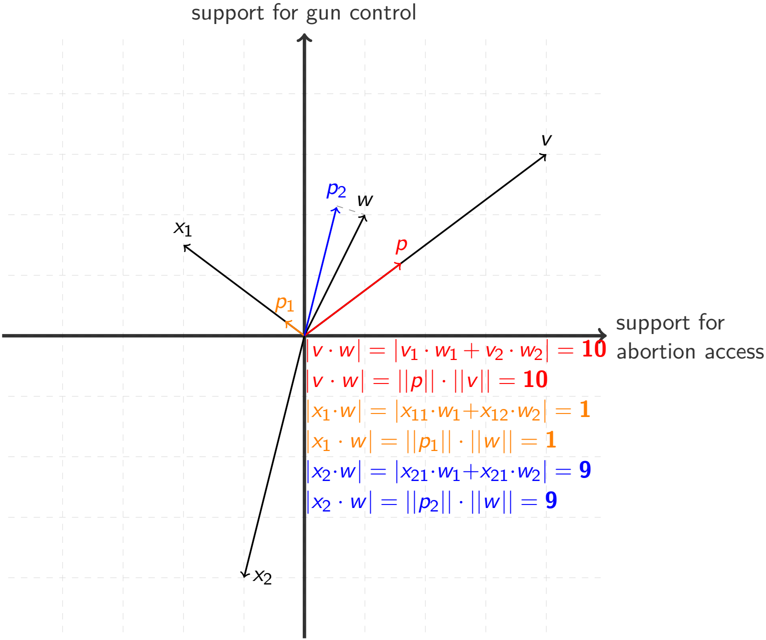

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We can also combine vectors by **addition**:

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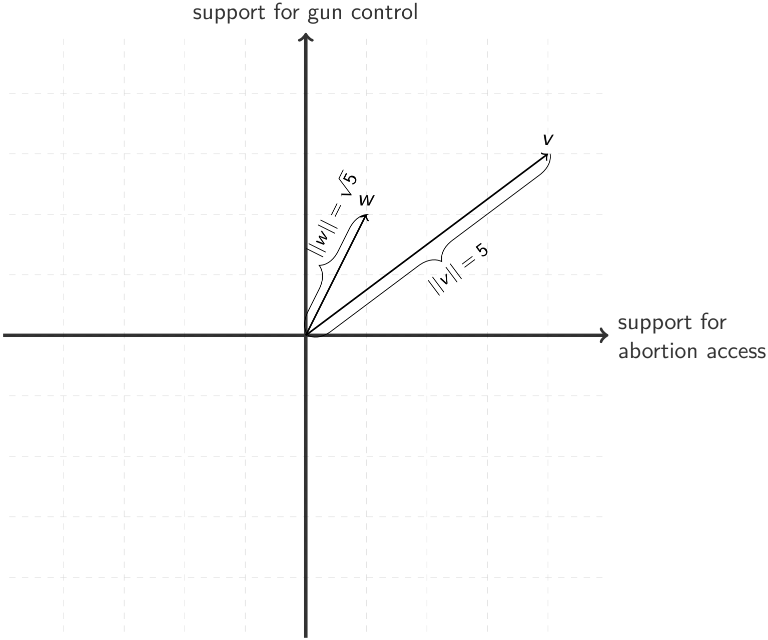
In our case:

$$||v + w|| = 5\sqrt{2} \leq 5 + \sqrt{5} = ||v|| + ||w||$$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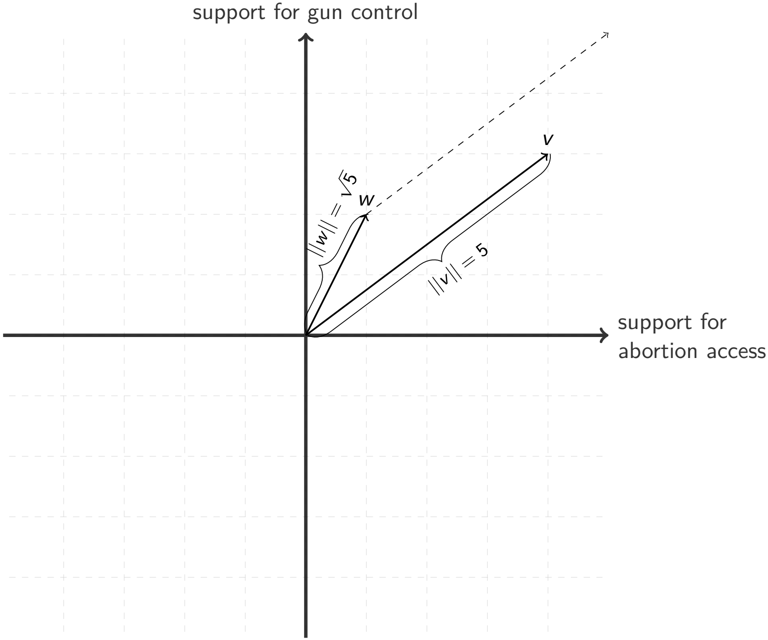

