AERO 7970 Midterm Exam

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1 Question 1

Given the system:

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix} \tag{1}$$

- Find a state space realization
- Determine the controllability and observability
- Find the transmission zeroes and eigenvalues of the A matrix using Matlab
- Find the H2 and Hinf norms and plot the Hinf vs frequency
- Plot the singular values vs frequency

1.1 State Space Realization

$G(s)$ can be split into the following transfer functions:

$$\frac{y_1(s)}{u_1(s)} = \frac{10(s+1)}{s^2+0.2s+100} \tag{2}$$

$$\frac{y_2(s)}{u_1(s)} = \frac{1}{s+1} \tag{3}$$

$$\frac{y_1(s)}{u_2(s)} = \frac{s+2}{s^2+0.1s+10} \tag{4}$$

$$\frac{y_2(s)}{u_2(s)} = \frac{5(s+1)}{(s+2)(s+3)} \tag{5}$$

By introducing the intermediate variables $z_1 - z_4$, these can be rewritten as:

$$\frac{y_1}{u_1} = \frac{10\dot{z}_1 + 10z_1}{\ddot{z}_1 + 0.2\dot{z}_1 + 100z_1} \tag{6}$$

$$\frac{y_1}{u_1} = \frac{\dot{z}_2 + 2z_2}{\ddot{z}_2 + 0.1\dot{z}_2 + 10z_2} \tag{7}$$

$$\frac{y_1}{u_1} = \frac{z_3}{\dot{z}_3 + z_3} \tag{8}$$

$$\frac{y_1}{u_1} = \frac{5\dot{z}_4 + 5z_4}{\ddot{z}_4 + 5\dot{z}_4 + 6z_4} \tag{9}$$

By choosing $(\dot{z}_1, z_1, \dot{z}_2, z_2, z_3, \dot{z}_4, z_4)^T$ as our state variables, we arrive at the realization:

$$\mathbf{A} = \begin{bmatrix} -0.2 & -100 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -10 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 10 & 10 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 & 5 \end{bmatrix}$$

$$\mathbf{D} = 0$$

1.2 Controllability and Observability

By using $>>$ $ctrb(A, B)$ and $>>$ $obsv(A, C)$, we find that both the controllability and observability matrices are full rank, so the system is controllable and observable.

1.3 Transmission Zeros and Eigenvalues

Via $>>$ $tzero(A, B, C, D)$, the transmission zeros of the system are:

$$tz = \begin{bmatrix} -0.4840 \pm 3.0020i \\ -1.4003 \pm 0.3046i \\ 0.7523 + 0.0i \end{bmatrix}$$

The eigenvalues of the A matrix are:

$$eig = \begin{bmatrix} -0.1 \pm 9.9995i \\ -0.05 \pm 3.1619i \\ -1.0 + 0.0i \\ -2.0 + 0.0i \\ -3.0 + 0.0i \end{bmatrix}$$

1.4 H2 and Hinf Norms

Via $>> h2norm(sys)$, the H_2 norm is:

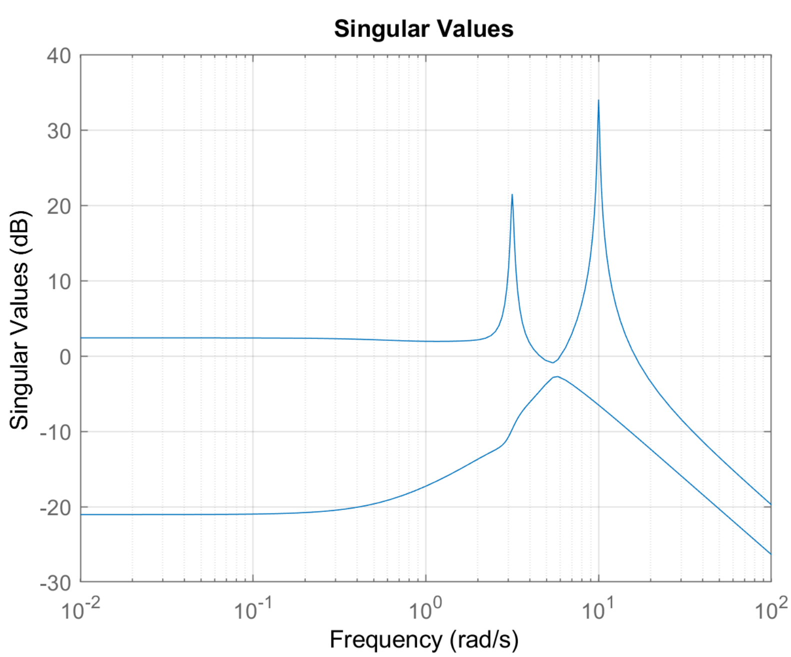
$$H_2 = 16.2147$$

Via $>> G = pck(A, B, C, D); >> hinfnorm(G)$, the H_{inf} norm is:

$$H_{\text{inf}} = 50.2496$$

1.5 Singular Values vs Frequency

The plot of singular values vs frequency is shown below:



2 Question 2

Using the class notes and the appropriate handouts, provide a 1 page max explanation of the relationship between Hamiltonian matrices and algebraic Riccati equations, concentrating on the LQR guaranteed closed loop stability.

2.1 Solution

For the LQR problem, we have the associated Hamiltonian

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q^T & -A^T \end{bmatrix}$$

and algebraic Riccati equation:

$$A^TS + SA + Q - SBR^{-1}B^TS = 0$$

The relationship between the Hamiltonian and the ARE is fundamentally that the Hamiltonian defines a unique, stabilizing solution S of the ARE (assuming a controllable and observable system), which then results in a stabilizing controller which minimizes the cost function associated with the Q, R matrices (and is therefore the solution to the LQR problem).
From the theorems given in the notes and chapter 13 of the Zhou book, the logic that defines this relationship is as such:

- A matrix which solves the ARE can be defined relative to a subspace of the Hamiltonian.
- There can be defined two subspaces of the Hamiltonian: $X_-(H)$ and $X_+(H)$ which correspond to its negative and positive eigenvalues, respectively.
- A matrix S which solves the ARE for $X_-(H)$ is unique, real symmetric, and results in a stable $A + RS$.

From these theorems, we can see that a matrix S that solves the ARE can be defined relative to the subspace of the Hamiltonian which corresponds to the negative eigenvalues of the Hamiltonian, and for this matrix, the matrix $A+RS$ is stable. As $A+RS$ is directly related to the feedback system $A-BK$, finding S is equivalent to finding the desired controller K.

3 Question 3

Given the system:

$$G(s) = \frac{y(s)}{r(s)} = \frac{s+1}{s+10 \pm 0.1\delta}, |\delta| \leq 1 \tag{10}$$

Show that it is equivalent to

$$P(s) = \begin{bmatrix} \frac{-0.1}{s+10} & \frac{-0.1(s+1)}{s(s+1)} \\ \frac{1}{s+10} & \frac{s+1}{s(s+10)} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = P(s) \begin{bmatrix} p \\ r \end{bmatrix} \tag{12}$$

We split $P(s)$ into its component equations:

$$\frac{y(s)}{p(s)} = \frac{1}{s+10} \tag{13}$$

$$\frac{y(s)}{r(s)} = \frac{s+1}{s(s+10)} \tag{14}$$

$$\frac{z(s)}{p(s)} = \frac{-0.1}{s+10} \tag{15}$$

$$\frac{z(s)}{r(s)} = \frac{-0.1(s_1 0)}{s(s+10)} \tag{16}$$

Recognizing that this is an upper fractional transformation, we apply:

$$\begin{aligned} F(M, \Delta) &= M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \\ &= \frac{s+1}{s(s+10)} + \frac{s+1}{s(s+10)}\Delta(1 - \frac{0.1\Delta}{s+10})^{-1}\frac{-0.1(s+1)}{s(s+10)} \\ &= \frac{s+1}{s+10} + \frac{-0.1\Delta(s+1)}{s(s+10)(s+10+0.1\Delta)} \\ &= \frac{(s+1)(s+10+0.1\Delta)}{s(s+10)(s+10+0.1\Delta)} + \frac{-0.1\Delta(s+1)}{s(s+10)(s+10+0.1\Delta)} \\ &= \frac{s+1}{s(s+10+0.1\Delta)} \end{aligned}$$

which is equivalent to the given system with the exception of an extra integrator.