

14. Algebra of Least Squares

Spring 2021

Matthew Blackwell

Gov 2002 (Harvard)

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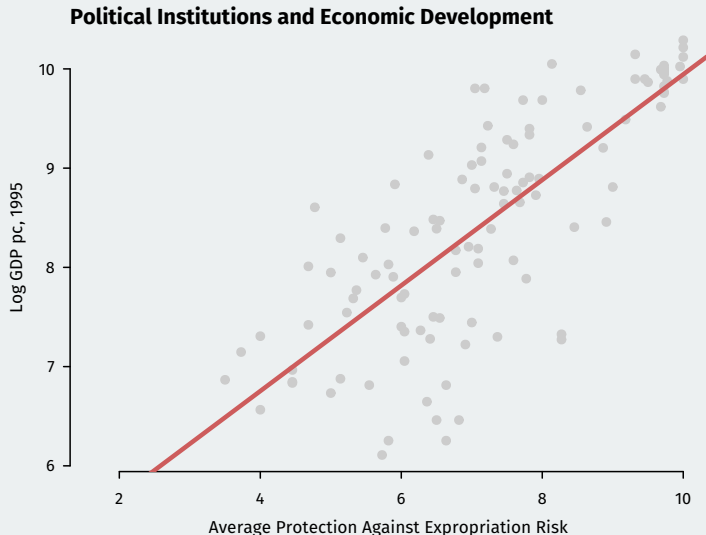
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- Now: least squares estimator and its algebraic properties.
- After that: the statistical properties of least squares.

Acemoglu, Johnson, and Robinson (2001)



Samples vs population

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- Violations include time-series data and clustered sampling.
 - Weakening i.i.d. usually complicates notation but can be done.

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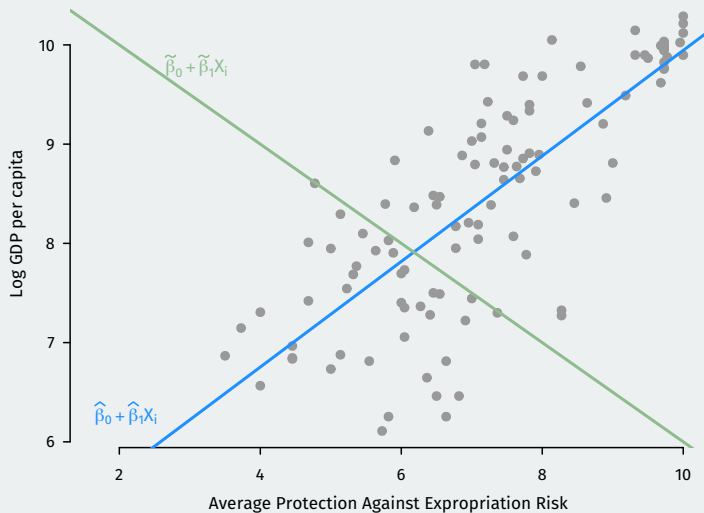
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Which line is better?