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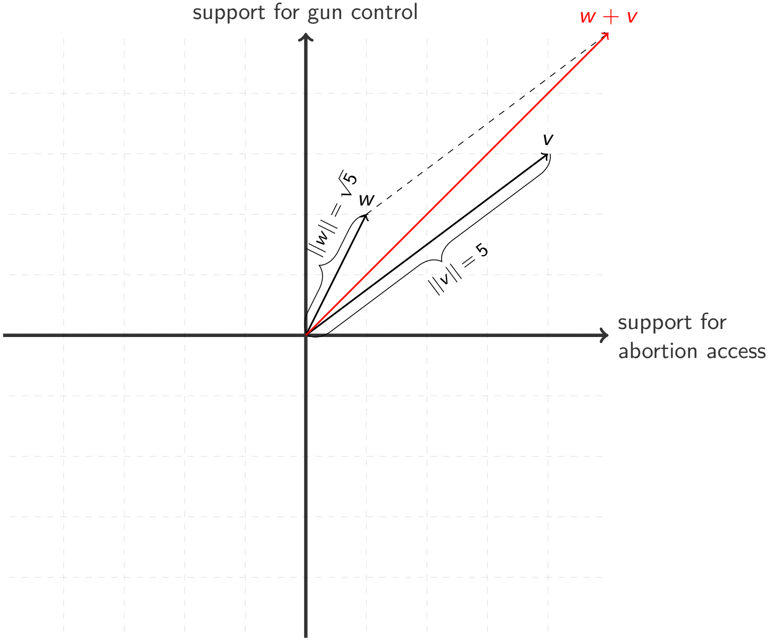
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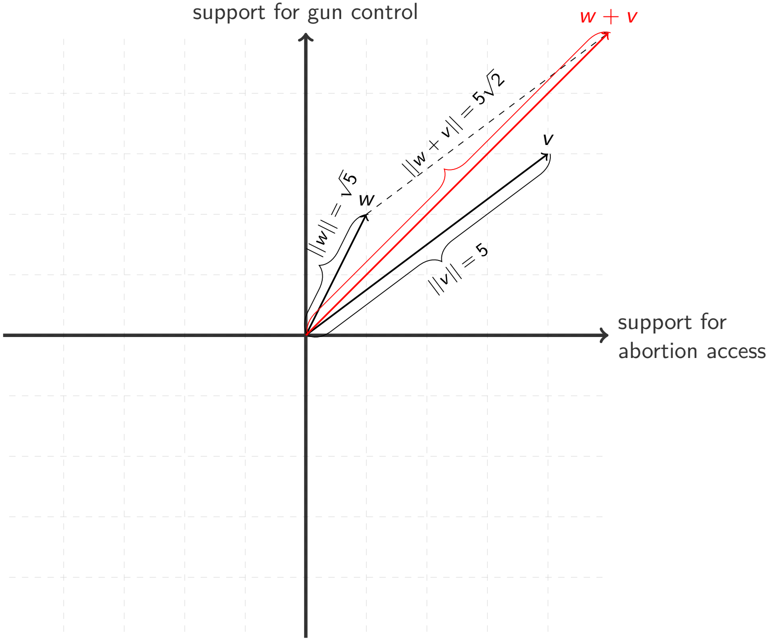
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$$\begin{array}{r c}
 & \begin{array}{ccc} \overbrace{\quad}^{\text{free trade}} & \overbrace{\quad}^{\text{abortion access}} & \overbrace{\quad}^{\text{gun control}} \end{array} \\
\begin{array}{r} \text{Maria}\{ \\ \text{Dev}\{ \\ \text{Jinyang}\{ \end{array} & \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} = \mathbf{A}
\end{array}$$

Suppose we want to scale down each person's views to a single ideology measure in $\mathbb{R}$ and collect it in a vector b. How would we do this?

