# 12. Algebra of Least Squares

Spring 2023

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Gov 2002 (Harvard)

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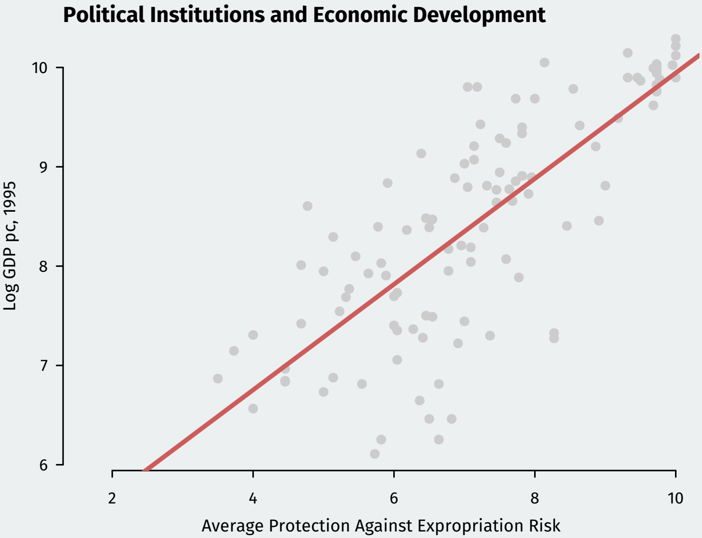
- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

# Acemoglu, Johnson, and Robinson (2001)



# **1/** Deriving the OLS estimator

# Samples vs population

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- Violations include time-series data and clustered sampling.
  - Weakening i.i.d. usually complicates notation but can be done.

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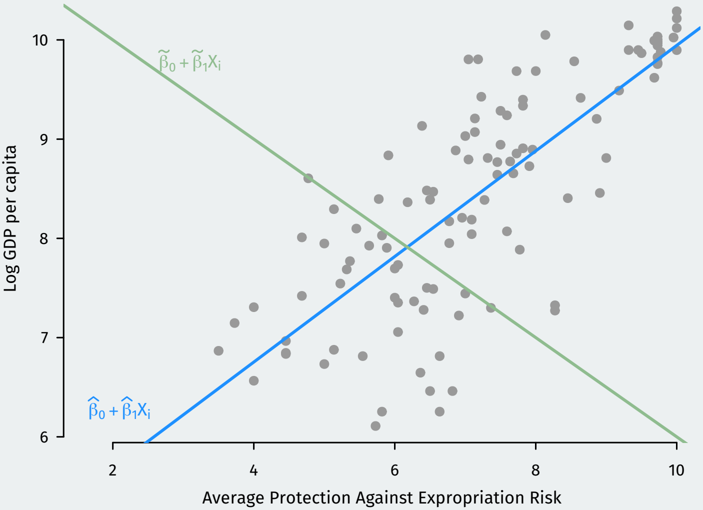

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- $\hat{S}(\mathbf{b})$ is an estimator of the expected squared error, $S(\mathbf{b})$.

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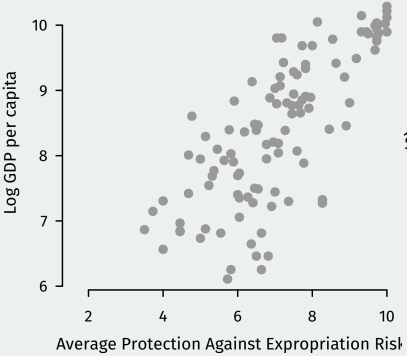
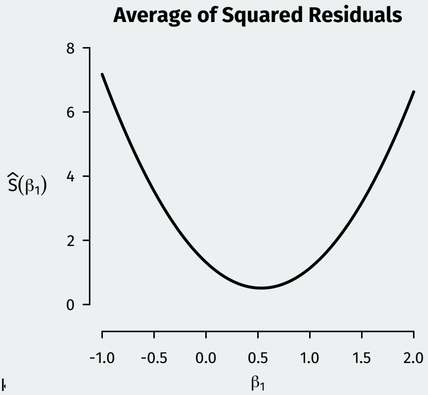
- We can show the OLS estimator of the slope is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\mathbb{V}}[X]}$$

