

# 7. Conditional Expectation

Spring 2023

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Gov 2002 (Harvard)

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- Time to preview a feature of these distributions we'll care a lot about: conditional expectations.
- At its core: how the average of one variable varies with others.

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## Definition

The **conditional expectation** of  $Y$  conditional on  $\mathbf{X} = \mathbf{x}$  is:

$$\mu(\mathbf{x}) = \mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}] = \begin{cases} \sum_y y \mathbb{P}(Y = y \mid \mathbf{X} = \mathbf{x}) & \text{discrete } Y \\ \int_{-\infty}^{\infty} y f_{Y|\mathbf{X}}(y \mid \mathbf{x}) dy & \text{continuous } Y \end{cases}$$

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- Viewed as a function of  $x$ , it is the **conditional expectation function (CEF)**
  - How does the average value of  $Y$  change given different levels of  $\mathbf{X}$ ?

# Conditional expectation example

	Support Gay Marriage $Y = 1$	Oppose Gay Marriage $Y = 0$
Female $X = 1$	0.30	0.21
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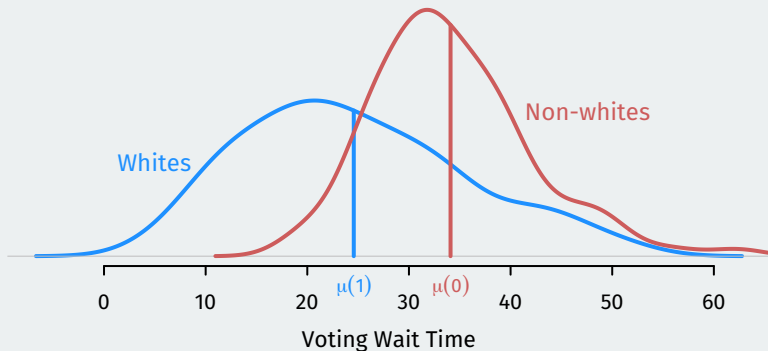
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  - $Y_i$  is the time respondent  $i$  waited in line to vote.
  - $X_i = 1$  for whites,  $X_i = 0$  for non-whites.
- Then the mean in each group is just a conditional expectation:

$$\mu(\text{white}) = E[Y_i | X_i = \text{white}]$$

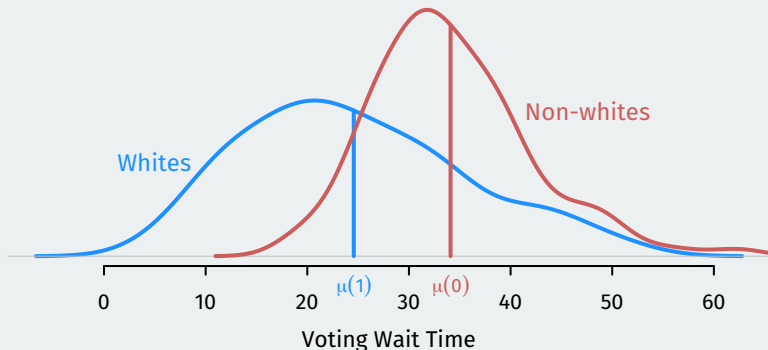
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# Why is the CEF useful?



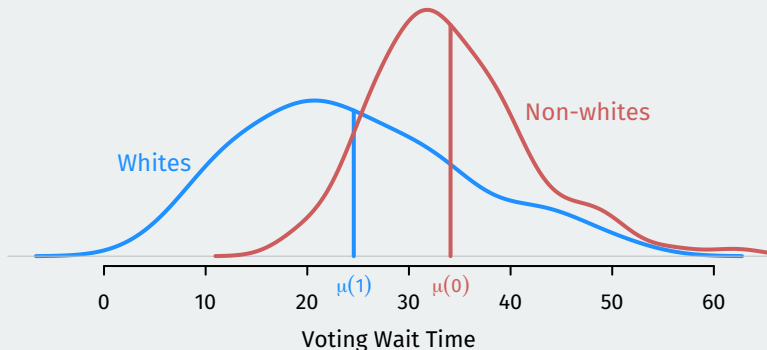
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- Indicates a relationship **in the population** between race and wait times.

