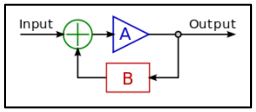
Chapter 2bis: System Theory Primer

- State Variables, Definition, Solution
- Stability
- Controllability
- Observability



State Variables – State Space Representation

- State variables representation is a convenient way (sometimes the only way) to describe multi-input multi-output nonlinear and linear dynamic systems.
- In the case of linear time invariant systems, matrix theory is widely used for the understanding of systems properties.

- Given a lumped parameter dynamic system with m inputs u_i and p outputs y_j, is always possible to describe it using:
  - Ordinary differential equations (nonlinear, linear)
  - Transfer functions (only for linear time invariant)
  - **State Variables (nonlinear, linear).**

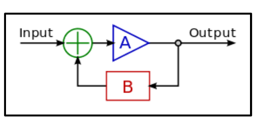
$$\begin{cases} \dfrac{d^{n_1} y_1}{dt^{n_1}} = f_1(u_1, u_2, \ldots) \\ \quad \ldots \\ \dfrac{d^{n_p} y_r}{dt^{n_p}} = f_r(u_1, u_2, \ldots) \end{cases}$$

$$G(s) = \frac{y(s)}{u(s)} = ??$$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) = g(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases}$$

$$\begin{cases} \boldsymbol{x}(t) \in \Re^n \\ \boldsymbol{u}(t) \in \Re^m \\ \boldsymbol{y}(t) \in \Re^p \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \end{cases}$$



State Variables – State Space Representation

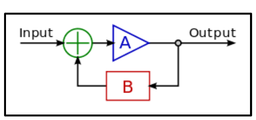
$$\begin{cases}\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) = g(\boldsymbol{x}(t), \boldsymbol{u}(t))\end{cases} \qquad \begin{cases}\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)\end{cases} \qquad \begin{cases}\boldsymbol{x}(t) \in \Re^n \\ \boldsymbol{u}(t) \in \Re^m \\ \boldsymbol{y}(t) \in \Re^p\end{cases}$$

- **Definition**: the vector $x(t)$ is called state vector and it consists of the smallest number **(*)** of independent variables necessary to uniquely describe the system in (with the input known) in the subspace $\Re^n$.

- **Note**: For linear systems the quadruple $(A,\ B,\ C,\ D)$ is not unique and, given an input vector, an infinite number of state vectors belonging to the same subspace exists, which yield the same output.

- A state space representation is a system of first order differential equations that can be obtained from a transfer function (matrix) or differential equations of any order.

(*) The minimum number of state variables equals the number of initial conditions necessary for the solution of the system.



State Variables – State Space Representation

- How to obtain a state variables representation from differential equations

- **Case 1**: Differential equations with NO derivative of the input (Linear)

$$\boxed{\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = u}$$

$$x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}, \ldots x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

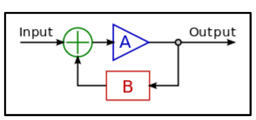
$$x \in \Re^n$$

$$x = [x_1, x_2, x_3, \ldots x_n]^T$$

$$\dot{x}_1 = \dot{y} = x_2, \dot{x}_2 = \ddot{y} = x_3, \ldots \dot{x}_n = \frac{d^n y}{dt^n}$$

$$\boxed{\begin{aligned} \dot{\boldsymbol{x}}(t) &= A_{n\times n}\boldsymbol{x}(t) + B_{n\times 1}u(t) \\ y(t) &= C_{1\times n}\boldsymbol{x}(t) \qquad (*) \end{aligned}}$$

(*) La matrice D è nulla se e solo se il sistema è strettamente proprio, come nel presente caso

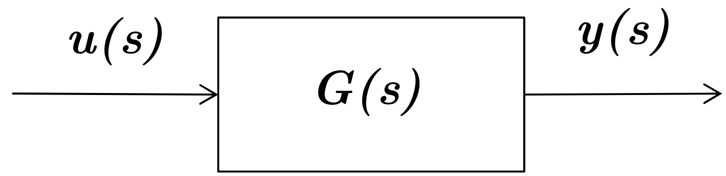


State Variables – State Space Representation

$$\dddot{y} + 16\ddot{y} - 4y = 18u$$

$$y(s) = G(s)u(s)$$

$$G(s) = \frac{18}{s^3 + 16s^2 - 4}$$



$$x_1 = y \qquad x_2 = \dot{y} \qquad x_3 = \ddot{y}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

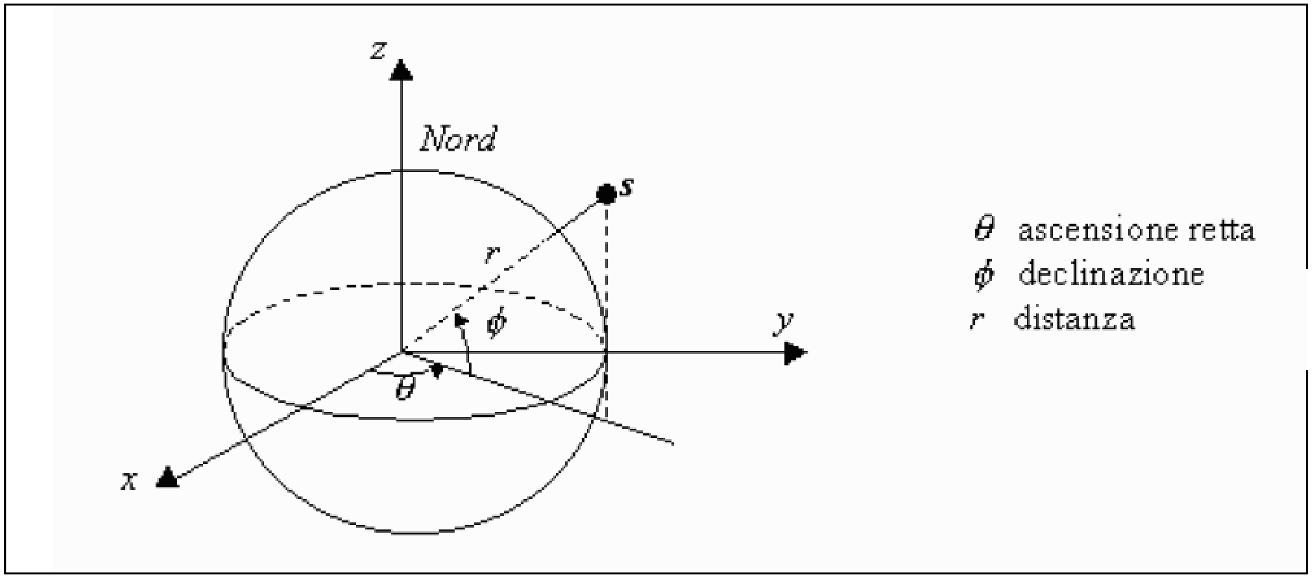
$$\dot{x}_3 = \dddot{y} = -16\ddot{y} + 4y + 18u = -16x_3 + 4x_1 + 18u$$

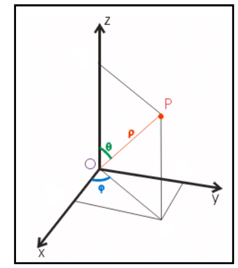
$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -16 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 18 \end{bmatrix} u = A\boldsymbol{x} + Bu$$

$$y = x_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \end{bmatrix} u = C\boldsymbol{x} + Du$$

State Variables – State Space Representation

- **Case 2 Example**: Differential equations with NO derivative of the input (Nonlinear)





$$\begin{cases} x = ho\sin(h\eta)\cos(\phi) \\ y = ho\sin(h\eta)\sin(\phi) \\ z = ho\cos(h\eta) \end{cases} \quad ho \in [0, +\infty) \;\; h\eta \in [0, \pi] \;\; \phi \in [0, 2\pi)$$

$$\begin{cases} ho = \sqrt{x^2 + y^2 + z^2} \\ \phi = \mathrm{arctg}\left(\frac{y}{x}\right) \\ h\eta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{cases}$$

Simplifying Assumptions:

- Spherical Earth with uniform Density
- Spacecraft as point mass m
- Constant gravitational attraction
- Two – Body Problem.

State Variables – State Space Representation

- Equations of motion based on energy balance

$$T = \frac{1}{2} m[\dot{r}^2 + (r\dot{\phi})^2 + (r\dot{\theta}\cos\phi)^2]$$

$$V = -\frac{km}{r} = -\frac{gMm}{r}$$

$$\begin{cases} \ddot{r} = r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2 - \dfrac{k}{r^2} + \dfrac{u_r}{m} \\ \ddot{\theta} = -\dfrac{2\dot{r}\dot{\theta}}{r} + \dfrac{2\dot{\theta}\dot{\phi}\sin\phi}{\cos\phi} + \dfrac{u_\theta}{mr\cos^2\phi} \\ \ddot{\phi} = -\dot{\theta}^2\cos\phi\sin\phi - \dfrac{2\dot{r}\dot{\phi}}{r} + \dfrac{u_\phi}{mr} \end{cases}$$

- The state of the system is desccribed by 6 variables

$$\boldsymbol{x} = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} & \phi & \dot{\phi} \end{bmatrix}^T$$

- Assume 3 input, one for each degree of freedom.

$$\boldsymbol{u} = \begin{bmatrix} u_r & u_\theta & u_\phi \end{bmatrix}^T$$

$$\longrightarrow \quad \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, \boldsymbol{u})$$

$$\begin{cases} r(t) = ? \\ \theta(t) = ? \\ \phi(t) = ? \end{cases}$$

No analytical solution, only numerical.

- **Choice of SW language**
- **Appropriate integration algorithm**
- **Choice of integration step**
- **Solution depends on choice of initial conditions**

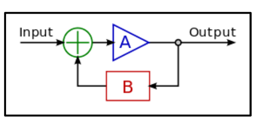
State Variables – State Space Representation

$$\dot{\boldsymbol{x}}^{6\times1}(t) = f^{6\times1}(\boldsymbol{x}^{6\times1}, \boldsymbol{u}^{3\times1}) \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 x_4^2 \cos^2 x_5 + x_1 x_6^2 - \dfrac{k}{x_1^2} + \dfrac{u_r}{m} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{2x_2 x_4}{x_1} + \dfrac{2x_4 x_6 \operatorname{sen} x_5}{\cos x_5} + \dfrac{u_\theta}{m x_1 \cos^2 x_5} \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = -x_4^2 \cos x_5 \operatorname{sen} x_5 - \dfrac{2x_2 x_6}{x_1} + \dfrac{u_\phi}{m x_1} \end{cases}$$

- How can we have an idea of the solution?: Select an equilibrium condition and linearize

$$f^{6\times1}(\boldsymbol{x}^{6\times1}, \boldsymbol{u}^{3\times1}) = 0 \Rightarrow \boldsymbol{x}_0^{6\times1}, \boldsymbol{u}_0^{3\times1}$$

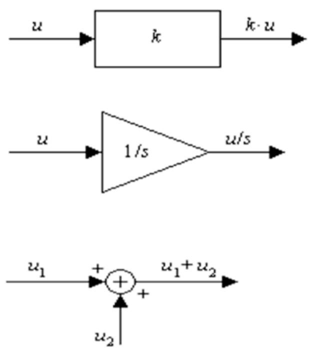
- Computation can be complex having to solve a nonlinear algebraic sysyem of order 6.

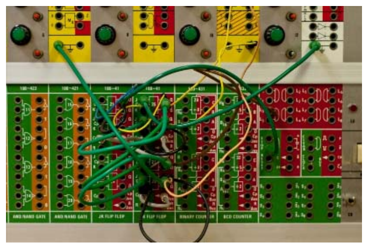


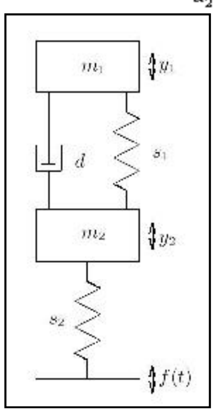
State Variables – State Space Representation

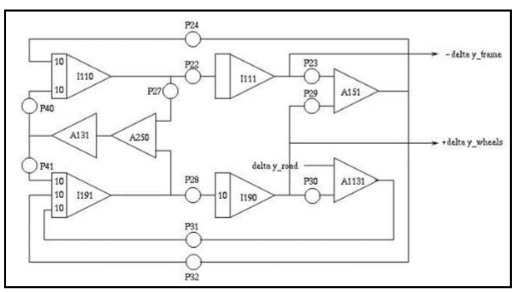
❑ **Case 2**: Differential equations with derivative of the input

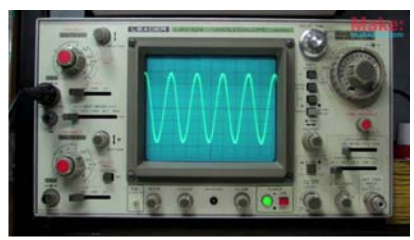
❑ Classical approach is the use of analog diagrams (from analog computing). The problem is feasible if the system is proper or strictly proper (principle of causality).

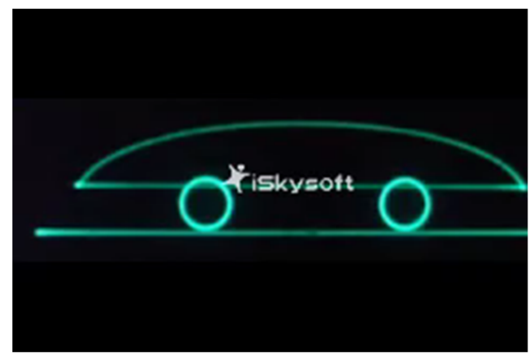


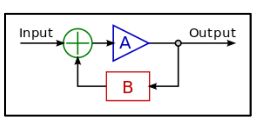












$$\ddot{y} + 2\dot{y} + y = 2u + 3\dot{u}$$

$$G(s) = \frac{3s + 2}{s^2 + 2s + 1}$$

- **Integral Operator:** $y = \frac{1}{s}\dot{y}$

Note: The integral operator is NOT a Laplace Transform!

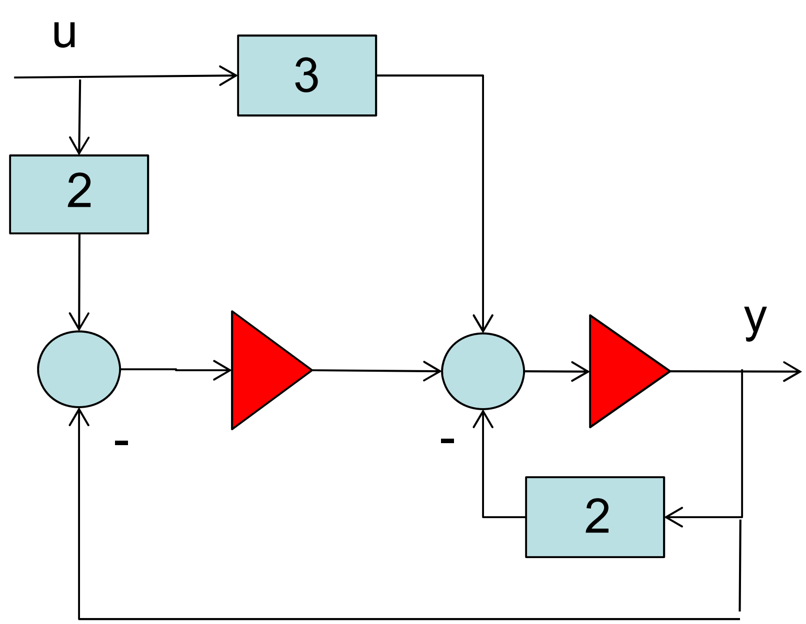
$$s^2 y + 2sy + y = 2u + 3su$$

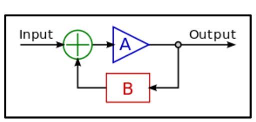
$$s^2 y = 2u + 3su - 2sy - y$$

$$y = \frac{1}{s^2}\left[2u + 3su - 2sy - y\right]$$

$$y = \frac{2u}{s^2} + \frac{3u}{s} - \frac{2y}{s} - \frac{y}{s^2}$$

$$y = \frac{1}{s}\left\{3u - 2y + \frac{1}{s}\left[2u - y\right]\right\}$$





State Variables – State Space Representation

Select as state variables the output of each integrator

$$\begin{aligned}\dot{x}_1 &= 3u - 2x_1 + x_2 \\ \dot{x}_2 &= 2u - x_1\end{aligned}$$

$$\begin{aligned}\dot{x}_1 &= 2u - x_2 \\ \dot{x}_2 &= 3u - 2x_2 + x_1\end{aligned}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 1 \\ -1 & 0\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}3 \\ 2\end{bmatrix}u = A\boldsymbol{x} + Bu$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & -2\end{bmatrix}\boldsymbol{x} + \begin{bmatrix}2 \\ 3\end{bmatrix}u = A\boldsymbol{x} + Bu$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\boldsymbol{x}$$

$$G(s) = \frac{3s + 2}{s^2 + 2s + 1}$$

State Variables – Solution

- **Solution**: Given the representation:

$$\boxed{\begin{aligned}\dot{\boldsymbol{x}}(t) &= A_{n\times n}\boldsymbol{x}(t) + B_{n\times m}\boldsymbol{u}(t)\\ y(t) &= C_{p\times n}\boldsymbol{x}(t) + D_{p\times m}\boldsymbol{u}(t)\end{aligned}}$$

given $\boldsymbol{u}(t)$ and $\boldsymbol{x}_0$, find $\boldsymbol{x}(t)$and $\boldsymbol{y}(t)$, for every $t > t_0$.

- Frequency domain approach only for LTI systems.

- Time domain approach for LTV and nonlinear systems.

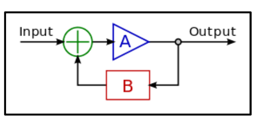
- **Frequency Domain (i.e. Use of Laplace Transform)**

$$\begin{aligned}\boldsymbol{s}\boldsymbol{x}(s) - \boldsymbol{x}_0 &= A\boldsymbol{x}(s) + B\boldsymbol{u}(s)\\ \boldsymbol{y}(s) &= C\boldsymbol{x}(s) + D\boldsymbol{u}(s)\end{aligned}$$

$$\begin{aligned}(sI - A)\boldsymbol{x}(s) &= \boldsymbol{x}_0 + B\boldsymbol{u}(s)\\ \boldsymbol{y}(s) &= C\boldsymbol{x}(s) + D\boldsymbol{u}(s)\end{aligned}$$

$$\begin{aligned}\boldsymbol{x}(s) &= (sI - A)^{-1}\boldsymbol{x}_0 + (sI - A)^{-1}B\boldsymbol{u}(s)\\ \boldsymbol{y}(s) &= C(sI - A)^{-1}\boldsymbol{x}_0 + C(sI - A)^{-1}B\boldsymbol{u}(s) + D\boldsymbol{u}(s)\end{aligned}$$

$$\boxed{\begin{aligned}\boldsymbol{y}(s) &= G(s)\boldsymbol{u}(s)\\ G_{p\times m}(s) &= [C(sI - A)^{-1}B + D]\end{aligned}}$$



State Variables – Solution

❑ Example

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = 0 \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

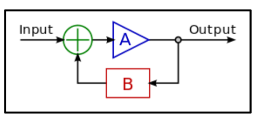
$$\boldsymbol{y} = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$$

$$sI - A = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} \quad \Rightarrow \quad (sI - A)^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \qquad (sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

$$C(sI - A)^{-1} = (sI - A)^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

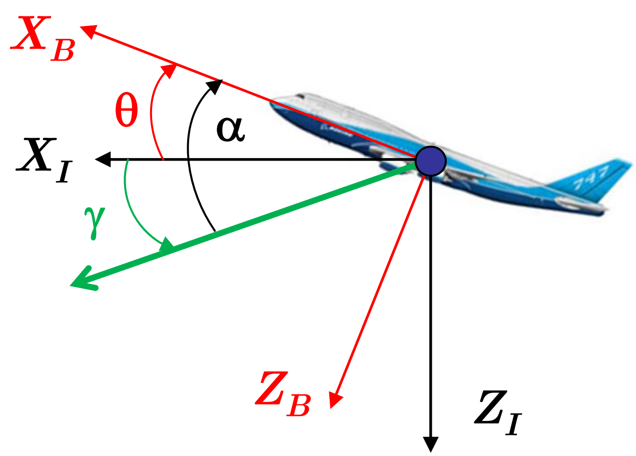
$$\boxed{C(sI - A)^{-1}B = \begin{bmatrix} \dfrac{3}{s+1} \\ \dfrac{-3}{s+2} \end{bmatrix} = H(s) \quad \text{matrice 2x1}}$$

$$\boldsymbol{y}(s) = \begin{bmatrix} \dfrac{1}{s+1} \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{3}{s+1} \\ \dfrac{-3}{s+2} \end{bmatrix} U(s)$$



State Variables – Solution

Example: Aircraft longitudinal linearized motion



Trim Conditions

$$\begin{aligned} Altitude &= Sea\ Level(ft) \\ Mach &= 0.2 \\ U_0 &= 221(ft/\sec) \\ \gamma_0 = \theta_0 - \alpha_0 &= -3.5^0 = -6.11 rad^{-2} \\ \alpha_0 &= 6^0 = 10.47 rad^{-2} \end{aligned}$$

State variables:

- u = translation velocity component along xB
- w = translation velocity component along zB
- q = pitch rate
- θ = pitch angle (kinematics)

$$\boldsymbol{x}_{AC}(t) = \begin{bmatrix} u \\ w = \alpha U_0 \\ q = \dot{\theta} \\ \theta \end{bmatrix}; u_{AC}(t) = \delta_E; u_T(t) = \delta_T; \boldsymbol{d}(t) = \begin{bmatrix} u_w \\ w_w \end{bmatrix}$$

$$\dot{\delta}_T = 0.25\delta_T + 0.25\delta_{TC}$$

$$\dot{h} = -w + 2.21\theta$$

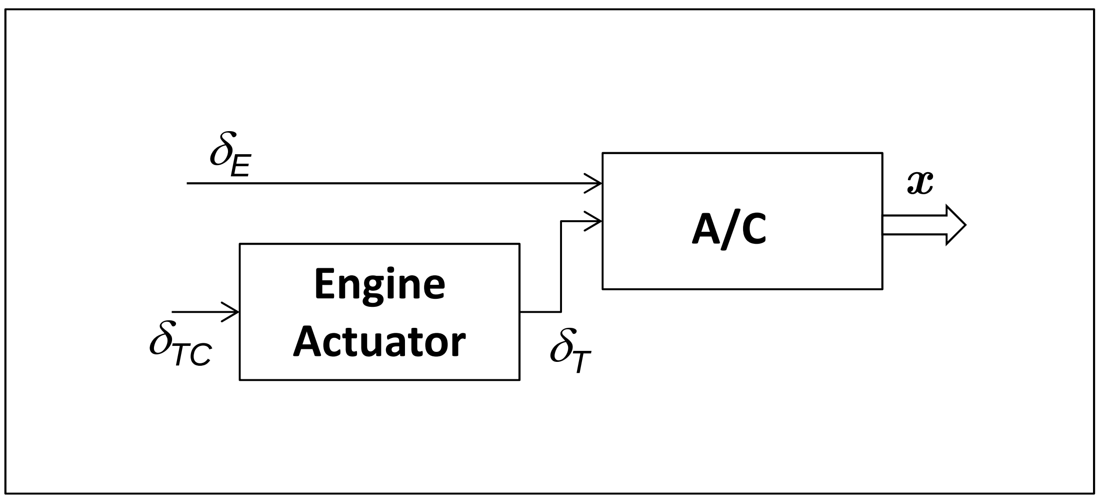
State Variables – Solution

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{x}}_{AC} \\ \dot{\delta}_T \end{bmatrix} = \left[\begin{array}{cccc|c} -.021 & .122 & 0 & -.322 & 1 \\ -.209 & -.53 & 2.21 & 0 & -.044 \\ .017 & -.164 & -.412 & 0 & .544 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -.25 \end{array}\right] \boldsymbol{x} + \left[\begin{array}{c|c} .01 & 0 \\ -.064 & 0 \\ -.378 & 0 \\ 0 & 0 \\ \hline 0 & .25 \end{array}\right] \boldsymbol{u} + \left[\begin{array}{c|c} .021 & -.122 \\ .209 & .530 \\ -.017 & .164 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right] \boldsymbol{d}$$

$$\boldsymbol{u}(t) = \begin{bmatrix} \delta_E \\ \delta_{TC} \end{bmatrix}$$

1. Engine dynamics decoupled from aircraft to engine (but not viceversa).

$$\dot{\delta}_T = 0.25\delta_T + 0.25\delta_{TC} \Rightarrow G_T(s) = \frac{0.25}{s + 0.25} \qquad \delta_{TC} = 1 \Rightarrow \delta_T = 1 - e^{-0.25t}$$