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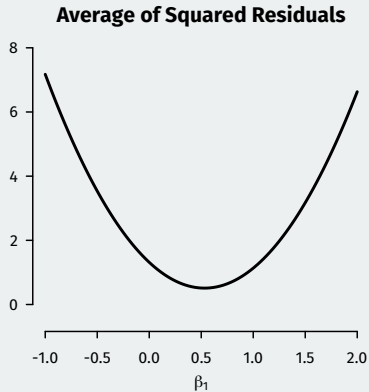
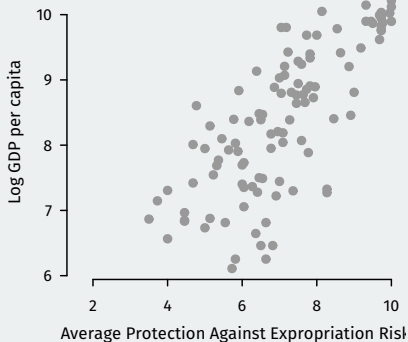
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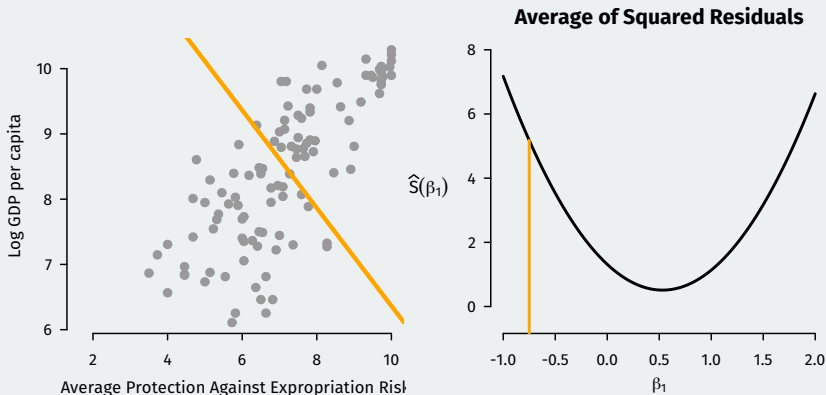
- We can show the OLS estimator of the slope is:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\mathbb{V}}[X]}$$

