POISSON PROCESS GAMMA DISTRIBUTION

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Outline

Recall the Poisson approximation of binomial

The assumptions are:

- $p \to 0$ (in practice p is very small),
- $n \to \infty$ (in practice n is very large),
- $np \to \lambda$ (in practice np is stable).

Then the count of "1"s in n trials in the limit becomes the count of "signals" in the unit interval $[0, 1]$:

$$\mathrm{P}(S_n = k) \quad \approx \quad \frac{\lambda^k}{k!} e^{-\lambda},$$

which is the Poisson pmf with intensity λ.

Recall - the Poisson process

What if we consider an interval of length t? Denote

$$N_t = \textbf{the number of signals in the time interval } [0, t].$$

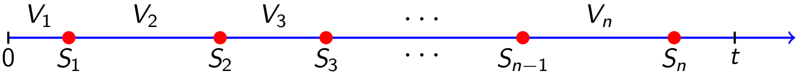
Then, by the additivity rule the intensity (the mean value in the unit interval) is replaced by λt (the mean value in the interval $[0, t]$)

$$\mathrm{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

We have factually arrived at a consistent system of Poisson random variables, which we call a **Poisson process with intensity λ**.

Poisson process - illustration

Signals (arrivals, events, random points, epochs, happenings, etc.) are recorded on the positive half-line:



signals	$S_n = V_1 + \cdots + V_n,$
inter-signal times :	$V_n = S_n - S_{n-1}$
the signal count in $[0, t]$:	N_t
the basic relation :	$\mathbf{\{S_n \le t\} = \{N_t \ge n\}}$

Poisson process - quick properties

1. The inter-signal times are iid exponential with mean $\theta = \frac{1}{\lambda}$.
2. The signal count N_t is Poisson with parameter λt,
3. The signal count $N(s,t] = N_t - N_s$ between times s and t $(s < t)$ is Poisson $(\lambda(t-s))$.

   That is, the Poisson process has **stationary increments**, i.e., the distribution of the count depends only on duration, not on location of the interval.

4. Denoting by $|A|$ the combined length of a set A (e.g., a union of several disjoint intervals), the distribution of the count $N(A)$ is Poisson $(\lambda|A|)$.

The distribution of the n^{th} signal

By the basic relation

$$F_n(t) = \mathrm{P}(S_n \le t) = \mathrm{P}(N_t \ge n) = e^{-\lambda t} \sum_{x=n}^{\infty} \frac{(\lambda t)^x}{x!}$$

Differentiating and performing telescopic summation, the density is

$$f_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}\, e^{-\lambda t},$$

often written with $\lambda = 1/\theta$.

It is named **Erlang distribution**.

The 1st signal

By the basic relation for $t \geq 0$,

$$\overline{F}_1(t) = \mathrm{P}(S_1 > t) = \mathrm{P}(N_t = 0) = e^{-\lambda t}$$

Differentiating, the density is exponential:

$$f_n(t) = \lambda\, e^{-\lambda t}, \quad t \geq 0.$$

Clearly, $\lambda = 1/\theta$ and $\theta = 1/\lambda$.

So, an Erlang distribution corresponds to a finite sum of iid exponential distributions.

The mean and variance

The mean and variance smartly follow from the summation formula:

$$\begin{aligned} \mathrm{E}\, S_n = &\ \mathrm{E}\,(V_1 + \cdots + V_n) = n\,\mathrm{E}\, V = n\,\theta, \\ \mathrm{Var}(S_n) = n &\ \mathrm{Var}(V_1 + \cdots + V_n) = n\,\mathrm{Var}(V) = n\,\theta^2. \end{aligned}$$

The pdf also could be used directly with the help of "**smart integration**" (to be shown on the scan or the blackboard).

Like the Poisson count is analogous to the binomial count in a BP, the exponential waiting time corresponds to the geometric discrete waiting tine in a BP, so the n^{th} signal is an analog of the negative binomial.

Gamma function

Recall the function defined as an improper integral in Calculus 2:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx, \quad \alpha > 0,$$

which extends the notion of the factorial to a real positive argument:

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha), \quad \text{and for an integer }, \Gamma(n) = (n-1)!.$$

Fractional signals

Replacing $n \mapsto \alpha$ in the pdf of the n^{th} signal we obtain the pdf

$$f_\alpha(t) = \frac{\lambda^\alpha\, t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\lambda t},$$

which could be interpreted as the pdf of the "α^{th}" signal.

All parameters follow the same replacement $n \mapsto \alpha$.

It is not just a metaphor because physical signals often are preceded by indicators predicting the actual arrival in the future. For example, $S_{1/2}$ would indicate that the first signal is about "half way" from incoming.

The additivity rule

Given the same intensity λ,

$$S_\alpha + S'_\beta = S_{\alpha+\beta},$$

where the prime indicates independent signals. For example, the sum of two independent "half-signals" yields one signal

$$S_{1/2} + S'_{1/2} = S_1 = V \quad \text{(exponential)}.$$

This phenomenon was absent in the discrete case because the continuous time line is infinitely divisible in contrast to the discrete time line.

Uniform and binomial

- A single signal in A is uniformly distributed in A.
- The Poisson counts are independent for disjoint sets.

For $B \subset A$, $p = |B|/|A|$ is the proportion of the measures.
Then n signals in A yield n independent Bernoulli trials:

1 - a signal is in B with probability p,
0 - a signal is in $A \setminus B$ with probability $1 - p$.

In particular, the count of signals in B is binomial (n, p).