positive definite, where Q and R are both constant, then the problem of minimizing the performance index

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} (u'Ru + x'Qx)\, dt \tag{4.1-4}$$

is solved by using a feedback law of the form of (4.1-2). (For the moment, we shall not be concerned with the scheme for calculating K.) We recall, too, from Sec. 3.3 that if (3) $[F, D]$ is completely observable, where D is any matrix such that $DD' = Q$, then the resulting closed-loop system is asymptotically stable.

There are, of course, other procedures for determining control laws of the form (4.1-2), which might achieve goals other than the minimization of a performance index such as (4.1-4). One other goal is to seek to have all the eigenvalues of the closed-loop system (4.1-3) taking prescribed values. The task of choosing an appropriate K has been termed the pole-positioning problem.

To solve the pole-positioning problem, the complete controllability of $[F, G]$ is required. (For a proof of this result, see [1].) The task of computing K in the single-input case is actually quite straightforward, if one converts F to companion matrix form and g to a vector containing all zeros except in the last position. (The complete controllability of $[F, g]$ is actually sufficient to ensure the existence of a basis transformation in the state space taking F and g to the special form; see Appendix B.) Now if

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

a choice of $k' = [k_1 \quad k_2 \quad \cdots \quad k_n]$ causes

$$F + gk' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 + k_1 & -a_2 + k_2 & -a_3 + k_3 & \cdots & -a_n + k_n \end{bmatrix}$$

and the eigenvalues of $F + gk'$ become the roots of the equation

$$s^n + (a_n - k_n)s^{n-1} + \cdots + (a_2 - k_2)s + (a_1 - k_1) = 0.$$

If these eigenvalues are prescribed, the determination of the feedback vector k, knowing a_1 through a_n, is immediate. For the multiple-input case, where G is no longer a vector but a matrix, the computational task is much harder but is nevertheless possible (see [2] through [6]).

From the practical point of view, the essential problem may not be to fix precisely the eigenvalues of $F + GK'$, but rather to ensure that these eigenvalues be within a certain region of the complex plane. Typical regions might be that sector of the left half-plane $Re[s] < 0$ bounded by straight lines extending from the origin and making angles $\pm\theta$ with the negative real axis, or, again, they might be that part of the left half-plane to the left of $Re[s] = -\alpha$, for some $\alpha > 0$.

We shall be concerned in this chapter with achieving the latter restriction. Moreover, we shall attempt to achieve the restriction not by selecting K through some modification of the procedure used for solving the pole-positioning problem, but by posing a suitable version of the regulator problem. Essentially what we are after is a solution of the regulator problem that gives a constant control law and that gives not merely an asymptotically stable closed-loop system, but one with a degree of stability of at least α. In other words, nonzero initial states of the closed-loop system (4.1-3) should decay at least as fast as $e^{-\alpha t}$. This is equivalent to requiring the eigenvalues of $F + GK'$ to have real parts less than $-\alpha$.

As we shall illustrate, commencing in the next section, it proves appropriate to replace the performance index (4.1-4) usually employed in the time-invariant regulator problem by a performance index

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} e^{2\alpha t}(u'Ru + x'Qx)\, dt \tag{4.1-5}$$

where R and Q are as before—i.e., constant, symmetric, and respectively positive and nonnegative definite. The constant α is nonnegative (of course, $\alpha = 0$ corresponds to the situation considered in Chapter 3).

Since the integrand in (4.1-5) can be rewritten as $u'\hat{R}u + x'\hat{Q}x$, where $\hat{R} = Re^{2\alpha t}$, $\hat{Q} = Qe^{2\alpha t}$, the results summarized in Sec. 3.3 will guarantee that a linear feedback law will generate the optimal control for (4.1-5). That this law should be constant is not at all clear, but it will be proved in the next section. However, supposing that this is proven, we can give plausible reasons as to why the closed-loop system should have a degree of stability of at least α. It would be expected that the optimal value of (4.1-5) should be finite. Now, the integrand will be of the form $e^{2\alpha t}x'(t)Mx(t)$ for some constant M when u is written as a linear constant function of x. For the integrand to approach zero as t approaches infinity, it is clearly sufficient, and probably necessary, for $x(t)$ to decay faster than $e^{-\alpha t}$. This is equivalent to requiring the closed-loop system to have a degree of stability of at least α. The results of this chapter first appeared in [7].

4.2 QUANTITATIVE STATEMENT AND SOLUTION OF THE REGULATOR PROBLEM WITH A PRESCRIBED DEGREE OF STABILITY

The remarks of the previous section justify the formal statement of the modified regulator problem in the following manner.

Modified regulator problem. Consider the system

$$\dot{x} = Fx + Gu \qquad x(t_0) \text{ given} \tag{4.2-1}$$

where F and G are constant and the pair $[F, G]$ is completely controllable. Consider also the associated performance index

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} e^{2\alpha t}(u'Ru + x'Qx)\, dt \tag{4.2-2}$$

where R and Q are constant, symmetric, and respectively positive definite and nonnegative definite. Let α be a nonnegative constant (which will turn out to be the minimum degree of stability of the closed-loop system). With D any matrix such that $DD' = Q$, let $[F, D]$ be completely observable. Define the minimization problem as the task of finding the minimum value of the performance index (4.2-2) and the associated optimal control.

The reason for the complete controllability condition is the same as explained in the last chapter; it ensures that the infinite, as distinct from finite, time problem has a solution. The observability condition will be needed to establish the constraint on the degree of stability of the closed-loop system.

The strategy we adopt in solving this modified problem is to introduce transformations that convert the problem to an infinite-time regulator problem of the type considered in the last chapter. Accordingly, we make the definitions

$$\hat{x}(t) = e^{\alpha t}x(t) \tag{4.2-3}$$

$$\hat{u}(t) = e^{\alpha t}u(t). \tag{4.2-4}$$

Just as $x(\cdot)$ and $u(\cdot)$ may be related [via Eq. (4.2-1)], so $\hat{x}(\cdot)$ and $\hat{u}(\cdot)$ may be related. Observe that

$$\begin{aligned}\dot{\hat{x}} &= \frac{d}{dt}(e^{\alpha t}x(t)) = \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t)\\ &= \alpha\hat{x} + e^{\alpha t}Fx + e^{\alpha t}Gu\\ &= (F + \alpha I)\hat{x} + G\hat{u}.\end{aligned} \tag{4.2-5}$$

Thus, given the relations (4.2-3) and (4.2-4), the system equation (4.2-1) implies the system equation (4.2-5). The converse is clearly true, too. Corresponding initial conditions for the two systems (4.2-1) and (4.2-5) are given by setting $t = t_0$ in (4.2-3)—i.e., $\hat{x}(t_0) = e^{\alpha t_0}x(t_0)$.

