MECH7710: Optimal Estimation and Control: Midterm Exam

Due on April 6, 2020

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Problem 1

Assume that t observations of a random variable are given:

$$\widetilde{y}_k = y_k \forall k = 1, ..., t$$

Find a recursive formula for calculating the mean at time t.

Solution Working through a series of examples,

$$\begin{aligned}
\overline{y}_1 &= y_1 \\
\overline{y}_2 &= \frac{1}{2}(\overline{y}_1 + y_2) \\
\overline{y}_3 &= \frac{1}{3}(2\overline{y}_2 + y_3)
\end{aligned}$$

the formula

$$\overline{y}_t = \frac{1}{t}[(t-1)\overline{y}_{t-1} + y_t]$$

becomes apparent.

Problem 2

Consider the continuous system blow

a Calculate the expected steady state Kalman filter estimation error variance

b Calculate the steady state Kalman gaussian

c Calculate the open loop and closed loop estimator eigenvalues

d Plot the location of all possible closed loop eigenvalues. What do you notice about the minimum Kalman gain and the slowest closed loop estimator the Kalman filter will use?

$$\begin{aligned}
\dot{x} &= x + w \\
y &= x + v \\
E[w] &= 0 \\
E[v] &= 0 \\
E[ww^T] &= Q \\
E[vv^t] &= R
\end{aligned}$$

Solution

Problem 3

Two random vectors X_1 and X_2 are uncorrelated if

$$E[(X_1 - \overline{X_1})(X_2 - \overline{X_2})] = 0$$

Show that:

Problem 3 continued on next page...

1. Independent random vectors are uncorrelated
2. Uncorrelated Gaussian random vectors are independent

Solution If two random vectors are independent, then $f(x_1, x_2) = f(x_1)f(x_2)$. From this we get

$$\begin{aligned} E[X_1 X_2] &= \sum\sum x_1 x_2 f(x_1) f(x_2) \\ &= \sum x_1 f(x_1) \sum x_2 f(x_2) \\ &= E[X_1]E[X_2] \end{aligned}$$

Since $Cov(X_1, X_2) = E[X_1X_2] - E[X_1]EX_2 = 0$, the vectors are uncorrelated.

If two gaussian random vectors are uncorrelated, then $\rho_{1,2} = 0$. From this, the exponential term in the PDF takes the form $\exp(\alpha_1 X_1^2 + \alpha_2 X_2^2)$, which can clearly be written as a separable product of exponentials, and the vectors are thus independent.

Problem 4

Consider a sequence created by throwing a pair of dice and summing the numbers, which are $[-2.5, -1.5, -0.5, 0.5, 1.5, 2.5]$. Call this $V_0(k)$.

a What is the PDF?

b What are the mean and variance?

If we generate a new random sequence

$$V_N(k+1) = (1-r)V_N(k) + rV_0(k),$$

$V_N(k)$ is serially correlated.

c What are the steady-state mean and variance of this new sequence?

d What is the covariance function $R(k) = E[V_N(k)V_N(k-L)]$?

e Are there any practical constrains on r?

Solution

$$\begin{aligned} f(v_0) &= \frac{1}{6^2}\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \in [-5, 5] \\ E[v_0] &= 0 \\ \sigma_{v_0}^2 &= 5.832 \end{aligned}$$

If we expand the sequence definition backwards in time a few steps, we find that it is described by the summation

$$V_n(k) = (1-r)^k V_0(0) + \sum_{i=0}^{k-1} r(1-r)^i V_0(i)$$

Our $V_0(0)$ term goes to 0 as $k \to \inf$, and

$$\begin{aligned} E[\sum_{i=0}^{k-1} r(1-r)^i V_0(i)] &= rE[V_0]\sum_{i=0}^{k-1}(1-r)^i \\ &= 0 \end{aligned}$$

Performing the same expansion for $V_n(k)^2$, we get

$$V_n(k)^2 = (1-r)^{2k} V_n(0)^k + \sum_{i=0}^{k-1} r^2 (1-r) V_0(i)^2$$

Again, our $V_0(0)$ term goes to 0, and

$$\begin{aligned} E[\sum_{i=0}^{k-1} r^2 (1-r)^{2i} V_0(i)^2] &= r^2 \sum_{i=0}^{k-1} (1-r)^{2i} E[V_0(i)^2] \\ &= [r^2 \sum_{i=0}^{k-1} (1-r)^{2i}](\sigma_{v0}^2 - 0) \end{aligned}$$

