AERO 7970 - Multivariable Control of Uncertain Systems
Homework 2

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1 Problem 1

Given the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = Cx$$

- Check the controllability of the system
- Check the stability of the system
- What is the response to initial conditions?

1.1 Solution

1.1.1 Controllability

We form the controllability matrix $\mathbf{C}$ using Matlab's $ctrb(A, B)$ command. The $\mathbf{A}$ matrix has $dim(\mathbf{A}) = 5$, and the $\mathbf{C}$ matrix has $rank(\mathbf{C} = 5$. Therefore the system is controllable.

1.1.2 Stability

From the $\mathbf{A}$ matrix being in Jordan form, we can see that $\lambda(\mathbf{A}) = 0 \forall \lambda$. Therefore the system is marginally stable.

1.1.3 Response to Initial Conditions

The homogeneous response of the system is denoted by:

$$\Phi(t) = expm(\mathbf{A}t)x_0 = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \frac{t^4}{24} \\ 0 & 1 & t & \frac{t^2}{2} & \frac{t^3}{6} \\ 0 & 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_0$$

As we can see, the system is unstable to nonzero initial conditions.

2 Problem 2

Given the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \mathbf{B}u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} x$$

- Check the observability of the system
- Check the stability of the system
- What is the response to initial conditions?

2.1 Solution

2.1.1 Observability

We form the observability matrix $\mathbf{O}$ using Matlab's $obsv(A, C)$ command. The $\mathbf{A}$ matrix has $dim(\mathbf{A}) = 5$, and the $\mathbf{O}$ matrix has $rank(\mathbf{O} = 5$. Therefore the system is observable.

2.1.2 Stability

As $\mathbf{A}$ is the same as in Problem 1, the same answer applies.

2.1.3 Response to Initial Conditions

As $\mathbf{A}$ is the same as in Problem 1, the same answer applies.

3 Problem 3

Given the system:

$$\dot{x} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{mr_0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{mr_0} \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \\ u_\phi \end{bmatrix}$$

- Determine the stability, controllability, and observability properties
- Find the following transfer functions:

$$\frac{r(s)}{u_r(s)}, \frac{\phi(s)}{u_\phi(s)}, \frac{\theta(s)}{u_\theta(s)}$$

3.1 Solution

3.1.1 Stability, Controllability, Observability

The system has six eigenvalues, all on the imaginary axis. Thus the system is marginally stable.

The system is 6-dimensional, and both the controllability and observability matrices are rank 6. Thus the system is both controllable and observable.

3.1.2 Transfer Functions

We split the **A** matrix into the differential equations it represents:

$$\begin{aligned} \dot{r} &= \dot{r} \\ \ddot{r} &= 2\omega_0^2 r + 2\omega_0 r_0 \dot{\theta} + \frac{1}{m} u_r \\ \dot{\theta} &= \dot{\theta} \\ \ddot{\theta} &= \frac{-2\omega_0}{r_0} \dot{r} + \frac{1}{mr_0} u_\theta \\ \dot{\phi} &= \dot{\phi} \\ \ddot{\phi} &= -\omega_0^2 \phi + \frac{1}{mr_0} u_\phi \end{aligned}$$

We solve for $\frac{\phi(s)}{u_\phi(s)}$ by taking the inverse laplace of the $\ddot{\phi}$ equation. Rearranging gives:

$$\frac{\phi(s)}{u_\phi(s)} = \frac{\frac{1}{mr_0}}{s^2 + \omega_0^2}$$

As the differential equation for $\ddot{r}$ includes a $\dot{\theta}$ term, we solve for $\theta(s)$ in terms of $r(s)$ and remove the input $u_\theta(s)$. Rearranging gives:

$$\frac{r(s)}{u_r(s)} = \frac{\frac{1}{m}}{s^2 + \omega_0^2}$$

Similarly, the differential equation for $\ddot{\theta}$ includes an $\dot{r}$ term. Following the same process and eliminating $u_r(s)$ gives:

$$\frac{\theta(s)}{u_\theta(s)} = \frac{s^2 - 3\omega_0^2}{mr_0 s^2(s^2 + \omega_0^2)}$$

4 Problem 4

Given the system:

$$G(s) = \frac{s+2}{s^2 + 7s + 12}$$

Show that $z = -2$ is a transmission zero using the generalized eigenvalue problem.

4.1 Solution

To show that $z = -2$ is a transmission zero, it must solve

$$\begin{bmatrix} z\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ u_0 \end{bmatrix} = 0$$

for the state space system:

$$\mathbf{A} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$
$$\mathbf{D} = 0$$

The constructed matrix system is then:

$$\begin{bmatrix} 5 & 12 & -1 \\ -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ u_0 \end{bmatrix} = 0$$

Clearly, rows 2 and 3 of this matrix are not linearly dependent, and as such the system has lost rank. Thus $z = -2$ is a transmission zero. Solving the matrix system for $\mathbf{x}_0$ and u_0 gives $x_1 = -2x_2$ and $u_0 = 2x_2$.