# NORMAL DISTRIBUTION

Jerzy Szulga

Department of Mathematics and Statistics
Auburn University

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# PROBABILITY I

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# Outline

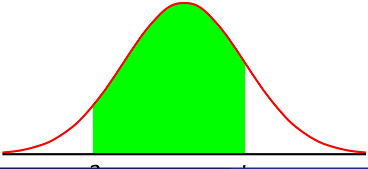

1. Normal
2. Tables
3. CLT

# The famous Gauss' bell

A **standard normal** (a.k.a. **Gaussian**) random variable, distribution, mgf, etc., has the

$$\text{pdf:} \quad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\text{cdf:} \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt \quad \text{(a special function).}$$



$$P(a \leq X \leq b) = \Phi(b) - \Phi(a)$$

# Ambiguous “bell curve”

There are infinitely many bell curves. Some may gave light tails, others have fat tails. For example, the **Cauchy** bell-shaped pdf

$$f(x) = \frac{1}{\pi(x^2+1)}$$

(named after French mathematician Augustin Cauchy) has no mean.

**Exercise**. Prove that the mean does not exist. Hence it has no (natural) moments. Its mgf does not exist.

**Exercise**. Explain where the constant $\pi$ comes from.

# More bells

**Exercise**. Let $r \geq 2$ be an integer. Introduce a power:

$$f(x) = \frac{c}{(x^2+1)^r}$$

Find $c$. for what $r$ does the mean exist? The variance?

As $r$ increases, the tails of the pdf become lighter and lighter.

It's still Pareto-like. The mgf doesn't exist.

# The constant $1/\sqrt{2\pi}$

**Problem.** Find $c$ to turn $c\,e^{-x^2}$ to a pdf.

**Solution**. We must compute the integral over $\mathbb{R}$. By symmetry, it suffices to compute $I = \int_0^\infty e^{-x^2/2}\,dx$. First, the "$x$" in the integral is a "dummy variable" - any reasonable letter can be used, e.g., $y$. So,

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2/2}\,dx\right)\cdot\left(\int_0^\infty e^{-y^2/2}\,dy\right) \\ &= \int_0^\infty\int_0^\infty e^{-x^2/2}\,e^{-y^2/2}\,dx\,dy = \iint\limits_Q e^{-(x^2+y^2)/2}\,dx\,dy, \end{aligned}$$

where $Q$ denotes the first quadrant of the plane.

Switching to polar coordinates:

$$I^2 \;\; = \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta = \frac{\pi}{2}.$$

So, $I = \sqrt{\pi/2}$, and thus

$$c = \frac{1}{2I} = \frac{1}{\sqrt{2\pi}}.$$

# Mean, variance, mgf of normal

Let $X$ have the standard normal density $\phi(x)$.

**Exercise**. Show that $\mathrm{E}\,X = 0$ (easy) and $\mathrm{Var}(X) = 1$ (integration by parts twice).

**Exercise**. Show that the mgf $M(t) = \mathrm{E}\,e^{tX} = e^{t^2/2}$.

**Hint.** Expand the formula for the mgf, use the HS Algebra (The Rule of Exponents, Completion of Squares), and basic Calculus (Substitution).

# Higher moments

**Exercise**. Compute a few higher moments $\mathrm{E}\, X^3$, $\mathrm{E}\, X^4$,...

Hint: Differentiate the mgf. It's rather tedious.

**Exercise**. Find easily the formula for any moment. Obviously, odd moments are 0 by symmetry.

Hint: Expand $e^{t^2/2}$ and $\mathrm{E}\, e^{tX}$ into the Maclaurin series. Compare.

**Exercise**. Show that the pdf of $X^2$ is $\Gamma(\alpha = \frac{1}{2}, \theta = 2)$.

**Hint**. Put $Y = X^2$. Then $G(y) = \mathrm{P}(Y \le y) = \mathrm{P}(X^2 \le y)$, $y \ge 0$. Solving the inequality yields $G(y) = 2\Phi(\sqrt{y}) - 1$. Differentiate.

# $\chi^2(r)$

**Exercise**. Let $X_1, \ldots, X_r$ be iid standard normal. Find the mgf of

$$X_1^2 + X_2^2 + \cdots + X_r^2,$$

and then show that its probability distribution is $\Gamma(\alpha = r/2, \theta = 2)$.

We call it **chi-squared with r degrees of freedom**.

Each independent r.v. provides one degree of freedom.

There are $r$ of them, so (acronym) $df = r$.