# Visualizing OLS



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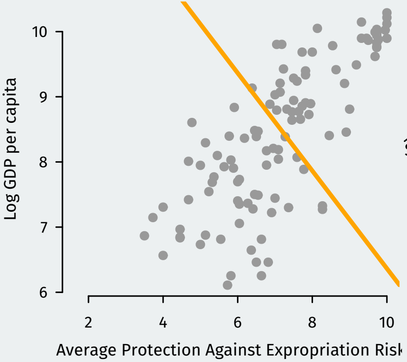


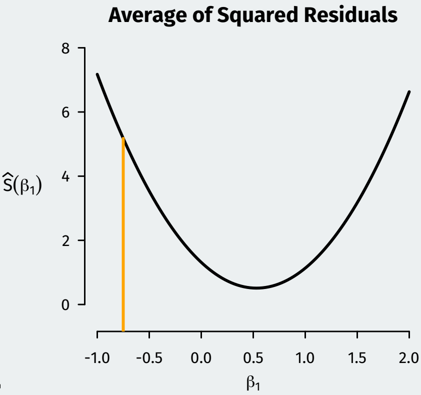

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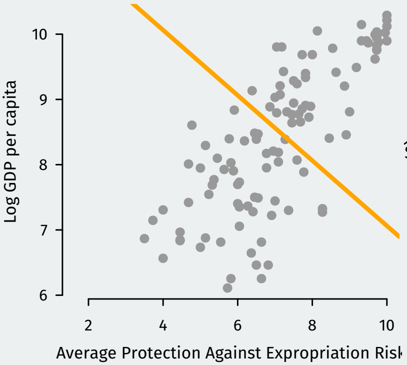


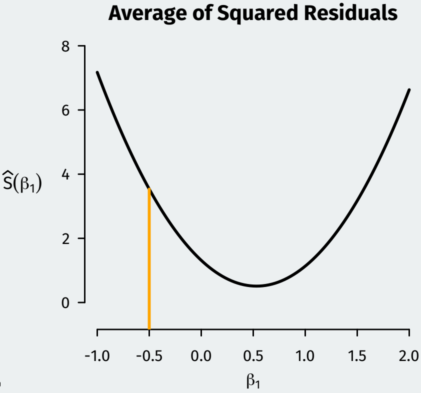

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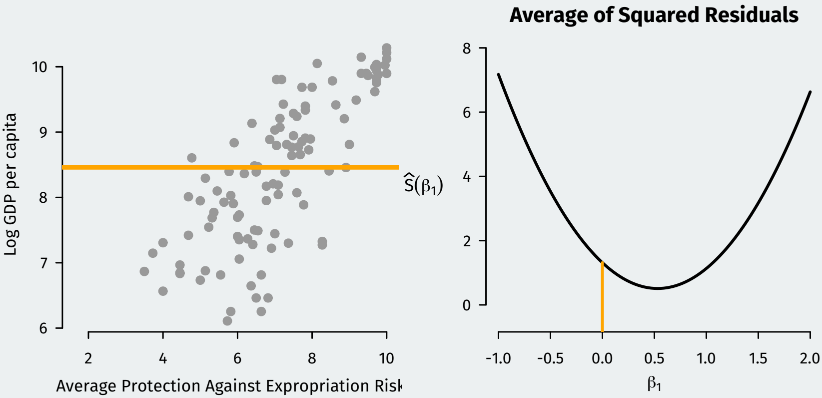

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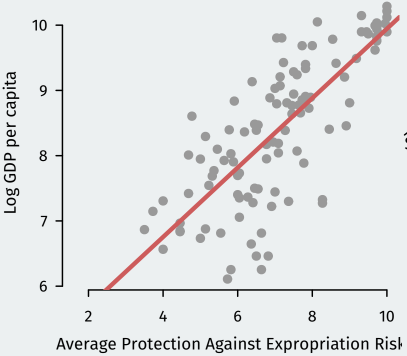


