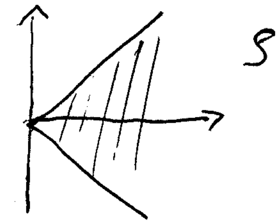
1. The bivariate density is given up to a constant:

$$f(x,y) = c\,(x^2 - y^2)\,e^{-x}, \quad x > 0, |y| \le x.$$



Remember to sketch the support on the plane. **It's crucial!**

(a) (10%) Find c.

(b) (10%) Compute exactly one of the conditional expectations $\mathsf{E}[Y|x]$ or $\mathsf{E}[X|y]$.

Your choice. *Remark: One is easy while the other is tedious. Choose wisely!*

You can do it even without c - your answer would have c in it if c weren't computed.

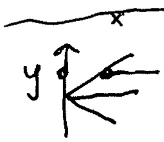
(a) $$\iint_S f(x,y)\,dA = 2\left(\iint_{S_+} x^2 e^{-x}\,dA - \iint_{S_+} y^2 e^{-x}\,dA\right) = \mathrm{I} - \mathrm{II}$$

$$\mathrm{I} = \int_0^\infty \int_0^x x^2 e^{-x}\,dy\,dx = \int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$$

$$\mathrm{II} = \int_0^\infty \int_0^x y^2 e^{-x}\,dy\,dx = \int_0^\infty \frac{x^3}{3} e^{-x}\,dx = \frac{6}{3} = 2$$

$$c = \frac{1}{6+2} = \frac{1}{8}$$

(b)  by symmetry $E[Y|x] = 0$



For $E[X|y]$ the marginal & conditional $\Big\}$ $f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$ needed

$$f_Y(y) = \frac{1}{8}\int_y^\infty (x^2-y^2)e^{-x}\,dx$$

$x = u + y$, $dx = du$

$x^2 - y^2 = u^2 + 2uy + y^2 - y^2 = u^2 + 2uy$

$$= \frac{1}{8}\int_0^\infty (u^2 + 2uy)e^{-u}\,du\; e^{-y}$$

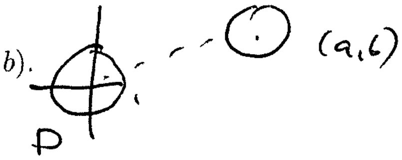
$$= \frac{1}{8}\Big[\underbrace{2}_{\Gamma(3)} + 2y \cdot \underbrace{1}_{\Gamma(2)}\Big]e^{-y} = \frac{1+y}{4}e^{-y}, \quad y > 0$$

$$E[X|y] = \frac{4e^y}{1+y}\int_0^\infty x(x^2 - y^2)e^{-x}\,dx =$$

$\Gamma(4)$ $\quad\Gamma(2)$

$$= \frac{4e^y}{1+y}\left[6 - y^2\right]. \quad \text{uff!}$$

2. Let (X, Y) be uniformly distributed on the disk D of radius 1 and center (a, b). That is, the support is described as $(x-a)^2 + (y-b)^2 \le 1$.



Hint: *Observe and use the shift:* $(X, Y) = (U + a, V + b)$, *where* (U, V) *is uniformly distributed on the unit desk centered at the origin. Otherwise you may get lost.*

$X = U + a$

$Y = V + b$

(a) Compute exactly the variances $\mathsf{Var}(X)$ and $\mathsf{Var}(Y)$. $\int_0^1 u^2\sqrt{1-u^2}\,du = \frac{\pi}{16}$.

(b) Compute exactly $\mathsf{Var}(X - Y)$.

(c) Compute exactly the conditional variances $\mathsf{Var}(X|Y = y)$ and $\mathsf{Var}(Y|X = x)$.

(d) Consider the mixed $V(x, y) = \mathsf{Var}(X|Y = y) + \mathsf{Var}(Y|X = x)$ for $(x, y) \in D$.

Find the points where $V(x, y)$ attains the maximum and the minimum value. What are these values?

$x = u + a$

$y = v + b$

$f(u) = \frac{1}{\pi}$ on D

(a) $Var(X) = Var(U)$. $EU = 0$ by symmetry, so $EV = 0$.