The integrand in (4.2-2) may also be written in terms of $\hat{u}$ and $\hat{x}$:

$$e^{2\alpha t}(u'Ru + x'Qx) = \hat{u}'R\hat{u} + \hat{x}'Q\hat{x}.$$

Consequently, we may associate with the system (4.2-5) the performance index

$$\hat{V}(\hat{x}(t_0), \hat{u}(\cdot), t_0) = \int_{t_0}^{\infty} (\hat{u}'R\hat{u} + \hat{x}'Q\hat{x})\, dt. \tag{4.2-6}$$

Moreover, there is a strong connection between the minimization problem associated with the equation pair (4.2-1), (4.2-2), and the pair (4.2-5), (4.2-6). Suppose $u^*(t)$ is the value of the optimal control at time t for the first problem, and that $x(t)$ is the resulting value of the state at time t when the initial state is $x(t_0)$. Then the value of the optimal control at time t for the second problem is $\hat{u}^*(t) = e^{\alpha t}u^*(t)$, and the resulting value of the state at time t is given by $\hat{x}(t) = e^{\alpha t}x(t)$, provided $\hat{x}(t_0) = e^{\alpha t_0}x(t_0)$. Moreover, the minimum performance index is the same for each problem.

Moreover, if the optimal control for the second problem can be expressed in feedback form as

$$\hat{u}^*(t) = k(\hat{x}(t), t), \tag{4.2-7}$$

then the optimal control for the first problem may also be expressed in feedback form; thus,

$$u^*(t) = e^{-\alpha t}\hat{u}^*(t) = e^{-\alpha t}k(e^{\alpha t}x(t), t). \tag{4.2-8}$$

[We know that the control law (4.2-7) should be a linear one; and, indeed, we shall shortly note the specific law; the point to observe here is that a *feedback* control law for the second problem readily yields a *feedback* control law for the first problem, irrespective of the notion of linearity.]

Our temporary task is now to study the system (4.2-5), repeated for convenience as

$$\dot{\hat{x}} = (F + \alpha I)\hat{x} + G\hat{u} \qquad \hat{x}(t_0) \text{ given} \tag{4.2-5}$$

and to select a control $\hat{u}^*(\cdot)$ that minimizes the performance index

$$\hat{V}(\hat{x}(t_0), \hat{u}(\cdot), t_0) = \int_{t_0}^{\infty} (\hat{u}'R\hat{u} + \hat{x}'Q\hat{x})\, dt \tag{4.2-6}$$

where R and Q have the constraints imposed at the beginning of the section.

As we discussed in the last chapter, this minimization problem may not have a solution without additional constraints of stability or controllability. One constraint that will guarantee existence of an optimal control is, however, a requirement that $[F + \alpha I, G]$ be completely controllable. As will be shown, *this is implied by the restriction that* $[F, G]$ *is completely con-*

trollable, which was imposed in our original statement of the modified regulator problem. To see this, we first need to observe a property of the complete controllability concept, derivable from the definition given earlier. This property, also given in Appendix B, is that $[F, G]$ is completely controllable if and only if the equation $w'e^{Ft}G = 0$ for all t where w is a constant vector implies $w = 0$.

The complete controllability of $[F + \alpha I, G]$ follows from that of $[F, G]$ (and vice versa) by observing the equivalence of the following four statements.

1. $[F, G]$ is completely controllable.
2. For constant w and all t, $w'e^{Ft}G = 0$ implies $w = 0$.
3. For constant w and all t, $w'e^{\alpha t}e^{Ft}G = w'e^{(F+\alpha I)t}G = 0$ implies $w = 0$.
4. $[F + \alpha I, G]$ is completely controllable.

Given this complete controllability constraint, we can define a solution to the preceding minimization problem with the most minor of modifications to the material of Sec. 3.3. Let $P(t, T)$ be the solution at time t of the equation

$$-\dot{P} = P(F + \alpha I) + (F' + \alpha I)P - PGR^{-1}G'P + Q \tag{4.2-9}$$

with boundary condition $P(T, T) = 0$. Define

$$\bar{P} = \lim_{t \to -\infty} P(t, T), \tag{4.2-10}$$

which is a constant matrix, satisfying the steady state version of (4.2-9):

$$\bar{P}(F + \alpha I) + (F' + \alpha I)\bar{P} - \bar{P}GR^{-1}G'\bar{P} + Q = 0. \tag{4.2-11}$$

Then the optimal control becomes

$$\hat{u}^*(t) = -R^{-1}G'\bar{P}\hat{x}(t). \tag{4.2-12}$$

It is interesting to know whether the application of the control law (4.2-12) to the open-loop system (4.2-5) results in an asymptotically stable closed-loop system. We recall from the results of the last chapter that a sufficient condition ensuring this asymptotic stability is that $[F + \alpha I, D]$ should be completely observable, where D is any matrix such that $DD' = Q$. Now, just as the complete controllability of $[F, G]$ implies the complete controllability of $[F + \alpha I, G]$, so by duality, the complete observability of $[F, D]$ implies the complete observability of $[F + \alpha I, D]$. Since the complete observability of $[F, D]$ was assumed in our original statement of the regulator problem with degree of stability constraint, it follows that $[F + \alpha I, D]$ is completely observable and that the closed-loop system

$$\dot{\hat{x}} = (F + \alpha I - GR^{-1}G'\bar{P})\hat{x} \tag{4.2-13}$$

is asymptotically stable.

We can now apply these results to the original optimization problem. Recall that we need to demonstrate, first, that the optimal control law is a constant, linear feedback law, and second, that the degree of stability of the closed-loop system is at least α. Furthermore, we need to find the minimum value of (4.2-2).

Equations (4.2-7) and (4.2-8) show us that

$$u^*(t) = -e^{-\alpha t}R^{-1}G'\bar{P}e^{\alpha t}x(t) = -R^{-1}G'\bar{P}x(t). \tag{4.2-14}$$

This is the desired constant control law; note that it has the same structure as the control law of (4.2-12).

To demonstrate the degree of stability, we have from (4.2-3) that $x(t) = e^{-\alpha t}\hat{x}(t)$. Since the closed-loop system (4.2-13) has been proved asymptotically stable, we know that $\hat{x}(t)$ approaches zero as t approaches infinity, and, consequently, that $x(t)$ approaches zero at least as fast as $e^{-\alpha t}$ when t approaches infinity.

The minimum value achieved by (4.2-2) is the same as the minimum value achieved by (4.2-6). As was shown in the previous chapter, the optimal performance index for (4.2-6) is expressible in terms of $\bar{P}$ as $\hat{x}'(t_0)\bar{P}\hat{x}(t_0)$. Consequently, the minimum value achieved by (4.2-2) is $x'(t_0)e^{-2\alpha t_0}\bar{P}x(t_0)$. Let us now summarize the results in terms of the notation used in this chapter.

> ***Solution of the regulator problem with prescribed degree of stability.*** The optimal performance index for the modified regulator problem stated at the start of this section is $x'(t_0)e^{-2\alpha t_0}\bar{P}x(t_0)$, where $\bar{P}$ is defined as the limiting solution of the Riccati equation (4.2-9) with boundary condition $P(T, T) = 0$. The matrix $\bar{P}$ also satisfies the algebraic equation (4.2-11). The associated optimal control is given by the constant linear feedback law (4.2-14), and the closed-loop system has degree of stability of at least α.

One might well ask if it is possible to construct a performance index with α equal to zero such that the control law resulting from the minimization is the same as that obtained from the preceding problem when α is nonzero. The answer is yes (see Problem 4.2-2). In other words, there are sets of pairs of matrices R and Q such that the associated regulator problem with zero α leads to a closed-loop system with degree of stability α. However, it does not appear possible to give an explicit formula for writing down these matrices without first solving a regulator problem with α nonzero.