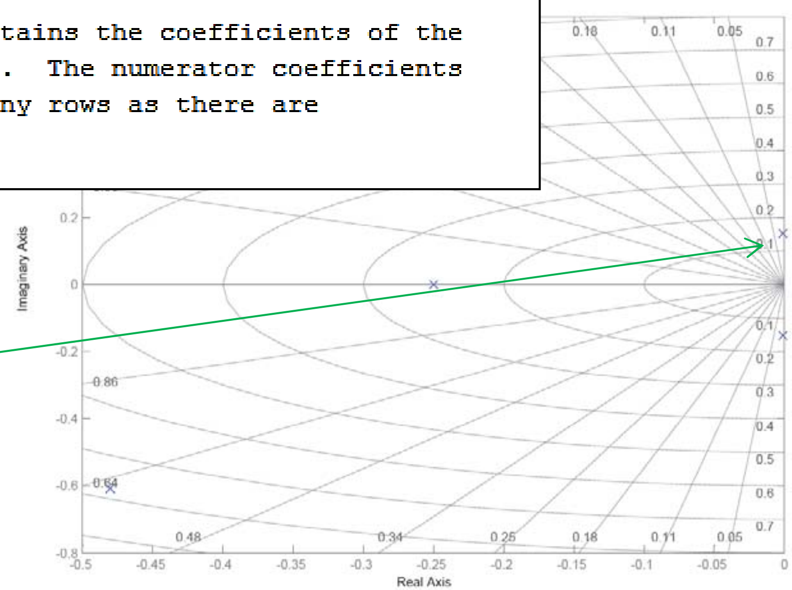
State Variables – Solution

$$G(s)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}\cdot\begin{bmatrix}s+.021&-.122&0&.322&-1\\.209&s+.53&-2.21&0&.044\\-.017&.164&s+.412&0&-.544\\0&0&-1&0&0\end{bmatrix}^{-1}\cdot\begin{bmatrix}.01&0\\-.064&0\\-.378&0\\0&0\\&.25\end{bmatrix}$$

0	0.0100	-	…	.1375	-0.0238
0	-0.0640	-	…	.0059	0.0091
0	-0.3780	-	…	.0158	0.0000
0	-0.0000	-	…	.0810	0.0158

```
ss2tf  State-space to transfer function conversion.
  [NUM,DEN] = ss2tf(A,B,C,D,iu)  calculates the transfer function:

            NUM(s)          -1
     H(s) = -------- = C(sI-A) B + D
            DEN(s)
  of the system:
      .
      x = Ax + Bu
      y = Cx + Du

  from the iu'th input.  Vector DEN contains the coefficients of the
  denominator in descending powers of s.  The numerator coefficients
  are returned in matrix NUM with as many rows as there are
  outputs y.
```

```
>> roots([1.0000    1.

ans =

  -0.4804 + 0.6083i
  -0.4804 - 0.6083i
  -0.2507
  -0.0008 + 0.1524i
  -0.0008 - 0.1524i

>> eig(a)

ans =

  -0.4804 + 0.6083i
  -0.4804 - 0.6083i
  -0.0011 + 0.1523i
  -0.0011 - 0.1523i
  -0.2500
```



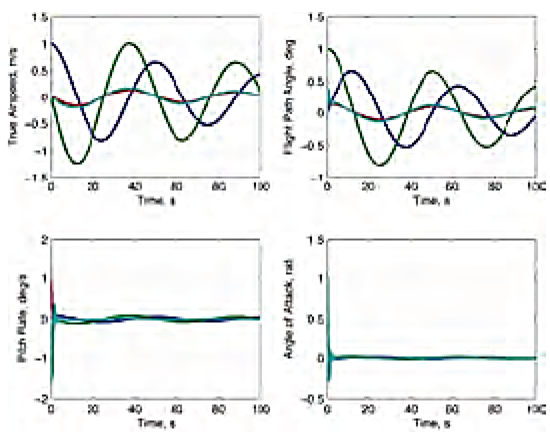
State Variables – Solution

$$\boldsymbol{y}(t) = \begin{bmatrix} u(t) \\ w(t) \\ q(t) \\ \theta(t) \end{bmatrix} = \boldsymbol{C}_1 e^{-0.4804t}\sin(0.6083t + \boldsymbol{C}_2) + \boldsymbol{C}_3 e^{-0.0011t}\sin(0.1543t + \boldsymbol{C}_4)$$

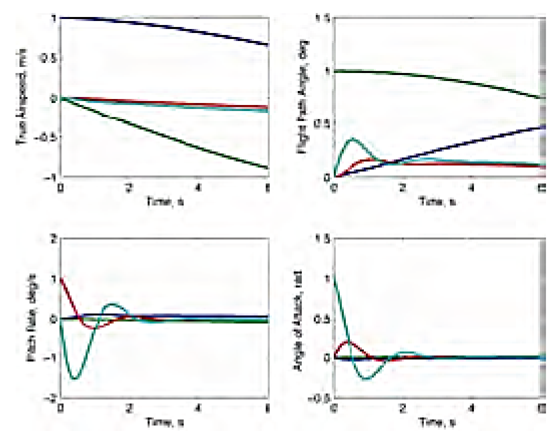
- 2 Natural Modes
- Short Period influenced primarily by $\boldsymbol{w}$ e $\boldsymbol{q}$
- Phugoid influenced primarily by $\boldsymbol{u}$, $\boldsymbol{w}$ e $\boldsymbol{q}$

- **LTI 4th-order responses viewed over different periods of time**
  - **4 initial conditions**

- **0 - 100 sec**
- **Reveals Phugoid Mode**



- **0 - 6 sec**
- **Reveals Short-Period Mode**



State Variables – Solution

❑ Time Domain Solution (limited to LTI systems)

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \end{cases}, \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

❑ Consider a scalar example

$$\begin{cases} \dot{x} = ax \\ x(t_0) = x_0 \end{cases} \qquad x(t) = ke^{at} \qquad x(t_0) = x_0 = ke^{at_0} \Rightarrow k = x_0 e^{-at_0} \qquad x(t) = e^{a(t-t_0)}x_0$$

$$e^{at} = 1 + at + a^2\frac{t^2}{2!} + a^3\frac{t^3}{3!} + \dots$$

- **<u>Globally convergent power series</u>**

Consider an n-th dimensional homogeneous system: $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}, A^{n\times n} \quad \boldsymbol{x}(t) \in \Re^n$

From Laplace Transform: $\boldsymbol{x}(s) = (sI - A)^{-1}\boldsymbol{x}_0$

- Analogy with scalar case: $\Rightarrow$ $\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0$

State Variables – Solution

- **Definition :** for LTI systems, we define the state **transition matrix** (or matrix exponential):

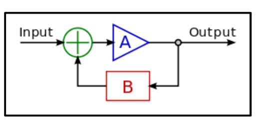
$$e^{A(t-t_0)}$$

- The solutiuon of the homogeneous system only requires the computation of the transition matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad e^{At} \neq \begin{bmatrix} e^{t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}$$

Method I: Power Series Expansion

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \ldots + A^k \frac{t^k}{k!} + \ldots \qquad A^i = A \cdot A \cdot A \ldots$$

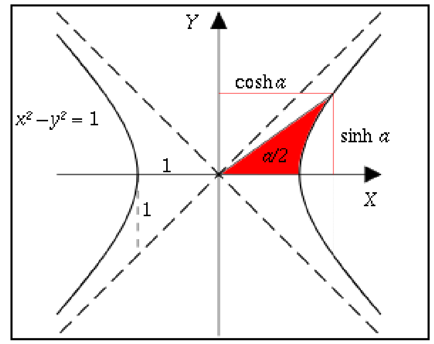


State Variables – Solution

- **Example:** $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^3 = A\ , \quad A^4 = I\ , \quad \ldots$

$$e^{At} = I\left[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \ldots\right] + A\left[t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots\right] =$$

$$= I\cosh t + A\sinh t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$



❑ Method II: Laplace Transform

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \;\Rightarrow\; s\boldsymbol{x}(s) - \boldsymbol{x}_0 = A\boldsymbol{x}(s) \;\Rightarrow\; \boldsymbol{x}(s) = [sI - A]^{-1}\boldsymbol{x}_0$$

$$\Rightarrow \quad \boldsymbol{x}(t) = L^{-1}\left\{ \left[sI - A\right]^{-1} \right\} \boldsymbol{x}_0 = e^{At}\boldsymbol{x}_0 \quad \Rightarrow \quad e^{At} = \boxed{L^{-1}\left\{ \left[sI - A\right]^{-1} \right\}}$$

State Variables – Solution

- **Example:** $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ $\quad [sI - A] = \begin{bmatrix} s+3 & -1 \\ -1 & s+3 \end{bmatrix}$

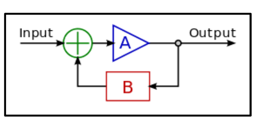
$$[sI - A]^{-1} = \frac{\begin{bmatrix} s+3 & -1 \\ -1 & s+3 \end{bmatrix}}{s^2 + 6s + 8} = \begin{bmatrix} \frac{s+3}{(s+2)(s+4)} & \frac{1}{(s+2)(s+4)} \\ \frac{1}{(s+2)(s+4)} & \frac{s+3}{(s+2)(s+4)} \end{bmatrix} \qquad e^{At} = \begin{bmatrix} \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} & -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} \\ -\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} & \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t} \end{bmatrix}$$

Method III: Jordan Form

Given a square Matrix A, it is always possible to transform it into a diagonal Matrix Λ or a Jordan Matrix J with a similitute transformation.

Case A: A with distinct eigenvalues

$$M^{-1}AM = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad A = M\Lambda\ M^{-1}$$

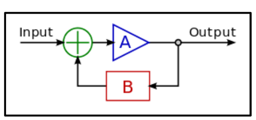


State Variables – Solution

$$
\begin{aligned}
e^{At} &= MM^{-1} + M\Lambda\ M^{-1}t + M\Lambda\ M^{-1}M\Lambda\ M^{-1}\frac{t^2}{2!} + \dots \\
&= MIM^{-1} + M\Lambda\ M^{-1}t + M\Lambda^2 M^{-1}\frac{t^2}{2!} + \dots = \\
&= M\left\{ I + \Lambda\ t + \Lambda^2\frac{t^2}{2!} + \dots \right\} M^{-1} = Me^{\Lambda t}M^{-1}
\end{aligned}
$$

$$
\begin{aligned}
e^{\Lambda t} &= I + \Lambda t + \Lambda^2\frac{t^2}{2!} + \Lambda^3\frac{t^3}{3!} + \dots + \Lambda^k\frac{t^k}{k!} + \dots = \\
&= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} t + \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix}\frac{t^2}{2!} + \dots + \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix}\frac{t^k}{k!} + \dots = \\
&= \begin{bmatrix} 1 + \lambda_1 t + \lambda_1^2 t^2 + \cdots + \lambda_1^k t^k + \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \lambda_n t + \lambda_n^2 t^2 + \cdots + \lambda_n^k t^k + \cdots \end{bmatrix} =
\end{aligned}
$$

$$
\boxed{e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}}
$$



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State Variables – Solution

- Since $e^{\Lambda t}$ is known once the eigenvalues of A are known, the original transition matrix can be found from M and M^{-1}.

$$M = [v_1, v_2, \ldots, v_n], \; (\lambda_i I - A)v_i = 0 \qquad M^{-1} = \begin{bmatrix} \mu_1^T \\ \mu_2^T \\ \vdots \\ \mu_n^T \end{bmatrix}, \; \mu_i^T \left(\lambda_i - A\right) = 0$$

$$e^{At} = M e^{\Lambda t} M^{-1} = \left[v_1, v_2, \ldots, v_n\right] \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \mu_1^T \\ \mu_2^T \\ \vdots \\ \mu_n^T \end{bmatrix}$$

$$\boxed{\boldsymbol{x}(t) = \sum_{i=1}^{n} \boldsymbol{v}_i \boldsymbol{\mu}_i^T e^{\lambda_i t} \boldsymbol{x}_0 = \sum_{i=1}^{n} \boldsymbol{\alpha}_i e^{\lambda_i t} = \boldsymbol{\alpha}_1 e^{\lambda_1 t} + \ldots + \boldsymbol{\alpha}_n e^{\lambda_n t}}$$

Case B: Repeated Eigenvalues

In this case, using an appropriate similitude transformation matrix P, A can be put in Jordan Form (P is made of eigenvectors and **generalized eigenvectors**).

$$P^{-1}AP = J$$

Consider, for example, a single Jordan block:

$$J = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} = \lambda_i I + \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = \lambda_i I + R$$

Thus:

$$\boxed{e^{Jt} = e^{(\lambda_i I + R)t} = e^{\lambda_i I t} e^{Rt}}$$

State Variables – Solution

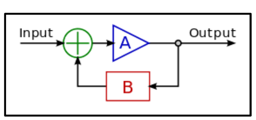
The product of any matrix C by R results in the same matrix with columns shifted to the right and the others equal to a zero column.

$$CR = [c_1, c_2, \ldots, c_n]\ R \;=\; [0, c_1, c_2, \ldots, c_{n-1}]$$

If R is (n x n), then R^n = 0, for instance: n = 3:

$$R_{3x3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R^4 = R^5$$

$$e^{Rt} = I + Rt + R^2 \frac{t^2}{2!} + \ldots + R^n \frac{t^n}{n!} + 0 \qquad = \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$



State Variables – Solution

Example:

$$\dot{\boldsymbol{x}} = \left[\begin{array}{cc:c} -1 & 1 & 0 \\ 0 & -1 & 0 \\ \hdashline 0 & 0 & -2 \end{array}\right]\boldsymbol{x} = \left[\begin{array}{c:c} J_1 & 0 \\ \hdashline 0 & J_2 \end{array}\right]\boldsymbol{x}$$

$$J_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \lambda_i I + R$$

$$e^{J_1 t} = e^{-t}e^{Rt} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}e^{-t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\boldsymbol{x}(t) = \left[\begin{array}{c:c} e^{J_1 t} & 0 \\ \hdashline 0 & e^{J_2 t} \end{array}\right]\boldsymbol{x}_0 = \left[\begin{array}{cc:c} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ \hdashline 0 & 0 & e^{-2t} \end{array}\right]\begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix}$$

$$\begin{cases} x_1(t) = x_{10}e^{-t} + x_{20}te^{-t} \\ x_2(t) = x_{20}e^{-t} \\ x_3(t) = x_{30}e^{-2t} \end{cases}$$

State Variables – Solution

❑ Methodo IV: From Cayley-Hamilton Theorem

$$\Delta(A) = A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0$$

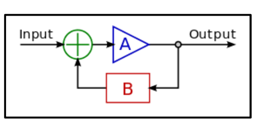
$$A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \dots - a_0 I$$

$$A^{n+1} = AA^n = -a_{n-1}\left[-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \dots - a_0 I\right] - a_{n-2}A^{n-1} - \dots - a_0 A$$

$$A^{n+2} = AA^{n+1} = -a_{n-1}\left[-a_{n-1}\left[-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \dots - a_0 I\right] - a_{n-2}A^{n-1} - \dots - a_0 A\right]$$
$$-a_{n-2}\left[-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \dots - a_0 I\right] - \dots - a_0 A^2$$

- From above we can say that any power n of A is given by a linear combination of the first n – 1 powers.

$$\boxed{e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \quad \dots \quad + \alpha_{n-1}(t)A^{n-1} = \sum_{i=0}^{n-1} A^i \alpha_i(t)}$$



State Variables – Solution

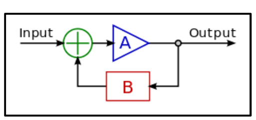
- From Caley-Hamilton we can find α_i for every eigenvalue:

$$\forall \lambda_i \Rightarrow e^{\lambda_i t} = 1 + \lambda_i t + \lambda_i^2 \frac{t^2}{2!} + \ \ldots \ = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2 + \ \ldots \ + \alpha_{n-1}(t)\lambda_i^{n-1}$$

- In the case of matrix A with distinct eigenvalues we can built the ***Vandermonde Matrix*** from which we can compute α_i

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

- **Note:** This can be done also for repeated eigenvalues.



State Variables – Solution

- **Example:**

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}, Eigenvalues : \left(\lambda_1 = -1 \quad \lambda_2 = +2\right)$$

$$e^{At} = \alpha_0(t) I + \alpha_1(t) A \qquad \begin{bmatrix} 1 & -1 \\ 1 & +2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{+2t} \end{bmatrix} \Rightarrow \begin{cases} \alpha_0(t) = \dfrac{e^{2t} + 2e^{-t}}{3} \\ \alpha_1(t) = \dfrac{e^{2t} - e^{-t}}{3} \end{cases}$$

$$e^{At} = \frac{e^{2t} + 2e^{-t}}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{e^{2t} - e^{-t}}{3} \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix}$$

$$\boldsymbol{x}(t) = e^{At} \boldsymbol{x}_0 = \begin{bmatrix} \frac{1}{3}\left(3x_{10} - 4x_{20}\right) e^{-t} + \frac{4}{3} x_{20} e^{2t} \\ \frac{1}{3}\left(x_{10} - 2x_{20}\right) e^{-t} + \frac{1}{3}\left(x_{10} + 2x_{20}\right) e^{2t} \end{bmatrix}$$

State Variables – Solution

- Consider the general case of a forced system:

$$\begin{cases}\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)\end{cases}, \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

$$e^{-At}\dot{\boldsymbol{x}} = e^{-At}A\boldsymbol{x} + e^{-At}B\boldsymbol{u} \quad \Rightarrow \quad \frac{d}{dt}\left[e^{-At}\boldsymbol{x}\right] = e^{-At}B\boldsymbol{u} \quad \Rightarrow \quad e^{-At}\boldsymbol{x} = e^{-At_0}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{-At}B\boldsymbol{u}(\tau)d\tau$$

$$\boxed{\begin{cases}\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + \displaystyle\int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau \\ \boldsymbol{y}(t) = Ce^{A(t-t_0)}\boldsymbol{x}_0 + C\displaystyle\int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau\end{cases}}$$

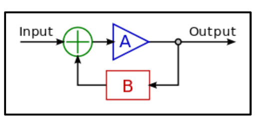
- **NOTE:** Transition matrix is the only element that needs to be computed.

- Given the system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ With transformation $\boldsymbol{q} = M^{-1}\boldsymbol{x}$

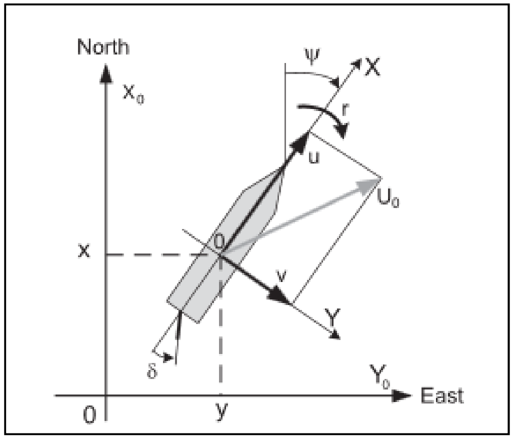
- $\boldsymbol{q}$ is called vector of modal coordinates

$$\dot{\boldsymbol{q}} = M^{-1}\dot{\boldsymbol{x}} = M^{-1}A\boldsymbol{x} = M^{-1}AM\boldsymbol{q} = \Lambda\boldsymbol{q}$$

$$\boldsymbol{q}(t) = e^{\Lambda t}\boldsymbol{q}_0, \begin{bmatrix} q_1(t) \\ \dots \\ q_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & .. & 0 \\ 0 & e^{\lambda_2 t} & .. & .. \\ .. & .. & .. & 0 \\ 0 & .. & 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} q_{10} \\ \dots \\ q_{n0} \end{bmatrix}$$



State Variables – Examples



$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} -0.589 & -3.847 \\ -0.763 & -0.351 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0.1374 \\ -1.3402 \end{bmatrix} \boldsymbol{u}$$
$$\boldsymbol{y} = C\boldsymbol{x}$$

- Find the response to initial conditions, the response to a unit step both in terms of vector $\boldsymbol{x}$ and modal coordinates $\boldsymbol{q}$

Structural Properties

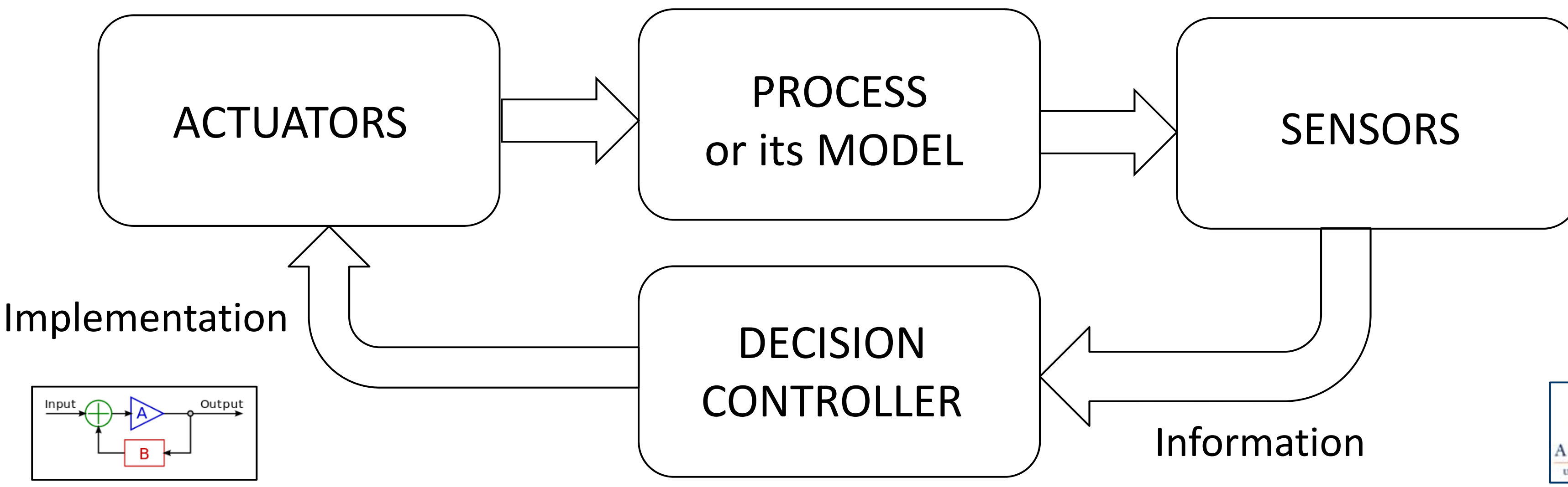
- Introduction
- Stability
- Controllability / Reachability
- Observability / Detectability
- Realizations

Structural Properties

- The analysis of dynamic systems is a process that allows the determination of physical characteristics based on their analytical model and NOT the nature of the process itself:

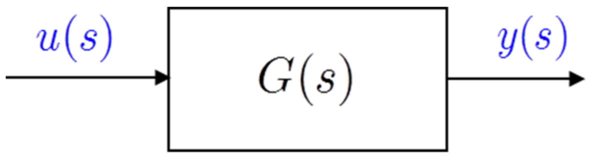
    - **Structural Properties**: depend on the nature of the system and not on external actions
    - **Esternal Properties**: depend of the input – output relationship

- The analysis allows the control engineer to apply a set of **decision methods** for controlling the system, which are not dependent on the physical nature of the system itself.



Stability

- The concept of stability is general and applicable to dynamic systems that are nonlinear, linear, continuous, discrete, time varying, time invariant, and even to systems not characterized by a mathematical model.



