

ELEC-7500

September 12, 2014

- Solution of the general
linear state equation

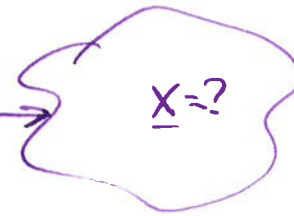
$$\dot{x} = Ax + Bu$$

- More methods for e^{At}

Laplace transform approach

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$\underline{u} \rightarrow$



$$\underline{x}(t) \xrightarrow{\mathcal{L}} \underline{X}(s) \quad \underline{u}(t) \xrightarrow{\mathcal{L}} \underline{U}(s)$$

$$\frac{d}{dt}\underline{x} = \underline{\dot{x}}(t) \xrightarrow{\mathcal{L}} s\underline{X}(s) - \underline{x}(0) \leftarrow \text{initial condition}$$

Transformed problem is

$$s\underline{X}(s) - \underline{x}(0) = A\underline{X}(s) + B\underline{U}(s)$$

Solve for $\underline{X}(s)$

$$(sI - A)\underline{X}(s) = \underline{x}(0) + B\underline{U}(s)$$

$$\underline{X}(s) = (sI - A)^{-1}\underline{x}(0) + (sI - A)^{-1}B\underline{U}(s)$$

①

$$\underline{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \underline{x}(0)}_{\text{zero input response}} + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} U(s)}_{\text{zero state response}}$$

- zero input response
- natural response
- response to initial state
- homogeneous solution
- zero state response
- particular solution
- response to input
- forced response

In time (apply inverse Laplace transform)
(element-by-element)

$$\underline{X}(t) = \underbrace{e^{\mathbf{A}t}}_{\substack{\text{matrix exponential} \\ \text{state transition matrix}}} \underline{x}(0) + \int_0^t \underbrace{e^{\mathbf{A}(t-\tau)}}_{\substack{\uparrow \\ e^{\mathbf{A}(t-\tau)}}} \mathbf{B} u(\tau) d\tau$$

(2)

$$\dot{x} = Ax + Bu, \quad x(0) \sim \text{given}$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

matrix exponential is important
convolution integral

Techniques discussed:

$$(1) \quad e^{At} = V e^{\Sigma t} V^{-1}$$

\uparrow matrix of eigenvectors \uparrow diagonal matrix of eigenvalues

$$e^{\Sigma t} = \begin{bmatrix} e^{s_1 t} & & \\ & e^{s_2 t} & \\ & & \ddots \\ & & & e^{s_n t} \end{bmatrix}$$

$s_i \sim \text{eigenvalues}$

Easy if eigenvalues are unique + real.
"distinct"

(3)

⑩ What if eigenvalues are not unique?

$$e^{At} = V e^{Jt} V^{-1}$$

J-Jordan matrix

$$J = \begin{bmatrix} \boxed{\lambda_1} & & & \\ & \boxed{\lambda_2} & & \\ & & \boxed{\lambda_3} & \\ & & & \boxed{\lambda_4} \end{bmatrix}$$

Diagram illustrating the structure of the Jordan matrix J for non-unique eigenvalues. The matrix is block diagonal, with blocks corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The blocks are arranged in a way that shows the Jordan form for each eigenvalue. The blocks are:

- λ_1 : A single block $\boxed{\lambda_1}$.
- λ_2 : A block $\boxed{\lambda_2}$ and a block $\boxed{\lambda_2}$ (indicated by a dashed box).
- λ_3 : A block $\boxed{\lambda_3}$ and a block $\boxed{\lambda_3}$ (indicated by a dashed box).
- λ_4 : A block $\boxed{\lambda_4}$ and a block $\boxed{\lambda_4}$ (indicated by a dashed box).

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④

② Laplace transform

$$e^{At} = \text{Inverse Laplace transform of } [sI - A]^{-1}$$

$$e^{At} = \mathcal{L}^{-1} \{ [sI - A]^{-1} \}$$

⑤

(3)

Series approximation. (Intind e)

Consider a scalar exponential function

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$$

power series. t^n — the basis functions

So we could write for the matrix case

$$e^{At} = \underline{I} + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots$$

identity matrix $\begin{bmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{bmatrix}$

↑
when do we
stop?
or truncate?

(6)

④ Finite series representation.

- When we used power series t, t^2, t^3, \dots
we need an infinite series to create
 e^{At} .

- Q: Can we ~~used~~ use different functions
 $\beta_1(t) \dots \beta_n(t)$ so that n finite?

$$e^{At} = \sum_{i=1}^n A^i \beta_i(t)$$

⑦