MATH 5670-6670 Fall 2019

Exam 1, in-class part, 15 points

Answer key
(Alternative methods possible)
At home part at the end

Eight one-pointers (Structured answers, a.k.a. "not computed", are acceptable.

1. A committee of size 5, including a chairman, is selected at random from among 20 people.

   Describe the outcome space Ω and find the number of possible committees.

Approach 1 (other possible). Enumerate people $\{1, \dots, 20\} = U$

1. Choose a committe of 5
2. choose a chairman among them

$$\Omega = \{ (C, n) : C \subset U, |C| = 5, n \in C \}$$

$\boxed{\text{example: } (\{1,5,6,7,20\}, 6)}$

By Mult. Rule

$$|\Omega| = \binom{20}{5} \cdot \binom{5}{1} = \frac{(20)_5}{5!} \cdot 5 = \frac{(20)_4}{4!}$$

combinations (sets)

combination of 1 {singleton}

Comment: Perhaps, this is the most common and natural approach.

2. 10 identical lollipops are given at random to 5 children but each child must get at least one lollipop.

   Describe the outcome space Ω and find the number of possible distributions of lollipops to children.

10 id items in 5 cells, at least one in each

$$\Omega = \{ (n_1, \dots, n_5) : n_1 + \dots + n_5 = 10, n_i \geq 1 \}$$

same as $10 - 5 = n_1 + \dots + n_5$, $n_i \geq 0$

5 id items

$$\binom{5+5-1}{5-1} = \binom{9}{4}$$

3. The Mahabharata is an ancient Indian epic.

11 letters

a) How many words are possible using all letters from MAHABHARATA?

b) How many words, if all A's must be next to each other?

a) $\binom{11}{1,5,2,1,1} = \frac{11!}{5!\,2!}$

b) AAAAA

$\rightarrow$ slide in 7 ways

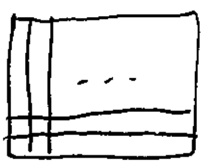
place other letters

$7 \cdot \binom{6}{1,2,1,1} = 7 \cdot \frac{6!}{2!}$

Comment: In New Zealand a place has a name with 85 letters. (See wikipedia)

4. A rook (castle) on an 8×8 chessboard may beat (or capture) another figure on the same row or column.

8 rooks are placed on a chessboard at random. Find the probability that they are all safe.



order rooks

1st rook $\quad 8^2 = 64$

2nd rook $\quad 64 - 1 = 63$ etc.

$|\Omega| = (64)_8$

etc. – when the pattern emerges

A: 1st: 8^2 now it's 7×7 chessboard

2nd 7^2

3rd 6^2 $\boxed{\text{etc.}}$

$P(A) = \frac{8^2 \cdot 7^2 \cdots 2^2 \cdot 1^2}{(64)_8}$

5. What is the probability that among 4 people, at least two were born in the same season? That is, in Winter, Spring, Summer, or Fall?

1 2 3 4

Ω - sequences $\quad |\Omega| = 4^4$

A' permutations $\quad |A'| = 4!$

$P(A) = 1 - \frac{4!}{4^4}$

previously: B'days per day or per month, or the elevator problem

6. What is the probability of two pairs in a five card poker hand?

$XXYYZ$, no order $\binom{52}{5}$

choose pairs · choose suits · choose the kicker

$$\frac{\binom{13}{2} \cdot \binom{4}{2}^2 \cdot \binom{44}{1}}{\binom{52}{5}}$$

$\boxed{\text{no order - crucial. order = a mess}}$

7. Five cards were drawn at random from a standard deck one by one.without replacement. What is the probability that there was at least one ace among the last two cards?

Like quiz

$\underset{A\ \ A'}{\boxed{4/48}}$

$\boxed{\text{exchangeability again.}}$

Ω: last two cards, ordered

$|\Omega| = (52)_2$, A': $|A'| = (48)_2$

$$P(A) = 1 - \frac{(48)_2}{(52)_2}$$

8. A sniffer dog at an airport detects a bad substance in a bag with probability 0.9. The dog barks then. It also raises a false alarm (e.g., barks although there is nothing bad in a bag) with probability 0.2. Say, one in 100 bags has bad stuff.

