# 6. Multivariate Distributions

Spring 2023

Matthew Blackwell

Gov 2002 (Harvard)

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- Today: **distributions of multiple variables** to describe relationships between variables.
- Later: use data to **learn** about probability distributions.

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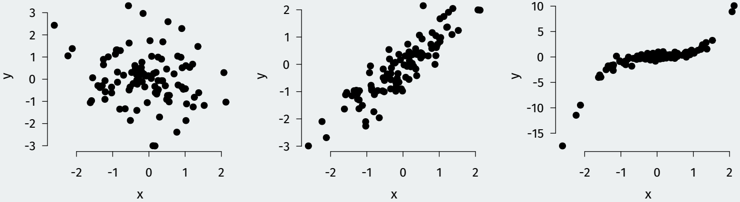
1. How to measure the relationship between two variables $X$ and $Y$?
2. What if we have many observations of the same variable, $X_1, X_2, \dots, X_n$?

# 1/ Distributions of Multiple Random Variables

# Joint distributions

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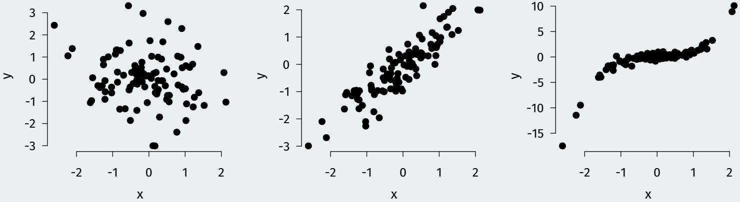


- The **joint distribution** of two r.v.s, $X$ and $Y$, describes what pairs of observations, $(x, y)$ are more likely than others.
- Shape of the joint distribution $\rightsquigarrow$ the relationship between $X$ and $Y$

# Discrete r.v.s

**Definition**

The **joint probability mass function (p.m.f.)** of a pair of discrete r.v.s, $(X, Y)$ describes the probability of any pair of values:

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  - $\sum_x$ is shorthand for sum over all possible values of $X$

# Example: Gay marriage and gender

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| Female $X = 1$ | 0.32 | 0.19 |
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$$p_{X,Y}(1, 1) = \mathbb{P}(X = 1, Y = 1) = 0.32$$

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  - Works because values of $X$ are disjoint.

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$$\mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x)}$$

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- This is a valid univariate probability distribution!
  - $P(Y = y \mid X = x) \geq 0$ and $\sum_y \mathbb{P}(Y = y \mid X = x) = 1$
- Can define the **conditional expectation** of this p.m.f.:

$$E[Y \mid X = x] = \sum_y y\mathbb{P}(Y = y \mid X = x)$$

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$$\mathbb{P}(Y = 1 \mid X = 0)$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20}$$

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$$\mathbb{P}(Y = 1 \mid X = 0) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(X = 0)} = \frac{0.29}{0.29 + 0.20} = 0.592$$

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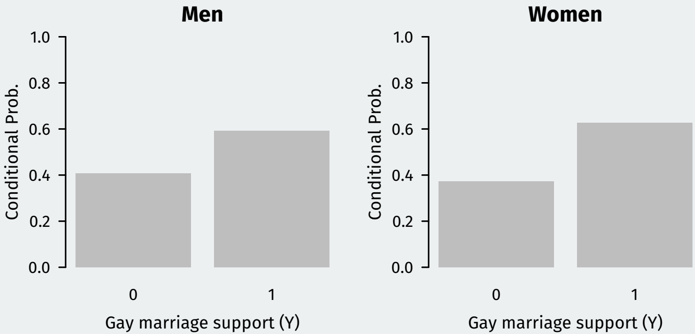


- Two values of $X \rightsquigarrow$ two **univariate** conditional distributions of $Y$

# Bayes and LTP

- Bayes' rule for r.v.s:

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- Law of total probability for r.v.s:

$$\mathbb{P}(X = x) = \sum_{y} \mathbb{P}(X = x \mid Y = y)\mathbb{P}(Y = y)$$

# Joint c.d.f.s

**Definition**

For two r.v.s $X$ and $Y$, the **joint cumulative distribution function** or joint c.d.f. $F_{X,Y}(x,y)$ is a function such that for finite values $x$ and $y$,

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- Well-defined for discrete and continuous $X$ and $Y$.
- For discrete we simply have:

$$F_{X,Y}(x, y) = \sum_{i \leq x} \sum_{j \leq y} \mathbb{P}(X = i, Y = j)$$

# Continuous r.v.s

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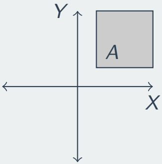
- Two continuous r.v.s: probability of being in some subset of the 2-dimensional plane.