From these,

$$\begin{aligned} \sigma_{vn}^2 &= E[V_n^2] - E[V_n]^2 \\ &= \sigma_{v0}^2 [r^2 \sum_{i=0}^{k-1} (1-r)^{2i}] \end{aligned}$$

To find $R(L)$ we'll start with a small example where $L = 1$.

$$\begin{aligned} E[V_N(k) V_N(k-1)] &= E[((1-r) V_N(k-1) + r V_0(k-1)) * V_N(k-1)] \\ &= E[(1-r) V_N(k-1)^2] + E[r V_0(k-1) V_N(k-1)] \\ &= (1-r) E[V_N(k-1)^2] \end{aligned}$$

From this, we expand the general solution to $R(L) = (1-r)^L E[V_N(k-L)^2]$. There's still some work to be done in exactly solving back to $V_N(1)$, but this gets the general idea that the sequence is less correlated with itself over larger time steps. A practical constraint on r is that $|r| \in [0, 1]$, otherwise the sequence goes to infinity.

Problem 5

A random variable x has a PDF given by

$$f_X = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

a What is the mean of x?

b What is the variance of x?

Solution

$$\begin{aligned} \mu_x &= \int_0^2 x \frac{x}{2} dx \\ &= \frac{x^3}{6} |_0^2 \\ &= \frac{4}{3} \end{aligned}$$

Problem 5 continued on next page. . .

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\begin{aligned}
\sigma_x^2 &= E[X^2] - E[X]^2 \\
&= \int_0^2 x^2 \frac{x}{2} dx - \frac{16}{9} \\
&= \frac{x^4}{8}|_0^2 - \frac{16}{9} \\
&= 2 - \frac{16}{9}
\end{aligned}
$$

Problem 6

Consider a normally-distributed 2D vector X, with mean 0 and

$$P_X = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

a Find the eigenvalues of P_X

b What are the principal axes?

c Plot the likelihood ellipses for $c = 0.25, 1, 1.5$

d What is the probability of finding X inside each of these ellipses?

Solution

```
[V, D] = eig(P_x);
% Eigenvalues:
D = 1.5858      0
    0           4.4142
% Principal axes:
V = -0.9239     0.3827
    0.3827      0.9239
```

We find our error ellipses via:

```
function [c] = error_ellipse(covariance, k)
    theta = linspace(0, 2*pi, 100);
    a = k*[cos(theta); sin(theta)];
    [V, D] = eig(covariance);
    A = D^(-1/2) * V;
    A_inv = pin(A);
    c = A_inv * a;
end
```

We expect these ellipses to contain X with the following probabilities:

```
c = [0.25; 1; 1.5];
alpha = c .^ 2
P = chi2cdf(alpha, 2);
```

$$
\begin{aligned}
C = 0.25 &: P(X) = 0.0308 \\
C = 1.00 &: P(X) = 0.3935 \\
C = 1.50 &: P(X) = 0.6753
\end{aligned}
$$

Problem 7

Given $x \sim N(0, \sigma_x^2)$ and $y = 2x^2$

a Find the PDF of y

b Draw the PDFs of x and y on the same plot for $\sigma_x^2 = 2.0$

c How is the density function changed by this transformation?

d Is y a normal random variable?

Solution For an RV X $N(0, \sigma_x^2)$ transformed as $Y = g(X) = 2X^2$, we slightly abuse the density transformation technique:

$$f_y(y) = |\frac{d}{dy} g^{-1}(y)| f_x(g^{-1}(y))$$

To use this, g must be invertible. Instead we will handle it piecewise for $x < 0$ and $x > 0$.

$$\begin{aligned} g^{-1}(y) &= \pm sqrt(\frac{y}{2}) \\ |\frac{d}{dy} g^{-1}(y)| &= \frac{1}{2sqrt(2y)} \\ f_x(x) &= \frac{1}{sqrt(2\pi\sigma_x^2)} exp(\frac{-1}{2}(\frac{x-\mu}{\sigma})^2) \end{aligned}$$

To handle our uninvertible transformation, we only performt the calculation for $x > 0$ and double it to account for $x < 0$ mapping to the same value as their positive equivalent.

$$\begin{aligned} f_y(y) &= 2 * |\frac{1}{2sqrt(2y)}| f_x(sqrt(\frac{y}{2})) \\ &= \frac{1}{2\sigma_x sqrt(\pi y)} exp(\frac{-y}{4\sigma_x^2}) \end{aligned}$$