ELEC-7500

September 12, 2014

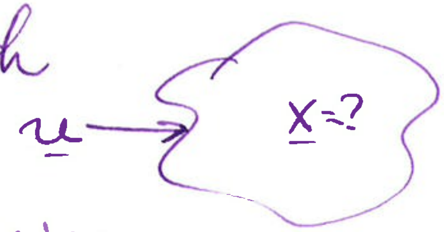
- Solution of the general linear state equation

  $\dot{x} = Ax + Bu$

- More methods for e^{At}

Laplace transform approach

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$



$$\underline{x}(t) \xrightarrow{\mathcal{L}} \underline{X}(s) \qquad \underline{u}(t) \xrightarrow{\mathcal{L}} \underline{U}(s)$$

$$\frac{d}{dt}x = \underline{\dot{x}(t)} \xrightarrow{\mathcal{L}} s\underline{X}(s) - \underline{x}(0) \leftarrow \text{initial condition}$$

Transformed problem is

$$sX(s) - x(0) = AX(s) + BU(s)$$

Solve for $X(s)$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$$

1

$$X(s) = \underbrace{(sI-A)^{-1}x(0)}_{} + \underbrace{(sI-A)^{-1}BU(s)}_{}$$

- zero input response
- natural response
- response to initial state
- homogeneous solution

- zero state response
- particular solution
- response to input
- forced response

In time (apply inverse Laplace transform)
(element-by-element)

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

- matrix exponential
- state transition matrix

$e^{A(t-\tau)}$

2

$\dot{x} = Ax + Bu$, $x(0)$ ~ given

$$x(t) = e^{At} x(0) + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{convolution integral}}$$

matrix exponential is important

Techniques discussed:

(1) $e^{At} = V e^{\Sigma t} V^{-1}$ diagonal matrix of eigenvalues

matrix of eigenvectors

$$e^{\Sigma t} = \begin{bmatrix} e^{s_1 t} & & \emptyset \\ & e^{s_2 t} & \\ \emptyset & & \ddots \; e^{s_n t} \end{bmatrix}$$

s_i ~ eigenvalues

Easy if eigenvalues are unique + real.
"distinct"

(1B) What if eigenvalues are not unique?

$$e^{At} = V e^{Jt} V^{-1}$$

J - Jorden matrix

$$J = \begin{bmatrix} \lambda_1 & & & \\ & \begin{matrix}\lambda_2 & \emptyset \\ \emptyset & \lambda_2\end{matrix} & & \emptyset \\ & & \begin{matrix}\lambda_3 & 1 \\ 0 & \lambda_3\end{matrix} & \\ \emptyset & & & \begin{matrix}\lambda_4 & 1 & 0 \\ & \lambda_4 & 1 \\ 0 & & \lambda_4\end{matrix} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_4 & 1 & 0 \\ 0 & \lambda_4 & 1 \\ 0 & 0 & \lambda_4 \end{bmatrix}$$

(2) Laplace transform

$$e^{At} = \text{Inverse Laplace transform of } [sI-A]^{-1}$$

$$e^{At} = \mathcal{L}^{-1}\left\{ [sI-A]^{-1} \right\}$$

(5)

③ Series approximation. (Infinite)

Consider a scalar exponential function

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

power series. t^n — the basis functions

So we could write for the matrix case

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} \cdots,$$

identity matrix $\begin{bmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{bmatrix}$

when do we stop?

or truncate?

⑥

(4) Finite series representation.

- When we used power series $t, t^2, t^3 \ldots$ we need an infinite series to create e^{At}.

- Q : Can we ~~used~~ use different functions $\beta_1(t) \ldots \beta_n(t)$ so that

$$e^{At} = \sum_{i=1}^{n} A^i \beta_i(t)$$

n finite ?

(7)