Signals and Systems

Tuesday, February 11, 2020 7:17 AM

Read 4.1-4.2

Two-dimensional signals + systems

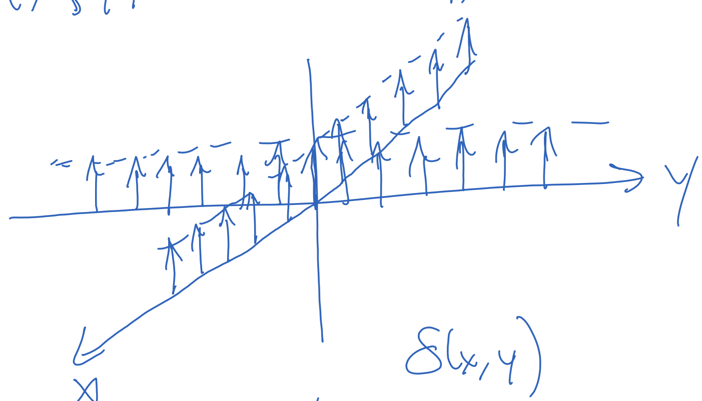
continuous signals + systems: basic signals

Dirac delta: $\delta(x,y) = 0 \quad , x, y \neq 0$

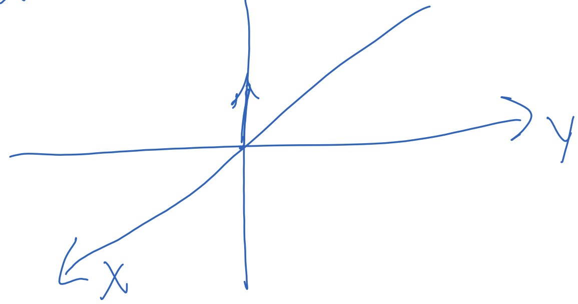
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x-x_0, y-y_0)\,dx\,dy = f(x_0,y_0)$$

(sifting property)

$$\delta(x,y) = \delta(x)\,\delta(y)$$



$\delta(x,y)$



line impulse — 1-D impulse plotted in 2-D — looks like a line

(x is horizontal) $\delta_v(x,y) = \delta(x)$ for all x, y

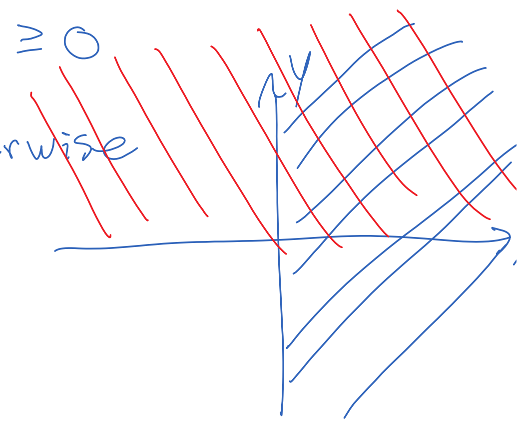
$\delta_h(x,y) = \delta(y)$ for all x, y

line impulse at angle θ:

$\delta_\theta(x,y) = \delta(x \tan\theta - y)$ for all x, y

$\Rightarrow$ turns on when $y = x \tan\theta$

step function: $\begin{cases} 1, & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$

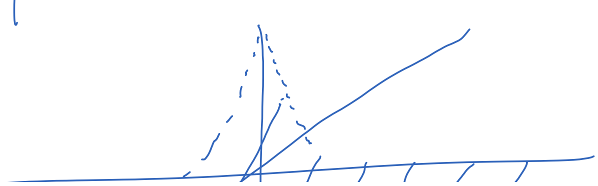


$= u(x)\,u(y)$

decaying exponential:

$$f(x,y) = \begin{cases} \exp\{-\alpha x - \beta y\}, & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$\alpha, \beta > 0$



Sinusoid:

$$f(x,y) = \sin(\omega_{0x} x + \omega_{0y} y)$$

ω_{0x} = freq. along x
ω_{0y} = freq. along y

A <u>separable</u> signal is one that can be factored into a product of two 1-D signals:

$$\delta(x,y) = \delta(x)\,\delta(y)$$

$$u(x,y) = u(x)\,u(y)$$

$$\exp[-\alpha x - \beta y]\,u(x,y) = [e^{-\alpha x} u(x)][e^{-\beta y} u(y)]$$

$$\exp[j(\omega_{0x} x + \omega_{0y} y)] = [e^{j\omega_{0x} x}][e^{j\omega_{0y} y}]$$

Linear systems

If $T[\cdot]$ is a system, it is linear iff

$$T[af(x,y) + bg(x,y)] = aT[f(x,y)] + bT[g(x,y)]$$

for all $a, b, f(x,y), g(x,y)$

— superposition

Ex: film with $T[I] = \log(1+I)$

— not linear

Proof:

$a=b=1$, $f(x,y) = g(x,y) = 1$

$$T[(1)(1)+(1)(1)] = T[2] = \log(1+2) = \log 3$$

$$(1)T[1] + (1)T[1] = 2T[1] = 2\log(1+1) = 2\log 2$$
$$= \log 4$$
$$\neq \log 3$$

$\Rightarrow$ not linear

Ex: optical system that blurs horizontally

$$T[f(x,y)] = \int_0^1 f(x-x', y)\,dx'$$

Show that $T[\cdot]$ is linear.

Must show that superposition holds in general.

$$T[af(x,y) + bg(x,y)] = \int_0^1 [af(x-x',y) + bg(x-x',y)]\,dx'$$

$$= a\int_0^1 f(x-x',y)\,dx' + b\int_0^1 g(x-x',y)\,dx'$$

$$= a\,T[f(x,y)] + b\,T[g(x,y)]$$

$\Rightarrow$ linear

Shift-invariant systems

$$g(x,y) = T[f(x,y)]$$

The system is SI iff

$$g(x-x_0, y-y_0) = T[f(x-x_0, y-y_0)]$$

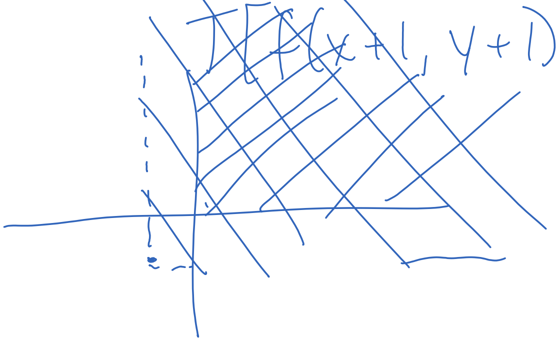
for all (x_0, y_0).

Ex: $g(x,y) = T[f(x,y)] = f(x,y)\,u(x,y)$

Show not SI.

