7 – Introduction to Analytical Mechanics

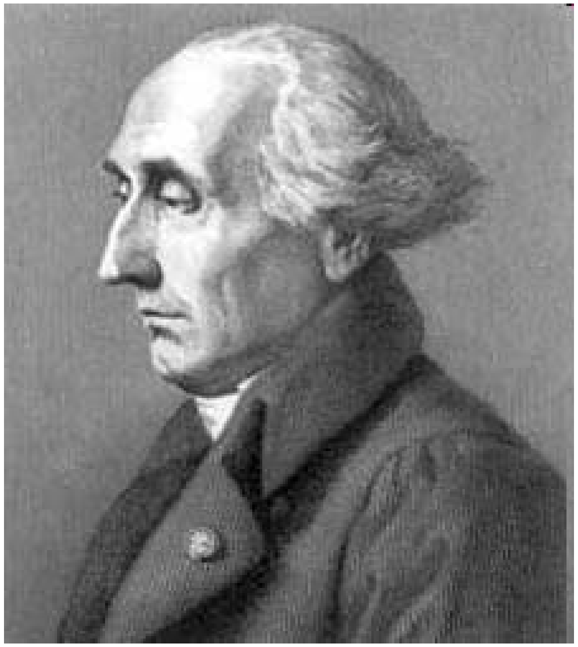
Outline

- Introduction
- Reviews
  - Work and energy
  - Virtual work
  - Generalized coordinates
- Lagrangian equations
- Hamilton's principle

1. Introduction

1. Introduction

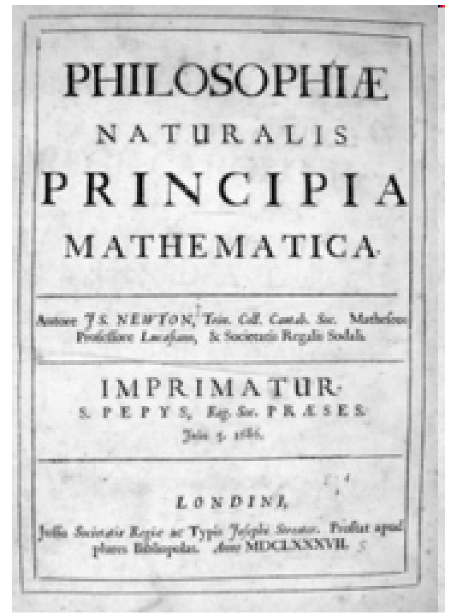
- Lagrange (*Mecanique Analytique*, 1788): “No diagrams will be found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field.”



Joseph-Louis Lagrange (1736-1813)

1. Introduction

- This was in marked contrast to Newton’s *Philosophiae Naturalis Principia Mathematica* which is full of elaborate geometrical constructions.



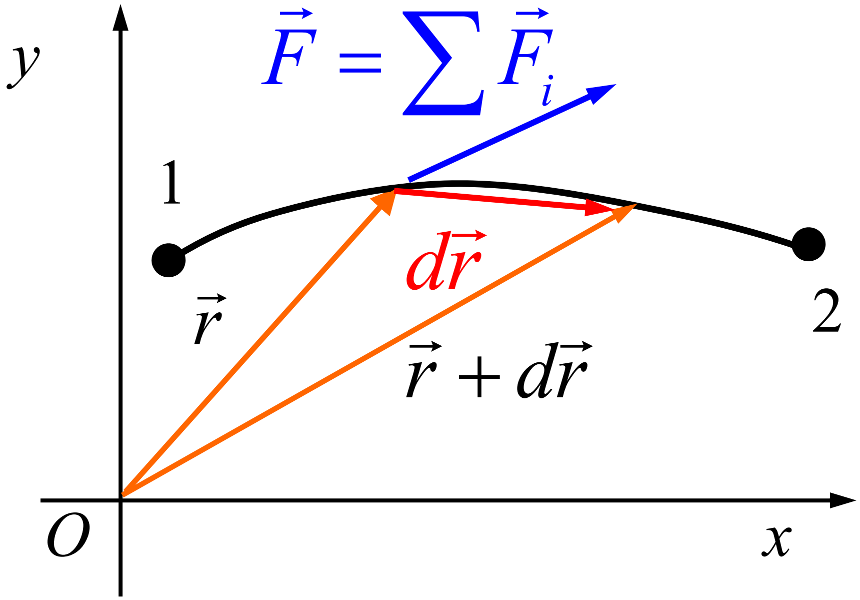
- The development of analytical dynamics begins with d’Alembert’s *Traite de dynamique* of 1743 and ends with Appell’s *Traite de Mechanique Rationelle* of 1896.

1. Introduction

- Among the 150 years, two works stand out:
  - Lagrange: Mecanique Analytique, 1788
  - Hamilton: General Method in Dynamics, 1835
- Unlike the Newton-Euler formulation which deals with the time derivatives of momentum principles, the Largrangian formulation takes a different view of systems, which is based on an overview of the system and its mechanical energy.
- The analytical methods are very useful when the system consists of more than one body.
- The reactions exerted by supports will usually not appear in the Lagrangian forumulation.

2. Reviews

2.1 Work and Energy



Definition:

$$dW = \vec{F} \bullet d\vec{r} \qquad (1)$$

$$W_{1\to 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \bullet d\vec{r} \qquad (2)$$

By Newton's 2nd law: $\vec{F} = m\vec{a} = m\dfrac{d^2\vec{r}}{dt^2} = m\dfrac{d\dot{\vec{r}}}{dt}$

Also, $d\vec{r} = \dfrac{d\vec{r}}{dt}dt = \dot{\vec{r}}dt$ We then can rewrite (2)

$$W_{1\to 2} = \int_{\vec{r}_1}^{\vec{r}_2} m\frac{d\dot{\vec{r}}}{dt}\bullet\dot{\vec{r}}dt = \frac{1}{2}\int_{\vec{r}_1}^{\vec{r}_2} m\frac{d}{dt}\left(\dot{\vec{r}}\bullet\dot{\vec{r}}\right)dt$$

$$= \frac{1}{2}m\dot{\vec{r}}_2\bullet\dot{\vec{r}}_2 - \frac{1}{2}m\dot{\vec{r}}_1\bullet\dot{\vec{r}}_1$$

$$W_{1\to 2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1 \qquad (3)$$

where $$T = \frac{1}{2}mv^2 \qquad (4)$$

For a particle, the work done by external forces is the change in kinetic energy. (always true)

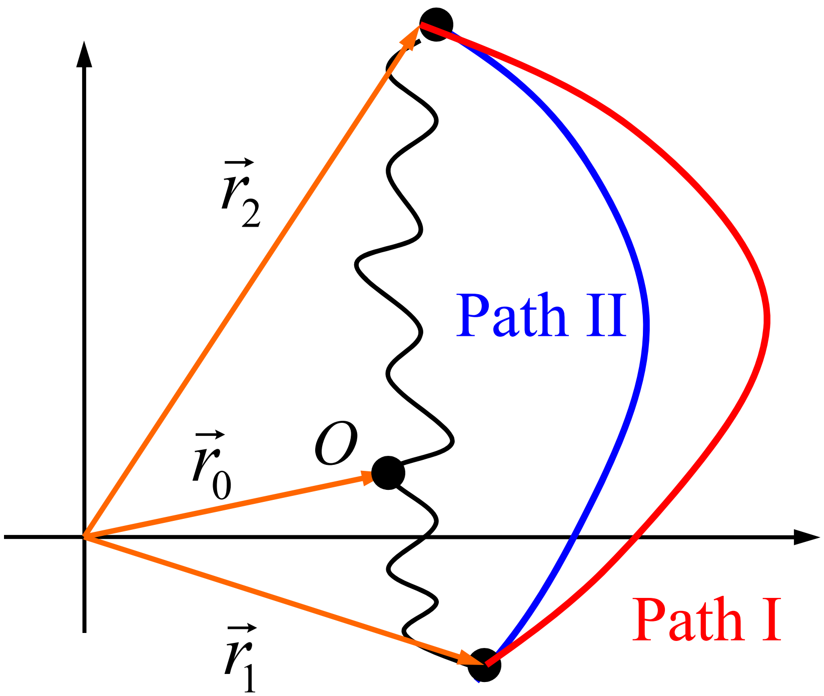
- Conservative force field: a force field in which the work done in moving a particle from one point to another is independent of path.

