MATH 5670-6670 Fall 2019

Exam 3: 20 points

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Problems are conceptual, not computational. Only minor computations are needed.

Heavy-duty approaches:
- integration by parts
- computation of moments from definition
- deriving simple from complex (e.g., exponential from Weibull)

Each of 7 problems is worth 3 points (extra 1 point). Bonus is worth up to 3 points.

- etc.

require a lot of time and energy and are very much error prone.

Basic algebra and calculus are necessities, and mostly suffice for computations.

1. Consider a pdf $f(x) = c\,x^3\,e^{-2x}$, $x > 0$, of a RV X.

   (a) Find c.

   (b) Sketch the pdf and find $\mathsf{E}X$.

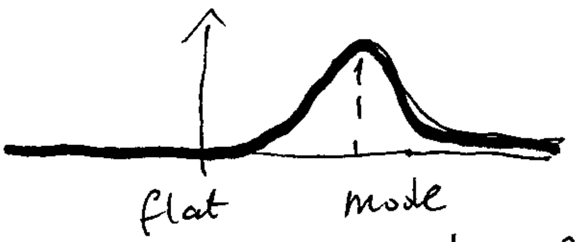
   (c) Find the mode of the pdf.

Recognize:

$$\Gamma\left(\alpha = 4, \theta = \frac{1}{2}\right)$$

$$c = \frac{1}{\theta^\alpha \Gamma(\alpha)} = \frac{2^4}{3!}$$

$$EX = \theta\alpha = \frac{1}{2} \cdot 4$$



$$(x^3 e^{-2x})' = 3x^2 e^{-2x} - 2x^3 e^{-2x}$$

$$= x^2 e^{-2x}(3 - 2x) = 0$$

mode $= \frac{3}{2}$. value: just plug in:

$$\frac{8}{3} x^3 e^{-2x} \Big|_{x = 3/2}$$

2. Let X have the pdf $f(x) = \frac{1}{3} \exp\left\{-\frac{x-b}{3}\right\}$, $x > b$.

   (a) Find b such that $\mathsf{E}X = 0$.

   (b) Sketch the pdf and find $\mathsf{Var}(X)$.

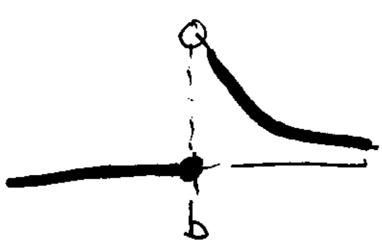
$$X = 3V + b$$

$V \sim \exp(1)$

$$EX = 3\,EV + b = 3 + b$$

$3 + b = 0$, $b = -3$

$$Var(X) = 9\,Var V = 9 \cdot 1$$



3. Let U be uniform on $[0,1]$. Put $X = \ln \frac{U}{1-U}$.

(a) Find the pdf of X.

(b) Sketch the pdf and find $\mathbb{E}X$.

algebra: $\frac{U}{1-U} \le e^x$, $U \le (1-U)e^x$

$(1+e^x)U \le e^x$

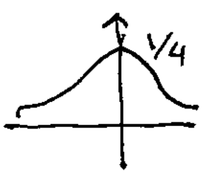
cdf $P(X \le x) = P\left(\ln \frac{U}{1-U} \le x\right) = P\left(U \le \frac{e^x}{1+e^x}\right) = \frac{e^x}{1+e^x}$ (in $(0,1)$)

$F(x) = \frac{e^x}{1+e^x} = \frac{e^x+1-1}{1+e^x} = 1 - \frac{1}{1+e^x}$

pdf $f(x) = F'(x) = \frac{e^x}{(1+e^x)^2}$ [Chain Rule]

a rough graph  Is it a bell?

$f(-x) = \frac{e^{-x}}{(1+e^{-x})^2} \cdot \frac{e^{2x}}{e^{2x}} = \frac{e^x}{(1+e^x)^2} = f(x)$. Yes!



So $EX = 0$

Alternative: $E \ln \frac{U}{1-U} = \int_0^1 \ln u \, du - \int_0^1 \ln(1-u)\, du = 0$ by symmetry

4. The Erlang ($n = 3, \theta = 3$) distribution serves as a possible model of lifetime of a battery pack that consists of three independent batteries. Each battery has exponential lifetime with mean $\theta = 3$ months and the next battery is connected automatically when the previous one expires.

- Find the probability that the battery pack won't last a year.

