

LTI State Variable Solutions

September 17, 2002

September 18, 2003

Sept 9, Sept 11, 2015/2016

The Problem

- State variable model

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad (\text{LTI})$$

- Initial Condition

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

- Desire $\mathbf{x}(t)$ for $t \geq t_0$

Lesson Goals

State variable model solutions by:

1. Laplace Transform
2. Series Approximation
3. Cayley-Hamilton Theorem
4. Similarity Transformation

Reference: Brogan, Chap. 8 and 9

Laplace Transform Approach

- Apply Laplace Transform

$$s\mathbf{X}(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

- Some matrix algebra

$$[s\mathbf{I} - \mathbf{A}]\mathbf{X}(s) = \mathbf{x}_0 + \mathbf{B}\mathbf{U}(s)$$

Solution in s-Domain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{x}_0 + [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}\mathbf{U}(s)$$

Solution in s-Domain

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{x}_0 + \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{B}\mathbf{U}(s)$$

"Natural"
"zero-input"
response

Solution in s-Domain

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{x}_0 + \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{B}\mathbf{U}(s)$$

"Natural"
"zero-input"
response

"Forced"
"zero-state"
response

Solution in s-Domain

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{x}_0 + \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{B}\mathbf{U}(s)$$

Call this matrix term
 $\Phi(s)$

The State Transition Matrix

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$$

Inverse Laplace
Transform



The State Transition Matrix

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$$

Inverse Laplace
Transform



$$\Phi(t) = L^{-1} \left\{ [s\mathbf{I} - \mathbf{A}]^{-1} \right\}$$

Solution in Time-Domain

$$\mathbf{x}(t) = \underbrace{\Phi(t - t_0)\mathbf{x}_0}_{\text{"Natural" response}} + \underbrace{\int_{t_0}^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau}_{\text{"Forced" response}}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 \\ -8 & s+2 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad [\mathbf{sI} - \mathbf{A}] = \begin{bmatrix} s & -1 \\ -8 & s+2 \end{bmatrix}$$

$$[\mathbf{sI} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{s+2}{(s+4)(s-2)} & \frac{1}{(s+4)(s-2)} \\ \frac{8}{(s+4)(s-2)} & \frac{s}{(s+4)(s-2)} \end{bmatrix}$$

Example

$$\left[s\mathbf{I} - \mathbf{A} \right]^{-1} = \begin{bmatrix} \frac{s+2}{(s+4)(s-2)} & \frac{1}{(s+4)(s-2)} \\ \frac{8}{(s+4)(s-2)} & \frac{s}{(s+4)(s-2)} \end{bmatrix}$$

Example

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{s+2}{(s+4)(s-2)} & \frac{1}{(s+4)(s-2)} \\ \frac{8}{(s+4)(s-2)} & \frac{s}{(s+4)(s-2)} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} \frac{1}{3}e^{-4t} + \frac{2}{3}e^{2t} & -\frac{1}{6}e^{-4t} + \frac{1}{6}e^{2t} \\ -\frac{4}{3}e^{-4t} + \frac{4}{3}e^{2t} & \frac{2}{3}e^{-4t} + \frac{1}{3}e^{2t} \end{bmatrix}$$

Comment

Laplace Transform approach:

- conceptually easy to grasp
- not easy for high-order systems

The Transfer Function Matrix

Substituting $\mathbf{X}(s)$ in the output

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \left\{ \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B} + \mathbf{D} \right\} \mathbf{U}(s)\end{aligned}$$

The Transfer Function Matrix

Substituting $\mathbf{X}(s)$ in the output

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \left\{ \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right\} \mathbf{U}(s)\end{aligned}$$

The Impulse Response Matrix

$$L^{-1} \left\{ \mathbf{C} \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \mathbf{B} + \mathbf{D} \right\}$$

Series Approximation Method

Consider the *matrix exponential* of **A**:

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k \\ &= \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \dots \end{aligned}$$

Claim ...

The solution of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0$$

and $e^{\mathbf{A}t}$ is the state transition matrix!

Check initial condition

Verify solution at $t = 0$:

$$\begin{aligned}\mathbf{x}(t = 0) &= e^{\mathbf{A}0} \mathbf{x}_0 \\ &= \mathbf{I} \mathbf{x}_0 \\ &= \mathbf{x}_0\end{aligned}$$

Check state equation

Expand $d\mathbf{x}/dt$ term-by-term

$$\begin{aligned}\dot{\mathbf{x}} &= \left(\mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3 \frac{t^2}{2!} + \cdots \right) \mathbf{x}_0 \\ &= \mathbf{A} \left(\mathbf{I} + \mathbf{A} t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots \right) \mathbf{x}_0 \\ &= \mathbf{A} \mathbf{x}\end{aligned}$$

Comment

Infinite series approach:

- also conceptually easy to grasp
- practically difficult to implement

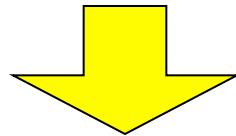
Cayley-Hamilton Theorem

Yields a method to replace
the infinite series solution
by a
FINITE sum!