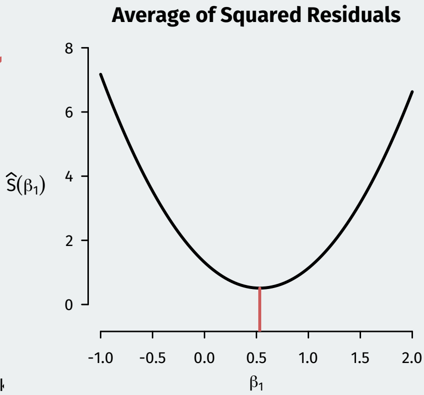

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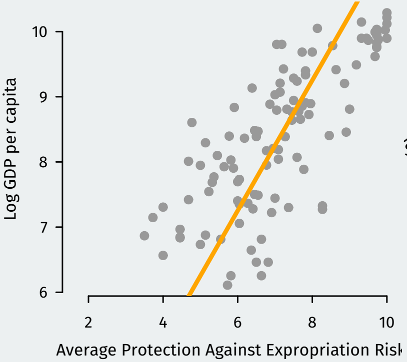


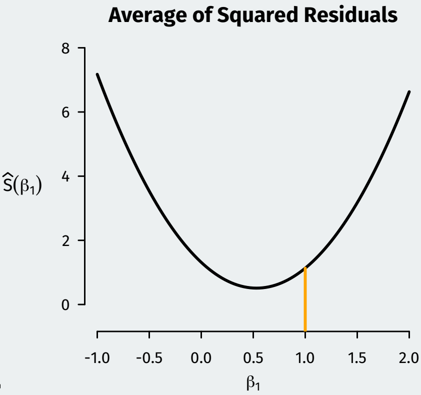

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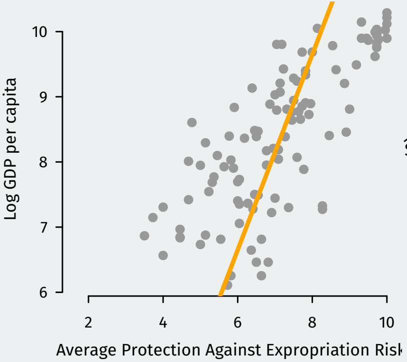


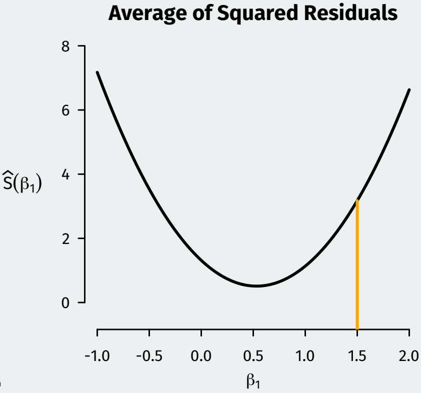

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- If $\mathbf{X}_i$ has a constant, then $n^{-1}\sum_{i=1}^{n} \widehat{e}_i = 0$

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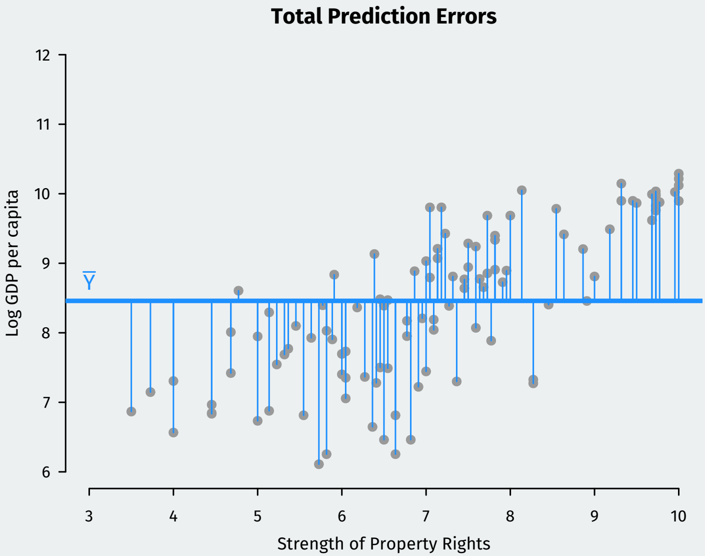
# Total SS vs SSR



