5 and 6. Newtonian Kinetics of Rigid Bodies

Outline

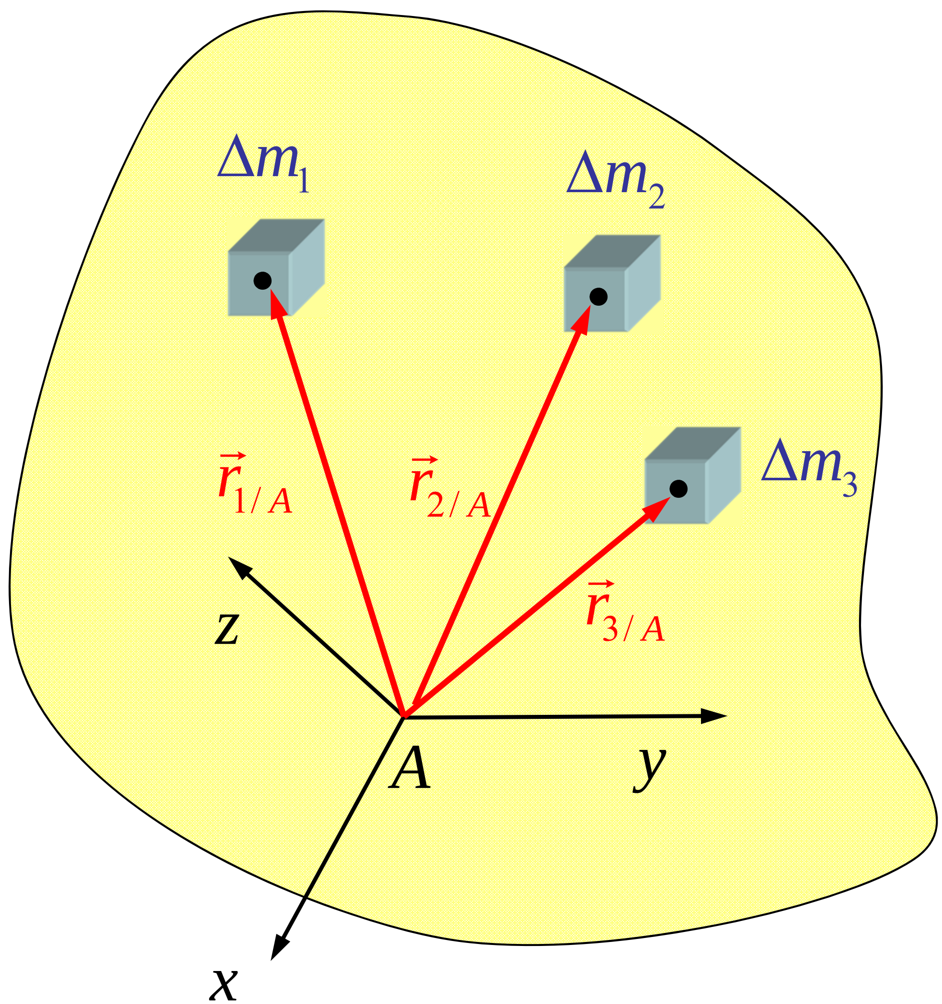
- Fundamental Principles
  - Newton's 2^{nd} Law
  - Resultant force and translation
  - Resultant moment and rotation
  - Kinetic energy
- Angular momentum and inertia properties
  - Moments and products of inertia
  - Transformation of inertia properties
  - Rate of change of angular momentum
- Equations of motion
- Planar motion

1. Fundamental Principles

1.1 Newton's second law

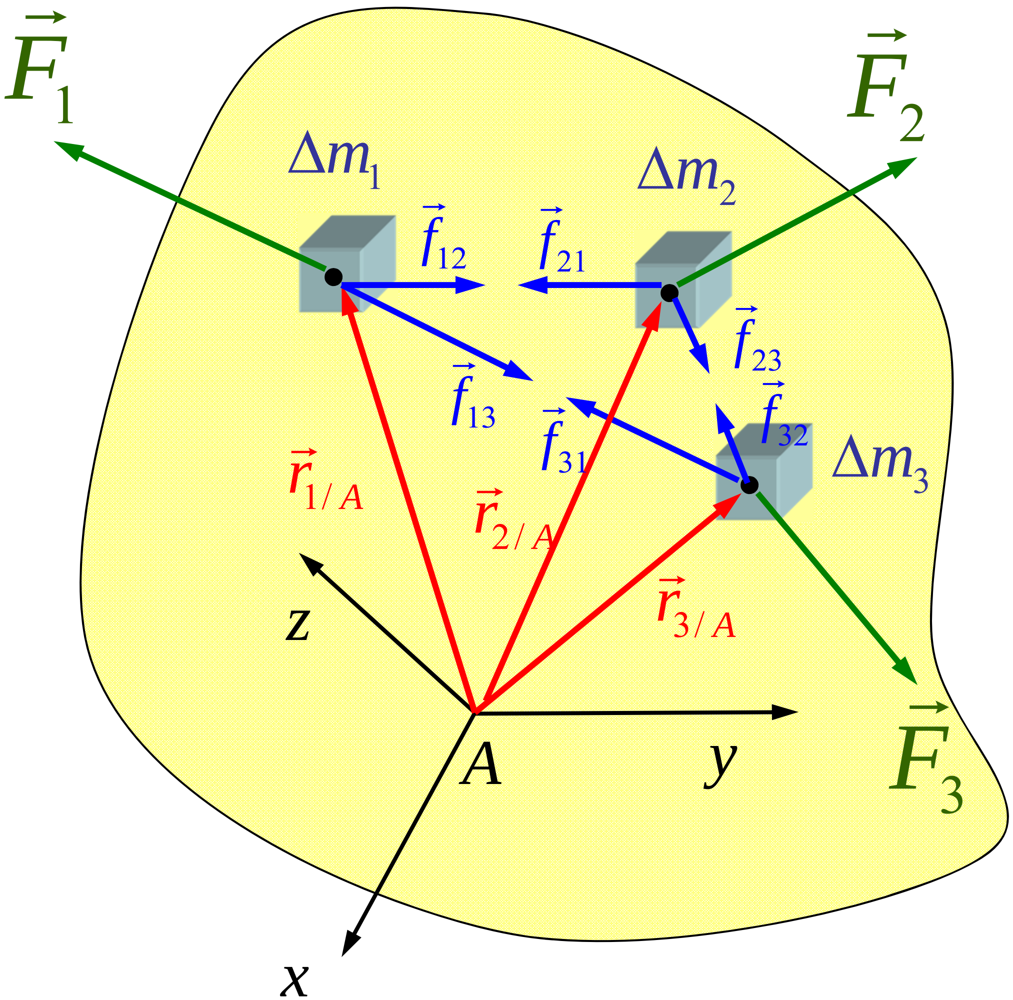
$$\vec{F} = m\vec{a}$$

- Note: Newton's second law is only applied to particles. We can treat a rigid body as a collection of infinitesimal particles whose motions are not independent.