Cayley-Hamilton Theorem

Consider the *characteristic equation*
of the matrix **A**:

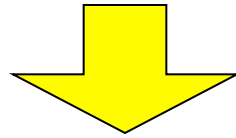
$$\det(s\mathbf{I}-\mathbf{A})=0$$



$$s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1s + \alpha_0 = 0$$

Cayley-Hamilton Theorem

The matrix **A** satisfies its own characteristic equation:



$$\mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \dots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \mathbf{0}$$

Consequences of C-H

1. \mathbf{A}^n can be expressed as a linear combination of lower order powers of \mathbf{A}
2. $e^{\mathbf{A}t}$ can be written as a *finite* sum!

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \beta_k(t) \mathbf{A}^k$$

Example

- Same **A** matrix as before ($n = 2$)
- By C-H Theorem

$$e^{At} = \beta_0(t)\mathbf{I} + \beta_1(t)\mathbf{A}$$

Q: How to find functions β_k ?

Finding functions $\beta(t)$

Apply C-H to eigenvalues of **A**

$$e^{-4t} = \beta_0(t) + \beta_1(t)(-4)$$

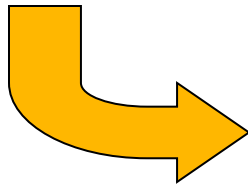
$$e^{2t} = \beta_0(t) + \beta_1(t)(2)$$

Finding functions $\beta(t)$

Apply C-H to eigenvalues of **A**

$$e^{-4t} = \beta_0(t) + \beta_1(t)(-4)$$

$$e^{2t} = \beta_0(t) + \beta_1(t)(2)$$



$$\beta_0(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-4t}$$

$$\beta_1(t) = \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}$$

Homework

Confirm

$$e^{At} = \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-4t} \right) \mathbf{I} + \left(\frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t} \right) \mathbf{A}$$

= same as first example??

The Similarity Transformation

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

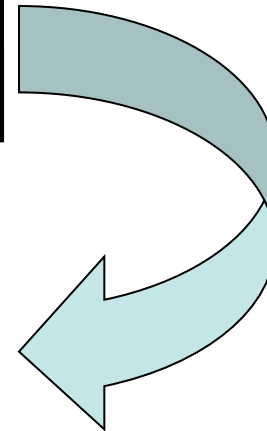
The Similarity Transformation

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

similar to

$$\mathbf{S} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$



Review & Homework

Use Matlab to find the modal matrix
M that satisfies

$$\mathbf{MS}=\mathbf{AM}$$

or

$$\mathbf{A}=\mathbf{MSM}^{-1}$$

State Transition Matrix for S

- Rows are decoupled
- Each row is easy to solve

$$e^{st} = \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Solution from Modal Solution

State model

$$\dot{\mathbf{x}} = \mathbf{Ax} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

similar to

$$\dot{\mathbf{z}} = \mathbf{Sz} \quad , \quad \mathbf{z}(0) = \mathbf{z}_0$$

by similarity transformation $\mathbf{x} = \mathbf{Mz}$



Solution from Modal Solution

State solution

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0$$

is similar to

$$\mathbf{z}(t) = e^{S t} \mathbf{z}_0$$

by similarity transformation $\mathbf{x} = \mathbf{M}\mathbf{z}$



Solution from Modal Solution

Therefore, the state solution is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{M}\mathbf{z}(t) \\ &= \mathbf{M}e^{St}\mathbf{z}_0 \\ &= \underbrace{\mathbf{M}e^{St}\mathbf{M}^{-1}}_{e^{At}}\mathbf{x}_0\end{aligned}$$

Summary of Modal Approach

- Find eigenvalues and eigenvectors
- Write state transition matrix e^{St}
- Then

$$\mathbf{x}(t) = \mathbf{M}e^{St}\mathbf{M}^{-1}\mathbf{x}_0$$

where \mathbf{M} is the modal matrix

Homework

Confirm e^{At} by transformation
from the modal form ...

Thought for the day

- "No act of kindness, no matter how small, is ever wasted."
 - *Aesop, "The Lion and the Mouse"*
Greek fabulist, 550 BC

Thought for the day

- "No act of kindness, no matter how small, is ever wasted."
– *Aesop, "The Lion and the Mouse"*
Greek fabulist, 550 BC
- "And to godliness, (*add*) brotherly kindness ..."
– *2 Peter 1:7*