Let $f(x,y) = u(x,y)$ Let $(x_0, y_0) = (-1,-1)$

$$T[f(x+1, y+1)] = f(x+1, y+1)\,u(x,y)$$
$$= u(x+1, y+1)\,u(x,y)$$
$$= u(x,y)$$



$$g(x,y) = f(x,y)\,u(x,y)$$
$$= u(x,y)\,u(x,y)$$
$$= u(x,y)$$

$$g(x+1, y+1) = ?$$
$$= u(x+1, y+1)$$

$\neq u(x,y)$

$\Rightarrow$ not SI

Linear shift-invariant (LSI) systems

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\delta(x-x',y-y')\,dx'dy'$$

(sifting

For linear system,

$$g(x,y) = T[f(x,y)]$$

$$= T\left[\iint f(x',y')\,\delta(x-x',y-y')\,dx'd\right.$$

$$= \iint f(x',y')\,T\left[\delta(x-x',y-y')\right]dx$$

If we define $h(x,y) = T[\delta(x,y)]$, we

$h(x-x',y-y') = T[\delta(x-x',y-y')]$ for

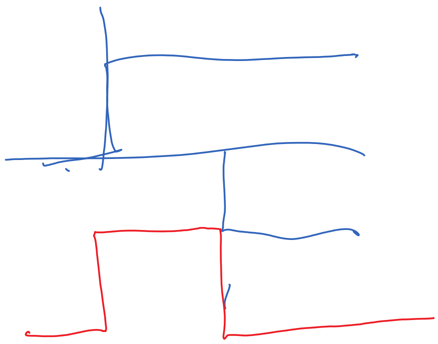
x', y' for a shift-invariant sy

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,h(x-x',y-y')\,dx'dy$$

$h(x,y)$ = is ~~f(x,y)~~ (the) ~~impulse~~ (impulse) response. convolut

Ex: horizontal blur

$$h(x,y) = ?$$



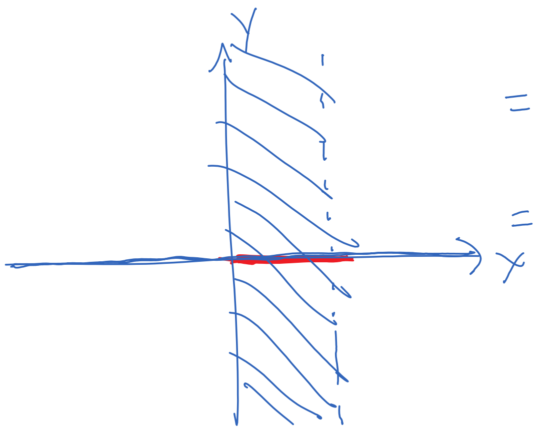
$$= T[\delta(x,y)]$$

$$= \int_0^1 \delta(x-x', y)\, dx'$$

$$= \int_0^1 \delta(x-x')\,\delta(y)\, dx'$$

$$= \delta(y) \int_0^1 \delta(x-x')\, dx'$$

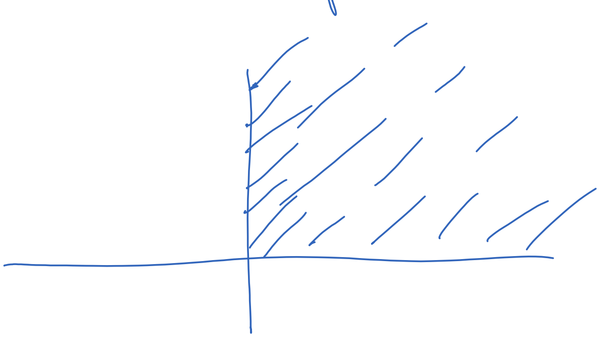
$$= \delta(y)\left[u(x) - u(x-1)\right]$$

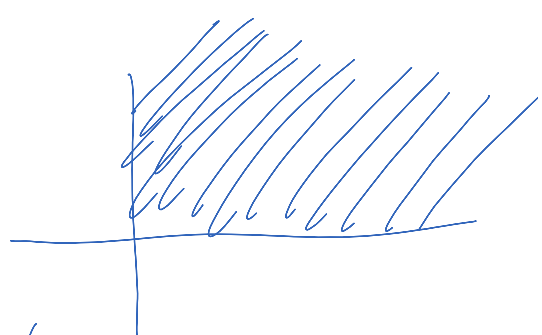


Read 4.3-4.5

HW #1 posted

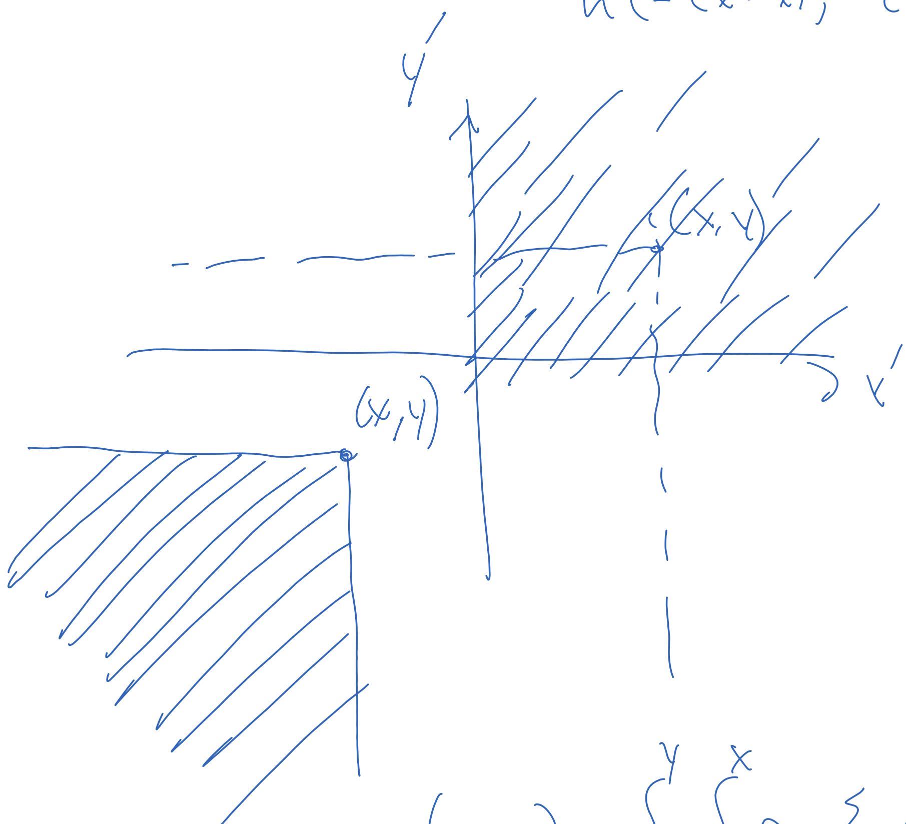
Ex: $\exp\{-\alpha x - \beta y\}\, u(x,y) * u(x,y)$





$$g(x,y) = \iint f(x',y')\, \underbrace{h(x-x', y-y')}\, dx'dy'$$

$h(-(x'-x), -(y'-y))$



$g(x,y) = 0,\ x<0$ or $y<0$

$$g(x,y) = \int_0^y \int_0^x \exp\{-\alpha x' - \beta y'\} dx' dy'$$

$$= \int_0^y e^{-\beta y'} dy' \int_0^x e^{-\alpha x'} dx'$$

$$= -\frac{1}{\beta} e^{-\beta y'}\Big|_0^y \; -\frac{1}{\alpha} e^{-\alpha x'}\Big|_0^x$$

$$= \frac{1}{\alpha\beta}\left[e^{-\beta y} - 1\right]\left[e^{-\alpha x} - 1\right]$$

$$g(x,y) = \frac{1}{\alpha\beta}\left[1 - e^{-\alpha x}\right]\left[1 - e^{-\beta y}\right] u(x,y)$$

Fourier transform

$\infty \quad \infty$

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \exp\{-j(\omega_x x + \omega_y y)\} dx\, dy$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x, \omega_y) \exp\{j(\omega_x x + \omega_y y)\} d\omega_x\, d\omega_y$$

$$f(x,y) \longleftrightarrow F(\omega_x, \omega_y)$$

$$\mathcal{F}\{f(x,y)\} = F(\omega_x, \omega_y)$$

Ex: FT of $\delta(x - x_0, y - y_0)$

$$F(\omega_x, \omega_y) = \int\int \delta(x - x_0, y - y_0) \exp\{-j(\omega_x x + \omega_y y)\} dx\, dy$$

$$= \exp\{-j(\omega_x x_0 + \omega_y y_0)\}$$

$\cancel{*}$ FT of $\delta(x-1, y) + \delta(x, y-2)$?

~~FT Properties~~

linearity

$$a f(x,y) + b g(x,y) \longleftrightarrow a F(\omega_x, \omega_y) + b G(\omega_x, \omega_y)$$

$(x_0, y_0) = (1, 0)$ $\quad (x_0, y_0) = (0, 2)$

Ex: ~~FT of $\delta(x-1, y) + \delta(x, y$~~$-2)$ by linearity is

$\exp\{-j1\omega_x\} + \exp\{-j2\omega_y\}$

$$f(x,y) * g(x,y) \longleftrightarrow F(\omega_x, \omega_y)\, G(\omega_x, \omega_y)$$

multiplication

$$f(x,y)\, g(x,y) \longleftrightarrow \frac{1}{4\pi^2} F(\omega_x, \omega_y) * G(\omega_x, \omega_y)$$

separability

$$F_y(\omega_x, y) = \int_{-\infty}^{\infty} f(x,y)\, e^{-j\omega_x x}\, dx$$

$$\int_{-\infty}^{\infty} F_y(\omega_x, y)\, e^{-j\omega_y y}\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, e^{-j\omega_x x}\, dx \right] e^{-j\omega_y y}\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \exp\{-j(\omega_x x + \omega_y y)\}\, dx\, dy$$

FT of separable signals

$$f(x,y) = f_x(x)\, f_y(y)$$

$$F(\omega_x, \omega_y) = F_x(\omega_x)\, F_y(\omega_y)$$

- FT is also separable

shifts

$$f(x - x_0, y - y_0) \longleftrightarrow \exp\{-j(\omega_x x_0 + \omega_y y_0)\} F(\omega_x, \omega_y)$$

$$\exp\{j(\omega_{x0} x + \omega_{y0} y)\} f(x, y) \longleftrightarrow F(\omega_x - \omega_{x0}, \omega_y - \omega_{y0})$$

scaling

$$f(ax, by) \longleftrightarrow \frac{1}{|ab|} F\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right)$$

Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 \, dx\, dy = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(\omega_x, \omega_y)|^2 \, d\omega_x\, d\omega_y$$

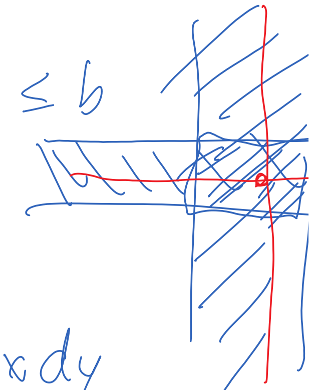
spatial derivatives

$$\frac{\partial f(x,y)}{\partial x} \longleftrightarrow -j\omega_x F(\omega_x, \omega_y)$$

Laplacian: $f(x,y)$

$$\cancel{\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}} - j\omega_y F(\omega_x, \omega_y) \longleftrightarrow -(\omega_x^2+\omega_y^2)F(\omega_x,\omega_y)$$

Ex: $f(x,y) = \begin{cases} 1, & 0 \le x \le a,\ 0 \le y \le b \\ 0, & \text{otherwise} \end{cases}$



$$F(\omega_x,\omega_y) = \int_0^b \int_0^a \exp\{-j(\omega_x x + \omega_y y)\}\,dx\,dy$$

$$= \left[\int_0^b e^{-j\omega_y y}\,dy\right]\left[\int_0^a e^{-j\omega_x x}\,dx\right]$$

$$= -\frac{1}{j\omega_y} e^{-j\omega_y y}\Big|_0^b \; \frac{-1}{j\omega_x} e^{-j\omega_x x}\Big|_0^a$$

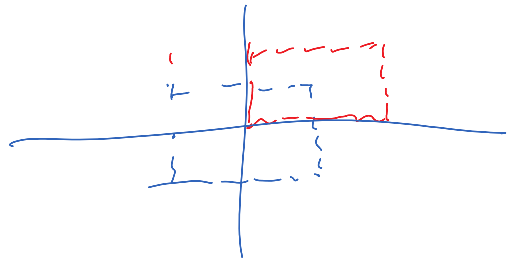
$$= -\frac{1}{\omega_x\omega_y}\left(1 - e^{-jb\omega_y}\right)\left(1 - e^{-ja\omega_x}\right)$$

$$= -\frac{1}{\omega_x\omega_y} e^{-j\left(\frac{b\omega_y}{2} + \frac{a\omega_x}{2}\right)} \times \left(e^{j\frac{b\omega_y}{2}} - e^{-j\frac{b\omega_y}{2}}\right)\left(e^{j\frac{a\omega_x}{2}} - e^{-j\frac{a\omega_x}{2}}\right)$$

$$= \frac{4}{\omega_x\omega_y} \sin\left(\frac{a\omega_x}{2}\right)\sin\left(\frac{b\omega_y}{2}\right) e^{-j\left(\frac{a\omega_x}{2} + \frac{b\omega_y}{2}\right)}$$

$$= ab\, e^{-j\left(\frac{a\omega_x}{2} + \frac{b\omega_y}{2}\right)} \frac{\sin\left(\frac{a\omega_x}{2}\right)}{\frac{a\omega_x}{2}} \cdot \frac{\sin\left(\frac{b\omega_y}{2}\right)}{\frac{b\omega_y}{2}}$$

Alternate approach:



- this is a 2-D pulse
  $\Rightarrow$ separable
- shifted from center, which introduces linear-term

Ex: IFT of $\delta(\omega_x - \omega_{x0}, \omega_y - \omega_{y0})$

$$f(x,y) = \frac{1}{4\pi^2}\int\int \delta(\omega_x - \omega_{x0}, \omega_y - \omega_{y0}) \exp[j(\omega_x x + \omega_y y)] d\omega_x$$

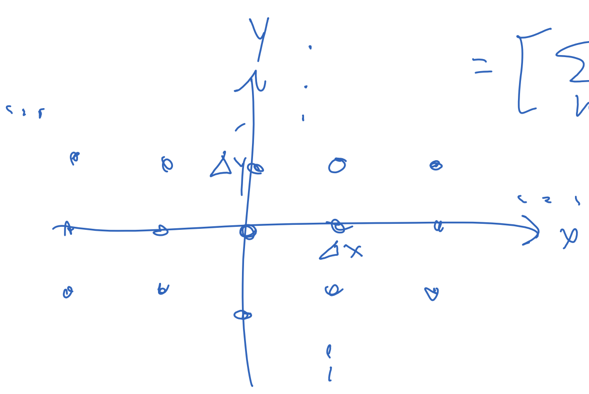
$$= \frac{1}{4\pi^2} \exp[j(\omega_{x0} x + \omega_{y0} y)]$$

Ex: $f(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$

$$= \sum_m \sum_n \delta(x - m\Delta x)\,\delta(y - n\Delta y)$$

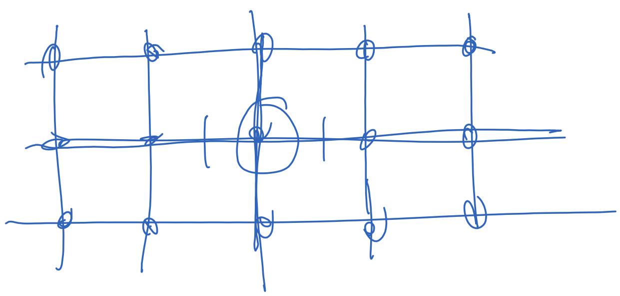
$$= \left[\sum_m \delta(x - m\Delta x)\right]\left[\sum_n \delta(y - n\Delta y)\right]$$

$$= \left[\sum_m a_m \exp\left\{j\frac{2\pi m}{\Delta x} x\right\}\right]\left[\sum_n b_n \exp\{\right.$$



$\frac{\Delta x}{2}$

$\frac{1}{\Delta x}\int$



$$a_m = \Delta x \int_{-\frac{\Delta x}{2}} \delta(x) \exp\left\{-j \frac{2\pi m}{\Delta x} x\right\}$$

$$= \frac{1}{\Delta x}, \quad \text{for all } m$$

$$b_n = \frac{1}{\Delta y}, \quad \text{for all } n$$

$$f(x,y) = \left[\sum_m \frac{1}{\Delta x} \exp\left\{j \frac{2\pi m}{\Delta x} x\right\}\right]\left[\sum_n \frac{1}{\Delta y} \exp\left\{j \frac{2\pi n}{\Delta y} y\right\}\right]$$

$$= \frac{1}{\Delta x \Delta y} \sum_m \sum_n \exp\left\{j 2\pi \left(\frac{m}{\Delta x} x + \frac{n}{\Delta y} y\right)\right\}$$

$$F(\omega_x, \omega_y) = \frac{4\pi^2}{\Delta x \Delta y} \sum_m \sum_n \delta\left(\omega_x - \frac{2\pi m}{\Delta x}, \omega_y - \frac{2\pi}{\Delta}\right.$$

Ex: $f(x,y) = \cos(\omega_{x0} x + \omega_{y0} y)$

$$= \frac{1}{2} \exp\{j(\omega_{x0} x + \omega_{y0} y)\} + \frac{1}{2} \exp\{-j(\ldots$$

$$F(\omega_x, \omega_y) = \frac{4\pi^2}{2} \delta(\omega_x - \omega_{x0}, \omega_y - \omega_{y0})$$

$$+ \frac{4\pi^2}{2} \delta(\omega_x + \omega_{x0}, \omega_y + \omega_{y0})$$

Ex: $f(x,y) = \underline{\underline{\cos(\omega_{x0} x + \omega_{y0} y)}} * [\delta(x - x_0, y - y_0) + \delta(x + x_0, y + y_0)$

$$= \cos(\omega_{x0}(x - x_0) + \omega_{y0}(y - y_0))$$

$$+ \cos(\omega_{x0}(x + x_0) + \omega_{y0}(y + y_0))$$

Or use convolution theorem:

$$g(x,y) = \delta(x-x_0, y-y_0) + \delta(x+x_0, y+y_0)$$

$$G(\omega_x,\omega_y) = \exp\{-j(\omega_x x_0 + \omega_y y_0)\} + \exp\{j(\omega_x x_0 + \omega_y y_0$$

$$= 2\cos(\omega_x x_0 + \omega_y y_0)$$

$$F(\omega_x,\omega_y) = [2\pi^2 \delta(\omega_x - \omega_{x_0}, \omega_y - \omega_{y_0}) + 2\pi^2 \delta(\omega_x + \omega_{x_0}, \omega_y +$$

$$2\cos(\omega_x x_0 + \omega_y y_0)$$

Sampling

$$s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

sampled signal:

$$f_s(x,y) = f(x,y)\, s(x,y)$$

$$= f(x,y) \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

$$= \sum_m \sum_n f(x,y)\, \delta(x - m\Delta x, y - m\Delta y)$$

$$= \sum_m \sum_n f(m\Delta x, n\Delta y)\, \delta(x - m\Delta x, y - n\Delta y)$$

$$F_s(\omega_x,\omega_y) = \frac{1}{4\pi^2} F(\omega_x,\omega_y) * S(\omega_x,\omega_y)$$

