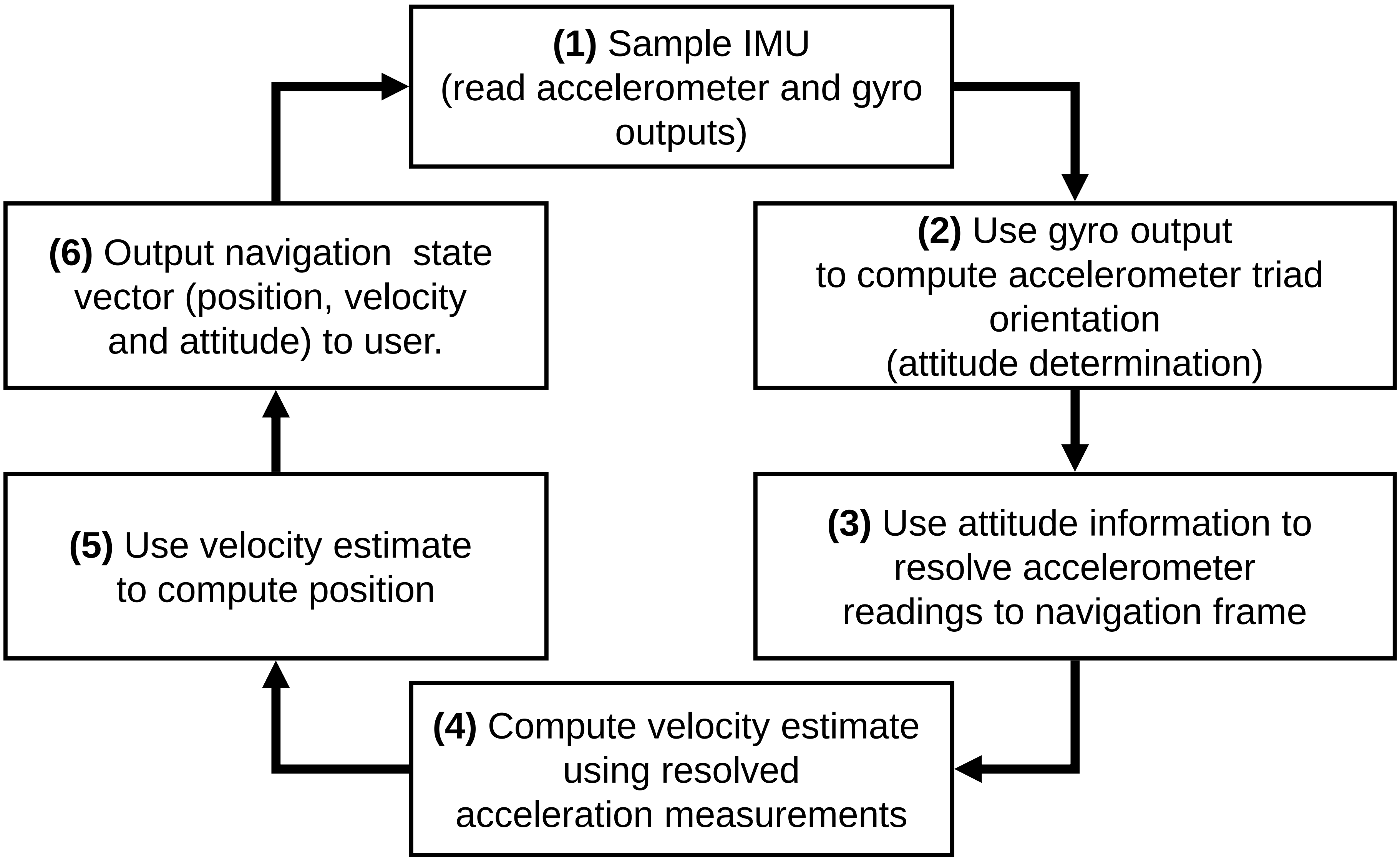
Spherical and Ellipsoidal Earth INS Equation

AEM 8442

Handout #19

Basic Inertial Navigation Computation Flow



INS Equations (Flat, non-Rotating Earth)

(1) Get IMU samples at $t = t_k$

$$\mathbf{f}^b(t_k) \text{ and } \boldsymbol{\omega}_{ib}^b(t_k)$$

(2) Determine accelerometer attitude

$$\psi(t_k), \quad \theta(t_k), \quad \phi(t_k), \rightarrow \mathbf{C}_b^n$$

(3) Resolve $\mathbf{f}^b$ to navigation frame

$$\mathbf{f}^n = \mathbf{C}_b^n \mathbf{f}^b$$

(4) Compute $\mathbf{v}^n$

$$\dot{\mathbf{v}}^n = \mathbf{f}^n + \mathbf{g}^n = \mathbf{a}^n \rightarrow \mathbf{v}^n(t_k) = \mathbf{v}^n(t_{k-1}) + \Delta t \times \mathbf{a}^n(t_{k-1})$$

(5) Compute $\mathbf{p}^n$

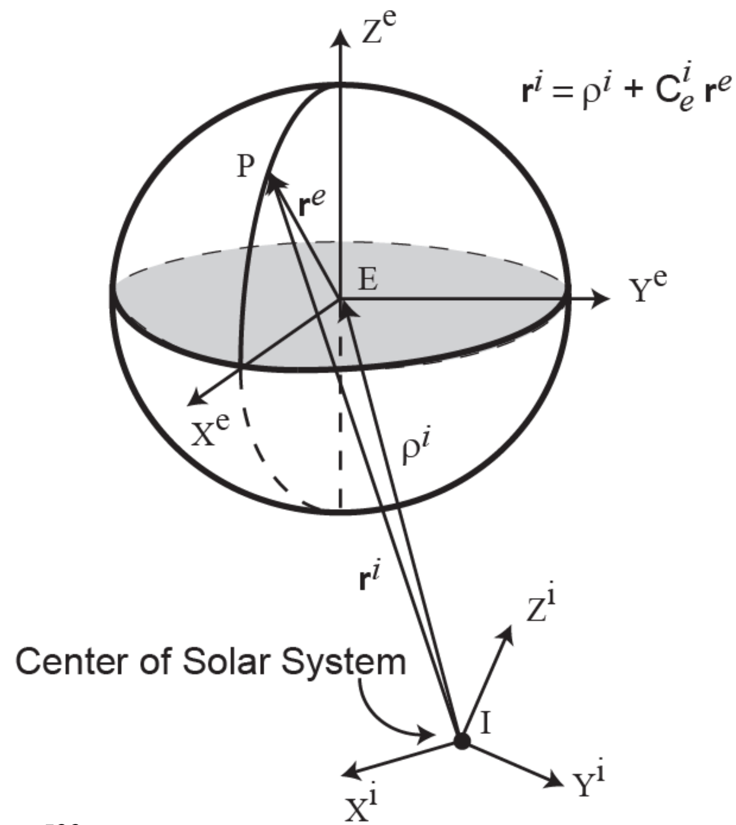
$$\mathbf{p}^n(t_k) = \mathbf{p}^n(t_{k-1}) + \Delta t \times \mathbf{v}^n(t_{k-1})$$

(6) Output navigation state vector (position, velocity and attitude) to user.