$X = S_3 \sim \Gamma(\alpha = 3, \theta = 3)$

$P(S_3 \le 12) = P(N_{12} \ge 3) = 1 - P(N_{12} \le 2)$

Poisson ($\lambda = \frac{1}{3}$, $t = 12$)

$\lambda t = 4$

$= 1 - e^{-4}\left(1 + 4 + \frac{4^2}{2!}\right) = 1 - \frac{13}{e^4}$

The variance of logistic distribution

Theorem 1 *For the logistic RV* $X = \ln\left(\dfrac{U}{1-U}\right)$, *where* $U \sim U(0,1)$,

$$\mathsf{Var}(X) = \frac{\pi^2}{3}.$$

Proof. We know that

$$\text{cdf}\ \ \mathsf{P}(X \le x) = \frac{1}{1+e^{-x}}, \quad \text{tail}\ \ \mathsf{P}(X > x) = \frac{1}{1+e^{x}}, \quad \text{pdf}\ \ \frac{e^x}{(1+e^x)^2}.$$

The variance equals the second moment since $\mathsf{E}\,X = 0$:

$$\mathsf{E}\,X^2 = 2\int_{\mathbb{R}} x\,\mathsf{P}(X > x)\,dx = 4\int_0^\infty \frac{x}{1+e^x}\,dx = 4\int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx.$$

Using geometric series with $r = e^{-x}$,

$$\frac{e^x}{1+e^{-x}} = \sum_{n=1}^{\infty}(-1)^n e^{-(n+1)x}.$$

Hence

$$\int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx = \sum_{n=1}^{\infty}(-1)^n \int_0^\infty e^{-(n+1)x}\,dx,$$

where the swap of the integral and the series was justified in Calculus 2.

Denote $c = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. Integrating term by term,

$$4\int_0^\infty \frac{xe^{-x}}{1+e^{-x}}\,dx = 4\sum_{n=0}^{\infty}(-1)^n\int_0^\infty x\,e^{-(n+1)x}\,dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 2c.$$

The variance formula follows. ■

Remark. The value of c.

Clearly $c = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n\ \text{odd}}^{\infty}\frac{1}{n^2} + \sum_{n\ \text{even}}^{\infty}\frac{1}{n^2} = a + b.$

Thus $b = \dfrac{c}{4}, \quad a = c - b = \dfrac{3c}{4}$, implying $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = a - b = \frac{c}{2}.$

A somewhat advanced argument is needed to compute $c = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$

5. The Weibull ($\beta = 0.5$, $\theta = 3$ [years]) distribution may serve as a liftetime model of some species of sea urchin (a spherical creature with spikes).

   (a) Verify whether its average lifetime exceeds or doesn't exceed 20 years.

   (b) Find the conditional probability of surviving 25 years when it already has survived 16 years.

$W = V^2$, where $V \sim \exp(\theta = 3)$

(a) $EW = E(3V_1)^2 = 9 \cdot \int_0^\infty v^2 e^{-v} dv = 9 \cdot \Gamma(3) = 9 \cdot 2 = 18$

$V_1 \sim \exp(\theta = 1)$

$\boxed{18 < 20}$

(b) $P(W > 25 \mid W > 16) = P(V^2 > 25 \mid V^2 > 16)$

$= P(V > 5 \mid V > 4) = P(V > 1) = e^{-1/3}$ $(\approx 72\,\%)$

lack of memory of exp

6. A discrete uniform RV on $\{1, ..., m\}$ has mean $\frac{m+1}{2}$ and variance $\frac{m^2-1}{12}$. $\mu = 4$, $\sigma^2 = 4$, $\sigma = 2$

A spinner with 7 equal angles with points 1,..., 7 is spun n times.

The outcomes $X_1, \ldots, X_n$ are noted and the average $\overline{X}$ is computed. $E(\bar{X}) = 4$, $Var(\bar{X}) = \frac{4}{n}$

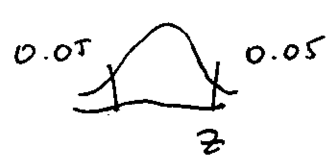
Approximately, how large must n be so that $\mathsf{P}\left(|\overline{X} - 4| > 0.1\right) < 0.1$?