Need $f_U(u) = \frac{1}{\pi}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv = \frac{2\sqrt{1-u^2}}{\pi}$

$Var(U) = EU^2 = \frac{2}{\pi}\int_{-1}^{1} u^2\sqrt{1-u^2}\,du = \frac{4}{\pi}\cdot\frac{\pi}{16} = \frac{1}{4} = Var(V) = Var(Y)$

(b) Need covariance, $Var(X-Y) = Var(U-V)$

$E(UV) = Cov(U,V) = \frac{1}{\pi}\iint_D uv\,dA = 0$ by symmetry

$Var(X-Y) = Var(X) + Var(Y) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(c) $Var(Y|X=x) = Var(V|U=u)$

$f(v|u) = \frac{1/\pi}{2\sqrt{1-u^2}/\pi} = \frac{1}{2\sqrt{1-u^2}}$, $u^2 \le 1$

Since $E(V|U=u) = 0$, $Var(Y|X=x) = E(V^2|U=u)$

$= \frac{1}{2\sqrt{1-u^2}}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} v^2\,dv = \frac{1-u^2}{3}$, $|u| \le 1$

$Var(X|Y=y) = \frac{1-v^2}{3}$

(d) $V(u,v) = \frac{1-u^2}{3} + \frac{1-v^2}{3} = \frac{2-u^2-v^2}{3} = \begin{cases} \max \frac{2}{3} & u^2+v^2=0 \\ \min \frac{1}{3} & u^2+v^2=1 \end{cases}$

3. For the sake of comparison we often study the quotient $Q = \dfrac{X}{Y}$ of two independent random variables with the same distribution.

Find the pdf of Q and sketch its graph when X, Y are iid

(a) (8%) exponential random variables with mean θ;

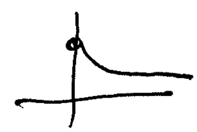
(b) (12%) uniform random variables on the interval $[0, L]$.

w.l.o.g $\theta = 1$, $L = 1$

(*Hint: Computations may be simplified without loosing generality.*)

(a) $P\left(\frac{X}{Y} \le q\right) = P(X \le qY) = \int_0^\infty (1 - e^{-qy}) e^{-y} \, dy$

$\frac{d}{dy}$: $\int_0^\infty y e^{-(q+1)y} dy = \frac{1}{(q+1)^2}$, $q > 0$



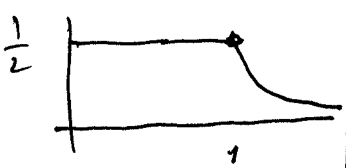
(b) $P(X \le qY) = \int_0^\infty P(X \le qy) f(y) dy$

For $q > 1$: Must split $qy \le 1$ $(y \le \frac{1}{q})$ or $qy > 1$ $(y > \frac{1}{q})$

$= \int_0^{1/q} qy \, dy + \int_{1/q}^1 dy = q \frac{1}{2q^2} + 1 - \frac{1}{q} = 1 - \frac{1}{2q}$.

For $q < 1$: $\int_0^1 qy \, dy = \frac{q}{2}$

$\frac{d}{dy}$ $\begin{cases} 1/2, & q < 1 \\ \frac{1}{2q^2}, & q > 1 \end{cases}$



By geometry

$q \le 1$ $\frac{q}{2}$

$q > 1$. $1 - \frac{1}{2q}$

S.a.a.

Both Pareto $P(Q > q) \approx \frac{c}{q}$

$EQ = \int_0^{\infty} P(Q > q) dq$ DNE

4. (a) Let $Z \sim$ N(0,1). Show that $A = |Z|$ and $S = \text{sign}(Z)$ are independent.

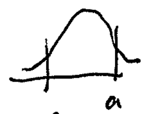
(b) Let (X, Y) be a random point on the Cartesian plane whose coordinates are independent N(0,1) random variables.

Show that the polar coordinates $R = \sqrt{X^2 + Y^2}$ and $T = \arctan\left(\frac{Y}{X}\right)$ are independent.

