6. Multivariate Distributions

Spring 2023

Matthew Blackwell

Gov 2002 (Harvard)

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- Distributions of one variable: how to describe and summarize uncertainty about one variable.
- Today: **distributions of multiple variables** to describe relationships between variables.
- Later: use data to **learn** about probability distributions.

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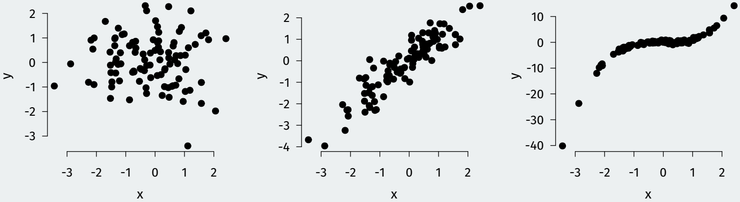
1. How to measure the relationship between two variables X and Y?
2. What if we have many observations of the same variable, $X_1, X_2, \dots, X_n$?

1/ Distributions of Multiple Random Variables

Joint distributions



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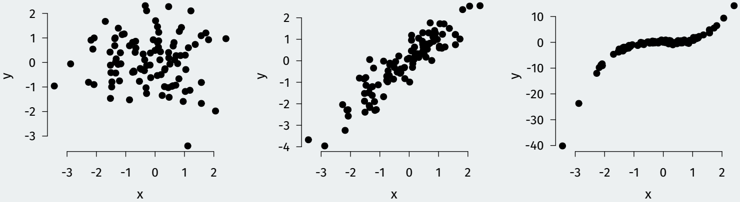
Joint distributions



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- Shape of the joint distribution $\rightsquigarrow$ the relationship between X and Y

Discrete r.v.s

Definition

The **joint probability mass function (p.m.f.)** of a pair of discrete r.v.s, (X, Y) describes the probability of any pair of values:

$$f_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$$

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    - $\sum_x$ is shorthand for sum over all possible values of X

Example: Gay marriage and gender

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$
Female $X = 1$	0.32	0.19
Male $X = 0$	0.29	0.20

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$$p_{X,Y}(1, 1) = \mathbb{P}(X = 1, Y = 1) = 0.32$$

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    - Works because values of X are disjoint.

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Marginal	0.32 + 0.29		

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Marginal	0.61	0.19 + 0.20	

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Conditional p.m.f.

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The **conditional probability mass function** or conditional p.m.f. of Y conditional on X is

$$\mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)}$$

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- This is a valid univariate probability distribution!
  - $P(Y = y \mid X = x) \geq 0$ and $\sum_y \mathbb{P}(Y = y \mid X = x) = 1$
- Can define the **conditional expectation** of this p.m.f.:

$$E[Y \mid X = x] = \sum_y y\mathbb{P}(Y = y \mid X = x)$$

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$$\mathbb{P}(Y = 1 \mid X = 0)$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20}$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20} = 0.592$$

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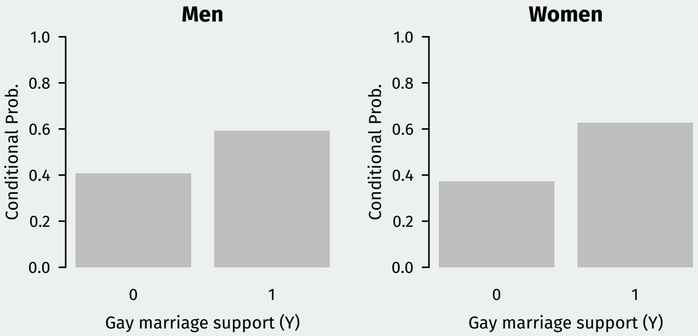

- Two values of $X \rightsquigarrow$ two **univariate** conditional distributions of Y

Bayes and LTP

- Bayes' rule for r.v.s:

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- Law of total probability for r.v.s:

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x \mid Y = y)\mathbb{P}(Y = y)$$

Joint c.d.f.s

Definition

For two r.v.s X and Y, the **joint cumulative distribution function** or joint c.d.f. $F_{X,Y}(x,y)$ is a function such that for finite values x and y,

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- Well-defined for discrete and continuous X and Y.
- For discrete we simply have:

$$F_{X,Y}(x,y) = \sum_{i \leq x} \sum_{j \leq y} \mathbb{P}(X = i, Y = j)$$

Continuous r.v.s

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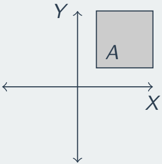
- Two continuous r.v.s: probability of being in some subset of the 2-dimensional plane.