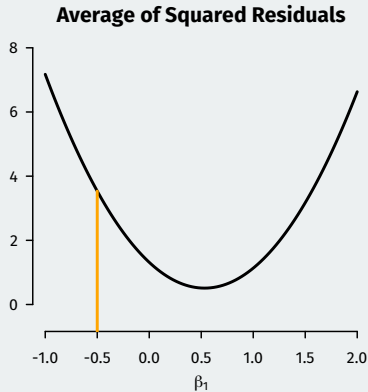
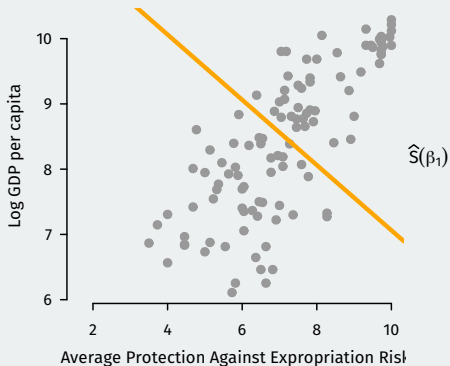
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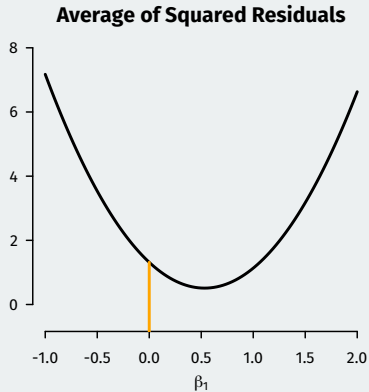
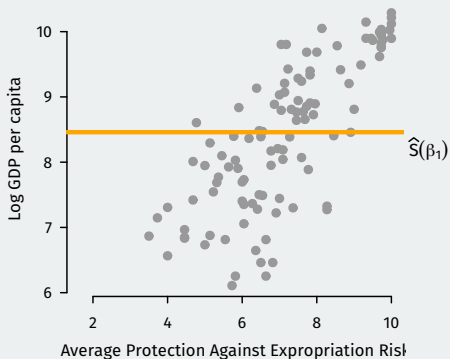
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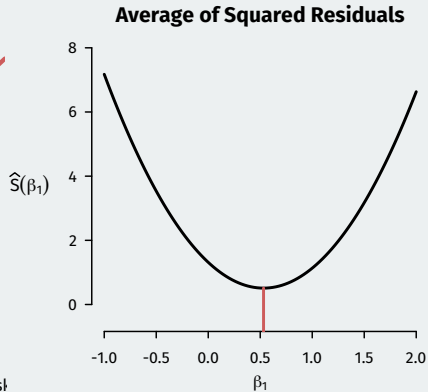
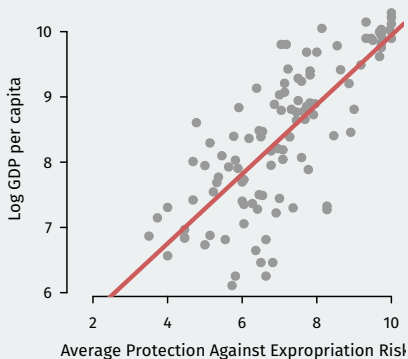
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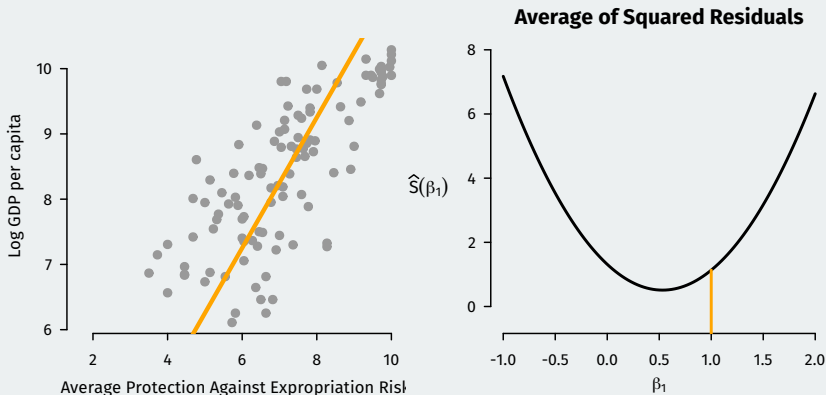
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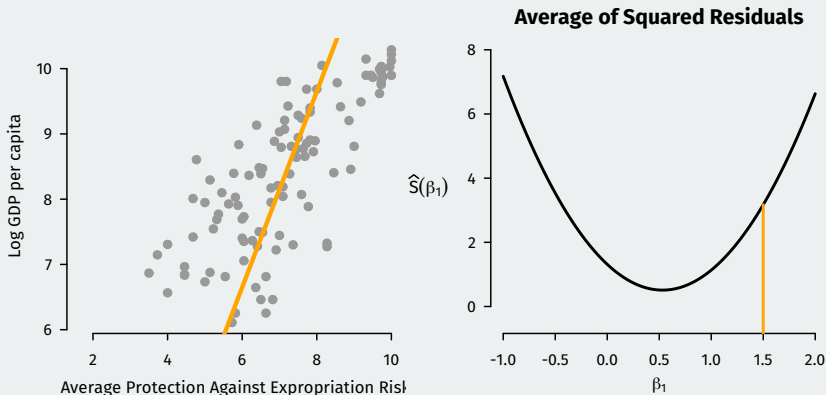
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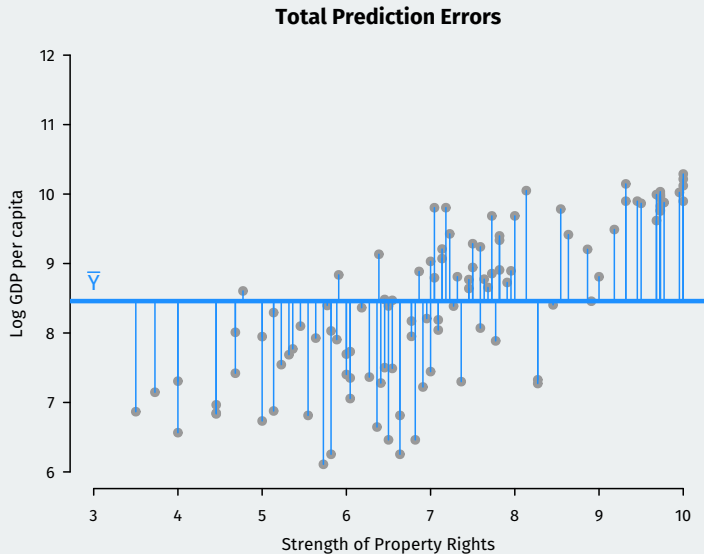
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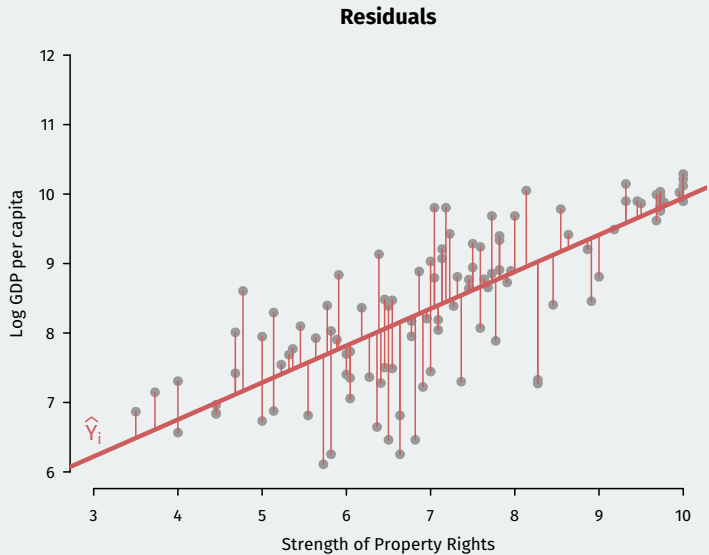
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- R^2 = fraction of the total prediction error eliminated by using \mathbf{X}_i .
- **Common interpretation:** R^2 is the fraction of the variation in Y_i is “explained by” \mathbf{X}_i .
 - $R^2 = 0$ means no relationship
 - $R^2 = 1$ implies perfect linear fit
- Mechanically increases with additional covariates (better fit measures exist)