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- $\{(x,y) : f_{X,Y}(x,y) > 0\}$ is called the **support** of the distribution.

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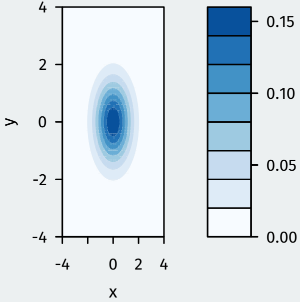
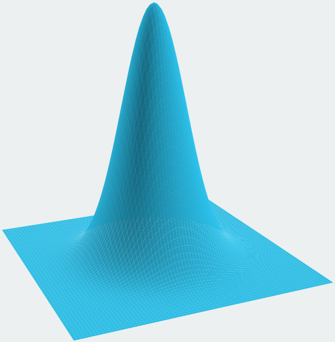
- $\mathbb{P}(X = x, Y = y) = 0$ for similar reasons as with single r.v.s.

# Joint densities are 3D



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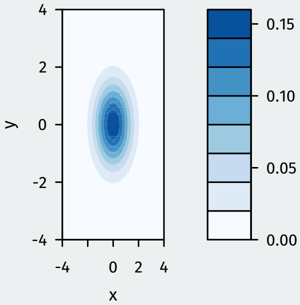
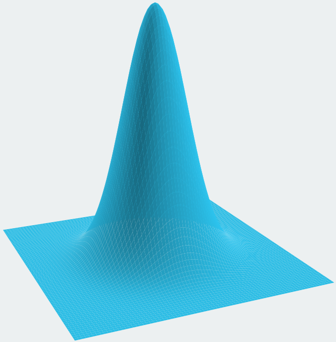
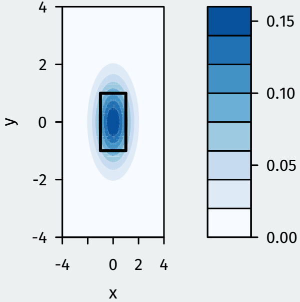
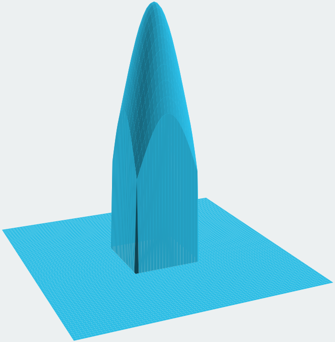


- $X$ and $Y$ axes are on the “floor,” height is the value of $f_{X,Y}(x, y)$.
- Remember $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$.

# Probability = volume



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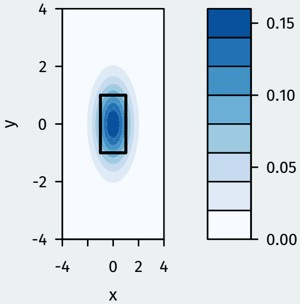


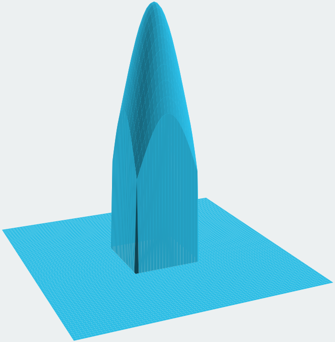

- $\mathbb{P}((X, Y) \in A) = \iint_{(x,y)\in A} f_{X,Y}(x, y)dxdy$
- Probability = volume above a specific region.

# Continuous marginal distributions

- We can recover the marginal PDF of one of the variables by integrating over the distribution of the other variable:

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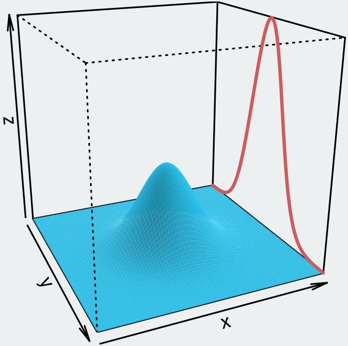
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

- Works for either variable:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

# Visualizing continuous marginals

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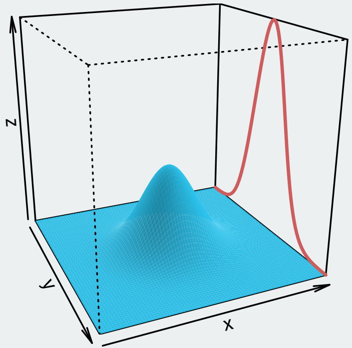


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- Pile up/flatten all of the joint density onto a single dimension.