Compute the work done in such a field

$$W_{1\to 2} = \int_{\substack{\vec{r}_1 \\ \text{Path I}}}^{\vec{r}_2} \vec{F} \bullet d\vec{r} = \int_{\substack{\vec{r}_1 \\ \text{Path II}}}^{\vec{r}_2} \vec{F} \bullet d\vec{r} \qquad (5)$$

- The potential Energy $V(r_1)$: associated with position r_1 is defined as the work done in a conservative force field in moving a particle from r_1 to a reference point r_0

$$V(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_0} \vec{F} \bullet d\vec{r} \qquad (6)$$



Now calculate the work done in moving the particle from position $\vec{r}_1$ to $\vec{r}_2$. Choosing an arbitrary path though O

$$\underset{\text{(conservative)}}{W_{1\to 2}} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \bullet d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_O} \vec{F} \bullet d\vec{r} + \int_{\vec{r}_O}^{\vec{r}_2} \vec{F} \bullet d\vec{r}$$

$$= \int_{\vec{r}_1}^{\vec{r}_O} \vec{F} \bullet d\vec{r} - \int_{\vec{r}_2}^{\vec{r}_O} \vec{F} \bullet d\vec{r}$$

$$\overset{\text{def.}}{=} -\left[V(\vec{r}_2) - V(\vec{r}_1)\right] \qquad (7)$$

<u>In a conservative force field</u>, the work done is negative of the change in potential energy.

$$\underset{\text{(conservative)}}{dW} = -dV \qquad (8)$$

In general both conservative and non-conservative forces act on the particle, let

$$\vec{F} = \vec{F}_c + \vec{F}_{nc} \qquad (9)$$

$$dW_c = \vec{F}_c \bullet d\vec{r} = -dV = -\left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right)$$

$$= -\underset{\sim}{\nabla} V \bullet d\vec{r} \qquad (10)$$

$$\text{where } \underset{\sim}{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \qquad (11)$$

$$\vec{F}_c = -\underset{\sim}{\nabla} V \tag{12}$$

All conservative forces can be derived from a potential function V and is equal to negative of its gradient.

But from vector algebra, for a smooth function Φ

$$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \Phi = 0$$

$$\underset{\sim}{\nabla} \times \vec{F} = -\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} V = \text{curl}\vec{F} = 0 \tag{13}$$

Read: http://en.wikipedia.org/wiki/Curl_(mathematics)

i.e. for a conservative force field, curl$\vec{F}$ is zero

Also from Eq. (5) $\oint \vec{F}_c \bullet d\vec{r} = 0$ (14)

The work done by conservative forces around a closed path is zero.

$$\text{Write } W_{1\to 2} = W_c + W_{nc} \tag{15}$$

$$\Rightarrow W_{nc} = \underbrace{W_{1\to 2}}_{T_2 - T_1} - \underbrace{W_c}_{V_1 - V_2} = \left(T_2 - T_1\right) + \left(V_2 - V_1\right)$$

$$\text{or } W_{nc} = \left(T_2 + V_2\right) - \left(T_1 + V_1\right) = E_2 - E_1 \tag{16}$$

$$\text{where } E = T + V \tag{17}$$

It indicates that the work done by non-conservative forces is responsible for the change in total energy of the particle.

If there are only conservative forces, then

$$W_{nc} \equiv 0 \tag{18}$$

$$\text{and } E_1 = E_1 = \text{const.} \tag{19}$$

This is the principle of conservation of energy.

Example: A particle moves around a circle C in the x-y plane

$$x^2 + y^2 = 3^2$$

Find the work done by a force

$$\vec{F} = (2x - y + z)\hat{i} + \left(x + y - z^2\right)\hat{j} + (3x - 2y + 4z)\hat{k}$$

Solution:

$$\vec{r} = x\hat{i} + y\hat{j} = (3\cos\omega t)\hat{i} + (3\sin\omega t)\hat{j}$$

In the plane $z = 0$, the force becomes

$$\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}$$

So the work done

$$W = \oint \vec{F} \bullet d\vec{r} = \oint \left[(2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}\right] \bullet \left[dx\hat{i} + dy\hat{j}\right]$$

$$= \oint \left[(2x - y)dx + (x + y)dy\right]$$

$$\text{Let}\quad x = 3\cos\theta; \qquad y = 3\sin\theta$$

$$dx = -3\sin\theta \cdot d\theta; \quad dy = 3\cos\theta \cdot d\theta$$

$$W = \int_0^{2\pi} \left[2\left(3\cos\theta\right) - 3\sin\theta\right]\left(-3\sin\theta\right) d\theta$$

$$+ \int_0^{2\pi} \left[3\cos\theta + 3\sin\theta\right]\left(3\cos\theta\right) d\theta$$

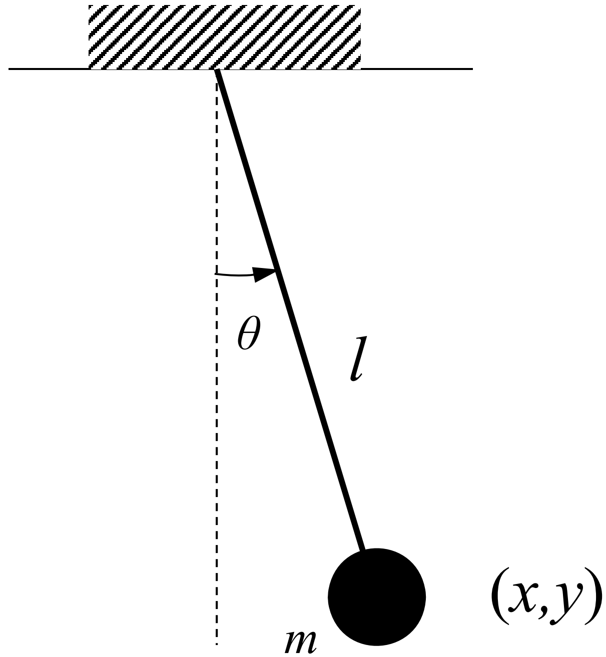
$$= \int_0^{2\pi} \left[9 - 9\sin\theta\cos\theta\right] d\theta = 18\pi$$

Since the work done around a closed path is not zero, the force field is non-conservative. Check:

$$\underset{\sim}{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(2x - y\right) & \left(x + y\right) & \left(3x - 2y\right) \end{vmatrix} = -2\hat{i} + \hat{k} + \hat{k} - 3\hat{j} \neq 0$$

2.2 DOF and constraints

- The minimum number of coordinates required to describe the configuration of the system is called the number of degrees of freedom.
- DOF and constraints are a pair of dialectical concepts
- The relations between constrained coordinates are called constraint equations.
- The number of DOF is the number of coordinates minus the number of constraint equations.

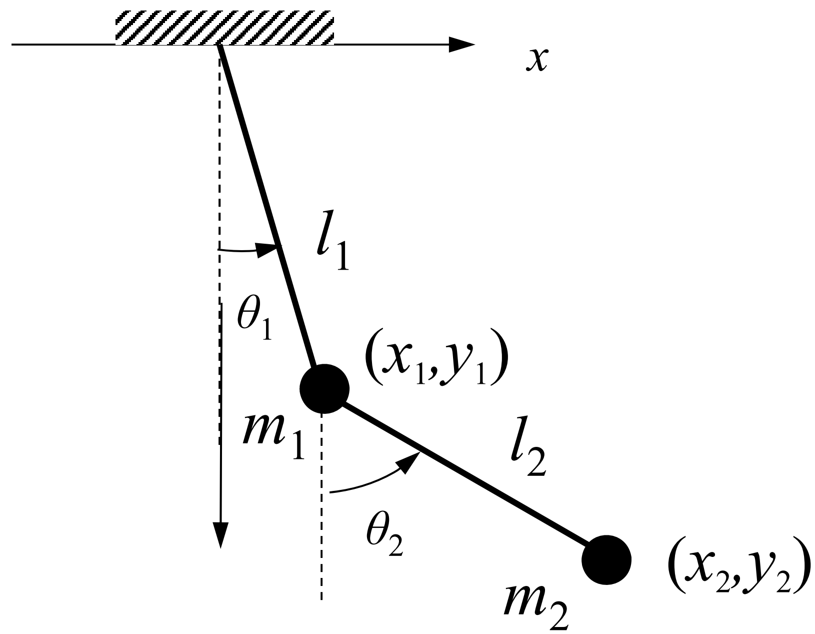


(x,y) are the coordinates, but

$$x^2 + y^2 = l^2 \qquad \text{constraint equation}$$

So simple pendulum is a one-DOF system.

