MATH 5670-6670 Fall 2019

Exam 2: 20 points

Eight one-pointers

1. A student would like to find a date with at least one interest similar to his own. He loves dogs, he is a fan of Star Trek (a SF movie series), and he loves to play computer games. He looks online for a date with at least one of these traits. However, in his social group, 50% of eligible dates like dogs, 20% are interested in SF movies, while 60% like computer games. Assume that the likings are independent.

   Are his chances for a successful date better than 80%? Compute.

A - dogs $\quad P(A)=0.5$

B - SF $\quad P(B)=0.2$

C - games $\quad P(C)=0.6$

Q: $P(A\cup B\cup C) = 1-P(A'B'C') = 1-P(A')P(B')P(C')$

$= 1 - 0.5 * 0.8 * 0.4$

2. Let X, Y, Z be iid random variables with the first moment 1 and the second moment 2. Calculate exactly $\mathsf{Var}(2X - Y - Z)$. $\quad Var(X) = E(x^2) - (EX)^2 = 2-1=1$

$\|$

$4\,Var(X) + Var(Y) + Var(Z) = 4+1+1 = 6$

all equal

3. In a town of 10,000 households there are 3,000 houses with pets, owned independently of each other. 10 randomly selected house owners were asked if they own a pet, yes or no.

   Find the probability that the third "yes" was heard when the 7^{th} house owner was asked.

   A "structured" answer suffices.

BT $\quad$ 1 - "yes" , 0 - "no" $\quad p=0.3$

X_r neg. bin. $r=3$, $n=7$

$P(X_r=7) = \binom{6}{2}\, 0.3^3\, 0.7^4$

4. When you play the game of rock-paper-scissors then a tie may happen.

RR
SS
PP
again 9

Suppose that you played 10 times in a row and there were no ties; a small chance but not impossible.

Given this information, what is the conditional probability that there will be no ties in the 11^{th} and 12^{th} game as well?

W - waiting time to break the tie
geometric, $p = 3/9 = 1/3$, $q = 2/3$

$$P(W > 12 \mid W > 10) = P(W > 2) = \left(\frac{2}{3}\right)^2$$

5. Give an example of a probability distribution that has a finite mean but infinite variance.

discrete $\quad p_n = \frac{c}{n^p}$ (p - to be decided, $p > 1$, $c = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^p}}$)

$n = 1, 2, \ldots$

$$EX = c\sum_n \frac{n}{n^p} = c \sum \frac{1}{n^{p-1}}. \quad \text{so } p-1>1 ; \boxed{p>2}$$

$$EX^2 = c\sum_n \frac{n^2}{n^p} = c\sum \frac{1}{n^{p-2}} = \infty, \quad \text{so } p-2 \le 1 \ \boxed{p \le 3}$$

continuous: Similar $f(x) = \frac{c}{x^p}, x \ge 1.$ $\boxed{2 < p \le 3}$

6. Let $Y = U^4$, where the random variable U is uniformly distributed on $[0, 1]$.

That is, its pdf is 1 on $[0, 1]$ and 0 elsewhere. Equivalently, $\mathsf{P}(U \le u) = u$, $0 \le u \le 1$.

Find exactly the standard deviation of Y.

$$EY = \int_0^1 u^4\,du = 1/5$$

$$EY^2 = \int_0^1 u^8\,du = 1/9$$

$$Var(Y) = \frac{1}{9} - \left(\frac{1}{5}\right)^2 = \frac{1}{9} - \frac{1}{25} = \frac{25-9}{225} = \frac{16}{225}$$

X_n - BT, $p = 1/6$

7. You roll a fair standard die over and over. If "six" occurs in the n^{th} roll you get $\frac{1}{2^n}$ dollars, otherwise you get nothing. Let X denote your total gain. Find $\mathsf{E}\,X$ and $\mathsf{Var}(X)$.

n^{th} roll, gain $\frac{X_n}{2^n}$ $\quad EX_n = p = \frac{1}{6}$, $\mathrm{Var}\,X_n = \frac{1}{6}\cdot\frac{5}{6} = \frac{5}{36}$

$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$, $\quad EX = \sum_{n=1}^{\infty} \frac{EX_n}{2^n} = \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{6}\cdot\frac{1/2}{1-1/2} = \boxed{\frac{1}{6}}$

$\mathrm{Var}(X) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \mathrm{Var}(X_n) = \frac{5}{36}\sum_{n=1}^{\infty}\frac{1}{4^n} = \frac{5}{36}\cdot\frac{1/4}{1-1/4} = \frac{5}{36\cdot 3}$
$= \frac{5}{108}$

8. An atom of some element may occasionally and randomly emit energy in the form of an alpha particle (He-4: 2 protons and 2 neutrons). Suppose that every nanosecond one atom in million shoots out an alpha particle. Now, a given substance consists of 2 million atoms. Compute approximately (<u>no binomial distribution answers accepted for the full credit</u>):

(a) probability that the substance emitted at least 2 alpha particles.

(b) the mean and the variance of the number of emitted alpha particles in 10 nanoseconds.