Δm_1
Δm_2
Δm_3
$\vec{r}_{1/A}$
$\vec{r}_{2/A}$
$\vec{r}_{3/A}$
z
A
y
x

1.2 Resultant force and translation



$\vec{f}_{ij}$ – internal force

$\vec{F}_i$ – arises from the action of other objects on the i^{th} particle

Newton's third law:

$$\vec{f}_{ij} + \vec{f}_{ji} = 0$$

Also

$$\vec{r}_{i/A} \times \vec{f}_{ij} + \vec{r}_{j/A} \times \vec{f}_{ji} = 0$$

For each element $$\vec{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{f}_{ij} = m_i \vec{a}_i$$

Sum them up…

$$\sum_{i=1}^{N} \vec{F}_i + \cancel{\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{f}_{ij}} = \sum_{i=1}^{N} m_i \vec{a}_i$$

$$\Rightarrow \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} m_i \vec{a}_i$$

Re-write it…

$$\sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} m_i \frac{d^2 \vec{r}_{i/A}}{dt^2} = \frac{d^2}{dt^2}\left(\sum_{i=1}^{N} m_i \vec{r}_{i/A} \right)$$

First moment of mass (subscript *G* stands for center of gravity)

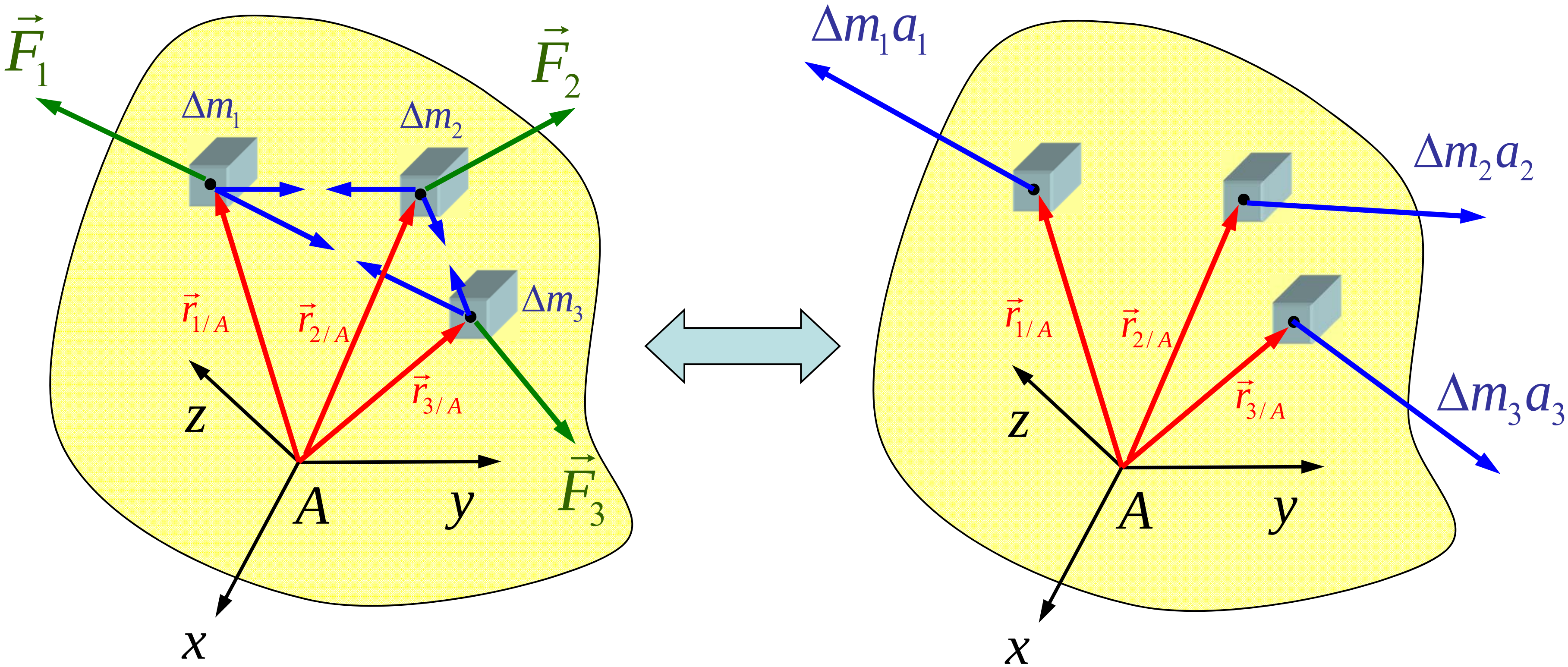
$$\sum_{i=1}^{N} m_i \vec{r}_{i/A} = m\vec{r}_{G/A}$$

$$\Longrightarrow \quad \boxed{\sum \vec{F} = \frac{d^2}{dt^2}\left(m\vec{r}_{G/A}\right) = m\vec{a}_G}$$

where ΣF is the resultant of external forces. This equation shows that modeling a rigid body as a particle is equivalent to focusing attention on the motion of the center of gravity. Notice that the acceleration of an arbitrary point is generally different than a_G.

According to Chasle's theorem, in order to complete describe the motion of a rigid body we also need to study the rotation.

