12. Algebra of Least Squares

Spring 2023

Matthew Blackwell

Gov 2002 (Harvard)

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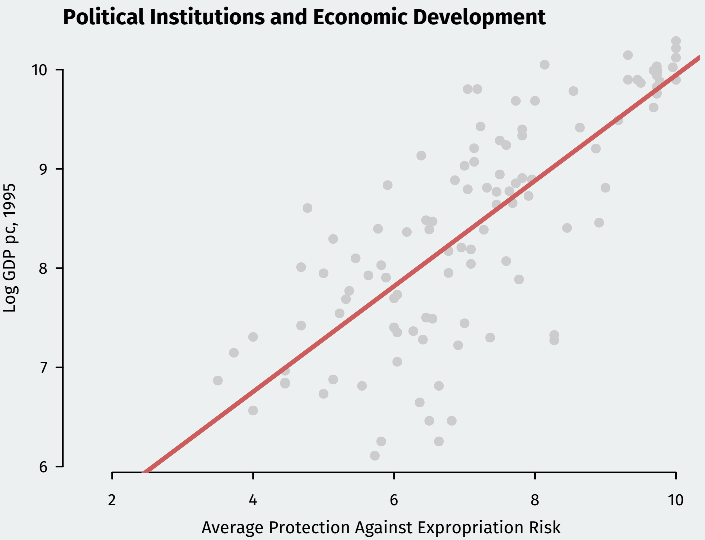
- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

Acemoglu, Johnson, and Robinson (2001)



1/ Deriving the OLS estimator

Samples vs population

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- Violations include time-series data and clustered sampling.
  - Weakening i.i.d. usually complicates notation but can be done.

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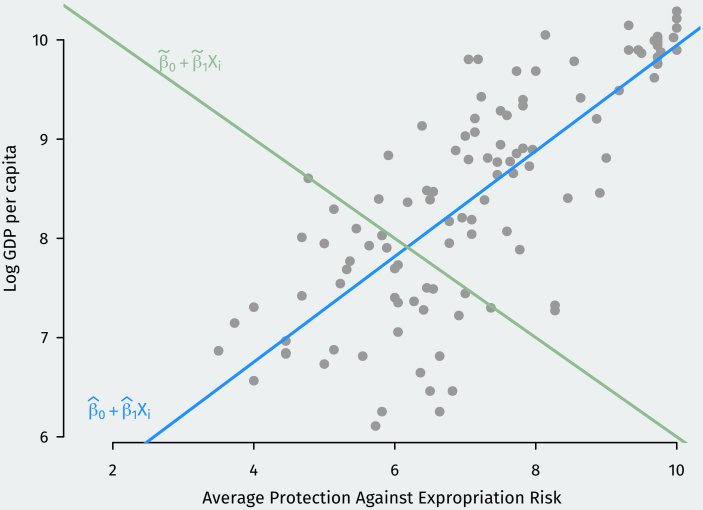

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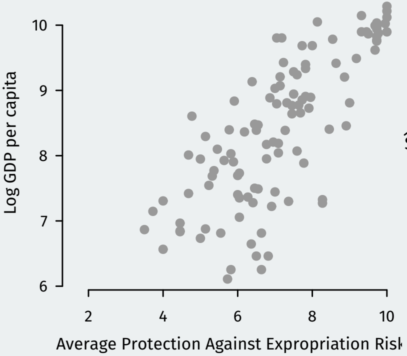
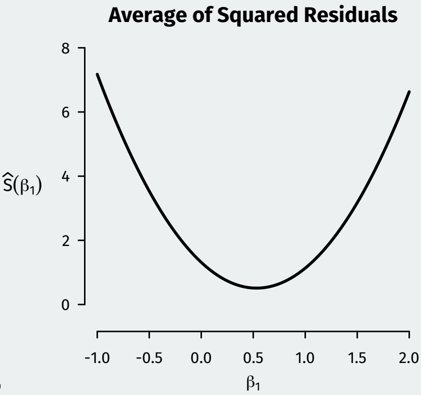
- We can show the OLS estimator of the slope is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\mathbb{V}}[X]}$$

Visualizing OLS





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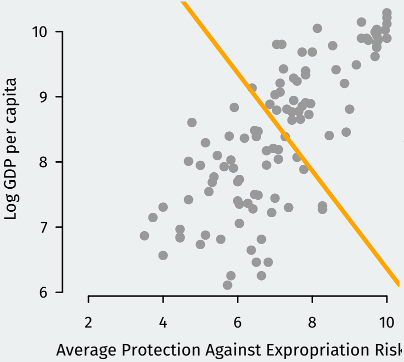