Equations in the Inertial (i) Frame

- Consider the user at point P. Let us divide the forces that are applied to the user into two groups
  - Forces due to gravitation (mass attraction as described by Newton's law) = $\mathbf{F}_\gamma$ (not gravity)
  - Other forces = $\mathbf{F}_r$
- Assuming a constant mass, Newton's 2nd now becomes:

$$\sum \mathbf{F} = \mathbf{F}_\gamma + \mathbf{F}_r = m\ddot{\mathbf{r}}^i$$



- But from Newton's *Law of Gravitation* we can write $\mathbf{F}_\gamma$ as follows:

$$\mathbf{F}_\gamma = k\frac{m \cdot m_{Earth}}{r^3}\mathbf{r} = m\boldsymbol{\gamma} \quad \text{where} \quad \boldsymbol{\gamma} = k\frac{m_{Earth}}{r^3}\mathbf{r}$$

- Rearranging,

$$\ddot{\mathbf{r}}^i = \frac{\mathbf{F}_r}{m} + \boldsymbol{\gamma} = \mathbf{f} + \boldsymbol{\gamma} \quad \text{where} \quad \mathbf{f} \equiv \text{Specific Force}$$

Equations in the Earth (e) Frame

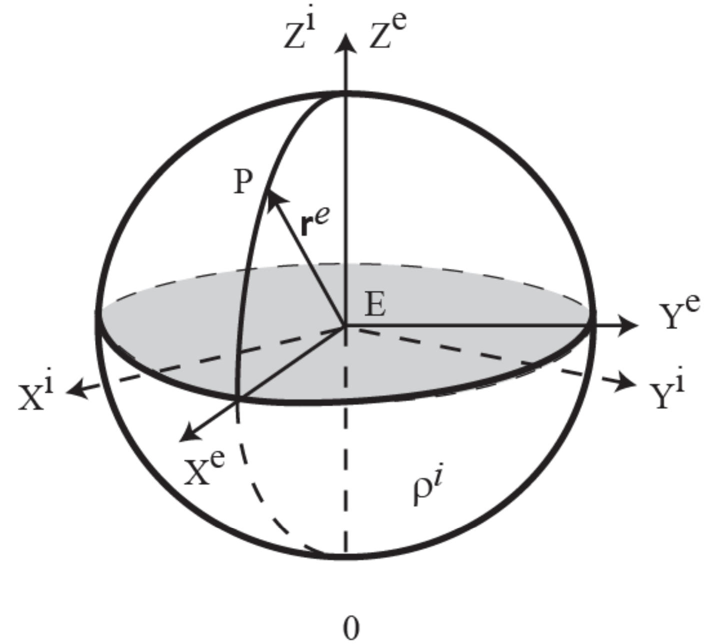
- Let us now observe the same situation from a coordinate frame that is moving with Earth. Summation of forces still remains the same.

$$\sum \mathbf{F} = \mathbf{F}_r + \mathbf{F}_g = m\ddot{\mathbf{r}}^i \rightarrow \ddot{\mathbf{r}}^i = \mathbf{f} + \boldsymbol{\gamma}$$

- The transport theorem from mechanics relates derivatives of vectors expressed in different coordinate frames.

$$\left(\frac{d(\bullet)}{dt}\right)^a = \left(\frac{d(\bullet)}{dt}\right)^b + \boldsymbol{\omega}_{ab}^b \times (\bullet)$$

- Moving the origin of the inertial frame to the center of Earth (i.e. $\rho^i = \mathbf{0}$) and applying the transport theorem to the equations on the previous slide



$\mathbf{r}^i = \rho^i + \mathbf{C}_e^i \mathbf{r}^e$

$$\left(\frac{d\mathbf{r}}{dt}\right)^i = \left(\frac{d\mathbf{r}}{dt}\right)^e + \boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e$$

$$\dot{\mathbf{r}}^i = \dot{\mathbf{r}}^e + \boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e$$

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)^i = \left(\frac{d^2\mathbf{r}}{dt^2}\right)^e + \boldsymbol{\omega}_{ie}^e \times \left(\frac{d\mathbf{r}}{dt}\right)^e$$

$$= \ddot{\mathbf{r}}^i$$

$$= \mathbf{f} + \boldsymbol{\gamma}$$

Equations in the Earth (e) Frame (2)

- Working with the second derivative equations some more:

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)^i = \ddot{\mathbf{r}}^i = \left(\frac{d^2\mathbf{r}}{dt^2}\right)^e + \boldsymbol{\omega}_{ie}^e \times \left(\frac{d\mathbf{r}}{dt}\right)^e$$

$$= \left(\ddot{\mathbf{r}}^e + \dot{\boldsymbol{\omega}}_{ie}^e \times \mathbf{r}^e + \boldsymbol{\omega}_{ie}^e \times \dot{\mathbf{r}}^e\right) + \boldsymbol{\omega}_{ie}^e \times \left(\dot{\mathbf{r}}^e + \boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right)$$

$$= \ddot{\mathbf{r}}^e + 2\boldsymbol{\omega}_{ie}^e \times \dot{\mathbf{r}}^e + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right) \quad \text{becasue } \dot{\boldsymbol{\omega}}_{ie}^e \approx \mathbf{0} \text{ (why?)}$$

$$= \mathbf{f}^e + \boldsymbol{\gamma}^e$$

- Noting that $\dot{\mathbf{r}}^e \equiv \mathbf{v}^e$ and, thus, $\ddot{\mathbf{r}}^e = \dot{\mathbf{v}}^e$ rearranging gives:

$$\dot{\mathbf{v}}^e = \mathbf{f}^e + 2\boldsymbol{\omega}_{ie}^e \times \mathbf{v}^e + \boldsymbol{\gamma} + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right)$$