If a dog barked at a bag, what is the conditional probability that it contained bad stuff?

Not. B - bad, D - detect

Data $P(B) = 0.01$, $P(D|B) = 0.9$, $P(D|B') = 0.2$

Question: $P(B/D)$

$\boxed{\text{Classical Bayes.}}$

$$P(B|D) = \frac{P(D|B)P(B)}{P(D)} = \frac{P(D|B)\,P(B)}{P(D|B)\,P(B) + P(D|B')\,P(B')}$$

$$= \frac{0.9 * 0.01}{0.9 * 0.01 + 0.2 * 0.99} = \frac{90}{90 + 2 \cdot 99}$$

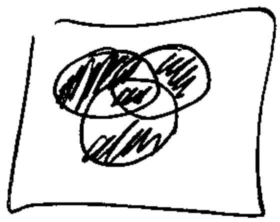
Two two-pointers

1. Is it true or false that $(A \oplus B \oplus C)' = ABC' \cup AB'C \cup A'BC$?

   Prove or disprove using a method of your choice. A guess without work bears no credit.

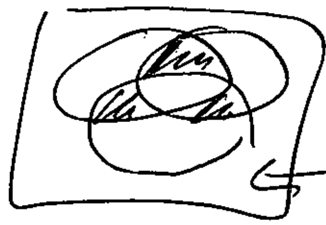
Venn

$A \oplus B \oplus C$



$(A \oplus B \oplus C)'$

LHS



missing

False

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01 coding

$A \oplus B$
either $|a-b|$ → easier!
or $a+b-2ab$

LHS $\quad 1 - \big|\,|a-b|-c\,\big|$

RHS $\quad ab(1-c) + a(1-b)c + (1-a)bc$

Hmm.... For $a=b=c=0$
$LHS = 1, \quad RHS = 0$

Set algebra ?

One must be really fluent!

Comment:

Venn diagrams seem to be easier, but one must know how the symmetric difference of 3 sets looks like.
If not, then it's tedious.

This solution, simpler & more transparent, replaced the previous one.

2. A student plans to take three consecutive exams and must pass the previous exam in order to be admitted to the next one. The probabilities of passing each exam are, respectively, p, q, r, where $p > q > r$.

A student failed the sequence of exams. What is the conditional probability that the second exam was failed?

(If desired, you may choose $p = 0.9$, $q = 0.6$, $r = 0.3$ for computations.)

(No, numbers would cloud the issue)

Ω first: coding : 1 (pass) & 0 (fail)
exams 1, 2, 3 : position

So, Ω : 4 atoms

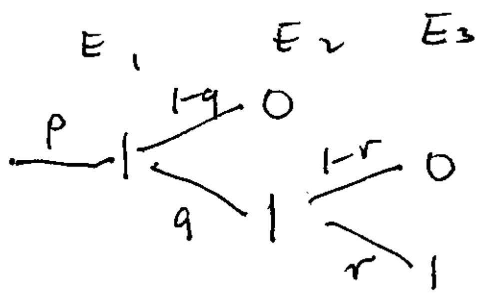
events			probabilities	
E_1	E_1 failed	0	$1-p$	process failed
E_2	E_2 failed	1 0	$p(1-q) = p - pq$	
E_3	E_3 failed	1 1 0	$pq(1-r) = pq - pqr$	
$E_1E_2E_3$,	all passed	1 1 1	pqr	
		total	Sum = 1 ✓	

Note that $(E_2 \text{ failed})^c \neq E_2$ passed
$\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad E_3 \cup E_1E_2E_3$

(There's the third state : "not taken")