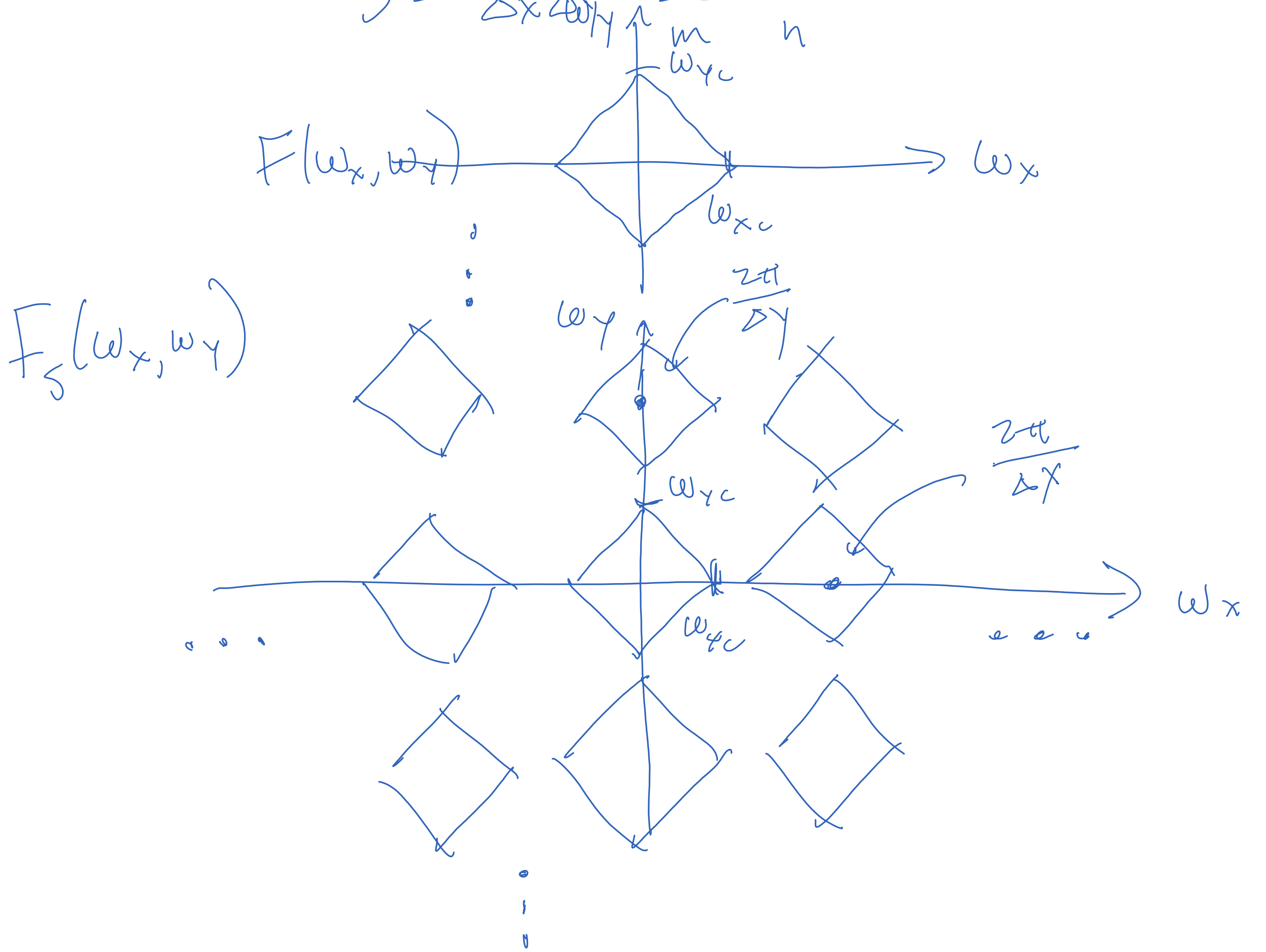
$$F(\ldots) = \frac{1}{\ldots} \sum \sum \delta(\omega_x - \frac{2\pi m}{\ldots}, \omega_y - \frac{2\pi}{\ldots}$$

$= F(\omega_x, \omega_y) * \frac{1}{\Delta x \Delta y} \sum_n \sum_n \delta(\omega_x - \ldots \Delta x, \omega_y \ldots$

Read 4.6-4.7

Will post Project 2 today

$$= \frac{1}{\Delta x \Delta y} \sum_m \sum_n F\left(\omega_x - \frac{2\pi m}{\Delta x}, \omega_y - \frac{2\pi n}{\Delta y}\right)$$



$$\frac{2\pi}{\Delta x} - \omega_{xc} > \omega_{xc}$$

$$\frac{2\pi}{\Delta x} > 2\omega_{xc}$$

$$\frac{2\pi}{\Delta y} > 2\omega_{yc}$$

We can recover $F(\omega_x, \omega_y)$ if the copies don't overlap. If Δx or Δy becomes too big, then copies overlap. The original can then no longer be recovered.

$\Rightarrow$ this is called spatial aliasing

Sampling theorem

A bandlimited image $f(x,y)$ sampled on a uniform rectangular grid with spacing $\Delta x, \Delta y$ can be recovered from the sample values $f(m\Delta x, n\Delta y)$ if

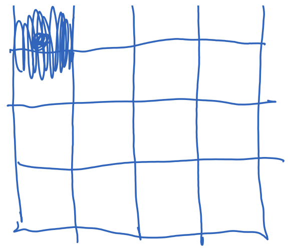
$$\frac{2\pi}{\Delta x} > 2\omega_{xc}$$

and

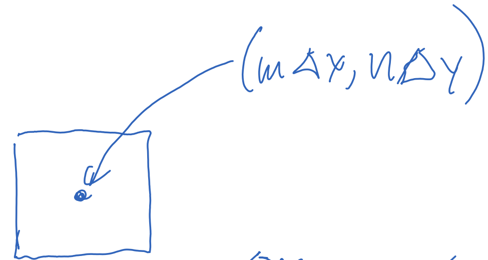
$$\frac{2\pi}{\Delta y} > 2\omega_{yc}$$

Non-ideal sampling

- A more realistic sampling model represents the sampling operation as integrating intensity over rectangular patches.



$$f_i(m\Delta x, n\Delta y) = \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} f(m\Delta x - x', n\Delta y - y')\,dx'\,dy'$$



$$\text{Let } \Pi(x,y) = \begin{cases} 1, & -\frac{\Delta x}{2} < x \le \frac{\Delta y}{2},\ -\frac{\Delta y}{2} < y \le \frac{\Delta y}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$f_i(m\Delta x, n\Delta y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(x', y')\, f(m\Delta x - x', n\Delta y - y')\,dx'\,dy'$$

$$f_i(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(x', y')\, f(x - x', y - y')\,dx'\,dy'$$

$$* \qquad = \underbrace{f(x,y)}_{\text{original scene}} * \Pi(x,y)$$

$$f_i(m\Delta x, n\Delta y) = f_i(x,y)\Big|_{(x,y) = (m\Delta x, n\Delta y)}$$

* $f_{i_s}(x,y) = f_i(x,y)\,\delta(x,y)$

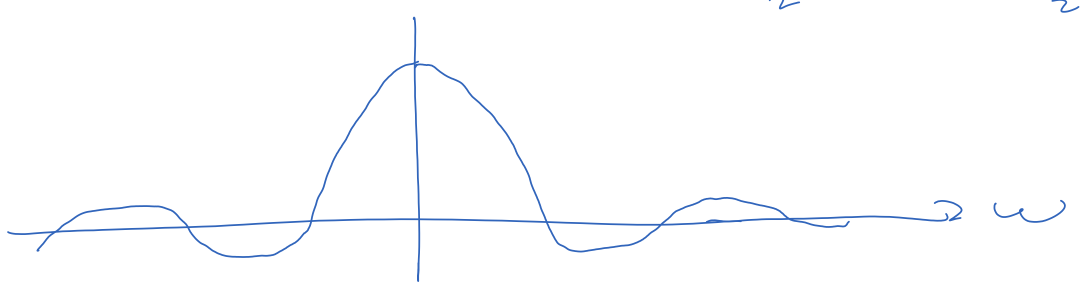
Skim 4.8 - 4.9

$$= \sum_m \sum_n f_i(m\Delta x, n\Delta y)\,\delta(x - m\Delta x, y - n\Delta y)$$

