MATH 6670 Take Home Exam 1

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1 Problem 1: Ch3 P31

Some notation:

- c is the event that Ms. Aquina has cancer
- n is the event that Ms. Aquina does not get a call
- β is $P(c|n)$
- α is $P(c)$

We can expand β to

$$\begin{aligned}
\beta &= \frac{P(n|c)P(c)}{P(n)} \\
&= \frac{P(n|c)P(c)}{P(n|c)P(c) + P(n|c')P(c')} \\
&= \frac{P(c)}{P(c) + 0.5(1 - P(c))} \\
&= \frac{\alpha}{0.5\alpha + 0.5} \\
&= \frac{2\alpha}{\alpha + 1}
\end{aligned}$$

We can then show that $\beta > \alpha$ by

$$\begin{aligned}
\frac{2\alpha}{\alpha + 1} &> \alpha \\
\frac{2}{\alpha + 1} &> 1 \\
2 &> \alpha + 1
\end{aligned}$$

which is true as $\alpha < 1$.

2 Problem 2: Ch3 TE3

Some notation:

- e_i is the event that a family with i children is randomly selected
- f is the event that a firstborn is chosen
- f_1 is the event that a firstborn is chosen using method 1
- f_2 is the event that a firstborn is chosen using method 2
- m is the total number of families (and then the total number of firstborns)
- C is the total number of children

Additionally, we have

$$m = \sum_{i=1}^{k} n_i$$
$$C = \sum_{i=1}^{k} i n_i$$

2.1 Method 1

If we randomly select a family and then a child, we are conditioning our second selection on the family chosen.

The probability that we select a given family size is

$$P(e_i) = \frac{n_i}{m}$$

$$P(f|e_i) = \frac{1}{i}$$

By summing across all possible family sizes, we get

$$\begin{aligned} P(f_1) &= \sum_{i=1}^{k} P(f|e_i)P(e_i) \\ &= \sum_{i=1}^{k} \frac{1}{i}\frac{n_i}{m} \\ &= \frac{1}{m}\sum_{i=1}^{k} \frac{n_i}{i} \end{aligned}$$

2.2 Method 2

The probability that a firstborn is selected by randomly choosing a child is the number of firstborns over the total number of children, so

$$P(f_2) = \frac{m}{\sum_{i=1}^{k} i n_i}$$

2.3 Comparison

Multiplying each by $m \sum_{j=1}^{k} j n_j$ gives the inequality shown in the problem.

$$\sum_{i=1}^{k} i n_i \sum_{j=1}^{k} \frac{n_j}{j} \geq \sum_{i=1}^{k} n_i \sum_{j=1}^{k} n_j$$

which can be proven via induction. For the base case of $k = 1$, we have

$$\begin{aligned} \sum_{i=1}^{k} i n_i \sum_{j=1}^{k} \frac{n_j}{j} &\geq \sum_{i=1}^{k} n_i \sum_{j=1}^{k} n_j \\ n_1^2 &\geq n_1^2 \end{aligned}$$

We then prove the inductive case, assuming this inequality holds for arbitrary k

$$\sum_{i=1}^{k+1} i n_i \sum_{j=1}^{k+1} \frac{n_j}{j} \geq \sum_{i=1}^{k+1} n_i \sum_{j=1}^{k+1} n_j$$

For the left hand side we have

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i n_i \sum_{j=1}^{k+1} \frac{n_j}{j} \\ &= \sum_{i=1}^{k} i n_i \sum_{j=1}^{k} \frac{n_j}{j} + n_{k+1}^2 + (\frac{k^2 + (k+1)^2}{k^2 + k}) n_k n_{k+1} \end{aligned}$$

For the right hand side we have

$$\begin{aligned} RHS &= \sum_{i=1}^{k+1} n_i \sum_{j=1}^{k+1} n_j \\ &= \sum_{i=1}^{k} n_i \sum_{j=1}^{k} n_j + n_{k+1}^2 + 2 n_k n_{k+1} \end{aligned}$$

which we reassemble as

$$\sum_{i=1}^{k} in_i \sum_{j=1}^{k} \frac{n_j}{j} + n_{k+1}^2 + (\frac{k^2+(k+1)^2}{k^2+k})n_k n_{k+1} \geq \sum_{i=1}^{k} n_i \sum_{j=1}^{k} n_j + n_{k+1}^2 + 2n_k n_{k+1}$$

The sums make up our assumed case, so we can subtract them without changing the sign of the inequality.

$$\begin{aligned} n_{k+1}^2 + (\frac{k^2+(k+1)^2}{k^2+k})n_k n_{k+1} &\geq n_{k+1}^2 + 2n_k n_{k+1} \\ (\frac{k^2+(k+1)^2}{k^2+k}) &\geq 2 \\ 2k^2+2k+1 &\geq 2k^2+2k \end{aligned}$$

which is true.