# POISSON PROCESS

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## PROBABILITY I

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# Outline


1. Poisson distribution
2. Poisson process
3. Natural extensions

# BP with rare 1

Imagine a typical outcome of BP with a small $p$:

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Replace 0 by – , and 1 by ▮ :

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In other words, when the inter-trial interval shrinks to 0, in the limit we obtain a process of **signals**, subject to the following assumptions that guarantee its existence. The discrete time-line becomes a continuous time line in the limit.

# The limit of BP

Suppose that the time axis is partitioned into subintervals of length $\frac{1}{n}$ each. We mark our "1"s at the partition points. In particular, the unit time interval $[0, 1]$ has $n$ points and recall that $np$ is the mean value of the number of "1" (now: signals in $[0, 1]$) in $n$ trials.

The assumptions are:

- $p \to 0$ (in practice $p$ is very small),
- $n \to \infty$ (in practice $n$ is very large),
- $np \to \Lambda$ (in practice $np$ is moderate).

# Poisson distribution emerges

What is the limit of the binomial distribution? Let us rewrite the expression, replacing $p$ by $\Lambda/n$ and letting $n \to \infty$:

$$\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} &\approx \frac{n!}{k!(n-k)!} \left(\frac{\Lambda}{n}\right)^k \left(1 - \frac{\Lambda}{n}\right)^{n-k} \\
&= \frac{\Lambda^k}{k!} \left(1 - \frac{\Lambda}{n}\right)^n \left[ \frac{(n)_k}{n^k} \left(1 - \frac{\Lambda}{n}\right)^{-k} \right] \\
&\approx \frac{\Lambda^k}{k!} e^{-\Lambda}.
\end{aligned}$$

In the limit the **Poisson distribution with parameter $\Lambda$** emerges.

# Poisson pmf and mgf

So this is the **Poisson pmf** with parameter $\Lambda > 0$:

$$f(x) = \frac{\Lambda^x}{x!} e^{-\Lambda}, \quad x = 0, 1, 2, \ldots$$

Let $N \sim Poisson(\Lambda)$. By definition,

$$\begin{aligned} M(t) = \mathrm{E}\, e^{tX} = & \sum_{x=0}^{\infty} e^{tx} \frac{\Lambda^x}{x!} e^{-\Lambda} = \sum_{x=0}^{\infty} e^{tx} \frac{\Lambda^x}{x!} e^{-\Lambda} \\ = & \; e^{-\Lambda} \sum_{x=0}^{\infty} \frac{(e^t \Lambda)^x}{x!} \\ = & \; e^{-\Lambda} e^{e^t \Lambda} = e^{\Lambda(e^t - 1)}. \end{aligned}$$

# mean and variance

By routine differentiation we find $M'(0) = \mathrm{E}\,X$ and $M''(0) = \mathrm{E}\,X^2$, and hence

$$\mathrm{E}\,(X) = \mathrm{Var}(X) = \Lambda,$$

and this coincidence is quite rare.

Hence the normalization of a Poisson variable $N$ with parameter $\Lambda$ is particularly simple:

$$\widetilde{N} = \frac{N - \Lambda}{\sqrt{\Lambda}}$$

# Poisson process

Consider an interval of length $t$ instead of the unit interval (cf. Slide 4). Denote

$$N_t = \textbf{the number of signals in the time interval } [0, t].$$

Then, the parameter $\lambda = \mathrm{E}\, N_1$, pertinent to the unit interval, is scaled by $t$, yielding the parameter $\Lambda = \lambda t$:

$$\mathrm{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

*We have arrived at a consistent system of Poisson random variables, which we call a* **Poisson process with intensity** $\lambda$.

# Additivity

The counting definition of "signals" in an interval $[0, t]$ can be extended to counting in any reasonable set $A \subset [0, \infty)$:

$$N(A) = \textbf{the number of signals in the set } A.$$

For example,

$$N(s, t] = N_t - N_s.$$

Obviously, counts add up:

$$N(A \cup B) = N(A) + N(B), \quad \text{if } A \text{ and } B \text{ are disjoint.} \tag{1}$$

# Independence of counts

The sum $S + S'$ of two independent binomial variables corresponding to the count of 1s in $n$ trials and $n'$ different trials is the count of 1s in $n + n'$ trials, i.e., a binomial random variable again. In symbols,

$$bin(p, n) \oplus bin(p, n') = bin(p, n + n').$$

Hence, since Poisson process is the limit of a Bernoulli process, and Poisson counts are limits of binomial counts,

$N(A)$ and $N(B)$ **are independent when $A$ and $B$ are disjoint**.