(a) Let $a > 0$, $s = \pm 1$, say $s = 1$.

$$P(A \le a, S = 1) = P(Z \le a, Z > 0) = P(0 < Z \le a) = \Phi(a) - \frac{1}{2}$$

$$P(A \le a) = P(|Z| \le a) = 2\Phi(a) - 1$$



So $P(A \le a)\, P(S = 1) = (2\Phi(a) - 1) \cdot \frac{1}{2} = \Phi(a) - \frac{1}{2}$ ✓

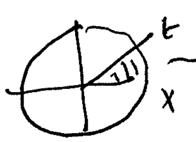
(b) Marginals

$$P(R \le x) = P(\underbrace{X^2 + Y^2}_{\chi^2_2} \le x^2) = 1 - e^{-x^2/2}$$

$$\chi^2_2 = \exp(\theta = 2)$$

$$P(T \le t) = \frac{t}{2\pi}, \quad 0 \le t < \pi$$

$x > 0$ $\quad P(R \le x, T \le t) = \iint_D \phi(x)\phi(y)\, dA$



$\to D \quad \phi(x)\phi(y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$

$D:$ $0 \le \theta \le t$, $0 \le r \le x$

$$= \int_0^t \int_0^x \frac{e^{-r^2/2}}{2\pi}\, r\, dr\, d\theta$$

$$= \frac{t}{2\pi} \int_0^x e^{-r^2/2}\, r\, dr = \frac{t}{2\pi} \cdot \left(1 - e^{-x^2/2}\right)$$

✓

4. (a) Let $Z \sim N(0,1)$. Show that $A = |Z|$ and $S = \text{sign}(Z)$ are independent.

(b) Let (X, Y) be a random point on the Cartesian plane whose coordinates are independent N(0,1) random variables.

Show that the polar coordinates $R = \sqrt{X^2 + Y^2}$ and $T = \arctan\left(\frac{Y}{X}\right)$ are independent.

An alternative for (b)
(for those who prefer the Jacobian approach)

We have $x = r\cos\theta$, $y = r\sin\theta$ or conversely $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Then $f_{(R,T)}(r,\theta) = \underbrace{f_{(X,Y)}(x,y)}_{\text{in terms of } (r,\theta)} \cdot \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$ ⊗

since $e^{-\frac{x^2+y^2}{2}} = e^{-r^2/2}$

$$= \frac{1}{2\pi} e^{-r^2/2} \cdot r, \quad 0 \le r < \infty, \ 0 \le \theta < 2\pi$$

$$= \left(r e^{-r^2/2}\right) \cdot \frac{1}{2\pi}$$

Clearly, the variables (r, θ) are separated,

Thus R, T are independent

Remark: ⊗ is factually formula (7.1) in Ross' Subsection 6.7

Indeed, $\left[\frac{\partial(x,y)}{\partial(r,\theta)}\right] = \left[\frac{\partial(r,\theta)}{\partial(x,y)}\right]^{-1}$.

5. Let (X,Y) be bivariate normal with $\mu_X = 1, \mu_Y = 1$, $\sigma_X^2 = 1$, $\sigma_Y^2 = 4$, and $\rho = \frac{1}{2}$.

(a) Is it possible to find c such that $X - Y$ and $X - cY$ are independent?

If no, then explain why not. If yes, then find such c.

$$\sigma_{XY} = \rho \sigma_X \sigma_Y = \tfrac{1}{2} \cdot 1 \cdot 2 = 1$$

(b) Compute the correlation coefficient $R(c) = \rho_{X-Y,X-cY}$.

Does $R(c)$ attain the maximum value? If so, what value and for what c? If not, why not?

Does $R(c)$ attain the minimum value? If so, what value and for what c? If not, why not?

(c) Compute $\mathsf{E}[Y|X = \frac{1}{2}]$ and $\mathsf{P}(X < Y|Y = 3)$.

(a) $Cov(X-Y, X-cY) = \sigma_X^2 - (c+1)\sigma_{XY} + c\sigma_Y^2 = 3c = 0$ $\quad \boxed{c=0}$

So $X - Y \perp X$, hence $X - Y \perp\!\!\!\perp X$ (normal !)