1.3 Resultant moment and rotation



$$\left(M_A\right)_i = \vec{r}_{i/A} \times m_i \vec{a}_i$$

since $$\vec{a}_i = \vec{a}_A + \frac{d\vec{v}_{i/A}}{dt}$$

$$(M_A)_i = \vec{r}_{i/A} \times m_i \vec{a}_A + \vec{r}_{i/A} \times m_i \frac{d\vec{v}_{i/A}}{dt}$$

$$= m_i \vec{r}_{i/A} \times \vec{a}_A + \frac{d}{dt} \underbrace{\left(\vec{r}_{i/A} \times m_i \vec{v}_{i/A} \right)}_{\substack{\text{angular momentum} \\ \text{of } i^{\text{th}} \text{ element}}}$$

Sum them up…

$$M_A = \sum_{i=1}^{N} \vec{r}_{i/A} \times F_i = \underbrace{\sum_{i=1}^{N} m_i \vec{r}_{i/A}}_{m\vec{r}_{G/A}} \times \vec{a}_A + \underbrace{\sum_{i=1}^{N} \frac{d}{dt} \left(\vec{r}_{i/A} \times m_i \vec{v}_{i/A} \right)}_{\dot{\vec{H}}}$$

Total angular momentum about point A

$$\boxed{\vec{H}_A = \sum_{i=1}^{N} \left(\vec{r}_{i/A} \times m_i \vec{v}_{i/A} \right) = \sum_{i=1}^{N} m_i \left[\vec{r}_{i/A} \times \left(\vec{\omega} \times \vec{r}_{i/A} \right) \right]}$$

$$\Longrightarrow \quad M_A = m\vec{r}_{G/A} \times \vec{a}_A + \dot{\vec{H}}$$

$m\vec{r}_{G/A} \times \vec{a}_A = 0$ if

1) point A is center of gravity G so that $\vec{r}_{G/A} = 0$

2) point A has no acceleration: $\vec{a}_A = 0$

3) $\vec{a}_A$ is parallel to $\vec{r}_{G/A}$ (read pp.168-169)

$$\Longrightarrow \quad \boxed{M_A = \dot{\vec{H}}}$$

1.4 kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\vec{v}_i \bullet \vec{v}_i\right)$$

w.r.t. an arbitrary point B

$$\vec{v}_i = \vec{v}_B + \cancel{\left(\vec{v}_i\right)_{xyz}} + \vec{v}_{i/B}$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\vec{v}_B + \vec{v}_{i/B}\right) \bullet \left(\vec{v}_B + \vec{v}_{i/B}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i \left(\vec{v}_B \bullet \vec{v}_B + 2\vec{v}_B \bullet \vec{v}_{i/B} + \vec{v}_{i/B} \bullet \vec{v}_{i/B}\right)$$

$$= \frac{1}{2} m \vec{v}_B \bullet \vec{v}_B + \vec{v}_B \bullet \sum_{i=1}^{N} m_i \vec{v}_{i/B} + \frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_{i/B} \bullet \vec{v}_{i/B}$$

2nd term: $$\sum_{i=1}^{N} m_i \vec{v}_{i/B} = \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \vec{r}_{i/B}\right] = m\vec{v}_{G/B}$$

3rd term: $$\frac{1}{2}\sum_{i=1}^{N} m_i \vec{v}_{i/B} \bullet \vec{v}_{i/B} = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\vec{\omega}\times\vec{r}_{i/B}\right)\bullet\left(\vec{\omega}\times\vec{r}_{i/B}\right)$$

Use identity: $$\left(\vec{a}\times\vec{b}\right)\bullet\vec{c} = \vec{a}\bullet\left(\vec{b}\times\vec{c}\right)$$

$$\frac{1}{2}\sum_{i=1}^{N} m_i \left(\vec{\omega}\times\vec{r}_{i/B}\right)\bullet\left(\vec{\omega}\times\vec{r}_{i/B}\right) = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{\omega}\bullet\left[\vec{r}_{i/B}\times\left(\vec{\omega}\times\vec{r}_{i/B}\right)\right] = \frac{1}{2}\vec{\omega}\bullet\vec{H}_B$$

$$T = \frac{1}{2} m\vec{v}_B \bullet \vec{v}_B + m\vec{v}_{G/B} + \frac{1}{2}\vec{\omega}\bullet\vec{H}_B$$

1. Fundamental Principles

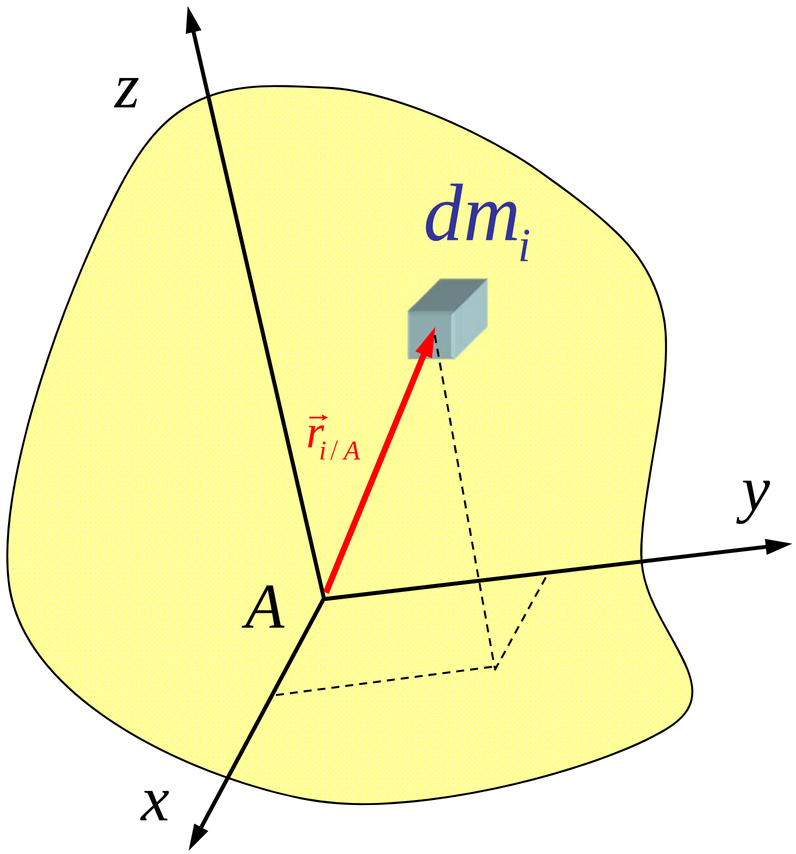
1.4 kinetic energy

If we select the center of mass as point B, the kinetic energy is simplified as

$$\boxed{\begin{aligned} T &= \frac{1}{2} m\vec{v}_G \bullet \vec{v}_G + \frac{1}{2}\vec{\omega} \bullet \vec{H}_G \text{ for all motions} \\ T &= \frac{1}{2}\vec{\omega} \bullet \vec{H}_O \text{ for pure rotation about point } O \end{aligned}}$$

2. Angular momentum and inertia properties

2.1 Moments and products of inertia



$$\vec{r}_{i/A} = x\hat{i} + y\hat{j} + x\hat{k}$$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\vec{H}_A = \sum_{i=1}^{N} m_i \left[\vec{r}_{i/A} \times \left(\vec{\omega} \times \vec{r}_{i/A} \right) \right]$$

$$\begin{aligned}
\vec{H}_A &= \iiint \left[\vec{r}_{i/A} \times \left(\vec{\omega} \times \vec{r}_{i/A} \right) \right] dm \\
&= \iiint \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \times \left[\left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \right) \times \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \right] dm
\end{aligned}$$

