MULTIPLE INTEGRALS

Jerzy Szulga

Department of Mathematics and Statistics
Auburn University

MATH 5670-6670 FALL 2019

PROBABILITY I

November 1, 2019

Outline

1. Double integrals - Overview
2. Probability
3. How to compute a double integral
   - Rectangular system
   - Polar system
4. HD

$\pm$ Volume

An equation $z = f(x, y)$ represents a surface in 3-space.

Let D be a region in the xy-plane within the domain of f.

If $f(x, y) \geq 0$ then D cuts off a 3D solid:

above D ("floor"), below the surface ("roof").

Its volume can be approximated like in the 1D case.

If f takes also negative values, then we assign "minus volume" for the part of the surface below the xy-plane.

The usual procedure

- partition the xy-plane into small simple shapes such as rectangles or other figures,
- on each small base of area ΔA build a narrow block reaching the surface with the volume $f(x, y)\,\Delta A$ (height $\times$ base),
- sum up the volumes: $\displaystyle\sum_x \sum_y f(x, y)\,\Delta A$,
- Increase the resolution and pass to the limit, denoted by

$$\iint_{\mathbf{D}} \mathbf{f(x, y)\,dA}$$

Normalized mass

Let $f(x,y) \geq 0$ be a density of the lamina of the plane.

It has a physical dimension, e.g. $\left[\frac{g}{cm^2}\right]$.

Denote $S = \{\,(x,y) : f(x,y) > 0\,\}$, called the **support**.. The mass of a lamina D equals

$$M(D) = \iint\limits_{D} f(x,y)\,dA = \iint\limits_{D\cap S} f(x,y)\,dA$$

The normalization yields the dimensionless (scalar) quantity

$$\mathrm{P}(D) \stackrel{\text{def}}{=} \frac{M(D\cap S)}{M(S)}, \quad \text{where, obviously} \quad \mathrm{P}(S) = 1.$$

Probability = normalized mass

We interpret $\mathrm{P}(D)$ as the probability that a random point (X, Y) falls into or occurs in the region D.

Typography: The capital X and Y indicate randomness of the coordinates or variables in contrast to lower case x and y that indicate non-random (deterministic) variables.

Normalization is equivalent to the assumption that a dimension-less function $f(x) \geq 0$ and $\displaystyle\iint_{\mathbb{R}^2} f \, dA = 1$.

Traditionally, $f(x, y)$ is called the **joint density**.
(However, the plain "density" would suffice).

Example

Let $f(x,y) = 3(x+y), \quad x \geq 0,\ y \geq 0,\ x+y \leq 1.$

The support S is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

The constant "3" is needed to ensure that $\displaystyle\iint_S f(x,y)\,dA = 1.$

Let us set up the double integral to compute $\mathrm{P}\left(X \leq \frac{3}{4},\ Y \leq \frac{3}{4}\right)$.

We see that $D = \left[0, \frac{3}{4}\right] \times \left[0, \frac{3}{4}\right]$, partially "sticking out" of S:

$$\mathrm{P}\left(X \leq \frac{3}{4},\ Y \leq \frac{3}{4}\right) = \iint_{D \cap S} 3(x+y)\,dA.$$

Why constant 3?

It takes a while to compute $\displaystyle\iint_S (x+y)\,dA = \frac{1}{3}$.

However, the given solid is a pyramid from the vertex $(0,0,0)$ to the rectangular base with four vertices

$$(1,0,0),\ (1,0,1),\ (0,1,1),\ (0,1,0).$$

The area of the base is $\sqrt{2}$ and the height of the pyramid is $\frac{\sqrt{2}}{2}$. Thus the volume equals

$$\frac{1}{3}\left(\sqrt{2} \times \frac{\sqrt{2}}{2}\right) = \frac{1}{3}.$$

Rectangles

If $D = [a, b] \times [c, d]$ is rectangular, then we also use the rectangular partition of the plane. Then

$$dA = dx\,dy \quad \text{or} \quad dA = dy\,dx,$$

and the double integral becomes either iterated integral

$$\int_c^d \left(\int_a^b f(x, y)\,dx \right)\,dy \quad \text{or} \quad \int_a^b \left(\int_c^d f(x, y)\,dy \right)\,dx.$$

That's Calculus 1 or 2 done twice.

A normal region

Let D lie between graphs of two functions, top and bottom:

$$a \le x \le b, \quad b(x) \le y \le t(x).$$

Then the double integral is again the iterated integral

$$\int_a^b \left(\int_{b(x)}^{t(x)} f(x,y)\, dy \right) dx.$$

Reverse order

By symmetry, let D lie between a left graph $x = l(y)$ and a right graph $x = r(y)$:

$$c \leq y \leq d, \quad l(y) \leq x \leq r(y).$$

Then the double integral is again the iterated integral

$$\int_c^d \left(\int_{l(y)}^{r(y)} f(x,y)\,dx \right) dy.$$

More complicated regions

Some regions are not normal but usually they can be divided into u union of normal regions.

