

13. Properties of Least Squares

Spring 2023

Matthew Blackwell

Gov 2002 (Harvard)

Where are we? Where are we going?

- Before: learned about CEFs and linear projections in the population.

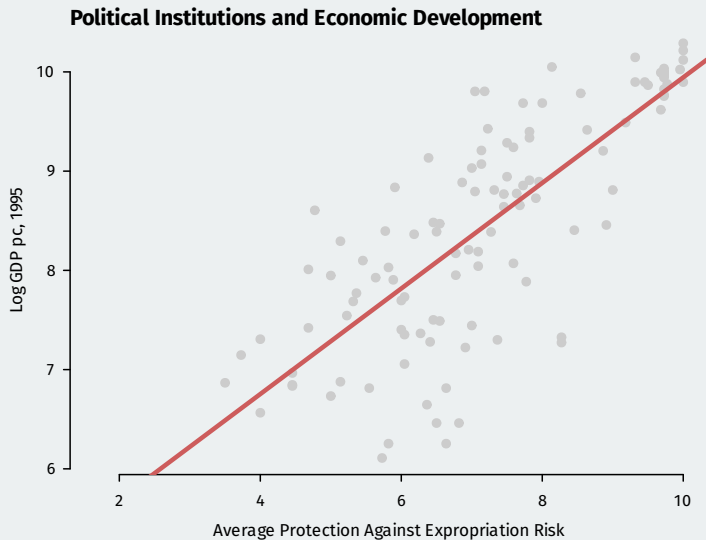
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- Before: learned about CEFs and linear projections in the population.
- Last time: OLS estimator, its algebraic properties.
- Now: its statistical properties, both finite-sample and asymptotic.

Acemoglu, Johnson, and Robinson (2001)



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- OLS is an estimator—we plug data into and we get out estimates.

