

Week 3 Review Session

Gov January Linear Algebra Review

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Plan

- ▶ Review of weeks 1 and 2
- ▶ Useful matrix utilities in R
- ▶ The **most** important concept in week 3
 - ▶ Problem 3 (4.2.13)
 - ▶ Problem 6 (4.2.23)
- ▶ Looking ahead to the semester

Review of weeks 1 and 2

$$\begin{array}{l} \text{Maria } (r_1)\{ \\ \text{Dev } (r_2)\{ \\ \text{Jinyang } (r_3)\{ \end{array} \left(\begin{array}{ccc} \overbrace{\text{free trade } (c_1)} & \overbrace{\text{abortion access } (c_2)} & \overbrace{\text{gun control } (c_3)} \\ 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{array} \right) = \mathbf{A}$$

Some important **techniques** and **concepts** we have learned so far:

- ▶ **Transformation:** $\mathbf{A}x = b$
 - ▶ ex: transforming vector of observed liberal party positions x by $\mathbf{A} \rightsquigarrow$ liberal-conservative summary of voters b .
- ▶ **Elimination:** solving for unknown x in $\mathbf{A}x = b$ by manipulating rows of \mathbf{A}
 - ▶ ex: solving for unobserved liberal party positions x , given observed voters positions \mathbf{A} and summaries b such that $\mathbf{A}x = b$.
- ▶ **Inversion:** solving for \mathbf{A} 's inverse to solve for x in $\mathbf{A}x = b$ as $x = \mathbf{A}^{-1}b$.
 - ▶ ex: inverting voters' positions by performing 'double elimination' on augmented matrix $[\mathbf{A} \mid \mathbf{I}]$.

Review of weeks 1 and 2

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Some important **techniques** and **concepts** we have learned so far:

- ▶ Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are **linearly independent** if $\sum_{j=1}^k \mathbf{v}_j a_j = 0$ only if $a_1 \dots a_k = 0$ (read: no redundant vectors).
 - ▶ ex: $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ not multiples/shifts of each other (unique voters).
 - ▶ ex: $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ not multiples/shifts of each other (unique issues).
- ▶ A **subspace** is a subset of a space that can be defined by the **span** (all possible linear combinations) of some minimal **basis** of independent vectors.
 - ▶ ex: **column space** of \mathbf{A} is all linear combos of issues $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$
 - ▶ ex: **row space** of \mathbf{A} is all linear combos of voters $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$
 - ▶ ex: **null space** of \mathbf{A} is all party positions \mathbf{x} such that $\mathbf{A}\mathbf{x} = 0$.

Review of weeks 1 and 2

$$\begin{array}{l} \text{Maria } (r_1) \{ \\ \text{Dev } (r_2) \{ \\ \text{Jinyang } (r_3) \{ \end{array} \left(\begin{array}{ccc} \overbrace{\text{free trade } (c_1)} & \overbrace{\text{abortion access } (c_2)} & \overbrace{\text{gun control } (c_3)} \\ 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{array} \right) = \mathbf{A}$$

- ▶ The **rank** of \mathbf{A} is (1) the max number of linearly independent columns that can be found (2) dimensions of column space, $\mathbf{C}(\mathbf{A})$.
- ▶ \mathbf{A} is **full rank** if it achieves the greatest possible number of linearly independent rows and columns.
 - ▶ \mathbf{A} is square and full rank \rightsquigarrow all rows & columns independent.
 - ▶ \mathbf{A} is wide and full rank \rightsquigarrow all rows independent.
 - ▶ \mathbf{A} is long and full rank \rightsquigarrow all columns independent.

Useful: If \mathbf{A} is square, we know it's invertible if it's full rank.