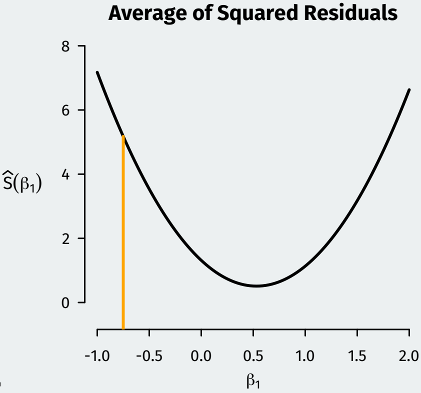

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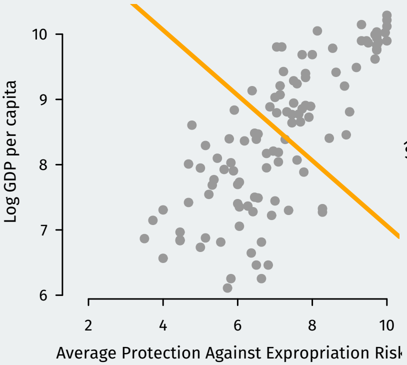

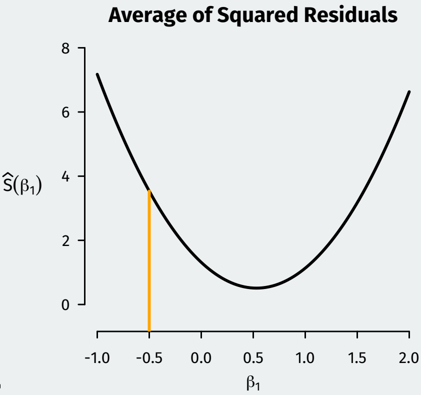

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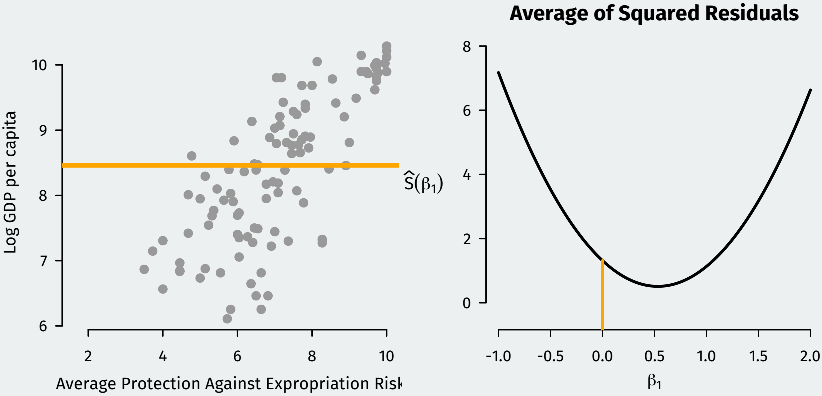

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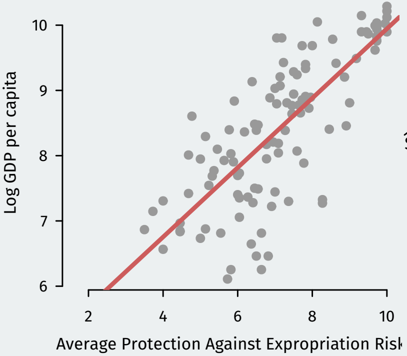

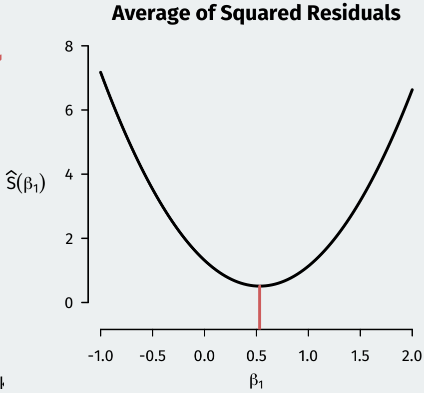

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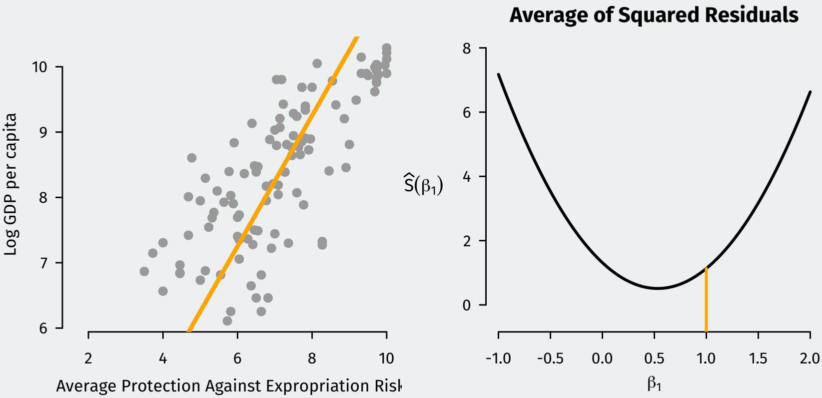