# Continuous conditional distributions

**Definition**

The **conditional p.d.f.** of a continuous random variable is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

for all values $x$ s.t. $f_X(x) > 0$.

- Implies

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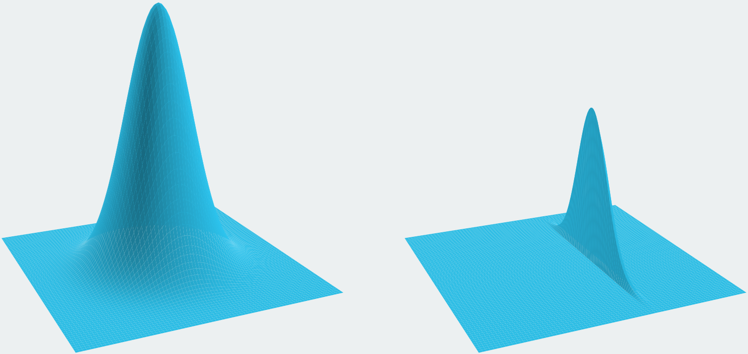
- Based on the definition of the conditional p.m.f./p.d.f., we have the following factorization:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x)$$

# Conditional distributions as slices



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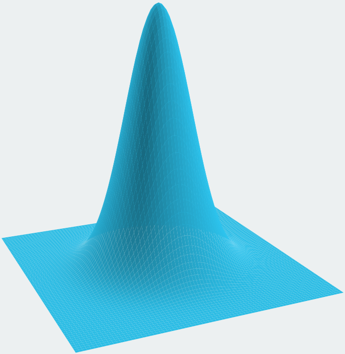

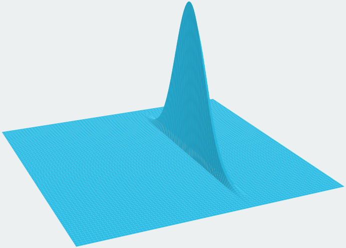

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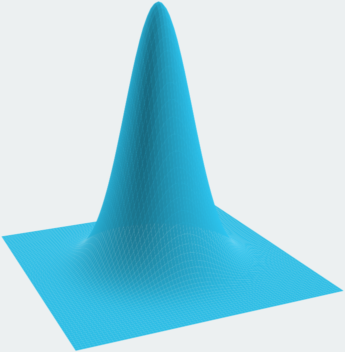

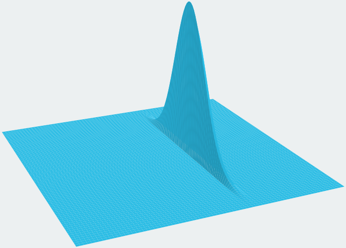

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- Normalize by dividing by $f_X(x_0)$ to ensure proper p.d.f.

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Two r.v.s $Y$ and $X$ are **independent** (which we write $X \perp\!\!\!\perp Y$) if for all sets $A$ and $B$:

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

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- **Conditional independence** implies similar to conditional distributions:

$$\mathbb{P}(X \in A, Y \in B \mid Z) = \mathbb{P}(X \in A \mid Z)\mathbb{P}(Y \in B \mid Z)$$

# 2/ Expectations of Joint Distributions

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# 3/ Covariance and Correlation

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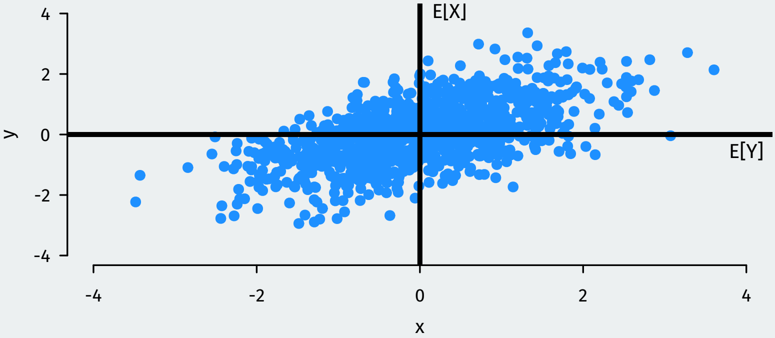

