The Singular Value Decomposition

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Introduction

The rank, range space, and null space of a linear operator (matrix A) can be revealed by the Singular Value Decomposition (or SVD). The SVD factors the $m \times n$ matrix A as:

$$A = USV^*$$

where

- U is an $m \times m$ unitary matrix,
- S is an $m \times n$ diagonal rectangular matrix, and
- V^* is an $n \times n$ unitary matrix.

"Unitary" means the inverse is the complex conjugate transpose, i.e. $HH^* = I$.

For real matrices, $HH^T = I$, i.e. the matrix is orthogonal.

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Geometric Interpretation

Ref: Wikipedia

Linear operator A represents a sequence of rotations and scaling

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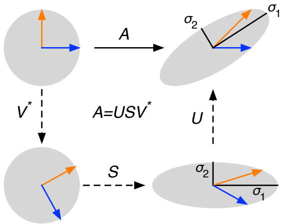
where

- V^* first performs a rotation,
- S then introduces scaling, and
- U executes a final rotation.



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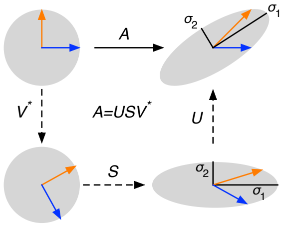

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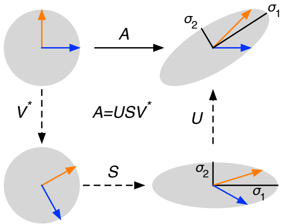


Example 1
from Wikipedia

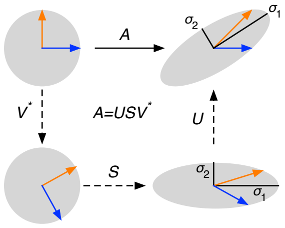
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = USV^*$$

where

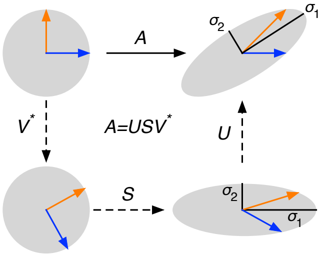
$$V^* = \begin{bmatrix} 0.526 & 0.851 \\ -0.851 & 0.526 \end{bmatrix}$$

$$S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.851 & -0.526 \\ 0.526 & 0.851 \end{bmatrix}$$



More explanation of the figure



1. Upper left unit circle represents the domain space, with basis vectors $[1 \ \ 0]^T$ (blue) and $[0 \ \ 1]^T$ (orange).
2. Lower left circle describes rotation performed by V^*.
3. Lower right ellipse illustrates scaling introduced by S. Major and minor axes are the singular values of A.
4. Upper right ellipse represents the range space, and shows rotational effect of U, and total effect of A.

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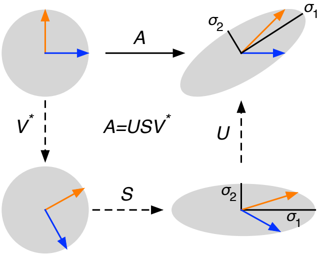

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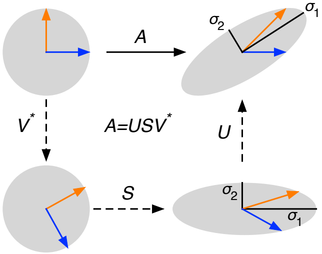

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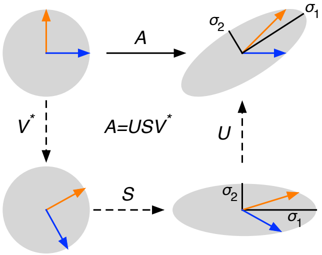

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Rank, range space, and null space from the SVD

$$A = USV^*$$

Rank The number r of non-zero singular values of A, or non-zero diagonal elements of S.

Range space Orthonormal basis set formed by the leftmost r columns of U.

Null space Orthonormal basis set formed by the rightmost $n - r$ columns of V^*.

Note: Matrix S is unique, but
U and V^* are not unique.

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Example 2: $\mathcal{A} : \mathcal{R}^4 \to \mathcal{R}^3$, but rank is 2.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = USV^*$$

$$U = \begin{bmatrix} 0.707 & 0.707 & 0 \\ 0.5 & -0.5 & -0.707 \\ 0.5 & -0.5 & 0.707 \end{bmatrix} \text{ Range space basis set}$$

$$S = \begin{bmatrix} 1.848 & 0 & 0 & 0 \\ 0 & 0.765 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ Singular values of } A$$

$$V^* = \begin{bmatrix} 0.383 & 0.924 & 0 & 0 \\ 0.924 & -0.383 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ Null space basis set}$$

[U,S,V]=svd(A) Returns matrices U, S, and V such that $A = USV^*$

orth(A) an orthonormal basis for the range of A

null(A) an orthonormal basis for the null space of A

Some MATLAB tools

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