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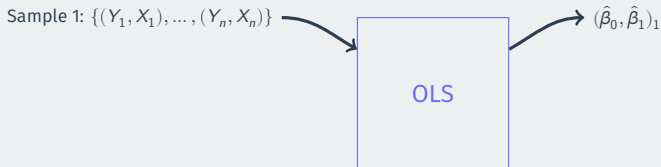
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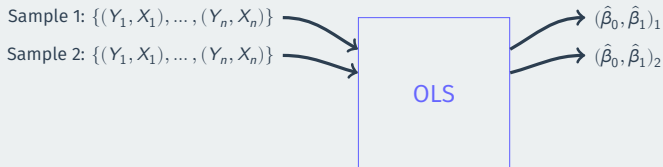
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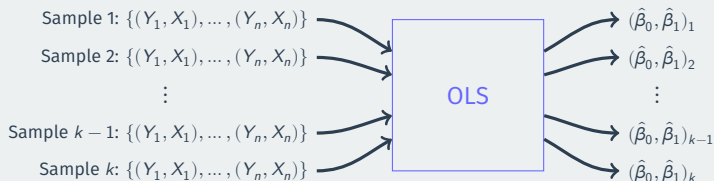
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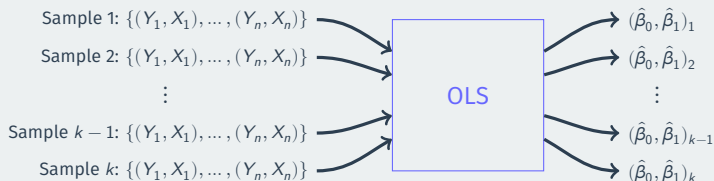
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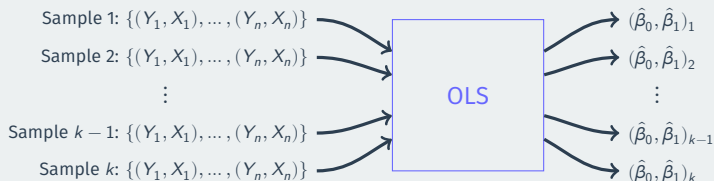
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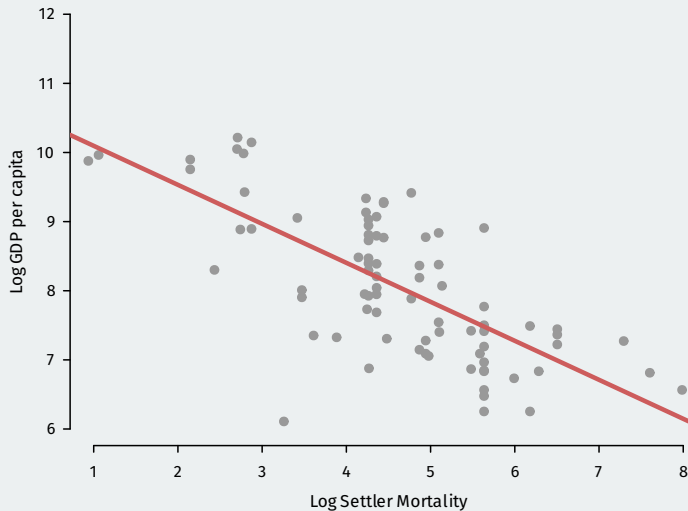
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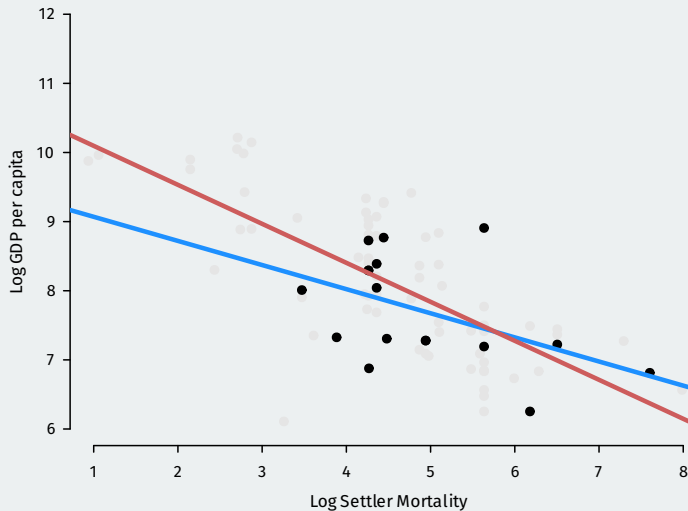
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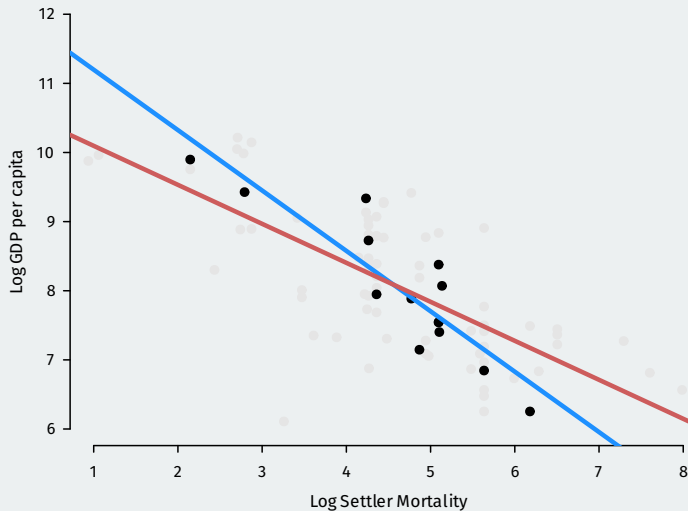
Population Regression