$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \\ \boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \end{cases}, \quad \boldsymbol{x}_0$$

- **Definition**: Given and initial time, initial state and an input signal $t_0,\ \boldsymbol{x}(t_0),\ \boldsymbol{u}(t)$: we define a movement, the time evolution of the state vector at the current time, as the solution of the goverrning differential equations.

$$\tilde{\boldsymbol{x}}(t) \quad \text{satisfies} \quad \begin{cases} \dot{\tilde{\boldsymbol{x}}} = f(\tilde{\boldsymbol{x}}, \boldsymbol{u}, t) \\ \tilde{\boldsymbol{y}} = g(\tilde{\boldsymbol{x}}, \boldsymbol{u}, t) \end{cases}, \quad \tilde{\boldsymbol{x}}_0$$

- **Definition**: A movement $\boldsymbol{x}_E(t)$ is of equilibrium if it satisfies:

$$\begin{cases} \dot{\boldsymbol{x}}_E = 0 = f(\boldsymbol{x}_E, \boldsymbol{u}_E, t) \\ \boldsymbol{y}_E = g(\boldsymbol{x}_E, \boldsymbol{u}_E, t) \end{cases}, \quad \boldsymbol{x}_{E0}$$

Stability

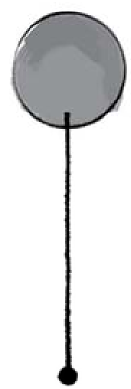
- **Definition**: A perturbation is the variation of a movement (even equilibrium) due to variations in initial conditions and/or input:

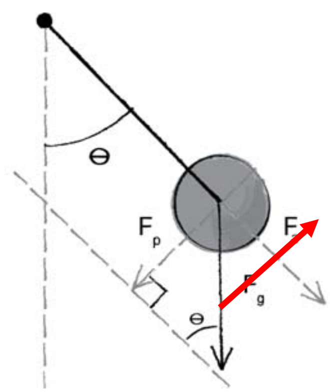
$$\tilde{\boldsymbol{x}}^*(t) = \tilde{\boldsymbol{x}}(t) + \delta \boldsymbol{x}(t)$$

$$\dot{\tilde{\boldsymbol{x}}}^*(t) = \dot{\tilde{\boldsymbol{x}}}(t) + \delta \dot{\boldsymbol{x}}(t) = f(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{u}}^*, t), \quad \tilde{\boldsymbol{x}}_0^* = \tilde{\boldsymbol{x}}_0 + \delta \tilde{\boldsymbol{x}}_0$$

$$\dot{\tilde{\boldsymbol{x}}}_E^*(t) = \dot{\tilde{\boldsymbol{x}}}_E(t) + \delta \dot{\boldsymbol{x}}(t) = \delta \dot{\boldsymbol{x}}(t) = f(\tilde{\boldsymbol{x}}^*, \tilde{\boldsymbol{u}}, t), \quad \tilde{\boldsymbol{x}}_E^* = \tilde{\boldsymbol{x}}_{E0} + \delta \tilde{\boldsymbol{x}}_E$$

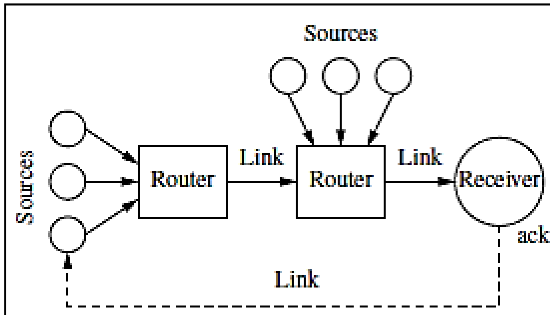
  - In general, at the equilibrium, we assume the input perturbation equal to zero, and we only consider the nominal input (if necessary) needed to maintain the equilibrium condition





  - Note: For a linear time invariant system, the only equilibrium condition (movement) is the origin.

Stability



(a) Block diagram

Figure A.12: Internet congestion control. (a) Source computers send information to routers, which forward the information to other routers that eventually connect to the receiving computer. When a packet is received, an acknowledgment packet is sent back through the routers (not shown). The routers buffer information received from the sources and send the data across the outgoing link. (b) The equilibrium buffer size b_e for a set of N identical computers sending packets through a single router with drop probability ρ.

Definitions:

ω = Source window size
b = Router buffer size
ρ = Probability Packet Drop
c = Link Capacity

Internet Congestion Control: The system has two control mechanisms called protocols: Transmission Control Protocol (TCP) and Internet Protocol (IP).

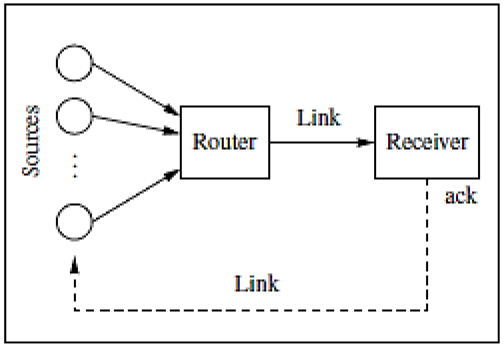
State Variables:

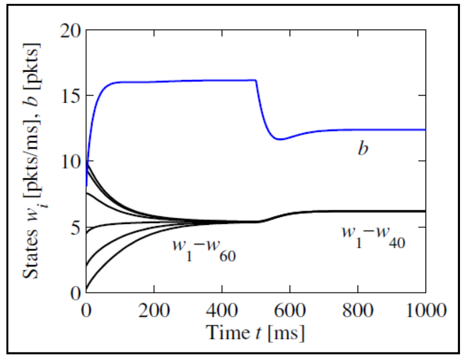
1. Window dynamic size of every source as function of the last packet received.
2. Router size memory buffer indicating the number of packets available to be sent.

Computation of equilibrium points: Equilibrium buffer size as balance between transmission speed of sources and link capacity.

Stability

Example: Single router, N equal sources, buffer not higher than 500 packets.





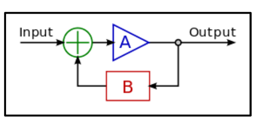
$$x_1 = \omega, x_2 = b$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{c}{x_2} - \rho c(1 + \dfrac{x_1^2}{2}) \\ \dfrac{Ncx_1}{x_2} - c \end{bmatrix}$$

- As shown on the left, multiple sources attempt to communicate through a router across a single link. An "ack" packet sent by the receiver acknowledges that the message was received; otherwise the message packet is resent and the sending rate is slowed down at the source.
- The simulation on the right is for 60 sources starting random rates, with 20 sources dropping out at t = 500 ms. The buffer size is shown at the top, and the individual source rates for 6 of the sources are shown at the bottom.

$$\dot{\boldsymbol{x}}_E = 0 \Rightarrow \begin{bmatrix} \dfrac{c}{x_{2E}} - \rho c(1 + \dfrac{x_{1E}^2}{2}) \\ \dfrac{Ncx_{1E}}{x_{2E}} - c \end{bmatrix} = 0$$

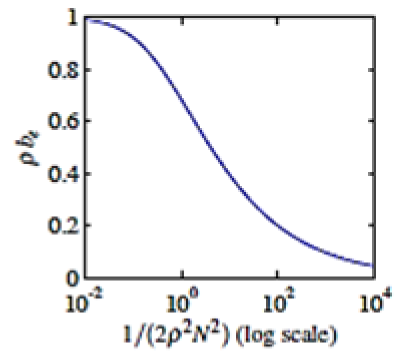
$$\begin{cases} \dfrac{c}{x_{2E}} - \rho c(1 + \dfrac{x_{1E}^2}{2}) = 0 \\ \dfrac{Ncx_{1E}}{x_{2E}} - c = 0 \end{cases} \qquad \begin{cases} 1 - \rho x_{2E}(1 + \dfrac{x_{2E}^2}{2N^2}) = 0 \\ x_{1E} = \dfrac{1}{N} x_{2E} \end{cases} \qquad \begin{cases} \dfrac{1}{2\rho^2 N^2}(\rho^3 x_{2E}^3) + (\rho x_{2E}) - 1 = 0 \\ x_{1E} = \dfrac{1}{N} x_{2E} \end{cases}$$



Stability

Numerical solution of the equilibrium equation for x_{2E} (size of buffer at equilibrium).

$$\frac{1}{2\rho^2 N^2}(\rho^3 x_{2E}^3) + (\rho x_{2E}) - 1 = 0$$



(b) Operating point

- **Question:** How does the solution differ if we consider the linearized system?

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{c}{x_2} - \rho c\left(1 + \dfrac{x_1^2}{2}\right) \\ \dfrac{N c x_1}{x_2} - c \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \Rightarrow \dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_E + \delta\dot{\boldsymbol{x}} \Rightarrow \delta\dot{\boldsymbol{x}} = J\Big|_{x_E} \delta\boldsymbol{x} \qquad J\Big|_{x_E} = \left.\begin{bmatrix} -\rho c x_1 & -\dfrac{c}{x_2^2} \\ \dfrac{Nc}{x_2} & -\dfrac{N c x_1}{x_2^2} \end{bmatrix}\right|_{x_E}$$

- Use Phase Plane representation

Stability

Movement/Condition

Stable

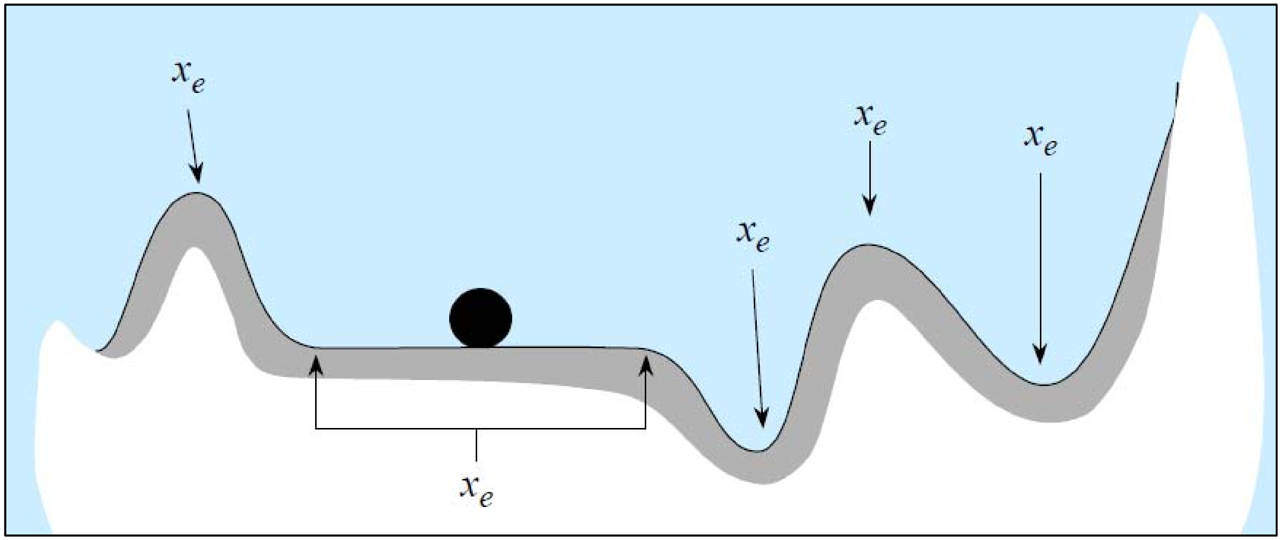
Unstable

Marginally stable

Asymptotically Stable

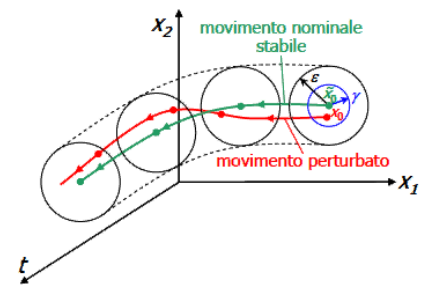
Locally Asymptotically Stable

Globally Asymptotically Stable

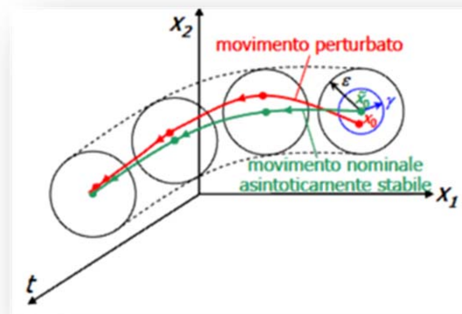


Stability

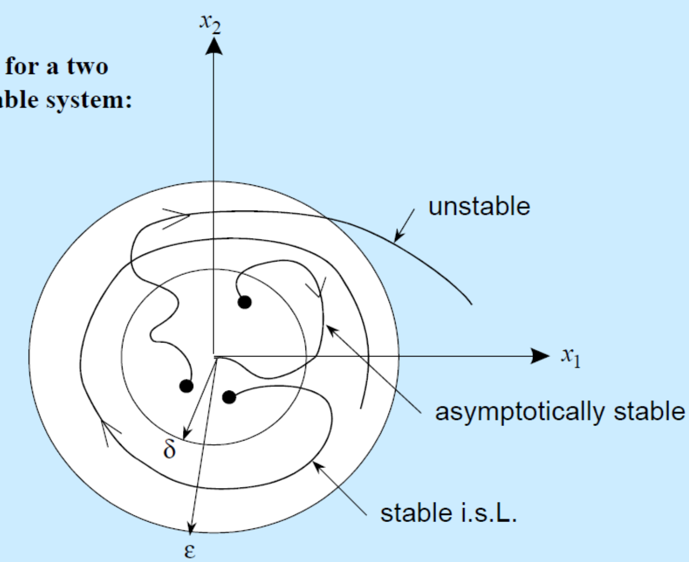
DEFINITION: The origin is a stable equilibrium if for $\varepsilon > 0$ any there exists a $\delta(\varepsilon, t_0) > 0$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t > t_0$.

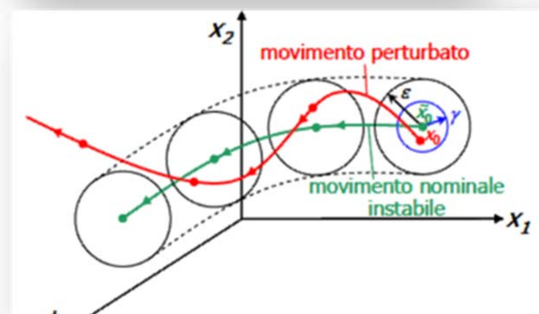


DEFINITION: The origin is *asymptotically stable* if it is stable *and*: there exists an $\delta'(t_0) > 0$ such that whenever $\|x(t_0)\| < \delta'(t_0)$ then $\lim_{t \to \infty} \|x(t)\| = 0$



Picture for a two variable system:

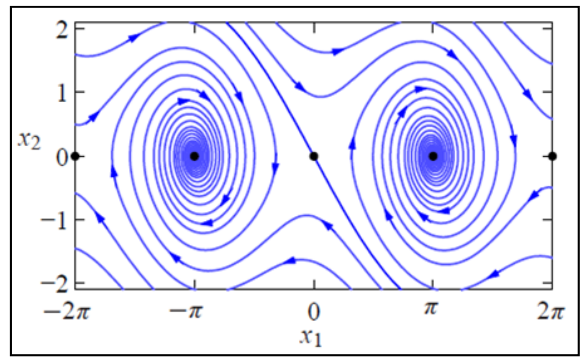




- **External Stability**: Function of the Input

movement
- unstable
- stable
  - bounded input bounded state (BIBS)
  - Bounded input bounded output (BIBO)

- **Comments:**
  - Stability of a movement of equilibrium does not usually imply stability of a system.



- **Theorem:** An equilibrium movement of a system is stable, asymptotically stable or unstable, if and only if all movements have the same property.

Stability

❑ Stability properties for linear systems

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \quad \boldsymbol{x}_0$$

- Stability of a movement $\tilde{\boldsymbol{x}}$

$\tilde{\boldsymbol{x}}_p = \tilde{\boldsymbol{x}} + \delta\boldsymbol{x}$ Perturbed movement

$$\dot{\tilde{\boldsymbol{x}}}_p - \dot{\tilde{\boldsymbol{x}}} = A(\tilde{\boldsymbol{x}} + \delta\boldsymbol{x}) + B\boldsymbol{u} - A\tilde{\boldsymbol{x}} - B\boldsymbol{u} \quad \Rightarrow \quad \delta\,\dot{\boldsymbol{x}} \; = \; A\,\delta\,\boldsymbol{x}$$

❑ The following are equivalent:

1. Stability can be studied by looking at the autonomous system only.
2. It is sufficient to study the stability of the origin.
3. Stability of the origin and stability of the system are the same.
4. The knowledge of the syste matrix A provides all necessary information about stability.

Stability

- **Summary**

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \qquad \boldsymbol{x}_0 \qquad \begin{cases} \boldsymbol{x}(t) = \Phi\left(t,\ t_0\right)\boldsymbol{x}_0 \\ \boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 \end{cases} \qquad \begin{array}{l} A = A(t) \\ A = \text{costante} \end{array}$$

$$\boldsymbol{x}(t) = 0 \Leftrightarrow \boldsymbol{x}_E = 0$$

$$\left\| \Phi(t,t_0)\right\| \leq M \quad \forall\ t \geq t_0 \qquad\qquad \lim_{t\to\infty} \left\| \Phi(t,t_0)\right\| = 0 \quad \forall\ t \geq t_0$$

$$\left\| e^{A(t-t_0)}\right\| \leq M \quad \forall\ t \geq t_0 \qquad\qquad \lim_{t\to\infty} \left\| e^{A(t-t_0)}\right\| = 0 \quad \forall\ t \geq t_0$$

  - For LTI systems: $\delta\boldsymbol{x}(t) = e^{A(t-t_0)}\delta\boldsymbol{x}_0$

- **Theorem**: A LTI system is asymptotically stable iff all eigenvalues of A have strictly negative real part.

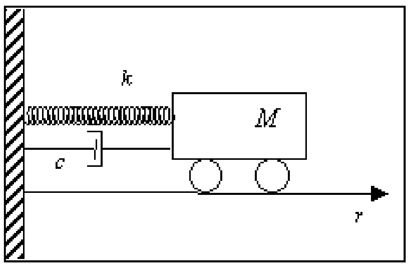
- **Theorem:** A LTI system is stable iff the eigenvalues of A have negative or zero part. The latter must have equal algebraic and geometric multiplicity.

- **Theorem:** L LTI system is unstable otherwise.

Stability

- Lyapunov Theory allows a systematic study of internal stability for dynamical sysyems (NL, LTV, LTI) and it is based on the dissipative properties of electromechanical systems.
- When the total energy is dissipated, the system evolves towards the equilibrium.
- The total energy change with time can be associated to stability.

1857 - 1918



- Potential Energy (spring)
- Kinetic energy (mass)
- Dissipative energy (friction)

$$M\ \ddot{r} + c\dot{r}\left|\dot{r}\right| + k_0 r + k_1 r^3 = 0$$

$$\boldsymbol{x} = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}; \dot{\boldsymbol{x}} = f(\boldsymbol{x}) \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{c}{M} x_2 \left|x_2\right| - \dfrac{k_0}{M} x_1 - \dfrac{k_1}{M} x_1^3 \end{cases}; \boldsymbol{x}_E = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V(\boldsymbol{x}) = \frac{1}{2} M\dot{r}^2 + \int_0^r (k_0\xi + k_1\xi^3)d\xi = \frac{1}{2} M\dot{r}^2 + \frac{1}{2} k_0 r^2 + \frac{1}{4} k_1 r^4$$

- Mechanical energy (conservative field) remains constant

Stability

- **Considerations on the mechanical energy $V(\boldsymbol{x})$:** $$V(\boldsymbol{x}) = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}k_0 r^2 + \frac{1}{4}k_1 r^4$$

- $V(\boldsymbol{x}) > 0$ — Is a positive function
- $V(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{x}_E = \left(r = 0, \dot{r} = 0\right)$ — It is equal to zero only at equilibrium
- $V(\boldsymbol{x}) \rightarrow 0 \Rightarrow \boldsymbol{x}(t) \rightarrow \boldsymbol{x}_E$ — The movement is asymptotically stable
- $V(\boldsymbol{x}) \rightarrow \infty \Rightarrow \boldsymbol{x}(t) \rightarrow \infty \in \Re^2$ — The movement is unstable

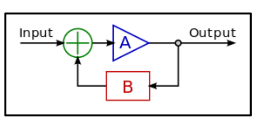
The mechanical energy gives the amplitude of the state vector, but its time variation characterizes the stability of the system.

$$\dot{V}(\boldsymbol{x}) = M\dot{r}\ddot{r} + (k_0 r + k_1 r^3)\dot{r}$$

$$\dot{V}(\boldsymbol{x}) = \dot{r}(-c\dot{r}\left|\dot{r}\right|\ ) = -c\left|\dot{r}\right|^3 < 0$$
$$c > 0$$

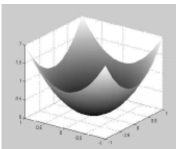
- The total energy continually decreases due to the damper, until the mass stops moving.

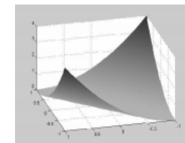
- **Can this property be generalized?**



Stability

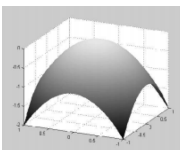
- **Definition – Lyapunov Function**: Consider a scalar function $V(\cdot) : \Re^n \longrightarrow \Re^1$ then:

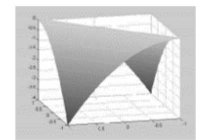




- V is positive definite (semi-definite) in $\boldsymbol{x}'$ if:

$$V(\boldsymbol{x}') = 0, \exists\delta > 0 \Rightarrow V(\boldsymbol{x}) > 0 \left(V(\boldsymbol{x}) \geq 0\right) \forall \boldsymbol{x} : \left\|\boldsymbol{x} - \boldsymbol{x}'\right\| < \delta, \boldsymbol{x} \neq \boldsymbol{x}'$$





- V is negative definite (semi-definite) in $\boldsymbol{x}'$ if:

$$V(\boldsymbol{x}') = 0, \exists\delta > 0 \Rightarrow V(\boldsymbol{x}) < 0 \left(V(\boldsymbol{x}) \leq 0\right) \forall \boldsymbol{x} : \left\|\boldsymbol{x} - \boldsymbol{x}'\right\| < \delta, \boldsymbol{x} \neq \boldsymbol{x}'$$

- V is undefined otherwise.

  - Example: Quadratic forms:

$$V(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} > 0 \quad \Leftrightarrow \quad A = A^T > 0$$
$$V(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x} < 0 \quad \Leftrightarrow \quad A = A^T < 0$$

Stability

First Theorem of Lyapunov:

Consider a nonlinear autonomous system with equilibrium state $\boldsymbol{x}_E$:

$$\dot{\boldsymbol{x}}_E = f(\boldsymbol{x}_E) = 0$$

Compute the linearized system: $\delta\dot{\boldsymbol{x}} = \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} \delta\boldsymbol{x} = A\delta\boldsymbol{x}$

- If the origin of the linearized system ($\delta\boldsymbol{x} = 0$) is asymptotically stable, then the equilibrium state $\boldsymbol{x}_E$ is asymptotically stable,
- If the origin of the linearized system ($\delta\boldsymbol{x} = 0$) is unstable, then the equilibrium state $\boldsymbol{x}_E$ is unstable,
- If the origin of the linearized system ($\delta\boldsymbol{x} = 0$) is stable, then nothing can be said about the stability of the equilibrium state $\boldsymbol{x}_E$.

A more useful theorem is the following

Stability

- **Direct Method of Lyapunov (second theorem):** – Let $V(.)$ be a Lyapunov function, continuous and with continuous partial derivatives. Consider the equilibrium state of an autonomous system:

$$\dot{\boldsymbol{x}}_E = f(\boldsymbol{x}_E) = 0$$

Let $V(\boldsymbol{x}) > 0$ except for at most $V(\boldsymbol{x}_E) = 0$ and compute its total time derivative

$$\dot{V}(\cdot) = \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x})$$

The equilibrium state is unstable if $\dot{V}(\cdot) > 0$

The equilibrium state is stable if $\dot{V}(\cdot) \leq 0$

The equilibrium state is asymptotically stable if $\dot{V}(\cdot) < 0$

The equilibrium state is globally asymptotically stable if $\dot{V}(\cdot) < 0,\ and\ \lim_{x\to\infty} V(x) = \infty$

- **Note:** if the Lyapunov function depend $V(\boldsymbol{x},\ t)$ depends explicitely on the time we have:

$$\dot{V}(\boldsymbol{x}, t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x})$$

Stability

❑ Notes

- Sufficient Conditions: what happens if there is no Lyapunov function?
- How do we find a Lyapunov function?

▪ **Example** $\dot{x} = (1 - x^5)$ One equilibrium point: $x_E = 1$

- **<u>Choose</u>** a Lyapunov Function

$$V(x) = (1-x)^2 > 0 \qquad \dot{V}(x) = \frac{\partial V}{\partial x} f(x) = -2(1-x)(1-x)^5 = -2(1-x)^6$$

- The equilibrium point is <u>asymptotically stable</u>.