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Let's say we have used some algorithms to compute a vector of the U.S. Democratic party platform's intensity of support/opposition of these three issues $x = [1, 3, 4]^T$.

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One method for ideologically scaling our voters is treating their positions as a **linear transformation** of the party's positions $\rightsquigarrow \mathbf{A}x$.

$$\mathbf{A}x = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$$

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There are two ways to conduct a linear transformation.

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1. As a dot product of row vectors with x:

$$\mathbf{A}x = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

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In either case, the results b of the linear transformation of the party positions (x) by our voters' issue positions ($\mathbf{A}$) reveals Maria to be the "strongest Democrat".

Problem 9 (2.2.1-2): solving unknowns between matrices

Now, suppose that we've collected two voters' issue positions, $\mathbf{A}$, *and* we already asked them to numerically scale the strength of their Republican-Democrat affiliation (in $\mathbb{R}$), collected as vector b.

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$$\begin{array}{r} \\ \text{Rashida}\{ \\ \text{Jeb}\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{}^{\text{public healthcare}} & \overbrace{}^{\text{military spending}} \end{array} \\ \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \end{array} = \mathbf{A}, \quad \begin{array}{r} \text{Rashida}\{ \\ \text{Jeb}\{ \end{array} \begin{pmatrix} 1 \\ 17 \end{pmatrix} = b$$

Problem 9 (2.2.1-2): solving unknowns between matrices

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This time, we *don't* have measures of x, the "Democratic-ness" of support for each issue. Can we solve for this, though?

$$\mathbf{A}x = b \quad \sim \quad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

Problem 9 (2.2.1-2): solving unknowns between matrices

Gaussian elimination: A generalized "linear algebraic" way of solving systems of equations.

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

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$$[\ \mathbf{A} \mid b\ ] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 17 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} U_{11} & U_{12} & b'_1 \\ 0 & U_{22} & b'_2 \end{array}\right]$$

via three row operations: (1) swapping rows, (2) multiplying a row by a real number, (3) adding multiple of one row to another.

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Problem 9 (2.2.1-2): solving unknowns between matrices

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rr} 2x_1 + & 3x_2 = 1 \\ & -5x_2 = 15 \end{array}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rcl} 2x_1 + & 3 \cdot -3 & = 1 \\ & x_2 & = -3 \end{array}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rcl} 2x_1 & & = 10 \\ & x_2 & = -3 \end{array}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right] \sim \begin{array}{rrl} x_1 & & = 5 \\ & x_2 & = -3 \end{array}$$

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Option b: Perform more row operations until the identity matrix appears on left-hand side (**Gauss-Jordan elimination**).

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & -3 \end{array}\right] \begin{array}{l} +(-3 \cdot \text{row } 2) \\ \end{array}$$

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$$\begin{array}{r} \\ \text{Rashida}\{ \\ \text{Jeb}\{ \end{array} \begin{pmatrix} \overbrace{}^{\text{public healthcare}} & \overbrace{}^{\text{military spending}} \\ 2 & 3 \\ 4 & 1 \end{pmatrix} \cdot \begin{bmatrix} \mathbf{5} \\ \mathbf{-3} \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

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In either case, the solution x tells us that:

Problem 9 (2.2.1-2): solving unknowns between matrices

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In either case, the solution x tells us that:

- support for public healthcare is a Democratic position $\rightsquigarrow$ increased support = increases Democratic affiliation by 5
- support for military spending is a Republican position $\rightsquigarrow$ increased support = decreases Democratic affiliation by -3

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In either case, the solution x tells us that:

- support for public healthcare is a Democratic position $\rightsquigarrow$ increased support = increases Democratic affiliation by $\color{blue}{5}$
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In many cases, there are not enough "independent" columns or rows in $\mathbf{A}$ $\rightsquigarrow$ we fail <u>Step 2</u> and cannot find exactly one solution (more this week).