A. $n \leq 100$. B. $100 < n \leq 500$. C. $50 < n \leq 1000$. D. $n > 1000$.

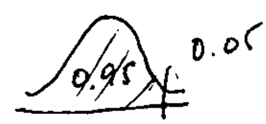
Show the computations for the full credit.

normalize $P\left(\left| \frac{\bar{X}-4}{2}\sqrt{n} \right| > \frac{0.1}{2}\sqrt{n} \right) < 0.1$

CLT $\approx P\left(|Z| > \frac{0.1}{2}\sqrt{n} \right) \approx 0.1$



Ross' Right Tail Tables for $p = 0.95$, $z \approx 1.65$



So $\frac{0.1}{2}\sqrt{n} \approx 1.65$

$\sqrt{n} \approx \frac{3.3}{0.1} = 33$

$n \approx 33^2 > 1000$

7. The square X^2 of a standard normal N(0,1) RV X may also serve as a model of lifetime.

(a) Show that it distributed as the half-signal in a Poisson process of half-unit intensity $\lambda = 0.5$.

(b) Show that the expected lifetime length equals 1.

(c) Let X and Y be independent N(0,1) RVs.

Consider the average $Z = \dfrac{X^2+Y^2}{2}$ of two lifetimes. Compute $\mathsf{P}(Z > 1)$.

(You must not use the χ^2 tables.)

(a) $P(X^2 \le y) = P(|X| \le \sqrt{y}) = 2\Phi(\sqrt{y}) - 1$

$\frac{d}{dy} \mapsto 2 \cdot \frac{1}{2\sqrt{y}} \varphi(\sqrt{y}) = \frac{1}{2\pi} y^{-1/2} e^{-y/2} \sim \Gamma(\alpha = \frac{1}{2}, \theta = 2)$

(b) $E X^2 = Var(X) = 1$

(c) $X^2 + Y^2$ half + half = one (signal)

so $X^2 + Y^2 \sim \exp(\theta = 2)$

$P\left(\frac{X^2+Y^2}{2} > 1\right) = P(X^2+Y^2 > 2) = e^{-2/2} = e^{-1}$

$V \sim \exp(\theta = 2)$

Bonus. Choose one for 2 points. The other may add extra 1 point.

Mark clearly your first choice (if you do both without a mark, you'll get half of the credit).

(a) Let X be normal N(0,4). Compute $\mathsf{E}\, X^8$. $Y \sim N(0,1)$ $X = 2Y$

(b) Let X be exponential with mean 2. Compute $\mathsf{E}\, X^8$.

(a) $E e^{tY} = e^{t^2/2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n!}$

$E e^{tY} = \sum_{n=0}^{\infty} \frac{EY^n}{n!} t^n = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} EY^{2n}$

0 for odd n

So $EY^{2n} = \frac{(2n)!}{2^n n!}$. Here $n = 4$: $\frac{8!}{2^4 \cdot 4!}$

$\cancel{EX^{2n}}$ $EX^8 = 2^8 EY^8 = 2^8 \cdot \frac{8!}{2^4 \cdot 4!} = 16 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

(b) $X = 2V$, $V \sim \exp(1)$. $EX^8 = 2^8 EV^8 = 2^8 \int_0^\infty v^8 e^{-v}\, dv = 2^8 \Gamma(9)$
$= 2^8 \cdot 8!$

Comments on Pb. 7

The "argument"
<< It's the special case of χ^2_r distribution for $r=1$ >>
represent a logical fallacy called a "circular reasoning".

How χ^2_r is derived?

* First, X^2, where $X \sim N(0,1)$.
It has $\Gamma(\alpha = \frac{1}{2}, \theta = 2)$ distribution
(that was the question)

** Second, Γ is additive for fixed θ.
That is, if $X \sim \Gamma(\alpha, \theta)$, $Y \sim \Gamma(\beta, \theta)$
and X and Y are independent, then
$X + Y \sim \Gamma(\alpha + \beta, \theta)$.
It can be seen through mgf

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = \frac{1}{(1-\theta t)^\alpha} \cdot \frac{1}{(1-\theta t)^\beta} = \frac{1}{(1-\theta t)^{\alpha+\beta}}$$

*** Third, $X_1^2 + \dots + X_n^2$ for iid $N(0,1)$
will have $\Gamma(\underbrace{\tfrac{1}{2} + \tfrac{1}{2} + \dots + \tfrac{1}{2}}_{r}, \theta = 2) = \Gamma(\frac{r}{2}, 2)$

Now, it's called
$$\chi^2_r$$
(r = # degrees of freedom)