- If we consider a static user for the moment then

$$\mathbf{0} = \mathbf{f}^e + \boldsymbol{\gamma}^e + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right) \rightarrow \mathbf{f}^e = \boldsymbol{\gamma}^e + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right)$$

$$\mathbf{g}^e \equiv \boldsymbol{\gamma}^e + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e\right) \quad \text{or}$$

$$\text{gavity} = \text{gavitation} + \text{centripetal acceleration}$$

Gravity and a Simple Gravity Model

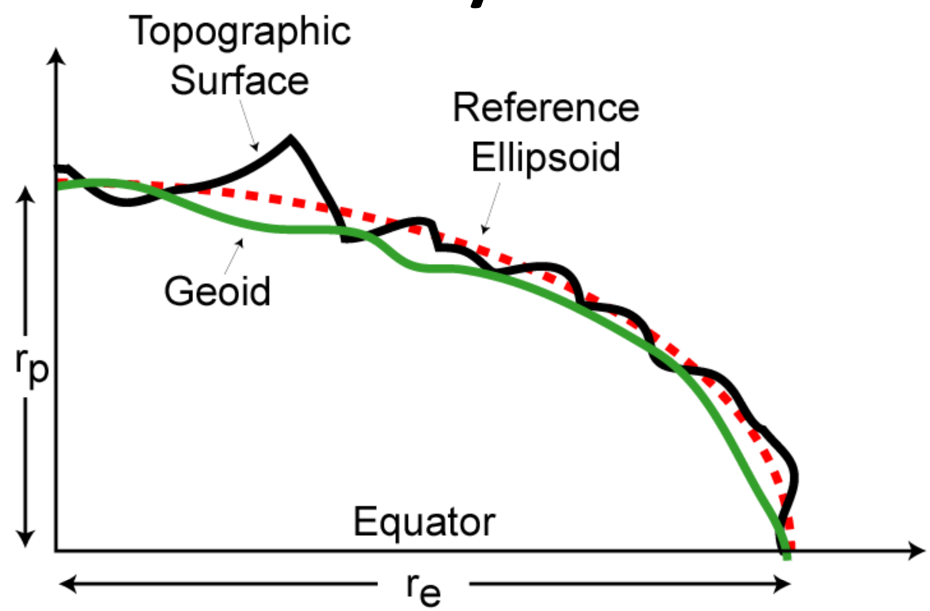
$$\mathbf{g}^e \equiv \boldsymbol{\gamma}^e + \boldsymbol{\omega}_{ie}^e \times \left(\boldsymbol{\omega}_{ie}^e \times \mathbf{r}^e \right)$$

- We note that the gravity vector is the combination of mass attraction (or gravitation) and centripetal acceleration.
- Both components are functions of location:
  - $\boldsymbol{\gamma}^e$ changes because Earth's density is not uniform.
  - $\mathbf{r}^e$ changes because Earth is not a sphere.
- WGS-84 provides a simple gravity model of the following form for a North East Down coordinate system:

$$g_d^n = 9.7803253359 \frac{1 + 1.931853 \times 10^{-3} \sin^2 L}{\sqrt{1 - e^2 \sin L}}$$

where L is geodetic latitude and e is the eccentricity of the WGS-84 reference ellipsoid

Geometry of Earth



- Topographic Surface
  - Shape assumed by Earth’s crust.
  - Very complicated shape not amenable to mathematical modeling
- Geoid
  - An equipotential surface of Earth's gravity field which best fits, in a least squares sense, global Mean Sea Level (MSL).
- Reference Ellipsoid
  - Mathematical fit to the geoid that happens to be an ellipsoid of revolution and minimizes the mean-square deviation of local gravity and the normal to the ellipsoid

Gravity Errors

- Since Earth is not a homogeneous ellipsoid, the magnitude of the local gravity vector may be different from what is predicted a model.
  - This is called the gravity anomaly.
- In addition, gravity is not always perpendicular to the local tangent.
  - This is called the "deflection of the vertical" or DOV
- Mathematically, these errors are as shown below:

$$\mathbf{g}^n_{corrected} = \mathbf{g}^n + \begin{bmatrix} \xi\|\mathbf{g}^n\| \\ -\eta\|\mathbf{g}^n\| \\ \Delta g \end{bmatrix}$$

ξ and η represents deflections in the North/South and East/West directions, respectively

- The deflections can be several arc-seconds. The combined effect can result in g errors on the order of ten μg
  - This is larger than the output errors of some high quality accelerometers

Velocity Equations (NED)

- The velocity equation is very similar to the one we derived in Week #2 except for the fact that now we are using the NED coordinates frame as our navigation frame. Thus, we have:

$\mathbf{C}_b^n \equiv$ Body to NED Direction Cosine Matrix

$\mathbf{f}^b \equiv$ Accelerometer Output

(Specific Force in Body Coordinates)

$\boldsymbol{\omega}_{ie}^n \equiv$ Earth Rate in NED Coordinates

$\boldsymbol{\omega}_{en}^n \equiv$ Transport Rate in NED Coordinates

$\mathbf{g}^n \equiv$ Local Gravity Vector in NED Coordinates

$$\boxed{\dot{\mathbf{v}}^n = \begin{bmatrix} \dot{v}_N \\ \dot{v}_E \\ \dot{v}_D \end{bmatrix} = \mathbf{C}_b^n \mathbf{f}^b - \left(2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n\right) \times \mathbf{v}^n + \mathbf{g}^n}$$

- The second term (in parenthesis) on the right hand side of the equation is a new term. It is represents the Coriolis acceleration:
  - It is present because the navigation frame is non-inertial (rotates)
  - The term $\boldsymbol{\omega}_{en}$ is the rotation rate (angular velocity) of the navigation frame relative to the Earth frame. It is known as the *transport rate.*

Using the DCM Directly

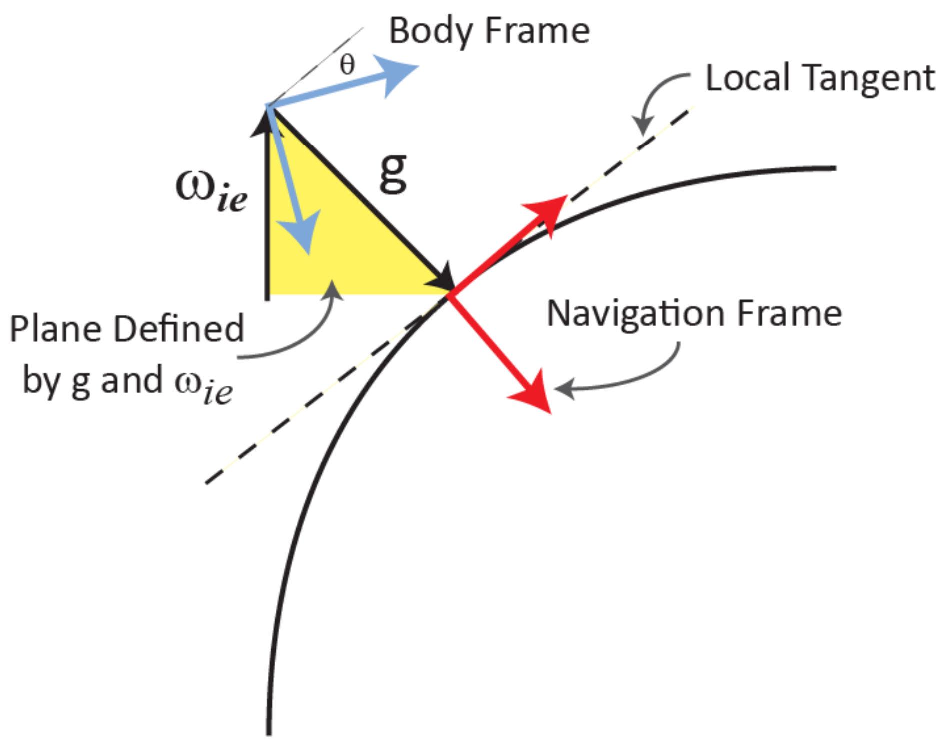
- While Euler angles are intuitive, they are cumbersome for actual computations because:
  - Singular at some points (3-2-1 sequence has θ = 90° singularity)
  - Equations relating Euler angles to gyro outputs (Euler angle kinematic differential equation) contains transcendental functions (relatively speaking, computationally expensive).
  - Two step process: (1) Compute Euler angles (2) Compute DCM or transformation matrix.
- Can we relate gyro outputs directly to the DCM?
  - Yes we can as shown below:

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \mathbf{\Omega}_{nb}^b \quad \text{where} \quad \mathbf{\Omega}_{nb}^b = \begin{bmatrix} 0 & -\omega_{nb,z}^b & \omega_{nb,y}^b \\ \omega_{nb,z}^b & 0 & -\omega_{nb,x}^b \\ -\omega_{nb,y}^b & \omega_{nb,x}^b & 0 \end{bmatrix}$$

$$\mathbf{C}_b^n(t_k) = \mathbf{C}_b^n(t_{k-1}) + \Delta t \times \dot{\mathbf{C}}_b^n = \left(\mathbf{I}_{3\times 3} + \Delta t \times \mathbf{\Omega}_{nb}^b(t_{k-1})\right)\mathbf{C}_b^n(t_{k-1})$$

Initial Condition for DCM (Alignment)

- Recall from last time when we discussed the alignment process
  - Alignment = Establishing initial conditions for attitude
  - Leveling = Determining pitch (θ) and roll (ϕ) from accelerometer readings
  - Gyrocompassing = Determining yaw/heading (ψ) from gyro's measurements of Earth's rotation rate.

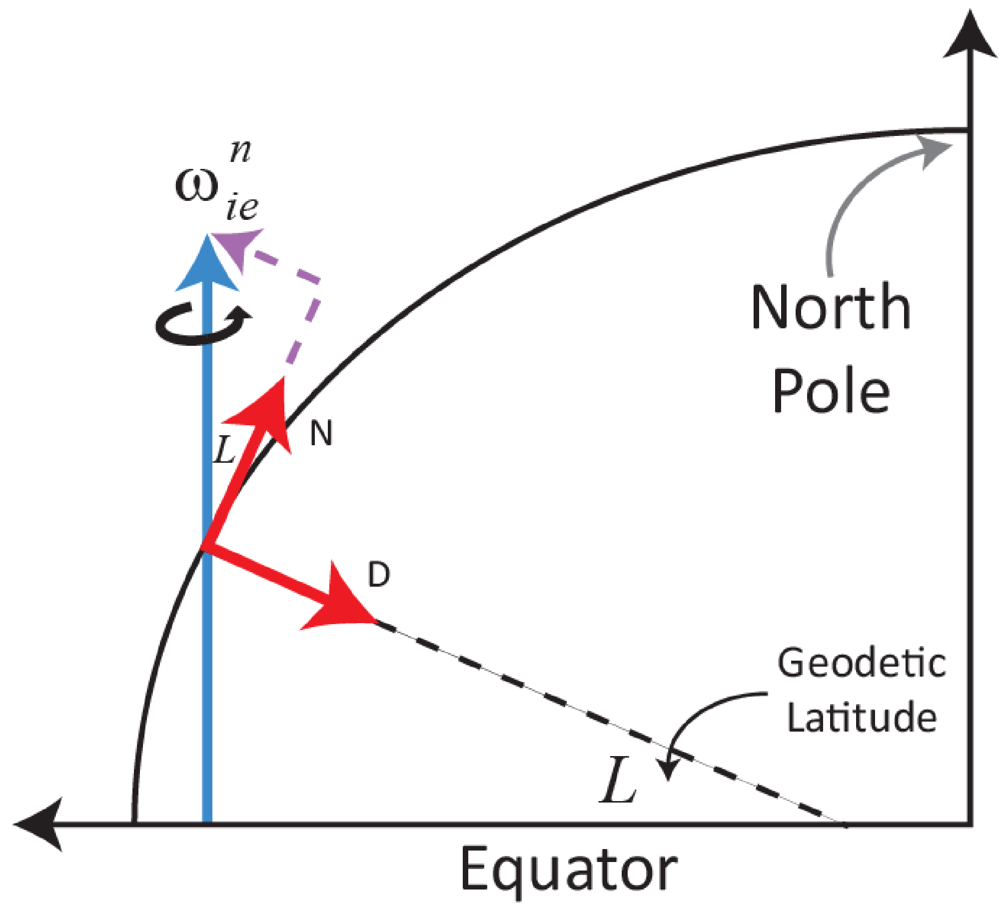


- We can establish the initial condition for attitude at once (i.e., without having to break up the process into leveling and gyro- compassing) by directly estimating the initial DCM.
  - Two non-collinear vectors define a plane.
  - Knowing components of these vectors in both frames allows attitude estimation.

Earth Rate

- Earth rate is a the rotation rate of Earth relative to inertial space.
- It is the combination of Earth's rotation about its axis and Earth's rotation about sun.
- Its magnitude, ω_{ie}, is equal to 7.292115 x 10^{-5} rad/sec = 15.042 deg/hr.
- In North-East-Down coordinates this is given by:

$$\boldsymbol{\omega}_{ie}^{n} = \begin{bmatrix} \omega_{ie}\cos(L) \\ 0 \\ -\omega_{ie}\sin(L) \end{bmatrix}$$



Attitude Estimation from $\mathbf{g}$ and ω_{ie}

- Assume vehicle (body frame) is static. Thus,

$$\mathbf{a} = \mathbf{f} + \mathbf{g} = 0 \rightarrow \mathbf{f} = -\mathbf{g}$$