How to linear algebra in R

Three convenient ways of making our running matrix **A**:

```
A <- matrix(c(1,2,4,-2,3,1,-4,1,2),  
            nrow=3, byrow=TRUE)
```

```
A <- rbind(c(1,2,4),  
           c(-2,3,1),  
           c(-4,1,2))
```

```
A <- cbind(c(1,-2,-4),  
          c(2,3,1),  
          c(4,1,2))
```

How to linear algebra in R

Example matrix operations:

```
t(A)
```

```
##      [,1] [,2] [,3]
## [1,]    1  -2  -4
## [2,]    2   3   1
## [3,]    4   1   2
```

```
rowMeans(A)
```

```
## [1]  2.333333  0.6666667 -0.3333333
```

```
colMeans(A)
```

```
## [1] -1.666667  2.000000  2.333333
```

How to linear algebra in R

Matrix multiplication:

```
x <- c(1,3,4)
A %*% x
```

```
##      [,1]
## [1,]   23
## [2,]   11
## [3,]    7
```

Solving for a system of equations:

```
b <- c(23,11,7)
solve(A, b)
```

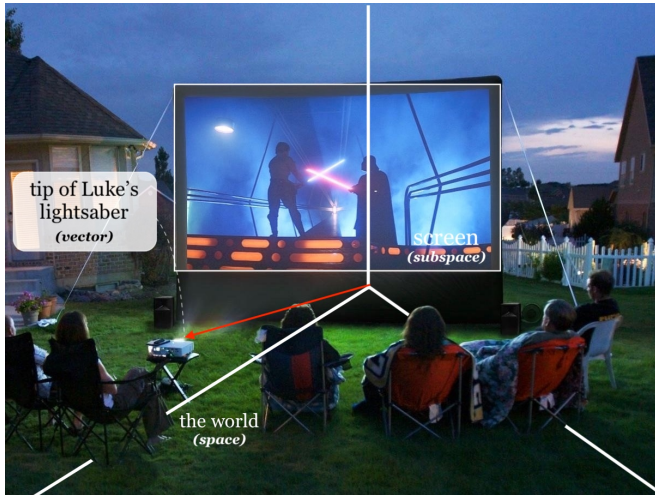
```
## [1] 1 3 4
```


Quick R exercise

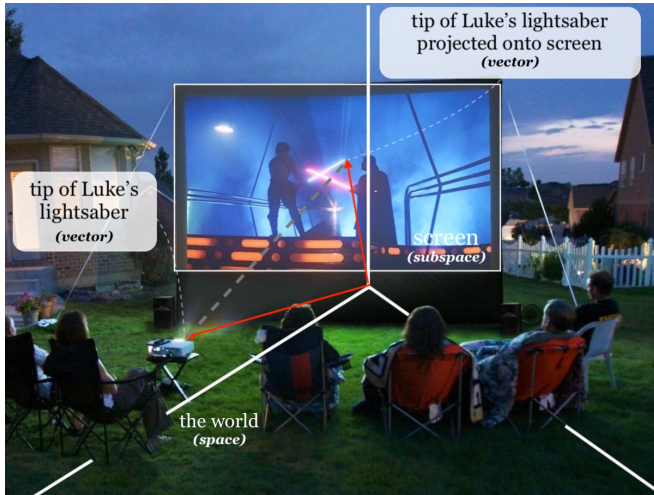
Find the solutions to these exercises from previous weeks using R:

1. Exercise 1 in Problem Set 1 (1.2.1): dot products
2. Exercise 8 in Problem Set 1 (2.2.1–2): elimination
3. Exercise 5 in Problem Set 2 (3.3.19): rank
 - ▶ hint: the function to find rank is highly google-able!

The **most** important concept in week 3



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The **most** important concept in week 3

The **projection** p (e.g. lightsaber on screen) of a vector b (e.g. lightsaber in projector) onto a subspace \mathbf{V} (screen) gives us the closest 'location' on \mathbf{V} to b .

(1) Here, \mathbf{V} is the horizontal plane in \mathbb{R}^3 , so $p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b$

(2) If \mathbf{V} is the line formed by vector a in \mathbb{R}^2 , $p = a \frac{a^T b}{a^T a}$.