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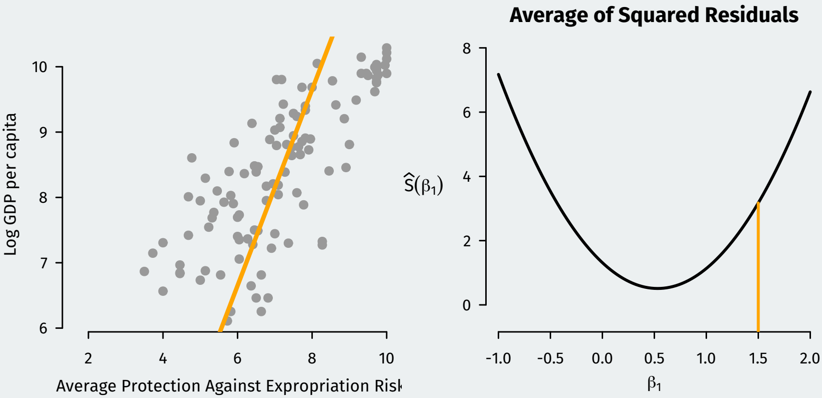

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2/ Model fit

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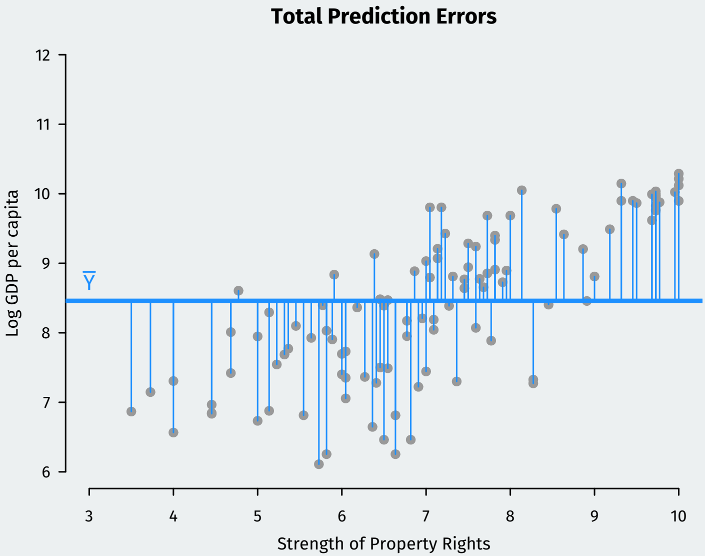
Total SS vs SSR



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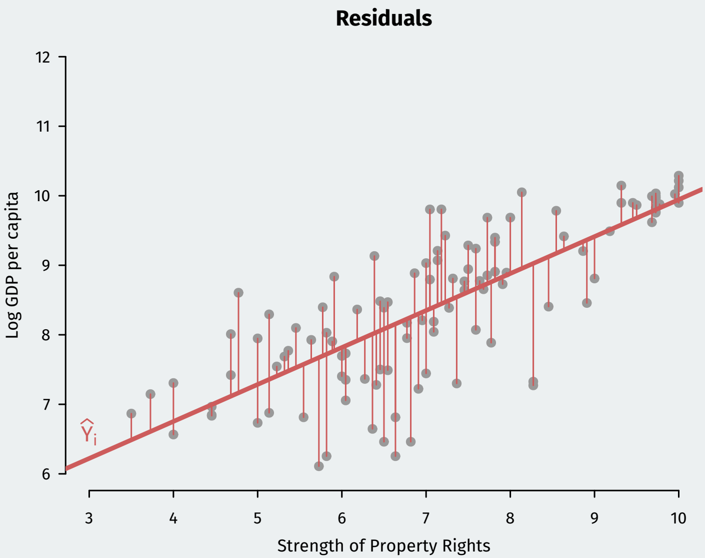

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- $R^2 =$ fraction of the total prediction error eliminated by using $\mathbf{X}_i$.