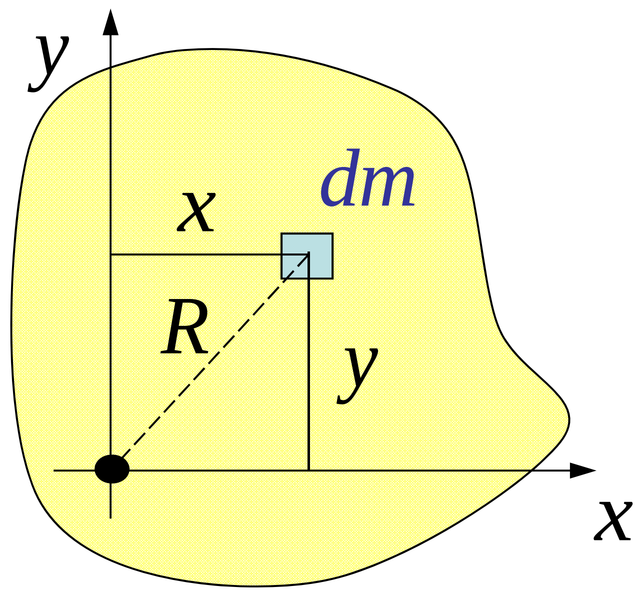
$$\begin{aligned}
\vec{H}_A = & \left[\iiint \left(\omega_x y^2 + \omega_x z^2 - \omega_y xy - \omega_z xz \right) dm \right] \hat{i} \\
& + \left[\iiint \left(\omega_y x^2 + \omega_y z^2 - \omega_x xy - \omega_z yz \right) dm \right] \hat{j} \\
& + \left[\iiint \left(\omega_z x^2 + \omega_z y^2 - \omega_x xz - \omega_y yz \right) dm \right] \hat{k}
\end{aligned}$$

Define *mass moments of inertia*:

$$I_{xx} = \iiint \left(y^2 + z^2 \right) dm, I_{yy} = \iiint \left(x^2 + z^2 \right) dm, I_{zz} = \iiint \left(x^2 + y^2 \right) dm$$

and *products of inertia*:

$$I_{xy} = \iiint xydm,\ I_{yz} = \iiint yzdm,\ \ I_{xz} = \iiint xzdm$$



$$\Rightarrow \quad \vec{H}_A = \left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right)\hat{i} + \left(I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z\right)\hat{j} + \left(I_{zz}\omega_z - I_{xz}\omega_x - I_{yz}\omega_y\right)\hat{k}$$

or
$$\vec{H}_A = \begin{Bmatrix} H_{Ax} \\ H_{Ay} \\ H_{Az} \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

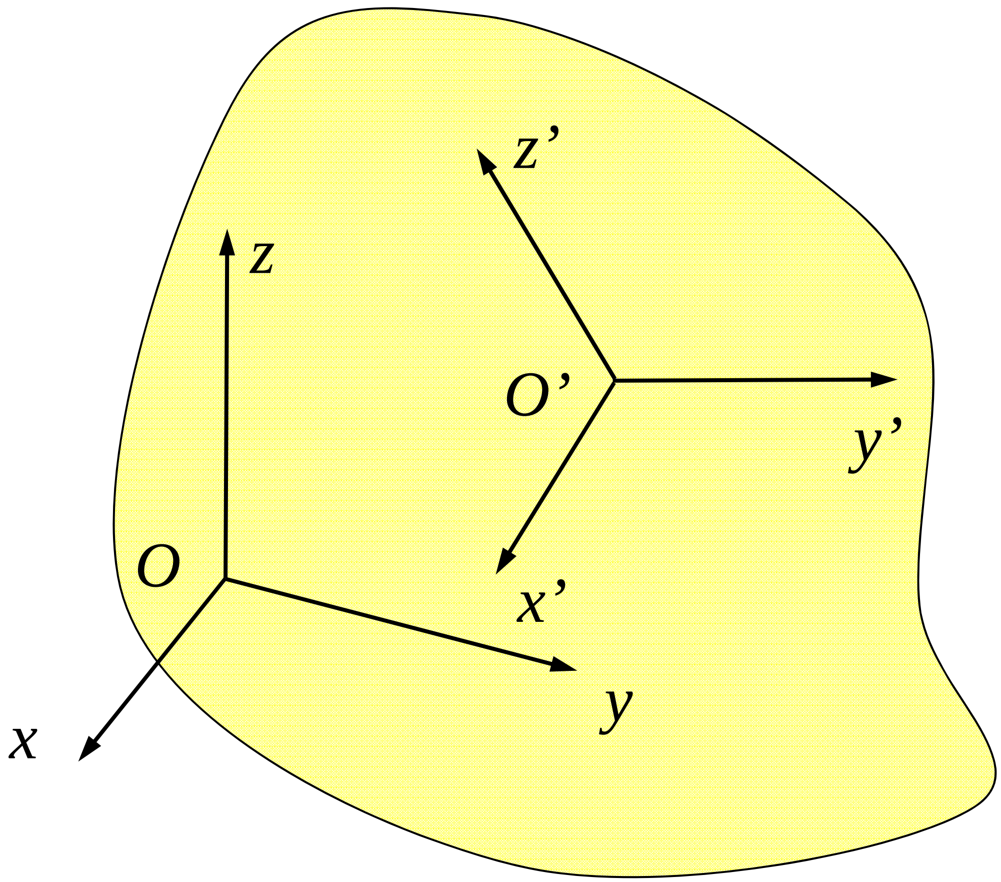
- If two coordinate axes form a plane of symmetry for a body, then all products of inertia involving the coordinate *normal to* that plane are zero.

- If at least two of the three coordinate planes are planes of symmetry for a body, then all products of inertia (all the off-diagonal terms) are zero.

- Whenever the coordinate axes correspond to vanishing values of al products of inertia, they are called **principal axes**.