By way of example, consider an idealized angular position control system where the position of the rotation shaft is controlled by the torque applied, with no friction in the system. The equation of motion is

$$J\ddot{\theta} = T$$

where θ is the angular position, T is the applied torque, and J is the moment of inertia of the rotating parts. In state-space form, this becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a \end{bmatrix} u$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, and $a = T/J$. As a performance index guaranteeing a degree of stability $\alpha = 1$, we choose

$$\int_0^\infty e^{2t}(u^2 + x_1^2)\, dt.$$

The appropriate algebraic equation satisfied by $\bar{P}$ is

$$\bar{P}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\bar{P} - \bar{P}\begin{bmatrix} 0 & 0 \\ 0 & a^2 \end{bmatrix}\bar{P} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

One possible way to solve this equation is to write down simultaneous equations for the entries of $\bar{P}$; thus,

$$\begin{aligned} 2\bar{p}_{11} - \bar{p}_{12}^2 a^2 + 1 &= 0 \\ \bar{p}_{11} + 2\bar{p}_{12} - \bar{p}_{12}\bar{p}_{22}a^2 &= 0 \\ 2\bar{p}_{12} + 2\bar{p}_{22} - \bar{p}_{22}^2 a^2 &= 0. \end{aligned}$$

These equations lead to

$$\begin{aligned} \bar{p}_{11} &= \frac{1}{a^2}\left[2 + 2\sqrt{1 + a^2} + \frac{1}{2}(2 + 2\sqrt{1 + a^2})^{3/2}\right] \\ \bar{p}_{12} &= \frac{1}{a^2}[1 + \sqrt{1 + a^2} + \sqrt{2 + 2\sqrt{1 + a^2}}] \\ \bar{p}_{22} &= \frac{1}{a^2}[2 + \sqrt{2 + 2\sqrt{1 + a^2}}]. \end{aligned}$$

The optimal control law is

$$\begin{aligned} u^* &= -g'\bar{P}x \\ &= \begin{bmatrix} -\frac{1}{a}(1 + \sqrt{1 + a^2} + \sqrt{2 + 2\sqrt{1 + a^2}}) \\ -\frac{1}{a}(2 + \sqrt{2 + 2\sqrt{1 + a^2}}) \end{bmatrix}' \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

This is implementable with proportional plus derivative (in this case, tacho, or angular velocity) feedback. The closed-loop system equation is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -(1 + \sqrt{1 + a^2} + \sqrt{2 + 2\sqrt{1 + a^2}}) & -(2 + \sqrt{2 + 2\sqrt{1 + a^2}}) \end{bmatrix} x$$

for which the characteristic polynomial is

$$s^2 + (2 + \sqrt{2 + 2\sqrt{1 + a^2}})s + (1 + \sqrt{1 + a^2} + \sqrt{2 + 2\sqrt{1 + a^2}}).$$

It is readily checked that the roots of this polynomial are complex for all a;

therefore, the real part of the closed-loop poles is

$$-1 - \tfrac{1}{2}\sqrt{2 + 2\sqrt{1 + a^2}} < -1.$$

Thus, the requisite degree of stability is achieved.

Problem 4.2-1. Consider the system (with constant F and G)

$$\dot{x} = Fx + Gu \qquad x(t_0) \text{ given}$$

and the associated performance index

$$\int_{t_0}^{\infty} e^{2\alpha t}[(u'u)^5 + (x'Qx)^5]\,dt$$

where Q is a constant nonnegative definite matrix. Find a related linear system and performance index where the integrand in the performance index is not specifically dependent on time (although, of course, it depends on u and x). Show that if an optimal feedback law exists for this related system and performance index, it is a constant law, and that from it an optimal feedback law may be determined for the original system and performance index.

Problem 4.2-2. Consider the system

$$\dot{x} = Fx + Gu \qquad x(t_0) \text{ given}$$

where F and G are constant and $[F, G]$ is completely controllable. Show that associated with any performance index of the form

$$\int_{t_0}^{\infty} e^{2\alpha t}(u'Ru + x'Qx)\,dt,$$

where R is constant and positive definite, Q is constant and nonnegative definite, and α is positive, there is a performance index

$$\int_{t_0}^{\infty} (u'Ru + x'\hat{Q}x)\,dt,$$

where $\hat{Q}$ is constant and nonnegative definite, such that the optimal controls associated with minimizing these indices are the same. [*Hint:* Define $\hat{Q}$ using the solution of the first minimization problem.]

Problem 4.2-3. Consider the system

$$\dot{x} = ax + u$$

with performance index

$$\int_{t_0}^{\infty} e^{2\alpha t}(u^2 + bx^2)\,dt$$

where a, b, and α are constants, with b and α nonnegative. Examine the eigenvalue of the closed-loop system matrix obtained by implementing an optimal feedback control, and indicate graphically the variation of this eigenvalue as one or more of a, b, and α vary.

Problem 4.2-4. Consider the modified regulator problem as stated at the beginning of the section, and let $\bar{P}$ be defined as in Eqs. (4.2-9) through (4.2-11).

Show that $V(x) = x'(t)\bar{P}x(t)$ is a Lyapunov function for the closed-loop system with the property that $\dot{V}/V \leq -2\alpha$. (This constitutes another proof of the degree of stability property.)

Problem 4.2-5. Suppose that you are given a linear time-invariant system with a feedback law K_α derived from the minimization of a performance index

$$\int_{t_0}^{\infty} e^{2\alpha t}(u'u + x'x)\,dt.$$

Suppose also that experiments are performed to determine the transient response of the system for the following three cases.

1. $K_\alpha = K_0$—i.e., α is chosen as zero.
2. $K_\alpha = K_{\alpha_1}$—i.e., α is chosen as a "large," positive constant.
3. $K_\alpha = K_0 + (K_{\alpha_1} - K_0)(1 - e^{-t})$.

Give sketches of the possible transient responses.

Problem 4.2-6. If F and G are constant matrices and F is $n \times n$, it is known that $[F, G]$ is completely controllable if and only if the matrix $[G\ FG \cdots F^{n-1}G]$ has rank n. Prove that complete controllability of $[F, G]$ implies complete controllability of $[F + \alpha I, G]$ by showing that if the matrix $[G\ FG \cdots F^{n-1}G]$ has rank n, so, too, does the matrix $[G\ (F + \alpha I)G \cdots (F + \alpha I)^{n-1}G]$.

Problem 4.2-7. Imagine two optimization problems of the type considered in this chapter with the same F, G, Q, and R but with two different α—viz., α_1 and α_2, with $\alpha_1 > \alpha_2$. Show that $\bar{P}_{\alpha_1} - \bar{P}_{\alpha_2}$ is positive definite.

4.3 EXTENSIONS OF THE PRECEDING RESULTS

We now ask if there are positive functions $f(t)$ other than $e^{\alpha t}$ with the property that the minimization of

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} f(t)(u'Ru + x'Qx)\,dt, \tag{4.3-1}$$

given that

$$\dot{x} = Fx + Gu \qquad x(t_0) \text{ given} \tag{4.3-2}$$

leads to a linear constant control law. (Here, the usual constraints on F, G, Q, and R, including constancy, are assumed to apply.) Reference [8] suggests that maybe $f(t) = t^k$ could lead to a constant control law.