▪ **Example**

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \boldsymbol{x} \qquad \Rightarrow \boldsymbol{x}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \boldsymbol{x}_0 \qquad \Rightarrow \qquad \textbf{unstable}$$

Stability

- **Example**

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(x_1^2 - 1)x_2 - x_1 \end{cases} ; \boldsymbol{x}_E = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Consider a Lyapunov function which is continuous, with continuous derivative, positive definite and equal to zero only at equilibrium.

$$V(\boldsymbol{x}) = x_1^2 + x_2^2 = \boldsymbol{x}^T P \boldsymbol{x}; P = I$$

- Compute its total time derivative:

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -(x_1^2 - 1)x_2 - x_1 \end{bmatrix} = -2x_2^2(x_1^2 - 1)$$

$$\forall |x_1| < 1 \Rightarrow \dot{V}(\boldsymbol{x}) > 0$$

- The origin is therefore an unstable equilibrium point

- **Can we stabilize the system around the equilibrium with an appropriate input?**

- **Example**

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(x_1^2 - 1)x_2 - x_1 + u \end{cases}$$

  - Perform the same analysis

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -(x_1^2 - 1)x_2 - x_1 + u \end{bmatrix} = -2x_2^2(x_1^2 - 1) + 2x_2 u$$

$$\dot{V}(\boldsymbol{x}) < 0 \Rightarrow u < x_2(x_1^2 - 1)$$

    - The origin is now an asymptotically equilibrium point

Stability

- Simplified Procedure for analysis of internal stability in the case of LTI systems.

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t) \end{cases} \Longrightarrow \quad \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) \qquad \boldsymbol{x}_E(t) = 0$$

Choose a Lyapunov function of the form: $V(\boldsymbol{x}) = \boldsymbol{x}^T P\ \boldsymbol{x} > 0\ \forall \boldsymbol{x} \neq 0 \quad P = P^T > 0$

Stability requires to verify the sign of $\dot{V}$

$$\dot{V}(\boldsymbol{x}) = \dot{\boldsymbol{x}}^T P\ \boldsymbol{x} + \boldsymbol{x}^T P\ \dot{\boldsymbol{x}} = \boldsymbol{x}^T A^T P\ \boldsymbol{x} + \boldsymbol{x}^T PA\boldsymbol{x} = \boldsymbol{x}^T (A^T P\ + PA)\boldsymbol{x}$$

Let:

$$A^T P + PA = -Q \Longrightarrow \dot{V}(\boldsymbol{x}) = -\boldsymbol{x}^T Q\ \boldsymbol{x}$$

- **Definition**: The above is called **Lyapunov matrix algebraic equation**:

- **Theorem:** A linear autonomous system is asymptotically stable if and only if, for every Q symmetric and positive definite, there exists a matrix P, also symmetric and positive definite, such that:

$$A^T P + PA = -Q$$

- **Theorem:** Given a matrix A, All its eigenvalues have strictly negative part if and only if there exists a matrix P symmetric and positive definite and a matrix Q symmetric and positive definite, such that:

$$A^T P + PA + Q = 0$$

Corollary: If P esiste, is unique.

- **Theorem**: Let $A \in \Re^{n \times n}$ the following are equivalent:

1. All eigenvalues of A have strictly negative real part
2. For every $Q = Q^T > 0$, there exists a unique solution $P = P^T > 0$ of:

$$A^T P + PA + Q = 0 \qquad (1)$$

Stability

❑ Proof 1 -> 2

Assume A to have all eigenvalues with strictly negative real part. Let us define:

$$P := \int_0^{+\infty} e^{A^\top t} Q e^{At}\, dt$$

The integral exists and is finite since: $\lim\limits_{t\to\infty} e^{At} = 0$

Verify that P is solution of (1).

$$A^T P + PA + Q = 0 \ (1)$$

$$A^\top P + PA = \int_0^{+\infty} (A^\top e^{A^\top t} Q e^{At} + e^{A^\top t} Q e^{At} A)\, dt$$

$$= \int_0^{+\infty} \frac{d}{dt}\left(e^{A^\top t} Q e^{At}\right)\, dt = \left[e^{A^\top t} Q e^{At}\right]_0^{+\infty}$$

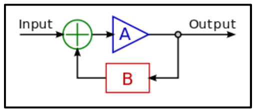
$$\lim_{t\to+\infty} e^{A^\top t} Q e^{At} = 0$$

The solution is also unique. Let P_1 e P_2 solutions of (1), ovvero:

$$A^\top P_1 + P_1 A + Q = 0$$
$$A^\top P_2 + P_2 A + Q = 0$$

Subtracting the second from the first: $A^\top(P_1 - P_2) + (P_1 - P_2)A = 0$

$$\begin{aligned} 0 &= e^{A^\top t}\left(A^\top(P_1 - P_2) + (P_1 - P_2)A\right) e^{At} \\ &= e^{A^\top t} A^\top (P_1 - P_2) e^{At} + e^{A^\top t}(P_1 - P_2) A e^{At} \\ &= \frac{d}{dt}\left(e^{A^\top t}(P_1 - P_2) e^{At}\right) \end{aligned}$$



Stability

the term $e^{A^\top t}(P_1 - P_2)e^{At}$ is constant, thus has the value acquired for t = 0

$$e^{A^\top t}(P_1 - P_2)e^{At} = P_1 - P_2$$

For $t \to +\infty \quad P_1 - P_2 = 0$

Therefore both solutions are coincident ■

- **Example:** Consider the following Lyapunov equation:

$$A^T P + PA + Q = 0; A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, P = P^T > 0, Q = Q^T > 0$$

- Select a symmetric and positive matrix Q:

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} = q_{12} & q_{22} \end{bmatrix} \Rightarrow q_{11}q_{22} - q_{12}^2 > 0 \Rightarrow \begin{bmatrix} q_{11} > 0 & 0 \\ 0 & q_{22} > 0 \end{bmatrix}$$

Stability

- Solve for P:

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = -\begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

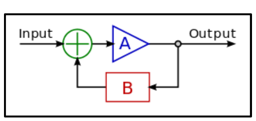
$$\begin{bmatrix} -2p_{11} & -3p_{12} \\ -3p_{12} & -4p_{22} \end{bmatrix} = -\begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

$$\begin{cases} p_{11} = \dfrac{q_{11}}{2} > 0 \\ p_{12} = 0 \\ p_{22} = \dfrac{q_{22}}{4} > 0 \end{cases}, \Rightarrow P = P^T > 0$$

- Repeat the procedure with: $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

$$Q = \begin{bmatrix} q_{11} > 0 & 0 \\ 0 & q_{22} > 0 \end{bmatrix}$$

$$\begin{cases} p_{11} = -\dfrac{q_{11}}{2} < 0 \\ p_{12} = 0 \\ p_{22} = \dfrac{q_{22}}{4} > 0 \end{cases}, \Rightarrow P = P^T < 0$$



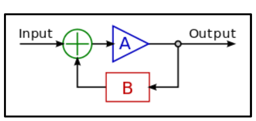
Reachability, Controllability

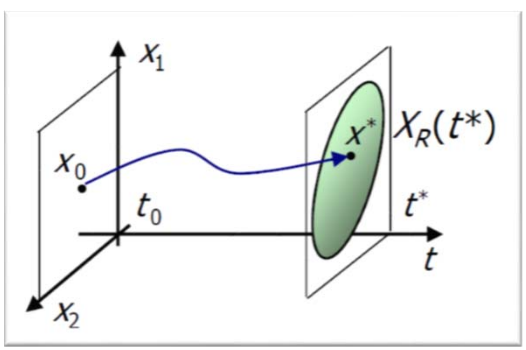
- The following properties are validi for dynamic systems in general (LTI, LTV, NL). We will limit ourselves to LTI systems of the form

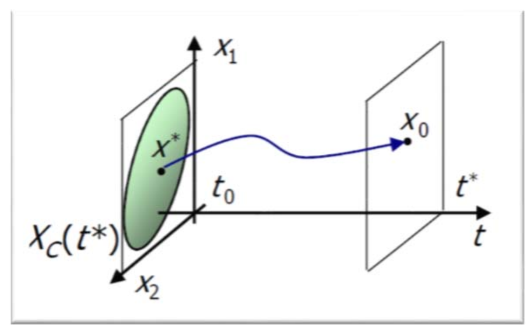
$$\begin{cases}\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ y = Cx\end{cases}, \quad \boldsymbol{x}_0, \quad \boldsymbol{x} \in \Re^n, \boldsymbol{u} \in \Re^m, \boldsymbol{y} \in \Re^p$$

- **Objective:** Analyze the intrinsic capability of a system to manipulate the entire systems state by an appropriate choice of input**.**

1. **Where can we transfer the state from $x(t_0)$ at some instant $t = t_1$?**
2. **How fast can we transfer the state to a desired value $x(t_1)$?**
3. **How can we compute the input $u(t)$ such that we can transfer the state from some $x(t_0)$ to a desired value $x(t_1)$?**
4. **How can we find the minimum (most efficient) $u(t)$ such that point 3 holds ?**







Reachability, Controllability

- **Definition – Reachable System**: A system is reachable from time t_0, if it is possible to find a **limited** input $\boldsymbol{u}(t)$ to bring the system from $\boldsymbol{x}(t_0) = 0$ to a final state $\boldsymbol{x}(t_1)$ in in time interval $[t_0,\ t_1]$.

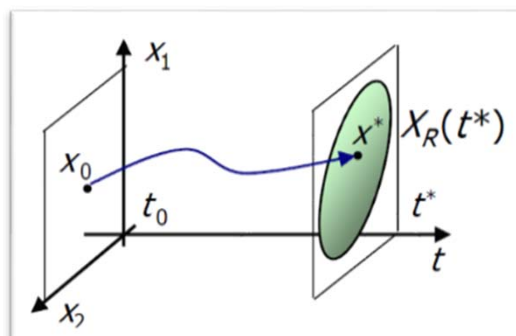
  - Given the state vector: $\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau;\ \boldsymbol{x}(t) \in \Re^n$

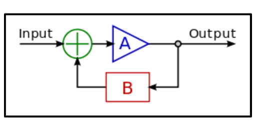
$$\forall t_1 > 0\ \ \exists\ \left\{\boldsymbol{u}(\tau), \left\|\boldsymbol{u}(\tau)\right\| < M : \int_{t_0}^{t_1} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau = \boldsymbol{x}(t_1)\right\}$$

- The set of all states for which the definition holds is given X_R (subspace of $\mathcal{R}^n$); if $X_R = R^n$ the system is said completely reachable.

- The set of non reachable states X_{NR} is the complement to X_R of the subspace $\mathcal{R}^n$ We have then:

$$X_R \cup X_{NR} = \Re^n$$





Reachability, Controllability

- **Definition – Controllable**: A system is controllable from time t_0, if it is possible to find a **limited** input $\boldsymbol{u}(t)$ such that we can bring the state from some initial condition $\boldsymbol{x}(t_0) \neq 0$ to the final state $\boldsymbol{x}(t_1) = 0$ in a **limited** time interval $[t_0,\ t_1]$.

  - Given the state vector:

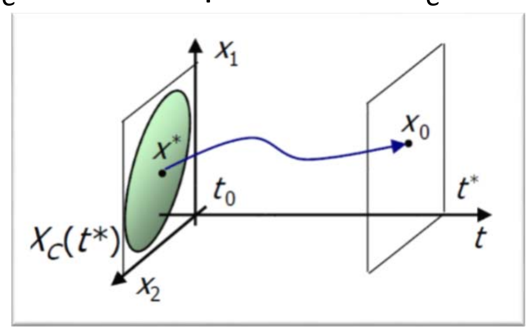
$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau; \boldsymbol{x}(t) \in \Re^n$$

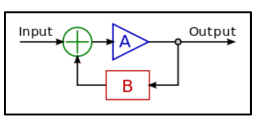
$$\forall t_1 > 0 \ \ \exists \left\{ \boldsymbol{u}(\tau), \left\| \boldsymbol{u}(\tau) \right\| < M : e^{At}\boldsymbol{x}_0 + \int_{t_0}^{t_1} e^{A(t-\tau)}B\boldsymbol{u}(\tau)d\tau = 0 \right\}$$

- The set of all the states for which the definition holds is X_C (subspace of R^n); if $X_C = R^n$ the system is said to be completely controllable.

- The set of all non controllable states X_{NC} is the complement to X_C of the subspace R^n thus:

$$X_C \cup X_{NC} = \Re^n$$





Reachability, Controllability

- **Theorem**: for LTI continuous systems we have: $X_C = X_R$. For LTI discrete systems we have : $X_C \leq X_R$
  - Proof: see, Proof of theorem

- Consider a completely controllable system $X_C = R^n$. From the definition, setting $t_0 = 0$:

$$\forall t_1 > 0 \;\; \exists \; \boldsymbol{u}(\tau) : e^{At_1}\boldsymbol{x}_0 + \int_0^{t_1} e^{A(t-\tau)} B\boldsymbol{u}(\tau)d\tau = 0$$

$$\forall t_1 > 0, \;\; \forall \boldsymbol{x}_0 \in \Re^n \;\; \exists \; \boldsymbol{u}(\tau) : \int_0^{t_1} e^{A(t_1-\tau)} B\boldsymbol{u}(\tau)d\tau = -e^{At_1}\boldsymbol{x}_0$$

- But Rank $\left(e^{At_1}\right) = n$, therefore the RHS gives a vector in R^n, for any initial condition

$$\boldsymbol{x} = -e^{At_1}\boldsymbol{x}_0 \;\; \in \Re^n \;\; \forall \; \boldsymbol{x}_0 \in \Re^n$$

Since this hold for any vector in R^n, every state is reachable, thus:

$$X_C \in \Re^n \rightarrow X_R \in \Re^n \rightarrow X_C = X_R \qquad \blacksquare$$

Reachability, Controllability

Given the system:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u, \qquad \boldsymbol{x}(0) = 0$$

Is completely controllable if

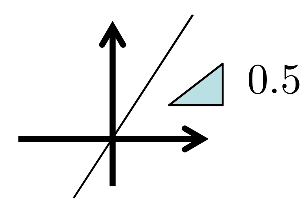
$$X_C \in \Re^2$$

$$\begin{cases} X_1(s) = \dfrac{2}{s+1} U(s) \\ X_2(s) = \dfrac{1}{s+1} U(s) \end{cases}$$

The system is **NOT controllable**, since for any $U(s)$, we have $X_1(s) = 2X_2(s)$ therefore $x_1(t) = 2x_2(t)$ and $X_C \in \Re^1$

- For any $u(t)$, we can reach all the states for which:

$$\boldsymbol{x} \in \Re^2 : \boldsymbol{x} = \begin{pmatrix} 2a \\ a \end{pmatrix}$$



- Any other state in R^2 can not be reached:

$$\boldsymbol{x} \in \Re^2 : \boldsymbol{x} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad a \neq b$$

Reachability, Controllability

Given the system:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad \boldsymbol{x}(0) = 0$$

Is completely controllable if

$$X_C \in \Re^2$$

$$\begin{cases} X_1(s) = \dfrac{1}{s+1} X_2(s) = \dfrac{1}{(s+1)^2} U(s) \\ X_2(s) = \dfrac{1}{s+1} U(s) \end{cases}$$

The system is **completely controllable**, since for $U(s)$, $X_1(s)$ is never a linear combination of $X_2(s)$ so, $x_1(t)$ and $x_2(t)$ are linearly independent.

- By proper choice of $u(t)$, we can always reach a state $\boldsymbol{x}(t)$ such that:

$$\boldsymbol{x} \in \Re^2 : \boldsymbol{x} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad a \neq b$$

Reachability, Controllability

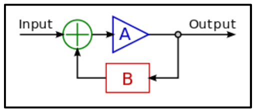
- Influence of controllability on stability

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{t} x_{10} \\ e^{-3t} x_{20} + \int_0^t e^{-3(t-\tau)} u(\tau)\, d\tau \end{pmatrix}$$

- The system is unstable in $x_1(t)$ and it can not be stabilized since the state $x_2(t)$ is not controllable.

- **Definition**: A system is siad to be **stabilizable** if all the unstable states are controllable.
  - If a system is controllable, there exists an input that makes the system evolve to the origin with a desired behavior.
  - If the system is stabilizable this can be achieved only partially (all states tend to the origin but only the controllable states can be manipulated).

- **Is there a general procedure that allows us to test controllability?**



$$\boldsymbol{x}(t_1) \in X_C \equiv \Re^n$$

Reachability, Controllability

❑ **Definition – Controllability Matrix :** Given a LTI system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{x} \in \Re^n, \quad \boldsymbol{u} \in \Re^m$$

We define controllability matrix $\mathcal{B}$, the $n\mathrm{x}nm$ matrix given by:

$$\mathcal{B} = (B, AB, A^2B, \ldots, A^{n-1}B)$$

❑ **Theorem (see notes):** A LTI continuous system is (completely) controllable **if and only if** its controllability matrix has full rank:

$$rank(\mathcal{B}) = rank\Big((B, AB, A^2B, \ldots, A^{n-1}B)\Big) = n$$

❑ **Trace of proof (only NC)**

- Without loss of generality, let us assume zero initial conditions

$$\boldsymbol{x}(t_1) = e^{At_1}\boldsymbol{x}(0) + \int_0^{t_1} e^{A(t_1-\tau)}B\boldsymbol{u}(\tau)d\tau = \int_0^{t_1} e^{A(t_1-\tau)}B\boldsymbol{u}(\tau)d\tau$$

Reachability, Controllability

- Necessary condition requires to prove that or any finite $u(t)$:

$$\boldsymbol{x}(t_1) \in \Re^n \Rightarrow rank(\mathcal{B}) = n$$

- From Cayley-Hamilton Theorem

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

$$\boldsymbol{x}(t_1) = \int_0^{t_1} \sum_{i=0}^{n-1} \alpha_i(t_1 - \tau) A^i B \boldsymbol{u}(\tau) d\tau \; = \sum_{i=0}^{n-1} A^i B \int_0^{t_1} \alpha_i(t_1 - \tau) \boldsymbol{u}(\tau) d\tau = \sum_{i=0}^{n-1} A^i B \beta_i(i, \boldsymbol{u}, t_1)$$

$$\beta_i(i, \boldsymbol{u}, t_1) = \int_0^{t_1} \alpha_i(t_1 - \tau) \boldsymbol{u}(\tau) d\tau$$

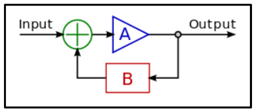
$$\left[B, AB, A^2B, \ldots, A^{n-1}B \right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = \boldsymbol{x}(t_1)$$

The solution is a linearly independent vector.,

$$\boldsymbol{x}(t_1) \in \Re^n$$

This means that the controllability matrix has full rank

$$Rank(\mathcal{B}) = n$$



Reachability, Controllability

❑ **Teorema:** Controllability is invariant with respect to a change in basis

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

- Consider a similarity transformation with a nonsingular matrix T

$$\boldsymbol{x} = T\ \boldsymbol{z} \Rightarrow \boldsymbol{z} = T^{-1}\boldsymbol{x} \quad \longrightarrow \quad \dot{\boldsymbol{z}} = T^{-1}AT\ \boldsymbol{z} + T^{-1}B\boldsymbol{u}$$

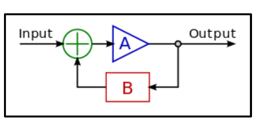
- Controllability property is maintained since both controllability matrices have the same rank.

$$\widetilde{\mathcal{B}} = \left[T^{-1}B, T^{-1}ATT^{-1}B, (T^{-1}AT)^2 T^{-1}B, \ldots, (T^{-1}AT)^{n-1}T^{-1}B\right]$$

$$(T^{-1}AT)^r = (T^{-1}AT)_1 \cdot (T^{-1}AT)_2 \cdot (T^{-1}AT)_3 \cdot \ldots \cdot (T^{-1}AT)_r = T^{-1}A^rT$$

$$\widetilde{\mathcal{B}} = \left[T^{-1}B, T^{-1}ATT^{-1}B, T^{-1}A^2B, \ldots, T^{-1}A^{n-1}B\right] = T^{-1}\mathcal{B}$$

$$\boxed{Rank(\widetilde{\mathcal{B}}) = \min\left[Rank(T^{-1}), Rank(\mathcal{B})\right] = Rank(\mathcal{B})}$$



Reachability, Controllability

- **Theorem:** A system is not controllable is at least one variable is not directly or indirectly influenced by the input.

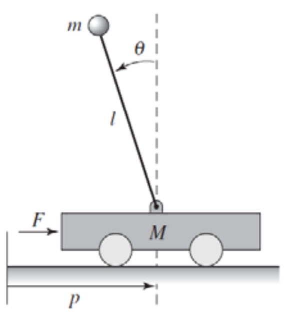
  - Consider the system: $$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \boldsymbol{u}; \boldsymbol{x} \in \Re^n, \boldsymbol{x}_2 \in \Re^p$$

  - The first n - p variabili are directly influenced by the input, whereas the others follow a free evolution giveen by:

$$\dot{\boldsymbol{x}}_2 = A_{22}\boldsymbol{x}_2 \Rightarrow \boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0$$

They therefore constitute an uncontrollable subsystem.

- **Example: Pendulum - Cart**



A linear approximation about the vertical is:

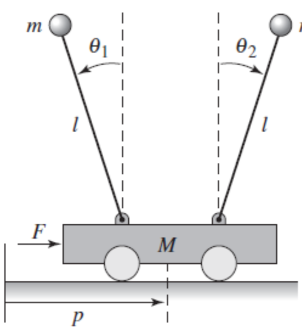
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2l^2g/\mu & 0 & 0 \\ 0 & M_t mgl/\mu & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ J_t/\mu \\ lm/\mu \end{bmatrix}$$

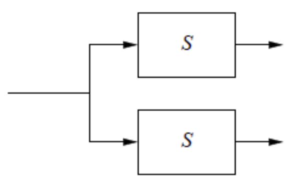
Reachability, Controllability

The controllability Matrix is:

$$\mathcal{B} = \begin{bmatrix} 0 & J_t/\mu & 0 & gl^3m^3/\mu^2 \\ 0 & lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 \\ J_t/\mu & 0 & gl^3m^3/\mu^2 & 0 \\ lm/\mu & 0 & gl^2m^2(m+M)/\mu^2 & 0 \end{bmatrix}$$

Rank [$\mathcal{B}$] = 4





Same dynamics for the two pendulums

$$\boldsymbol{x} = \begin{bmatrix} p & \theta_1 & \theta_2 & \dot{p} & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T$$

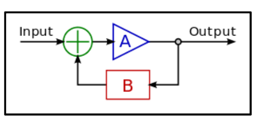
$$\dot{\boldsymbol{x}} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & \dfrac{m^2l^2g}{\mu} & \dfrac{m^2l^2g}{\mu} & 0 & 0 & 0 \\ 0 & \dfrac{M_t mgl}{\mu} & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{M_t mgl}{\mu} & 0 & 0 & 0 \end{array}\right] \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{J_t}{\mu} \\ \dfrac{lm}{\mu} \\ \dfrac{lm}{\mu} \end{bmatrix} F$$

Reachability, Controllability

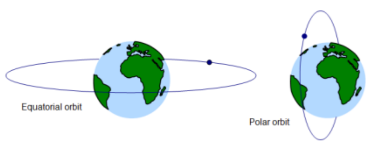
$$
B = \begin{bmatrix}
0 & J_t / \mu & 0 & \dfrac{gl^3m^3}{\mu^2} & 0 & a_{42}B_{42} \\
0 & lm / \mu & 0 & \dfrac{gl^2m^2(m+M)}{\mu^2} & 0 & a_{52}B_{42} \\
0 & lm / \mu & 0 & \dfrac{gl^2m^2(m+M)}{\mu^2} & 0 & a_{63}B_{43} = a_{52}B_{42} \\
J_t / \mu & 0 & \dfrac{gl^3m^3}{\mu^2} & 0 & a_{42}B_{42} & 0 \\
lm / \mu & 0 & \dfrac{gl^2m^2(m+M)}{\mu^2} & 0 & a_{52}B_{42} & 0 \\
lm / \mu & 0 & \dfrac{gl^2m^2(m+M)}{\mu^2} & 0 & a_{63}B_{43} = a_{52}B_{42} & 0
\end{bmatrix}
$$

Rank($\mathcal{B}$) = 4

Uncontrollable system



Reachability, Controllability



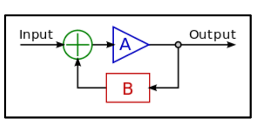
$$
\dot{\bar{x}} = \begin{bmatrix} \dot r \\ \ddot r \\ \dot\theta \\ \ddot\theta \\ \hdashline \dot\phi \\ \ddot\phi \end{bmatrix} = \left[\begin{array}{cccc:cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega_0/r_0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_0^2 & 0 \end{array}\right] \begin{bmatrix} r \\ \dot r \\ \theta \\ \dot\theta \\ \phi \\ \dot\phi \end{bmatrix} + \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/mr_0 & 0 \\ \hdashline 0 & 0 & 0 \\ 0 & 0 & 1/mr_0 \end{array}\right] \begin{bmatrix} u_r \\ u_\theta \\ \hdashline u_\phi \end{bmatrix}
$$