- That is, the accelerometer readings will be equal the negative of $\mathbf{g}$. Also let us define a third vector, β, perpendicular to both ω_{ie} and $\mathbf{g}$: $\beta = \omega_{ie} \times \mathbf{g}$

$$\mathbf{g}^n = \mathbf{C}_b^n \mathbf{g}^b = -\mathbf{C}_b^n \mathbf{f}^b$$

$$\boldsymbol{\omega}_{ie}^n = \mathbf{C}_b^n \boldsymbol{\omega}_{ie}^b \quad \text{where } \boldsymbol{\omega}_{ie}^b \text{ is the gyro output.}$$

$$\boldsymbol{\beta}^n = \mathbf{C}_b^n \boldsymbol{\beta}^b$$

- We can arrange all of this into the following matrix equation where $\mathbf{K}$ = known quantities and $\mathbf{M}$ = Measured quantities (inertial sensor outputs):

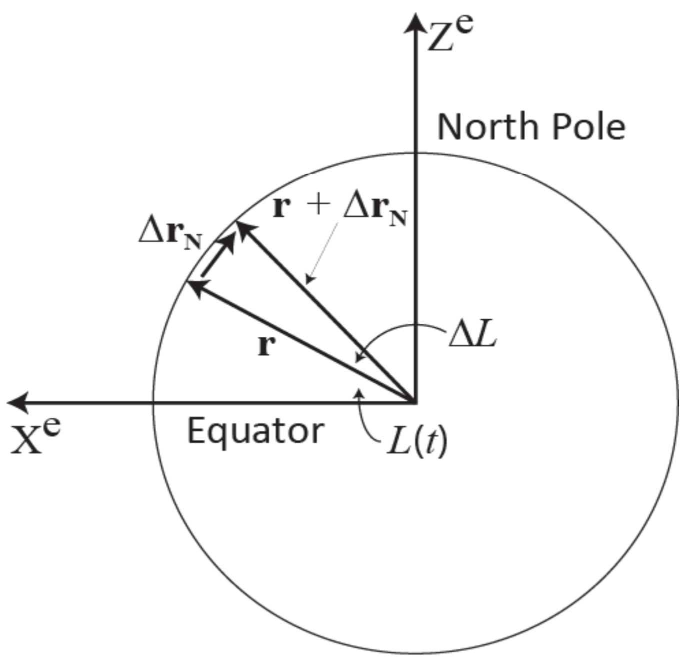
$$\begin{bmatrix} \mathbf{g}^n & \boldsymbol{\omega}_{ie}^n & \boldsymbol{\beta}^n \end{bmatrix} = \mathbf{C}_b^n \begin{bmatrix} -\mathbf{f}^b & \boldsymbol{\omega}_{ie}^b & \boldsymbol{\beta}^b \end{bmatrix}$$

$$\mathbf{K} = \mathbf{C}_b^n \mathbf{M}$$

$$\mathbf{C}_b^n = \mathbf{K} \times \mathbf{M}^{-1}$$

Spherical Earth: Latitude Rate

- Since we are interested in navigation on or close to the surface of Earth, the position vector is (directly or indirectly) relative to the e frame.
- Thus, it makes sense to use geodetic or ECEF coordinates for position.
- Given velocities relative to the e frame expressed in an NED coordinate system, the latitude rate differential equation becomes.



Latitude Rate

$$\Delta r_N = R \cdot \Delta L$$

$$\lim_{\Delta t \to 0} \frac{\Delta r_N}{\Delta t} = \lim_{\Delta t \to 0} R \frac{\Delta L}{\Delta t}$$

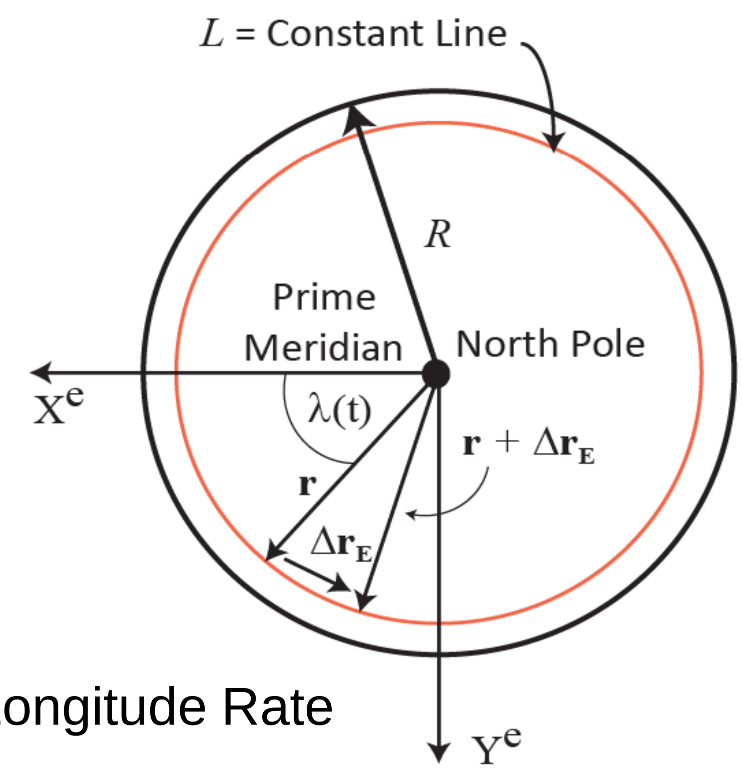
$$\dot{r}_N = R \times \dot{L} \equiv v_N$$

$$\dot{L} = \frac{v_N}{R}$$

R = Radius of a spherical Earth

Spherical Earth: Longitude Rate

- The longitude rate equation can be derived in a similar manner as the latitude rate equation.
- The only difference is to note that instead of the radius of Earth, R, we use $R\cos(L)$ because the radius of curvature gets smaller as we travel north or south away from the equator.