Continuous joint p.d.f.

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If two continuous r.v.s X and Y with joint c.d.f. $F_{X,Y}$, their **joint p.d.f.** $f_{X,Y}(x, y)$ is the derivative of $F_{X,Y}$ with respect to x and y,

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- Integrate over both dimensions to get the probability of a region:

$$\mathbb{P}((X,Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x,y) dx dy.$$

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- $\{(x,y) : f_{X,Y}(x,y) > 0\}$ is called the **support** of the distribution.

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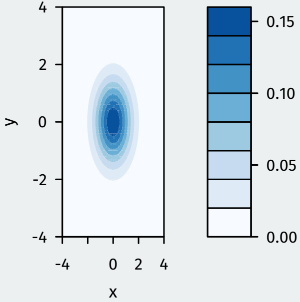
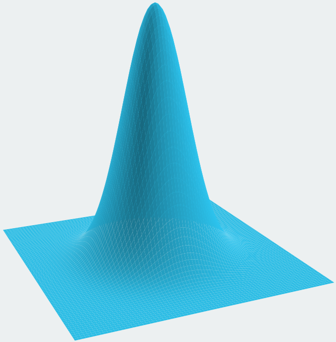
- $\mathbb{P}(X = x, Y = y) = 0$ for similar reasons as with single r.v.s.

Joint densities are 3D





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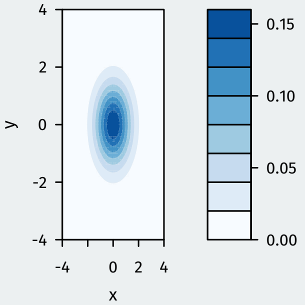
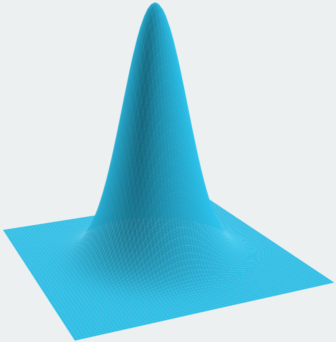
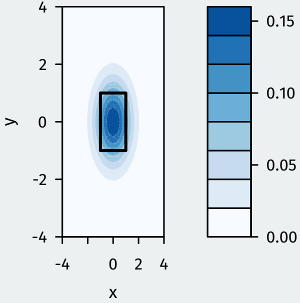
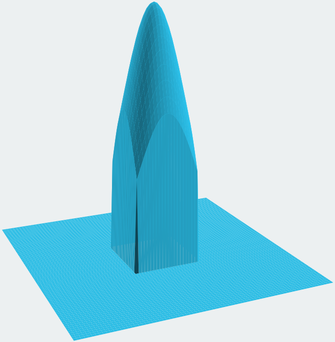




- X and Y axes are on the "floor," height is the value of $f_{X,Y}(x,y)$.
- Remember $f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$.

Probability = volume





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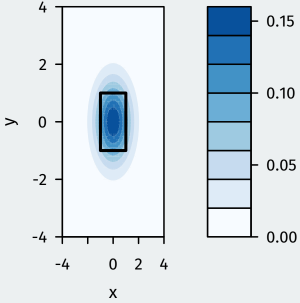

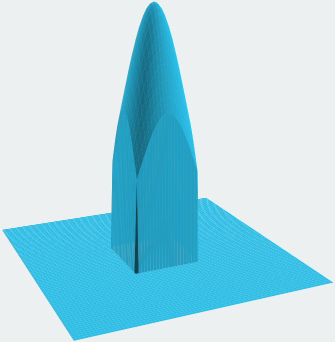

- $\mathbb{P}((X, Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x, y)dxdy$
- Probability = volume above a specific region.

Continuous marginal distributions

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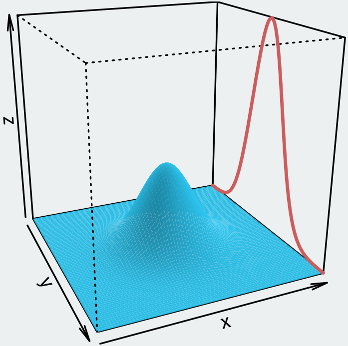
- Works for either variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

Visualizing continuous marginals



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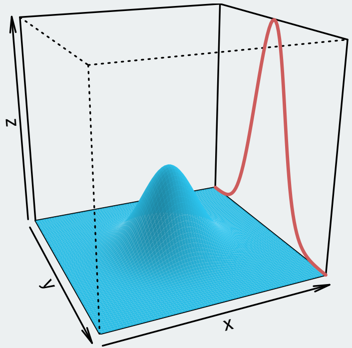

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- Pile up/flatten all of the joint density onto a single dimension.