2. Angular momentum and inertia properties

2.2 Transformation of inertia properties

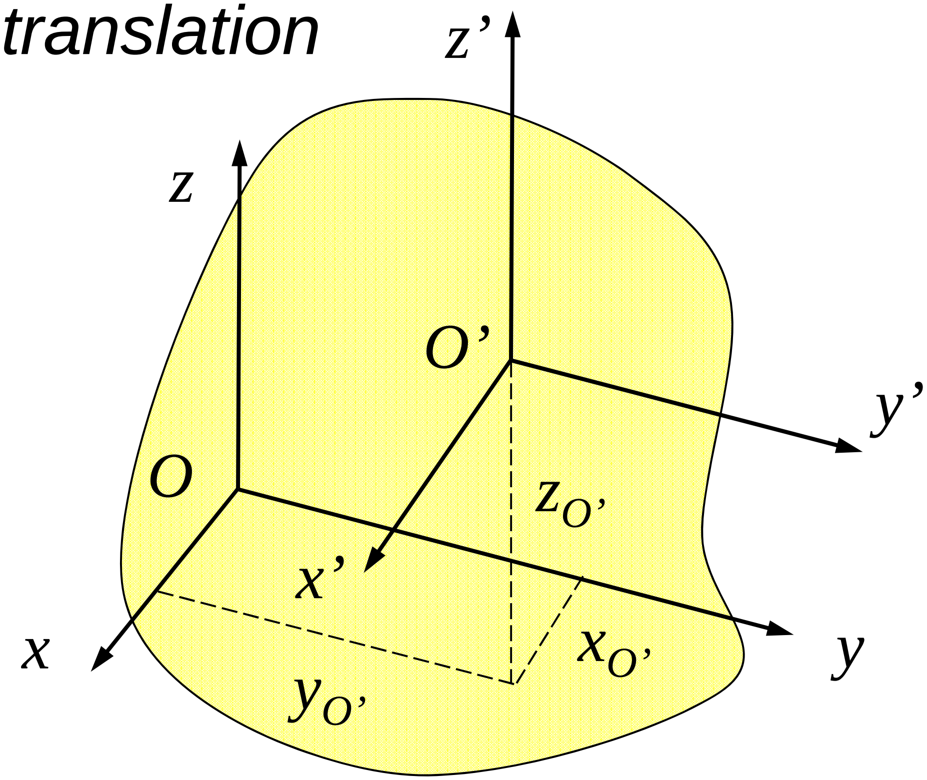


Transformation due to pure translation

$$x' = x - x_{O'}$$

$$y' = y - y_{O'}$$

$$z' = z - z_{O'}$$



$$I_{x'x'} = \iiint \left[(y')^2 + (z')^2 \right] dm$$

$$= \iiint \left[y^2 + z^2 - 2y_{O'}y - 2z_{O'}z + (y_{O'})^2 + (z_{O'})^2 \right] dm$$

$$= I_{xx} - 2y_{O'} \iiint y\,dm - 2z_{O'} \iiint z\,dm + m\left(y_{O'}^2 + z_{O'}^2 \right)$$

$$= I_{xx} - 2my_{O'}y_G - 2mz_{O'}z_G + m\left(y_{O'}^2 + z_{O'}^2 \right)$$

This translation transformation may be simplified if the origin O is restricted to being the center of mass. Then $x_\mathrm{G} = y_\mathrm{G} = z_\mathrm{G} = 0$.

$$I_{x'x'} = I_{xx} + m\left(y_{O'}^2 + z_{O'}^2\right)$$

$$I_{y'y'} = I_{yy} + m\left(x_{O'}^2 + z_{O'}^2\right)$$

$$I_{z'z'} = I_{zz} + m\left(x_{O'}^2 + y_{O'}^2\right)$$

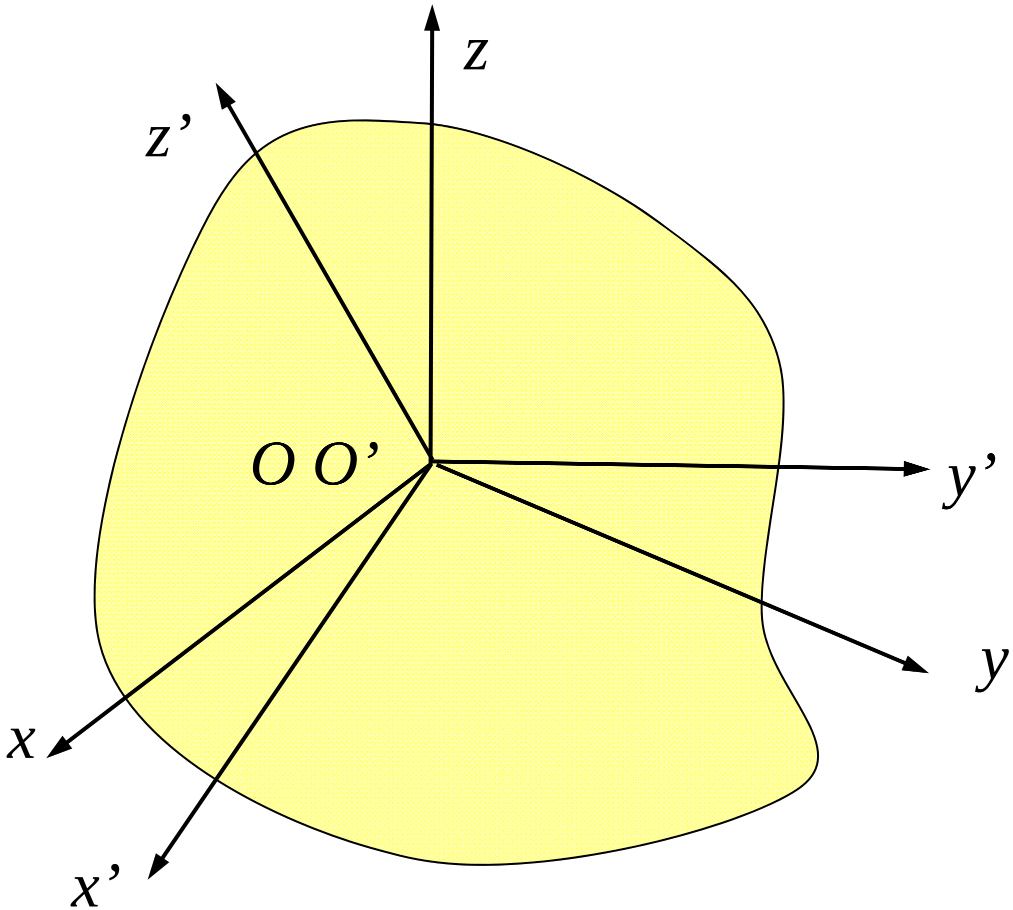
$$I_{x'y'} = I_{xy} + m x_{O'} y_{O'}$$

$$I_{y'z'} = I_{yz} + m y_{O'} z_{O'}$$

$$I_{x'z'} = I_{xz} + m x_{O'} z_{O'}$$

This is why the inertia formulas of common rigid bodies listed in any textbook are all about their centers of mass.