$\Longrightarrow$ filtering followed by ideal sampling

$$F_i(\omega_x, \omega_y) = F(\omega_x, \omega_y)\,\mathcal{F}\{\Pi(x,y)\}$$

$$= F(\omega_x, \omega_y)\,\frac{\sin\frac{\Delta x}{2}\omega_x}{\frac{\Delta x}{2}\omega_x} \cdot \frac{\sin\frac{\Delta y}{2}\omega_y}{\frac{\Delta y}{2}\omega_y}$$



* we sample the filtered signal rather than the original

* filtered signal has higher frequencies suppressed

- aliasing will be reduced (but not eliminated)

- image will be slightly blurred

Display/reconstruction

In DSP, reconstruction from samples is done in concept by creating an impulse train from a sequence, then lowpass filtering. This is implemented using electronics.

In image processing, the display is the filter. (The HVS is also a filter) Samples are projected as rectangular patches (LCD display), Gaussian spots (CRT), etc.

optical display "filter" can be modeled as —

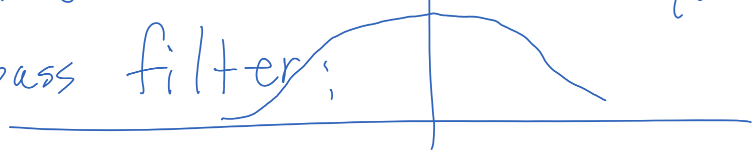
- $f_s(x, y) \longrightarrow \boxed{p(x, y)} \longrightarrow f_s(x, y) * p(x, y)$

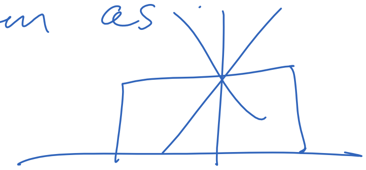
(ω_x, ω_y) $P(\omega_x, \omega_y)$

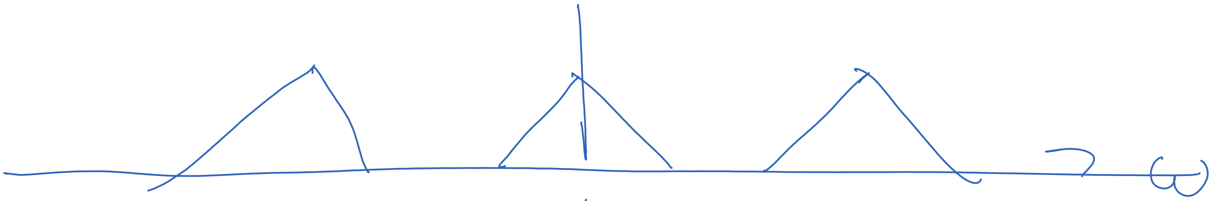
reduces/removes spectral replicas

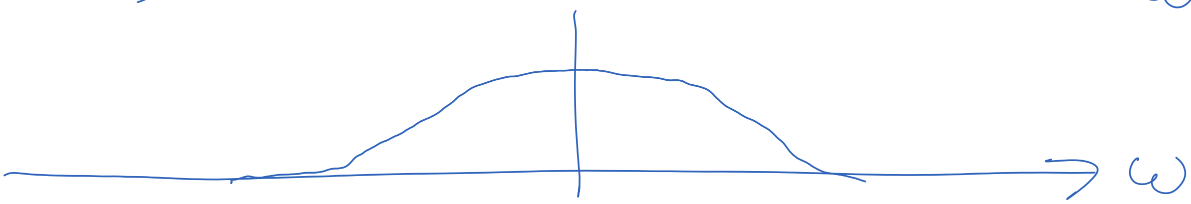
periodically replicated

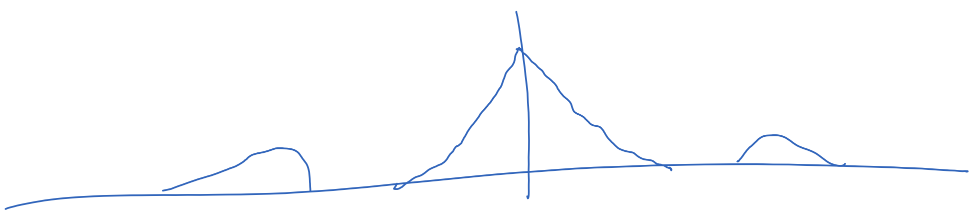
Think of human visual system as a lowpass filter:











- can interpolate to spread out spectral copies

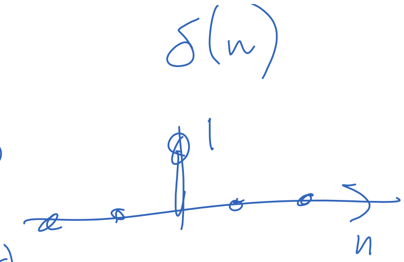
Basic discrete 2-D signals

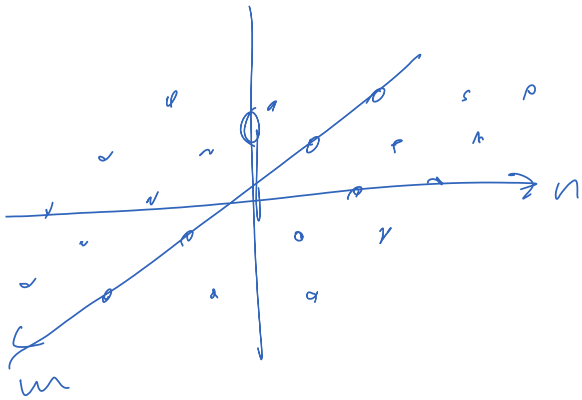
- delta (impulse, unit sample)
  Kronecker delta, not Dirac

$$\delta(m,n) = \begin{cases} 1, & m=n=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\delta(m,n) = \delta(m)\,\delta(n)$$

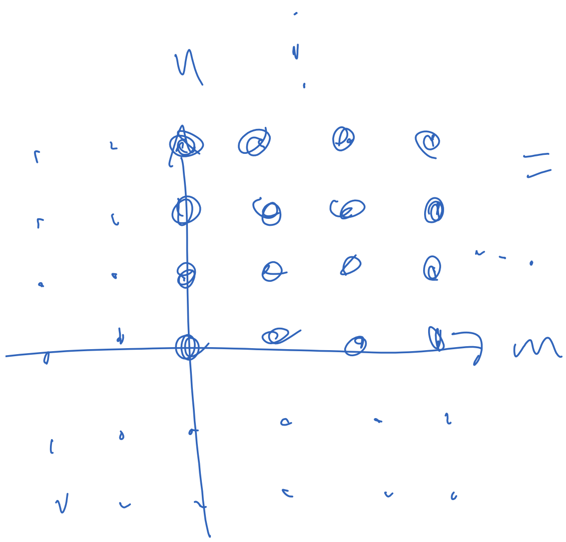
$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



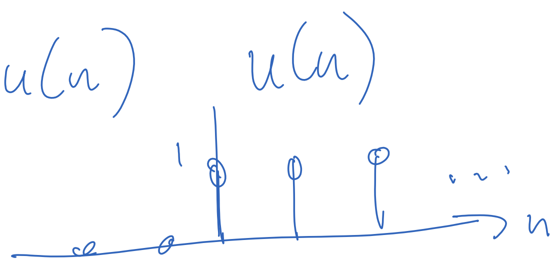


- step

$$u(m,n) = \begin{cases} 1, & m,n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



$$= u(m)\,u(n)$$



exponential

$$f(m,n) = \exp\{-(\alpha m + \beta n)\}\,u(m,n)$$
$$= \left[e^{-\alpha m} u(m)\right]\left[e^{-\beta n} u(n)\right]$$