Now, it's clear : $\dfrac{p - pq}{1 - pqr}$

Remark: Ω may be given by a "tree" with terminating branches



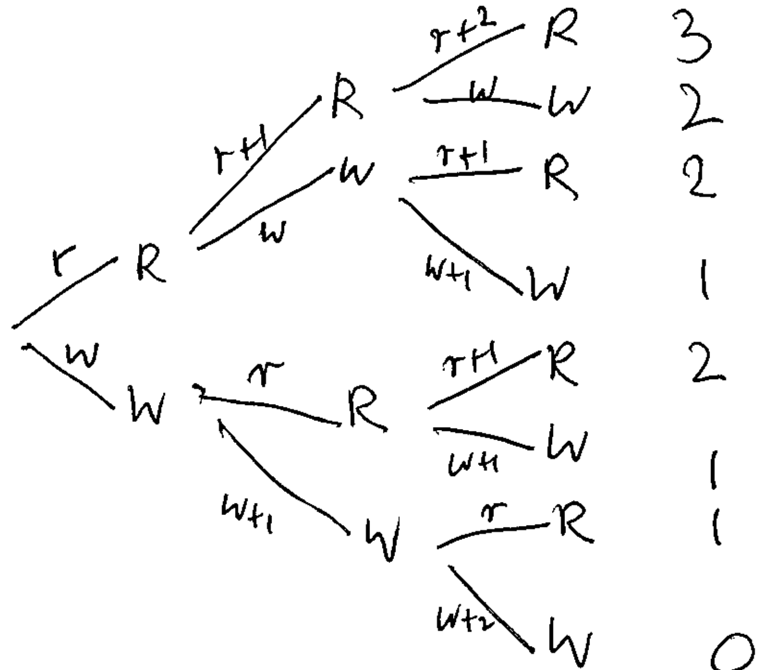
A three-pointer

$r \mid w$

An urn has initially r red balls and w white balls. Each day one ball is selected at random and then returned to the urn along with an additional ball of the same color. The process goes on and on day after day.

Find the probability that exactly i red balls were selected during n days.

For simplicity, let $r = 5$, $w = 7$, $n = 3$. However, consider all possibilities $i = 0, 1, 2, 3$.



$\leftarrow i$

Comment: a tree is the best; Also exchangeability!

over $r+w$ $\quad r+w+1 \quad r+w+2$

$d = (r+w)(r+w+1)(r+w+1)$

Comment: (concrete numbers ~~might~~ will cloud the clear issue)

$0: \quad \dfrac{w(w+1)(w+2)}{d}$

$1: \quad 3 \cdot \dfrac{w(w+1)r}{d}$

$2: \quad 3 \cdot \dfrac{w\, r(r+1)}{d}$

$3: \quad \dfrac{r(r+1)(r+2)}{d}$

4^{th} basket = floor

Bonus (2 points).

9 identical balls are thrown at random to 3 baskets but some may fall to the floor.

How many possible outcomes are possible? Compute exactly (e.g. 60, or 170, or 220, or so).

$\binom{9+\cancel{3}4-1}{\cancel{3}4-1} = \binom{12}{3} = \frac{(12)_3}{3!} = \frac{\cancel{12}^{2}\cdot 11\cdot 10}{\cancel{6}} = 220$

3.31 "Ms. Aquina..."

C cancer H/T

1 0

Method 1

coin	disease		p-ties
T	1	no cell	$\alpha/2$
T	0	no cell	$\frac{1-\alpha}{2}$
H	1	no cell	$\alpha/2$
H	0		$\frac{1-\alpha}{2}$

$$\text{sum} = \frac{\alpha}{2} + \frac{1-\alpha}{2} + \frac{\alpha}{2} = \frac{\alpha+1}{2}$$

cancer: $\frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$

ratio $\beta = \frac{\alpha}{\frac{\alpha+1}{2}} = \frac{2\alpha}{\alpha+1} > \alpha$ $\left(\frac{2}{\alpha+1} > 1,\ 2 > \alpha+1,\ 1 > \alpha \right)$