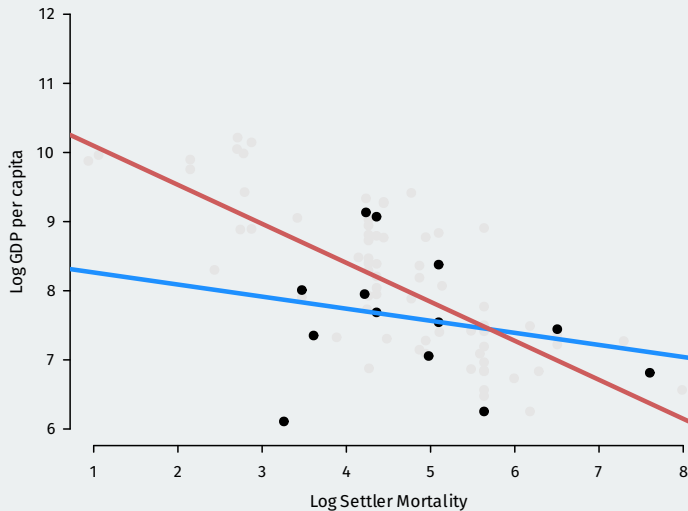
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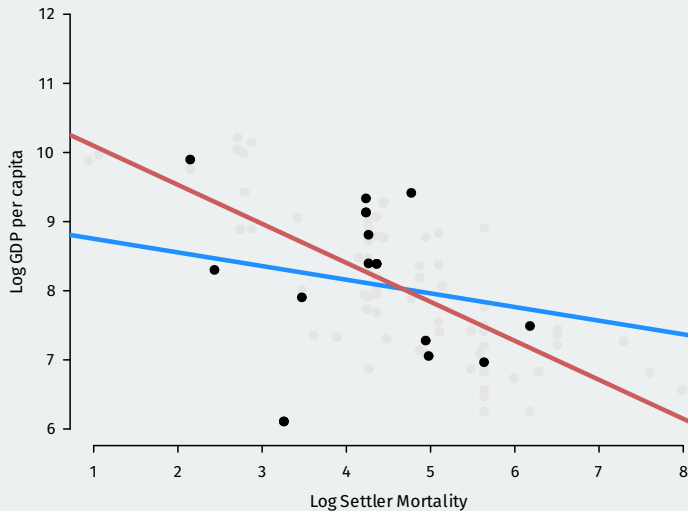
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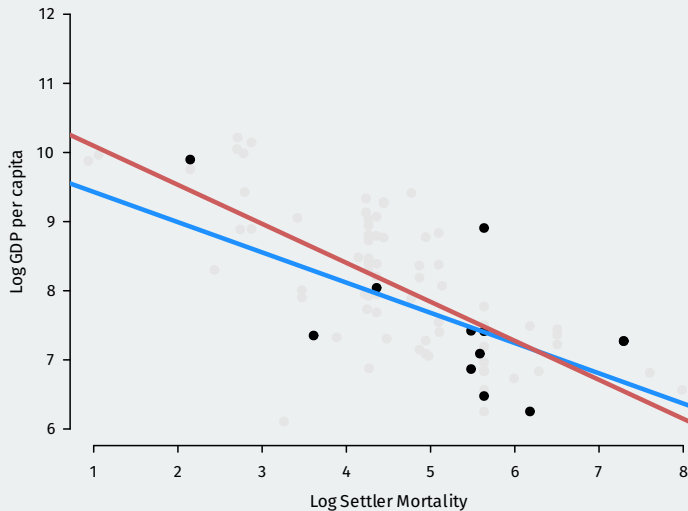
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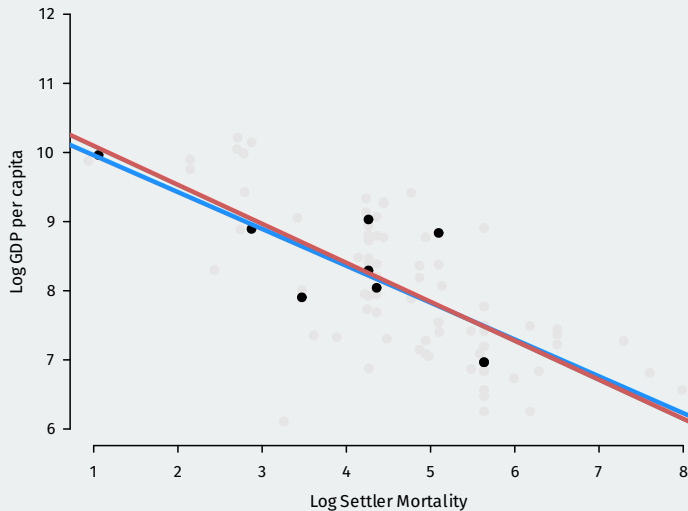
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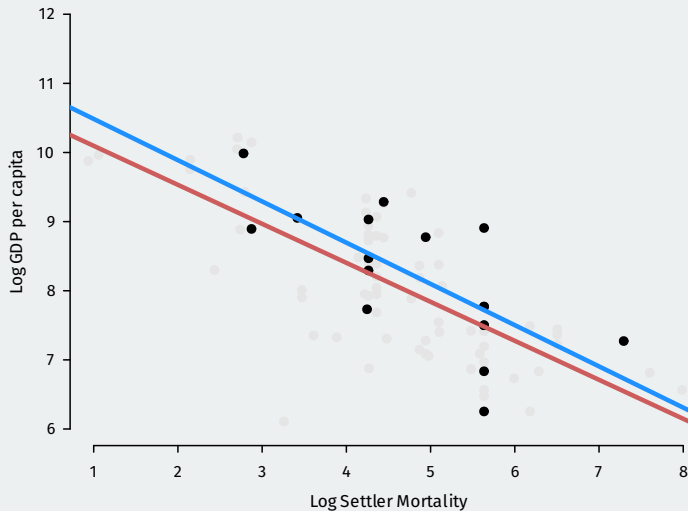
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 - **Linear regression/CEF model** for finite samples.

