ELEC - 7500

August 25, 2014

- Liniarization

Dynamics are usually <u>not</u> linear

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$\underline{u}$: vector (set) of inputs

$\underline{x}$: vector of variables

$$\underline{f} = \begin{bmatrix} f_1(\underline{x}, \underline{u}) \\ \vdots \\ f_n(\underline{x}, \underline{u}) \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

Example

$$\dot{x}_1 = \sin(x_1 x_2) \quad f_1(\underline{x}, u)$$

$$\dot{x}_2 = \underbrace{\cos(x_2) + u^2}_{f_2(\underline{x}, u)}$$

Consider scalar analysis.

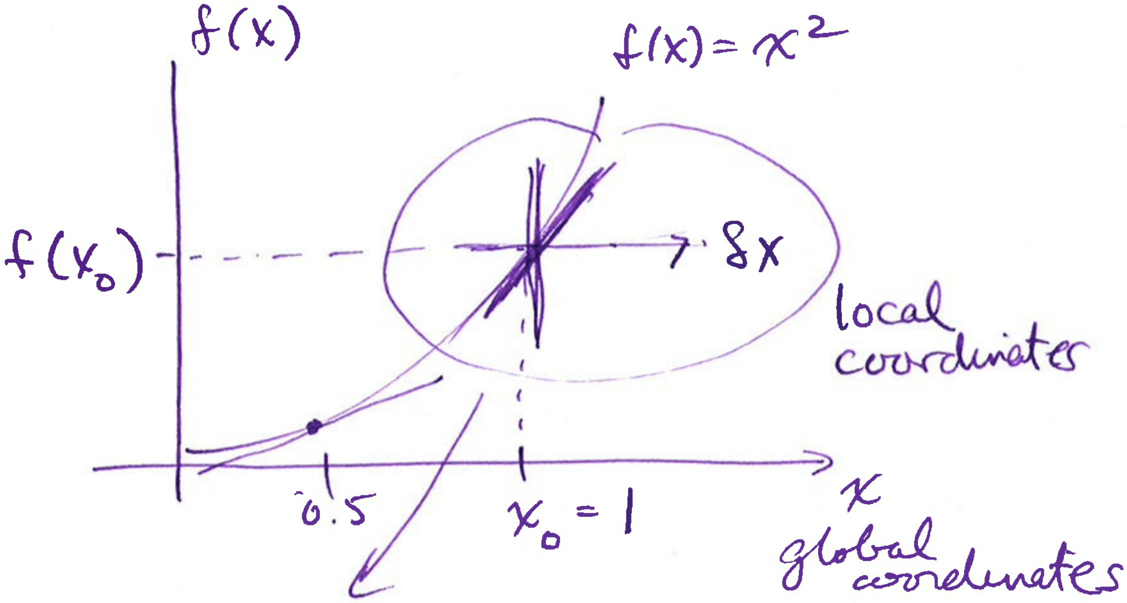
$$x(t) = \underbrace{x_0}_{\substack{\text{nominal value} \\ \text{or operating point}}} + \underbrace{\delta x}_{\text{change}}$$

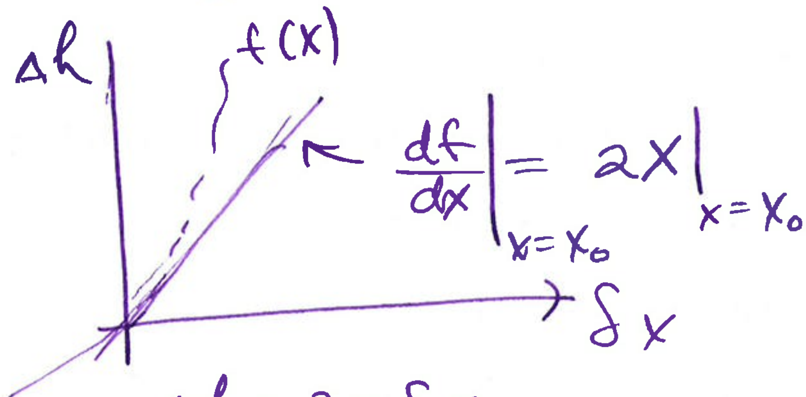
$$\frac{dx}{dt} = \frac{d}{dt}(x_0 + \delta x)$$