Linear model in matrix form

- Linear model is a system of n linear equations:

$$Y_1 = \mathbf{X}_1' \boldsymbol{\beta} + e_1$$

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- We can write this more compactly using matrices and vectors:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} \mathbf{X}'_1 \\ \mathbf{X}'_2 \\ \vdots \\ \mathbf{X}'_n \end{pmatrix} = \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{nk} \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

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- Model is now just:

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OLS estimator in matrix form

- Key relationship: sample sums can be written in matrix notation:

$$\sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i' = \mathbb{X}'\mathbb{X}$$

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Least squares in matrix form

- OLS still minimizes sum of the squared residuals

$$\arg \min_{\mathbf{b} \in \mathbb{R}^{k+1}} \hat{\mathbf{e}}' \hat{\mathbf{e}} = \arg \min_{\mathbf{b} \in \mathbb{R}^{k+1}} (\mathbf{Y} - \mathbb{X}\mathbf{b})' (\mathbf{Y} - \mathbb{X}\mathbf{b})$$

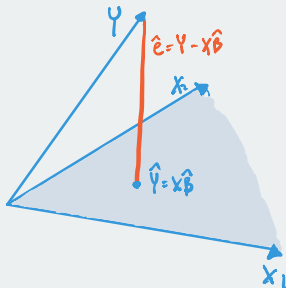
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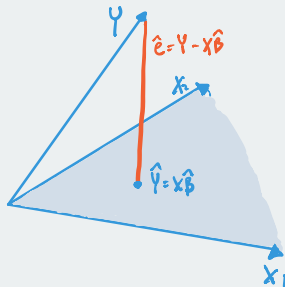
- We can write the covariate-residual orthogonality as $\mathbb{X}'\hat{\mathbf{e}} = 0$.

Projection



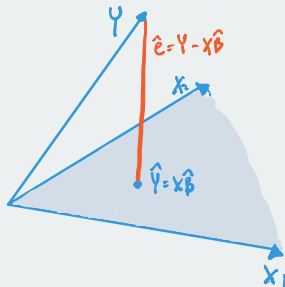
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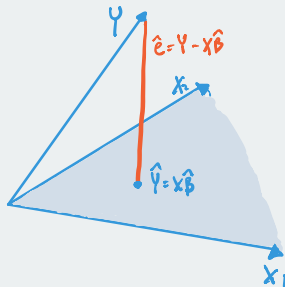
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 - Picture with $n = 3$ and $k = 2$: points in 3D space,
 - Column space of \mathbf{X} is a plane in this space.
- Intuition: $\hat{\boldsymbol{\beta}}$ defines the projection that gets is shortest distance between \mathbf{Y} and prediction.