Transformation due to pure rotation



The kinetic energy of a rigid body is independent of the choice of the C.S. For pure rotation

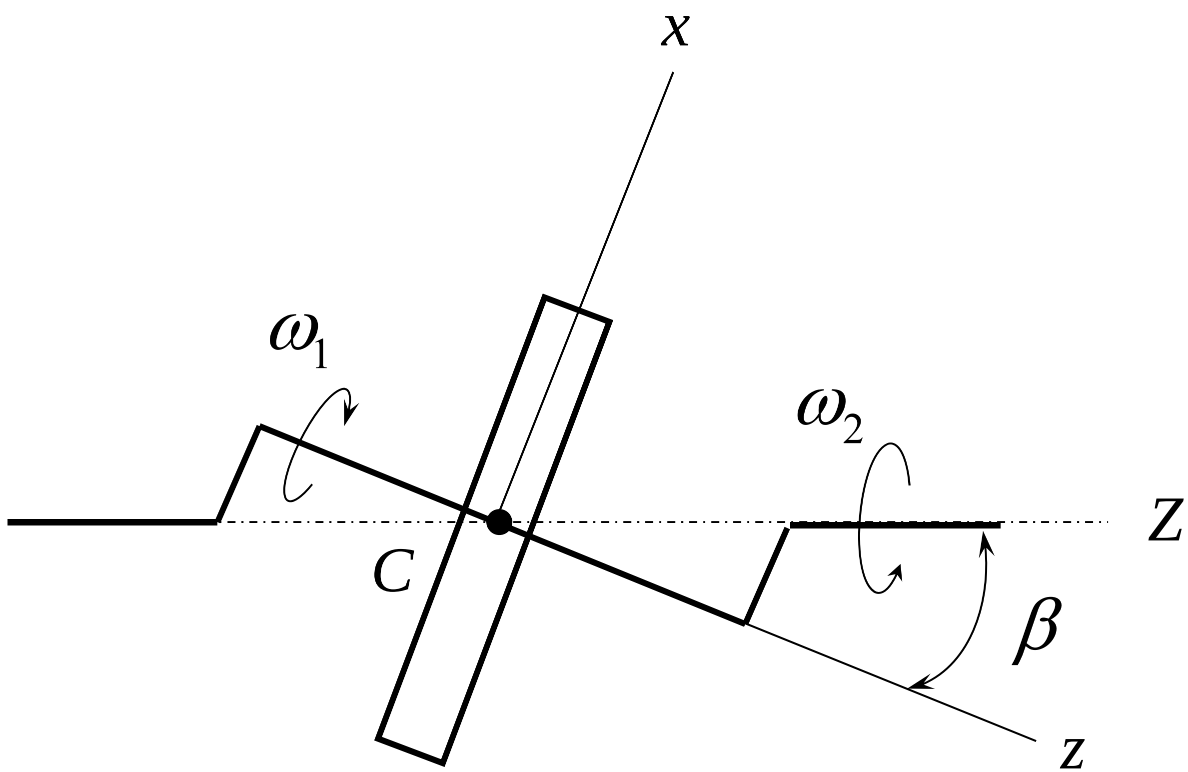
Let $\{\omega\}_{xyz} = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}_{xyz}^T$ notice $\{\omega\}_{x'y'z'} = [R]\{\omega\}_{xyz}$

$$T = \frac{1}{2}\vec{\omega} \bullet \vec{H}_O \text{ for pure rotation}$$

$$= \frac{1}{2}\{\omega\}_{xyz}^T [I]_{xyz} \{\omega\}_{xyz} = \frac{1}{2}\{\omega\}_{x'y'z'}^T [I]_{x'y'z'} \{\omega\}_{x'y'z'}$$

$$\{\omega\}_{x'y'z'}^T [I]_{x'y'z'} \{\omega\}_{x'y'z'} = \{\omega\}_{xyz}^T \underbrace{[R]^T [I]_{x'y'z'} [R]}_{=[I]_{xyz}} \{\omega\}_{xyz}$$

$$\Rightarrow \boxed{[I]_{x'y'z'} = [R][I]_{xyz}[R]^T}$$



x
ω1
ω2
Z
C
β
z

2. Angular momentum and inertia properties

2.3 Rate of change of angular momentum

$$\dot{\vec{H}}_A = \frac{\delta \vec{H}_A}{\delta t} + \vec{\omega} \times \vec{H}_A$$

where

$$\frac{\delta \vec{H}_A}{\delta t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z \right)\hat{i} + \left(I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z \right)\hat{j}$$

$$+ \left(I_{zz}\alpha_z - I_{xz}\alpha_z - I_{yz}\alpha_y \right)\hat{k}$$

3. Equation of motion

$$\sum \vec{F} = \frac{d^2}{dt^2}\left(m\vec{r}_{G/A}\right) = m\vec{a}_G \qquad \sum M_A = \frac{\delta \vec{H}_A}{\delta t} + \vec{\omega} \times \vec{H}_A$$

$$\vec{H}_A = \left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right)\hat{i} + \left(I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z\right)\hat{j}$$
$$+\left(I_{zz}\omega_z - I_{xz}\omega_z - I_{yz}\omega_y\right)\hat{k} = \left[I\right]_{xyz}\left\{\omega\right\}_{xyz}$$

$$\frac{\delta \vec{H}_A}{\delta t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z\right)\hat{i} + \left(I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z\right)\hat{j}$$
$$+\left(I_{zz}\alpha_z - I_{xz}\alpha_z - I_{yz}\alpha_y\right)\hat{k} = \left[I\right]_{xyz}\left\{\alpha\right\}_{xyz}$$

1) point A is center of gravity G, or 2) point A has no acceleration

3. Equation of motion

Matrix form

$$\begin{Bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{Bmatrix} = m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ a_{Gz} \end{Bmatrix}$$

$$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = [I] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} [I] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