$\alpha, \beta \geq 0$ to prevent

blowing up

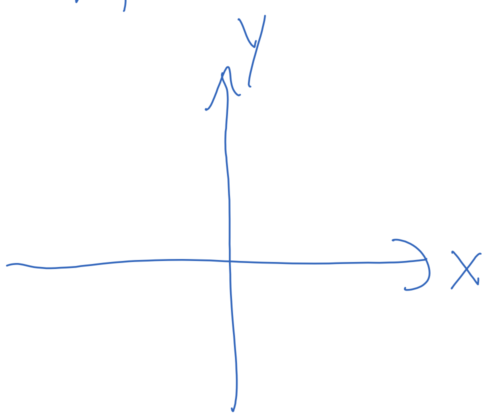
Sinusoid

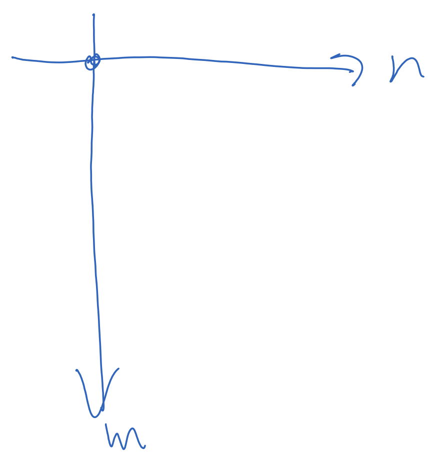
$$f(m,n) = f(x,y)\Big|_{(x,y) = (m\Delta x,\, n\Delta y)}$$

$$= \sin\left(\underbrace{\omega_{0x}\Delta x}\, m + \underbrace{\omega_{0y}\Delta y}\, n\right)$$

$$= \sin\left(\omega_{0m}\, m + \omega_{0n}\, n\right)$$

ω_{0m}, ω_{0n} are in rad/sample





discrete systems

$$f(m,n) \longrightarrow \boxed{\begin{array}{c}\text{2-D}\\ \text{system}\\ T[\cdot]\end{array}} \longrightarrow g(m,n)$$

(m,n) are integer index values

- linearity

$$T[a f_1(m,n) + b f_2(m,n)] = a T[f_1(m,n)] + b T[f_2(m,n)]$$

for all a, b, f_1, f_2

- shift-invariance

$$T[f(m-k, n-\ell)] = g(m-k, n-\ell)$$

for all $k, \ell, f(m,n)$

(k, ℓ) must be integers

inear shift-invariant

$$f(m,n) = \sum_k \sum_\ell f(k,\ell)\, \delta(m-k, n-\ell)$$

$$g(m,n) = T[f(m,n)]$$

$$= T\left[\sum_k \sum_\ell f(k,\ell)\, \delta(m-k, n-\ell)\right]$$

linear) $$= \sum_k \sum_\ell f(k,\ell)\, T[\delta(m-k, n-\ell)]$$

(SI) $$= \sum_k \sum_\ell f(k,\ell)\, h(m-k, n-\ell)$$

$$= f(m,n) * h(m,n)$$

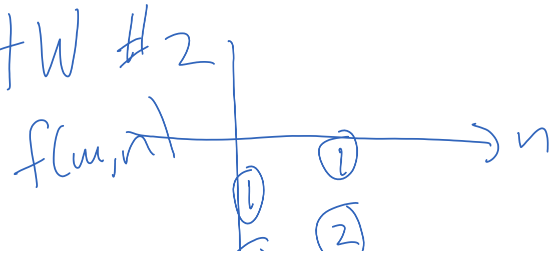
(2-D discrete convolution)

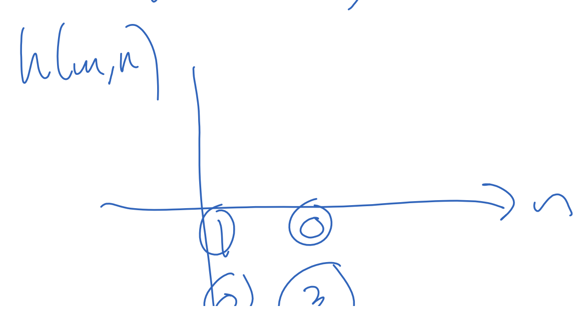
$,m) = T[\delta(m,n)]$

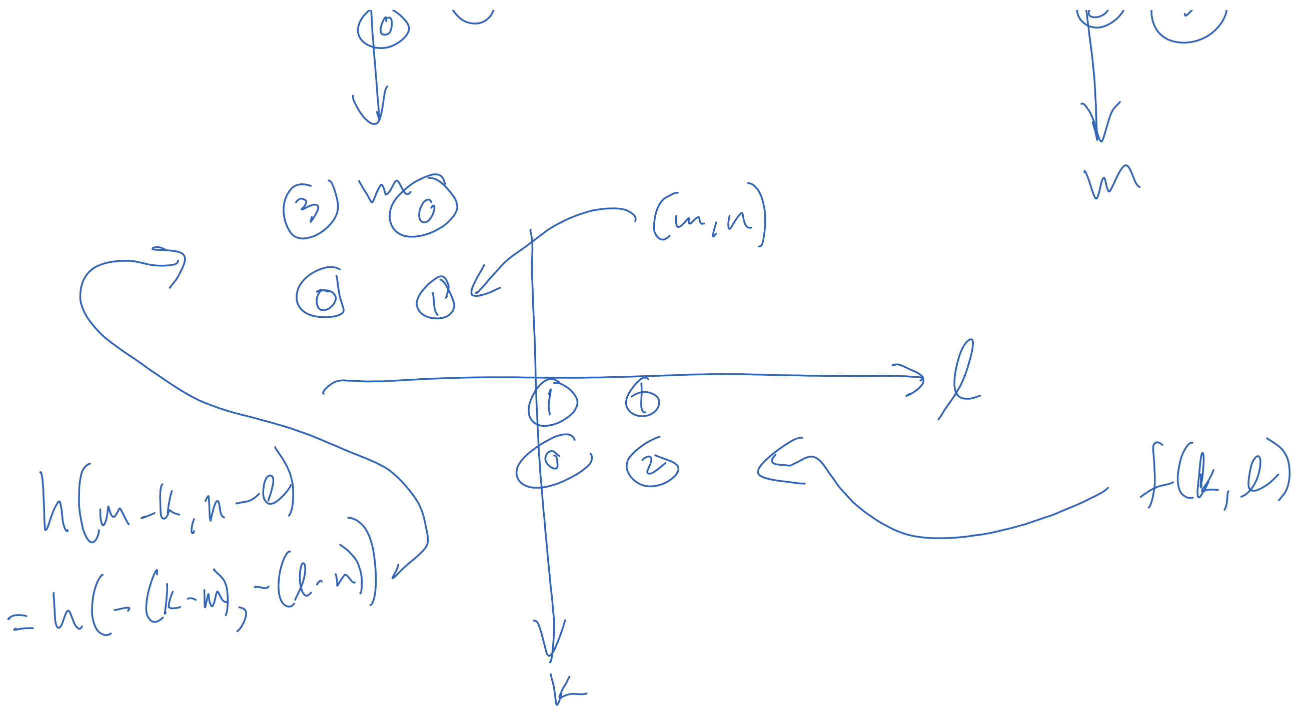
impulse response)

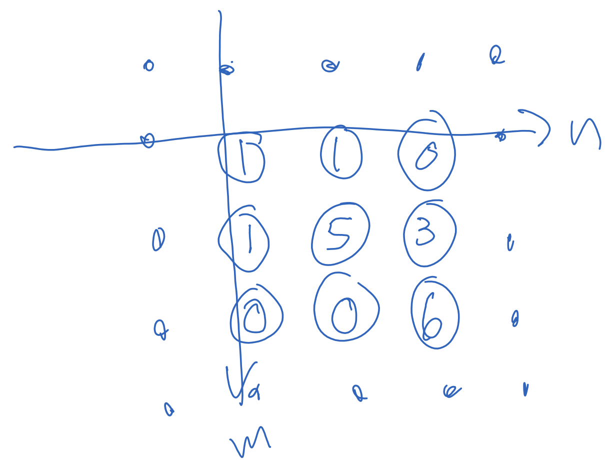
$\Rightarrow h(m-k, n-\ell) = T[\delta(m-k, n-\ell)]$