- **Controllable equatorial motion**

$$
\mathcal{B} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \left[\begin{array}{cc:cc:cc:cc} 0 & 0 & 1/m & 0 & 0 & \dfrac{2\omega_0}{m} & .. & .. \\ 1/m & 0 & 0 & \dfrac{2\omega_0}{m} & \dfrac{3\omega_0^2}{m} - \dfrac{2\omega_0^2}{mr_0} & 0 & .. & .. \\ 0 & 0 & 0 & 1/mr_0 & \dfrac{-2\omega_0}{mr_0^2} & 0 & .. & .. \\ 0 & 1/mr_0 & \dfrac{-2\omega_0}{mr_0^2} & 0 & 0 & \dfrac{-2\omega_0}{mr_0^2} & .. & .. \end{array}\right]
$$

- **Controllable off equatorial motion**

$$
\mathcal{B} = \begin{bmatrix} B & AB \end{bmatrix} = \left[\begin{array}{c:c} 0 & 1/mr_0 \\ 1/mr_0 & 0 \end{array}\right]
$$



Reachability, Controllability

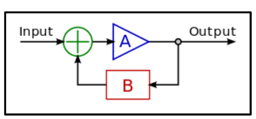
- ❑ Numerical example

$$Unità:\begin{cases} m & 10^3\,Kg & =1 \\ \omega_0 & rad/day & =2\pi \\ r_0 & 10^4\,Km & =3.6 \end{cases}$$

$$\dot{\boldsymbol{x}}(t)=\left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 118.4 & 0 & 0 & 22.68 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -.3537 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -39.48 \end{array}\right]\boldsymbol{x}(t)+\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & .28 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & .28 \end{array}\right]\boldsymbol{u}(t)$$

```
a =

         0    1.0000         0         0         0         0
  118.4000         0         0   22.6200         0         0
         0         0         0    1.0000         0         0
         0   -0.3537         0         0         0         0
         0         0         0         0         0    1.0000
         0         0         0         0         0  -39.4800

>> b

b =

         0         0         0
    1.0000         0         0
         0         0         0
         0    0.2800         0
         0         0         0
         0         0    0.2800

>> rank(ctrb(a,b))

ans =

     6
```

- ❑ The system is controllable
- ❑ **What is the time evolution ?**



Reachability, Controllability

- **Problem:** Consider a point mass subjected to second Newton's law:

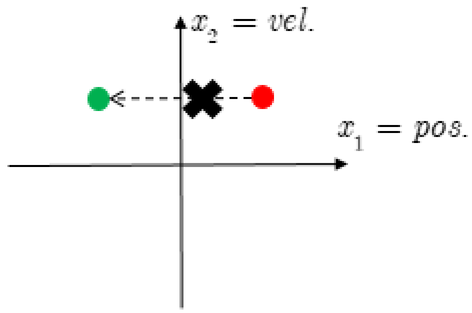
$$\begin{cases} x_1 = pos. \\ x_2 = vel. \end{cases} \qquad \dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \boldsymbol{x}(s) = \begin{bmatrix} \dfrac{1}{s^2} \\ \dfrac{1}{s} \end{bmatrix} u(s)$$

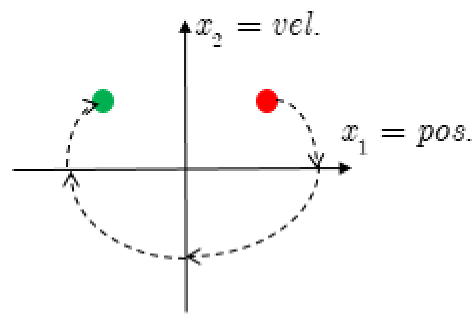
$$Rank\left\{\left[B, AB\right]\right\} = Rank\left\{\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right\} = 2 \qquad e^{At} = e^{Jt} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

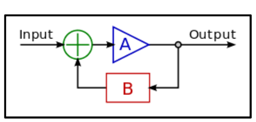
$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_0 + \int_0^t \begin{bmatrix} t-\tau \\ 1 \end{bmatrix} u(\tau) d\tau$$

- Find a finite input $u(t)$ that brings the state vector from $\boldsymbol{x}(0) = \boldsymbol{x}_0$ to $\boldsymbol{x}(t_f) = \boldsymbol{x}_f$.

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_f = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$







Reachability, Controllability

- **Additional Note:** Consider a linear time varying system

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t)$$

- At some time t_1, the solution is:

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

<u>Theorem</u>: The time-varying system given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is completely controllable iff the matrix $G(t_0, t_1)$ **defined below is** ***positive definite*** **for every** t_0 **and every** $t_1 > t_0$.

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau)d\tau$$

Reachability, Controllability

This matrix is called the *Controllability Grammian*. Its being positive definite also implies that the columns of $\Phi(t_0, \tau)B(\tau)$ are *linearly independent* time functions. It also happens to be the solution to the Lyapunov equation: $AG_c + G_c A^* = -BB^*$

- Recall for a LTI system:

$$G_C = \int_{t_0}^{t} e^{A\tau} B B^T e^{A^T \tau} d\tau \longrightarrow AG_c + G_c A^* = -BB^*$$

- If the system matrix A has all eigenvalues with strictly negative real part, the solution of the Lyapunov equation is unique, and the controllability Grammian is positive definite.

Observability

- This property gives the capability of determining the initial state (Observability) or the current state(Reconstructability) of the system, knowing the output.

  - The two definitions coincide for continuous systems and can be seen as **DUAL** of controllability/reachability

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ \boldsymbol{y} = C\boldsymbol{x} \end{cases}, \quad \boldsymbol{x}_0, \quad \boldsymbol{x} \in \Re^n, \boldsymbol{y} \in \Re^p$$

- **Definition**: A LTI system is **NOT observable** if:

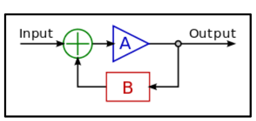
$$\boldsymbol{y}(t) = Ce^{At}\boldsymbol{x}_0 = 0, \qquad \forall t \geq 0, \quad \forall \boldsymbol{x}_0 \neq 0 \in \Re^n$$

- Given an LTI system, identify the Not observable subspace as:

$$X_{NO} = \left\{ \boldsymbol{x}(t) \in \Re^n : \boldsymbol{y}(t) = Ce^{At}\boldsymbol{x}_0 = 0, \forall t \geq 0 \right\}$$

- The system is (completely) observable, if and only $X_{NO} = 0$. Define the observability subspace as X_O, yields:

$$X_{NO} \cup X_O = \Re^n$$



Observability

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} \qquad y(\mathrm{t}) = x_1(t)$$

- From the output we have the first state directly and we can compute the second state, thus the system is observable.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} y(t) \\ 2\dot{y}(t) - 4y(t) \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{x}$$

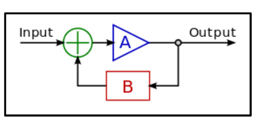
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$$

- In this case, the first variable can not be obtained from the output, and the system is **not observable**.

$$\forall \mathrm{t} \geq 0, \forall \boldsymbol{x} \in \Re^n \Rightarrow Ker(Ce^{At}) \in \Re^n$$

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \qquad \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} e^{-2t} & -t \\ 0 & e^{-2t} \end{bmatrix} \begin{pmatrix} x_1 \\ y \end{pmatrix}$$

$$y(\mathrm{t}) = x_2(t)$$



Observability

- Observability is also related to the capability of stabilizing a system.

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{t} x_{10} \\ e^{-3t} x_{20} \end{pmatrix}, y(t) = x_2(t) = e^{-3t} x_{20}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$$

- The first state is not observable and also unstable!

- **Definition – Observability Matrix**: Given a LTI system, we define the observability matrix $\mathcal{C}$ as:

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C^T & A^T C^T & \left(A^2\right)^T CT & \dots & \left(A^{n-1}\right)^T C^{T} \end{bmatrix}$$

Observability

- **Theorem:** A LTI system is (completely) observable if and only if the observability matrix has maximum rank:

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} \\ \boldsymbol{y} = C\boldsymbol{x} \end{cases}, \quad \boldsymbol{x}_0, \quad \boldsymbol{x} \in \Re^n, \boldsymbol{y} \in \Re^p \quad \Rightarrow \quad Rank(C) = Rank\left((C, CA, CA^2, \ldots, CA^{n-1})^T\right) = n$$

- **Proof (trace)**

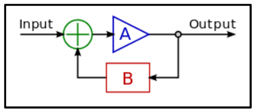
- Let us assume that there are r unobservable states:

$$\boldsymbol{y}(t) = Ce^{At}\boldsymbol{x}_{NO} = 0\,,\ \boldsymbol{x}_{NO} \in \Re^r$$

- From Cayley – Hamilton theorem:

$$e^{At} = \sum_{i=0}^{n-1} A^i \alpha_i(t) \quad \longrightarrow \quad \boldsymbol{y}(t) = C\sum_{i=0}^{n-1} A^i \alpha_i(t)\boldsymbol{x}_{NO} = 0$$

- Since $\alpha_i(t)$ are *n* linearly independent functions, we have



Observability

$$\boldsymbol{y}(t) = C\alpha_0(t)\boldsymbol{x}_{NO} + CA\alpha_1(t)\boldsymbol{x}_{NO} + CA^2\alpha_2(t)\boldsymbol{x}_{NO} + \ldots + CA^{n-1}\alpha_{n-1}(t)\boldsymbol{x}_{NO} = 0$$

$$CA^i\boldsymbol{x}_{NO} = 0 \qquad \forall \ i$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \boldsymbol{x}_{NO} = \boldsymbol{C}\boldsymbol{x}_{NO} = 0$$

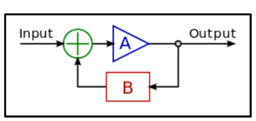
- Since the vector $\boldsymbol{x}_{NO}$ is not zero, the observability matrix must have rank less than the maximum

- If the observability matrix rank is maximum (n), the only solution of the equation is the zero vector, therefore the set X_{NO} is an empty subspace and the syste is observable.

❑ **Theorem:** Observability is invariant with respect to a change of basis.

❑ **Theorem:** A system is unobservable if at least one does not influence directly or indirectly by theoutput.

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \boldsymbol{x} \end{cases}, \quad \boldsymbol{x} \in \Re^n, \quad A_{11} = (m x n)$$



Realizations

- **Theorem (Controllable Canonical Form) :** Given an uncontrollable system, define a non singular transformation matrix T = (T_1, T_2) with the columns of T_1 a basis of the controllable subspace of dimension m and the columns of T_2 a basis of the uncontrollable subspace, then:

$$\boldsymbol{x} = T\boldsymbol{z} \Rightarrow \boldsymbol{z} = T^{-1}\boldsymbol{x}$$

$$\begin{cases} \dot{\boldsymbol{z}} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]\boldsymbol{z} + \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right]\boldsymbol{u} \\ \boldsymbol{y} = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right]\boldsymbol{z} \end{cases} \qquad \boldsymbol{z} = \left[\begin{array}{c} \boldsymbol{z}_{CO} \\ \hline \boldsymbol{z}_{NC} \end{array}\right]$$

The subsystem (A_{11}, B_1, C_1) of size m is controllable, while the (A_{22}, 0, C_2) of size $p = n\text{-}m$, is uncontrollable.

- **Theorem (Observable Canonical Form) :** Given an unobservable system, define a non singular transformation matrix T = (T_1, T_2) with the columns of T_1 a basis of the observable subspace of dimension m and the columns of T_2 a basis of the unobservable subspace, then:

$$\boldsymbol{x} = T\boldsymbol{z} \Rightarrow \boldsymbol{z} = T^{-1}\boldsymbol{x}$$

$$\begin{cases} \dot{\boldsymbol{z}} = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right]\boldsymbol{z} + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right]\boldsymbol{u} \\ \boldsymbol{y} = \left[\begin{array}{c|c} C_1 & 0 \end{array}\right]\boldsymbol{z} \end{cases} \qquad \boldsymbol{z} = \left[\begin{array}{c|c} \boldsymbol{z}_O & \boldsymbol{z}_{NO} \end{array}\right]^T$$

Realizations

- **Kalman Decomposition**: Kalman decomposition is a similarity transformation, which separates the state variables in theri 4 structural subspaces.

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x} \end{cases} \quad \boldsymbol{x}_0, \quad \boldsymbol{x} \in \Re^n, \boldsymbol{u} \in \Re^m, \boldsymbol{u} \in \Re^p$$

- Define a similarity matrix T composed by a basis each of the 4 structural subspaces. Then by similarity transformation, we obtain:

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \hat{A}\hat{\boldsymbol{x}}(t) + \hat{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \hat{C}\hat{\boldsymbol{x}}(t) \end{cases}$$

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \\ \hat{\boldsymbol{x}}_3 \\ \hat{\boldsymbol{x}}_4 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ 0 & \hat{A}_{22} & 0 & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & \hat{A}_{34} \\ 0 & 0 & 0 & \hat{A}_{44} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & \hat{C}_2 & 0 & \hat{C}_4 \end{bmatrix}$$

Realizations

$$\dot{\bar{\hat{x}}}_3(t) = \hat{A}_{33}\,\bar{\hat{x}}_3(t) + \hat{A}_{34}\,\bar{\hat{x}}_4(t)$$

- Uncontrollable and Unobservable subsystem

$$\begin{cases} \begin{bmatrix} \dot{\hat{\boldsymbol{x}}}_1(t) \\ \dot{\hat{\boldsymbol{x}}}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \hat{C}_2 \hat{\boldsymbol{x}}_2(t) \end{cases}$$

- Controllable subsystem

$$\begin{cases} \begin{bmatrix} \dot{\hat{\boldsymbol{x}}}_2(t) \\ \dot{\hat{\boldsymbol{x}}}_4(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{22} & \hat{A}_{24} \\ 0 & \hat{A}_{44} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}}_2(t) \\ \hat{\boldsymbol{x}}_4(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_2 \\ 0 \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \hat{C}_2 \hat{\boldsymbol{x}}_2(t) + \hat{C}_4 \hat{\boldsymbol{x}}_4(t) \end{cases}$$

- Observable subsystem

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}_2(t) = \hat{A}_{22} \hat{\boldsymbol{x}}_2(t) + \hat{B}_2 \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \hat{C}_2 \hat{\boldsymbol{x}}_2(t) \end{cases}$$

- **Controllable and Observable subsystem**

Relation with Frequency Response

$$\begin{cases}\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} = C\boldsymbol{x}\end{cases} \quad \boldsymbol{x}_0, \quad \boldsymbol{x} \in \Re^n, \boldsymbol{u} \in \Re^m, \boldsymbol{u} \in \Re^p$$

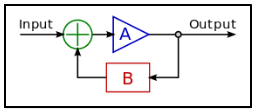
- Compute the Transfer Funcyion Matrix **(TFM)**: $G(s) = C\left[sI - A\right]^{-1} B + D$

❑ TFM is invariant with respect to a similarity transformation

$$\boldsymbol{z} = T^{-1}\boldsymbol{x} \qquad \begin{cases}\dot{\boldsymbol{z}} = T^{-1}AT\boldsymbol{z} + T^{-1}B\boldsymbol{u} = A'\boldsymbol{x} + B'\boldsymbol{u} \\ \boldsymbol{y} = CT\boldsymbol{z} = C'\boldsymbol{z}\end{cases} \Rightarrow G'(s)$$

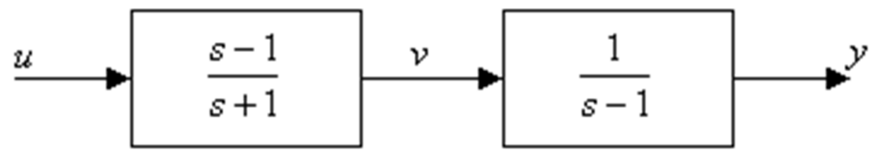
$$G'(s) = C'(sI - A')^{-1} + B' = C'\left[T^{-1}(sI - A)T\right]^{-1} B' =$$
$$= CTT^{-1}(sI - A)^{-1}TT^{-1}B = C(sI - A)^{-1}B = G(s)$$

- **Observation**: A TFM contains ONLY the subsystem which is simultaneously controllable and observable
- **Observation**: A system is completely controllable and observable if and only if the poles are ALL the eigenvalues of A.

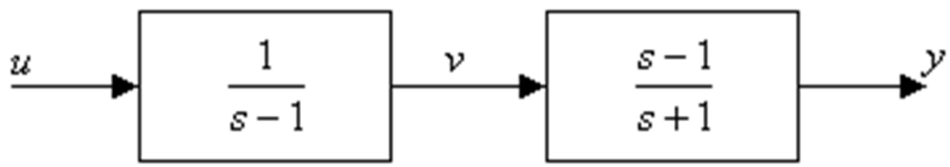


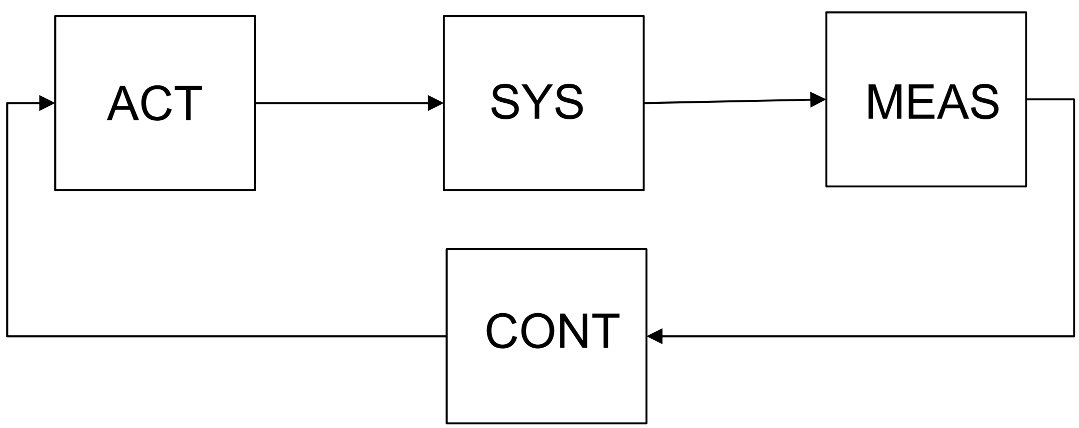
Relation with Frequency Response

- **Example: Zero – Pole cancellation (uncontrollable – observable)**



- **Example: Pole – Zero cancellation (controllable – unobservable)**

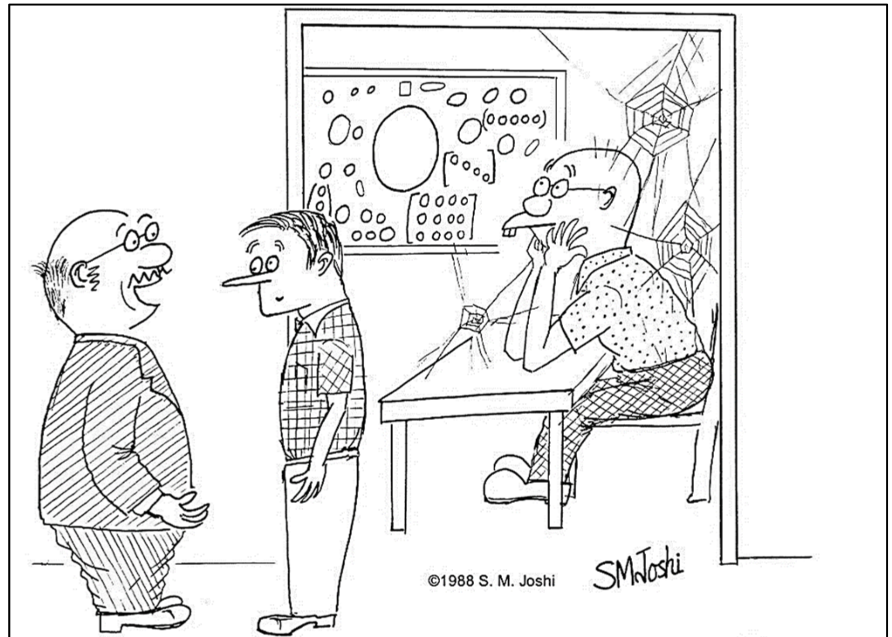




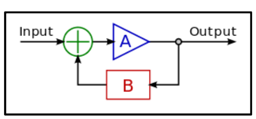
Relation with Frequency Response

❑ Summary Comments

- Controllability is associated with actuators
- Observability is associated with sensors
- Stability is associated with system's eigenvalues (and/or poles depending on controllability and/or observability)
- Controllability and observability are associated with zeros and loop gain
- Zeros define the controller limits of performance



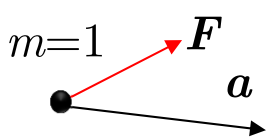
"...and this is the office of our resident expert on zeros!"



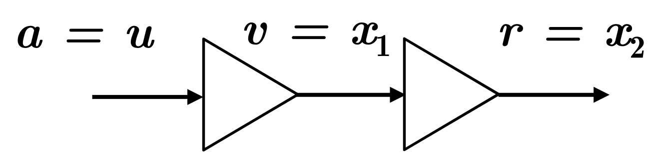
Relation with Frequency Response

Example:

- Point Mass System subjected to external force (unit mass for simplicity)



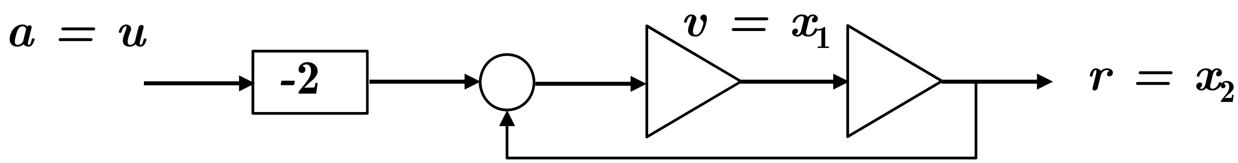
$$a = F$$



$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}, G(s) = \frac{1}{s^2}$$

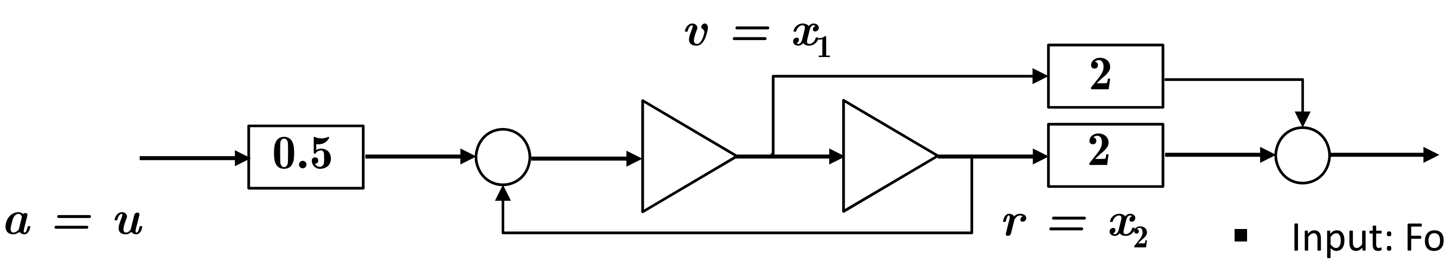
- Input: Force/Acceleration
- Output: Position

Relation with Frequency Response



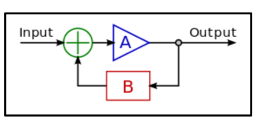
- Input: Force/Acceleration + Position
- Output: Position

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} \end{cases}, G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1 - 2s}{s^2}$$



- Input: Force/Acceleration + Position
- Output: Position + Velocity

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} .5 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 2 & 2 \end{bmatrix} \boldsymbol{x} \end{cases}, G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 2 & 2 \end{bmatrix} \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} .5 \\ 1 \end{bmatrix} = \frac{2 + 3s}{s^2}$$



Relation with Frequency Response

$$\begin{cases} \dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x} \end{cases}, G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

- Exercise: Draw the Block Diagram

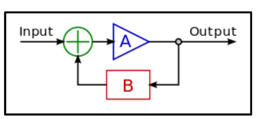
Note:

- Same system, 4 different Transfer Functions depending on the number and location of sensors and actuators

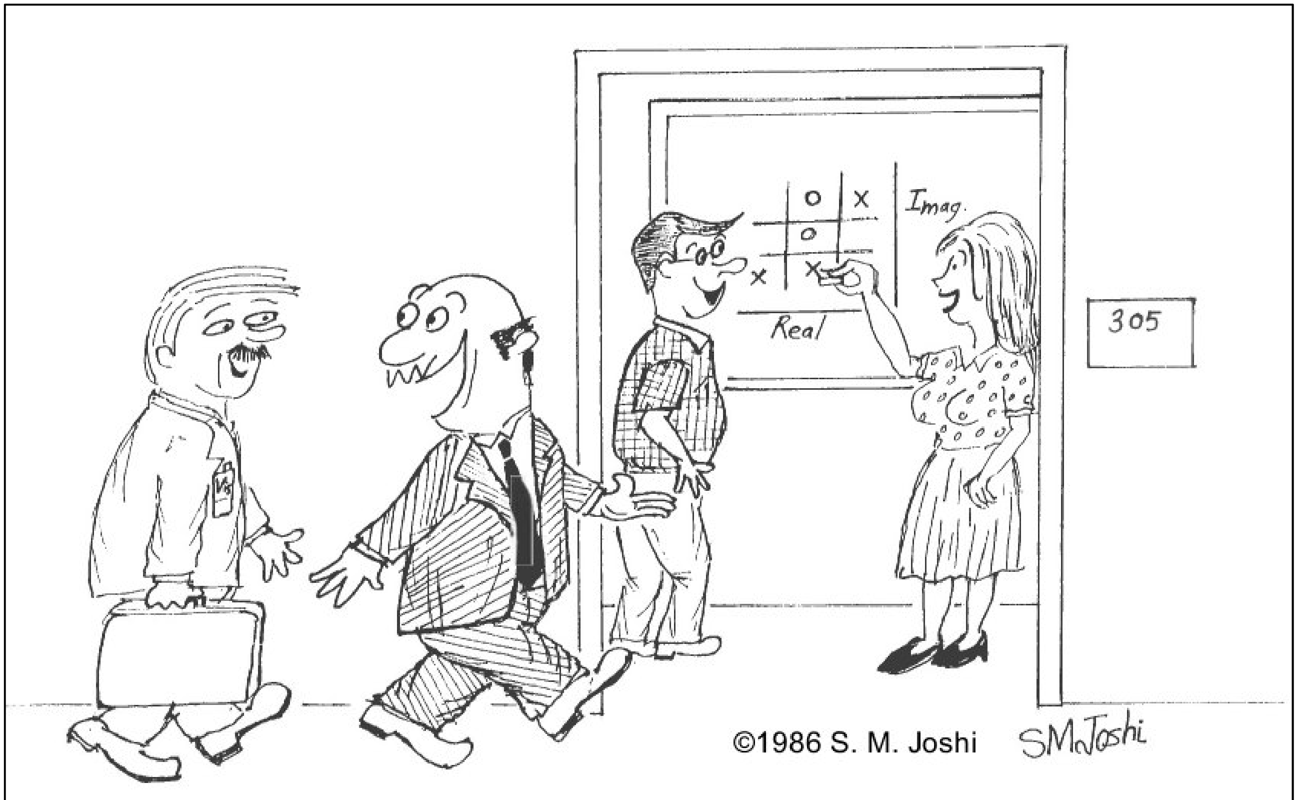
$$\frac{1}{s^2}, \frac{-2(s - \frac{1}{2})}{s^2}, \frac{3(s + \frac{2}{3})}{s^2}, 0$$

Comments:

- What are the stability properties?
- What is the simplest controller for asymptotic stability, and the ‘best’ transfer function to work with?



