LTI State Variable Solutions

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The Problem

- State variable model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \text{(LTI)}$$

- Initial Condition

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

- Desire $\mathbf{x}(t)$ for $t \geq t_0$

Lesson Goals

State variable model solutions by:

1. Laplace Transform
2. Series Approximation
3. Cayley-Hamilton Theorem
4. Similarity Transformation

Reference: Brogan, Chap. 8 and 9

Laplace Transform Approach

- Apply Laplace Transform

$$s\mathbf{X}(s)-\mathbf{x}_0=\mathbf{A}\mathbf{X}(s)+\mathbf{B}\mathbf{U}(s)$$

- Some matrix algebra

$$\left[s\mathbf{I}-\mathbf{A}\right]\mathbf{X}(s)=\mathbf{x}_0+\mathbf{B}\mathbf{U}(s)$$

Solution in s-Domain

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{x}_0 + \left[s\mathbf{I} - \mathbf{A}\right]^{-1} \mathbf{B}\mathbf{U}(s)$$

Solution in s-Domain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}_0 + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

"Natural"
"zero-input"
response

Solution in s-Domain

$$\mathbf{X}(s) = \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{x}_0 + \left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{BU}(s)$$

"Natural"
"zero-input"
response

"Forced"
"zero-state"
response

Solution in s-Domain

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}_0 + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

Call this matrix term

$\Phi(s)$

The State Transition Matrix

$$\Phi(s) = \left[s\mathbf{I} - \mathbf{A} \right]^{-1}$$

Inverse Laplace Transform

The State Transition Matrix

$$\Phi(s) = \left[s\mathbf{I} - \mathbf{A} \right]^{-1}$$

Inverse Laplace Transform

$$\Phi(t) = L^{-1}\left\{ \left[s\mathbf{I} - \mathbf{A} \right]^{-1} \right\}$$

Solution in Time-Domain

$$\mathbf{x}(t) = \underbrace{\Phi(t-t_0)\mathbf{x}_0}_{\text{"Natural" response}} + \underbrace{\int_{t_0}^{t} \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau}_{\text{"Forced" response}}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 \\ -8 & s+2 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad \left[s\mathbf{I} - \mathbf{A} \right] = \begin{bmatrix} s & -1 \\ -8 & s+2 \end{bmatrix}$$

$$\left[s\mathbf{I} - \mathbf{A} \right]^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+4)(s-2)} & \dfrac{1}{(s+4)(s-2)} \\ \dfrac{8}{(s+4)(s-2)} & \dfrac{s}{(s+4)(s-2)} \end{bmatrix}$$

Example

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+4)(s-2)} & \dfrac{1}{(s+4)(s-2)} \\ \dfrac{8}{(s+4)(s-2)} & \dfrac{s}{(s+4)(s-2)} \end{bmatrix}$$

Example

$$\left[s\mathbf{I}-\mathbf{A}\right]^{-1}=\begin{bmatrix} \dfrac{s+2}{(s+4)(s-2)} & \dfrac{1}{(s+4)(s-2)} \\ \dfrac{8}{(s+4)(s-2)} & \dfrac{s}{(s+4)(s-2)} \end{bmatrix}$$

$$\boxed{\Phi(t)=\begin{bmatrix} \dfrac{1}{3}e^{-4t}+\dfrac{2}{3}e^{2t} & -\dfrac{1}{6}e^{-4t}+\dfrac{1}{6}e^{2t} \\ -\dfrac{4}{3}e^{-4t}+\dfrac{4}{3}e^{2t} & \dfrac{2}{3}e^{-4t}+\dfrac{1}{3}e^{2t} \end{bmatrix}}$$

Comment

Laplace Transform approach:

- conceptually easy to grasp
- not easy for high-order systems

The Transfer Function Matrix

Substituting $\mathbf{X}(s)$ in the output

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ &= \left\{\mathbf{C}\left[s\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}\right\}\mathbf{U}(s)\end{aligned}$$

The Transfer Function Matrix

Substituting $\mathbf{X}(s)$ in the output

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{CX}(s) + \mathbf{DU}(s)\\ &= \mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{BU}(s) + \mathbf{DU}(s)\\ &= \left\{\mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}+\mathbf{D}\right\}\mathbf{U}(s)\end{aligned}$$

The Impulse Response Matrix

$$\mathcal{L}^{-1}\left\{\mathbf{C}\left[s\mathbf{I}-\mathbf{A}\right]^{-1}\mathbf{B}+\mathbf{D}\right\}$$

Series Approximation Method

Consider the *matrix exponential* of **A**:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k$$

$$= \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \cdots$$

Claim …

The solution of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

and $e^{\mathbf{A}t}$ is the state transition matrix!