We show here that essentially the only possible $f(t)$ are those we have already considered—i.e., those of the form $e^{\alpha t}$—but with the constraint that α should simply be real (rather than nonnegative, as required earlier in this chapter).

By writing the integrand of (4.3-1) as $u'[f(t)R]u + x'[f(t)Q]x$, it is evident that the associated optimal control is

$$u^*(t) = -f^{-1}(t)R^{-1}G'\bar{P}(t)x(t) \tag{4.3-3}$$

where $\bar{P}(\cdot)$ is a solution of

$$-\dot{\bar{P}} = \bar{P}F + F'\bar{P} - f^{-1}(t)\bar{P}GR^{-1}G'\bar{P} + f(t)Q. \tag{4.3-4}$$

Now (4.3-3) is to be a constant control law. Hence, the matrix $\bar{P}(t)G$ must be of the form $f(t)M$, where M is some constant matrix. Premultiply (4.3-4) by G' and postmultiply by G to obtain

$$-G'M\dot{f} = [M'FG + G'F'M - G'MR^{-1}M'G + G'QG]f. \tag{4.3-5}$$

Suppose $G'\bar{P}(t)G = G'M$ is nonzero. Then there is at least one entry—say, the i-j entry—that is nonzero. Equating the i-j entries on both sides of (4.3-5), we obtain

$$\dot{f} = \alpha f$$

for some constant α. Thus, the claim that f has the form $e^{\alpha t}$ is established for the case when $G'\bar{P}(t)G \neq 0$ for some t. Now suppose $G'\bar{P}(t)G \equiv 0$. Then, since $\bar{P}(t)$ is nonnegative definite, it follows that $\bar{P}(t)G \equiv 0$, and thus the optimal control is identically zero. In this clearly exceptional case, it is actually possible to tolerate $f(t)$ differing from $e^{\alpha t}$. Thus, we have the trivial example of

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} f(t)u'Ru\,dt,$$

which for arbitrary positive $f(t)$ and arbitrary F, G, and R has the constant optimal feedback law $u \equiv 0$.

Since the preceding two sections have restricted attention to the case of nonnegative α, it might well be asked why this restriction was made. The answer is straightforward: Constant control laws certainly result from negative α, but the resulting closed-loop systems may be unstable, although no state grows faster than $e^{-\alpha t}$. (Note that $e^{-\alpha t}$ is a growing exponential, because α is negative.)

An open problem for which a solution would be of some interest (if one exists) is to set up a regulator problem for an arbitrary system (4.3-2) such that the eigenvalues of the closed-loop system matrix $F - GR^{-1}G'\bar{P}$ possess a prescribed *relative stability*—i.e., the eigenvalues have negative real part—and if written as $\sigma + j\omega$ (σ, ω real, $j = \sqrt{-1}$), the constraint $|\omega|/|\sigma| < k$ for some prescribed constant k is satisfied.

A second open problem is to set up a regulator problem such that the closed-loop system matrix possesses a dominant pair of eigenvalues (all but one pair of eigenvalues should have a large negative real part). Partial solutions to this problem will be presented in Chapter 5.

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PART III

PROPERTIES AND APPLICATION OF THE OPTIMAL REGULATOR

CHAPTER 5

PROPERTIES OF REGULATOR SYSTEMS WITH A CLASSICAL CONTROL INTERPRETATION

5.1 THE REGULATOR FROM AN ENGINEERING VIEWPOINT

We have earlier intimated a desire to point out what might be termed the "engineering significance" of the regulator. Until now, we have exposed a mathematical theory for obtaining feedback laws for linear systems. These feedback laws minimize performance indices that reflect the costs of control and of having a nonzero state. In this sense, they may have engineering significance. Furthermore, we have indicated in some detail for time-invariant systems a technique whereby the closed-loop system will be asymptotically stable, and will even possess a prescribed degree of stability. This, too, has obvious engineering significance.

Again, there is engineering significance in the fact that, in distinction to most classical design procedures, the techniques are applicable to multiple-input systems, and to time-varying systems. (We have tended to avoid discussion of the latter because of the additional complexity required in, for example, assumptions guaranteeing stability of the closed-loop system. However, virtually all the results presented hitherto and those to follow are applicable in some way to this class of system.)

But there still remains a number of unanswered questions concerning the engineering significance of the results. For example, we might well wonder to what extent it is reasonable to think in terms of state feedback when the states of a system are not directly measurable. All the preceding, and most of the following, theory is built upon the assumption that the system states are available; quite clearly, if this theory is to be justified, we shall have to indicate some technique for dealing with a situation where no direct measurement is possible. We shall discuss such techniques in a subsequent chapter. Meanwhile, we shall continue with the assumption that the system states are available.

In classical control, the notions of gain margin and phase margin play an important role. Thus, engineering system specifications will often place lower bounds on these quantities, since it has been found, essentially empirically, that if these quantities are too small system performance will be degraded in some way. For example, if for a system with a small amount of time delay a controller is designed neglecting the time delay, and if the phase margin of the closed loop is small, there may well be oscillations in the actual closed loop. The natural question now arises as to what may be said about the gain margin and phase margin (if these quantities can, in fact, be defined) of an optimal regulator.

Of course, at first glance, there can be no parallel between the dynamic feedback of the output of a system, as occurs in classical control, and the memoryless feedback of states, as in the optimal regulator. But both schemes have associated with them a closed loop. Figure 5.1-1 shows the classical

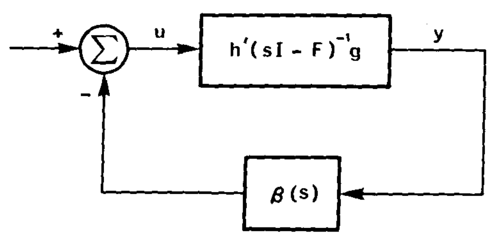

Fig. 5.1-1 Classical feedback arrangement with dynamic controller driven by system output.

feedback arrangement for a system with transfer function $h'(sI - F)^{-1}g$, where the output is fed back through a dynamic controller with transfer function $\beta(s)$. Figure 5.1-2 shows a system with transfer function $h'(sI - F)^{-1}g$ but with memoryless state-variable feedback. Here a closed loop is formed; however, it does not include the output of the open-loop system, merely the states. This closed loop is shown in Fig. 5.1-3.

Now it is clear how to give interpretations of the classical variety to the optimal feedback system. The optimal feedback system is like a classical

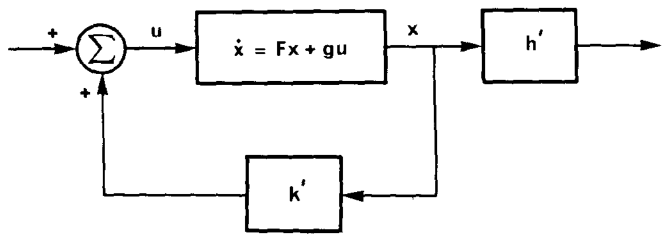

Fig. 5.1-2 Modern feedback arrangement with memoryless controller driven by system states.