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- Admits a nice expression for the residual vector: $\hat{\mathbf{e}} = \mathbf{M}\mathbf{e}$

Partitioned regression

- Partition covariates and coefficients $\mathbb{X} = [\mathbb{X}_1 \ \mathbb{X}_2]$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)'$:

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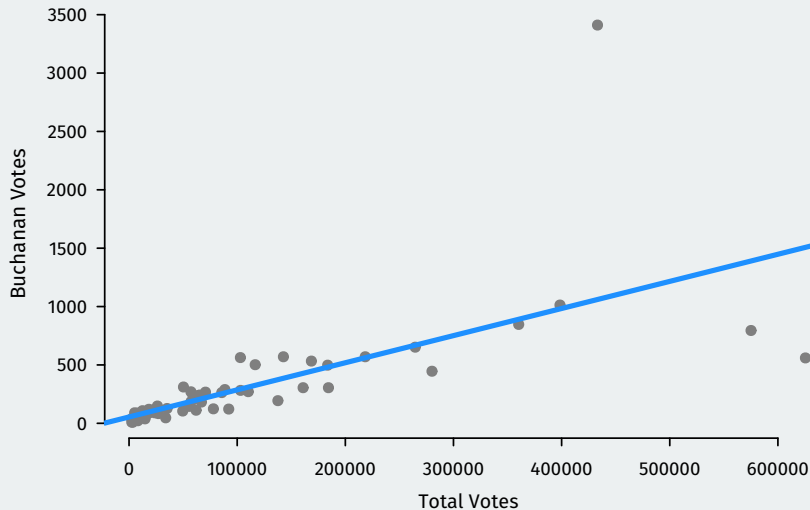
Example: Buchanan votes in Florida, 2000

- 2000 Presidential election in FL (Wand et al., 2001, APSR)

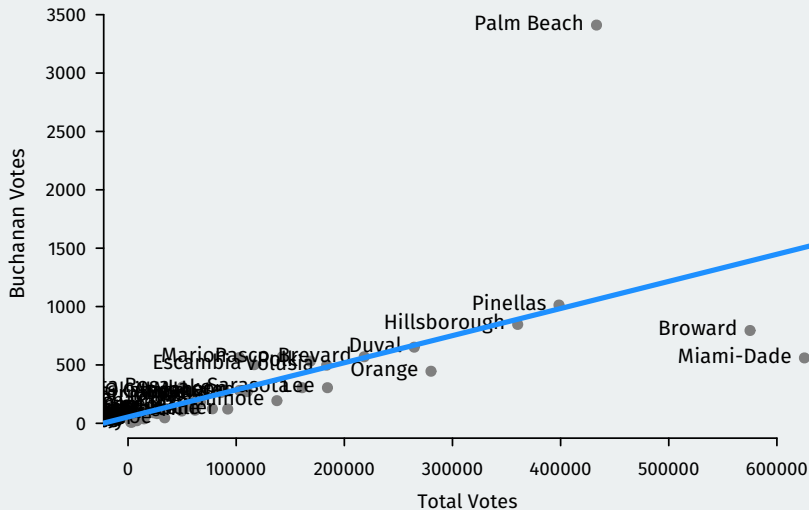
OFFICIAL BALLOT, GENERAL ELECTION PALM BEACH COUNTY, FLORIDA NOVEMBER 7, 2000		
es will electors.)	(REPUBLICAN)	
	GEORGE W. BUSH - PRESIDENT	3 ➡
	DICK CHENEY - VICE PRESIDENT	
	(DEMOCRATIC)	
	AL GORE - PRESIDENT	5 ➡
	JOE LIEBERMAN - VICE PRESIDENT	
	(LIBERTARIAN)	
	HARRY BROWNE - PRESIDENT	7 ➡
	ART OLIVIER - VICE PRESIDENT	
	(GREEN)	
	RALPH NADER - PRESIDENT	9 ➡
	WINONA LaDUKE - VICE PRESIDENT	
	(SOCIALIST WORKERS)	
JAMES HARRIS - PRESIDENT	11 ➡	
MARGARET TROWE - VICE PRESIDENT		
(NATURAL LAW)		
JOHN HAGELIN - PRESIDENT	13 ➡	
NAT GOLDBABER - VICE PRESIDENT		

