ELEC-7500

September 10, 2014

- matrix exponential
  state transition matrix
- Eigenvector approach
- Laplace approach

The larger picture...

$$\underline{\dot{x}} = A\underline{x}$$ → what is the state response?

$$\underline{x}(t) = ?$$

Just study the <u>form</u> of the scalar (1st order) problem

$$\dot{x}(t) = a\,x(t)$$

$$\dot{x} = a x$$

$$\frac{dx}{dt} = a x$$

Separation of variables leads to

$$\frac{1}{x}\,dx = a\,dt$$

$$\int_{x(0)}^{x(t)} \frac{1}{x}\,dx = \int_0^t a\,dt$$

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$$\int_{x(0)}^{x(t)} \frac{1}{x}\,dx = \int_0^t a\,dt$$

dust clears

$$x(t) = e^{at}\,x(0)$$

exponential function

We desire (hope for)

$$\underline{x}(t) = e^{At}\,\underline{x}(0)$$

What is this matrix exponential?

Also called state transition matrix.

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More generally $\underline{x}(t_0) \rightarrow \underline{x}(t)$

$$\underline{x}(t) = \underbrace{e^{A(t-t_0)}}\, x(t_0)$$

– state transition matrix

– matrix exponential

Google: author Moler, Golub, Van Loan

"21 dubious solutions for the matrix exponential"

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Recall Similarity Transformation

$$AV = V\Sigma$$

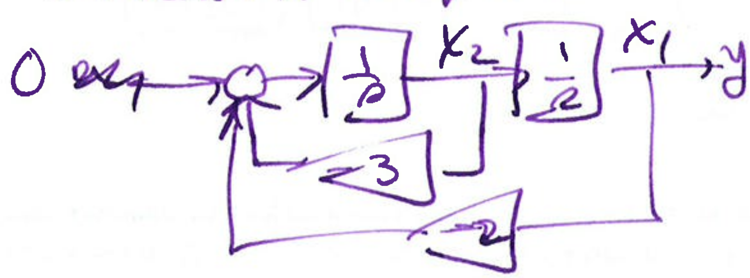
or $$A = V\Sigma V^{-1}$$

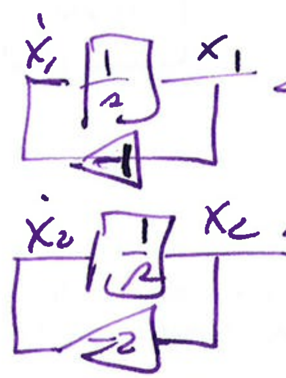
"A" is similar to "Σ"

decoupled

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \text{ is related to } \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{\Sigma}$$

Simulation diagram





So, given $\dot{x} = A x$ — $\dim(x) = n \times 1$

A — matrix $n \times n$

Recall A has eigenvectors & eigenvalues

$$A V = V \Sigma$$

V — matrix of eigenvectors

Σ — diagonal matrix of eigenvalues

Last meeting $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $V = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Claim Since $A = V \Sigma V^{-1}$

Then $e^{At} = V e^{\Sigma t} V^{-1}$

Easy to compute

$$\Sigma = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad e^{\Sigma t} = \begin{bmatrix} e^{-1t} & \emptyset \\ \emptyset & e^{-2t} \end{bmatrix}$$

$$e^{At} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{\substack{v_1 \quad v_2 \\ V}} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix}$$

$$V^{-1} = \frac{1}{|V|} \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}}_{\text{cofactor}}$$

Finish steps for homework

$$\begin{bmatrix} e^{-t}, e^{-2t} & \text{—} \\ \text{—} & \text{—} \end{bmatrix}$$

Another approach

$$\dot{\underline{x}}(t) = A\,\underline{x}(t)$$

Apply Laplace transform to both sides

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

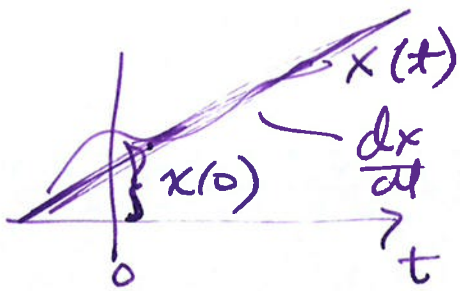
Laplace transform

$$X(s) = \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix}$$

If $\underline{x}(t) \xrightarrow{\mathcal{L}} \underline{X}(s)$

then $\frac{d}{dt}\underline{x}(t) \xrightarrow{\mathcal{L}} s\,\underline{X}(s) - \underline{x}(0)$

initial state



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Therefore $\dot{\underline{x}}(t) = A\,\underline{x}(t)$

transforms to

$$s\,\underline{X}(s) - \underline{x}(0) = A\,X(s)$$

Solve for $X(s)$

$$s\,X(s) - A\,X(s) = \underline{x}(0)$$

$$(sI - A)\,X(s) = \underline{x}(0)$$

Solution in transform domain

$$\boxed{\underline{X}(s) = (sI - A)^{-1}\,x(0)}$$

Inverse Laplace

$$\underline{x}(t) = e^{At}\,\underline{x}(0)$$

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Claim

$$e^{At} = \mathcal{L}^{-1}\left\{ (sI - A)^{-1} \right\}$$

matrix inverse

transform inverse

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Example 2 (compare result later to first method uses $e^{\Sigma t}$ ← eigenvalues)

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{\underbrace{s^2+3s+2}_{|sI-A|}} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2 - -} \\ \dfrac{-2}{\sim} & \dfrac{s}{\sim} \end{bmatrix}$$

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$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{\sim} \\ \dfrac{-2}{} & \dfrac{s}{\sim} \end{bmatrix} \overset{?}{=} e^{At}$$

inverse Laplace transform, element-by-element

$$\frac{s+3}{s^2+3s+2} \rightarrow ?$$

residues

$$\frac{s+3}{\underbrace{s^2+3s+2}_{(s+1)(s+2)}} = \underbrace{\frac{}{s+1} + \frac{}{s+2}}_{\text{partial fraction expansion}} + K$$

In MATLAB

```
>> num = [1  3]
>> den = [1  3  2]
>> [R ,P, K] = residue (num,den)
```

residues poles