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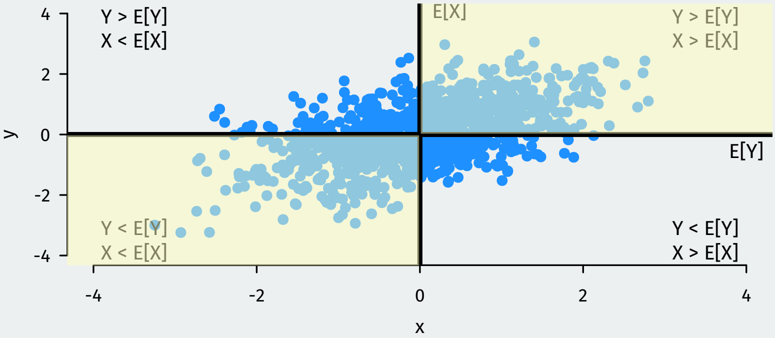


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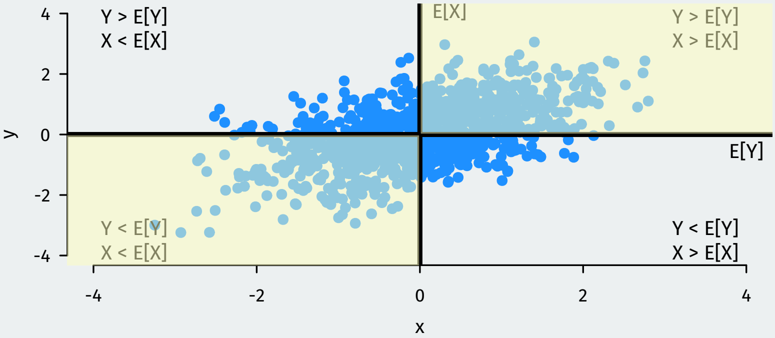


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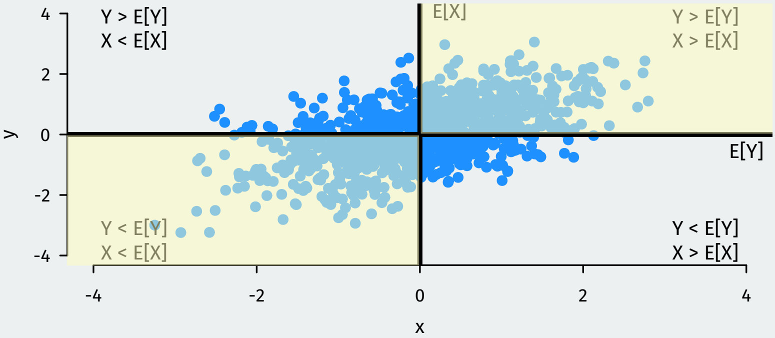


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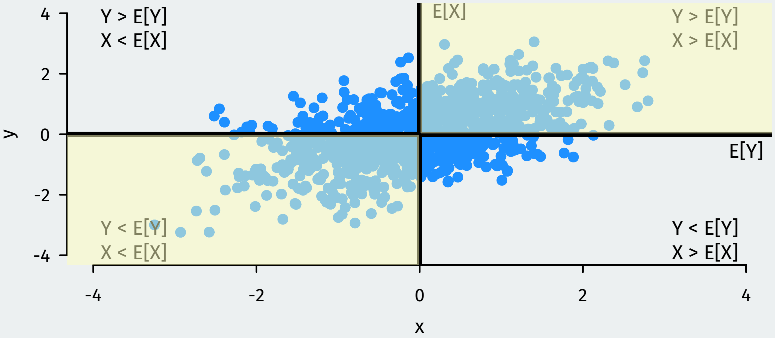


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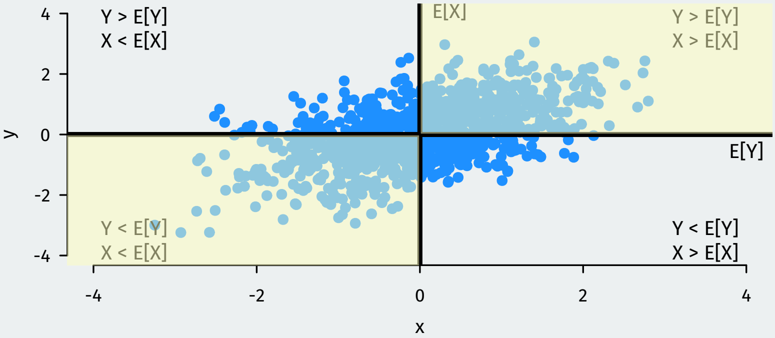