Generally, n particles in space have $3n$ coordinates. If there are m constraints, then the DOF is $3n - m$.



(x_1,y_1) and (x_2,y_2) are four coordinates, but there are two constraints

$$x_1^2 + y_1^2 = l_1^2; \quad \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 = l_2^2$$

θ_1 and θ_2 can be taken as generalized coordinates.

Definition: Consider a system of n particles. Such a system has a $3n$ configuration space. A *holonomic* constraints on the motion of the particles is one that can be expressed in the form

$$f(x_1, x_2, \ldots x_{3n}, t) = 0$$

Otherwise the constraint is *nonholonomic*.

If all the constraints are holonomic, the system is holonomic:

$$f_i(x_1, x_2, \ldots x_{3n}, t) = 0,\ i = 1,2,\ldots, m$$

For example, the constraint for a planar motion is

$$f(x, y, z, t) = z = 0 \Rightarrow \dot{z} = 0, \ddot{z} = 0$$

For the simple pendulum, the constraint can be written as

$$f(x, y, t) = x^2 + y^2 - l^2 = 0$$

Definition: The differential form of the displacement constraints, whether integrable or not, is called the *Pfaffian form*.

For example, the constraint for the simple pendulum example

$$\dot{f}(x, y, t) = x\dot{x} + y\dot{y} = 0$$

Note this velocity constraint condition does not have l. However,

$$\frac{\dot{y}}{\dot{x}} = -\frac{x}{y} = -\frac{1}{\tan\theta}$$

We observe that

(1) The velocity is perpendicular to the rod;

(2) x and y must be initialized using the rod length l.

Starting with a holonomic constraint

$$f_i\,(x_1, x_2, \ldots x_{3n}, t) = 0$$

If $x_1, x_2, \ldots x_{3n}$ are functions of a parameter α, we differentiate it and get

$$\sum_{s=1}^{3n} \frac{\partial f_i}{\partial x_s} \frac{dx_s}{d\alpha} + \frac{\partial f_i}{\partial t} \frac{dt}{d\alpha} = 0$$

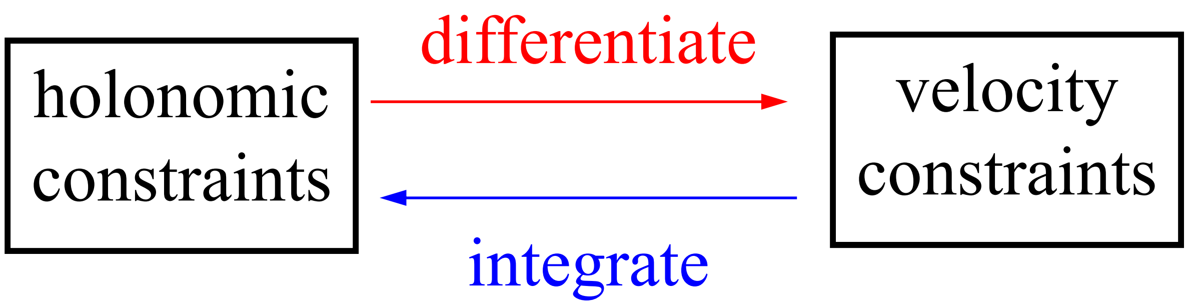
Specially, when $\alpha = t$

$$\sum_{s=1}^{3n} \frac{\partial f_i}{\partial x_s} \dot{x}_s + \frac{\partial f_i}{\partial t} = 0$$

where $\dot{x}_s$ is velocity. We can integrate it and go back to the finite displacement form.

If a velocity constraint is holonomic, then there exists an integrating function g_i for which the Pfaffian form of the constraint equation becomes a perfect differential.

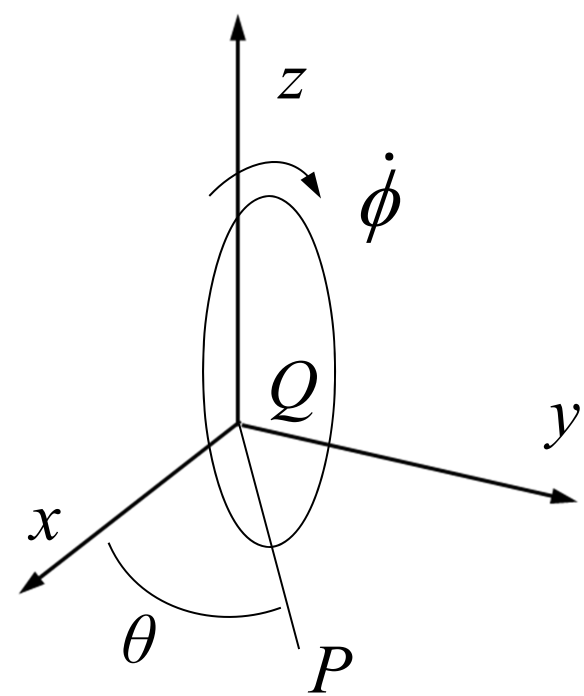
Note: Geometric constraints are holonomic constraints.
Most problems in mechanics are holonomic systems.



Holonomic constraints usually come in integrated form.
Sometimes they come in Pfaffian form.

A *nonholonomic* constraint is one which is not holonomic. It can happen in many ways. One way that a nonholonomic constraint occur is the Pfaffian form is not integrable to get a displacement constraint .

Example: A disk rolls w/o slipping on the *xy* plane. At the instant shown the line *QP* is tangent to the path. The disk has an angular velocity $\dot{\phi}$ that is directed about a line that passes through the disk center and is perpendicular to line *QP*.



There are two constraints:

1) the edge remains in contact with the plane (holonomic);

2) no slipping condition (non-holonomic). The x and y component of velocity of the center of the disk satisfy the constraints

$$\dot{x} - r\dot{\phi}\cos\theta = 0; \dot{y} - r\dot{\phi}\sin\theta = 0$$

In differential form these constraints are

$$dx - r \cdot d\phi \cdot \cos\theta = 0; \quad dy - r \cdot d\phi \cdot \sin\theta = 0$$

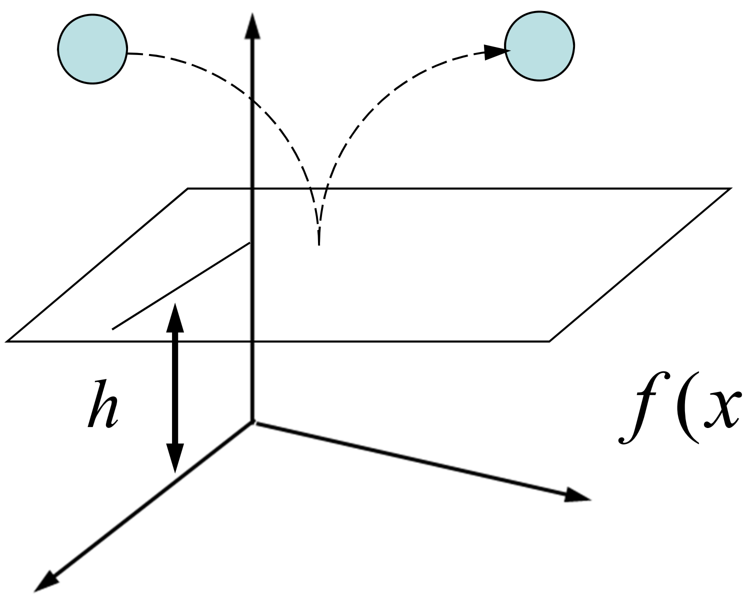
Clearly these constraints are in Pfaffian form where x, y, ϕ and θ are all displacement variables. Morever, there are no integrating factors that will reduce these equations with displacement variables alone. Therefore, these constraints are nonholonomic. Unlike holonomic constraints, these equations cannot be used to eliminate two of the four displacement variables. Hence, all four displacement variables must be used.