OFFICIAL BALLOT, GENERAL ELECTION PALM BEACH COUNTY, FLORIDA NOVEMBER 7, 2000	
4 ⬅	(REFORM) PAT BUCHANAN - PRESIDENT EZOLA FOSTER - VICE PRESIDENT
6 ⬅	(SOCIALIST) DAVID McREYNOLDS - PRESIDENT MARY CAL HOLLIS - VICE PRESIDENT
8 ⬅	(CONSTITUTION) HOWARD PHILLIPS - PRESIDENT J. CURTIS FRAZIER - VICE PRESIDENT
10 ⬅	(WORKERS WORLD) MONICA MOOREHEAD - PRESIDENT GLORIA La RIVA - VICE PRESIDENT
WRITE-IN CANDIDATE To vote for a write-in candidate, follow the directions on the long stub of your ballot card.	

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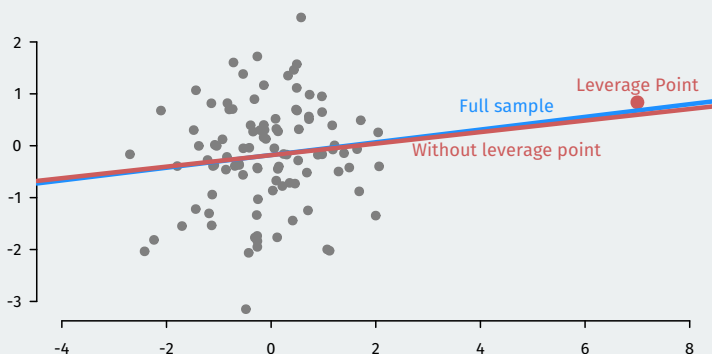