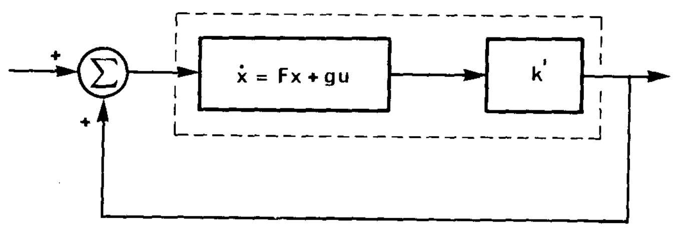

Fig. 5.1-3 The closed-loop part of a feedback system using modern feedback arrangement.

situation where unity negative feedback is applied around a (single-input, single-output) system with transfer function $-k'(sI - F)^{-1}g$. Thus, the gain margin of the optimal regulator may be determined from a Nyquist, or some other, plot of $W(j\omega) = -k'(j\omega I - F)^{-1}g$ in the usual manner.

We may recall the attention given in classical control design procedures to the question of obtaining satisfactory transient response. Thus, to obtain for a second-order system a fast response to a step input, without excessive overshoot, it is suggested that the poles of the closed-loop system should have a damping ratio of about 0.7. For a higher order system, the same sort of response can be achieved if two dominant poles of 0.7 damping ratio are used. We shall discuss how such effects can also be achieved using an optimal regulator; the key idea revolves around appropriate selection of the weighting matrices (Q and R) appearing in the performance index definition.

A common design procedure for systems containing a nonlinearity is to replace the nonlinearity by an equivalent linear element, and to design and analyze with this replacement. One then needs to know to what extent the true system performance will vary from the approximating system performance. As will be seen, a number of results involving the regulator can be obtained, giving comparative information of the sort wanted.

We shall also consider the question of incorporating relay control in otherwise optimal systems. There may often be physical and economic advantages in using relay rather than continuous control, provided stability problems can be overcome; accordingly, it is useful to ask whether any general remarks may be made concerning the introduction of relays into optimal regulators.

A further question of engineering significance arises when we seek to discover how well an optimal regulator will perform with variations in the parameters of the forward part of the closed-loop system. One of the common aims of classical control (and particularly that specialization of classical control, feedback amplifier design) is to insert feedback so that the input-output performance of the closed-loop system becomes less sensitive to variations in the forward part of the system. In other words, one seeks to desensitize the performance to certain parameter variations.

The quantitative discussion of many of the ideas just touched upon depends on the application of one of several basic formulas, which are derived in the next section. Then we pass on to the real meat of the regulator ideas in this and the next two chapters.

5.2 SOME FUNDAMENTAL FORMULAS

To fix ideas for the remainder of this chapter, we shall restrict attention to closed-loop systems that are completely controllable, time invariant, and asymptotically stable. Thus, we shall take as our fundamental open-loop system

$$\dot{x} = Fx + Gu \tag{5.2-1}$$

with $[F, G]$ completely controllable. As the performance index, we take

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{\infty} e^{2\alpha t}(u'Ru + x'Qx)\,dt \tag{5.2-2}$$

with the usual constraints on Q and R, including that $[F, D]$ be completely observable for any D such that $DD' = Q$. At times, α will be zero. Since we shall be considering different values of α, we define P_α, rather than $\bar{P}$, as the unique positive definite solution of

$$P_\alpha(F + \alpha I) + (F' + \alpha I)P_\alpha - P_\alpha GR^{-1}G'P_\alpha + Q = 0. \tag{5.2-3}$$

The optimal control law is

$$u = K'_\alpha x = -R^{-1}G'P_\alpha x. \tag{5.2-4}$$

Using the notation $A_0 = A_\alpha|(\alpha = 0)$, we shall prove first that

$$\begin{aligned}&[I - R^{1/2}K'_0(-sI - F)^{-1}GR^{-1/2}]'[I - R^{1/2}K'_0(sI - F)^{-1}GR^{-1/2}]\\ &\quad = I - R^{-1/2}G'(-sI - F')^{-1}Q(sI - F)^{-1}GR^{-1/2}.\end{aligned} \tag{5.2-5}$$

This is an identity that has appeared in a number of places (often merely in scalar form)—e.g., [1], [2], [3]. From (5.2-3), it follows that

$$P_0(sI - F) + (-sI - F')P_0 + K_0RK'_0 = Q.$$

Multiplying on the left by $R^{-1/2}G'(-sI - F')^{-1}$, and on the right by $(sI - F)^{-1}GR^{-1/2}$, yields

$$\begin{aligned} R^{-1/2}G'(-sI - F')^{-1}P_0GR^{-1/2} &+ R^{-1/2}G'P_0(sI - F)^{-1}GR^{-1/2} \\ &+ R^{-1/2}G'(-sI - F')^{-1}K_0RK_0'(sI - F)^{-1}GR^{-1/2} \\ &= R^{-1/2}G'(-sI - F')^{-1}Q(sI - F)^{-1}GR^{-1/2}. \end{aligned}$$

By adding I to each side, and observing that $P_0GR^{-1/2} = -K_0R^{1/2}$, Eq. (5.2-5) follows.

Let us now examine some variants of (5.2-5). With $s = j\omega$, Eq. (5.2-5) becomes

$$\begin{aligned} [I - R^{1/2}K_0'(-j\omega I - F)^{-1}GR^{-1/2}]'[I - R^{1/2}K_0'(j\omega I - F)^{-1}GR^{-1/2}] \\ = I + R^{-1/2}G'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}GR^{-1/2}. \end{aligned} \tag{5.2-6}$$

With * denoting the complex conjugate, the left-hand side is Hermitian, whereas the right-hand side is of the form $I + B'^*(j\omega)QB(j\omega)$ for a matrix B. If we adopt the notation $C_1 \geq C_2$ for arbitrary Hermitian matrices C_1 and C_2 to indicate that $C_1 - C_2$ is nonnegative, and if we use the fact that $B'^*QB \geq 0$, Eq. (5.2-6) implies that

$$[I - R^{1/2}K_0'(-j\omega I - F)^{-1}GR^{-1/2}]'[I - R^{1/2}K_0'(j\omega I - F)^{-1}GR^{-1/2}] \geq I. \tag{5.2-7}$$

For a single-input system (5.2-1), there is no loss of generality in taking R as unity. Equations (5.2-5), (5.2-6), and (5.2-7) then become

$$\begin{aligned} [1 - k_0'(-sI - F)^{-1}g][1 - k_0'(sI - F)^{-1}g] \\ = 1 + g'(-sI - F')^{-1}Q(sI - F)^{-1}g, \end{aligned} \tag{5.2-8}$$

or

$$|1 - k_0'(j\omega I - F)^{-1}g|^2 = 1 + g'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}g. \tag{5.2-9}$$

Therefore, we have that

$$|1 - k_0'(j\omega I - F)^{-1}g| \geq 1. \tag{5.2-10}$$

Problem 5.2-1 asks for the establishment of the result that the equality sign in (5.2-7) and (5.2-10) can hold only for isolated values of ω.