Longitude Rate

$$\Delta r_E = R\cos(L)\cdot\Delta\lambda$$

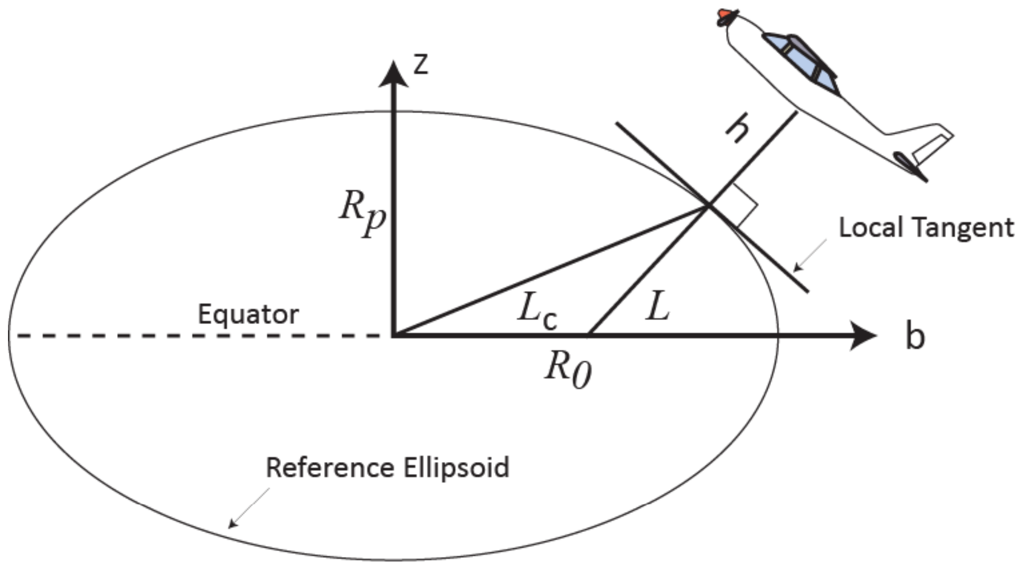
$$\lim_{\Delta t\to 0}\frac{\Delta r_E}{\Delta t} = R\cos(L)\cdot\lim_{\Delta t\to 0}\frac{\Delta\lambda}{\Delta t}$$

$$\dot{r}_E = R\cos(L)\dot{\lambda}$$

$$\equiv v_E$$

$$\dot{\lambda} = \frac{v_E}{R\cos(L)}$$

Reference Ellipsoid and Radii of Curvature



$$\tan L_c = \left(1 - e^2\right)\tan L$$

$$R_N = \frac{R_0\left(1-e^2\right)}{\left(1-e^2\sin^2 L\right)^{3/2}}$$

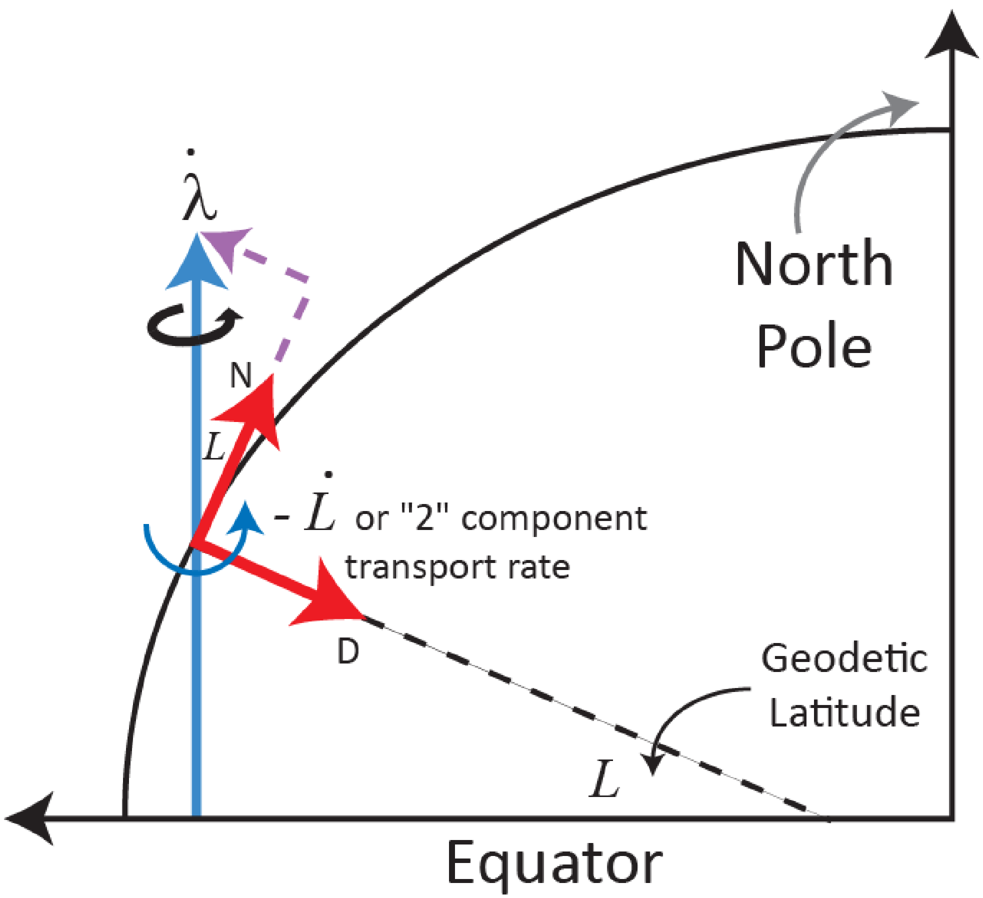
$$R_E = \frac{R_0}{\left(1-e^2\sin^2 L\right)^{1/2}}$$

$$\dot{L} = \frac{v_N}{R_N}, \quad \dot{\lambda} = \frac{v_E}{R_E\cos(L)}, \quad \dot{h} = -v_D$$

- Since Earth is not a sphere but an oblate spheroid, we must use the appropriate radii of curvature for the latitude and longitude rate equations.
- For the latitude rate we use the North/South radius of curvature, R_N, and for the longitude rate we use East/West radius of curvature, R_E.
  - R_0 = Equatorial Radius of WGS-84 Reference Ellipsoid = 6378137 m
  - e = Eccentricity of the WGS-84 reference ellipsoid = 0.0818191908426

The Transport Rate (ω_{en})

- The transport rate is the angular velocity of the navigation frame relative to the Earth frame.

