ELEC-7500

September 18, 2013

- general solution of state equations
- other approaches to find e^{At}
- Cayley-Hamilton Theorem

Laplace transform approach to solutions $x(t), t>0$ of the state equation (continued)

$$\dot{x} = Ax + Bu, \quad x(0) \text{ - initial condition}$$

(Previously we got this): Solution in transform domain

$$X(s) = [sI - A]^{-1} x(0) + [sI - A]^{-1} BU(s)$$

inverse Laplace transform

"temptation"

$$x(t) = e^{At} x(0) + \boxed{e^{At} B u(t)}$$

matrix exponential

Laplace transform of $e^{At} \Rightarrow [sI - A]^{-1}$

or $e^{At} = \mathcal{L}^{-1}\{[sI - A]^{-1}\}$

inverse Laplace transform

matrix inverse

But temptations are usually not good.

$$[sI-A]^{-1}BU(s) \xrightarrow{\mathcal{L}^{-1}} \cancel{e^{At}Bu(t)}$$

Recall from transform theory:

If: $x(t) \longrightarrow X(s)$

$y(t) \longrightarrow Y(s)$

Then $$\underbrace{X(s)Y(s)}_{\text{multiplication in transform domain}} \xrightarrow{\mathcal{L}^{-1}} \underbrace{\int_0^t x(t-\tau)y(\tau)d\tau}_{\text{convolution in time domain}}$$

So, the correct answer:

$$X(s) = [sI-A]^{-1}x(0) + [sI-A]^{-1}BU(s)$$

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

General solution of $\dot{x} = Ax + Bu$, $x(0)$ given

$$x(t) = \boxed{e^{At}x(0)} + \boxed{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

- response to initial condition
- "homogeneous solution"
- completely characterized by matrix A.
- "zero input response"

- response to an input
- "particular solution"
- will depend on $u(t)$
- "zero state response"

initial state

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Summary of recent days:

e^{At} ~ matrix exponential

① ~ can be found by coordinate change involves eigenvectors of A

$$e^{At} = \underset{\substack{\uparrow \\ \text{matrix of eigenvectors}}}{M}\, e^{St} M^{-1}$$

S ~ matrix of eigenvalues of A

② ~ can also use $e^{At} = \mathcal{L}^{-1}\left\{[sI-A]^{-1}\right\}$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{convolution integral.}}$$

Today: Another approach to find e^{At}?
(there are many!)

Infinite series? e^t ~ scalar function

$$e^x \text{ or } e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots$$

Matrix:
$$e^{At} = I_{n\times n} + At + A^2\left(\frac{t^2}{2!}\right) + A^3\left(\frac{t^3}{3!}\right) + \cdots$$

identity matrix

power series in t

An approximation can be found by ending the series (truncate).

Q: Does there exist a finite series that is not an approximation?

$$e^{At} = \sum_{k=0}^{\infty} A^k \left(\frac{t^k}{k!}\right)$$ ← infinite!

power series in t

$$e^{At} \stackrel{?}{=} \sum_{k=0}^{p} A^k \left(\beta_k(t)\right)$$ ← finite

some other functions?

Something from linear algebra:

$$\boxed{\text{"Cayley–Hamilton Theorem" A matrix } A \text{ satisfies its own characteristic equation.}}$$

Example $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\boxed{\text{Characteristic equation is } |\lambda I - A| = 0}$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = 0 \rightsquigarrow \lambda^2 + 3\lambda + 2 = 0$$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \longrightarrow |\lambda I - A| = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

Cayley-Hamilton theorem says:

$$A^2 + 3A + 2I_{2\times 2} = O_{2\times 2}$$

$$A^2 = -2I - 3A$$

In the general case A is $n\times n$

A^n = a linear combination of

$I, A, A^2, \dots, A^{n-1}$

How can we use C-H theorem?

Power series

$$e^{At} = \sum_{k=0}^{\infty} A^k \left(\frac{t^k}{k!}\right)$$

Claim: A^k for $k = n, n+1, \cdots$ can be replaced by combinations of $I, A, A^2, \cdots, A^{n-1}$.

Next meeting: The finite series solution for e^{At}. $\boxtimes$