Example: Buchanan votes

```
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

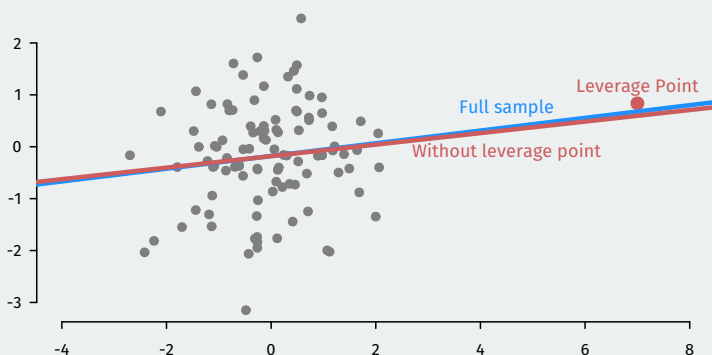
```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  54.22945    49.14146    1.10    0.27
## edaytotal     0.00232     0.00031    7.48 2.4e-10 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 333 on 65 degrees of freedom
## Multiple R-squared:  0.463, Adjusted R-squared:  0.455
## F-statistic: 56 on 1 and 65 DF, p-value: 2.42e-10
```

Leverage point definition



- Values that are extreme in the X dimension

Leverage point definition



- Values that are extreme in the X dimension
- That is, values far from the center of the covariate distribution

Leverage values

- Let h_{ij} be the (i, j) entry of \mathbf{P} . Then:

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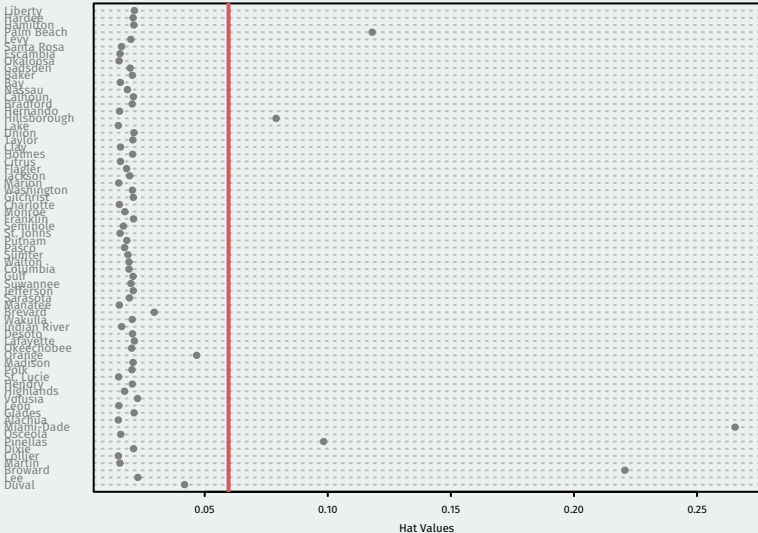
- \rightsquigarrow how far i is from the center of the X distribution
- Rule of thumb:** examine hat values greater than $2(k+1)/n$

Buchanan hats

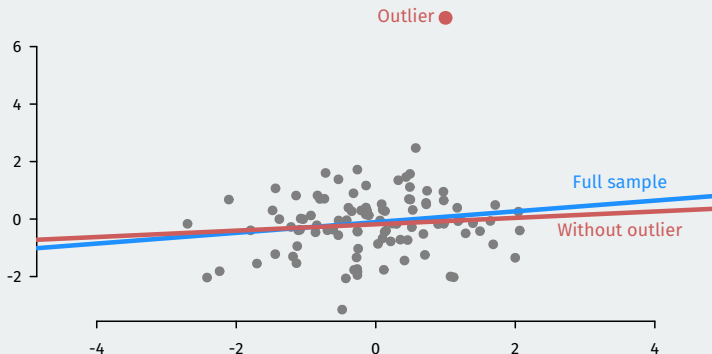
```
head(hatvalues(mod), 5)
```