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- Mechanically increases with additional covariates (better fit measures exist)

3/ Geometry of OLS

Linear model in matrix form

- Linear model is a system of n linear equations:

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Y_1 &= \mathbf{X}_1'\boldsymbol{\beta} + e_1 \\
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- Model is now just:

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\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \mathbf{e}
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OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

$$\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' = \mathbb{X}'\mathbb{X}$$

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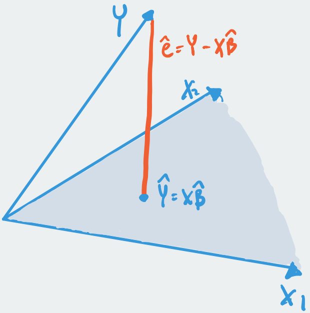

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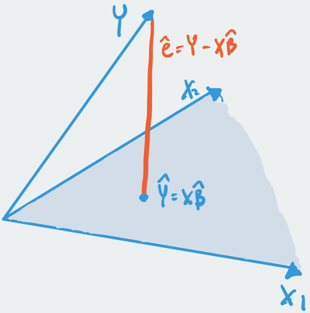

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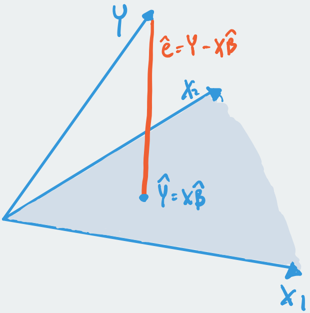

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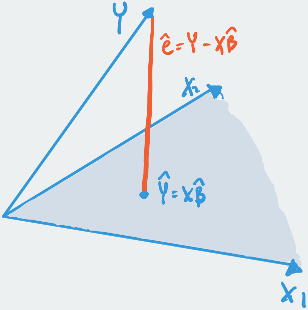

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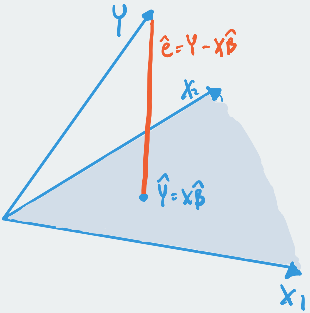

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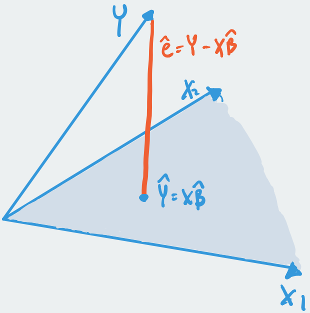

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  - $\mathbf{P}$ is **idempotent**: $\mathbf{PP} = \mathbf{P}$

Projection/hat matrix

- We can define the transformation of $\mathbf{Y}$ that does the projection.

$$\mathbb{X}\hat{\boldsymbol{\beta}} = \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbf{Y}$$

- **Projection matrix**

$$\mathbf{P} = \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'$$

- Also called the **hat matrix** it puts the "hat" on $\mathbf{Y}$:

$$\mathbf{P}\mathbf{Y} = \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbf{Y} = \mathbb{X}\hat{\boldsymbol{\beta}} = \widehat{\mathbf{Y}}$$

- Key properties:
  - $\mathbf{P}$ is an $n \times n$ symmetric matrix
  - $\mathbf{P}$ is **idempotent**: $\mathbf{P}\mathbf{P} = \mathbf{P}$
  - Projecting $\mathbb{X}$ onto itself returns itself: $\mathbf{P}\mathbb{X} = \mathbb{X}$

Annihilator matrix

- **Annihilator matrix** projects onto the space spanned by the residual:

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P} = \mathbf{I}_n - \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'$$

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- Properties:
    - $\mathbf{M}$ is a symmetric $n \times n$ matrix.
    - $\mathbf{M}$ is idempotent so that $\mathbf{MM} = \mathbf{M}$
    - Admits a nice expression for the residual vector: $\hat{\mathbf{e}} = \mathbf{Me}$

Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \ \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

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- Suppose we add an **orthogonal covariate** $\mathbf{Y} = \mathbf{X}\beta + \mathbf{Z}\gamma + \mathbf{e}$ with $\langle \mathbf{X}, \mathbf{Z} \rangle = 0$.

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  - With exactly orthogonal covariates, multivariate OLS is the same as univariate OLS.
  - Only holds in balanced, designed experiments.

Adding the intercept

- Consider the OLS slope with an intercept:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} - \overline{Y}\mathbf{1} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle} = \frac{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{Y} \rangle}{\langle \mathbf{X} - \overline{X}\mathbf{1}, \mathbf{X} - \overline{X}\mathbf{1} \rangle}$$

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  1. Regress **X** on **1** to get coefficient $\overline{X}$

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- How can we get this?