Problem 11 (2.5.12): inverting matrices

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Another method for solving $\mathbf{A}x = b$ would be eliminating $\mathbf{A}$ from the left-hand side of the equation completely via its **inverse**:

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Noting that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$, we can do this sort of elimination to create helpful formulas such as in Problem 11:

$$\mathbf{C} = \mathbf{AB}$$

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Another method for solving $\mathbf{A}x = b$ would be eliminating $\mathbf{A}$ from the left-hand side of the equation completely via its **inverse**:

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Problem 11 (2.5.12): inverting matrices

If $\mathbf{A}^{-1}$ does not exist, $\mathbf{A}$ is called **singular**. How would we know $\mathbf{A}^{-1}$ singular?

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- if there aren't enough independent rows or columns.
- if the "volume" $\mathbf{A}$ takes up in space (determinant) is zero.
- if there is more than one solution to the system $\mathbf{A}x = \mathbf{0}$.

Problem 11 (2.5.12): inverting matrices

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- if there is more than one solution to the system $\mathbf{A}x = \mathbf{0}$.

Ok, so how exactly do you invert a non-singular matrix?

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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We can split these into two systems and solve each via Gauss-Jordan elimination:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right]$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4{-4} & 1{-6} & 0{-2} & 1 \end{array}\right] +(-2 \cdot \text{row } 1)$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] \cdot -1/5$$

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can split these into two systems and solve each via Gauss-Jordan elimination:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$\left[\begin{array}{cc|cc} 2 & 3{\color{red}-3} & 1{\color{red}-6/5} & 0{\color{red}+3/5} \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] {\color{red}+(-3\cdot\text{row } 2)}$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 2 & 0 & -1/5 & 3/5 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right]$$

Problem 11 (2.5.12): inverting matrices

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$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 1 & 0 & -1/10 & 3/10 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right] \cdot 1/2$$

Problem 11 (2.5.12): inverting matrices

All we have to do is to solve:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 0 \end{array}\right], \qquad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 1 & 1 \end{array}\right]$$

But, it turns out performing row reduction separately is the same as performing row reduction on the joint augmented matrix ☺:

$$\left[\begin{array}{cc|cc} 1 & 0 & \mathbf{-0.1} & \mathbf{0.3} \\ 0 & 1 & \mathbf{0.4} & \mathbf{-0.2} \end{array}\right]$$

Problem 11 (2.5.12): inverting matrices

We can verify this in R:

```
A <- matrix(c(2,4,3,1), ncol=2, nrow=2)
solve(A)
```

```
##      [,1] [,2]
## [1,] -0.1  0.3
## [2,]  0.4 -0.2
```

Important Takeaways

1. A vector's length captures its' magnitude
   - ex: intensity of a voter's public opinion on select issues
2. The dot product between two vectors roughly captures their covariance
   - ex: how well-aligned (positive or negative) two voters' public opinions are
3. A matrix (a convenient way of collecting vectors) can be interpreted as a transformation on a (known or unknown) vector
   - ex: scaling voters' public opinions according to party positions
4. Matrix multiplication is just a bunch of vector dot products
5. Solving for unknown vectors (e.g. system of equations) or matrices (e.g. inversion) can be done via row reduction
6. Order of operations matters in linear algebra!
   - Triangle and Schwarz Inequality broadly tells us that combination of parts is usually greater than their whole
   - Cancellations in Problem 11 only work if we multiply $\mathbf{C}^{-1}$ to right-hand side

This Coming Week

Suggested concepts to focus on:

- Space and subspace
  - column space
  - row space
  - nullspace
- Linear independence
  - rank
  - basis

Questions?

Appendix: Covariance and Dot Product

The **sample covariance** between two vectors $v = [4, 3]$ and $w = [1, 2]$ is:

$$S_{v,w} = (v_1 - \overline{v}) \cdot (w_1 - \overline{w}) + (v_2 - \overline{v}) \cdot (w_2 - \overline{w})$$

where $\overline{v}$ and $\overline{w}$ are the averages across all entries in each vector respectively. More generally for generic vectors x, y in some d-dimensional space:

$$S_{x,y} = \frac{1}{d-1} \sum_{j=1}^{d} (x_j - \overline{x}) \cdot (y_j - \overline{y})$$

$\rightsquigarrow$ same as taking the **dot product** of the de-meaned (or centered) vectors, $x - \overline{x}$ and $y - \overline{y}$ and dividing by $\frac{1}{d-1}$, so they measure $\approx$ same thing.

However, sample covariance implies that v and w are observations of random variables (which have **covariances**), whereas dot product applies to *any* two vectors $\rightsquigarrow$ we'll return to covariance of random variables in Gov 2002!