```
##      1      2      3      4      5  
## 0.0418 0.0228 0.2207 0.0156 0.0149
```

Buchanan hats

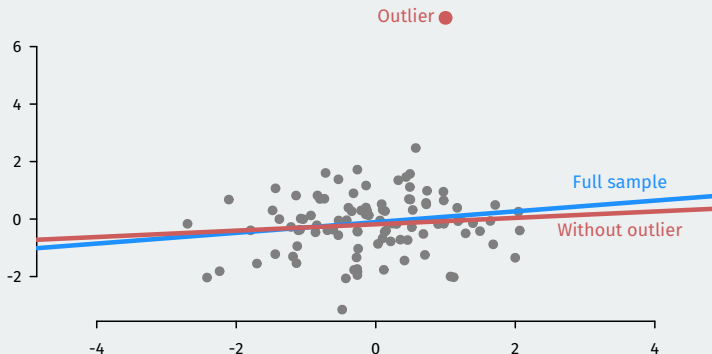


Outlier definition



- An **outlier** is far away from the center of the Y distribution.

Outlier definition



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- Intuitively: a point that would be poorly predicted by the regression.

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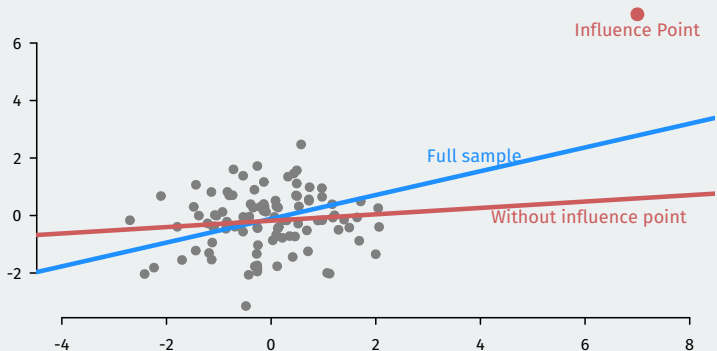
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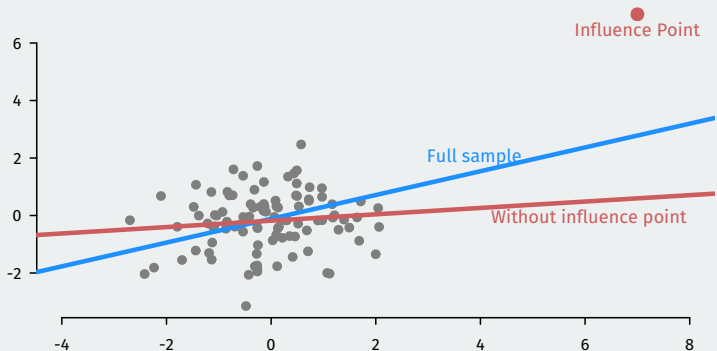
$$\hat{\beta}_{(-i)} = \hat{\beta} - (\mathbb{X}'\mathbb{X})^{-1} \mathbf{x}_i \tilde{e}_i \quad \tilde{e}_i = \frac{\hat{e}_i}{1 - h_{ii}}$$

Influence points



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Influence points



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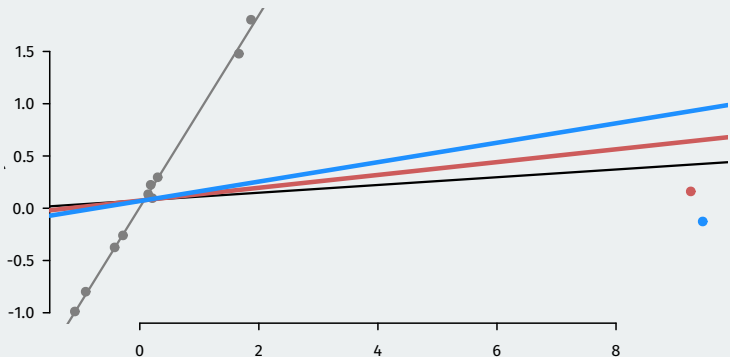
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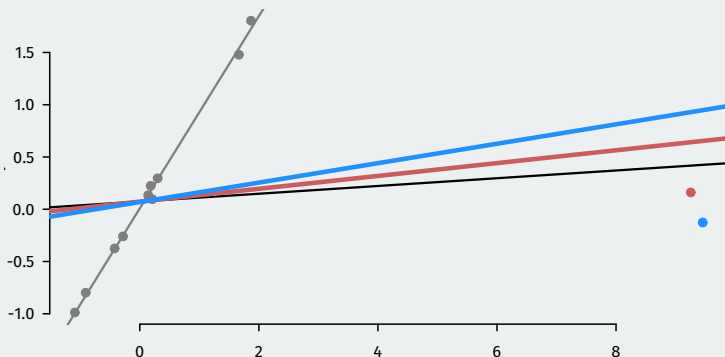
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 - Does removing the point change a coefficient by a lot?

Limitations of the standard tools



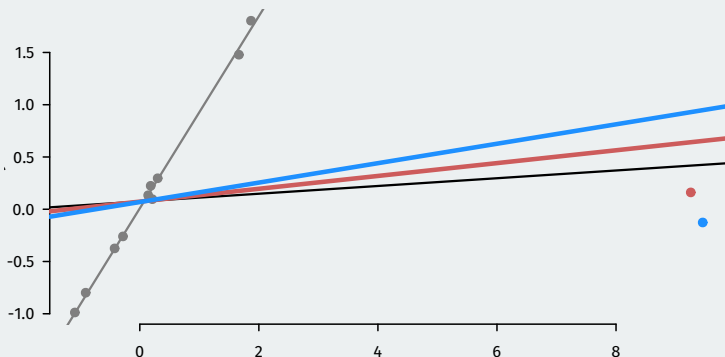
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 - Use a method that is robust to outliers (robust regression, least absolute deviations)