Check initial condition

Verify solution at $t = 0$:

$$\mathbf{x}(t=0) = e^{\mathbf{A}0}\mathbf{x}_0$$

$$= \mathbf{I}\mathbf{x}_0$$

$$= \mathbf{x}_0$$

Check state equation

Expand d**x**/dt term-by-term

$$\dot{\mathbf{x}} = \left(\mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3 \frac{t^2}{2!} + \cdots \right) \mathbf{x}_0$$

$$= \mathbf{A} \left(\mathbf{I} + \mathbf{A} t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots \right) \mathbf{x}_0$$

$$= \mathbf{A}\mathbf{x}$$

Comment

Infinite series approach:

- also conceptually easy to grasp
- practically difficult to implement

Cayley-Hamilton Theorem

Yields a method to replace
the infinite series solution
by a
FINITE sum!

Cayley-Hamilton Theorem

Consider the *characteristic equation* of the matrix **A**:

$$\det(s\mathbf{I}-\mathbf{A})=0$$

$$\Downarrow$$

$$s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0 = 0$$

Cayley-Hamilton Theorem

The matrix **A** satisfies its own characteristic equation:

$$\mathbf{A}^{n} + \alpha_{n-1}\mathbf{A}^{n-1} + \ldots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = \mathbf{0}$$

Consequences of C-H

1. $\mathbf{A}^n$ can be expressed as a linear combination of lower order powers of **A**
2. $e^{\mathbf{A}t}$ can be written as a *finite* sum!

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \beta_k(t)\mathbf{A}^k$$

Example

- Same **A** matrix as before (n = 2)
- By C-H Theorem

$$e^{\mathbf{A}t} = \beta_0(t)\mathbf{I} + \beta_1(t)\mathbf{A}$$

Q: How to find functions β_k?

Finding functions $\beta(t)$

Apply C-H to eigenvalues of **A**

$$e^{-4t} = \beta_0(t) + \beta_1(t)(-4)$$

$$e^{2t} = \beta_0(t) + \beta_1(t)(2)$$

Finding functions $\beta(t)$

Apply C-H to eigenvalues of **A**

$$e^{-4t} = \beta_0(t) + \beta_1(t)(-4)$$

$$e^{2t} = \beta_0(t) + \beta_1(t)(2)$$

$$\beta_0(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-4t}$$

$$\beta_1(t) = \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}$$

Homework

Confirm

$$e^{\mathbf{A}t} = \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-4t}\right)\mathbf{I} + \left(\frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}\right)\mathbf{A}$$

= same as first example??

The Similarity Transformation

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

The Similarity Transformation

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}$$

similar to

$$\mathbf{S} = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$$

Review & Homework

Use Matlab to find the modal matrix **M** that satisfies

$$\mathbf{MS=AM}$$

or

$$\mathbf{A}=\mathbf{MSM}^{-1}$$

State Transition Matrix for **S**

- Rows are decoupled
- Each row is easy to solve

$$e^{\mathbf{S}t} = \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Solution from Modal Solution

State model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

similar to

$$\dot{\mathbf{z}} = \mathbf{S}\mathbf{z} \quad , \quad \mathbf{z}(0) = \mathbf{z}_0$$

by similarity transformation **x=Mz**

Solution from Modal Solution

State solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

is similar to

$$\mathbf{z}(t) = e^{\mathbf{S}t}\mathbf{z}_0$$

by similarity transformation **x=Mz**

Solution from Modal Solution

Therefore, the state solution is

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{M}\mathbf{z}(t) \\
&= \mathbf{M}e^{\mathbf{S}t}\mathbf{z}_0 \\
&= \underbrace{\mathbf{M}e^{\mathbf{S}t}\mathbf{M}^{-1}}_{e^{\mathbf{A}t}}\mathbf{x}_0
\end{aligned}
$$

Summary of Modal Approach

- Find eigenvalues and eigenvectors
- Write state transition matrix $e^{\mathbf{S}t}$
- Then

$$\boxed{\mathbf{x}(t) = \mathbf{M}e^{\mathbf{S}t}\mathbf{M}^{-1}\mathbf{x}_0}$$

where **M** is the modal matrix

Homework

Confirm $e^{\mathbf{A}t}$ by transformation from the modal form …

Thought for the day

- "No act of kindness, no matter how small, is ever wasted."
  - *Aesop, "The Lion and the Mouse"*
    Greek fabulist, 550 BC

Thought for the day

- "No act of kindness, no matter how small, is ever wasted."
  - *Aesop, "The Lion and the Mouse"*
    Greek fabulist, 550 BC
- "And to godliness, (*add*) brotherly kindness …"
  - *2 Peter 1:7*