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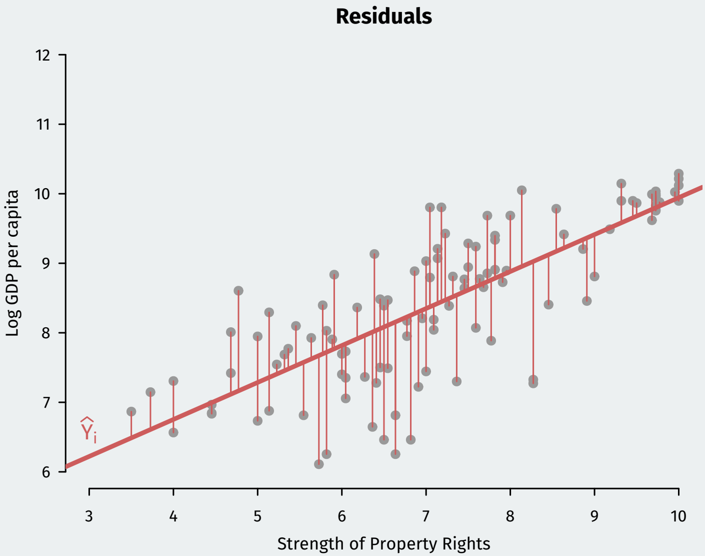

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- Mechanically increases with additional covariates (better fit measures exist)

# 3/ Geometry of OLS

# Linear model in matrix form

- Linear model is a system of $n$ linear equations:

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Y_1 &= \mathbf{X}_1'\boldsymbol{\beta} + e_1 \\
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- Model is now just:

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# OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

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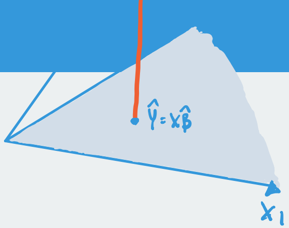


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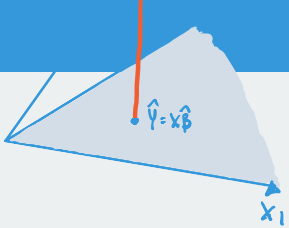


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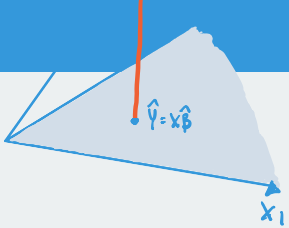


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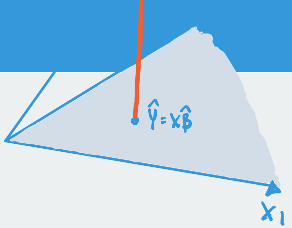


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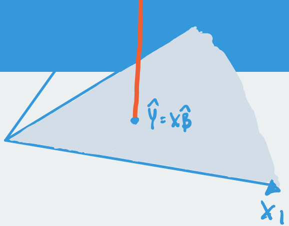


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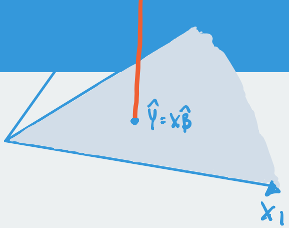


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$$\mathbf{P} = \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'$$

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    - $\mathbf{P}$ is **idempotent**: $\mathbf{P}\mathbf{P} = \mathbf{P}$
    - Projecting $\mathbb{X}$ onto itself returns itself: $\mathbf{P}\mathbb{X} = \mathbb{X}$

# Annihilator matrix

- **Annihilator matrix** projects onto the space spanned by the residual:

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    - Admits a nice expression for the residual vector: $\hat{\mathbf{e}} = \mathbf{Me}$

# Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \ \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

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    - With exactly orthogonal covariates, multivariate OLS is the same as univariate OLS.
    - Only holds in balanced, designed experiments.