(b) $Var(X-cY) = \sigma_X^2 - 2c\sigma_{XY} + c^2\sigma_Y^2 = 1 - 2c + 4c^2$

$Var(X-Y) = (c=1) = 3$

$$R(c) = \frac{3c}{\sqrt{3}\sqrt{1-2c+4c^2}} = \sqrt{3}\,\frac{1}{\sqrt{\frac{1}{c^2} - \frac{2}{c} + 4}}$$

$h(c) = 4 - \frac{2}{c} + \frac{1}{c^2}$

$h'(c) = \frac{2}{c^2} - \frac{2}{c^3} = \frac{2}{c^3}(c-1)$

only one critical point

$\boxed{\begin{array}{l} \max = 1 \text{ at } c = 1 \\ R(c) > -\frac{\sqrt{3}}{2}, \text{ no min} \end{array}}$

(c) $E[Y|X = \frac{1}{2}] \underset{\text{regression}}{=} \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)\Big|_{x=\frac{1}{2}} = \frac{1}{2}$

Put $W = [X - Y | Y = 3] = [X | Y = 3] - 3$

$E[X|Y=3] \underset{\text{regression}}{=} \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y) = \frac{3}{2}$. $\quad EW = \frac{3}{2} - 3 = -\frac{3}{2}$

$Var(W) = Var[X|Y=3] = (1-\rho^2)\sigma_X^2 = 3/4$

So $W \sim N(-3/2, 3/4)$

$$P(W < 0) = P\left(\frac{W + 3/2}{\sqrt{3}/2} < \frac{3/2}{\sqrt{3}/2}\right) = \Phi(\sqrt{3})$$

Bonus

Choose one for 2 points and another one for 1 point, and the third also for 1 point.

(B1) Let U, V be independent exponential with mean θ. Find the surprising distribution of

$$|U - V| = \max(U, V) - \min(U, V).$$

(B2) We know that the quotient $Q = \dfrac{X}{Y}$ of two independent N(0,1) RVs is Cauchy.
Is is true that the product $X = QY$ is N(0,1), with independent Cauchy Q and $Y \sim$ N(0,1)? Explain.

(B3) Let X have a Beta distribution, i.e., the pdf is $f(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$.
Compute exactly $\mathsf{E}\,(X^3)$. If you prefer, you may use $\alpha = \beta = \frac{1}{2}$ (recall that $\Gamma(1/2) = \sqrt{\pi}$).

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2 pointer

B2 Moments!

$E|X| < \infty$ but $E|QY| = E|Q| \cdot E|Y| = \infty$

contradiction

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1 pointers

B1 $F(x) = P(U \vee V - U \wedge V \le x) = P(U - V \le x, U > V) + P(V - U \le x, V > U)$

$= 2\, P(U - V \le x, U > V) = 2\int_0^\infty P(U \le x + v)\, e^{-v}\, dv$

$\boxed{\text{w.l.o.g. } \theta = 1}$ $\quad = 2\int_0^\infty (1 - e^{-x-v})\, e^{-v}\, dv$

$f(x) = F'(x) = 2\int_0^\infty e^{-x-2v}\, dv = e^{-x} \underbrace{\int_0^\infty 2e^{-2v}\, dv}_{1} = e^{-x}$

Same! Wow!

B3

$$EX^3 = \int_0^\infty x^3 f(x)\, dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\cancel{\Gamma(\beta)}\Gamma(3+\alpha)}{\Gamma(3+\alpha+\beta)} \boxed{\underbrace{\frac{\cancel{\Gamma(3+\alpha+\beta)}}{\cancel{\Gamma(3+\alpha)\Gamma(\beta)}} \int_0^\infty x^{3+\alpha-1}(1-x)^{\beta-1}\, dx}}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+3)} \cdot \frac{\Gamma(\alpha+3)}{\Gamma(\alpha)} = \frac{\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} \frac{(\alpha+2)(\alpha+1)\alpha\,\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$= \frac{(\alpha+2)(\alpha+1)\alpha}{(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)}$$