THE LAW OF LARGE NUMBERS

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PROBABILITY I

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Outline

1. LLN
2. Chebyshev's inequality and WLLN
3. SLLN

Another "Iron Rule" of probability

A **random sample** is just a sequence $X_1, \ldots, X_n$ of iid random variables. **LLN** - the **Law of Large Numbers** states that **sample average** in a large sample is close to the ideal average

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n} \approx \mu = \mathrm{E}\, X_i, \quad n \text{ large.}$$

Theorem

For a random sample with mean μ, $\mathrm{Var}(\overline{X} - \mu) \to 0$ as $n \to \infty$.

Proof: easy.

Closeness

The symbol "$A \approx B$" means that that two quantities A and B are close under some circumstances.

If A and B are RVs with equal means, then

$$||A - B||_2 = \sqrt{\mathrm{Var}(A - B)}$$

is a natural measurement of the closeness, akin to the Euclidean distance or norm in an n-space.

Typically, we use a **metric** (Google it up) to measure the distance. Other methods do exist. One appears below.

An obvious simple inequality

Denote a Boolean (a.k.a. "logic") function by $\mathbb{1}_{\{statement\}}$.

Its value is 1 if the statement is true, 0 if it is false.

Consider a random variable $X \geq 0$ and a number $c > 0$. Then

$$X \geq X\mathbb{1}_{\{X>c\}} \geq c\mathbb{1}_{\{X>c\}}$$

Apply the expectation:

$$\mathrm{E}\,(X) \geq \mathrm{E}\left(X\mathbb{1}_{\{X>c\}}\right) \geq \mathrm{E}\left(c\mathbb{1}_{\{X>c\}}\right) = c\mathrm{P}(X > c)$$

Chebyshev's inequality

Rewriting,

$$P(X > c) \leq \frac{E\,X}{c}.$$

Theorem (Chebyshev's inequality)

$$P(|X-\mu| > \epsilon) \leq \frac{\mathrm{Var}(X)}{\epsilon^2}.$$

Proof In the top inequality, substitute $X := |X-\mu|^2 \geq 0$ and put $c = \epsilon^2$. Algebra:

$$P(|X-\mu| > \epsilon) = P(|X-\mu|^2 > \epsilon^2) \leq \frac{E\,|X-\mu|^2}{\epsilon^2} = \frac{\mathrm{Var}(X)}{\epsilon^2}.$$

WLLN

Corollary (Weak Law of Large Numbers)

Let $\mathrm{E}\, X^2 < \infty$. *For the sample average of a sample of size n:*

$$\mathrm{P}(|\overline{X} - \mu| > \epsilon) \to 0$$

Proof: $\mathrm{P}(|\overline{X} - \mu| > \epsilon) \leq \dfrac{\mathrm{Var}(\overline{X})}{\epsilon^2} = \dfrac{\sigma^2}{n\epsilon^2} \to 0.$ ■

In words: *the probability that the deviation of the sample mean from the ideal mean exceeds any* ϵ *is almost 0 when n is large.*

Remark. The WLLN is valid when only the mean exists.

A potential proof exceeds the math prerequisites for this course.

SLLN

Why this LLN is called “weak”?

First, it is derived from a stronger LLN based on the variance.

Secondly, the term “strong” is reserved for the following LLN whose proof is rather advanced.

Theorem (**Strong Law of Large Numbers**)

For the sample average of a sample of size n:

$$\mathrm{P}\left(\lim_n \overline{X} = \mu\right) = 1$$

Illustration: WEAK but not STRONG

Consider a sequence of Bernoulli random variables with $p_n \to 0$.

The following “wandering” variables do not converge at all.

Yet, the probability of any ϵ-deviation from 0 converges to 0.