Relations (5.2-5) through (5.2-10) are all specialized to the case $\alpha = 0$. For the case $\alpha \neq 0$, there are effectively two ways of developing the corresponding relations. The first relies on observing, with the aid of (5.2-3), that P_α satisfies the same equation as P_0, save that $F + \alpha I$ replaces F. Consequently, Eqs. (5.2-5) through (5.2-10), derived from (5.2-3), will be supplanted by equations where $F + \alpha I$ replaces F, and K_α replaces K_0. For example, (5.2-9) yields

$$\begin{aligned} |1 - k_0'(j\omega I - F - \alpha I)^{-1}g|^2 \\ = 1 + g'(-j\omega I - F' - \alpha I)^{-1}Q(j\omega I - F - \alpha I)^{-1}g, \end{aligned} \tag{5.2-11}$$

and, evidently, one could also regard this as coming from (5.2-9) by replacing k_0 by k_α and $\pm j\omega$ by $\pm j\omega - \alpha$ (leaving F invariant).

The second way of deriving relations corresponding to (5.2-5) through (5.2-10) for the case when α is nonzero is to observe, again with the aid of (5.2-3), that P_α satisfies the same equation as P_0, save that $Q + 2\alpha P_\alpha$ replaces Q. Then Eq. (5.2-9), for example, is replaced by

$$|1 - k'_\alpha(j\omega I - F)^{-1}g|^2 = 1 + g'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}g + 2\alpha g'(-j\omega I - F')^{-1}P_\alpha(j\omega I - F)^{-1}g. \tag{5.2-12}$$

The other equations in the set (5.2-5) through (5.2-10) will also yield new relations by simply replacing K_0 by K_α and Q by $Q + 2\alpha P_\alpha$.

Problem 5.2-1. Show that inequalities (5.2-7) and (5.2-10) become equalities only for isolated values of ω.

Problem 5.2-2. Show that (5.2-9) implies

$$1 - |1 + k'_0(j\omega I - F - gk'_0)^{-1}g|^2 = g'(-j\omega I - F' - k_0 g')^{-1}Q(j\omega I - F - gk'_0)^{-1}g.$$

State two equivalent equations applying for $\alpha \neq 0$.

Problem 5.2-3. Derive an equation analogous to (5.2-8) when $u = k'_0 x$ is the optimal control for the system $\dot{x} = Fx + gu$ with a performance index $\int_{t_0}^{\infty}(u^2 + 2x'Su + x'Qx)\,dt$, and when $Q - SS'$ is nonnegative definite symmetric.

Problem 5.2-4. Assume that for a specific value of α—say, $\alpha = \bar{\alpha}$—the matrix $P_{\bar{\alpha}}$ has been calculated as the positive definite solution of (5.2-3). As an approximate technique for computing P_α for values of α near $\bar{\alpha}$, one could set $P_\alpha = P_{\bar{\alpha}} + (\alpha - \bar{\alpha})(\partial P_\alpha/\partial\alpha)|(\alpha = \bar{\alpha})$. Show that $\partial P_\alpha/\partial\alpha$ satisfies a linear algebraic matrix equation.

5.3 GAIN MARGIN, PHASE MARGIN AND TIME-DELAY TOLERANCE

In this section, we shall restrict attention to the regulator with a scalar input, and we shall examine certain properties of the closed-loop scheme of Fig. 5.3-1, which is a redrawn version of Fig. 5.1-3 (discussed earlier in this chapter), with the gain k replaced by k_α. We shall be especially interested in the gain margin, phase margin, and time-delay tolerance of the scheme.

We recall that the gain margin of a closed-loop system is the amount by which the loop gain can be increased until the system becomes unstable.

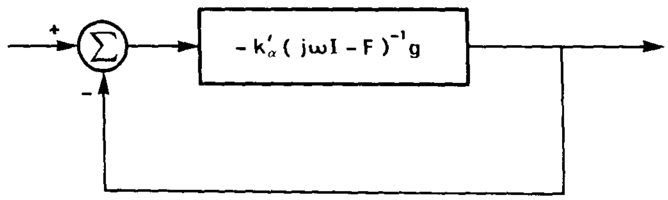

Fig. 5.3-1 Closed-loop optimal scheme with degree of stability α drawn as a unity negative feedback system.

If the loop gain can be increased without bound—i.e., instability is not encountered, no matter how large the loop gain becomes—then the closed-loop system is said to possess an infinite gain margin.

Of course, no real system has infinite gain margin. Such parasitic effects as stray capacitance, time delay, etc., will always prevent infinite gain margin from being a physical reality. Some mathematical models of systems may, however, have an infinite gain margin. Clearly, if these models are accurate representations of the physical picture—save, perhaps, for their representation of parasitic effects—it could validly be concluded that the physical system had a very large gain margin.

We shall now show that the optimally designed regulator possesses the infinite gain margin property, by noting a characteristic feature of the Nyquist diagram of the open-loop gain of the regulator. The scheme of Fig. 5.3-1 is arranged to have unity *negative* feedback, so that we may apply the Nyquist diagram ideas immediately. The associated Nyquist plot is a curve on the Argand diagram, or complex plane, obtained from the complex values of $-k'_\alpha(j\omega I - F)^{-1}g$ as ω varies through the real numbers from minus to plus infinity. Now the Nyquist plot of $-k'_\alpha(j\omega I - F)^{-1}g$ is constrained to avoid a certain region of the complex plane, because Eq. (5.2-12) of the last section yields that

$$|1 - k'_\alpha(j\omega I - F)^{-1}g| \geq 1, \tag{5.3-1}$$

which is to say that the distance of any point on the Nyquist plot from the point $-1 + j0$ is at least unity. In other words, the plot of $-k'_\alpha(j\omega I - F)^{-1}g$ avoids a circle of unit radius centered at $-1 + j0$.

See Fig. 5.3-2 for examples of various plots. (The transfer functions are irrelevant.) The arrow marks the direction of increasing ω. Note that the plots end at the origin, which is always the case when the open-loop transfer function, expressed as a numerator polynomial divided by a denominator polynomial, has the numerator degree less than the denominator degree.

There is yet a further constraint on the Nyquist plot, which is caused by the fact that the closed-loop system is known to be asymptotically stable. This restricts the number of *counterclockwise encirclements* of the point $-1 + j0$ by the Nyquist plot to being precisely the number of poles of the