    1. Regress $\mathbf{X}$ on $\mathbf{1}$ to get coefficient $\overline{X}$
    2. Regress $\mathbf{Y}$ on residuals from step 1, $\mathbf{X} - \overline{X}\mathbf{1}$

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  1. Regress **Z** on $\widetilde{\mathbf{X}} = \mathbf{X} - \overline{X}\mathbf{1}$ on and obtain coefficient $\langle \mathbf{Z}, \widetilde{\mathbf{X}} \rangle / \langle \widetilde{\mathbf{X}}, \widetilde{\mathbf{X}} \rangle$

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  2. Regress $\mathbf{Y}$ on residual from

Visualizing orthogonalization

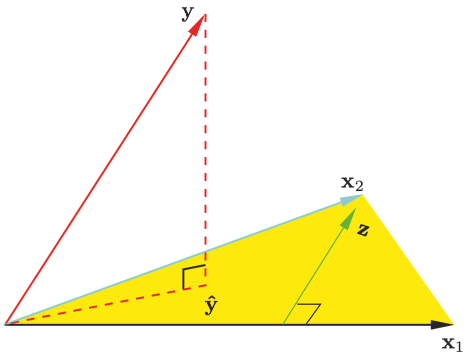

FIGURE 3.4. *Least squares regression by orthogonalization of the inputs. The vector $\mathbf{x}_2$ is regressed on the vector $\mathbf{x}_1$, leaving the residual vector $\mathbf{z}$. The regression of $\mathbf{y}$ on $\mathbf{z}$ gives the multiple regression coefficient of $\mathbf{x}_2$. Adding together the projections of $\mathbf{y}$ on each of $\mathbf{x}_1$ and $\mathbf{z}$ gives the least squares fit $\hat{\mathbf{y}}$.*

Why does residual regression work?

- We can find $\hat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\hat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left(\min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1\boldsymbol{\beta}_1 - \mathbb{X}_2\boldsymbol{\beta}_2\|^2 \right)$$

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- Solving the nested minimization gives:

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$$\hat{\boldsymbol{\beta}}_1 = (\mathbb{X}_1'\mathbf{M}_2\mathbb{X}_1)^{-1}(\mathbb{X}_1'\mathbf{M}_2\mathbf{Y})$$

- When will $\hat{\boldsymbol{\beta}}_1$ will be the same regardless of whether $\mathbb{X}_2$ is included?

Why does residual regression work?

- We can find $\hat{\boldsymbol{\beta}}_1$ by nested minimization:

$$\hat{\boldsymbol{\beta}}_1 = \arg\min_{\boldsymbol{\beta}_1} \left(\min_{\boldsymbol{\beta}_2} \|\mathbf{Y} - \mathbb{X}_1\boldsymbol{\beta}_1 - \mathbb{X}_2\boldsymbol{\beta}_2\|^2 \right)$$

  - First find the minimum of the SSR over $\boldsymbol{\beta}_2$ fixing $\boldsymbol{\beta}_1$
  - Then find $\boldsymbol{\beta}_1$ that minimizes the resulting SSR.

- The projection and annihilator matrices are defined only by covariates.
  - $\mathbf{M}_2 = \mathbf{I}_n - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'$
  - Creates residuals from a regression on or $\mathbb{X}_2$

- Solving the nested minimization gives:

$$\hat{\boldsymbol{\beta}}_1 = (\mathbb{X}_1'\mathbf{M}_2\mathbb{X}_1)^{-1} (\mathbb{X}_1'\mathbf{M}_2\mathbf{Y})$$

- When will $\hat{\boldsymbol{\beta}}_1$ will be the same regardless of whether $\mathbb{X}_2$ is included?
  - If $\mathbb{X}_1$ and $\mathbb{X}_2$ are orthogonal so $\mathbb{X}_2'\mathbb{X}_1 = 0$ so $\mathbf{M}_2\mathbb{X}_1 = \mathbb{X}_1$

Residual regression

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  - Called the **Frisch-Waugh-Lovell Theorem**
  - Sample version of the results we saw for the linear projection.

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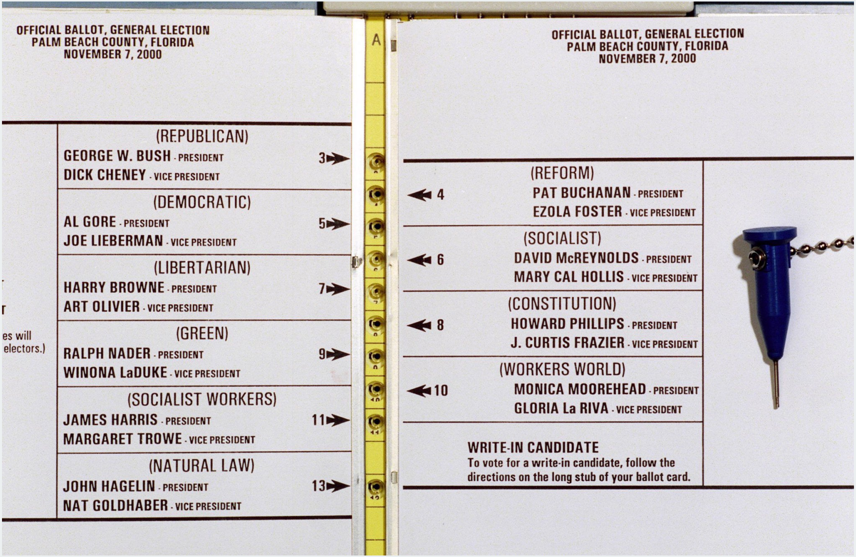
Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)