# Adding the intercept

- Consider the OLS slope with an intercept:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} - \overline{Y}\mathbf{1} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle}$$

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    2. Regress **Y** on residuals from step 1, $\mathbf{X} - \overline{X}\mathbf{1}$

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  2. Regress $\mathbf{Y}$ on residual from

# Visualizing orthogonalization

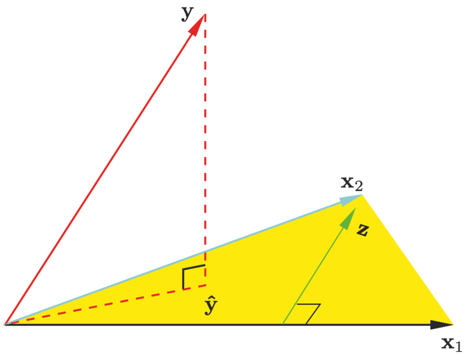


**FIGURE 3.4.** *Least squares regression by orthogonalization of the inputs. The vector $\mathbf{x}_2$ is regressed on the vector $\mathbf{x}_1$, leaving the residual vector $\mathbf{z}$. The regression of $\mathbf{y}$ on $\mathbf{z}$ gives the multiple regression coefficient of $\mathbf{x}_2$. Adding together the projections of $\mathbf{y}$ on each of $\mathbf{x}_1$ and $\mathbf{z}$ gives the least squares fit $\hat{\mathbf{y}}$.*

# Why does residual regression work?

- We can find $\hat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\hat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left( \min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1 \boldsymbol{\beta}_1 - \mathbb{X}_2 \boldsymbol{\beta}_2\|^2 \right)$$

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  - Creates residuals from a regression on or $\mathbb{X}_2$

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  - Sample version of the results we saw for the linear projection.

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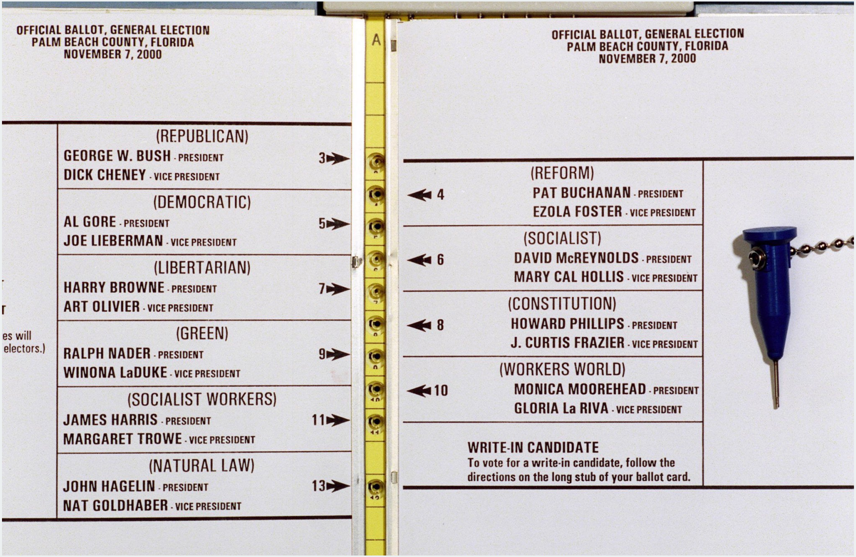
# Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)



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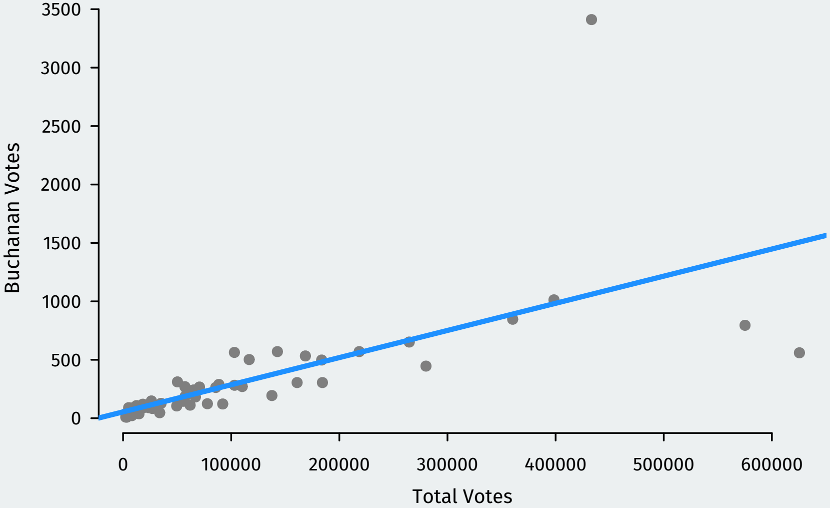