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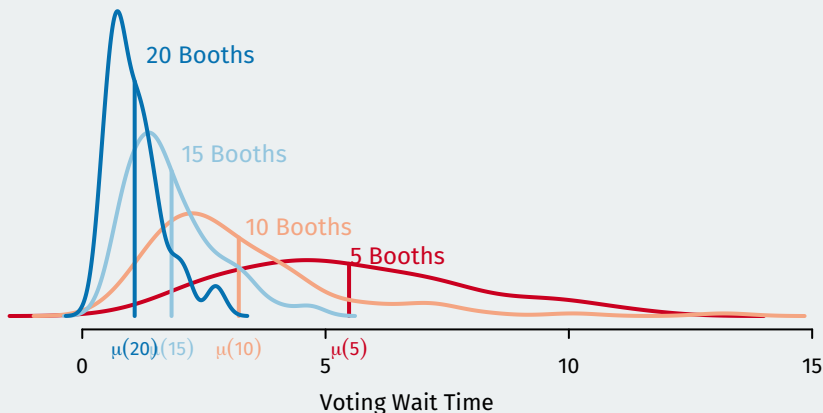
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- Why? Allows more credible **all else equal** comparisons (ceteris paribus).
- Ex: average difference in wait times between white and non-white citizens **of the same gender**:

$$\mu(\text{white, man}) - \mu(\text{non-white, man})$$

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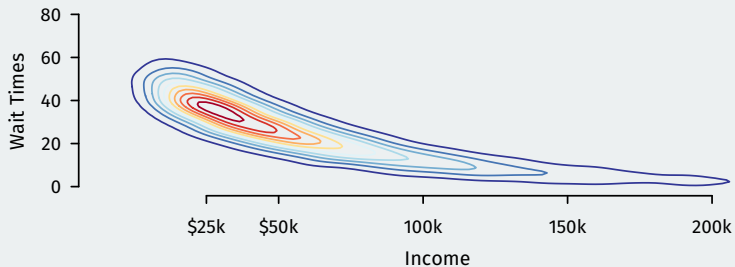
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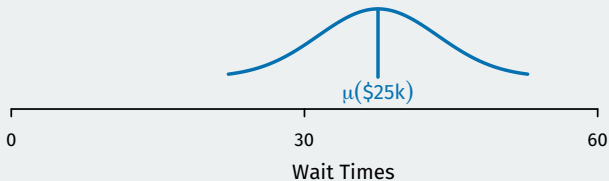
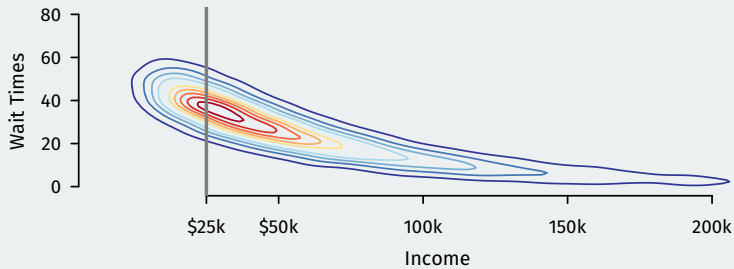
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- These are **unknown functions in the population!** This is going to make producing an estimator  $\hat{\mu}(x)$  very difficult!

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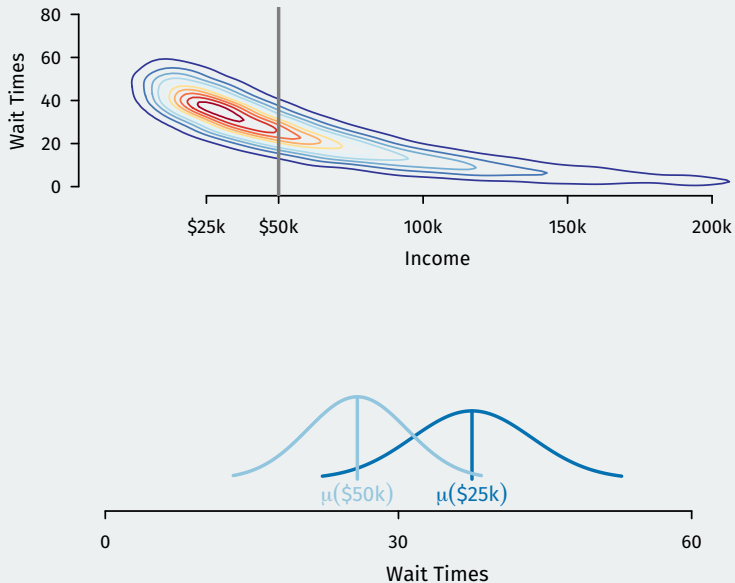