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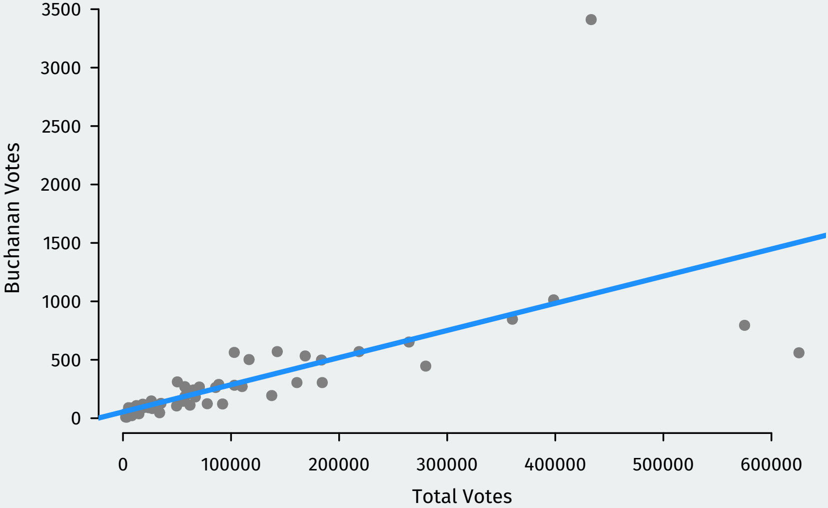

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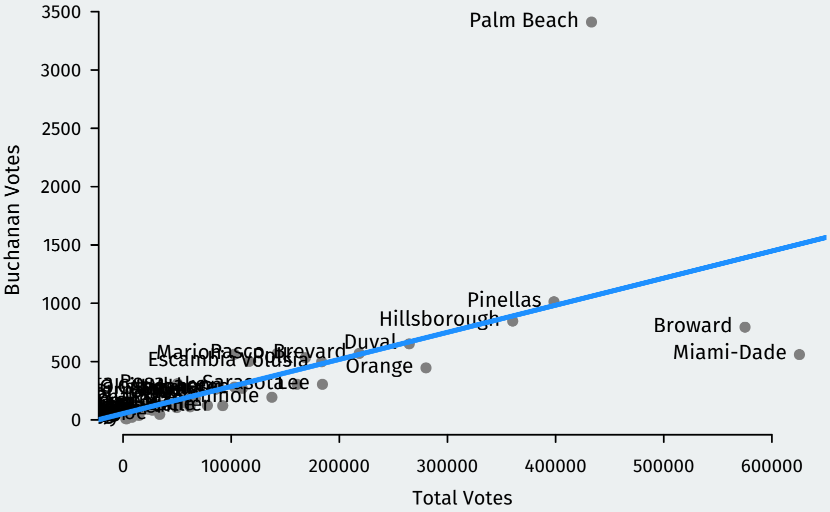
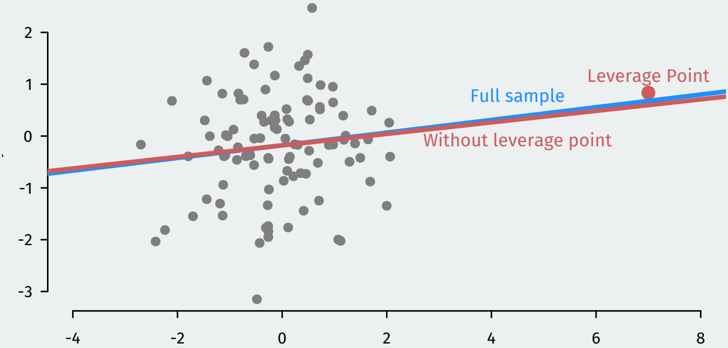


Example: Buchanan votes

```r
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 54.22945   49.14146    1.10     0.27
## edaytotal    0.00232    0.00031    7.48  2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463,  Adjusted R-squared:  0.455
## F-statistic:   56 on 1 and 65 DF,  p-value: 2.42e-10
```