Here is another example of a *nonholonomic* constraint .

A constraint reduced to an *inequality* in the configuration space

$$f(x_1, x_2, \ldots x_{3n}, t) \le 0$$

For example: an object must stay on or above a plane surface



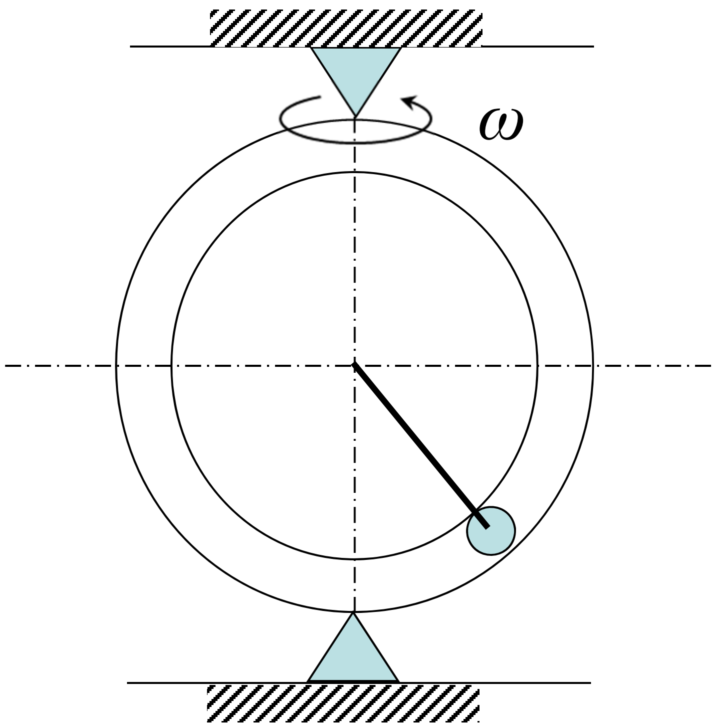
$$f(x, y, z, t) = z - h \ge 0$$

The holonomic constraints can be further classified as *scleronomic* and *rheonomic*. ("Sclero" and "rheo" are Greek phrases meaning "rigid" and "flowing", respectively).

Definition: If a holonomic constraint can be written as

$$f(x_1, x_2, \ldots x_{3n}) = 0$$

it is called *scleronomic*; otherwise, it is *rheonomic*.



$$\omega = 0 \rightarrow \textit{scleronomic}$$

$$\omega \neq 0 \rightarrow \textit{rheonomic}.$$

2.3 Virtual Work

Consider a system of n particles whose configuration is defined in terms of the Cartesian coordinates $x_1, x_2, \ldots x_{3n}$ ($x_1, x_2, \ldots x_n$, $y_1, y_2, \ldots y_n$, $z_1, z_2, \ldots z_n$) where, of course, three coordinates are required to specify the position of each particle. Suppose that the forces $F_1, F_2, \ldots F_{3n}$ are applied at the corresponding coordinates in the positive direction. Now let us imagine that, at a given instant (i.e. <u>holding the time constant</u>), the system is <u>given arbitrary</u> <u>small displacements $\delta x_1, \delta x_2, \ldots \delta x_{3n}$</u> in their corresponding directions. The work done by the applied forces:

$$\delta W = \sum_{i=1}^{3n} F_i \delta x_i \tag{20}$$

An alternate form:

$$\delta W = \sum_{i=1}^{n} \vec{F}_i \bullet \delta \vec{r}_i \tag{21}$$

This is known as *virtual work*. The small displacements are called virtual displacements because they are imaginary in the sense that they are assumed to occur without the passage of time, the applied forces remaining constant. Usually the virtual displacements conform to the kinematic constraints and are designated by $\underline{\delta x}$ in order to distinguish from the real displacement dx which occur during the time interval dt.

The difference between δx and dx is that dx refers to an actual infinitesimal change in position and can be integrated, whereas δx refers to an infinitesimal virtual (assumed) movement and cannot be integrated. Mathematically they are both the first-order differentials.

Now we consider the case where the coordinates x_1, x_2, … x_{3n} are subjected to holonomic constraints (the number of constraints is $m < 3n$)

$$\Phi_k\left(x_1,\ x_2,\ \cdots, x_{3n}, t\right) = C_k,\ k = 1, 2, \cdots, m < 3n \qquad (22)$$

Note that for a virtual displacement consistent with the constraints, the δx's are no longer completely arbitrary but are related by the m equations by differentiating Eq. (22):

$$\frac{\partial \Phi_k}{\partial x_1}\delta x_1 + \frac{\partial \Phi_k}{\partial x_2}\delta x_2 + \cdots \frac{\partial \Phi_k}{\partial x_{3n}}\delta x_{3n} = 0$$
$$(k = 1, 2, \cdots, m < 3n) \qquad (23)$$

Note the time remains constant during a virtual displacement: $\delta t = 0$

- **The principle of virtual work**

Now let's consider the conditions required for the static equilibrium of a system of particles. We separate the total force acting on a given particle m_i into a constraint force R_i and the applied force F_i. If a system of n particles in static equilibrium then for each particle

$$\vec{F}_i + \vec{R}_i = 0 \qquad (24)$$

and the virtual work from (21) becomes

$$\delta W = \sum_{i=1}^{n} \left(\vec{F}_i + \vec{R}_i \right) \bullet \delta \vec{r}_i$$

$$= \sum_{i=1}^{n} \vec{F}_i \bullet \delta \vec{r}_i + \sum_{i=1}^{n} \vec{R}_i \bullet \delta \vec{r}_i = 0 \qquad (25)$$

But, in general, constraint forces do not perform work, because the displacements do not have any components in the direction of the constraint forces. As an example, consider a particle moving on a smooth surface. The constraint forces is normal to the surface and the displacements are parallel to the surface. Hence,

$$\sum_{i=1}^{n} \vec{R}_i \bullet \delta \vec{r}_i = 0 \tag{26}$$

$$\delta W = \sum_{i=1}^{n} \vec{F}_i \bullet \delta \vec{r}_i = 0 \tag{27}$$

If a system is in static equilibrium then the work done by the externally applied forces through virtual displacements compatible with the constrains must be zero.

Now, consider a system of particles, with same type of constraints, which is initially motionless but not in equilibrium. Then one or more of the particles must have a net force applied to it, and in accordance with Newton's law, it will tend to move in the direction of that force. We can always choose a virtual displacement in the direction of the net force at each point. In this case the virtual work is positive, that is

$$\sum_{i=1}^{n} \vec{F}_i \bullet \delta\vec{r}_i + \sum_{i=1}^{n} \vec{R}_i \bullet \delta\vec{r}_i > 0 \tag{28}$$

$$\delta W = \sum_{i=1}^{n} \vec{F}_i \bullet \delta\vec{r}_i > 0 \tag{29}$$

In other words, if the system is not in equilibrium then it is always possible to find a set of virtual displacements consistent with the constraints for which the **virtual work of the applied forces is positive.**

Let's now consider a system of n particles where all the applied forces are <u>conservative</u>. Then we can write the potential energy function in the form

$$V = V\left(x_1,\ x_2,\ \cdots, x_{3n}\right) \tag{30}$$

Using equation (12) we see that the component of the applied force in x_i direction is

$$\vec{F}_c = -\frac{\partial V}{\partial x_i} \tag{31}$$

From (20), the virtual work is then

$$\delta W = \sum_{i=1}^{3n} F_i \delta x_i = -\sum_{i=1}^{3n} \frac{\partial V}{\partial x_i} \delta x_i \tag{32}$$

By definition

$$\delta V = \sum_{i=1}^{3n} \frac{\partial V}{\partial x_i} \delta x_i \tag{33}$$

Therefore, for the system to be in static equilibrium, (33) and (27) yield:

$$\delta V = 0 \tag{34}$$

If δx_i are linearly independent, (33) and (34) are satisfied only if each partial derivative is zero, i.e.

$$\frac{\partial V}{\partial x_i} = 0, \text{ for } i = 1, 2, \cdots 3n \tag{35}$$

Notice these are also the conditions for V to have an extreme value. That means the potential energy is either maximum (unstable) or minimum (stable).