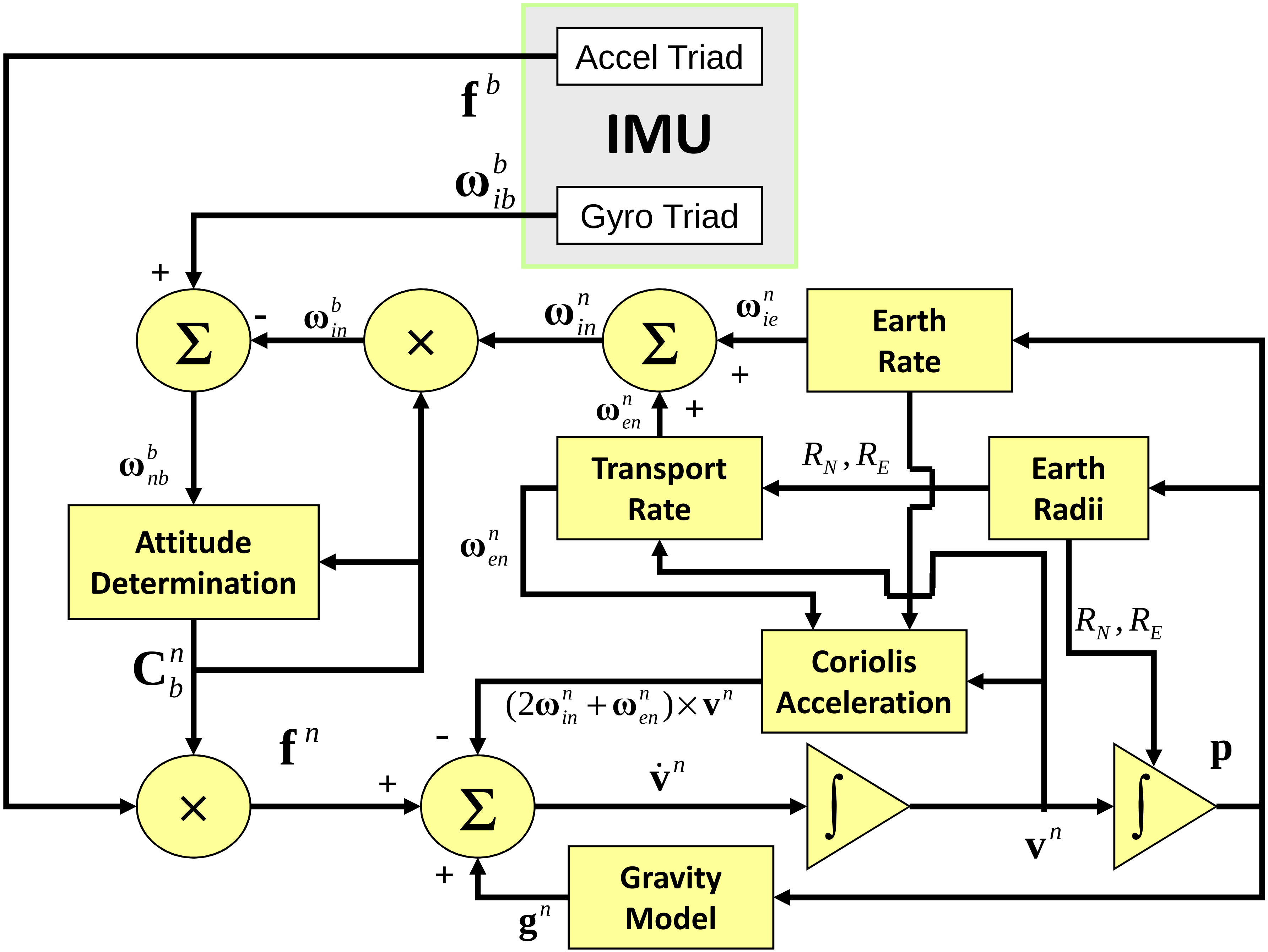


$$\boldsymbol{\omega}_{en}^{n} = \begin{bmatrix} \omega_{en,N} \\ \omega_{en,E} \\ \omega_{en,D} \end{bmatrix}$$

$$\omega_{en,N} = \frac{v_E}{R_E} = \dot{\lambda}\cos(L)$$

$$\omega_{en,E} = -\frac{v_N}{R_N} = -\dot{L}$$

$$\omega_{en,D} = -\frac{v_E}{R_E}\tan(L) = -\dot{\lambda}\sin(L)$$


Accel Triad
IMU
Gyro Triad
$\mathbf{f}^b$
$\boldsymbol{\omega}_{ib}^b$
+
-
$\boldsymbol{\omega}_{in}^b$
$\boldsymbol{\omega}_{in}^n$
$\boldsymbol{\omega}_{ie}^n$
+
Earth Rate
$\boldsymbol{\omega}_{en}^n$
+
Transport Rate
R_N, R_E
Earth Radii
$\boldsymbol{\omega}_{nb}^b$
Attitude Determination
$\boldsymbol{\omega}_{en}^n$
R_N, R_E
Coriolis Acceleration
$\mathbf{C}_b^n$
$(2\boldsymbol{\omega}_{in}^n + \boldsymbol{\omega}_{en}^n) \times \mathbf{v}^n$
$\mathbf{f}^n$
-
+
$\dot{\mathbf{v}}^n$
$\mathbf{p}$
$\mathbf{v}^n$
+
$\mathbf{g}^n$
Gravity Model

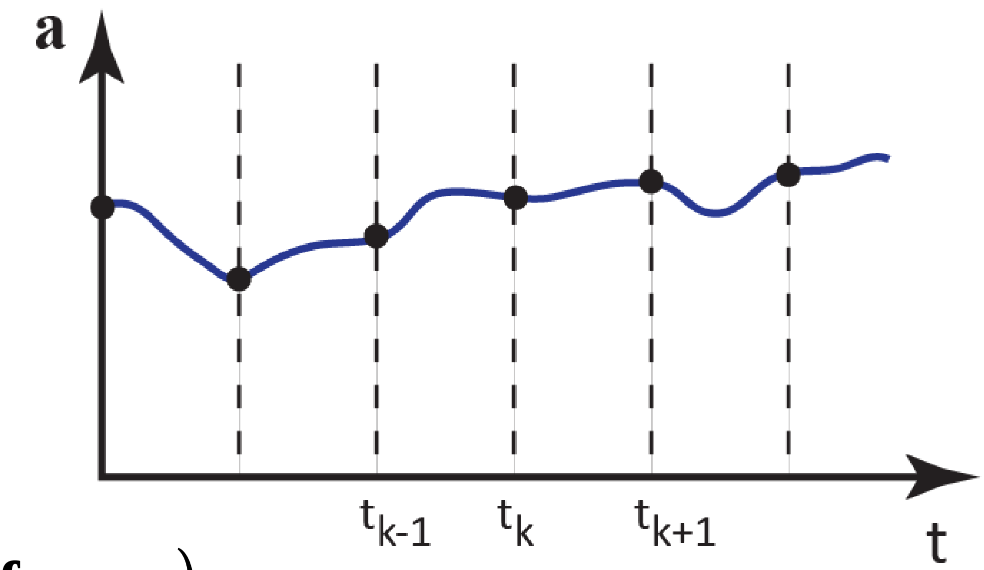
Euler Integration

- The simplest type of integration you can use.
  - Not very accurate
  - Do not use unless inertial sensor sampling rate is very high compared to the frequency content of motion.
- Consider the velocity equation.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$$

