MATH 6670 Quiz 1

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September 8, 2019

1 Problem 2.29

1.1 Part A

The sample space Ω_a consists of all 2-permutations of the balls in the urn, so $|\Omega_a| = (n+m)_2$. The two options for drawing balls of the same color are drawing two white balls or two black balls, which we denote as events W_a and B_a, respectively. As they are drawn without replacement, $|W_a| = (n)_2$ and $|B_a| = (m)_2$. Thus $P(W_a) = \frac{|W_a|}{|\Omega_a|}$ and $P(B_a) = \frac{|B_a|}{|\Omega_a|}$, and since W_a and B_a are disjoint, $P(W_a \cup B_a) = P(W_a) + P(B_a)$.

1.2 Part B

Since we now sample with replacement, Ω_b consists of all possible pairings of balls in the urn, including repetitions, so $|\Omega_b| = (n+m)^2$. We denote drawing a white ball, replacing it, and drawing another as the event W_b, and the equivalent with black balls B_b. Since the balls are sampled with replacement, $|W_b| = n^2$ and $|B_b| = m^2$ by the multiplication rule. Thus $P(W_b) = \frac{|W_b|}{|\Omega_b|}$ and $P(B_b) = \frac{|B_b|}{|\Omega_b|}$, and since W_b and B_b are disjoint, $P(W_b \cup B_b) = P(W_b) + P(B_b)$.

1.3 Part C

Since events W and B are disjoint, we'll prove that $P(W_a) < P(W_b)$ and $P(B_a) < P(B_b)$ separately, from which it will be clear that $P(W_a \cup B_a) < P(W_b \cup B_b)$.

We will prove this by contradiction. We assume $P(W_a) \geq P(W_b)$.

$$\begin{aligned} P(W_a) &\geq P(W_b) \\ \frac{(n)_2}{(n+m)_2} &\geq \frac{n^2}{(n+m)^2} \\ \frac{n}{n+m}\frac{n-1}{n+m-1} &\geq \frac{n}{n+m}\frac{n}{n+m} \end{aligned}$$

Since n and m are positive, we can divide both sides by $\frac{n}{n+m}$.

$$\begin{aligned} \frac{n-1}{n+m-1} &\geq \frac{n}{n+m} \\ (n-1)(n+m) &\geq (n)(n+m-1) \\ n^2+nm-n-m &\geq n^2+nm-n \end{aligned}$$

We subtract n^2+nm-n from both sides to get

$$-m \geq 0$$

Since m is positive, this contradicts our assumption, and thus $P(W_a) < P(W_b)$. The proof for $P(B_a) < P(B_b)$ is clear by the same method.

We now have

$$\begin{aligned} P(W_a \cup B_a) &= P(W_a) + P(B_a) \\ P(W_b \cup B_b) &= P(W_b) + P(B_b) \\ P(W_a) &< P(W_b) \\ P(B_a) &< P(B_b) \end{aligned}$$

from which it is clear that $P(W_a \cup B_a) < P(W_b \cup B_b)$.

2 Theoretical Exercise 2.18

We notate the set of sequences of order n that begin with t to be t_n, the set of sequences of order n that begin with h to be h_n, and $t(\circ)$ and $h(\circ)$ as functions that prepend t or h to all sequences in a set, i.e. $t_2 = (tt, th)$, and $h(t_2) = (htt, hth)$. Note that these functions do not change the size of the set they operate on. Also, note that (t_n, h_n) is the set of all sequences of order n, so $|t_n, h_n| = f_n$.

$$\begin{aligned} f_4 &= |(tttt, ttht, ttth, thtt, thth, httt, htht, htth)| \\ &= |(t(ttt, tht, tth, htt, hth), h(ttt, tht, tth))| \\ &= |(t(t_3, h_3), h(t_3))| \\ &= f_3 + |h(t_3)| \\ &= f_3 + |h(t(t_2, h_2))| \\ &= f_3 + f_2 \end{aligned}$$

We can see that f_n is equal to $2|t_{n-1}| + |h_{n-1}| = f_{n-1} + |t_{n-1}|$, and that $|t_{n-1}| = f_{n-2}$, so $f_n = f_{n-1} + f_{n-2}$.

This can be explained as all valid sequences will remain valid if a t is prepended, while only valid sequences beginning with a t will remain valid if prepended by an h. Then f_n consists of all sequences in t_{n-1} and h_{n-1} prepended by a t plus the sequences in t_{n-1} prepended by an h. By our previous reasoning, t_{n-1} is equal to all of the sequences in f_{n-2} prepended by a t, so $f_n = f_{n-1} + f_{n-2}$.