This also means the components of (31) are all zero. In another word, every force component is zero, which just satisfies the definition of static equilibrium.

2.4 Generalized Coordinates

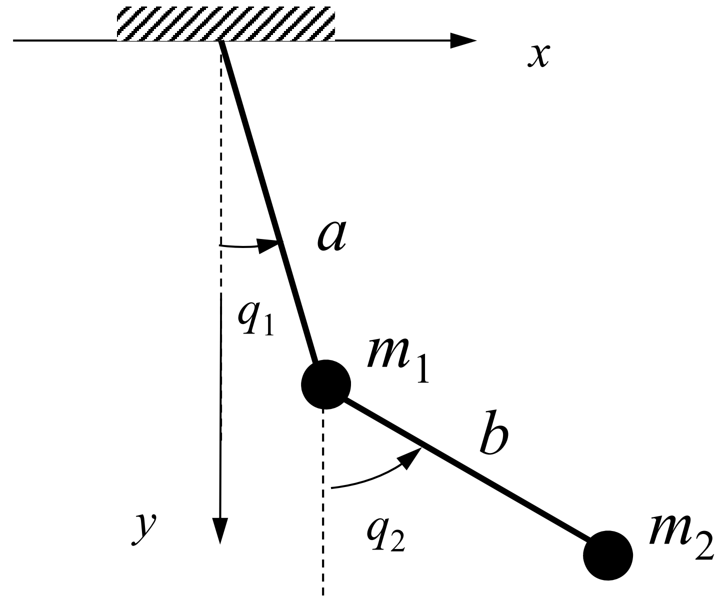
Consider the motion of n particles in three dimensional space which satisfy m independent equations of constraints. Hence the system has $(3n - m)$ DOF and a new set of $(3n - m)$ independent coordinates can be introduced. There are called the <u>generalized coordinates</u>.

Notes: (1) Generalized coordinates do not necessarily have the dimension of length

(2) Generalized coordinates do not form a unique set. (see Figures 6.1 through 6.3)

Example:

Double-pendulum



We can choose q_1 and q_2 as generalized coordinates. Then the transformation relations are

$$x_1 = a\sin q_1 ; y_1 = a\cos q_1$$

$$x_2 = a\sin q_1 + b\sin q_2$$

$$y_2 = a\cos q_1 + b\cos q_2$$

In general the transformation relations are given in the form

$$\vec{r}_i = \vec{r}_i\left(q_1,\ q_2,\ \cdots, q_{3n-m}, t\right), i = 1, 2, \cdots, n \qquad (36)$$

Note that δt is zero for any virtual displacement since virtual displacements are defined at a certain time t. We obtain

$$\delta\vec{r}_i = \sum_{j=1}^{3n-m} \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j, \quad i = 1, 2, \cdots, n \qquad (37)$$

Note q_i are not vectors.

2.5 Generalized forces

In conjunction with the generalized coordinates we can define generalized forces by equating the total virtual work in both systems of coordinates, i.e.

$$\delta W = \sum_{i=1}^{3n} \vec{F}_i \delta \vec{r}_i = \sum_{j=1}^{3n-m} Q_j \delta q_j \qquad (38)$$

Q_j is defined as the generalized force for the coordinate q_j. From (37) and (38)

$$\sum_{j=1}^{3n-m} Q_j \delta q_j = \sum_{i=1}^{3n} \vec{F}_i \delta \vec{r}_i = \sum_{i=1}^{3n} \vec{F}_i \left(\sum_{j=1}^{3n-m} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \right)$$

or $$\sum_{j=1}^{3n-m}\left(\sum_{i=1}^{3n}\vec{F}_i\frac{\partial\vec{r}_i}{\partial q_j}-Q_j\right)\delta q_j=0 \tag{39}$$

Since all δq_j are arbitrarily chosen, we obtain

$$Q_j=\sum_{i=1}^{3n}\vec{F}_i\frac{\partial\vec{r}_i}{\partial q_j},\quad j=1,2,\cdots,3n-m \tag{40}$$

Example: let us calculate the generalized forces for the double pendulum in the coordinates $q_1$and q_2

$$\vec{F}_1=m_1g\hat{j};\quad \vec{F}_2=m_2g\hat{j}$$

$$\vec{r}_1=x_1\hat{i}+y_1\hat{j}=a\sin q_1\hat{i}+a\cos q_1\hat{j}$$

$$\vec{r}_2=x_2\hat{i}+y_2\hat{j}=\left(a\sin q_1+b\sin q_2\right)\hat{i}+\left(a\cos q_1+b\cos q_2\right)\hat{j}$$

$$\vec{F}_1\delta\vec{r}_1 = m_1 g\hat{j}\left(a\cos q_1\cdot\delta q_1\hat{i} - a\sin q_1\cdot\delta q_1\hat{j}\right)$$

$$= -m_1 g a\sin q_1\cdot\delta q_1$$

$$\vec{F}_2\delta\vec{r}_2 = m_2 g\hat{j}\begin{bmatrix}\left(a\cos q_1\cdot\delta q_1 + b\cos q_2\cdot\delta q_2\right)\hat{i} \\ -\left(a\sin q_1\cdot\delta q_1 + b\sin q_2\cdot\delta q_2\right)\hat{j}\end{bmatrix}$$

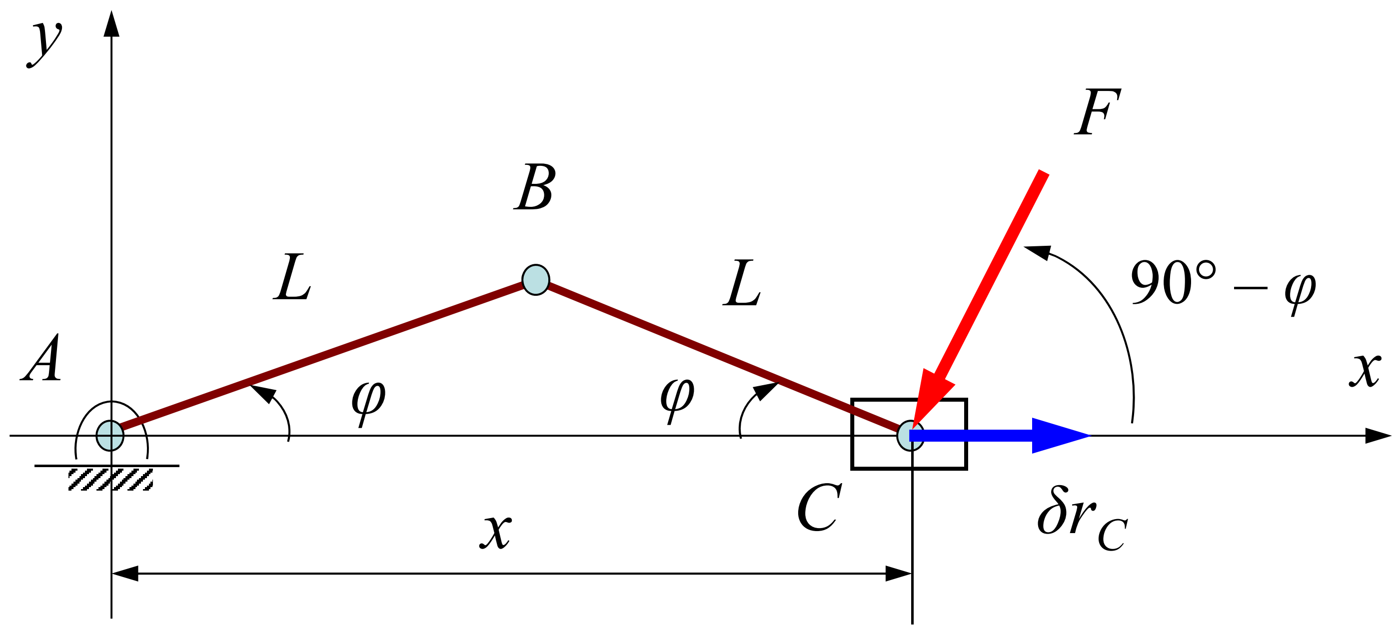
$$= -m_2 g\left(a\sin q_1\cdot\delta q_1 + b\sin q_2\cdot\delta q_2\right)$$

$$Q_1\delta q_1 + Q_2\delta q_2 = \vec{F}_1\delta\vec{r}_1 + \vec{F}_2\delta\vec{r}_2$$

$$= -\left(m_1 + m_2\right) g a\sin q_1\cdot\delta q_1 - m_2 g b\sin q_2\cdot\delta q_2$$

$$\Rightarrow Q_1 = -\left(m_1 + m_2\right) g a\sin q_1$$

$$Q_2 = -m_2 g b\sin q_2$$



DOF = ?