Continuous conditional distributions

Definition

The **conditional p.d.f.** of a continuous random variable is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

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Continuous conditional distributions

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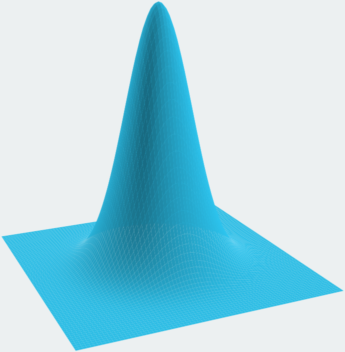
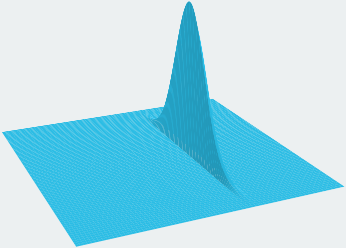
Conditional distributions as slices





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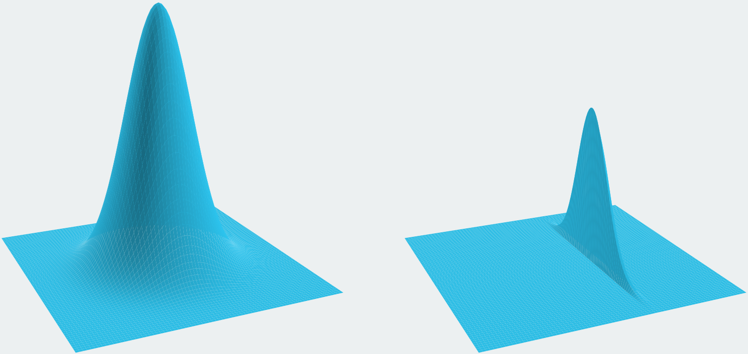

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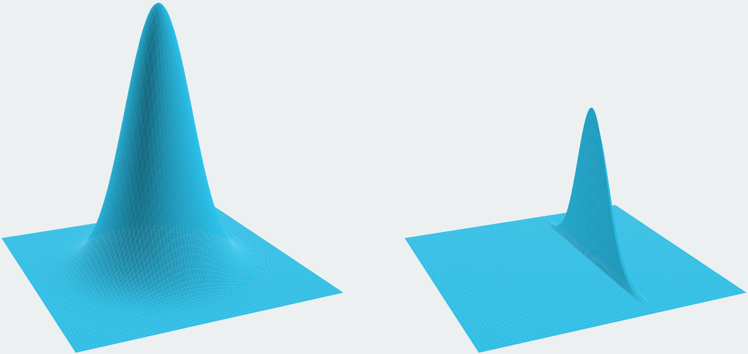

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- **Conditional independence** implies similar to conditional distributions:

$$\mathbb{P}(X \in A, Y \in B \mid Z) = \mathbb{P}(X \in A \mid Z)\mathbb{P}(Y \in B \mid Z)$$

2/ Expectations of Joint Distributions

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3/ Covariance and Correlation

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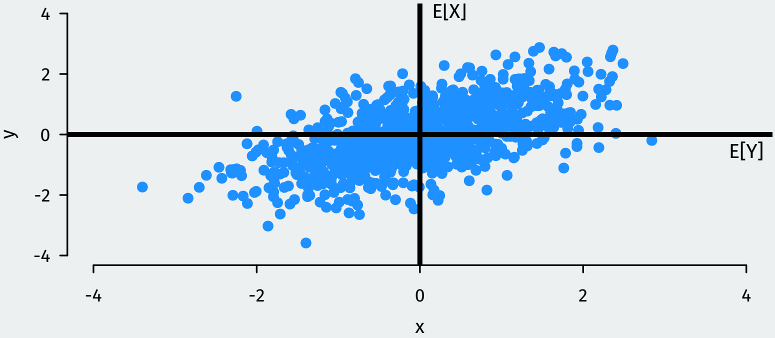

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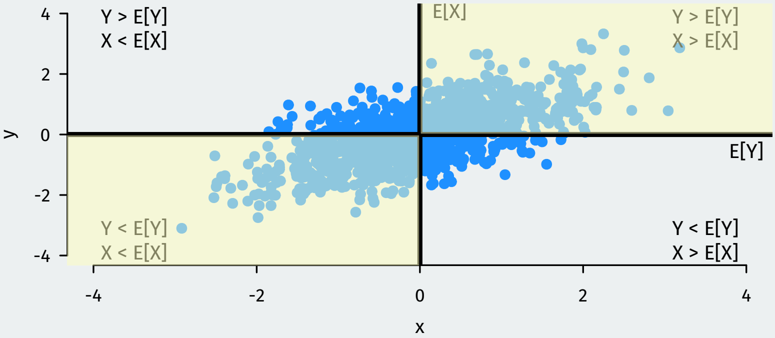