Leverage point definition



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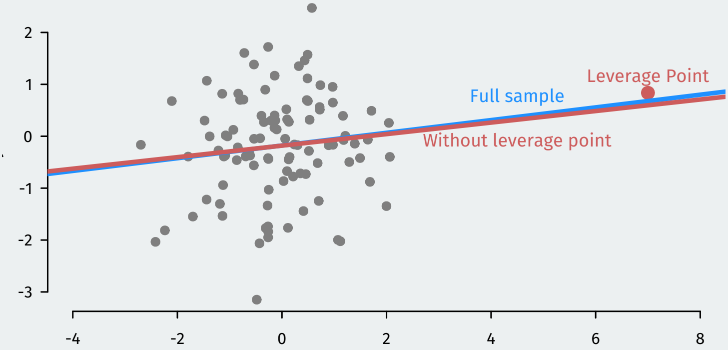

- Values that are extreme in the X dimension
- That is, values far from the center of the covariate distribution

Leverage values

- Let h_{ij} be the (i, j) entry of $\mathbf{P}$. Then:

$$\widehat{\mathbf{Y}} = \mathbf{P}\mathbf{Y} \qquad \Longrightarrow \qquad \widehat{Y}_i = \sum_{j=1}^{n} h_{ij} Y_j$$

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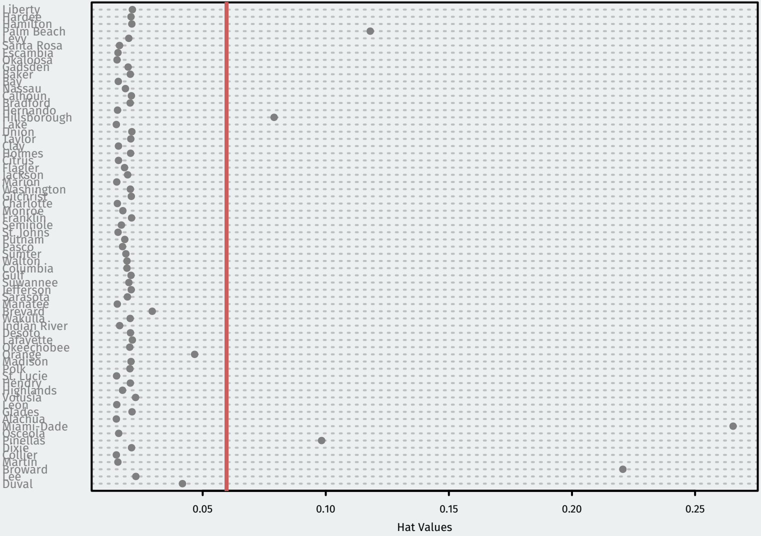
  - $\rightsquigarrow$ how far i is from the center of the X distribution
- **Rule of thumb:** examine hat values greater than $2(k+1)/n$