$$\mathbf{a}_k \approx \frac{\mathbf{v}_{k+1} - \mathbf{v}_k}{t_{k+1} - t_k}$$

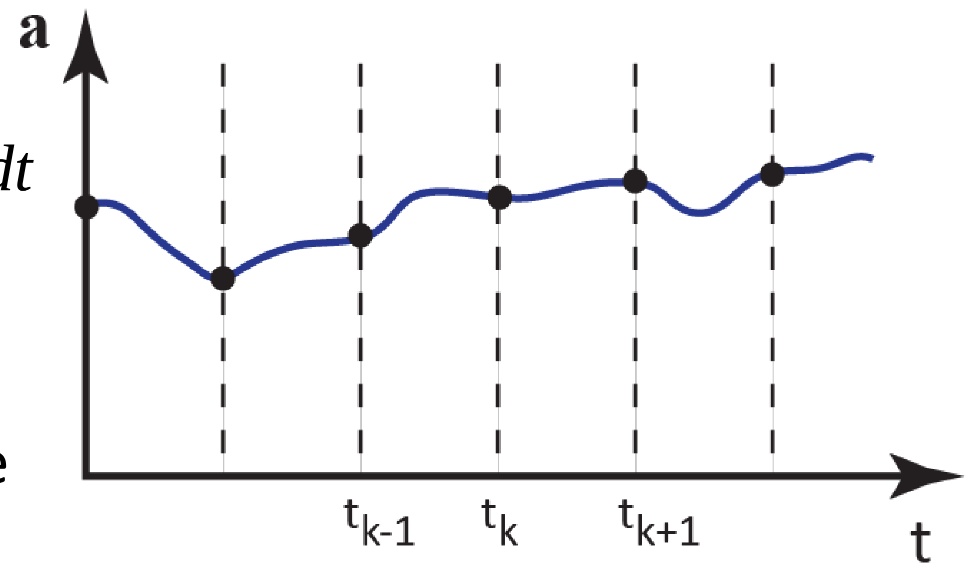


$$\mathbf{v}_{k+1} \approx \mathbf{v}_k + \Delta t\, \mathbf{a}_k = \mathbf{v}_k + \Delta t \left(\mathbf{f}_k + \mathbf{g}_k \right)$$

Trapezoidal Integration

- An improvement over the Euler integration.
- Uses two samples of the input (in this case, acceleration) to generate an estimate.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \rightarrow \mathbf{v}(t_{k+1}) - \mathbf{v}(t_k) = \int_{t_k}^{t_{k+1}} \mathbf{a}\, dt$$



- Noting that

$$\int_{t_k}^{t_{k+1}} \mathbf{a}\, dt = \text{Area under } \mathbf{a}(t) \text{ curve}$$

$$\approx \frac{\Delta t}{2}\left(\mathbf{a}_k + \mathbf{a}_{k+1}\right)$$

$$\mathbf{v}(t_{k+1}) \approx \mathbf{v}(t_k) + \frac{\Delta t}{2}\left(\mathbf{a}_k + \mathbf{a}_{k+1}\right) = \mathbf{v}(t_k) + \frac{\Delta t}{2}\left[\left(\mathbf{f}_{k+1} + \mathbf{g}_{k+1}\right) + \left(\mathbf{f}_k + \mathbf{g}_k\right)\right]$$

4^{th} Order Runge-Kutta

- An improvement over the Euler and Trapezoidal integration.
- Consider the following generic vector differential equation:

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(\mathbf{v}, \mathbf{u}(t), t)$$

  – The vector $\mathbf{u}$(t) is a known input. Given time, we know its value. In our case, it is the IMU outputs

- The 4^{th} order Runge-Kutta integration algorithm generates an estimate for v in the following way:

$$\mathbf{v}_{m+1} = \mathbf{v}_m + \frac{h}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right), \quad h = t_{m+1} - t_m$$

$$\mathbf{k}_1 = \mathbf{a}(\mathbf{v}(t_m), \mathbf{u}(t_m), t_m)$$

$$\mathbf{k}_2 = \mathbf{a}(\mathbf{v}(t_m) + \frac{h\mathbf{k}_1}{2}, \mathbf{u}(t_m + \frac{h}{2}), t_m + \frac{h}{2})$$

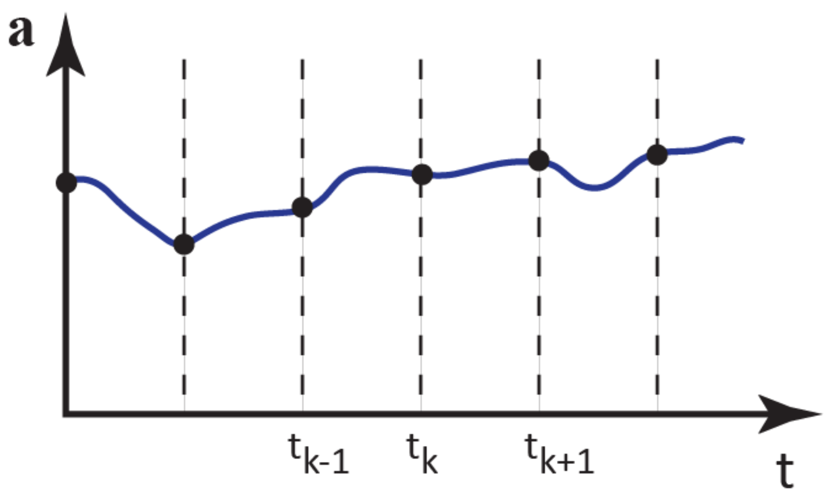
$$\mathbf{k}_3 = \mathbf{a}(\mathbf{v}(t_m) + \frac{h\mathbf{k}_2}{2}, \mathbf{u}(t_m + \frac{h}{2}), t_m + \frac{h}{2})$$

$$\mathbf{k}_4 = \mathbf{a}(\mathbf{v}(t_m) + k\mathbf{k}_3, \mathbf{u}(t_m + h), t_m + h)$$

4^{th} Order Runge-Kutta

- An improvement over the Euler and Trapezoidal integration.
- Uses three samples of the input (in this case, acceleration) to generate an estimate.

$$\mathbf{v}(t_{k+1}) = \mathbf{v}(t_{k-1}) + \frac{\Delta t}{2}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right)$$



$$\mathbf{k}_1 = \left.\frac{d\mathbf{v}}{dt}\right|_{k-1} = \mathbf{a}_{k-1}$$

$$\mathbf{k}_2 = \mathbf{a}_k = \mathbf{k}_3$$

$$\mathbf{k}_4 = \mathbf{a}_{k+1}$$

- Note that while for this simple case $\mathbf{k}_2$ and $\mathbf{k}_3$ are equal, that is not always the case.
- Also, note the relation between sampling rate and navigation state vector output rate.

Summary

- There are several ways to parameterize attitude. The ultimate purpose of attitude determination, however, is to determine the transformation matrix that enables expressing specific forces measured in the body frame in navigation frame components.
- Alignment is the process of establishing initial conditions for an INS. Leveling establishes the local vertical and gyrocompasing establishes heading.
- Navigating on the reference ellipsoid introduces more terms into the velocity equations. These terms are transport rate and Coriolis acceleration.
- Once velocity is known, latitude rate and longitude rate can be computed.