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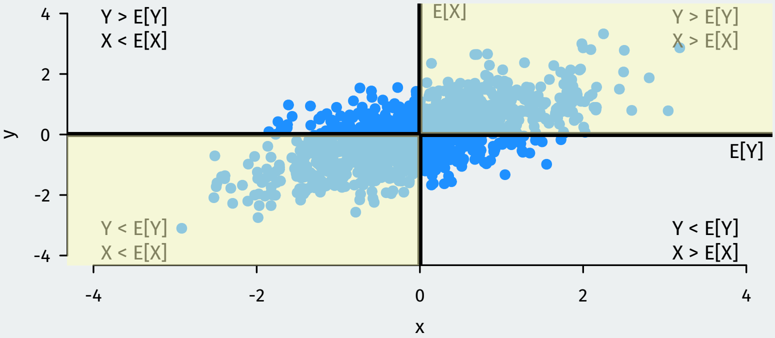

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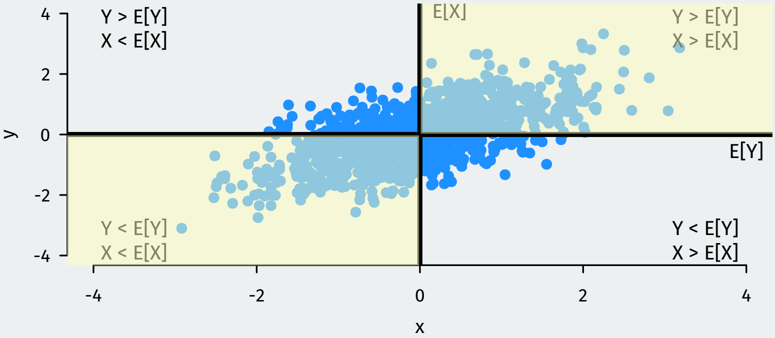

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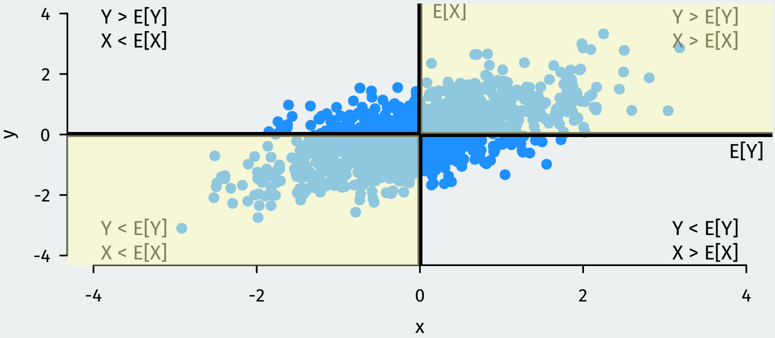

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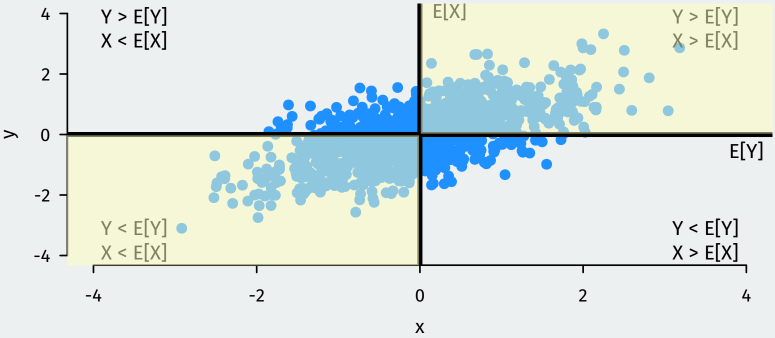

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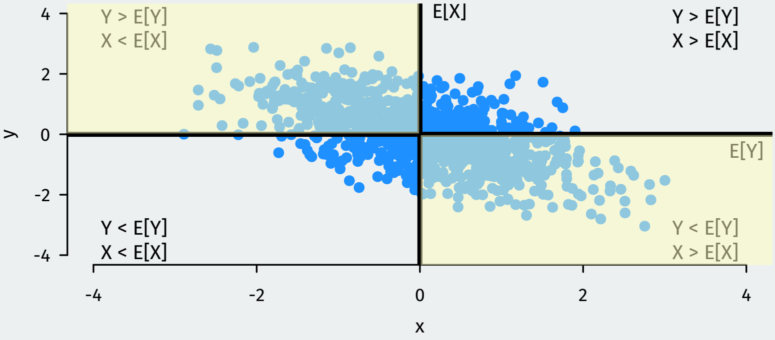