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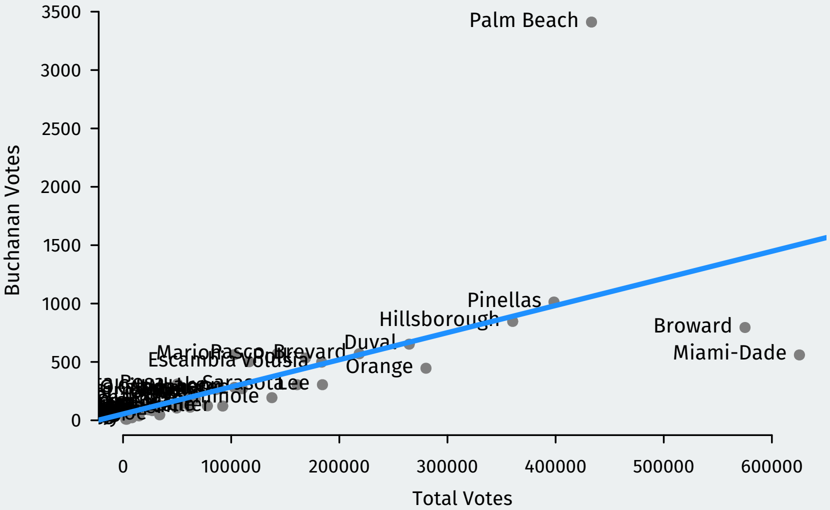
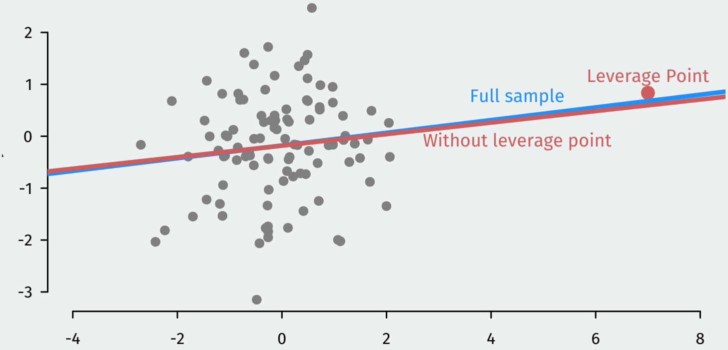


# Example: Buchanan votes

```r
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 54.22945   49.14146    1.10     0.27
## edaytotal    0.00232    0.00031    7.48  2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463,  Adjusted R-squared:  0.455
## F-statistic:   56 on 1 and 65 DF,  p-value: 2.42e-10
```

# Leverage point definition

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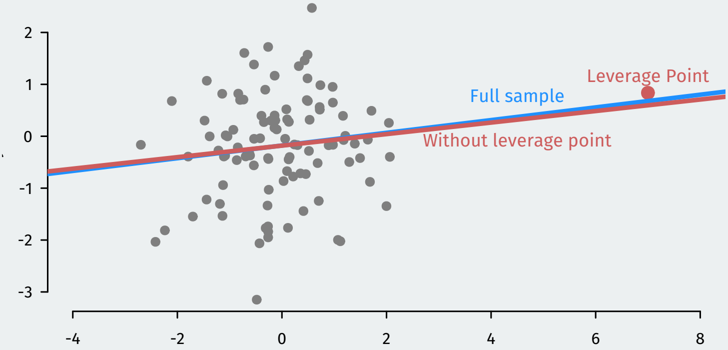


- Values that are extreme in the $X$ dimension
- That is, values far from the center of the covariate distribution

# Leverage values

- Let $h_{ij}$ be the $(i, j)$ entry of $\mathbf{P}$. Then:

$$\widehat{\mathbf{Y}} = \mathbf{P}\mathbf{Y} \qquad \Longrightarrow \qquad \widehat{Y}_i = \sum_{j=1}^{n} h_{ij} Y_j$$