Relation with Frequency Response



"...and our control engineers apply highly advanced, state-of-the-art techniques for selecting pole-zero locations."

Extras

Chapter 3

Controllability and Observability

In this chapter, we study the controllability and observability concepts. Controllability is concerned with whether one can design control input to steer the state to arbitrarily values. Observability is concerned with whether without knowing the initial state, one can determine the state of a system given the input and the output.

3.1 Some linear mapping concepts

The study of controllability and observability for linear systems essentially boils down to studying two linear maps: the reachability map L_r which maps, for zero initial state, the control input $u(\cdot)$ to the final state $x(t_1)$; and the observability map, L_o which maps, for zero input, from the initial state $x(t_0)$ to the output trajectory $y(\cdot)$. The *range* and the *null* spaces of these maps respectively are critical for their studies.

Let $L : \mathcal{X}(= \Re^p) \to \mathcal{Y}(= \Re^q)$ be a linear map: i.e.

$$L(\alpha x_1 + \beta x_2) = \alpha(Lx_1) + \beta \cdot (Lx_2)$$

$\mathcal{R}(L) \subset \mathcal{Y}$ denotes the range of L, given by:

$$\mathcal{R}(L) = \{y \in \mathcal{Y} | y = Lx \text{ for some } x \in \mathcal{X}\}$$

$\mathcal{N}(L) \subset \mathcal{X}$ denotes the null space of L, given by:

$$\mathcal{N}(L) = \{x \in \mathcal{X} | Lx = \mathbf{0}0\}$$

Note: Both $\mathcal{R}(L)$ and $\mathcal{N}(L)$ are linear subspaces, i.e. for scalar α, β,

$$y_1, y_2 \in \mathcal{R}(L) \Rightarrow (\alpha y_1 + \beta y_2) \in \mathcal{R}(L)$$
$$x_1, x_2 \in \mathcal{N}(L) \Rightarrow (\alpha x_1 + \beta x_2) \in \mathcal{R}(L)$$

- L is called onto or surjective if $\mathcal{R}(L) = \mathcal{Y}$ (everything is within range).
- L is called into or one-to-one (cf. many-to-one) if $\mathcal{N}(L) = \{0\}$. Thus,

$$Lx_1 = Lx_2 \Leftrightarrow x_1 = x_2$$

3.1.1 The Finite Rank Theorem

Proposition 3.1.1 *: Let $L \in \Re^{n\times m}$, so that $L : u \in \Re^m \mapsto y \in \Re^n$.*

1. $$\mathcal{R}(L) = \mathcal{R}(LL^T)$$

2. $$\mathcal{N}(L) = \mathcal{N}(L^T L)$$

3. *Any point $u \in \Re^n$ can be written as*
$$u = L^T y_1 + u_n \in \Re^n; \textit{where} \quad y_1 \in \Re^n, \quad Lu_n = 0.$$
In other words, u can always be written as a sum of $u_1 \in \mathcal{R}(L^T)$ and $u_n \in \mathcal{N}(L)$.

4. *Similarly, any point $u \in \Re^n$ can be written as*
$$y = Lu_1 + y_n \in \Re^n; \quad \textit{where} \;\; u_1 \in \Re^m, \quad L^T y_n = 0.$$
In other words, u can always be written as a sum of $y_1 \in \mathcal{R}(L)$ and $y_n \in \mathcal{N}(L^T)$.

Note:

- If $L \in \Re^{n\times m}$ and $m > n$ (short and fat), then $LL^T \in \Re^{n\times n}$ is both short and thin.
- To check whether the system is controllable we need to check if LL^T is full rank - e.g. check its determinant.
- If $m < n$ (tall and thin), then $L^T L \in \Re^{n\times n}$ is also both short and thin.
- Checking $\mathcal{N}(L^T L)$ will be useful when we talk about observerability.

3.2 Controllability

Consider a state determined dynamical system with a transition map $s(t_1, t_0, x_0, u(\cdot))$ such that
$$x(t_1) = s(t_1, t_0, x(t_0), u(\cdot)).$$
This can be continuous time or discrete time.

Definition 3.2.1 (Controllability) *: The dynamical system is controllable on $[t_0, t_1]$ if $\forall$ initial and final states x_0, x_1, $\exists u(\cdot)$ so that $s(t_1, t_0, x_0, u) = x_1$.*

It is said to controllable at t_0 if $\forall x_0, x_1$, $\exists t_1 \geq t_0$ and $u(\cdot) \in \mathcal{U}$ so that $s(t_1, t_0, x_0, u) = x_1$

A system that is **not** controllable probably means the followings:

- If state space system is realized from input-output data, then the realization is redundant. i.e. some states have no relationship to either the input or the output.
- If states are meaningful (physical) variables that need to be controlled, then the design of the actuators are deficient.
- The effect of control is limited. There is also a possibililty of instability

Remarks:

- The followings are equivalent:
  1. A system is controllable at $t = t_0$.
  2. The system can be transferred from any state at $t = t_0$ to any other state in finite time.
  3. For each $x_0 \in \Sigma$,
  $$s(\cdot, t_0, x_0, \cdot) : [t_0, \infty) \times u(\cdot) \mapsto s(t_1, t_0, x_0, u(\cdot))$$
  is surjective (onto).
- Controllability does **not** say that $x(t)$ remains at x_1. e.g. for
$$\dot{x} = Ax + Bu$$
to make $x(t) = x_1 \ \forall t \geq t_0$, it is necessary that $Ax_1 \in \texttt{Range}(B)$. This is generally not true.

Example - An Uncontrollable System

- An linear actuator pushing 2 masses on each end of the actuator in space.
$$m_1\ddot{x}_1 = u; \qquad m_2\ddot{x}_2 = -u$$
- State: $X = [x_1, \dot{x}_1, x_2, \dot{x}_2]^T$.
- Action and reaction are equal and opposite, and act on different bodies.
- Momentum is conserved:
$$m_1\dot{x}_1 + m_2\dot{x}_2 = \text{constant} = m_1\dot{x}_1(t_0) + m_2\dot{x}_2(t_0)$$
- E.g. if both masses are initially at rest: it is not possible to choose $u(\cdot)$ to make $\dot{x}_1 = 0$, $\dot{x}_2 = 1$.

3.2.1 Effects of feedback on controllability

Consider a system
$$\dot{x} = f(t, x(t), u(t)).$$
Suppose that the system is under **memoryless (static) state feedback**:
$$u(t) = v(t) - g(t, x(t))$$
with $v(t)$ being the new input.

- A system is controllable if and only if the system under memoryless feedback with $v(t)$ as the input is controllable.

Proof: Suppose that the original system is controllable. Given a pair of initial and final states, x_0 and x_1, suppose that the control $u(\cdot)$ transfers the state from x_0 at t_0 to x_1 at t_1. For the new system, which has also $x(t)$ as the state variables, we can use the control $v(t) = u(t) + g(t, x(t))$ to transfer the state from x_0 at t_0 to x_1 at t_1. Therefore, the new system is also controllable.

Suppose that the feedback system is controllable, and $v(\cdot)$ is the control that can achieve the desired state transfer. Then for the original system, we can use the control $u(t) = v(t) - g(t, x(t))$ to achieve the same state transfer. Thus, the original system is also controllable. $\diamond$

Consider now the system under **dynamic state feedback**:

$$\dot{z}(t) = \alpha(t, x(t), z(t))$$
$$u(t) = v(t) - g(t, x(t), z(t))$$

where $v(t)$ is the new input and $z(t)$ is the state of the controller.

- If a system under **dynamic feedback** is controllable then the original system is.
- The converse is not true, i.e. the original system is controllable does not necessarily imply that the system under dynamic state feedback is.

Proof: The same proof for the memoryless state feedback works for the first statement. For the second statement, we give an example of the controller:

$$\dot{z} = z; \qquad u(t) = v(t) - g(t, x(t), z(t)).$$

The state of the new system is (x, z) but no control can affect $z(t)$. Thus, the new system is not controllable. $\diamond$

These results show that feedback control cannot make a uncontrollable system become controllable, but can even make a controllable system, uncontrollable. To make a system controllable, one must redesign the actuators.

3.3 Controllability of Linear Systems

Continuous time system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x \in \Re^n$$
$$x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, t)B(t)u(t)dt.$$

The **reachability map** on $[t_0, t_1]$ is defined to be:

$$L_{r,[t_0,t_1]}(u(\cdot)) = \int_{t_0}^{t_1} \Phi(t_1, t)B(t)u(t)dt$$

Thus, it is *controllable on* $[t_0, t_1]$ if and only if $L_{r,[t_0,t_1]}(u(\cdot))$ is surjective (onto).

Notice that $L_{r,[t_0,t_1]}$ determines the set of states that can be reached from the origin at $t = t_1$.

The study of the range space of the linear map:

$$L_{r,[t_0,t_1]} : \{u(\cdot)\} \to \Re^n$$

is central to the study of controllability.

Proposition 3.3.1 *If a continuous time, possibly time varying, linear system is controllable on* $[t_0, t_1]$ *then it is controllable on* $[t_0, t_2]$ *where* $t_2 \geq t_1$.

Proof: To transfer from $x(t_0) = x_0$ to $x(t_2) = x_2$, define

$$x_1 = \Phi(t_1, t_2)x_2$$

Suppose that $u^*_{[t_0,t_1]}$ transfers $x(t_0) = x_0$ to $x(t_1) = x_1$. Choose

$$u(t) = \begin{cases} u^*(t) & t \in [t_0, t_1] \\ 0 & t \in (t_1, t_2] \end{cases}$$

$\diamond$. $\diamond$

This result does **not** hold for $t_0 < t_2 < t_1$. e.g. The system

$$\dot{x}(t) = B(t)u(t) \qquad B(t) = \begin{cases} 0 & t \in [t_0, t_2] \\ \mathbf{I}_n & t \in (t_2, t_1] \end{cases}.$$

is controllable on $[t_0, t_1]$ but $x(t_2) = x(t_0)$ for any control $u(\cdot)$.

3.3.1 Discrete time system

$$x(k+1) = A(k)x(k) + B(k)u(k), \qquad x \in \Re^n, u \in \Re^m$$

$$x(k_1) = \Phi(k_1, k_0)x(k_0) + \sum_{k=k_0}^{k_1-1} \Phi(k_1, k+1)B(k)u(k)$$

The transition function of the discrete time system is given by:

$$x(k_1) = s(k_0, k_1, x(k_0), u(\cdot)) = \Phi(k_1, k_0)x(k_0) + \sum_{k=k_0}^{k_1-1} \Phi(k_1, k+1)B(k)u(k) \tag{3.1}$$

$$\Phi(k_f, k_i) = \Pi_{k=k_i}^{k_f-1} A(k) \tag{3.2}$$

The reachability map of the discrete time system can be written as:

$$L_{r,[k_0,k_1]}(u(k_0), \ldots, u(k_1-1)) = \sum_{k=k_0}^{k_1-1} \Phi(k_1, k+1)B(k)u(k) = L_r(k_0, k_1)\mathbf{U}$$

where $L_r(k_0, k_1) \in \Re^{n \times (k_1-k_0)m}$, $\mathbf{U} = \begin{pmatrix} u(k_0) \\ \vdots \\ u(k_1-1) \end{pmatrix}$.

Thus, the system is controllable on $[k_0, k_1]$ if and only if $L_{r,[k_0,k_1]}$ is surjective, which is true if and only if $L_r(k_0, k_1)$ has rank n

Proposition 3.3.2 *Suppose that* $A(k)$ *is nonsingular for each* $k_1 \leq k < k_2$*, then the discrete time linear system is controllable on* $[k_0, k_1]$ *implies that it is controllable on* $[k_0, k_2]$ *where* $k_2 \geq k_1$.

Proof: : The proof is similar to the continuous case. However for a discrete time system, we need $A(k)$ nonsingular for $k_1 \leq k < k_2$ to ensure that $\Phi(k_2, k_1)$ is invertible. Q.E.D. $\diamond$

Example

Consider a unit point mass under control of a force:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

This is the same as $\ddot{x} = u$.

Suppose that $x(0) = \dot{x}(0) = 0$, we would like to translate the mass to $x(T) = 1$, $\dot{x}(T) = 0$.

The reachability map $L : u(\cdot) \to x(T)$ is

$$x(T) = \int_0^T \Phi(T,t)B(t)u(t)dt$$

Let use try to solve this problem using piecewise constant control:

$$u(t) = \begin{cases} u_0 & 0 \le t < T/10 \\ u_1 & T/10 \le t < 2T/10 \\ \vdots & \\ u_{10} & 9T/10 \le t \le T \end{cases}$$

and find $U = [u_1, u_2, \dots, u_{10}]^T$.

The reachability map becomes:

$$x(T) = \underbrace{\begin{pmatrix} L_1 & L_2 & \dots & L_{10} \end{pmatrix}}_{L_r} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{10} \end{pmatrix}$$

where

$$L_1 = \left[\int_0^{T/10} \Phi(T,t)B(t)dt\right] = \frac{T}{10}\begin{pmatrix} T \\ 1 \end{pmatrix},$$

$$L_2 = \left[\int_{T/10}^{2T/10} \Phi(T,t)B(t)dt\right] = \frac{T}{10}\begin{pmatrix} 0.9T \\ 1 \end{pmatrix},$$

$$\vdots$$

$$L_{10} = \frac{T}{10}\begin{pmatrix} 0.1T \\ 1 \end{pmatrix}$$

Whether the matrix denoted by $L_r \in \Re^{2\times 10}$ is surjective (onto) (i.e. $\mathcal{R}(L_r) = \Re^2$ determines whether we can achieve our task using piecewise constant control.

In our case, let $T = 10$ for convenience,

$$L_r = \begin{pmatrix} 10 & 9 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

and

$$L_r L_r^T = \begin{pmatrix} 385 & 55 \\ 55 & 10 \end{pmatrix}$$

which is non-singular (full rank). Thus, from the finite rank theorem $\mathcal{R}(L_r) = \mathcal{R}(L_r L_r^T)$, the particle transfer problem is solvable using piecewise constant control.

If we make the number of pieces in the piecewise constant control larger and larger, then the multiplication of L_r by L_r^T becomes an integral:

$$L_r L_r^T \to \int_{t_0}^{t_1} \Phi(t_1, t)B(t)B(t)^T \Phi(t_1, t)^T dt.$$

Therefore for the *continuous time* linear system is controllable over the interval $[t_0, t_1]$ if and only if the matrix,

$$\int_{t_0}^{t_1} \Phi(t_1, t)B(t)B(t)^T \Phi(t_1, t)^T dt \tag{3.3}$$

is full rank.

3.4 Reachability Grammian

The matrix $W_r = L_r L_r^T \in \Re^{n\times n}$ is called the *Reachability Grammian* for the interval that L_r is defined. For the continuous time system

$$\dot{x} = A(t)x + B(t)u$$

the Reachability Grammian on the time interval $[t_0, t_1]$ is defined to be:

$$W_{r,[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(t_1, t)B(t)B(t)^T \Phi(t_1, t)^T dt$$

For time invariant systems,

$$\begin{aligned} W_r([t_1, t_0]) = L_r L_r^* &= \int_{t_0}^{t_1} e^{A(t_1-t)} B B^T e^{A^T(t_1-t)} dt \\ &= \int_{t_1-t_0}^{0} e^{A\tau} B B^T e^{A^T\tau} d\tau. \end{aligned}$$

For the discrete time system,

$$x(k+1) = A(k)x(k) + B(k)u(k),$$

the Reachability Grammian on the interval $[k_0, k_1]$ is

$$\sum_{k=k_0}^{k_1-1} \Phi(k_1, k+1)B(k)B(k)^T \Phi(k_1, k+1)^T.$$

From the finite rank theorem, we know that the controllability of a system on $[t_0, t_1]$ (or on $[k_0, k_1]$) is equivalent to Reachability Grammian on the same time interval being full rank.

3.4.1 Reduction theorem: Controllability in terms of Controllability to Zero

Thus far, we have been deciding controllability by considering the reachability map L_r which concerns what states we are able to steer to from $x(t_0) = 0$. The **reduction theorem** allows us to formulate the question in terms of whether a state can be steered to $x(t_f) = 0$ in finite time.

Theorem 3.4.1 *The followings are equivalent for a linear time varying differential system:*

$$\dot{x} = A(t)x + B(t)u$$

1. *It is controllable on* $[t_0, t_1]$.
2. *It is controllable to* 0 *on* $[t_0, t_1]$*, i.e.* $\forall x(t_0) = x_0$*, there exists* $u(\tau), \tau \in [t_0, t_1)$ *such that final state is* $x(t_1) = 0$.
3. *It is reachable from* $x(t_0) = 0$*, i.e.* $\forall\ x(t_1) = x_f$*, there exists* $u(\tau), \tau \in [t_0, t_1)$ *such that for* $x(t_0) = 0$*, the final state is* $x(t_1) = x_f$.

Proof:

- Clearly, (1) $\Rightarrow$ (2) and (3).
- To prove (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1), use

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau.$$

and $\Phi(t_1, t_0)$ is invertible, to construct the appropriate $x(t_0)$ and $x(t_f)$.

Controllability map: $L_{c,[t_0,t_1]}$

The controllability (to zero) map on $[t_0, t_1]$ is the map between $u(\cdot)$ to the *initial state* x_0 such that $x(t_1) = 0$.

$$L_{c,[t_0,t_1]}(u(\cdot)) = -\int_{t_0}^{t_1} \Phi(t_0, t)B(t)u(t)dt$$

Proof:

$$\begin{aligned}
0 = x(t_1) &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
x_0 &= -\Phi(t_1, t_0)^{-1} \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
x_0 &= -\int_{t_0}^{t_1} \Phi(t_0, t_1)\Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\
x_0 &= -\int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau
\end{aligned}$$

Notice that

$$\mathcal{R}(L_{c,[t_0,t_1]}) = \Phi(t_1, t_0)\mathcal{R}(L_{r,[t_0,t_1]})$$

These two ranges are typically not the same for time varying systems. However, one is full rank if and only if the other is, since $\Phi(t_1, t_0)$ is invertible.

For time invariant continuous time systems, $\mathcal{R}(L_{c,[t_0,t_1]}) = \mathcal{R}(L_{r,[t_0,t_1]})$.

The above two statements are not true for time invariant discrete time systems. Can you find counter-examples?

Because of the reduction theorem, the linear continuous time system is controllable on the interval $[t_0, t_f]$ if $\mathcal{R}(L_{c,[t_0,t_1]}) = \Re^n$. From the finite rank theorem, $\mathcal{R}(L_{c,[t_0,t_1]}) = \mathcal{R}(L_{c,[t_0,t_1]}L^*_{c,[t_0,t_1]}) = \Re^n$. The controllability-to-zero grammian on the time interval $[t_0, t_1]$ is defined to be:

$$W_c = L_c L_c^* = \int_{t_0}^{t_1} \Phi(t_0,t)B(t)B(t)^T\Phi(t_0,t)^T dt$$

For time invariant system,

$$\begin{aligned} W_c = L_c L_c^* &= \int_{t_0}^{t_1} e^{A(t_0-t)}BB^T e^{A^T(t_0-t)}dt \\ &= -\int_{t_1-t_0}^{0} e^{-A\tau}BB^T e^{-A^T\tau}d\tau. \end{aligned}$$

Since controllability of a system on $[t_0, t_1]$ is equivalent to controllability to 0 which is determined by the controllability map being surjective, it is also equivalent to controllability grammian on the same time interval $[t_0, t_1]$ being full rank.

3.5 Minimum norm control

3.5.1 Formulation

The Controllability Grammian is related to the cost of control.

Given $x(t_0) = x_0$, find a control function or sequence $u(\cdot)$ so that $x(t_1) = x_1$. Let $x_d = x_1 - \Phi(t_1,t_0)x_0$. Then we must have

$$x_d = L_{r,[t_0,t_1]}(u)$$

where $L_{r,[t_0,t_1]} : \{u(\cdot)\} \to \Re$ is the reachability map which, for continuous time system is:

$$L_{r,[t_0,t_1]}(u) = \int_{t_0}^{t_1} \Phi(t_1,t)B(t)u(t)dt.$$

and for discrete time system:

$$L_{r,[k_0,k_1]}(u(\cdot)) = \sum_{k=k_0}^{k_1-1} \Phi(k_1,k+1)B(k)u(k) = \mathbf{C}(k_0,k_1)\mathbf{U}$$

where $\mathbf{C} \in \Re^{n\times(k_1-k_0)m}$, $\mathbf{U} = \begin{pmatrix} u(k_0) \\ \vdots \\ u(k_1-1) \end{pmatrix}$.

Let us focus on the discrete time system.

Since $x_d \in \texttt{Range}(L_{r,[k_0,k_1]})$ for solutions to exist, we assume $\texttt{Range}(L_{r,[k_0,k_1]}) = \Re^n$. This implies that $W_{r,[k_0,k_1]}$ is invertible.

Generally there are multiple solutions. To resolve the non-uniqueness, we find the solution so that

$$J(U) = \frac{1}{2}\sum_{k=k_0}^{k_1-1} u(k)^T u(k) = U^T U$$

is minimized. Here $U = [u(k_0); u(k_0+1); \ldots; u(k_1-1)]$.

3.5.2 Least norm control solution

This is a constrained optimization problem (with $J(U)$ as the cost to be minimized, and $L_rU - x_d = 0$ as the constraint). It can be solved using the Lagrange Multiplier method of converting a constrained optimization problem into an unconstrained optimization.

Define an augmented cost function, with the Lagrange multipliers $\lambda \in \Re^n$

$$J'(U,\lambda) = J(U) + \lambda^T(L_rU - x_d)$$

The optimal given by (U^*,λ^*) must satisfy:

$$\frac{\partial J'}{\partial U}(U^*,\lambda^*) = 0; \qquad \frac{\partial J'}{\partial \lambda}(U^*,\lambda^*) = 0$$

The second condition is just the constraint: $L_rU = x_d$.