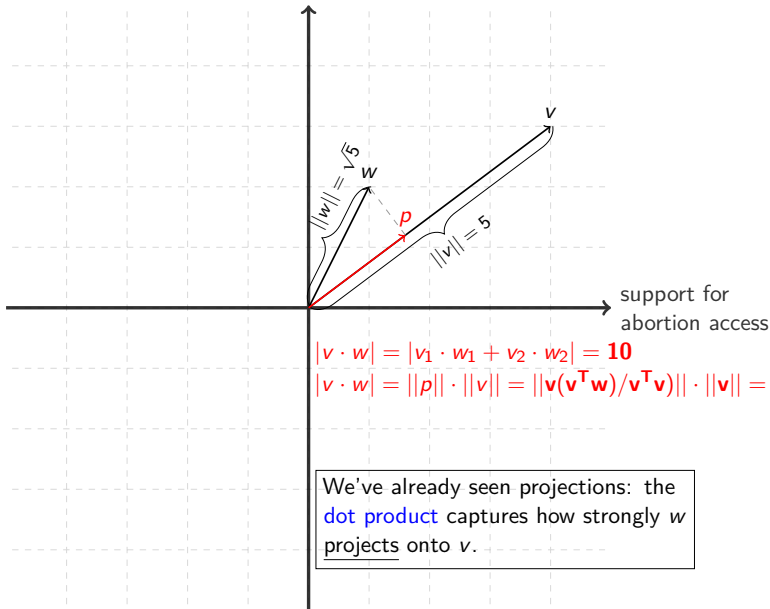
► ex: a and b are two voters in a 2-d ideological space.

(3) If \mathbf{V} is the column space of matrix \mathbf{A} , $p = \underbrace{\mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T}_{\text{projection matrix}} b$

► ex: \mathbf{A} is a stack of voters' positions on 3 issues, b is their single-dimensional ideology.

Projecting onto a line in \mathbb{R}^2 (example from week 1)

support for gun control



We've already seen projections: the **dot product** captures how strongly w projects onto v .

Problem 3 (4.2.13)

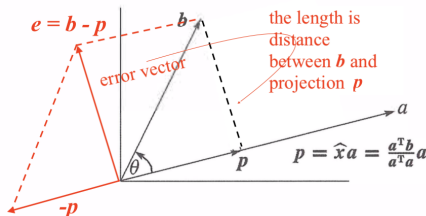
Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A . What shape is the projection matrix P and what is P ?

Let's find the projection matrix and try out the projection in R.

Hint: to create an identity matrix, look up `diag`.

Problem 6 (4.2.23)

For projection \mathbf{p} of \mathbf{b} on \mathbf{a} where $\mathbf{p} = \hat{\mathbf{x}}\mathbf{a} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a}$, the error vector is $\mathbf{e} = \mathbf{p} - \mathbf{b}$.



Which of the following is the error vector \mathbf{e} orthogonal (producing a dot product of 0 with) to?

- ▶ \mathbf{b}
- ▶ \mathbf{p}
- ▶ \mathbf{e}
- ▶ $\hat{\mathbf{x}}$

Looking ahead to the semester

- ▶ Know the important terms from JLAR review sessions, how they conceptually **relate** to each other, and what they tell us about data.
- ▶ When learning methods, always helpful to use meaningful but simple examples in interpretable dimensions.
- ▶ A preview of why projection is extremely useful:
 - ▶ With real world data (IVs as columns of \mathbf{A} and a DV b) there is often not one solution for $\mathbf{A}x = b$.
 - ▶ In other words, b is not in the column space of \mathbf{A} !
 - ▶ This is fine since we can always project b onto column space of \mathbf{A} \rightsquigarrow finds the combination $p = \mathbf{A}\hat{x}$ closest to a given vector b .