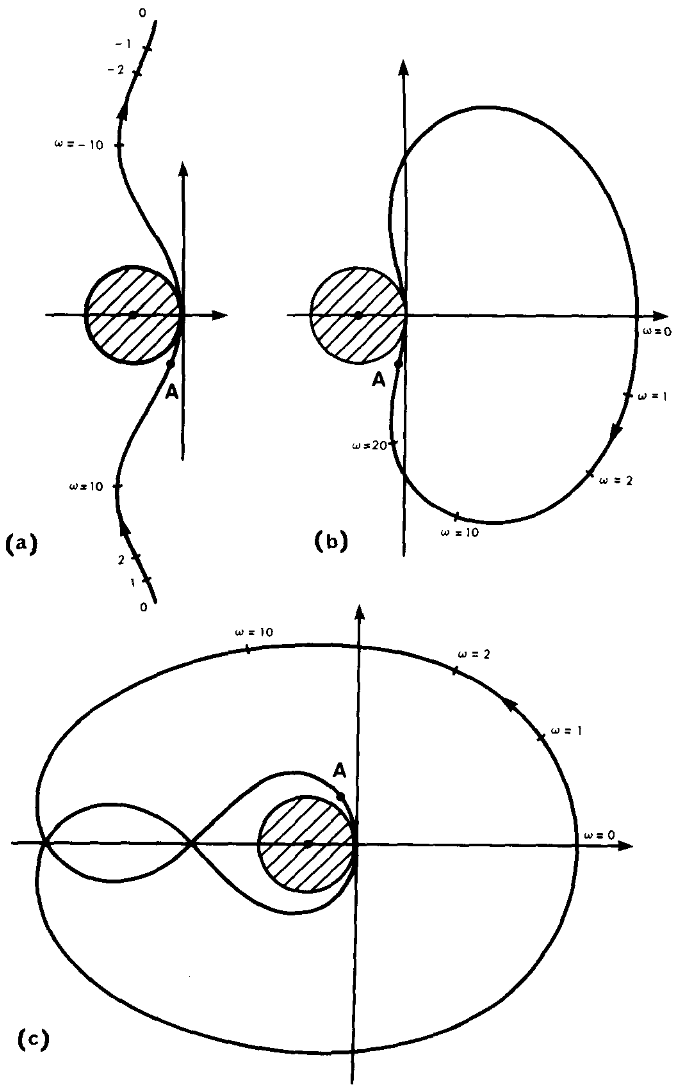

Fig. 5.3-2 Nyquist plots of $-k'_\alpha(j\omega I - F)^{-1}g$ avoiding a unit critical disc center $(-1, 0)$. Points A are at unity distance from the origin.

transfer function $-k'_\alpha(sI - F)^{-1}g$ lying in $Re[s] \geq 0$. We understand that if a pole lies on $Re[s] = 0$, the Nyquist diagram is constructed by making a small semicircular indentation into the region $Re[s] < 0$ around this pole, and plotting the complex numbers $-k'(sI - F)^{-1}g$ as s moves around this semicircular contour. (For a discussion and proof of this basic stability result, see, for example, [4].)

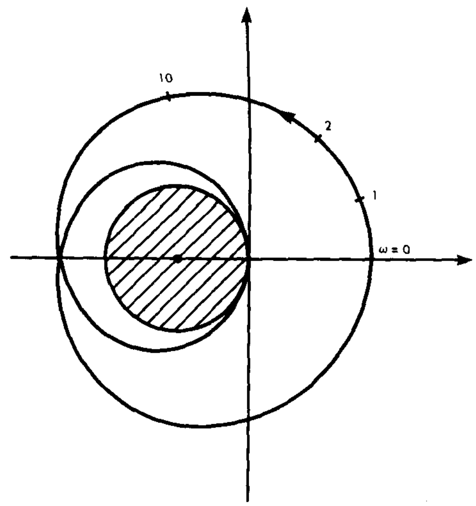

Fig. 5.3-3 Nyquist plot for which open-loop transfer function has two poles in $\mathrm{Re}[s] > 0$, and closed-loop is stable.

Thus, if $-k'_\alpha(sI - F)^{-1}g$ has no poles in $Re[s] \geq 0$, diagrams such as those in Fig. 5.3-2 may be obtained. Fig. 5.3-3 illustrates a case where $-k'_\alpha(sI - F)^{-1}g$ has two poles in $Re[s] > 0$.

The key observation we now make is that *the number of encirclements of the point* $-1 + j0$ *is the same as the number of encirclements of any other point inside the circle of unit radius and center* $-1 + j0$. The best way to see this seems to be by visual inspection; a few experiments will quickly show that if the preceding remarks were not true, the Nyquist diagram would have to enter the circle from which it has been excluded.

It is known that the closed-loop system with gain multiplied by a constant factor β will continue to be asymptotically stable if the Nyquist diagram of $-\beta k'_\alpha(j\omega I - F)^{-1}g$ encircles $-1 + j0$ in a counterclockwise direction a number of times equal to the number of poles of $-\beta k'_\alpha(sI - F)^{-1}g$ lying in $Re[s] \geq 0$. Equivalently, asymptotic stability will follow if the Nyquist diagram of $-k'_\alpha(j\omega I - F)^{-1}g$ encircles $-(1/\beta) + j0$ this number of times. But our previous observation precisely guarantees this for a range of β. The points $-(1/\beta) + j0$ for all real $\beta > \frac{1}{2}$ lie inside the critical circle, and thus are encircled counterclockwise the same number of times as the point $-1 + j0$. As we have argued, this number is the same as the number of poles of $-k'_\alpha(sI - F)^{-1}g$ in $Re[s] \geq 0$. Consequently, with asymptotic stability following for all real $\beta > \frac{1}{2}$, we have established the infinite gain margin property.

Let us now turn to consideration of the phase margin property. First, we recall the definition of phase margin. It is the amount of negative phase shift that must be introduced (without gain increase) to make that part of the Nyquist plot corresponding to $\omega \geq 0$ pass through the $-1 + j0$ point. For example, consider the three plots of Fig. 5.3-2; points A at unit distance from the origin on the $\omega \geq 0$ part of the plot have been marked. The negative phase shift that will need to be introduced in the first and second case is about 80°, and that in the third case about 280°. Thus, 80° is approximately the phase margin in the first and second case, and 280° in the third.

We shall now show that the phase margin of an optimal regulator is always at least 60°. The phase margin is determined from that point or those points on the $\omega \geq 0$ part of the Nyquist plot which are at unit distance from the origin. Since the Nyquist plot of an optimal regulator must avoid the circle with center $-1 + j0$ and unity radius, the points at unit distance from the origin and lying on the Nyquist plot of an optimal regulator are restricted to lying on the shaded part of the circle of unit radius and center the origin, shown in Fig. 5.3-4. The smallest angle through which one of the allowable

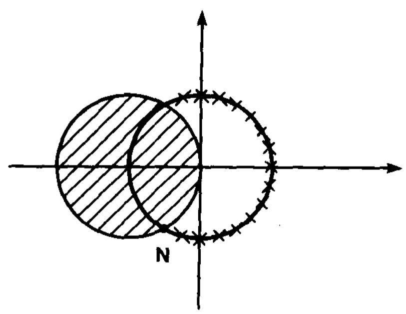

Fig. 5.3-4 Shaded points denote permissible points on Nyquist plot of optimal regulator at unit distance from origin.

points could move in a clockwise direction to reach $-1 + j0$ is 60°, corresponding to the point N of Fig. 5.3-4. Any other point in the allowed set of points (those outside the circle of center $-1 + j0$, and unity radius, but at unit distance from the origin) must move through more than 60° to reach $-1 + j0$. The angle through which a point such as N must move to reach $-1 + j0$ is precisely the phase margin. Consequently, the lower bound of 60° is established.