Buchanan hats

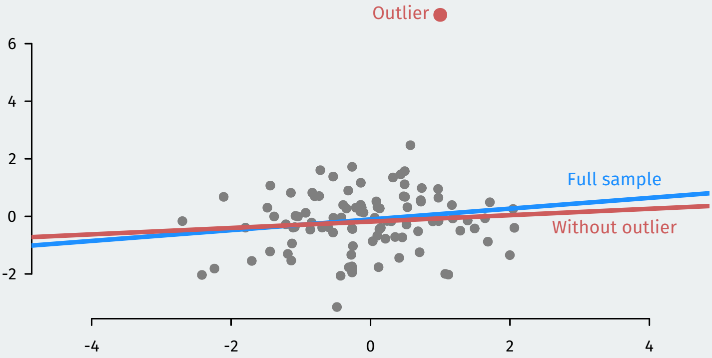
```r
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

Buchanan hats



Outlier definition



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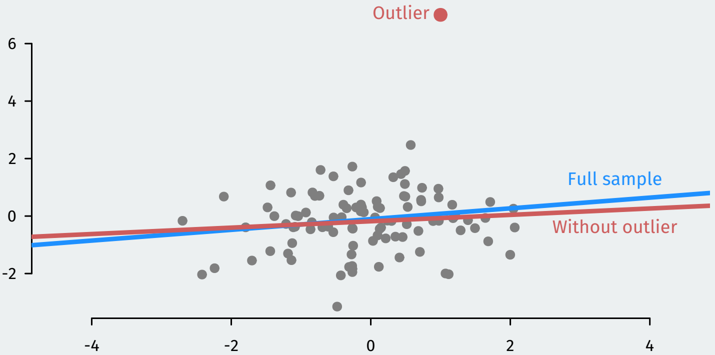

- An **outlier** is far away from the center of the Y distribution.
- Intuitively: a point that would be poorly predicted by the regression.

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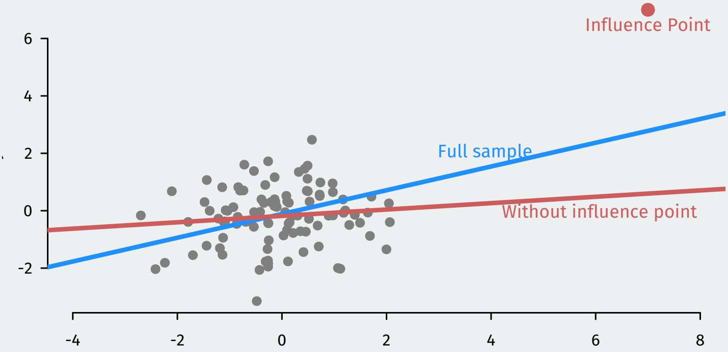
- Simple closed-form expressions:

$$\widehat{\boldsymbol{\beta}}_{(-i)} = \widehat{\boldsymbol{\beta}} - \left(\mathbb{X}'\mathbb{X}\right)^{-1}\mathbf{X}_i\tilde{e}_i \qquad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

Influence points



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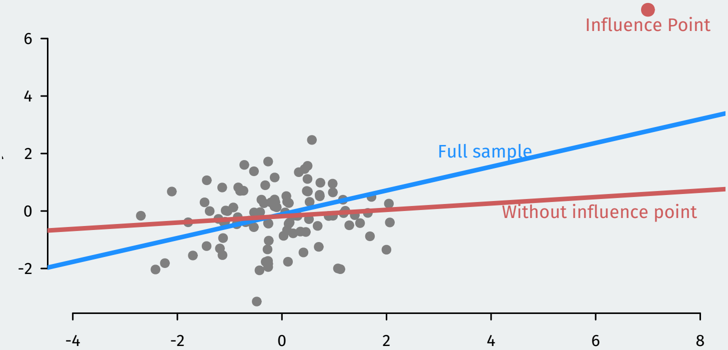
Influence points



- An **influence point** is one that is both an outlier and a leverage point.
- Extreme in both the X and Y dimensions

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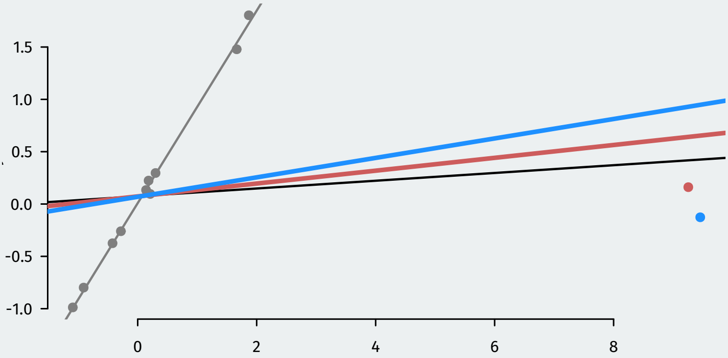
  - Does removing the point change a coefficient by a lot?

Limitations of the standard tools



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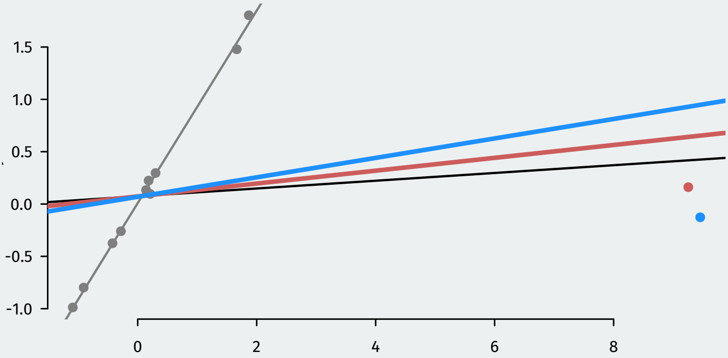

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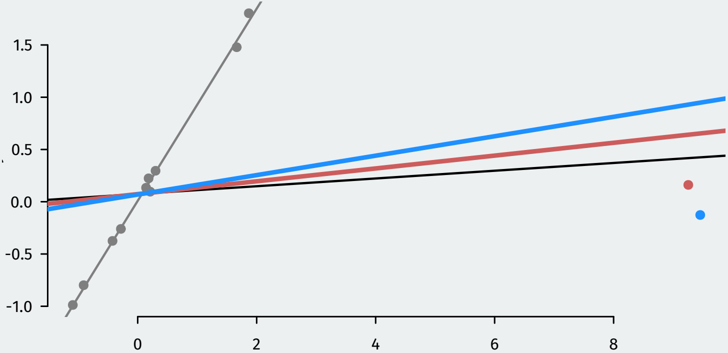

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point

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    - Fix the observation (obvious data entry errors)
    - Remove the observation
    - Be transparent either way
- Is the outlier part of the data generating process?
    - Transform the dependent variable ($\log(y)$)
    - Use a method that is robust to outliers (robust regression, least absolute deviations)