Choice #1: φ as the generalized coordinate

$$\vec{F} = -F\sin\phi\hat{i} - F\cos\phi\hat{j}$$

$$\vec{r}_C = 2L\cos\phi\hat{i}$$

$$\Rightarrow \delta\vec{r}_C = \frac{\partial\vec{r}_C}{\partial\phi}\delta\phi = -2L\cdot\sin\phi\cdot\delta\phi\hat{i}$$

$$\delta W = \vec{F}\bullet\delta\vec{r}_C = \left(-F\sin\phi\hat{i} - F\cos\phi\hat{j}\right)\bullet\left(-2L\cdot\sin\phi\cdot\delta\phi\hat{i}\right)$$

$$= 2FL\sin^2\phi\cdot\delta\phi = Q_\phi\cdot\delta\phi$$

$$\Rightarrow Q_\phi = 2FL\sin^2\phi$$

Choice #2: x as the generalized coordinate. We must express F using x instead of φ.

$$\delta \vec{r}_C = \delta x \hat{i}, \quad \cos\phi = \frac{x}{2L}, \quad \sin\phi = \sqrt{1-\cos^2\phi} = \frac{1}{L}\sqrt{L^2 - \frac{x^2}{4}}$$

$$F = -\frac{F}{L}\left[\sqrt{L^2 - \frac{x^2}{4}}\hat{i} + \frac{x}{2}\hat{j}\right]$$

$$\delta W = \vec{F} \bullet \delta \vec{r}_C = -2\frac{F}{L}\sqrt{L^2 - \frac{x^2}{4}} \cdot \delta x = Q_x \delta x$$

$$\Rightarrow Q_x = -2\frac{F}{L}\sqrt{L^2 - \frac{x^2}{4}}$$

3. Lagrangian Equations

Consider the expression

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \bullet \delta \vec{r}_i$$

With Eq. (37)

$$\sum_{i,j} m_i \ddot{\vec{r}} \bullet \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j,$$

where $i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, 3n - m$

can be written as

$$\sum_{i,j} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}} \bullet \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}} \bullet \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j \qquad (41)$$

Since $$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \cancel{\frac{\partial \vec{r}_i}{\partial t}} + \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j \qquad (42)$$

where $\dot{q}_j$ are the generalized velocities. Then differentiating (42) with respect to $\dot{q}_j$ yields

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \qquad (43)$$

Next, we transform the term $\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$ in (41) by interchanging the order of differentiation. Hence

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial \vec{v}_i}{\partial q_j} \qquad (44)$$

Substitute (43) and (44) into (41)

$$\sum_{i,j} m_i \ddot{\vec{r}} \bullet \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{v}_i \bullet \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \bullet \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j \tag{45}$$

Since i and j are independent, the right-hand-side can be rearranged as

$$\sum_j \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i \left| \vec{v}_i \right|^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i \left| \vec{v}_i \right|^2 \right) \right\} \delta q_j \tag{46}$$

By definition, the kinetic energy is

$$T = \frac{1}{2} \sum_i m_i \left| \vec{v}_i \right|^2 \tag{47}$$

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}} \bullet \delta\vec{r}_i = \sum_j \left\{ \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_j}\right] - \frac{\partial T}{\partial q_j} \right\} \delta q_j \tag{48}$$

In the meantime, by Newton's law

$$\sum_{i=1}^{n} \vec{F}_i \bullet \delta\vec{r}_i = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \bullet \delta\vec{r}_i$$

$$\text{or} \quad \sum_{i=1}^{n} \left(\vec{F}_i - m_i \ddot{\vec{r}}_i\right) \bullet \delta\vec{r}_i = 0 \tag{49}$$

where $\vec{F}_i$ are external forces only. Recall Eq. (38)

$$\delta W = \sum_i \vec{F}_i \delta\vec{r}_i = \sum_j Q_j \delta q_i$$

Then from (48), (49) and (38), we have

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0 \qquad (50)$$

since δq_j are arbitrary, we have a set of scalar equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j,\ j = 1, 2, \cdots, 3n - m, \qquad (51)$$

These equations are called **Lagrangian equations** (or Lagrange's equations) of motion. Note that Q_j are generalized forces, including the contributions from both force field types, conservative and non-conservative.

Let us separate the conservative and non-conservative parts

$$Q_j = Q_{jc} + Q_j^* \tag{52}$$

Using Eq. (31), then the conservative force is

$$Q_{jc} = -\frac{\partial V}{\partial q_j}$$

where V is the potential energy. We also observe from (12) that $\underset{\sim}{V}$ is only functions of q_j, but not $\dot{q}_j$. Then (51) becomes

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right) - \frac{\partial(T-V)}{\partial q_j} = Q_j^* \tag{53}$$

Define the **Lagrangian function** L

$$L = T - V \qquad (54)$$

then (53) $\Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j}\right) - \dfrac{\partial L}{\partial q_j} = Q_j^* \qquad (55)$

If there is no non-conservative force, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (56)$$

Note: set $-\dfrac{\partial L}{\partial q_j} = Q_j^*$ to find the static equilibrium

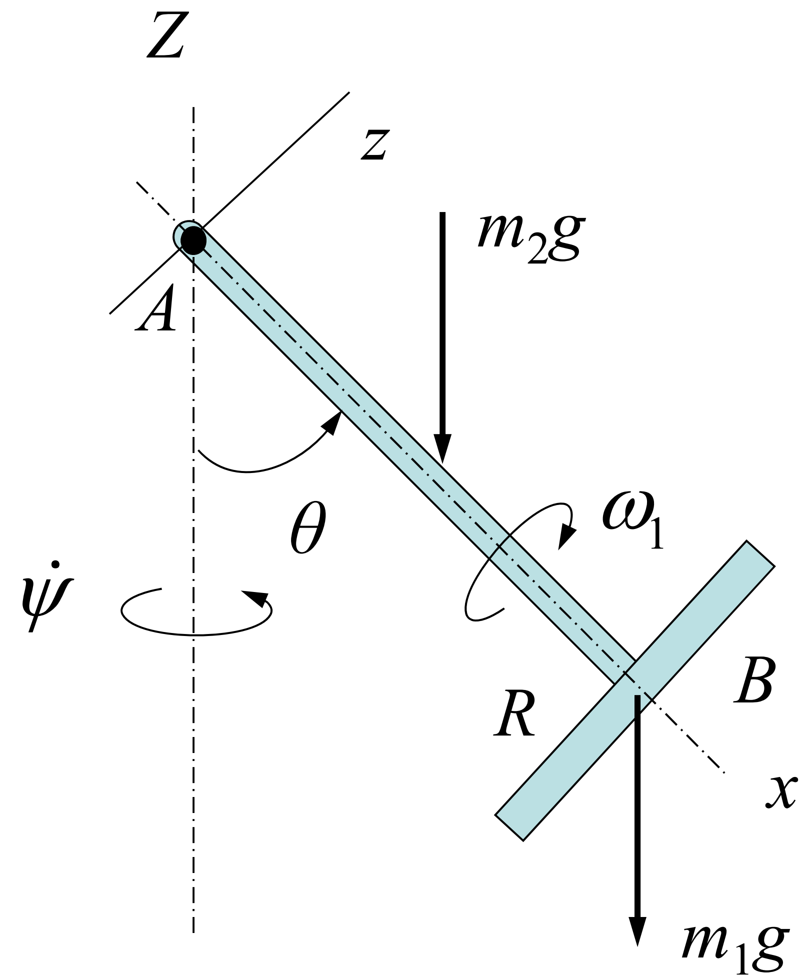
Historical remarks:

Lagrange was born in 1736 in France. After having mastered calculus on his own, his abilities were recognized early and he was appointed professor of mathematics at the local technical college at age 16. At that time he began development of the calculus of variations. This subject was originated in correspondence with Euler. Euler and other leading scientists arranged to have him appointed to the Berlin Academy of Sciences. He returned to France in 1787 and was appointed the French Academy of Sciences.