HW #2









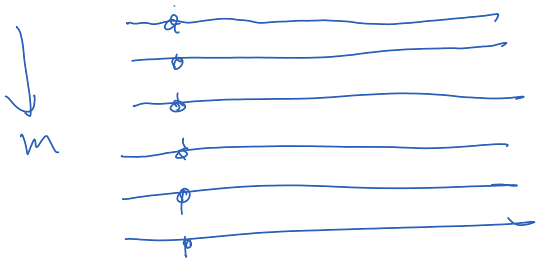
Fourier transform of 2-D sequence

$$\text{FT of } x(n) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

ω is in rad/sample

$$X_n(m;\omega_n) = \sum_{n=-\infty}^{\infty} x(m,n)\, e^{-j\omega_n n} \quad : \text{ 1-D FT of rows of } x(m,n)$$

$\rightarrow \omega_n$



$$X(\omega_m, \omega_n) = \sum_{m=-\infty}^{\infty} X_n(m;\omega_n)\, e^{-j\omega_m m}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n)\, e^{-j(\omega_m m + \omega_n n)}$$

- FT of 2-D sequence

$$m,n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_m, \omega_n)\, e^{j(\omega_m m + \omega_n n)}\, d\omega_m\, d\omega_n$$

- Inverse FT

$$\omega_m, \omega_n) = X(\omega_m - 2\pi, \omega_n) = X(\omega_m, \omega_n - 2\pi) = X(\omega_m - 2\pi, \omega_n - 2\pi)$$

FT of impulse response is the <u>frequency response</u> of system.

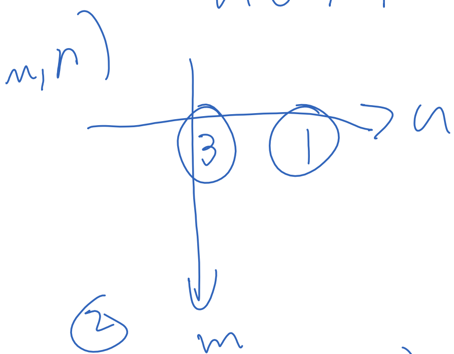
$$H(\omega_m, \omega_n) = \mathcal{F}\{h(m,n)\}$$

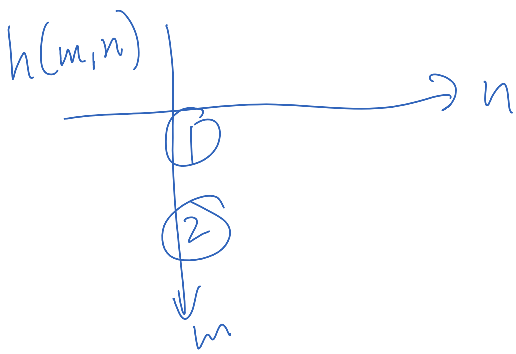
$$G(\omega_m, \omega_n) = F(\omega_m, \omega_n) H(\omega_m, \omega_n)$$

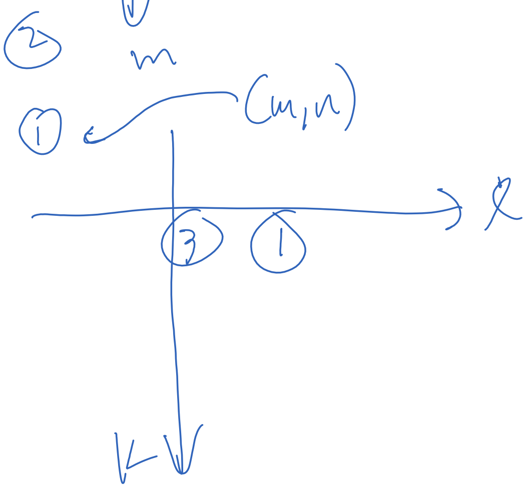
$f(m,n) * h(m,n)$ convolution ~~theorem~~

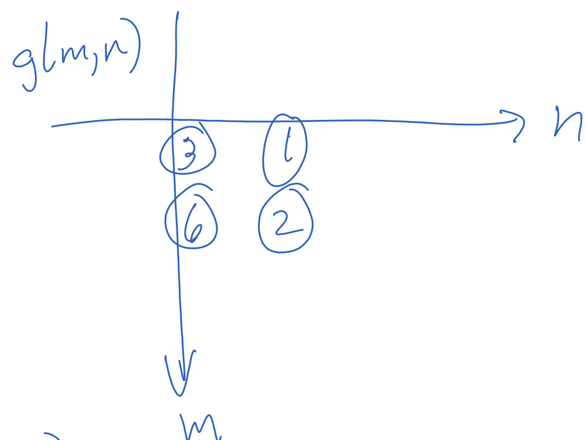
Ex: $f(m,n) = 3\delta(m,n) + \delta(m,n-1)$ $F(\omega_m,\omega_n) H(\omega_m, \omega_n)$

$h(m,n) = \delta(m,n) + 2\delta(m-1,n)$









$$\sum_m \sum_n \delta(m-k, n-l) e^{-j(\omega_m m + \omega_n n)}$$

$$= e^{-j(\omega_m k + \omega_n l)}$$

$$F(\omega_m, \omega_n) = 3 + e^{-j\omega_n}$$

$$H(\omega_m, \omega_n) = 1 + 2e^{-j\omega_m}$$

$$= FH = (3 + e^{-j\omega_n})(1 + 2e^{-j\omega_m})$$

$$= 3 + e^{-j\omega_n} + 6e^{-j\omega_m} + 2e^{-j(\omega_m + \omega_n)}$$

$$h(m,n) = 3\delta(m,n) + \delta(m,n-1) + 6\delta(m-1,n) + 2\delta(m-1,n-1)$$

Properties of discrete-space FT:

linear

separable — can decompose into two 1-D FTs

9y32

ad 4.11

shift $f(m-k, n-\ell) \longleftrightarrow F(\omega_m, \omega_n) e^{-j(\omega_m k + \omega_n \ell)}$

modulation $f(m,n) g(m,n) \longleftrightarrow \frac{1}{4\pi^2} F(\omega_m, \omega_n) * G(\omega_m, \omega_n)$

convolution thm

Parseval's theorem

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(m,n)|^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |F(\omega_m, \omega_n)|^2 d\omega_m \, d\omega_n$$

T is a function of continuous coordinates

and can't be stored in a computer. Integrals an't be calculated perfectly in a computer.

~~) discrete Fourier transform (2-D DFT)~~

, $M \times N$ image,

$$F(k,l) = F(\omega_m, \omega_n)\Big|_{(\omega_m,\omega_n) = \left(\frac{2\pi k}{M}, \frac{2\pi l}{N}\right)}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left[-j\left(\frac{2\pi k m}{M} + \frac{2\pi l n}{N}\right)\right]$$

$$0 \le k \le M-1, \quad 0 \le l \le N-1$$

DFT

$$f(m,n) = \frac{1}{\ } \quad {}^{M-1} \sum_{l=0}^{N-1} F(k,l) \exp\left[j\left(\frac{2\pi k m}{M} + \frac{2\pi l n}{N}\right)\right]$$

<u>operties of DFT</u>

linear

Parseval's theorem

$\sum\sum |f(m,n)|^2 = \frac{1}{MN} \sum\sum |F(k,l)|^2$ inverse $f(m+M, n+N) = f(m+M, n) = f(m, n+N) = f(m,$

separability - can take 1-D DFT in one direction + then 1-D DFT on perpendicular direction

- circular shift

$f(m-m_0, n-n_0) \longleftrightarrow F(k,l)\, e$

$$f(((m-m_0))_M, ((n-n_0))_N) e^{-j\frac{2\pi k n}{N}}$$

If ~~seq~~ where separable $((m))_M$ = then mod ~~D~~FT is separable

$$f(m,n) = f_M(m) f_N(n) \rightleftharpoons F_M(k) F_N(\ell)$$

$2\pi k m_0$

IDFT of $\{1\} \longrightarrow \delta(m,n)$

periodicity

(m_0, n_0) both small positive

$$F(k+M, \ell+N) = F(k+M, \ell) = F(k, \ell+N) = F(k, \ell)$$

(m_0, n_0) both small negative

$-j\left(\frac{2\pi k m_0}{M} + \frac{2\pi}{}\right.$

- circular convolution

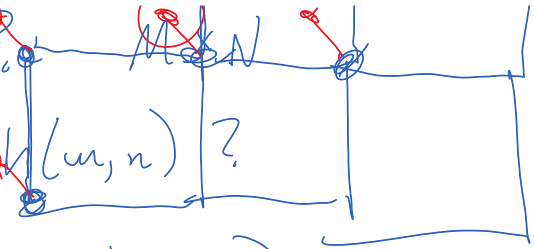
$$f(m,n) * h(m,n) \overset{?}{\longleftrightarrow} F(k,\ell) H(k,\ell)$$

$$g(m,n) = \sum_{k=0} \sum_{\ell=0} f(k,\ell)\, h(((m-k))_M, ((n-\ell))_N)$$

- like a linear convolution but with linear shifts replaced by circular shifts

What size is g(m,n)?