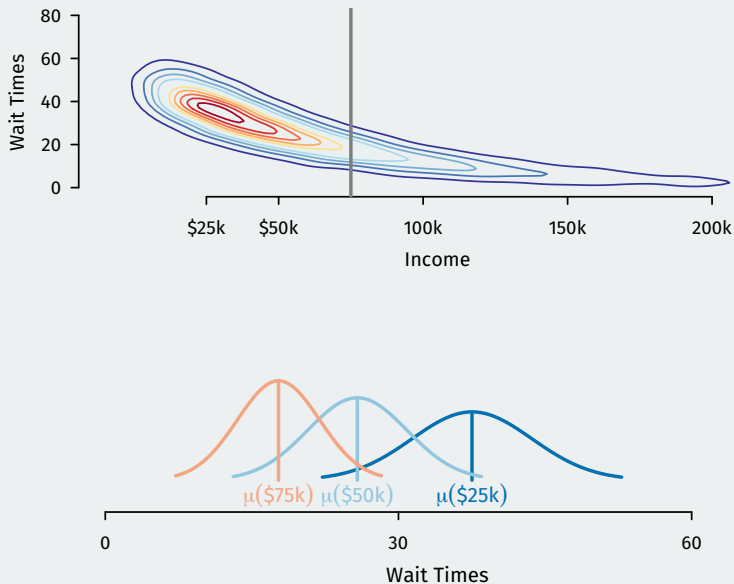
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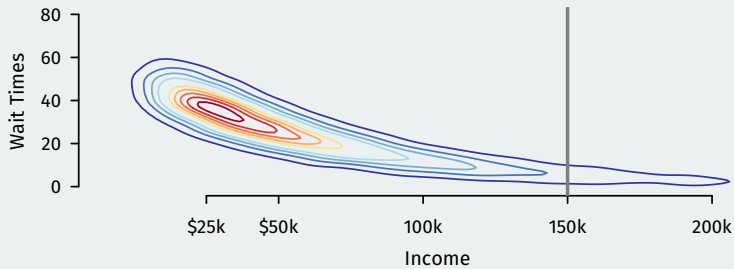
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- Has an expectation,  $\mathbb{E}[\mathbb{E}[Y \mid X]]$ , and a variance,  $\mathbb{V}[\mathbb{E}[Y \mid X]]$ .

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If  $\mathbb{E}|Y| < \infty$ , for any random vector  $\mathbf{X}$ ,  $\mathbb{E}\{\mathbb{E}[Y \mid \mathbf{X}]\} = E[Y]$ .

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- “Averaging” over what is not constant ( $\mathbf{X}_2$ ).

## Example: law of iterated expectations

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The **conditional variance** of a  $Y$  given  $\mathbf{X} = \mathbf{x}$  is defined as:

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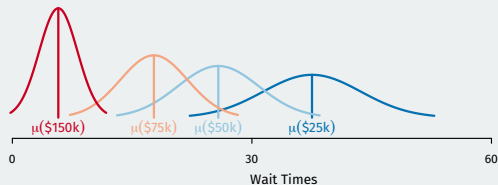
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- Can re-express in the usual way:

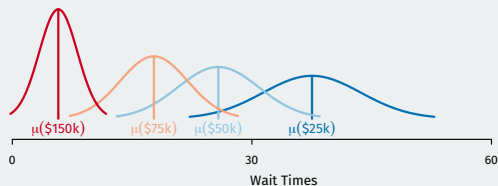
$$\mathbb{V}[Y \mid \mathbf{X} = \mathbf{x}] = \mathbb{E}[Y^2 \mid \mathbf{X} = \mathbf{x}] - (\mathbb{E}[Y \mid \mathbf{X} = \mathbf{x}])^2$$

# Skedasticity



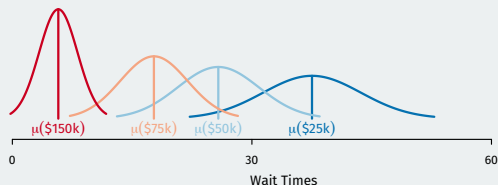
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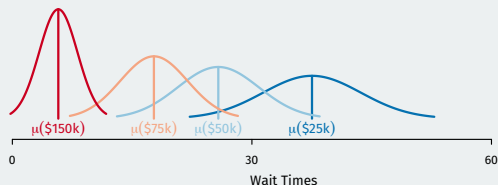
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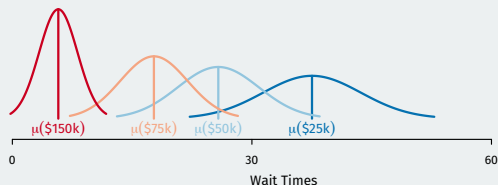
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- Default assumption should be the less restrictive one: heteroskedastic

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