# The general normal

Let $Z$ be a standard normal. Let's scale and shift it:

$$X = \sigma Z + \mu.$$

Clearly,

$$\mathrm{E}\, X = \mu, \quad \mathrm{Var}(X) = \sigma^2.$$

We shall write $\mathbf{X \sim N(\mu, \sigma^2)}$.

# Affine transformation

The general normal r.v. is more complex but in a simple way.

Any r.v. $Z$ (not necessarily normal) with a pdf $f(z)$ and mgf $M(t)$ entails the following pdf and mgf of the transformed r.v.

$$X = \sigma Z + \mu.$$

The parameters represent just the scale and shift, and are not necessarily mean and standard deviation:

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad e^{\mu t} M(\sigma t)$$

# General normal pdf

In the normal case

$$\phi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad M_{\mu,\sigma}(t) = e^{\mu\, t+\sigma^2 t^2/2}$$

Most of textbooks and other sources begin with complex, and derive the simple ($\mu = 0$, $\sigma = 1$) from the complex.

Indeed: Google up "*normal distribution*". Check the order of introduction in Wikipedia, Wolfram's Mathworld, etc.

# Normal tables

Let us normalize (or standardize) $X \sim N(\mu, \sigma^2)$:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Then the values up to 4 decimal places of the cdf $\Phi(z) = \mathrm{P}(Z \leq z)$ are given in Tables 5.1, p. 189. There are many online normal (or other) calculators.

Tables can be read "in reverse": given the probability, find the $z$ (usually, approximately).

# $\chi^2$ tables

No tables in Ross. A chi-squared calculator is available online.

However, for even $r$ we can compute probabilities ourselves.
Put $n = r/2$ for an even $r$. Then $X \sim \chi^2_r = \Gamma(n, 2)$.

Hence, $X = S_n$, the $n^{\text{th}}$ signal in a Poisson process with intensity $\lambda = 1/\theta = 1/2$. By Poisson-Gamma duality

$$\begin{aligned} P(X \le x) = P(S_n \le x) = P(N_x \ge n) = 1 - P(N_x \le n-1) \\ = 1 - e^{-x/2} \sum_{k=1}^{n-1} \frac{x^k}{2^k k!} \end{aligned}$$

$\boxed{k=0}$

# Old Bernoulli's CLT

The original Bernoulli's layout involved 01 Bernoulli random variables.

The bell-shaped pmf's or pdf's emerge for many distributions. They may be skewed but the skewness diminishes when their shape parameters grow large:

binomial for large $n$

negative binomial for large $r$

Poisson for large $\lambda$

Gamma for large $\alpha$

# Symmetric Bernoulli's CLT

We will use symmetric $\pm 1$ iid random variables, calling them $R_k$.
Clearly, $\mathrm{E}\, R_k = 0, \mathrm{Var}(R_k) = 1$, and the mgf

$$M_R(t) = \frac{e^t + e^{-t}}{2} \approx 1 + \frac{t^2}{2}, \quad \text{for small } t.$$

The latter follows from the Maclaurin series for the exponentials where the odd powers cancel out, then higher order terms are omitted.

The sum represents a random walk along the 1D integer grid and yields the simple normalization

$$S_n = R_1 + \cdots + R_n \quad \mapsto \quad \widetilde{S}_n = \frac{S_n}{\sqrt{n}},$$

since the mean is zero. Then, the mgf and the limit as $n \to \infty$ are as follows:

$$\mathrm{E}\, e^{t\widetilde{S_n}} \approx \left(1 + \frac{t^2}{2n}\right)^n \to e^{t^2}$$

We recognize the mgf of the standard normal in the limit.

# General CLT

The observed phenomenon is not accidental but "normal".

The normalized sum of $n$ iid random variable approaches the standard normal N(0,1) distribution in the limit as $n \to \infty$.

**Theorem (CLT - Central Limit Theorem)**

*Let $X_k$ be iid with mean $\mu$ and variance $\sigma^2$. Standardize*

$$S_n = X_1 + \cdots + X_n \quad \mapsto \quad \widetilde{S}_n = \frac{S_n - \mathrm{E}\, S_n}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

*Then* $\mathrm{P}(\widetilde{S}_n \le x) \approx \Phi(x)$ *as* $n \to \infty$.

# Practical use of CLT

If a probability distribution stems from a sum of many iid random variables then in the computation of probabilities we may apply the normal probabilities in lieu of the original, often tedious, distribution.

The procedure is simple:

1. standardize the sum,
2. work the basic algebra,
3. use normal tables or normal calculator.

Still, there is a question when "large" is factually large enough.

Practitioners of statistics have empirical answers.

Rigorous mathematical answers require quite advanced methods.