1/ Linear projection model and Large-sample Properties

Linear projection model

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1. For the variables (Y, \mathbf{X}) , we assume the linear projection of Y on \mathbf{X} is defined as:

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- What properties can we derive under such weak assumptions?

A very useful decomposition

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- $\mathbf{Q}_{\mathbf{xx}}$ is invertible by assumption, so by the continuous mapping theorem:

$$\left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i' \right)^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{xx}}^{-1} \implies \hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta} + \mathbf{Q}_{\mathbf{xx}}^{-1} \cdot \mathbf{0} = \boldsymbol{\beta},$$

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- Valid with no restrictions on Y : could be binary, discrete, etc.
- Not guaranteed to be unbiased (unless CEF is linear, as we'll see...)

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- Remember that $(n^{-1} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i')^{-1} \xrightarrow{p} \mathbf{Q}_{\mathbf{xx}}^{-1}$ so we have

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 - Rewrite as \sqrt{n} times an average of i.i.d. mean-zero random vectors.
- Let $\boldsymbol{\Omega} = \mathbb{E}[e_i^2 \mathbf{x}_i \mathbf{x}_i']$ and apply the CLT:

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbf{x}_i e_i \right) \xrightarrow{d} \mathcal{N}(0, \boldsymbol{\Omega})$$

Asymptotic normality

Theorem (Asymptotic Normality of OLS)

Under the linear projection model,

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- Allows us to formulate (approximate) confidence intervals, tests.

Estimating OLS variance

$$\mathbf{V}_\beta = \mathbf{Q}_{\mathbf{X}\mathbf{X}}^{-1} \mathbf{\Omega} \mathbf{Q}_{\mathbf{X}\mathbf{X}}^{-1}$$

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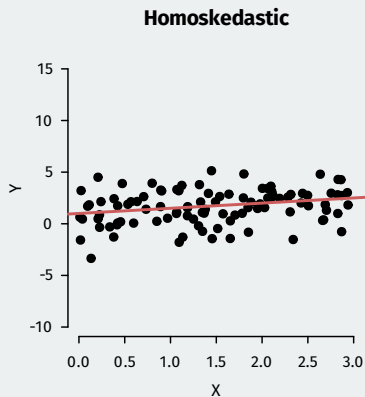
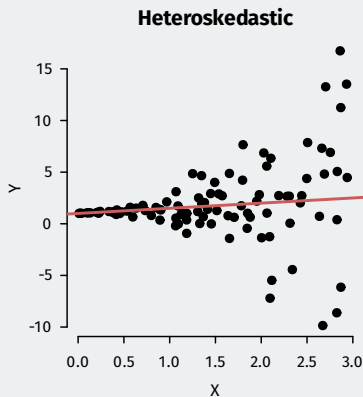
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heteroskedasticity-consistent (HC) SEs

Homoskedasticity

Assumption: Homoskedasticity

The variance of the error terms is constant in \mathbf{X} , $\mathbb{E}[e^2 | \mathbf{X}] = \sigma^2(\mathbf{X}) = \sigma^2$.