This means that

$$L_r^T\lambda^* + U^* = 0; \qquad L_rU^* = x_d$$

Solving, we get:

$$\begin{aligned}\lambda^* &= -(L_rL_r^T)^{-1}x_d = -W_r^{-1}x_d\\ U^* &= -L_r^T\lambda^* = L_r^TW_r^{-1}x_d.\end{aligned}$$

The **optimal cost** of control is:

$$J(U^*) = x_d^TW_r^{-1}L_rL_r^TW_r^{-1}x_d = x_d^TW_r^{-1}x_d$$

Thus, the inverse of the Reachability Grammian tells us how difficult it is to perform a state transfer from $x = 0$ to x_d. In particular, if W_r is not invertible, for some x_d, the cost is infinite.

3.5.3 Geometric interpretation of least norm solution

Geometrically, we can think of the cost as $J = U^TU$, i.e. the inner product of U with itself. In notation of inner product,

$$J = <U,U>_R$$

The advantage of the notation is that we can change the definition of inner product, e.g. $<U,V>_R = U^TRV$ where R is a positive definite matrix. The usual inner (dot) product has $R = I$.

We say that U and V are normal to each other if $<U,V>_R = 0$.

Any solution that satisfies the constraint must be of the form

$$(U - U^p) \in Null(L_r)$$

where $L_rU^p = x_f$ is any *particular* solution.

Claim: Let U^* be the optimal solution, and U is any solution that satisfies the constraint. Then, $(U - U^*) \perp U^*$, i.e.

$$<(U - U^*), U^*>_R = 0$$

which is the normal equation for the least norm solution problem.

Proof: Direct substitution.

$$(U - U^*)^TR_UR_U^{-1}L_{pc}^T\left(L_rR_U^{-1}L_r^T\right)^{-1}x_f = (L_r(U - U^*))^T\left(L_rR_U^{-1}L_r^T\right)^{-1}x_f = 0.$$

◇

3.5.4 Open-loop least norm control

Applying the above result to linear continuous time system using standard norms, i.e.

$$J = \int_{t_0}^{t_1} u^T(\tau)u(\tau)d\tau.$$

$$\begin{aligned}
(L_{r,[t_0,t_1]})^*(t) &= B^T(t)\Phi(t,t_1)^T \\
W_{r,[t_0,t_1]} &= L_r L_r^* = \left[\int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi(t_1,\tau)^T d\tau\right]
\end{aligned}$$

$$u_{opt}(t) = B(t)^T\Phi(t,t_1)^T W_{r,[t_0,t_1]}^{-1}(x_1 - \Phi(t_1,t_0)x_0) \tag{3.4}$$

Eq, (3.4) solves $u_{opt}(\cdot)$ in a batch form given x_0, i.e. it is openloop.

3.5.5 Recursive formulation

- Openloop control does not make use of feedback. It is not robust to disturbances or model unceratinty. Suppose now that $x(t)$ is measured. Conceptually, we can solve (3.4) for $u_{opt}(t)$ and let $t_0 = t$.

- Computing $W_{r,[t,t_1]}$ is expensive. Try recursion:

$$\begin{aligned}
u_{opt}(t) &= B(t)^T\Phi(t,t_1)^T W_{r,[t,t_1]}^{-1}(x_1 - \Phi(t_1,t)x(t)) \\
&= \left(B(t)^T\Phi(t,t_1)^T W_{r,[t,t_1]}^{-1}\right)x_1 - \left(B(t)^T\Phi(t,t_1)^T W_{r,[t,t_1]}^{-1}\Phi(t_1,t)\right)x(t)
\end{aligned}$$

  For any invertible matrix $Q(t) \in \Re^{n\times n}$, $Q(t)Q^{-1}(t) = I$, therefore

$$\frac{d}{dt}Q^{-1}(t) = -Q^{-1}(t)\dot{Q}(t)Q^{-1}(t)$$

  Since

$$\begin{aligned}
W_{r,[t,t_1]} &= \int_t^{t_1} \Phi(t_1,\tau)B(\tau)B(\tau)^*\Phi(t_1,\tau)^* d\tau \\
\frac{d}{dt}W_{r,[t,t_1]} &= -\Phi(t_1,t)B(t)B(t)^*\Phi(t_1,t)^*
\end{aligned}$$

- Thus we can solve for $W_{r,[t,t_1]}^{-1}$ dynamically by integrating on-line

$$\frac{d}{dt}W_{r,[t,t_1]}^{-1} = -W_{r,[t,t_1]}^{-1}\Phi(t_1,t)B(t)B(t)^*\Phi(t_1,t)^*W_{r,[t,t_1]}^{-1}$$

- **Issue**: Does not work well when $t \to t_1$ because $W_{r,[t,t_1]}$ can be close to singular.

3.5.6 Optimal cost

Optimal cost as a function of initial time t_0

Since

$$W_{r,[t_0,t_f]} = \int_{t_0}^{t_f} \Phi(t_f,\tau)B(\tau)B(\tau)^*\Phi(t_f,\tau)^* d\tau$$

and the integrand is a positive semi-definite term, for each $v \in \Re^n$, $v^T W^{-1}_{r,[t_0,t_f]} v$ decreases as t_0 decreases, i.e. given more time, the cost of control decreases.

To see this, fix the final time t_f, and consider the minimum cost function,

$$E_{min}(x_f,t_0) = x_f^T W^{-1}_{r,[t_0,t_f]} x_f$$

and consider

$$\frac{d}{dt_0} E_{min}(x_f,t_f) = v^T \frac{d}{dt_f} W^{-1}_{r,[t_0,t_f]} x_f$$

Since

$$\begin{aligned}
\frac{d}{dt_f} W^{-1}_{r,[t_0,t_f]} &= -W^{-1}_{r,[t_0,t_f]} \frac{d}{dt_f} W_{r,[t_0,t_f]} W^{-1}_{r,[t_0,t_f]} \\
&= -\left\{ W^{-1}_{r,[t_0,t_f]} \Phi(t_f,t_0)B(t_0)B(t_0)^*\Phi(t_f,t_0)^* W^{-1}_{r,[t_0,t_f]} \right\}
\end{aligned}$$

is negative semi-definite, $\frac{d}{dt_0} E_{min}(v,t_f) \leq 0$. Hence, the optimal control increases as the initial time decreases, increasing the allowable time interval.

In other words, the set of states reachable with a cost less than a given value grows as t_0 decreases.

If the system is unstable, in that, $\Phi(t_f,t_0)$ becomes unbounded as $t_0 \to -\infty$, then one can see that $W_{r,[t_0,t_f]}$ also becomes unbounded. This means that for some x_f, $E_{min}(x_f,t_0) \to 0$ and $t_f \to \infty$. This means that some directions do not require any cost of control. Which direction do these correspond to?

For time invariant system, decreasing t_0 is equivalent to increasing t_f. Thus for time-invariant system, increasing the time interval increases W_r, thus decreasing cost.

Optimal cost as a function final state x_f

Since $W_{r,[t_0,t_f]}$ is symmetric and positive semi-definite, has n orthogonal eigenvectors, $V = [v_1, \ldots, v_n]$, and associated eigenvalues $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, such that $VV^T = I$. Then

$$W_{r,[t_0,t_f]} = V\Lambda V^T \Rightarrow W^{-1}_{r,[t_0,t_f]} = V\,\text{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})V^T$$

This shows that most expensive direction to transfer to is the eigenvector associated with the minimum λ_i, and the least expensive direction is the eigenvector associated with the maximum λ_i

3.6 Linear Time Invariant (LTI) Continuous Time Systems

Consider first the LTI continuous time system:

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned} \tag{3.5}$$

Note: Because of time invariance, the system is controllable (observable) at t_0 means that it is controllable (observable) at any time.

We will develop tests for controllability and observability by using directly the matrices A and B.

The three tests for controllability of system (3.5) is given by the following theorem.

Theorem For the continuous time system (3.5), the followings are equivalent:

1. The system is controllable over the interval $[0, T]$, for some $T > 0$.
2. The system is controllable over any interval $[t_0, t_1]$ with $t_1 > t_0$.
3. The reachability grammian,
$$\begin{aligned} W_{r,T} &:= \int_0^T \Phi(T,t)B(t)B(t)^T\Phi(T,t)^T dt \\ &= \int_0^T e^{At}BB^* e^{A^*t}dt \end{aligned}$$
is full rank n. (This test works for time varying systems also)
4. The controllability matrix
$$\mathcal{C} := \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$
has rank n. Notice that the controllability matrix has dimension $n \times nm$ where m is the dimension of the control vector u.
5. **(Belovich-Popov-Hautus test)** For each $s \in \mathcal{C}$, the matrix,
$$\begin{pmatrix} sI - A & B \end{pmatrix}$$
has rank n.

   Note that rank of $(sI - A, B)$ is less than n only if s is an eigenvalue of A.

We need the Cayley-Hamilton Theorm to prove this result:

Theorem 3.6.1 *(Cayley Hamilton) Let $A \in \Re^{n\times n}$. Consider the polynomial*
$$\Delta(s) = det(sI - A)$$
Then $\Delta(A) = 0$.

Notice that if λ_i is an eigenvalue of A, then $\Delta(\lambda_i) = 0$.

Proof: This is true for any A, but is easy to see using the eigen decomposition of A when A is semi-simple.
$$A = T\Lambda T^{-1}$$
where Λ is diagonal with eigenvalues on its diagonal. Since $\psi(\lambda_i) = 0$ by definition,
$$\Delta(A) = T\psi(\Lambda)T^{-1} = 0.$$

Proof: Controllability test
(1) $\Longleftrightarrow$ (3): We have already seen before that controllability over the interval $[0,T]$ means that the range space of the reachability map,

$$L_r(u) = \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

must be the whole state space, $\Re^n$. Notice that

$$W_{r,T} := \int_0^T e^{At}BB^*e^{A^*t}dt = L_rL_r^*$$

where L_r^* is the adjoint (think transpose) of L_r.

From the finite rank linear map theorem, $\mathcal{R}(L_r) = \mathcal{R}(L_rL_r^*)$ but, $L_rL_r^*$ is nothing but $W_{r,T}$.

(3) $\Rightarrow$ (4): We will show this by showing "not (4) $\Rightarrow$ not (3)". Suppose that (4) is not true. Then, there exists a $1 \times n$ vector v^T so that,

$$v^TB = v^TAB = v^TA^2B \cdots = v^TA^{n-1}B = 0.$$

Consider now,

$$v^TW_{r,T}v = \int_0^T \|v^Te^{At}B\|_2^2dt.$$

Since

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^{n-1}t^{n-1}}{n-1!} + \cdots$$

and by the Cayley Hamilton Theorem, A^k is a linear combination of I, A, ... A^{n-1}, therefore,

$$v^Te^{At}B = 0$$

for all t. Hence, $v^TW_{r,T}v = 0$ or $W_{r,T}$ is not full rank.

(3) $\Rightarrow$ (2): We will show this by showing "not (2) $\Rightarrow$ not (3)". If (2) is not true, then there exists $1 \times n$ vector v^T so that

$$v^TW_{r,T}v = \int_0^T \|v^Te^{At}B\|_2^2dt = 0.$$

Because e^{At} is continuous in t, this implies that $v^Te^{At}B = 0$ for all $t \in [0,T]$.

Hence, the all time derivatives of $v^Te^{At}B = 0$ for all $t \in [0,T]$. In particular, at $t = 0$,

$$\begin{aligned}
v^Te^{At}B\,|_{t=0} &= v^TB = 0 \\
v^T\frac{d}{dt}e^{At}B\,|_{t=0} &= v^TAB = 0 \\
v^T\frac{d^2}{dt^2}e^{At}B\,|_{t=0} &= v^TA^2B = 0 \\
&\vdots \\
v^T\frac{d^{n-1}}{dt^{n-1}}e^{At}B\,|_{t=0} &= v^TA^{n-1}B = 0
\end{aligned}$$

Hence,

$$v^T\begin{pmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{pmatrix} = \begin{pmatrix}0 & 0 & \cdots 0\end{pmatrix}$$

i.e. the controllability matrix of (A, B) is not full rank.

(4) $\Rightarrow$ (5): We will show " not (5) implies not (4)". Suppose (5) is not true so that there exists a $1 \times n$ vector v, and $\lambda \in \mathcal{C}$,

$$v^T(\lambda I - A) = 0, \quad v^T B = 0.$$

for some λ. Then $v^T A = \lambda v^T$, $v^T A^2 = \lambda^2 v^T$ etc. Because $v^T B = 0$ by assumption, we have $v^T AB = \lambda v^T B = 0$, $v^T A^2 B = \lambda^2 v^T B = 0$ etc. Hence,

$$v^T B = v^T AB = v^T A^2 B = \cdots v^T A^{n-1} B = 0.$$

Therefore the controllability is not full rank.

The proof of (4) $\Rightarrow$ (3) requires the 2nd representation theorem, and we shall leave it afterward.

Note that (2) $\Rightarrow$ (1) is obvious. To see that (1) $\Rightarrow$ (2), notice that the controllability matrix $\mathcal{C}$ does not depend on T, so controllability does not depend on duration of time interval. Since the system is time invariant, it does not depend on the initial time (t_0). $\diamond$

Proof of (4) $\Rightarrow$ (3) - Optional

To prove (4) $\Rightarrow$ (3), we need the so called 2nd Representation Theorem. It is included here for completeness. We will not go through this part.

Definition Let $A : \Re^n \to \Re^n$ and $W \subset \Re^n$ a subspace with the property that for each $w \in W$, $Aw \in W$. We say that W is $A-$invariant.

Theorem *(2nd Representation theorem) Let*

- $A : \Re^n \to \Re^n$ *is a linear map (i.e. a matrix)*
- $W \subset \Re^n$ *be an* $A-$*invariant* $k-$ *dimensional subspace of* $\Re^n$.

There exists a nonsingular $T = \begin{pmatrix}e_1 & e_2 & \cdots & e_n\end{pmatrix}$ *s.t.*

- $e_1, \cdots, e_k$ *form a basis of* W,
- *in the basis of* $e_1, e_2, \cdots, e_n$, A *is represented by:*

$$T^{-1}AT = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22}\end{pmatrix}$$

  $\tilde{A}_{11} \in \Re^{k \times k}$ *where* $k = \texttt{dim}\ W$.

- *For any vector* $B \in W$, *the representation of* B *is given by*

$$B = T\begin{pmatrix}\tilde{B} \\ 0\end{pmatrix}.$$

The point of the theorem is that A can be put in a form with 0 in strategic locations ($\tilde{A}_{21}$ are zeros), similarly for any vector $B \in W$ ($\tilde{B}_2$ are zeros).
Proof: Take $e_1, \cdots, e_k$ to be a basis of W, and choose $e_{k+1}, \cdots e_n$ to complete the basis of $\Re^n$. Because $Ae_i \in W$ for each $i = 1, \cdots k$,

$$Ae_i = \sum_{l=1}^{k} \alpha_{li} e_l.$$

Therefore, α_{li} are the coefficients of $\tilde{A}_{11}$ and the coefficients of $\tilde{A}_{21}$ are all zero.

Let $B \in W$, then

$$B = \sum_{l=1}^{k} \beta_l e_l$$

so that β_l are simply the coefficients in $\tilde{B}_1$ and the coefficients of $\tilde{B}_2$ are zeros. ◇

Proof of (4) ⇒ (3) in controllability test: Notice that

$$\begin{aligned} &A\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} \\ =&\begin{pmatrix} AB & A^2B & \cdots & A^{n-1}B & A^nB \end{pmatrix} \end{aligned}$$

therefore, by the Cayley Hamilton Theorem, the range space of the controllability matrix is $A-$invariant.

Suppose that (3) is not true, so that the rank of

$$\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$

is less than n. Since the controllability matrix is A-invariant, by the 2nd representation theorem, there is an invertible matrix T so that in the new coordinates, $z = T^{-1}x$,

$$\dot{z} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} u \tag{3.6}$$

where $B = T\begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix}$, and

$$A = T\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} T^{-1}.$$

and the dim of $\tilde{A}_{22}$ is non-zero (it is n - rank($\mathcal{C}$)).

Let v_2^T be a left eigenvector of $\tilde{A}_{22}$ so that

$$v_2^T(\lambda I - \tilde{A}_{22}) = 0$$

for some $\lambda \in \mathcal{C}$. Define $v^T := \begin{pmatrix} 0 & v_2^T \end{pmatrix} T^{-1}$. Evaluating $v^T B$, $v^T(\lambda I - A)$ gives

$$\begin{aligned} v^T B &= \begin{pmatrix} 0 & v_2^T \end{pmatrix} T^{-1} T \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} = 0 \\ v^T(\lambda I - A) &= \lambda\begin{pmatrix} 0 & v_2^T \end{pmatrix} T^{-1}(\lambda T T^{-1} - T\tilde{A}T^{-1}) \\ &= \begin{pmatrix} 0 & v_2^T \end{pmatrix}(\lambda I - \tilde{A})T^{-1} = 0. \end{aligned}$$

This shows that $[\lambda I - A, B]$ has rank less than n.
Remarks:

1. Notice that the controllability matrix is not a function of the time interval. Hence, if a LTI system is controllable over some interval, it is controllable over any (non-zero) interval. *c.f.* with result of linear time varying system.

2. Because of the above fact, we often say that the pair (A, B) is controllable.

3. Controllability test can be done by just examining A and B without computing the grammian. The test in (4) is attractive in that it enumerates the vectors in the controllability subspace. However, numerically, since it involves power of A, numerical stability needs to be considered.

4. The PBH Test in (5), or the Hautus test for short, involves simply checking the condition at the eigenvalues. It is because for $(sI - A, B)$ to have rank less than n, s must be an eigenvalue.

5. The range space of the controllability matrix is of special interests. It is called the **controllable subspace** and is the set of all states that can be reached from zero-initial condition. **This is $A-$ invariant.**

6. Using the basis for the **controllable subspace** as part of the basis for $\Re^n$, the controllability property can be easily seen in (3.6).

3.7 Linear Time Invariant (LTI) discrete time system

For the discrete time system (3.7),

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{3.7}$$

where $x \in \Re^n$, $u \in \Re^m$, the conditions for controllability is fairly similar except that we need to consider controllability over period of length greater than n.

First, recall that the reachability map for a linear discrete time system $L_{r,[k,k_1]} : u(\cdot) \mapsto s(k, k_0, x = 0, u(\cdot))$ is given by:

$$L_{r,[k_0,k_1]}[u(\cdot)] = \sum_{i=k_0}^{k-1} \Pi_{j=i+1}^{k-1} A(j)B(i)u(i) \tag{3.8}$$

Theorem For the discrete time system (3.7), the followings are equivalent:

1. The system is controllable over the interval $[0, T]$, with $T \geq n$ (i.e. it can transfer from any arbitrary state at $k = 0$ to any other arbitrary state at $k = T$).

2. The system is controllable over the interval $[k_0, k_1]$, with $k_1 - k_0 \geq n$

3. The controllability grammian,
$$W_{r,T} := \sum_{l=0}^{T-1} A^l BB^* A^{*l}$$
is full rank n.

4. The controllability matrix
$$\begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}$$
has rank n. Notice that the controllability matrix has dimension $n \times nm$ where m is the dimension of the control vector u.

5. For each $z \in \mathcal{C}$, the matrix,
$$\begin{pmatrix} zI - A & B \end{pmatrix}$$
has rank n.

Proof: The proof of this theorem is exactly analogous to the continuous time case. However, in the case of the equivalence of (2) and (3), we can notice that

$$x(k) = A^k x(0) + [B, AB, A^2B, \cdots A^{n-1}B] \begin{pmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{pmatrix}$$

so clearly one can transfer to arbitrary states at $k = n$ iff the controllability matrix is full rank. Controllability does not improve for $T > n$ is again the consequence of Cayley-Hamilton Theorem. $\diamond$

3.8 Observability

The observability question deals with the question of while not knowing the initial state, whether one can determine the state from the input and output. This is equivalent to the question of whether one can determine the initial state $x(0)$ when given the input $u(t), t \in [0, T]$ and the output $y(t), t \in [0, T]$.

Definition 3.8.1 *A state determined dynamical system is called* observable *on* $[t_0, t_1]$ *if for all inputs,* $u(\tau)$*, and outputs* $y(\tau)$*,* $\tau \in [t_0, t_1]$*, the state* $x(t_0) = x_0$ *can be uniquely determined.*

For observability, the null space of the observability map on the interval $[t_0, t_1]$, given by:

$$L_o : x_0 \mapsto y(\cdot) = C(\cdot)\Phi(\cdot, t_0)x_0, \qquad y(\tau) = C(\tau)\Phi(\tau, t_0)x_0\Phi(\tau, 0)x, \quad \forall \tau \in [t_0, t_1]$$

must be trivial (i.e. only contains 0). Otherwise, if $L_o(x_n)(t) = y(t) = 0$, for all $t \in [0, T]$, then for any $\alpha \in \Re$,
$$y(t) = C(t)\Phi(t, 0)x = \Phi(t, 0)(x + \alpha x_n).$$

Recall from the finite rank theorem that if the observability map L_o is a matrix, then the null space of L_o, $\mathcal{N}(L_o) = \mathcal{N}(L_o^T L_o)$. For continuous time system, L_o can be thought of as an infinitely long matrix, then the matrix product operation in $L_o^T L_o$ becomes an integral, and is written as $L_o^* L_o$ where L^* denotes the adjoint of the L.

The observability Grammian is given by:

$$W_{o,[t_0,t_1]} = L_o^* L_o = \int_{t_0}^{t_1} \Phi(t, t_0)^T C^T(t) C(t) \Phi(t, t_0) \in \Re^{n \times n}$$

where L_o^* is the adjoint (think transpose) of the observability map.

3.9 Least square estimation

3.9.1 Formulation and solution

Consider the continuous time varying system:

$$\dot{x}(t) = A(t)x(t) \qquad\qquad y(t) = C(t)x(t).$$

The observability map on $[0, t_1]$ is given by: $\forall \tau \in [0, t_1]$

$$L_o : x(0) \mapsto y(\tau) = C(\tau)\Phi(\tau, 0)x(0) \tag{3.9}$$

The so-called Least squares estimate of $x(0)$ given the output $y(\tau)$, $\tau \in [0, t_1]$ is:

$$\hat{x}(0|t_1) = \text{arg min}_{x_0} \int_0^{t_1} e^T(\tau)e(\tau)d\tau.$$

The least squares estimation problem can be cast into a set of linear equations. For example, we can stack all the outputs $y(t), t \in [0, t_1]$ into a vector $Y = [Y(0), \ldots, Y(0.1), \ldots, Y(0.2), \ldots, Y(t_1)]^T \in \Re^{pp}$, and represent L_o as a matrix $\in \Re^{pp \times n}$, then,

$$\begin{aligned} Y &= L_o x_0 + E; \\ x_o^* &= argmin_{x_0} E^T E = (L_o^T L_o)^{-1} L_o^T Y \end{aligned}$$

Here, $E = [E(0), \ldots, E(t_1)]^T = \Re^{pp}$ is the modeling error or noise. The optimal estimate x_0^* uses the minimum amount of noise E to explain the data Y.

Notice that the resulting E satisfies $E^T L_o = 0$, i.e. E is normal to Range(L_o).

Using the observability map for the LTI continuous time system (3.9) as L_o,

$$\begin{aligned} \hat{x}(0|t_1) &= W_o^{-1}(t_1) \int_0^{t_1} \Phi^T(\tau, 0)C^T(\tau)y(\tau)d\tau \\ W_o(t_1) = L_o^* L_o &= \int_0^{t_1} \Phi^T(\tau, 0)C^T(\tau)C(\tau)\Phi(\tau, 0)d\tau. \end{aligned}$$

One can also find estimates of the states at time t_2 given data up to time t_1:

$$\hat{x}(t_2|t_1) = \Phi(t_2, 0)\hat{x}(0|t_1)$$

This refers to three classes of esimation problems:

- $t_2 < t_1$: Filtering problem
- $t_2 > t_1$: Prediction problem
- $t_2 = t_1$: Observer problem

The estimate $\hat{x}(t|t)$ is especially useful for control systems as it allows us to effective do state feedback control.