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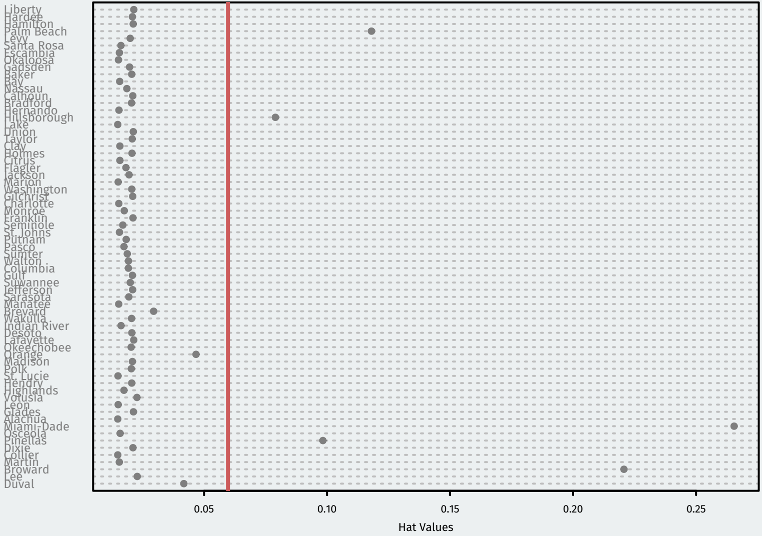
  - $\rightsquigarrow$ how far $i$ is from the center of the $X$ distribution
- **Rule of thumb:** examine hat values greater than $2(k+1)/n$