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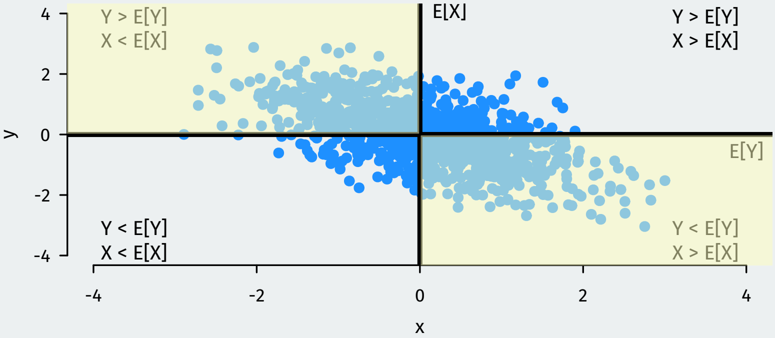

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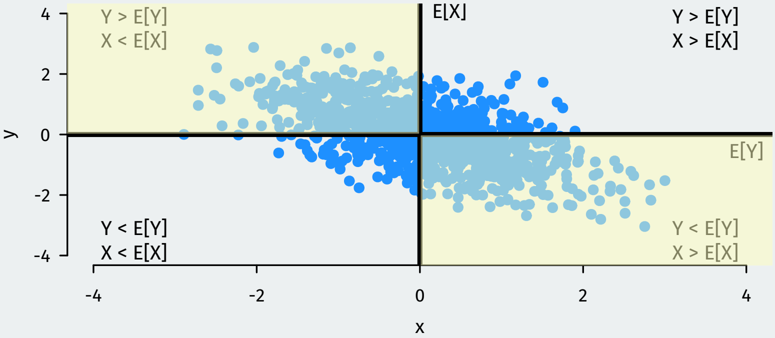

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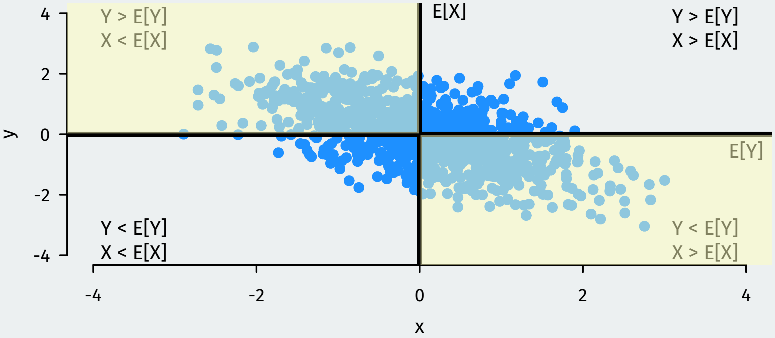

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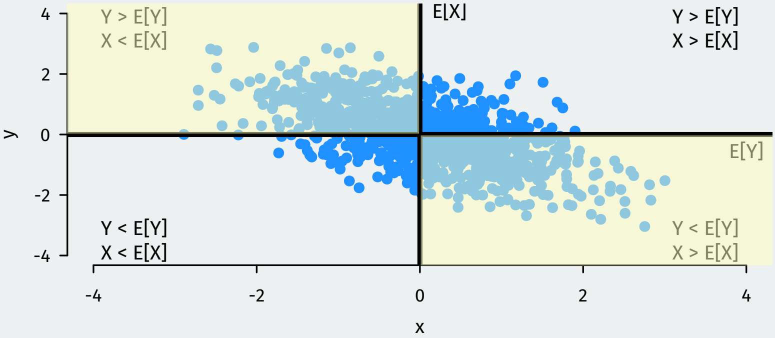

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    - Beware: $\mathbb{V}[X-Y] = \mathbb{V}[X] + \mathbb{V}[Y]$ as well.

Zero covariance doesn't imply independence

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  - $|\rho(X, Y)| = 1$ if and only if X and Y are perfectly correlated with a deterministic linear relationship: $Y = a + bX$.

4/ Random vectors

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- More generally, if $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $\mathbf{Y} = \mathbf{a} + \mathbf{BX} \sim \mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')$

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