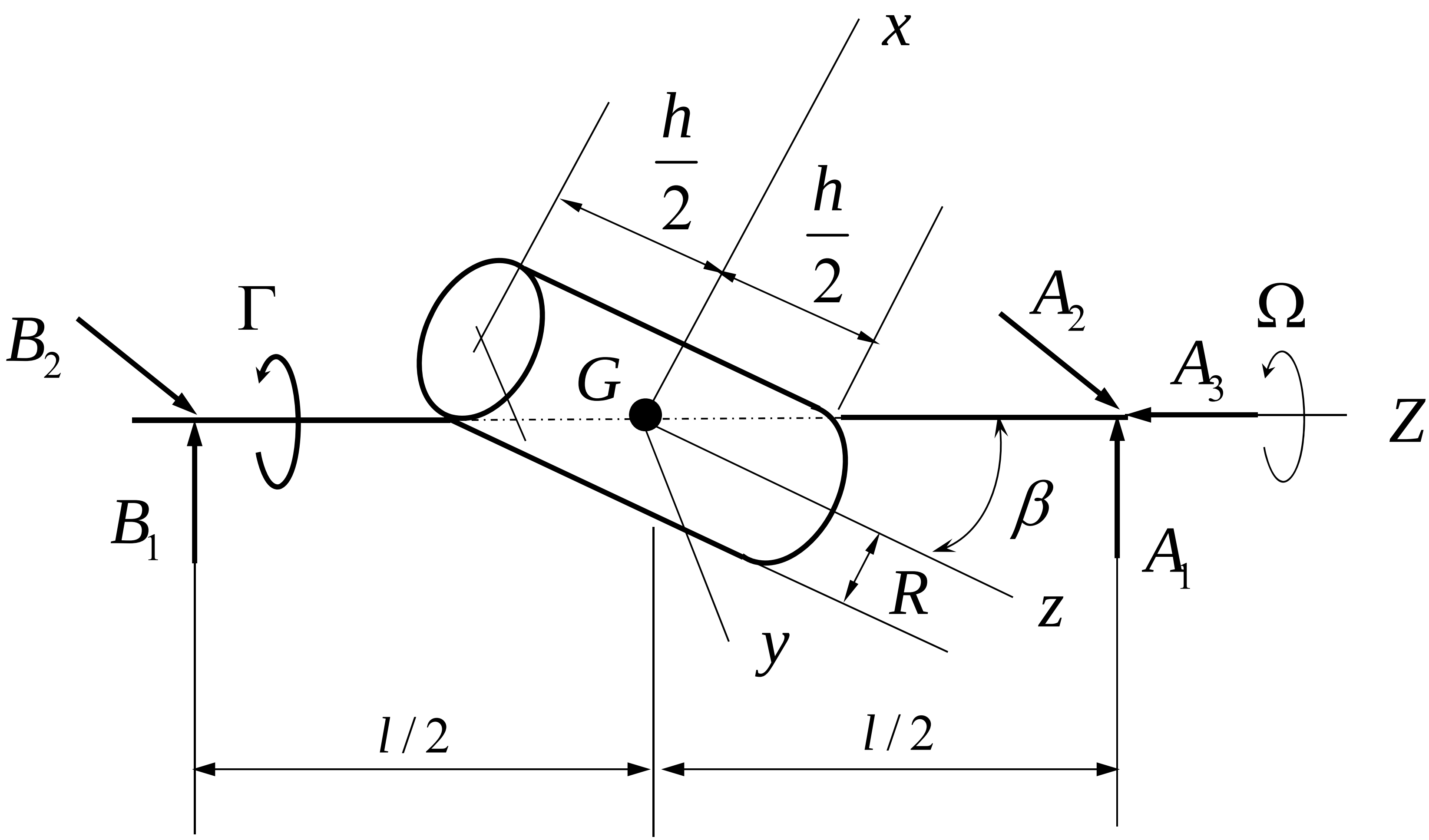
Coordinates attached to the body

The special case where *xyz* are principal axes leads to *Euler's equation*:

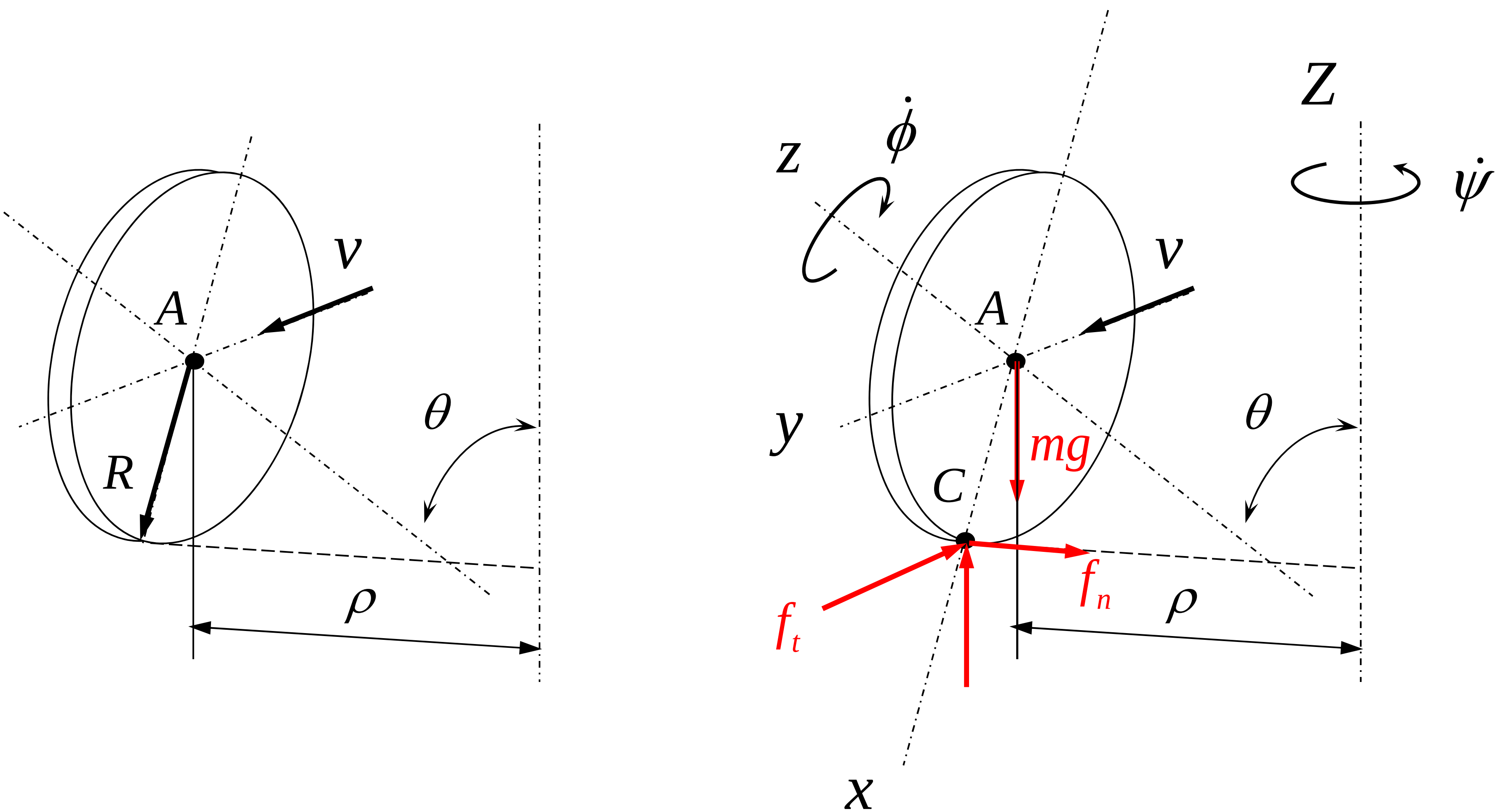
$$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = \begin{Bmatrix} I_{xx}\alpha_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z \\ I_{yy}\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\omega_z \\ I_{zz}\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y \end{Bmatrix}$$