Let us now consider the effect of varying α away from zero. Reference to Eq. (5.2-11) shows that

$$|1 - k'_\alpha(j\omega I - F - \alpha I)^{-1}g| = |1 - k'_\alpha[(j\omega - \alpha)I - F]^{-1}g| \geq 1. \tag{5.3-2}$$

Consequently, if a modified Nyquist plot is made of $-k'_\alpha[(j\omega - \alpha)I - F]^{-1}g$ rather than of $-k'_\alpha(j\omega I - F)^{-1}g$, this modified plot will also avoid the circle of center $-1 + j0$ and radius 1. Then it will follow that if a gain $\beta > \frac{1}{2}$ is inserted in the closed loop, the degree of stability α of the closed-loop system will be retained for all real β. If a negative phase shift of up to 60° is introduced, a degree of stability α will also still be retained.

Yet another interpretation of the case of nonzero α follows by comparing Eqs. (5.2-9) and (5.2-12) of Sec. 5.2, repeated here for convenience:

$$|1 - k_0'(j\omega I - F)^{-1}g|^2 = 1 + g'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}g$$

and

$$|1 - k_\alpha'(j\omega I - F)^{-1}g|^2 = 1 + g'(-j\omega I - F')^{-1}Q(j\omega I - F)^{-1}g + 2\alpha g'(-j\omega I - F')^{-1}P_\alpha(j\omega I - F)^{-1}g,$$

whence

$$|1 - k_\alpha'(j\omega I - F)^{-1}g|^2 = |1 - k_0'(j\omega I - F)^{-1}g|^2 + 2\alpha g'(-j\omega I - F')^{-1}P_\alpha(j\omega I - F)^{-1}g. \tag{5.3-3}$$

The second term on the right side is nonnegative, being of the form $2\alpha b'^*(j\omega)P_\alpha b(j\omega)$ for a vector b. In fact, one can show that b is never zero, and so

$$|1 - k_\alpha'(j\omega I - F)^{-1}g| > |1 - k_0'(j\omega I - F)^{-1}g|. \tag{5.3-4}$$

This equation says that any two points on the Nyquist plots of $-k_\alpha'(j\omega I - F)^{-1}g$ and $-k_0'(j\omega I - F)^{-1}g$ corresponding to the same value of ω are such that the point on the first plot is further from $-1 + j0$ than the point on the second plot. In loose terms, the whole plot of $-k_\alpha'(j\omega I - F)^{-1}g$ is further from $-1 + j0$ than the plot of $-k_0'(j\omega I - F)^{-1}g$. This does not, however, imply that the phase margin for the former is greater than that for the latter. Problem 5.3-1 asks for a proof of this somewhat surprising result.

Problem 5.3-2 asks for a proof of the result that if $\alpha_1 > \alpha_2 > 0$, the Nyquist plot of $-k_{\alpha_1}'(j\omega I - F)^{-1}g$ is further, in the preceding sense, from $-1 + j0$ than the plot of $-k_{\alpha_2}'(j\omega I - F)^{-1}g$.

We now turn to a discussion of the tolerance of time delay in the closed loop. Accordingly, we shall consider the scheme of Figure 5.3-5, where T is a certain time delay. (The delay block could equally well be anywhere else in the loop, from the point of view of the following discussion.) We shall be concerned with the stability of the closed-loop system.

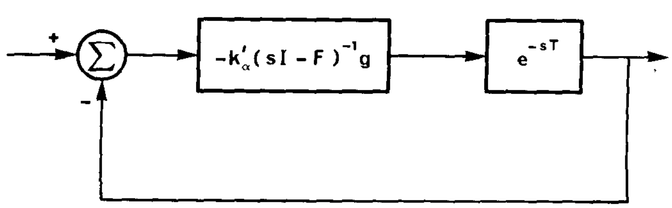

Fig. 5.3-5 Optimal regulator with time delay T inserted.

The effect of the time delay is to insert a frequency-dependent negative phase shift into the open-loop transfer function. Thus, instead of $-k'_\alpha(j\omega I - F)^{-1}g$ being the open-loop transfer function, it will be $-k'_\alpha(j\omega I - F)^{-1}ge^{-j\omega T}$. This has the same magnitude as $-k'_\alpha(j\omega I - F)^{-1}g$, but a negative phase shift of ωT radians.

It is straightforward to derive allowable values for the time delay which do not cause instability. Considering first the $\alpha = 0$ case, suppose the transfer function $-k'_0(j\omega I - F)^{-1}g$ has unity gain at the frequencies $\omega_1, \omega_2, \ldots, \omega_r$ with $0 < \omega_1 < \omega_2 < \cdots < \omega_r$, and let the amount of negative phase shift that would bring each of these unity gain points to the $-1 + j0$ point be $\phi_1, \phi_2, \ldots, \phi_r$, respectively. Of course, $\phi_i \geq \pi/3$ for all i. Then, if a time delay T is inserted, so long as $\omega_i T < \phi_i$ or $T < \phi_i/\omega_i$ for all i, stability will prevail. In particular, if $T < \pi/3\omega_r$, stability is assured.

For the case of α nonzero, since the modified Nyquist plot of $-k'_\alpha[(j\omega - \alpha)I - F]^{-1}g$ has the same properties as the Nyquist plot of $-k'_0(j\omega I - F)^{-1}g$, we see (with obvious definition of $\omega_1, \ldots, \omega_r$ and $\phi_1, \ldots, \phi_r$) that $T < \phi_i/\omega_i$ for all i, and, in particular, $T < \pi/3\omega_r$ will ensure maintenance of the degree of stability α. Greater time delays can be tolerated, but the degree of stability will be reduced.

The introduction of time delay will destroy the infinite gain margin property. To see this, observe that as ω approaches infinity, the phase shift introduced—viz., ωT radians—becomes infinitely great for any nonzero T. In particular, one can be assured that the Nyquist plot of $-k'_\alpha(j\omega I - F)^{-1}g$ will be rotated for suitably large ω such that the rotated plot crosses the real axis just to the left of the origin. (In fact, the plot will cross the axis infinitely often.) If for a given T, the leftmost point of the real axis that is crossed is $-(1/\beta) + j0$, then the gain margin of the closed loop with time delay is $20 \log_{10} \beta$ db.

Of course, the introduction of a multiplying factor, phase shift, or time delay in the loop will destroy the optimality of the original system. But the important point to note is that the optimality of the original system allows the introduction of these various modifications while maintaining the required degree of stability. Optimality may not, in any specific situation, be of direct engineering significance, but stability is; therefore, optimality becomes of engineering significance indirectly.

Problem 5.3-1. Show with suitable sketches that the relation $|1 - k'_\alpha(j\omega I - F)^{-1}g| > |1 - k'_0(j\omega I - F)^{-1}g|$ does not imply that the phase margin associated with the transfer function $-k'_\alpha(j\omega I - F)^{-1}g$ is greater than that associated with the transfer function $-k'_0(j\omega I - F)^{-1}g$.

Problem 5.3-2. Suppose $\alpha_1 > \alpha_2 > 0$. Show that $|1 - k'_{\alpha_1}(j\omega I - F)^{-1}g| > |1 - k'_{\alpha_2}(j\omega I - F)^{-1}g|$. [*Hint:* Use the result of Problem 4.2-7 of Chapter 4 that $P_{\alpha_1} - P_{\alpha_2}$ is positive definite.]