It's BT but we use Poisson approx.

$p = 10^{-6}$, $n = 2\times 10^6$, $\lambda = np = 2$, $X \sim \mathrm{Pois}(2)$

(a) 1 nanosecond $\quad P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - e^{-2}(1+2) = 1 - \frac{3}{e^2}$

(b) time $t = 10$. $\Lambda = \lambda t = 20$

$E\,N_{10} = \mathrm{Var}\,N_{10} = \Lambda = 20$

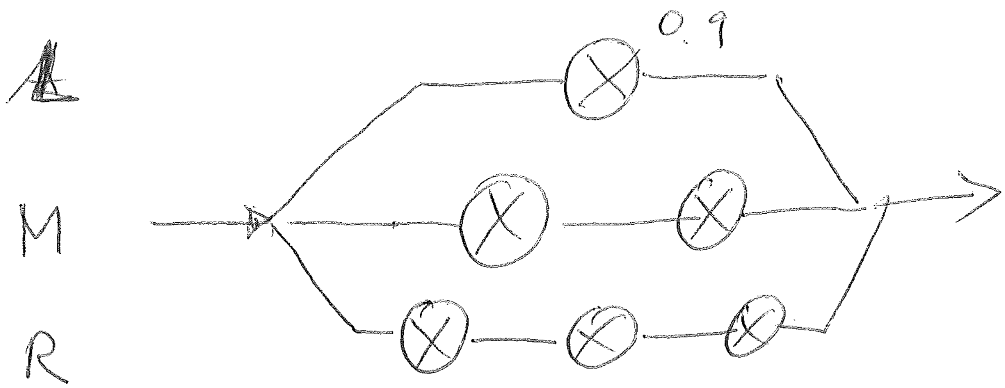
Three two-pointers

1. (2P1)

   While hiking in the mountains you came to the fork of the road. There are three paths to choose. There might be obstacles that would prevent you to pass through (avalanches, quick sand, men eating bears, hostile locals, IRS squads, falling trees - just name it!).

   On the path to the left, an obstacle may be present with probability 0.9. On the path to the right, three consecutive obstacles are possible, each with probability 0.3. The middle path may have two obstacles, each in place with probability 0.5. You flip a three sided coin, and choose a path at random.

   What is the probability that you will pass through? Assume that obstacles act independently.



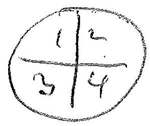
passing (no obstacle) : depending on path, average

$$\frac{0.1 + 0.5^2 + 0.7^3}{3}$$

2. (2P2) A fair four-faced die is a regular tetrahedron with numbers 1, 2, 3, 4. The outcome shows on the bottom wall when the rolled die lands on a table. Two such dice are rolled.

Simon says that the variance of the difference of outcomes equals exactly the mean outcome of one die.

Strange, isn't it? Is Simon right? You must show computations, a mere guess bears no credit.



(equivalent)

X, Y - outcomes

$$EX = \frac{1+2+3+4}{4} = \frac{10}{4} = \frac{5}{2}$$

$$E(X^2) = \frac{1+4+9+16}{4} = \frac{30}{4} = \frac{15}{2}$$

$$Var(X) = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{15}{2} - \frac{25}{4} = \frac{30-25}{4} = \frac{5}{4}$$

$$Var(Y) = Var(X) = 5/4$$

$$Var(X-Y) = Var(X) + Var(Y) = 2 \cdot \frac{5}{4} = \frac{5}{2}$$

Yes, indeed.

3. (2P2) It suffices to set up probabilities without finishing computations.

a) A fair coin was flipped 10 times and 5 heads showed. $n=10$ BT , $p=\frac{1}{2}$

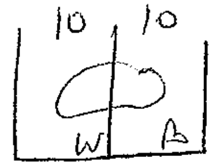
What is the probability that the seventh flip was a head?

What is the probability that the last flip was a head?

b) An urn has 10 white balls and 10 black balls. Ten balls were selected one by one <u>without replacement</u> and there were 5 white balls.

What is the probability that the 7^{th} ball was white?

What is the probability that the 10^{th} ball was white?



$n=10$

BT a) independent b) dependent but exchangeable

$X_i = 1$ (heads or white) $P(X_i=1)=\frac{1}{2}$

$$Q: P(X_k = 1 \mid X_1 + \cdots + X_{10} = 5)$$

↑ specific 7 or 10

in both cases $=$ $P(X_{10}=1 \mid X_1+\cdots+X_{10}=5)$

$$= \frac{P(X_{10}=1,\ X_1+\cdots+X_{10}=5)}{P(X_1+\cdots+X_{10}=5)} = \frac{P(X_{10}=1,\ X_1+\cdots+X_9=4)}{P(X_1+\cdots+X_{10}=5)}$$

(a) by independence

$$\frac{\frac{1}{2}\cdot\binom{9}{4}\frac{1}{2^9}}{\binom{10}{5}\frac{1}{2^{10}}} = \frac{1}{2}$$

(b) by Mult Rule & exchangeability

$$\frac{\dfrac{\frac{1}{2}\cdot\binom{9}{4}\binom{10}{5}}{\binom{19}{9}}}{\dfrac{\binom{10}{5}\binom{10}{5}}{\binom{20}{10}}} = \frac{1}{2}$$

hmm... $\longrightarrow$

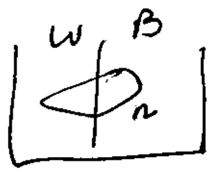
3.2P3. This simple and smart solution was found by one of the students.