3. Equation of motion

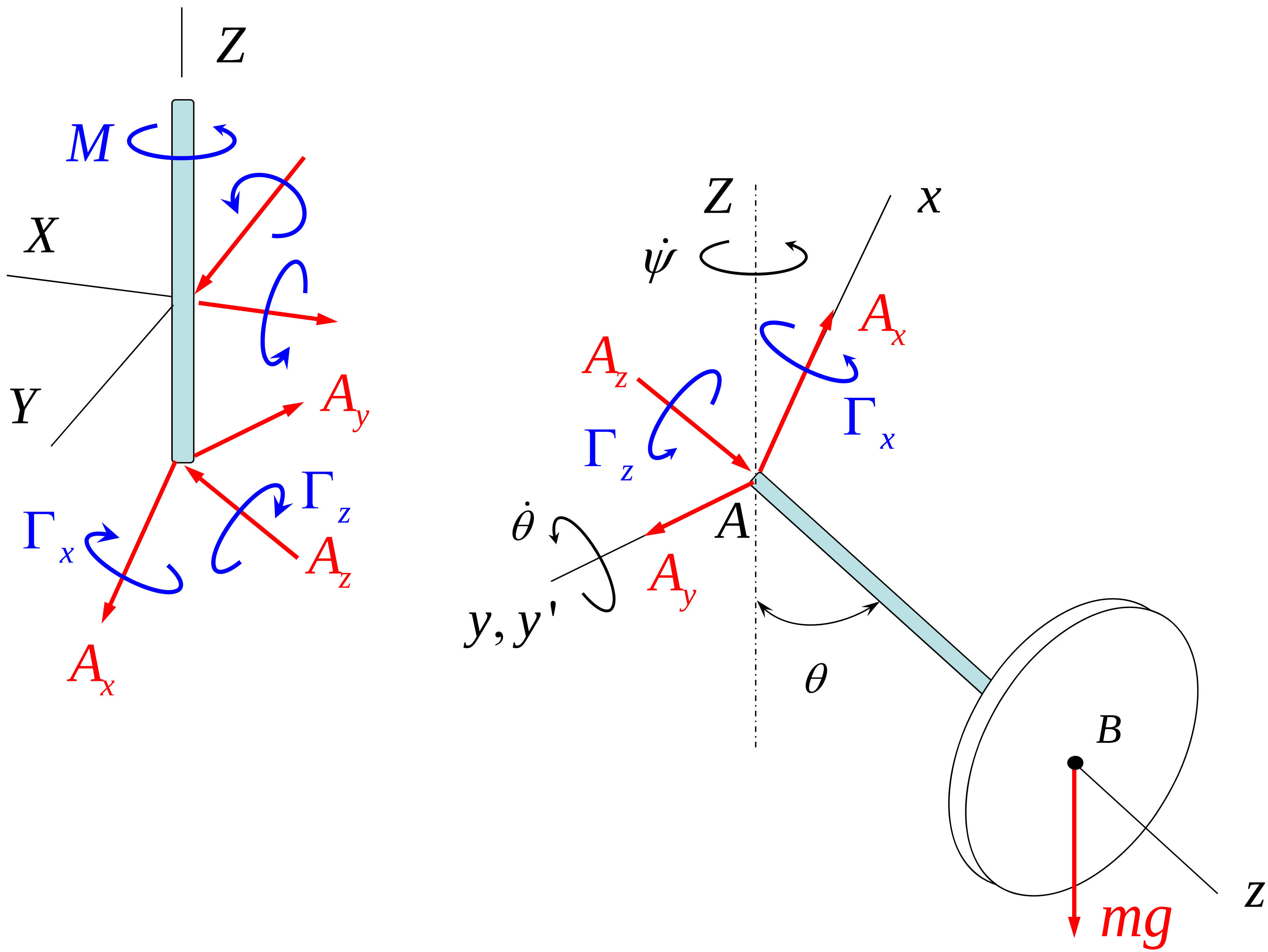
- Free-body-diagram
  - If a support prevents a point in the body from moving in a certain direction, then at that point there must be a reaction force exerted on the body in that direction.
  - Similarly, a kinematical constraint on rotation about an axis is imposed by a reaction couple exerted about this axis.



x
h/2
h/2
B2
Γ
A2
Ω
A3
G
Z
B1
β
A1
R
z
y
l / 2
l / 2



A
v
R
θ
ρ
z
$\dot{\phi}$
Z
$\dot{\psi}$
A
v
y
mg
C
θ
f_t
f_n
ρ
x



Z
M
X
Y
A_y
Γ_z
Γ_x
A_z
A_x
Z
x
$\dot{\psi}$
A_x
A_z
Γ_x
Γ_z
$\dot{\theta}$
A
A_y
y, y'
θ
B
z
mg

4. Planar motion

$$\vec{a}_G = a_{Gx}\hat{i} + a_{Gy}\hat{j}, \quad \vec{\omega} = \omega\hat{k}, \;\; \vec{\alpha} = \alpha\hat{k} = \dot{\omega}\hat{k}$$

$$\vec{H}_A = -I_{xz}\omega\hat{i} - I_{yz}\omega\hat{j} + I_{zz}\omega\hat{k}$$

$$\sum \vec{F}_x = ma_{Gx}, \;\; \sum \vec{F}_y = ma_{Gy}, \;\; \sum \vec{F}_z = 0$$

$$\sum M_{Ax} = -I_{xz}\dot{\omega} + I_{yz}\omega^2, \;\; \sum M_{Ay} = -I_{yz}\dot{\omega} - I_{xz}\omega^2, \;\; \sum M_{Az} = -I_{zz}\dot{\omega}$$