Method 2 table

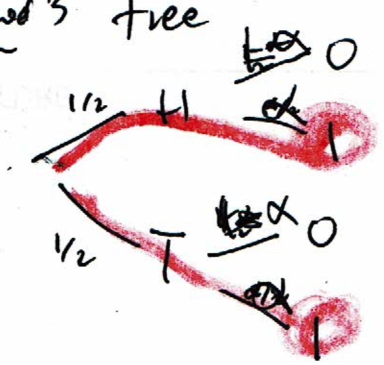
		coin H	coin T	
C	1	$\alpha/2$	$\alpha/2$	α
	0	$\frac{1-\alpha}{2}$	$\frac{1-\alpha}{2}$	$1-\alpha$
		$\frac{1}{2}$	$\frac{1}{2}$	

framed – no cell, sum $= \frac{\alpha+1}{2}$

cancer $\frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$

$$\beta = \frac{\alpha}{\frac{\alpha+1}{2}} = \frac{2\alpha}{\alpha+1}$$

Method 3 tree



no cell

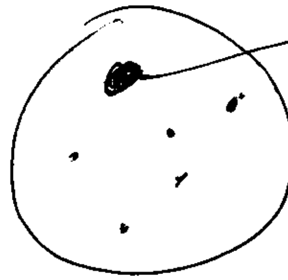
$$\frac{\frac{\alpha}{2} + \frac{\alpha}{2}}{\frac{\alpha}{2} + \frac{\alpha}{2} + \frac{1-\alpha}{2}}$$

s.a.a.

Chap 3 TE3

Analysis : first born = oldest = special

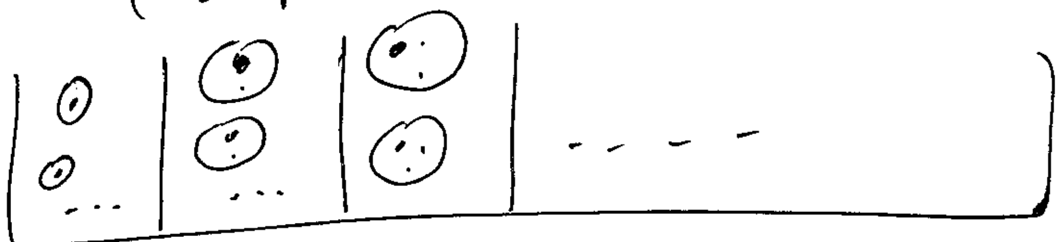
a family



i children

children

k - max # children in a family
(depends on a community)



n_1 n_2 n_3 n_k

$$n_1 + \dots + n_k = \sum_{i=1}^{k} n_i = m$$

(# of families

① **Notation** F_i - a family with i children

$$P(F_i) = \frac{n_i}{m}$$

S - special (firstborn)

$$P(S \mid F_i) = \frac{1}{i}$$

$$P_1 = P(S) = \sum_{i=1}^{k} P(S \mid F_i) P(F_i) = \sum_{i=1}^{k} \frac{1}{i} \frac{n_i}{m}$$

$$P_1 = \frac{1}{m} \sum_{i=1}^{k} \frac{n_i}{i}$$

(2) $N = \sum_{i=1}^{k} i n_i$ is # children

m - # specials (firstborns)

$$p_2 = P(S) = \frac{m}{N} = \frac{m}{\sum_{i=1}^{k} i n_i}$$

Task Show $p_1 > p_2$, i.e.,

$$\frac{1}{m} \sum_{i=1}^{k} \frac{n_i}{i} > \frac{m}{\sum_{i=1}^{k} i n_i}$$

Algebra: $\left(\sum_{i=1}^{k} \frac{n_i}{i}\right) \cdot \left(\sum_{i=1}^{\cancel{k}} i n_i\right) > m^2 = \left(\sum_{i=1}^{k} n_i\right)^2$

Rewrite the sums using another "dummy" j

$$LHS = \sum_i \frac{n_i}{i} \sum_j j n_j = \sum_i \sum_j \frac{j}{i} n_i n_j$$

$$RHS = \sum_i n_i \sum_j n_j = \sum_i \sum_j n_i n_j$$

Two cases $i = j$. RHS: $n_i n_i = n_i^2$ (same)
LHS: n_i^2

$i \neq j$. Combine $n_i n_j$ and $n_j n_i$

RHS $\left(\frac{j}{i} + \frac{i}{j}\right) n_i n_j = \frac{j^2 + i^2}{ij} n_i n_j$

LHS $2 n_i n_j$

Now $\frac{j^2 + i^2}{ij} \overset{?}{>} 2$. $j^2 + i^2 \overset{?}{>} 2ij$, $(j-i)^2 > 0$ yes.