# Buchanan hats

```r
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5 
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

# Buchanan hats



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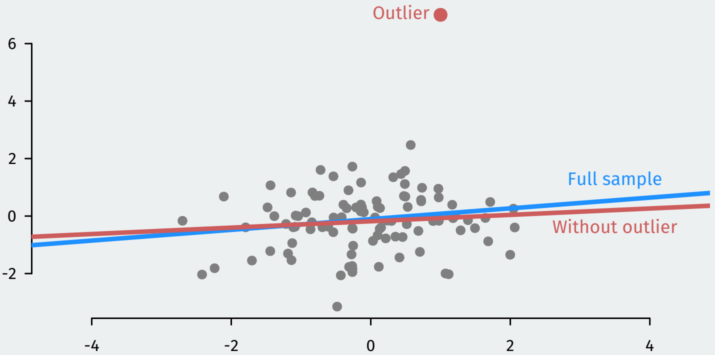


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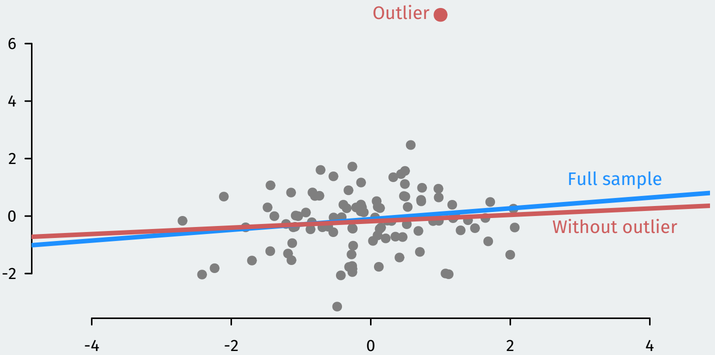


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- Intuitively: a point that would be poorly predicted by the regression.

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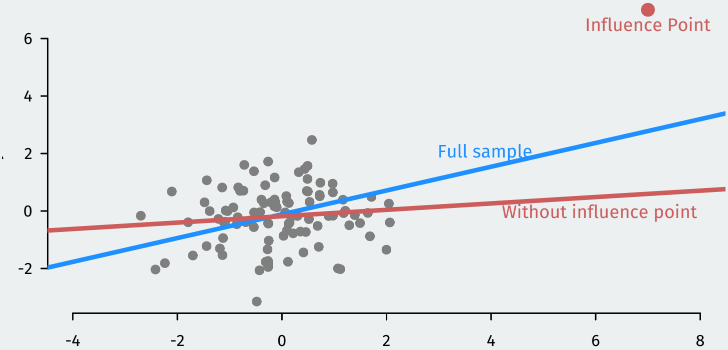
- Simple closed-form expressions:

$$\hat{\boldsymbol{\beta}}_{(-i)} = \hat{\boldsymbol{\beta}} - \left(\mathbb{X}'\mathbb{X}\right)^{-1}\mathbf{X}_i\tilde{e}_i \qquad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

# Influence points

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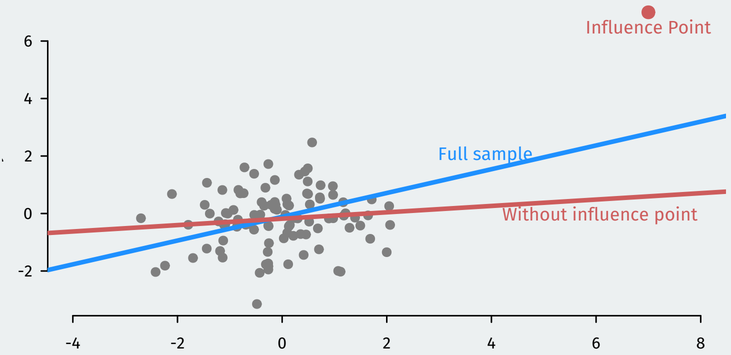


- An **influence point** is one that is both an outlier and a leverage point.
- Extreme in both the $X$ and $Y$ dimensions

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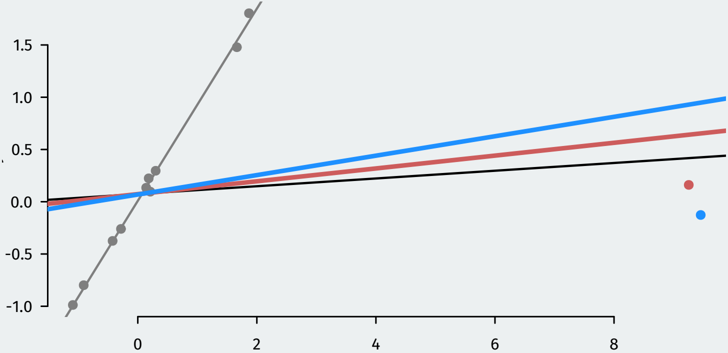
  - Does removing the point change a coefficient by a lot?

# Limitations of the standard tools

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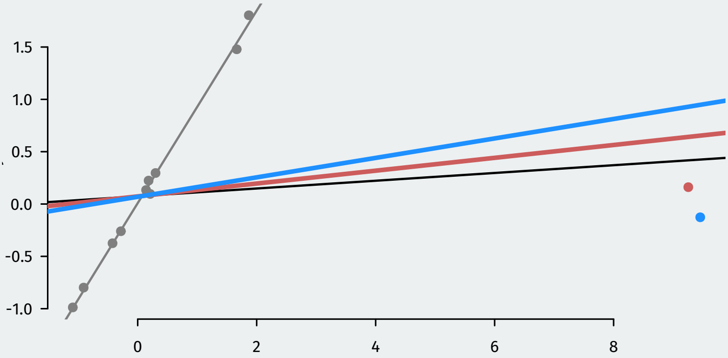


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- Red line drops the red influence point

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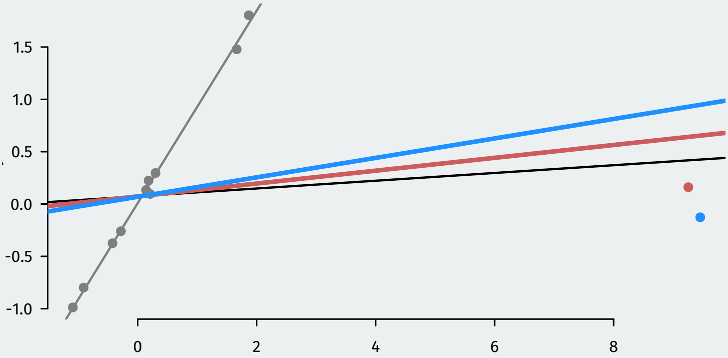


- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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- Is the outlier part of the data generating process?
    - Transform the dependent variable ($\log(y)$)
    - Use a method that is robust to outliers (robust regression, least absolute deviations)