3.9.2 Recursive Least Squares Observer

The least squares solution computed above is expensive to compute:

$$\hat{x}(t|t) = \Phi(t,0)W_o^{-1}(t)\int_0^t \Phi^T(\tau,0)C^T(\tau)y(\tau)d\tau$$
$$W_o(t) = L_o^* L_o = \int_0^t \Phi^T(\tau,0)C^T(\tau)C(\tau)\Phi(\tau,0)d\tau.$$

It is desirable if the estimate $\hat{x}(t|t)$ can be obtained recursively via a differential equation.

$$\begin{aligned}\frac{d}{dt}\hat{x}(t|t) = A(t)\,&\underbrace{\Phi(t,0)W_o^{-1}(t)\int_0^t \Phi^T(\tau,0)C^T(\tau)y(\tau)d\tau}_{\hat{x}(t|t)}\\ &+\Phi(t,0)\frac{d}{dt}W_o^{-1}(t)\int_0^t \Phi^T(\tau,0)C^T(\tau)y(\tau)d\tau\\ &+\Phi(t,0)W_o^{-1}(t)\Phi^T(t,0)C^T(t)y(t)\end{aligned}$$

Now by differentiating $W_o(t)W_o^{-1}(t) = I$, we have:

$$\frac{d}{dt}W_o^{-1}(t) = -W_o^{-1}(t)\frac{dW_o}{dt}(t)W_o^{-1}(t)$$

where

$$\frac{d}{dt}W_o(t) = \Phi^T(t,0)C^T(t)C(t)\Phi(t,0)$$

$$\begin{aligned}\frac{d}{dt}\hat{x}(t|t) &= A(t)\hat{x}(t|t) - \Phi(t,0)W_o^{-1}(t)\Phi^T(t,0)C^T(t)C(t)\hat{x}(t|t) + \Phi(t,0)W_o^{-1}(t)\Phi^T(t,0)C^T(t)y(t)\\ &= A(t)\hat{x}(t|t) - \underbrace{\Phi(t,0)W_o^{-1}(t)\Phi^T(t,0)}_{P(t)}C^T(t)[\underbrace{C(t)\hat{x}(t|t)}_{\hat{y}(t|t)} - y(t)] \qquad (3.10)\end{aligned}$$

$P(t)$ satisfies:

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)C(t)P(t)$$

Typically, we would initialize $P(0)$ to be a large non-singular matrix, e.g. $P(0) = \alpha \cdot I$, where α is large.

In the recursive least squares observer in (3.10), the first term is an openloop prediction term, and the second term is called an output injection where an output prediction error is used to correct the estimation error.

We can think of the observer feedback gain as:

$$L(t) = P(t)C^T(t)$$

3.9.3 Covariance Windup

Let us rewrite $P(t)$ as follows:

$$\begin{aligned}P(t) &= \Phi(t,0)\left[\int_0^t \Phi^T(\tau,0)C^T(\tau)C(\tau)\Phi(\tau,0)d\tau\right]^{-1}\Phi^T(t,0)\\ &= \left[\int_0^t \Phi^T(\tau,t)C^T(\tau)C(\tau)\Phi(\tau,t)d\tau\right]^{-1} =: K^{-1}(t)\end{aligned}$$

The matrix $K(t) := P^{-1}(t)$ is known as the *Information Matrix* and satisfies:

$$\dot{K} = C^T(t)C(t) - A^T(t)K(t) - K(t)A(t)$$

Notice that $W_o(t)$ increases with time. Thus, $W_o-1(t)$ decreases with time, as would $K^{-1}(t)$. This means that the observer gain $L(t)$ will decrease. The observer will asymptotically rely more and more on open loop estimation:

$$\frac{d}{dt}\hat{x}(t|t) = A(t)\hat{x}(t|t)$$

Forgetting factor

Forget old information (τ far away from current time): Choose $\lambda > 0$:

$$K(t) = \int_0^t \Phi^T(\tau,t)C^T(\tau)C(\tau)\Phi(\tau,t)e^{-\lambda(t-\tau)}d\tau;$$

$$\dot{K} = C^T(t)C(t) - A^T(t)K(t) - K(t)A(t) - \lambda K$$

Covariance reset

If $K(t)$ becomes large, reset it to small number.

- Check maximum singular value of $K(t)$, $\bar{\sigma}K(t)$.
- If $\bar{\sigma}K(t) \geq \lambda_{max}$, $K(t) = \epsilon I$, i.e. $P(t) = I/\epsilon$.

3.10 Observability Tests

The observability question deals with the question of whether one can determine what the initial state $x(0)$ is, given the input $u(t), t \in [0,T]$ and the output $y(t), t \in [0,T]$.

For observability, the null space of the observability map,

$$L_o : x \mapsto y(\cdot) = \Phi(\cdot,0)x, \qquad \tau \in [0,T],$$

must be trivial (i.e. only contains 0). Otherwise, if $L_o(x_n)(t) = y(t) = 0$, for all $t \in [0,T]$, then for any $\alpha \in \Re$,

$$y(t) = \Phi(t,0)x = \Phi(t,0)(x + \alpha x_n).$$

Hence, one cannot distinguish the various initial conditions from the output.

Just like for controllability, it is inconvenient to check the rank (and the null space) of the L_o which is tall and thin. We can instead check the rank and the null space of the observability grammian given by:

$$W_{o,T} = L_o^* L_o$$

where L_o^* is the adjoint (think transpose) of the observability map.

Proposition 3.10.1 $\texttt{Null}(L_o) = \texttt{Null}(L_o^* L_o)$.

Proof:

- Let $x \in \texttt{Nul}(L_o)$ so, $L_o x = 0$. Then, clearly, $L_o^T L_o x = L^T 0$. Therefore, $\texttt{Null}(L_o) \subset \texttt{Null}(L_o^* L_o)$.
- Let $x \in \texttt{Nul}(L_o^* L_o)$. Then, $x^T L_o^* L_o x = 0$. Or, $(L_o x)^T (L_o x) = 0$. This can only be true if $L_o x = 0$. Therefore, $\texttt{Null}(L_o^* L_o) \subset \texttt{Null}(L_o)$.

$\diamond$

3.11 Observability Tests for Continuous time LTI systems

Theorem For the LTI continuous time system (3.5), the followings are equivalent:

1. The system is observable over the interval $[0, T]$
2. The observability grammian,
$$W_{o,T} := \int_0^T e^{A^* t} C^* C e^{At} dt$$
is full rank n.
3. The observability matrix $\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$ has rank n. Notice that the observability matrix has dimension $np \times n$ where p is the dimension of the output vector y.
4. For each $s \in \mathcal{C}$, the matrix,
$$\begin{pmatrix} sI - A \\ C \end{pmatrix}$$
has rank n.

Proof: The proof is similar to the ones as in the controllable case and will not be repeated here. Some differences are that instead of considering $1 \times n$ vector v^T multiplying on the left hand sides of the controllability matrix and the grammians, we consider $n \times 1$ vector multiplying on the RHS of the observability matrix and the grammian etc.

Also instead of considering the range space of the controllability matrix, we consider the NULL space of the observability matrix. Its null space is also A-invariant. Hence if the observability matrix is not full rank, then using basis for its null space as the last k basis vectors of $\Re^n$, the system can be represented as:

$$\dot{z} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} z$$

where $C = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} T^{-1}$, and
$$A = T \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} T^{-1}.$$

and the dim of $\tilde{A}_{22}$ is non-zero (and is the dim of the null space of the observability matrix). $\diamond$

Remark

1. Again, observability of a LTI system does not depend on the time interval. So, theoretically speaking, if observing the output and input for an arbitrary amount of time will be sufficient to figure out $x(0)$. In reality, when more data is available, one can do more averaging to eliminate effects of noise (e.g. using the Least square ie. Kalman Filter approach).

2. The subspace of particular interest is the null space of the controllability matrix. An initial state lying in this set will generate identically 0 zero-input response. This subspace is called the **unobservable subspace**.

3. Using the basis of the **unobservable subspace** as part of the basis of $\Re^n$, the observability property can be easily seen.

3.12 Observability Tests for Discrete time LTI systems

The tests for observability of the discrete time system (3.7) is given similarly by the following theorem.

Theorem 3.12.1 *For the discrete time system* (3.7), *the followings are equivalent:*

1. *The system is observable over the interval* $[0, T]$, *for some* $T \geq n$.

2. *The observability grammian,*
$$W_{o,T} := \sum_{k=0}^{T-1} A^{*k} C^* C A^k$$
is full rank n.

3. *The observability matrix* $\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$ *has rank* n. *Notice that the observability matrix has dimension* $np \times n$ *where* p *is the dimension of the output vector* y.

4. *For each* $z \in \mathcal{C}$, *the matrix,*
$$\begin{pmatrix} zI - A \\ C \end{pmatrix}$$
has rank n.

The controllability matrix can have the following interpretation: zero-input response is given by:
$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(n-1) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x(0).$$

Thus, clearly if the observability matrix is full rank, one can reconstruct $x(0)$ from measurements of $y(0), \cdots, y(n-1)$. One might think that by increasing the number of output measurement times (e.g. $y(n)$) the system can become observable. However, because of Cayley-Hamilton theorem, $y(n) = CA^n x(0)$ can be expressed as $\sum_{i=0}^{n-1} a_i CA^i x(0)$ consisting of rows already in the observability matrix.

3.13 PHB test and Eigen/Jordan Forms

Consider

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Suppose first that A is semi-simple, then $A = TDT^{-1}$ and $D = \operatorname{diag} \lambda_1, \ldots, \lambda_n$ where λ_i might be repeating.

The PBH test in the transformed eigen-coordinates is to check

$$rank(\lambda_i D, T^{-1}B)$$

for each i if it is n. This shows that $T^{-1}B$ must be non-zero in each row. Also, if $\lambda_i = \lambda_j$ for some $i \neq j$, then a single input system cannot be controllable.

Now suppose that $A = TJT^{-1}$ and J is in Jordan form,

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

This Jordan form has 3 generalized eigenvector chains.

Controllability

- From homework, we saw that if the entries of $B \in \Re^4$ at the **beginning** of some *chain* is 0, i.e..
$$T^{-1}B = \begin{pmatrix} b_1 \\ b_2 \\ b3 \\ 0 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ b_4 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} b_1 \\ 0 \\ b3 \\ b_4 \end{pmatrix}$$
then the system is uncontrollable.
- Thus, $T^{-1}B$ being nonzero at the beginning of each chain is a necessary condition for controllability.
- **PBH test shows that it is in fact a sufficient condition**.

Observability

- Similarly, if $C \cdot T \in \Re^{1\times n}$ is a zero entry at the **end** of any *chain* (these correspond to the coordinates for the eigenvector), i.e.
$$CT = \begin{pmatrix} 0 & c_2 & c_3 & c_4 \end{pmatrix}, \quad CT = \begin{pmatrix} c_1 & c_2 & 0 & c_4 \end{pmatrix},$$
$$or, \quad CT = \begin{pmatrix} c_1 & c_2 & c_3 & 0 \end{pmatrix}$$
then we saw from the homework that the system is not observable.

- Hence, CT having non-zero entries at the **end** of each *chain* is necessary for observability.
- PHB test shows that it is also sufficient.

Note If the system is either uncontrollable or unobservable, then the order of the transfer function will be less than the order of A.

However, for a multi-input-multi-output system, the fact that the order of each transfer function in the transfer function matrix has order less than the order of A, does NOT imply that the system is uncontrollable or unobservable. e.g.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives,

$$Y(s) = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} U(s).$$

3.14 Kalman Decomposition

3.14.1 Controllable / uncontrollable decomposition

Suppose that the controllability matrix $\mathcal{C} \in \Re^{n\times n}$ of a system has rank $n_1 < n$. Then there exists an invertible transformation, $T \in \Re^{n\times n}$ such that:

$$z = T^{-1}x,$$

$$\dot{z} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix} u \tag{3.11}$$

where $B = T\begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix}$, and

$$A = T\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} T^{-1}.$$

and the dim of $\tilde{A}_{22}$ is $n - n_1$.

3.14.2 Observable / unobservable decomposition

Hence if the observability matrix is not full rank, then using basis for its null space as the last k basis vectors of $\Re^n$, the system can be represented as:

$$\dot{z} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} z + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} z$$

where $C = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} T^{-1}$, and

$$A = T\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} T^{-1}.$$

and the dim of $\tilde{A}_{22}$ is non-zero (and is the dim of the null space of the observability matrix).

3.14.3 Kalman decomposition

The Kalamn decomposition is just the combination of the observable/unobservable, and the controllable/uncontrollable decomposition.

Theorem 3.14.1 *There exists a coordinate transformation $z = T^{-1}x \in \Re^n$ such that*

$$\dot{z} = \begin{pmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & A_{23} & A_{24} \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{pmatrix} z + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} \tilde{C}_1 & 0 & \tilde{C}_2 & 0 \end{pmatrix} z$$

Let $T = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 \end{pmatrix}$ (with compatible block sizes), then

- *$T_2 \to (C - \bar{O})$: $T_2 =$ basis for Range($\mathcal{C}$) $\cap$ Null($\mathcal{O}$).*
- *$T_1 \to (C - O)$: $T_1 \cup T_2 =$ basis for Range($\mathcal{C}$)*
- *$T_4 \to (\bar{C} - \bar{O})$: $T_2 \cup T_4 =$ basis for Null($\mathcal{O}$).*
- *$T_3 \to (\bar{C} - O)$: $T_1 \cup T_2 \cup T_3 \cup T_4 =$ basis for $\Re^n$.*

Note: Only T_2 is uniquely defined.
Proof: This is just combination of the obs/unobs and cont/uncont decomposition. $\diamond$

3.14.4 Stabilizability and Detectability

If the uncontrollable modes $\bar{A}_{33}$, $\bar{A}_{44}$ are stable (have eigenvalues on the Left Half Plane), then, system is called *stabilizable*.

- One can use feedback to make the system stable;
- Uncontrollable modes decay, so not an issue.

If the unobservable modes $\bar{A}_{22}$, $\bar{A}_{44}$ are stable (have eigenvalues on the Left Half Plane), then, system is called *detectable*.

- The states z_2, z_4 decay to 0
- Eventually, they will have no effect on the output
- State can be reconstructed by ignoring the unobservable states (eventually).

The PBH tests can be modified to check if the system is stabilizable or detectable, namely, check

$$rank\begin{pmatrix} sI - A & B \end{pmatrix} \qquad rank\begin{pmatrix} sI - A \\ C \end{pmatrix}$$

for all s with non-negative real parts if they lose rank. Specifically, one needs to check only s which are the unstable eigenvalues of A.

3.15 Realization

3.15.1 Degree of controllability / observability

Formal definitions for controllability and observability are black and white. In reality, some states are very costly to control to, or have little effect on the outputs. They are therefore, effectively uncontrollable or unobservable.

Degree of controllability and observability can be evaluated by the sizes of W_r and W_o with infinite time horizon:

$$\begin{aligned} W_r(0,\infty) &= \lim_{T\to\infty}\int_0^T e^{A(T-\tau)}BB^T e^{A^T(T-\tau)}d\tau \\ &= \int_0^\infty e^{At}BB^T e^{A^T t}dt \\ W_o(0,\infty) &= \lim_{T\to\infty}\int_0^T e^{A^T\tau}C^T C e^{A\tau}d\tau \end{aligned}$$

Notice that W_r satisfies differential equation:

$$\frac{d}{dt}W_r(0,t) = AW_r(0,t) + W_r(0,t)A^T + BB^T$$

Thus, if all the eigenvalues of A have strictly negative real parts (i.e. A is stable), then, as $T\to\infty$,

$$0 = AW_r + W_rA^T + BB^T$$

which is a set of linear equations in the entries of W_r, and can be solved effectively. The equation is called a Lyapunov equation. Matlab can solve for W_r using `gram`.

Similarly, if A is stable, as $T\to\infty$, W_o satisfies the Lyapunov equation

$$0 = A^TW_o + W_oA + C^TC$$

which can be solved using linear method (or using Matlab via `gram`).

W_r is related to the minimum norm control. To reach $x(0)=x_0$ from $x(-\infty)=0$ in infinite time,

$$\begin{aligned} u &= L_r^T W_c^{-1}x_0 \\ \min_{u(\cdot)} J(u) = \min_{u(\cdot)} &\int_{-\infty}^0 u^T(\tau)u(\tau)d\tau = x_0^T W_c^{-1}x_0. \end{aligned}$$

If state x_0 is difficult to reach, then $x_0^TW_c^{-1}x_0$ is large meaning that x_0 is practically uncontrollable.

Similarly, if the initial state is $x(0)=x_0$, then, with $u(\tau)\equiv 0$

$$\int_0^\infty y^T(\tau)y(\tau)d\tau = x_0^TW_ox_0.$$

Thus, $x_0^TW_ox_0$ measures the effect of the initial condition on the output. If it is small, the effect of x_0 in the output $y(\cdot)$ is small, thus, it is hard to observe, or practically unobservable.

Generally, one can look at the smallest eigenvalues W_c and W_o, the span of the associated eigenvectors will be difficult to control, or difficult to observe.

3.15.2 Relation to Transfer Function

For the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

The transfer function from $U(s) \to Y(s)$ is given by:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{Cadj(sI - A)B}{det(sI - A)} + D. \end{aligned}$$

Note that transfer function is not affected by similarity transform.

From Kalman decomposition, it is obvious that

$$G(s) = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 + D$$

where $\tilde{A}_{11}$, $\tilde{B}_1$, $\tilde{C}_1$ correspond to the controllable and observable component.

Poles

- they are values of $s \in C$ s.t. $G(s)$ becomes infinite.
- poles of $G(s)$ are clearly the eigenvalues of $\tilde{A}_{11}$.

Zeros

- For the SISO case, $z \in \mathcal{C}$ (complex numbers) is a zero if it makes $G(z) = 0$.
- Thus, zeros satisfy
$$Cadj(zI - A)B + Ddet(zI - A) = 0$$
assuming system is controllable and observable, otherwise, need to apply Kalman decomposition first.
- In the general MIMO case, a zero implies that there can be non-zero inputs $U(s)$ that produce output $Y(s)$ that is identically zero. Therefore, there exists $X(z)$, $U(z)$ such that:
$$\begin{aligned} zI - A\ X(z) &= B\ U(z) \\ 0 &= C\ X(z) + D\ U(z) \end{aligned}$$
This will be true if and only if at $s = z$
$$\text{rank}\begin{pmatrix} sI - A & -B \\ -C & -D \end{pmatrix}$$
is less than the normal rank of the matrix at other s.

We shall return to the multi-variable zeros when we discuss the effect of state feedback on the zero of the system.

3.15.3 Pole-zero cancellation

Cascade system with input/output (u_1, y_2):

- System 1 with input/output (u_1, y_1) has a zero at α and a pole at β.

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ y_1 &= C_1 x_1\end{aligned}$$

- System 2 with input/output (u_2, y_2)has a pole at α and a zero at β.

$$\begin{aligned}\dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y_2 &= C_2 x_2\end{aligned}$$

- Cascade interconnection: $u_2 = y_1$.

Then

1. The system pole at β is not observable from y_2
2. The system pole at α is not controllable from u_1.

Combined system:

$$A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}; \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & C_2 \end{pmatrix}$$

Consider $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

- Use PBH test with $\lambda = \beta$.

$$\begin{pmatrix} \beta I - A_1 & 0 \\ -B_2 C_1 & \beta I - A_2 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0???$$

- Let x_1 be eigenvector associated with β for A_1.
- Let $x_2 = (\beta I - A_2)^{-1} B_2 C_2 x_1$ (i.e. solve second row in PBH test).
- Then since

$$Cx = C_2 x_2 = (C_2(\beta I - A_2)^{-1} B_2) C_2 x_1$$

  and β is a zero of system 2, PHB test shows that the mode β is not observable.

Similar for the case when α mode in system 2 not controllable by u.

These example shows that when cascading systems, it is important not to do unstable pole/zero cancellation.

3.16 Balanced Realization

The objective is to reduce the number of states while minimizing effect on I/O response (transfer function). This is especially useful when the state equations are obtained from distributed parameters system, P.D.E. via finite elements methods, etc. which generate many states. If states are unobservable or uncontrollable, they can be removed. The issue at hand is if some states are lightly controllable while others are lightly observable. Do we cancel them?

In particular,

- Some states can be lightly controllable, but heavily observable.
- Some states can be lightly observable, but heavily controllable.
- Both contribute to significant component to input/output (transfer function) response.

Idea: Exhibit states so that they are simultaneously lightly (heavily) controllable and observable.

Consider a stable system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

If A is stable, reachability and observability grammians can be computed by solving the Lyapunov equations:

$$0 = AW_r + W_r A^T + BB^T$$
$$0 = A^T W_o + W_o A + C^T C$$

- If state x_0 is difficult to reach, then $x_0^T W_r^{-1} x_0$ is large $\Rightarrow$ practically uncontrollable.
- If $x_0^T W_o x_0$ is small, the signal of x_0 in the output $y(\cdot)$ is small, thus, it is hard to observe $\Rightarrow$ practically unobservable.
- Generally, one can look at the smallest eigenvalues W_r and W_o, the span of the associated eigenvectors will be difficult to control, or difficult to observe.

Transformation of grammians: (beware of which side T goes !):

$$z = Tx$$

Minimum norm control should be invariant to coordinate transformation:

$$x_0^T W_r^{-1} x_0 = z^T W_{r,z}^{-1} z = x^T T^T W_{r,z}^{-1} T x$$

Therefore

$$W_r^{-1} = T^T W_{r,z}^{-1} T \Rightarrow W_{r,z} = T \cdot W_r T^T.$$

Similarly, the energy in a state transmitted to output should be invariant:

$$x^T W_o x = z W_{o,z} z \Rightarrow W_{o,z} = T^{-T} W_o T^{-1}.$$

Theorem 3.16.1 *(Balanced realization) If the linear time invariant system is controllable and observable, then, there exists an invertible transformation $T \in \Re^{n\times n}$ s.t. $W_{o,z} = W_{r,z}$.*

The balanced realization algorithm:

1. Compute controllability and observability grammians:

   >> W_r = gram(sys, 'c');
   >> W_o = gram(sys, 'o')

2. Write $W_o = R^T R$ [One can use SVD]

   ```
   >> [U,S,V]=svd(Wo);
   >> R = sqrtm(S)*V';
   ```

3. Diagonalize RW_rR^T [via SVD again]:

   $$RW_rR^T = U\Sigma^2U^T,$$

   where $UU^T = I$ and Σ is diagonal.

   >> [U_1,S_1,V_1]=svd(R*W_r*R');
   >> Σ = sqrtm(S_1';

4. Take $T = \Sigma^{-\frac{1}{2}}U^T R$

   >> T = inv(sqrtm(Σ))*U_1'*R;

5. This gives $W_{r,z} = W_{o,z} = \Sigma$.

   >> $W_{r,z} = T * W_r * T'$,
   >> $W_{o,z} = inv(T)' * W_o * inv(T)$

Proof: Direct substitution!

- Once $W_{r,z}$ and $W_{o,z}$ are diagonal, and equal, one can decide to eliminate states that correspond to the smallest few entries.
- Do not remove unstable states from the realization since these states need to be controlled (stabilized)
- It is also possible to maintain some D.C. performance.

Consider the system in the balanced realization form:

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = C_1 z_2 + C_2 z_2 + Du$$

where $z_1 \in \Re^{n_1}$ and $z_2 \in \Re^{n_2}$, and $n_1 + n_2 = n$. Suppose that the z_1 are associated with the simultaneously lightly controllable and light observable mode.

The naive truncation approach is to remove z_2 to form the reduced system:

$$\dot{z}_1 = A_{11} z_1 + B_1 u$$
$$y = C_1 z_1 + Du$$

However, this does not retain the D.C. (steady state) gain from u to y (which is important from a regulation application point of view).

To maintain the steady state response, instead of eliminating z_2 completely, replace z_2 by its steady state value:

In steady state:

$$\begin{aligned} \dot{z}_2 &= 0 = A_{21}z_1 + A_{22}z_2^* + B_2 u \\ \Rightarrow z_2^* &= -A_{22}^{-1}(A_{21}z_1 + B_2 u) \end{aligned}$$

This is feasible if A_{22} is invertible (0 is not an eigenvalue).

A truncation approach that maintains the steady state gain is to replace z_2 by its steady state value:

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2^* + B_1 u \\ y &= C_1 z_1 + C_2 z_2^* + Du \end{aligned}$$

so that,

$$\begin{aligned} \dot{z}_1 &= \underbrace{\left[A_{11} - A_{12}A_{22}^{-1}A_{21}\right]}_{A_r} + \underbrace{\left[B_1 - A_{12}A_{22}^{-1}B_2\right]}_{B_r} u \\ y &= \underbrace{\left[C_1 - C_2 A_{22}^{-1}A_{21}\right]}_{C_r} z_1 + \underbrace{\left[D - C_2 A_{22}^{-1}B_2\right]}_{D_r} u \end{aligned}$$

The truncated system with state $z_1 \in \Re^{n_1}$ is

$$\begin{aligned} \dot{z}_1 &= A_r z_1 + B_r u \\ y &= C_r z_1 + D_r u \end{aligned}$$

which with have the same steady state response as the original system.