Lagrange first conceived of his book *Mecanique Analytique* when he was 19, but did not finish until he was 52 years old, because he was a perfectionist.

Lagrange also contributed to many other branches of mathematics, including theories of limits, probability, numbers arithmetic, and algebraic equations. He was president of the committee to reform the system of weights and measures during the Revolution. The result of this committee's deliberations is today's metric system.

Example



$$T = \frac{1}{2}\left(\vec{\omega} \bullet \vec{H}_A\right)_{\text{disk}} + \frac{1}{2}\left(\vec{\omega} \bullet \vec{H}_A\right)_{\text{shaft } AB}$$

$$\left.\begin{aligned}\vec{\omega}_{\text{shaft}} &= \dot{\psi}\hat{K} - \dot{\theta}\hat{j} \\ \hat{K} &= \sin\theta\hat{k} - \cos\theta\hat{i}\end{aligned}\right\} \Rightarrow \vec{\omega}_{\text{shaft}} = -\dot{\psi}\cos\theta\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k}$$

$$\vec{\omega}_{\text{disk}} = \dot{\psi}\hat{K} - \dot{\theta}\hat{j} - \omega_1\hat{i} = -\left(\dot{\psi}\cos\theta + \omega_1\right)\hat{i} - \dot{\theta}\hat{j} + \dot{\psi}\sin\theta\hat{k}$$

If the shaft is thin compared to the disk, ignore I_{xx} of the shaft, then about point A (using parallel axis theorem)

$$\left(\vec{H}_A\right)_{\text{shaft}} = m_2L^2\begin{bmatrix}0 & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/3\end{bmatrix}\begin{Bmatrix}-\dot{\psi}\cos\theta\\ -\dot{\theta}\\ \dot{\psi}\sin\theta\end{Bmatrix}$$

$$\left(\vec{H}_A\right)_{\text{disk}} = \begin{bmatrix}m_1R^2/2 & 0 & 0\\ 0 & m_1R^2/4 + m_1L^2 & 0\\ 0 & 0 & m_1R^2/4 + m_1L^2\end{bmatrix}\begin{Bmatrix}-\left(\dot{\psi}\cos\theta + \omega_1\right)\\ -\dot{\theta}\\ \dot{\psi}\sin\theta\end{Bmatrix}$$

$$T = \underbrace{\frac{1}{2}\left(\frac{1}{2}m_1R^2\right)\left(\dot{\psi}\cos\theta + \omega_1\right)^2 + \frac{1}{2}\left(\frac{1}{4}m_1R^2 + m_1L^2\right)\left[\dot{\theta}^2 + \left(\dot{\psi}\sin\theta\right)^2\right]}_{T_{\text{disk}}}$$

$$+\underbrace{\frac{1}{2}\left(\frac{1}{3}m_2L^2\right)\left[\dot{\theta}^2 + \left(\dot{\psi}\sin\theta\right)^2\right]}_{T_{\text{shaft}}}$$

$$T = \frac{1}{2}\left(\frac{1}{2}m_1R^2\right)\left(\dot{\psi}\cos\theta + \omega_1\right)^2 + \frac{1}{2}\left[\frac{1}{4}m_1R^2 + \left(m_1 + \frac{1}{3}m_2\right)L^2\right]\left[\dot{\theta}^2 + \left(\dot{\psi}\sin\theta\right)^2\right]$$

For simplicity, define

$$I_1 = \frac{1}{2}m_1R^2, \quad I_2 = \frac{1}{4}m_1R^2 + \left(m_1 + \frac{1}{3}m_2\right)L^2$$

then

$$T = \frac{1}{2}I_1\left(\dot{\psi}\cos\theta + \omega_1\right)^2 + \frac{1}{2}I_2\left[\dot{\theta}^2 + \left(\dot{\psi}\sin\theta\right)^2\right]$$

$$V = m_1 g\left(-L\cos\theta\right) + m_2 g\left(-\frac{L}{2}\cos\theta\right)$$

$$L = T - V$$

$$= \frac{1}{2} I_1\left(\dot{\psi}\cos\theta + \omega_1\right)^2 + \frac{1}{2} I_2\left[\dot{\theta}^2 + \left(\dot{\psi}\sin\theta\right)^2\right]$$

$$+ m_1 g L\cos\theta + \frac{1}{2} m_2 g L\cos\theta$$

$$\frac{\partial L}{\partial\dot{\psi}} = I_1\left(\dot{\psi}\cos\theta + \omega_1\right)\cos\theta + I_2\left(\dot{\psi}\sin^2\theta\right)$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} = I_1\left(\ddot{\psi}\cos^2\theta - 2\dot{\psi}\dot{\theta}\cos\theta\sin\theta - \omega_1\dot{\theta}\sin\theta\right)$$

$$+ I_2\left(\ddot{\psi}\sin^2\theta + 2\dot{\psi}\dot{\theta}\cos\theta\sin\theta\right)$$

$$\frac{\partial L}{\partial\psi} = 0$$

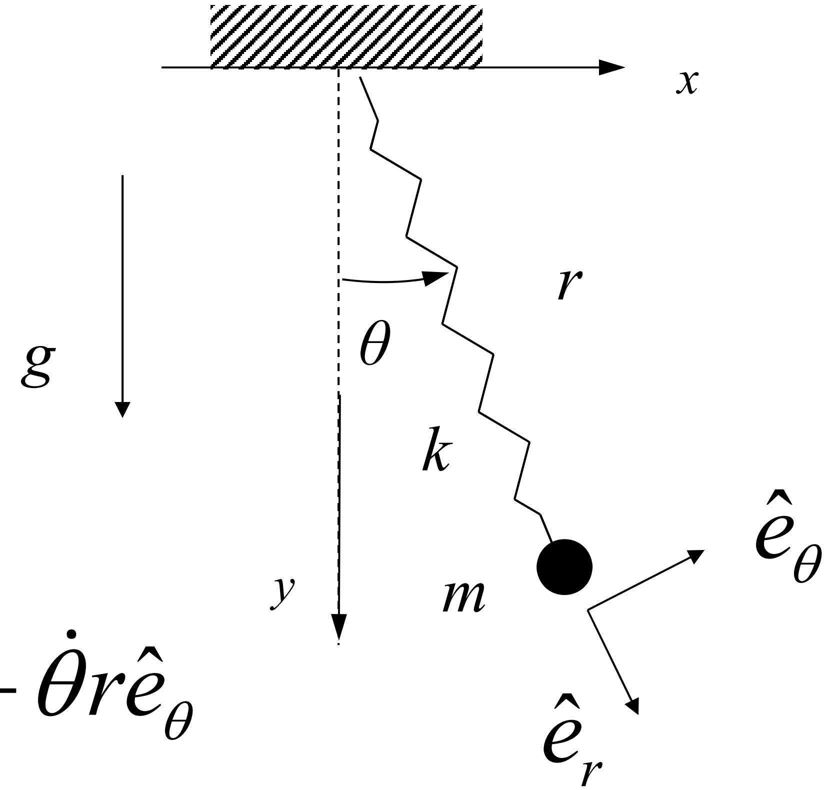
$$\frac{\partial L}{\partial \dot{\theta}} = I_2 \dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I_2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -I_1 \left(\dot{\psi}\cos\theta + \omega_1\right)\dot{\psi}\sin\theta + I_2 \dot{\psi}^2 \sin\theta\cos\theta - m_1 g L \sin\theta - \frac{1}{2} m_2 g L \sin\theta$$

equations of motion:

$$\begin{cases} \left(I_1 \cos^2\theta + I_2 \sin^2\theta\right)\ddot{\psi} - 2\left(I_1 - I_2\right)\dot{\psi}\dot{\theta}\cos\theta\sin\theta - I_1 \omega_1 \dot{\theta}\sin\theta = 0 \\ I_2 \ddot{\theta} + \left(I_1 - I_2\right)\dot{\psi}^2 \sin\theta\cos\theta + I_1 \omega_1 \dot{\psi}\sin\theta + \left(m_1 + \frac{1}{2} m_2\right) g L \sin\theta = 0 \end{cases}$$