What size is f(m,n) * h(m,n)?



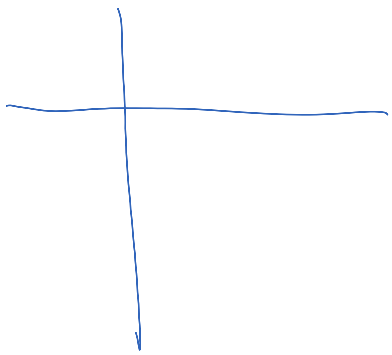
$$(M_f + M_h - 1) \times (N_f + N_h - 1)$$

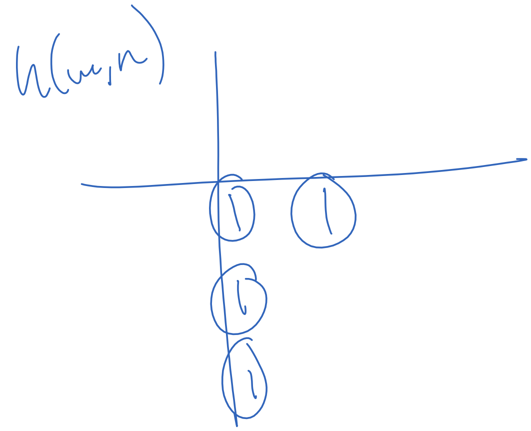
Read 2.6, 3.1 - 3.2

HW posted

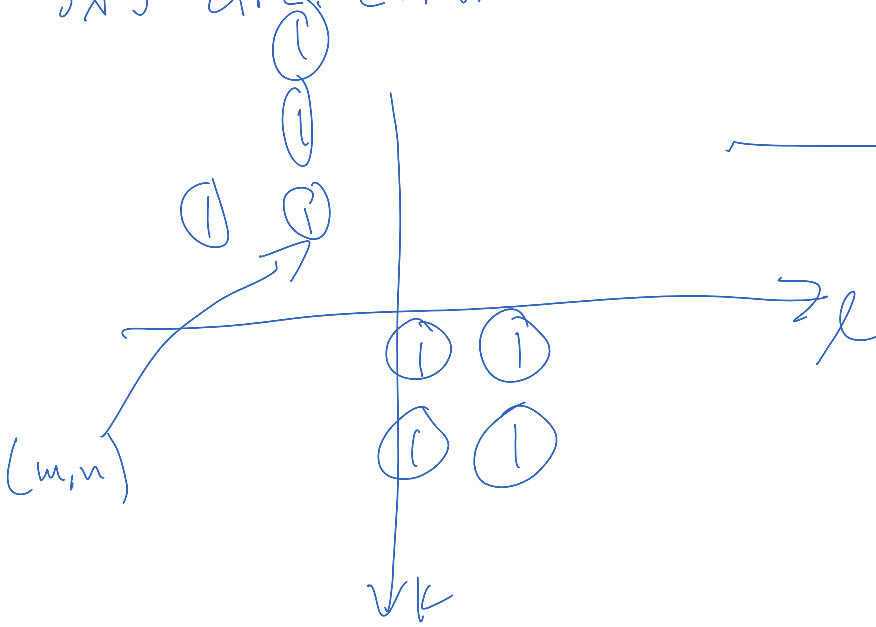
- can implement circular convolution by periodic extension of linear convolut.

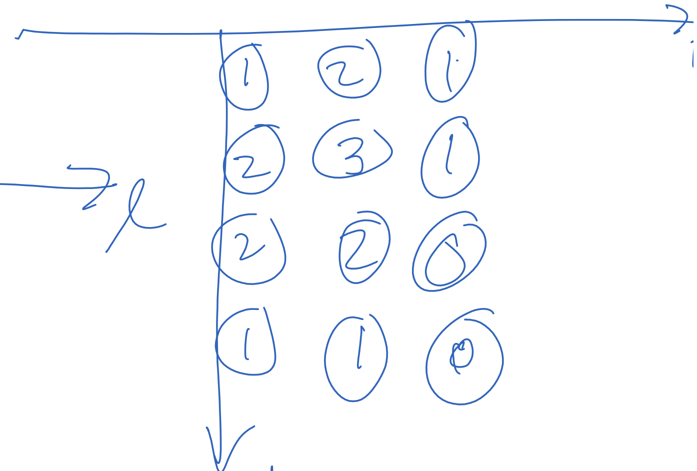
f(m,n)



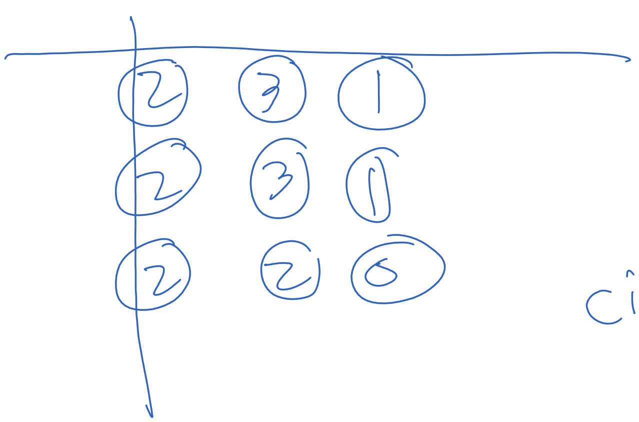


3x3 circ. conv.





lin. conv.



circ. conv.

linear convolution w/ DFTs

Let $h(m,n)$ be $M_h \times N_h$

" $f(m,n)$ be $M_f \times N_f$

1) zeropad $h(m,n)$ and $f(m,n)$ to be
$\geq (M_h + M_f - 1) \times (N_h + N_f - 1)$

2) Take DFTs of zeropadded seq's.

3) Multiply DFTs pointwise.
$H .* F$

4) Take IDFT of result.

Fast Fourier transform (FFT)

- efficient DFT implementation
- constructed by decomposing DFT into a sum of small DFTs
- DFT requires $M^2 N^2$ multiplies
- FFT requires $MN \log MN$ multiplies

For 1024×1024,

$$\text{DFT} = 10^{12} \text{ mults}$$

$$\text{FFT} = 10^{7} \quad "$$

(1 day vs. 1 sec.)

$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left\{-j\left(\frac{2\pi k m}{M} + \frac{2\pi l n}{N}\right)\right\} \qquad e^{-j\frac{2\pi k m}{M}}$$

$$= \sum_{m=0}^{M-1} \left\{ \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi l n}{N}} \right\}$$

$\Rightarrow$ row-column decomposition

1-D DFT of all rows

Then 1-D DFT of columns of result of previous step

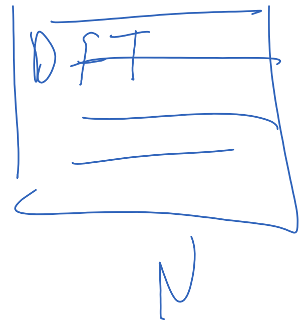
* can also replace 1-D DFTs w/ 1-D FFTs

1-D DFT requires N^2 mults

1-D FFT " $N \log_2 N$ mults.

multiplies:

for row-col w/



$$MN^2 + NM^2 = MN(M+N)$$

For full 2-D DFT $\Rightarrow M^2N^2$

1-D N-length FFT requires $\frac{1}{2}N\log_2 N$ mults

- for Row-Col w/ FFTs:

$$\frac{1}{2}MN\log_2 N + \frac{1}{2}NM\log_2 M$$

For 1024×1024 $\frac{1}{2}MN\log_2 MN$

direct DFT $= 2^{40} \approx 10^{12}$ mults

R-C DFT $= 2^{31} \approx 2\times 10^{9}$ "

R-C FFT $= 10\cdot 2^{20} \approx 10^{7}$ "

vector-radix FFT

(divide-and-conquer)

- 2-D DFT is divided into successively smaller 2-D DFTs

- similar to decimation-in-time or decimation-in-frequency 1-D FFT

\# mults is $\frac{3}{8}N^2 \log_2 N^2$ $(N \times N)$

compared to $\frac{1}{2}N^2 \log_2 N^2$ for R-C FFT