Then the overall integral is just the sum of partial integrals.

Which order of integration is better - **"**$dx\,dy$**"** **or** **"**$dy\,dx$**"**?

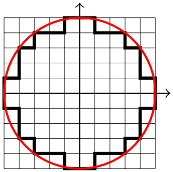
It depends.

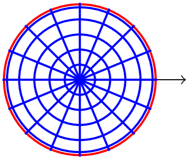
Sometimes it doesn't matter.
Sometimes one order yields impossible or hard integrals
while the other order yields easy ones.

Polar regions

Some regions D cooperate poorly with the rectangular partition.

A round pizza or cake is cut not into rectangles but to slices!





Concentric circles and polar semi-axes yield the polar partition (a.k.a. **tessellation**) of the plain.

Polar dA

The pair (x, y) of rectangular coordinates is replaced by the pair (r, θ) of polar coordinates, $r \geq 0$ and $\theta \in [0, 2\pi)$:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad \text{or} \qquad \begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \arctan\frac{y}{x} \end{aligned}$$

The polar infinitesimal piece of the area:

$$\mathbf{dA = r\,dr\,d\theta}$$

WARNING

A quite **frequent error**:

writing dA in polar coordinates as "$dr\, d\theta$", skipping the factor "r".

Apart from a common negligence, this error may have semantic and psychological reasons. The phrase

"*replace* (x, y) *by* (r, θ)"

might be incorrectly understood as

"*replace* x *by* r *and replace* y *by* θ"

Example 1

Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$.

Answer. It's $\sqrt{2\pi}$. See Slides 6-7 in the presentation 'BP_11_normal'.

This example shows that sometimes double integrals are much simpler than single integrals.

Exercise 2a

Find c to make $f(x,y) = \dfrac{c}{(1+x^2+y^2)^3}$ a 2D (a.k.a. joint) pdf of a random pair (X,Y).

Solution. The support is the whole plane: $r \geq 0, 0 \leq \theta < 2\pi$.

$$1 = \iint\limits_{\mathbb{R}^2} \frac{c}{(1+x^2+y^2)^3}\,dA = c\int_0^{2\pi}\int_0^{\infty} \frac{1}{(1+r^2)^3}\,r\,dr\,d\theta$$

$$= c \cdot 2\pi \cdot \left.\frac{-1}{2(1+r^2)^2}\right|_0^{\infty} = \pi c$$

Hence $c = 1/\pi$.

Exercise 2b

Find $\mathrm{P}(X/\sqrt{3} < Y < X\sqrt{3})$.

Solution. The event of interest:

$$\left\{ x/\sqrt{3} < y < x\sqrt{3} \right\} : \quad r \geq 0, \pi/6 < \theta < \pi/3.$$

Hence

$$\mathrm{P}(X/\sqrt{3} < Y < X\sqrt{3}) = \frac{1}{\pi} \int_{\pi/6}^{\pi/3} \int_{0}^{\infty} \frac{1}{(1+r^2)^3}\, r dr\, d\theta = \frac{1}{12}$$

A quick alternative. Angles of the same measure have equal probability. Ours measures $\pi/6$ radians, i.e., $1/12$ of the full 2π.

Comment on the setup

The set is framed by two straight lines $y = \frac{1}{\sqrt{3}}x$ and $y = x\sqrt{3}$, lying in the 1^{th} and 3^{rd} quadrant and passing through the origin.

However, the implicit inequality

$$\frac{x}{\sqrt{3}} < x \quad \text{yields} \quad x > 0, \quad \text{so } y > 0 \text{ also.}$$

Therefore, the region lies entirely in the first quadrant.

Exercise 2c: Find $\mathrm{E}\, X$, EY, $\mathrm{E}\, XY$.

Quick Solution: All quantities seem to be 0 by symmetry.

Warning: They might not exist! The existence needs proof!

Hard Solution: Three double integrals must be set up and calculated. For example, for the third quantity,

$$
\begin{aligned}
\mathrm{E}\, XY &= \iint\limits_{\mathbb{R}^2} xy \, \frac{1}{\pi(1+x^2+y^2)^3} \, dA \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{r\cos\theta \cdot r\sin\theta}{\pi(1+r^2)^3} \, r \, dr \, d\theta \\
&= \int_0^{2\pi} \left(\int_0^{\infty} \frac{r^3}{\pi(1+r^2)^3} \, dr \right) \cos\theta \sin\theta \, d\theta = \cdots = 0
\end{aligned}
$$

Triple and multiple integrals and pdfs

The concept: continues without change.

Probabilities - like diamonds - are forever.

Formalization: still the same but longer, more tedious or technical

Visualization: gone, no more.

No graphing in 4D or HD, only limited projections possible.