# Stationarity

The Poisson count $N(s,t]$ has the same distribution as the Poisson count $N[0,t-s] = N_{t-s}$ (see the next two slides for details).

That is, **the distribution of the Poisson count $N(A)$ depends only on the total length of the set $A$, not on its location or scatter**.

**Example**. Let $A = (1,2) \cup (7,10] \cup [15,20]$. The total length $|A| = 1+3+5 = 9$. Then, $N(A)$ is distributed as $N_9$. In particular

$$\mathrm{E}\,(N(A)) = 9\lambda = \mathrm{Var}(N(A)).$$

# Computations for stationarity (1)

We use the mgf. Since $t$ serves now as time, we must change the variable in the definition of the mgf. Let

$$M_X(u) = \mathrm{E}\, e^{uX}.$$

Let $s < t$. We know that $N[0,s] = N_s$ is Poisson$(\lambda s)$ and $N[0,t] = N_t$ is Poisson$(\lambda t)$. Further,

$$N[0,t] = N[0,s) + N(s,t], \quad \text{and } N(s,t] \text{ is independent of } N_s,$$

# Computations for stationarity (2)

Let's find the mgf of $N(s,t]$, using the fact that the mgf of the sum of two independent r.vs. is the product of their mgf's

$$e^{\lambda t(e^u-1)} = M_{N_t}(u) = M_{N_s}(u) \cdot M_{N(s,t]}(u) = e^{\lambda s(e^u-1)} \cdot M_{N(s,t]}(u),$$

Hence

$$M_{N(s,t]}(u) = \frac{e^{\lambda t(e^u-1)}}{e^{\lambda s(e^u-1)}} = e^{\lambda(t-s)(e^u-1)},$$

and we recognize the mgf of $N_{t-s}$.

# Summary - the point process

The Poisson process as a stream of random points on the positive half-line. Disjoint sets yield independent counts.

Denote by $A$ the measure (length) of a set $A \subset [0, \infty)$.

The count $N(A)$ of these random points that fall into $A$ by $N(A)$ has the Poisson distribution with the varying parameter $\lambda|A|$.

The intensity parameter $\lambda$ is the expected number of random points that fall into a set of unit measure.

The shape or location of $A$ is irrelevant, only the measure matters.

# Beyond the line and length

The same definition is valid for point processes on the plane or a surface, 3-space or a spatial region.

Only then $|A|$ will denote the surface area or volume.

Note the mgf and the probability of emptiness:

$$\mathrm{E}\, e^{tN(A)} = e^{\lambda |A| (e^t - 1)}, \quad \mathrm{P}(N(A) = 0) = e^{-\lambda |A|}.$$

The stationarity lies in the dependence only on the measure of a set $A$, regardless of its position or shape.

# Beyond stationarity

Stationarity implies the uniform distribution of random points in $A$.

In reality a physical set $A$ may have an internal structure that defies the latter property.

**Example 1**. The intensity of a stream of customer in a diner varies in time, with peaks around breakfast, lunch, or dinner.

**Example 2**. The count of a specific plant (e.g., wild flower) may depend on the distribution of nutrients in the soil and its density or structure, light, moisture, etc.

# Intensity measure

To model these more realistic phenomena we just replace the simple measure $\lambda|A|$ by more general measure

$$\Lambda(A) = \int_A \lambda(x)\, dx \quad \text{(use multiple integral if necessary).}$$

The previous stationary case corresponds to the constant intensity measure $\lambda(x) = \lambda$.

These processes are called **abstract Poisson processes** (many other terms are used). However they are more real (tangible or realistic) than the original one which is an idealization or abstraction.

# In Ross

The (standard) Poisson process appears late in Chapter 9.
It is introduced differently, mimicking the historical discovery of Poisson process in telephony. There are many (hundreds?) ways of defining the Poisson process.

In fact, the discovery was made by telecommunications engineers, not professional mathematicians. The name "Poisson process" was given to honor the French mathematician, engineer, and physicist Siméon Poisson of the 18th and 19th century.

Wikipedia has an article on Poisson process. However, beware!
It contains many inaccuracies and even utterly false statements.