We place 5 "1" and "5" on 10 positions.
If one position has to have "1", then we choose 4 positions from 9.
So $P(1 \text{ on a specific position}) = \dfrac{\binom{9}{4}}{\binom{10}{5}} = \dfrac{5}{10} = \dfrac{1}{2}$.
Doesn't matter where "1" or "0" come from.
So it's $\frac{1}{2}$ in (a) and (b)

In fact in general, when we place k "1" on n positions, $0 \le k \le n$, then by the same argument (Chap 1-2)

$$P(1 \text{ on a specific position}) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

(whether 1s come from BT, or from any urn



, with or without replacement.

Two three-pointers

1. (3P1) Suppose that a quiz has 11 true-false questions. A quiz taker answers questions independently and at random. Let X be the number of questions answered correctly.

   Show that $\mathsf{E}(X) \geq 5$ and $\mathsf{Var}(X) < 2$, if the chance of answering correctly the i^{th} question grows according to the formula $\frac{i}{11}$, $i = 1, \ldots, 11$.

BT with varying $p_i = \frac{i}{11}$, $i = 1 \ldots 11$

$X_i = 1$ if correct

$EX_i = p_i = \frac{i}{n}$, $\quad Var\, X_i = p_i q_i = \frac{i}{n}\left(1 - \frac{i}{n}\right) = \frac{i(n-i)}{n^2}$

$X = X_1 + \ldots + X_{11}$

$$EX = \sum_{i=1}^{11} EX_i = \frac{1}{11} \sum_{i=1}^{11} i = \frac{1}{11} \cdot \frac{11 \cdot 12}{2} = 6$$

$$Var\, X = \sum_{i=1}^{11} Var\, X_i = \frac{1}{11^2} \sum_{i=1}^{11} i(11-i) = \frac{1}{11^2}\left[11 \sum_{i=1}^{11} i - \sum_{i=1}^{11} i^2\right]$$

$= \cancel{\frac{1}{11^2}\left[11 \cdot \frac{11 \cdot 12}{2}\right.} -$ let's work with n

$$\frac{1}{n^2}\left[n \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2\right] = \frac{1}{n^2}\left[n \cdot \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right]$$

$\boxed{\text{pull out} = \frac{n(n+1)}{6}}$

$$\frac{n(n+1)}{6n^2}\left[3n - 2n - 1\right] = \frac{n+1}{6n}(n-1) = \frac{n^2-1}{6n}$$

for $n = 11$ $\quad \frac{\cancel{120}\ 20}{6 \cdot 11} = \frac{20}{11} < 2$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$1 + 4 + 9 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1 + 8 + 27 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2, \qquad \sum_{k=1}^{n} k^p = \text{known but more complicated}$$

2. (3P2) A game of dice: keep rolling a die, if a "one" shows before the 6th roll then you loose, otherwise you win.

ambiguous

How many games in average would you play in order to loose 6 times?

A. At most 6. B. Between 7 and 10. C. Between 11 and 14. D. at least 15.

(a lucky guess without supporting calculations is worth 0.5 point)

BT embedded into BD L - Loss

game: BT $p = \frac{1}{6}$. $P(L) = 1 - \left(\frac{5}{6}\right)^6$

playing multiple games (BT with $p_1 = 1 - \left(\frac{5}{6}\right)^6$)

negative binomial

$$E(T_1 + \cdots + T_6) = 6 \cdot \frac{1}{1 - \left(\frac{5}{6}\right)^6} \approx 9$$

so B

Bonus (2 points).

You have a fair quarter and nothing else. You can only flip it but as many times as you want. Design an experiment that would result in

(a) 3 equally likely outcomes (4 would be easy).

(b) 7 equally likely outcomes (8 would be easy).

(c) two outcomes, one with probability 5/12, and the other with probability 7/12.

(5+7=12, this is a favorite Ross' urn with 5 while balls and 7 black balls).

(d) Is it true that with just one fair quarter you can simulate sampling from any urn with any number of balls of any number of colors?

Explain briefly, yes or no.

(a) flip twice, disregard HH

(b) flip three times, disregard HHH

(c) flip four times, disregard four, e.g. one T: THHH, HTHH, HHTH, HHHT

choose 5 configurations to make the 1st event other 7 will make the other

(d) $\underbrace{|\ |\ |\ \cdots\ |\ |}_{N = n_1 + \cdots + n_k}$, #colors.

Let n be the smallest to make $N \le 2^n$. Disregard $2^n - N$. flip n times. Then partition N into groups, as shown.