Consequences of homoskedasticity

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- LLN implies $s^2 \xrightarrow{P} \sigma^2$ and so $n \widehat{\mathbf{V}}_{\hat{\boldsymbol{\beta}}}^{\text{lm}}$ is consistent for $\mathbf{V}_{\boldsymbol{\beta}}^{\text{lm}}$

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2/ Inference for Multiple Parameters

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 - Need to normalize like the t-statistic.

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- If this covariance matrix were identity, then these would be standard normal and $\hat{\beta}_1^2 + \hat{\beta}_3^2$ would be χ^2_2 under the null

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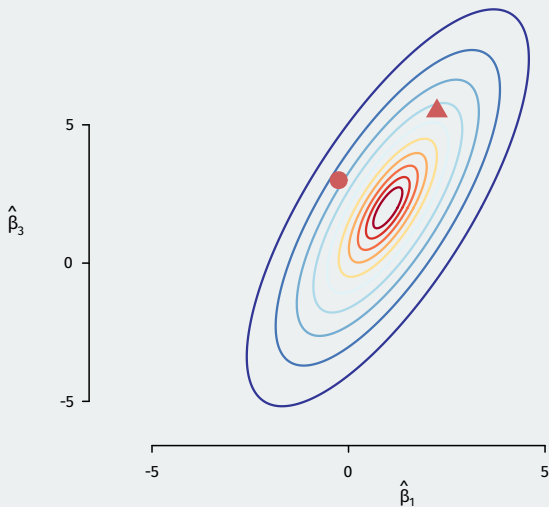
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Weighting by the distribution



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 - Use packages like `{aod}` or `{clubSandwich}` in R.

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- By design, no effect of any variable on any other.

Multiple test example

```
noise <- data.frame(matrix(rnorm(2100), nrow = 100, ncol = 21))
summary(lm(noise))
```

```
##
## Coefficients:
##           Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.028039   0.113820  -0.25  0.8061
## X2          -0.150390   0.112181  -1.34  0.1839
## X3           0.079158   0.095028   0.83  0.4074
## X4          -0.071742   0.104579  -0.69  0.4947
## X5           0.172078   0.114002   1.51  0.1352
## X6           0.080852   0.108341   0.75  0.4577
## X7           0.102913   0.114156   0.90  0.3701
## X8          -0.321053   0.120673  -2.66  0.0094 **
## X9          -0.053122   0.107983  -0.49  0.6241
## X10          0.180105   0.126443   1.42  0.1583
## X11          0.166386   0.110947   1.50  0.1377
## X12          0.008011   0.103766   0.08  0.9387
## X13          0.000212   0.103785   0.00  0.9984
## X14          -0.065969   0.112214  -0.59  0.5583
## X15          -0.129654   0.111575  -1.16  0.2487
## X16          -0.054446   0.125140  -0.44  0.6647
## X17          0.004335   0.112012   0.04  0.9692
## X18          -0.080796   0.109853  -0.74  0.4642
## X19          -0.085806   0.118553  -0.72  0.4713
## X20          -0.186006   0.104560  -1.78  0.0791 .
## X21          0.002111   0.108118   0.02  0.9845
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.999 on 79 degrees of freedom
## Multiple R-squared:  0.201, Adjusted R-squared: -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.48
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 - Example: $0.05/20 = 0.0025$
 - Ensures that the family-wise error rate (probability of making at least 1 Type I error) is less than α .

3/ Linear Regression Model and Finite-sample Properties

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- We continue to maintain $\{(Y_i, \mathbf{X}_i)\}$ are i.i.d.

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- Useful when linearity holds by default (discrete X in experiments, etc)

Linear CEF under homoskedasticity

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- Upshot: OLS will have the smaller SEs than any other linear estimator.

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- With reasonable n , asymptotic normality has the same effect.