If x_0 is constant, then $\frac{d}{dt}x_0 \approx 0$

So study $\frac{d}{dt}\delta x$ or $\dot{\delta x}$

$$\boxed{\text{Given } \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}), \text{ what is } \dot{\underline{\delta x}} = A\,\underline{\delta x} + B\,\underline{\delta u}}$$





$$\Delta h = 2x_0 \, \delta x$$
$$= 2 \, \delta x$$

Extend to the multivariable model

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$\underline{x} = \underline{x}_0 + \delta\underline{x} \quad , \quad \underline{u} = \underline{u}_0 + \delta\underline{u}$$

$$\delta\dot{\underline{x}} = A\,\delta\underline{x} + B\,\delta\underline{u}$$

Jacobian of $\underline{f}(\underline{x}, \underline{u})$ with respect to $\underline{x}$

$$\nabla_{\underline{x}}\underline{f}(\underline{x}, \underline{u}) = \begin{bmatrix} \nabla_{\underline{x}} f_1(\underline{x}, \underline{u}) \\ \vdots \\ \nabla_{\underline{x}} f_n(\underline{x}, \underline{u}) \end{bmatrix}$$

Related to the gradient of $f(\underline{x})$

scalar-valued

$$f(\underline{x}) = x_1^2 + x_2^2$$

$$\nabla_{\underline{x}} f(\underline{x}) : \left(\frac{\partial f}{\partial \underline{x}}\right) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}$$

$$\underline{f}(\underline{x},\underline{u}) = \begin{array}{c} \begin{bmatrix} x_1^2 + x_2^2 \\ \sin x_1 + u^3 \end{bmatrix} \end{array} \begin{array}{l} f_1 \\ f_2 \end{array}$$

$$\nabla_{\underline{x}} \underline{f}(\underline{x},\underline{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2x_2 \\ \cos x_1 & 0 \end{bmatrix}_{2\times 2}$$

vector x

$$\nabla_{\underline{u}} \underline{f}(\underline{x},\underline{u}) = \begin{bmatrix} 0 \\ 3u^2 \end{bmatrix}_{2\times 1}$$

only a scalar u

Given $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$

and operating point $\underline{x}_o, \underline{u}_o$

Then a linear approximate model is given by

$$\underline{x} = \underline{x}_o + \underline{\delta x}$$

$$\underline{u} = \underline{u}_o + \underline{\delta u}$$

$$\dot{\delta \underline{x}} = A\, \delta\underline{x} + B\, \delta\underline{u}$$

where

$$A = \underbrace{\nabla_{\underline{x}} \underline{f}(\underline{x}, \underline{u})}_{\text{matrix of } \frac{\partial f_i}{\partial x_j}} \Big|_{\underline{x}_o, \underline{u}_o}$$

evaluated at $\underline{x}_o, \underline{u}_o$

$$B = \underbrace{\nabla_{\underline{u}} \underline{f}(\underline{x}, \underline{u})}_{\text{matrix } \frac{\partial f_i}{\partial u_k}} \Big|_{\underline{x}_o, \underline{u}_o}$$

Summary: Linearization

- based on Taylor Series approximation
- gradient $\nabla_{\underline{x}} f$
- jacobian $\nabla_{\underline{x}} \underline{f}$

What operating point is chosen?

$\underline{x}_0 = ?$ depends on application

$\underline{u}_0 = ?$

One possible operating point is called the "equilibrium" state $\underline{x}_e$, $\underline{u}_e$

Equilibrium state $(\underline{x}_e, \underline{u}_e)$ satisfies

$\dot{\underline{x}} = 0$, that is $\underline{f}(\underline{x}, \underline{u}) = \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Describe equilibria for

$$\underline{f}(\underline{x}, u) = \begin{bmatrix} x_1^2 + x_2^2 \\ \sin x_1^2 + u^2 \end{bmatrix}$$

$$x_1^2 + x_2^2 = 0 \qquad \underline{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; u_e = 0$$

$$\sin x_1^2 + u^2 = 0$$

$$\dot{x} = \underline{0} \qquad \dot{x} = f(\underline{x}, u)$$