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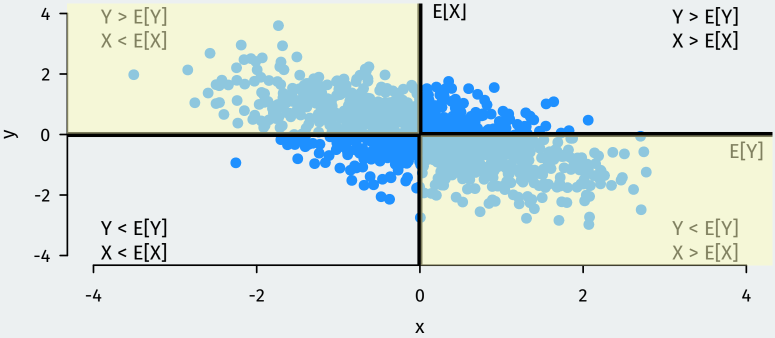


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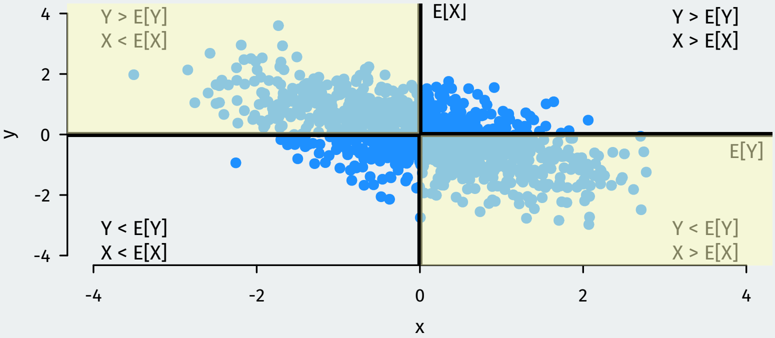


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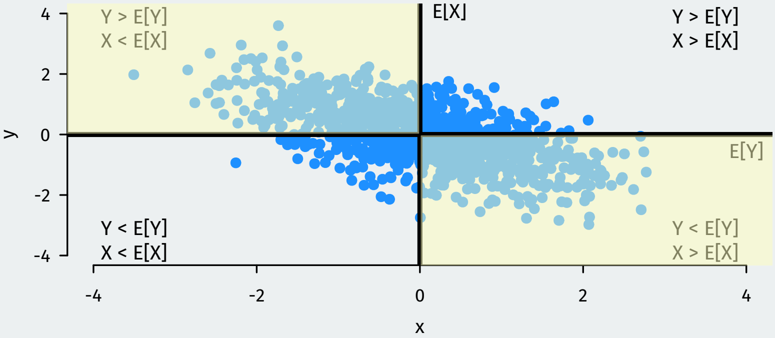


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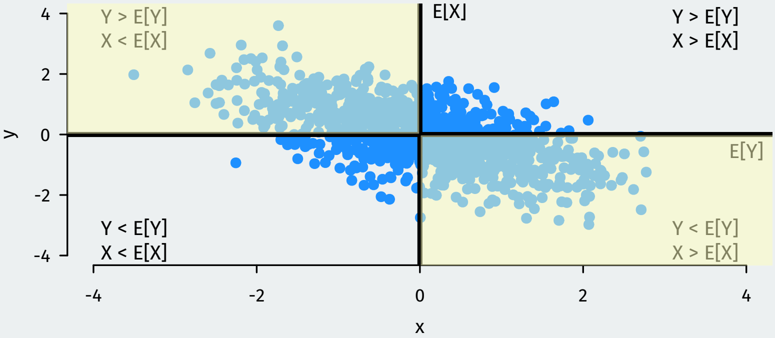


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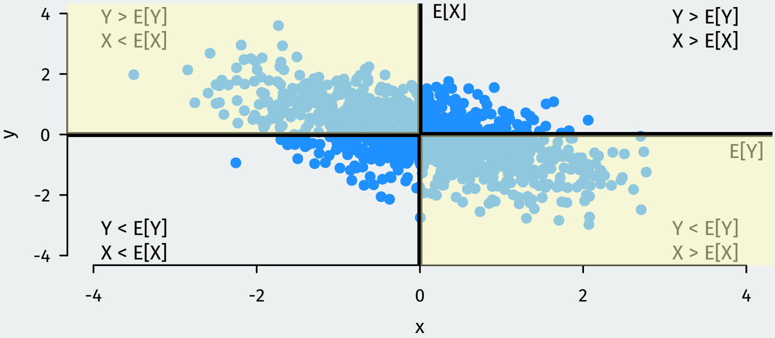


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  - $|\rho(X, Y)| = 1$ if and only if $X$ and $Y$ are perfectly correlated with a deterministic linear relationship: $Y = a + bX$.

# 4/ Random vectors

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- More generally, if $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $\mathbf{Y} = \mathbf{a} + \mathbf{BX} \sim \mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')$

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