Example



$$\vec{v} = \dot{r}\hat{e}_r + \dot{\theta} r \hat{e}_\theta$$

Kinetic energy: $$T = \frac{1}{2} m \vec{v} \bullet \vec{v} = \frac{1}{2} m \left(\dot{r}^2 + \dot{\theta}^2 r^2 \right)$$

Potential energy:

Lagrangian function L:

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + \dot{\theta}^2 r^2\right) - \frac{1}{2}k\left(r - r_0\right)^2 + mgr\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{\partial L}{\partial r} = m\dot{\theta}^2 r - k\left(r - r_0\right) + mg\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}r^2 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\ddot{\theta}r^2 + 2m\dot{\theta}r\dot{r}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$

Apply Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \Rightarrow m\ddot{\theta}r^2+2m\dot{\theta}r\dot{r}+mgr\sin\theta=0 \qquad \text{(a)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=m\ddot{r}-m\dot{\theta}^2 r+k\left(r-r_0\right)-mg\cos\theta=0 \qquad \text{(b)}$$

This is a set of non-linear equations of motion. We need to linearize them. Equilibrium or constant solutions are given by

$$\begin{cases} r_E\sin\theta_E=0 \\ k\left(r_E-r_0\right)-mg\cos\theta_E=0 \end{cases} \Rightarrow \begin{cases} \theta_E=0, r_E\neq 0 \\ r_E=\dfrac{1}{k}\left(mg+kr_0\right) \end{cases}$$

Define the disturbance from the equilibrium solutions:

$$\begin{cases} \theta = \theta_E + \phi = \phi \\ r = r_E + \eta \end{cases}$$

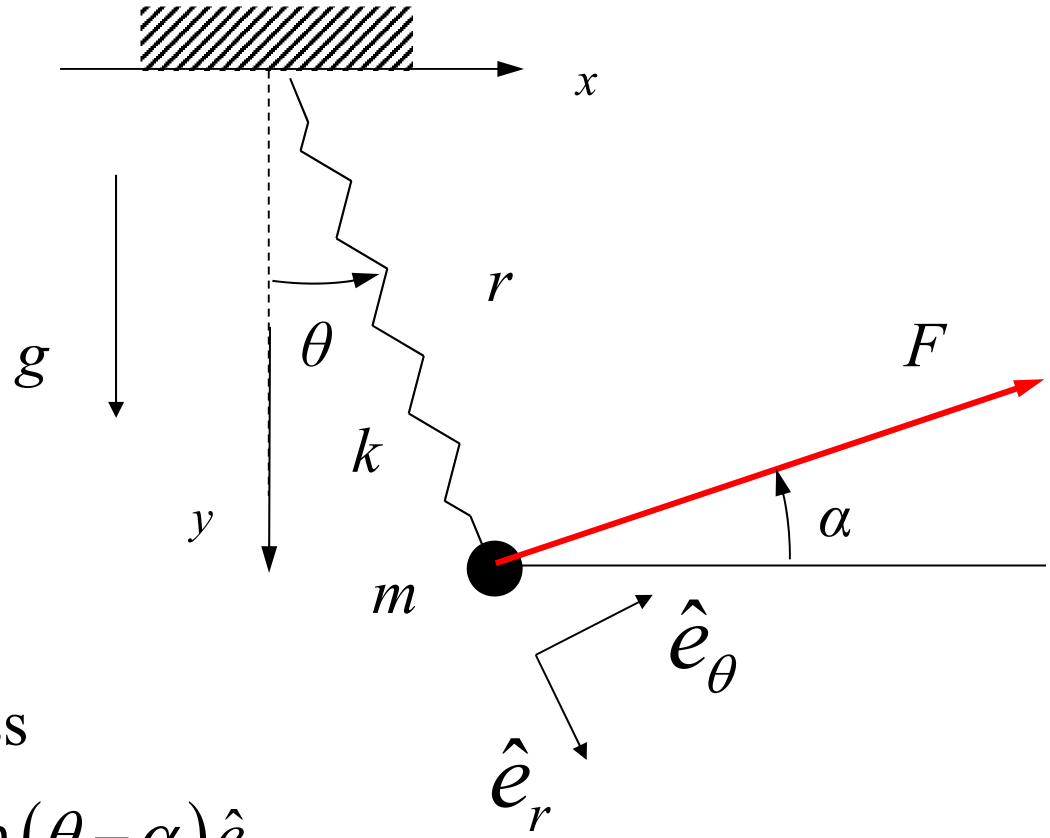
Substitute them back to (a) and (b), and neglect the second order terms, then

$$m\left(r_E + \eta\right)^2 \ddot{\theta} + \cancel{2m\dot{\theta}r\dot{r}} + mgr\theta = 0$$

$$m\ddot{\eta} \cancel{- m\dot{\theta}^2 r} + k\left(r_E + \eta - r_0\right) - mg = 0$$

$$\begin{cases} mr_E^2\ddot{\theta} + mgr_E\theta = 0 & \text{(c)} \\ m\ddot{\eta} + k\eta = 0 & \text{(d)} \end{cases}$$

Now suppose there was a force F acting on the mass as shown having an angle α with the horizon. We have to compute the generalized forces.



$$dW = \sum_{i=1}^{n} \vec{F}_i \bullet d\vec{r}_i = \vec{F} \bullet d\vec{r}$$

since there is only one mass

$$\vec{F} = F\cos(\theta - \alpha)\hat{e}_\theta + F\sin(\theta - \alpha)\hat{e}_r$$

$$\vec{r} = r\hat{e}_r \Rightarrow d\vec{r} = dr \cdot \hat{e}_r + r \cdot d\theta \cdot \hat{e}_\theta$$

Then $\vec{F} \bullet d\vec{r} = F\cos(\theta - \alpha) r \cdot d\theta + F\sin(\theta - \alpha) \cdot dr$

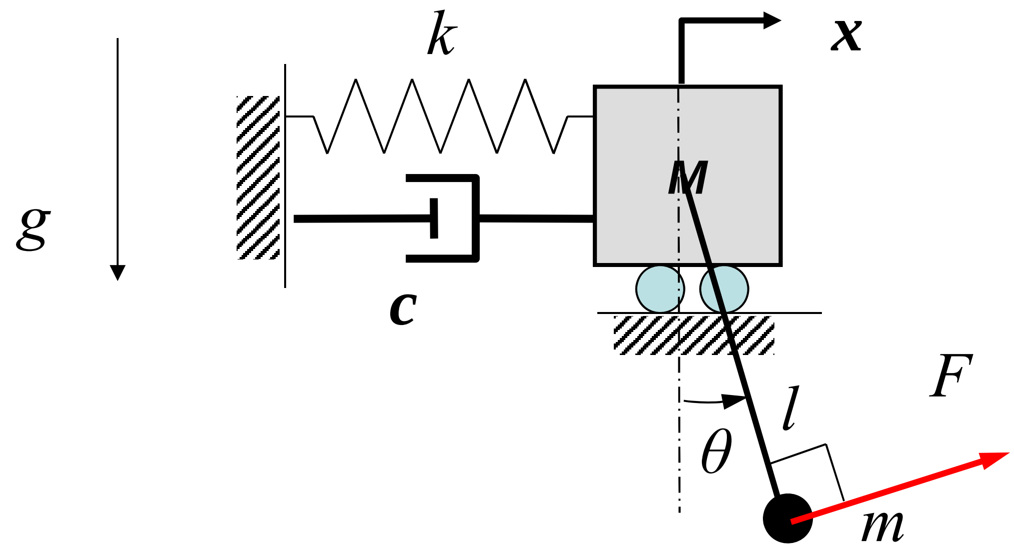
Since $dW = \sum Q_j q_j = Q_\theta d\theta + Q_r dr$

$$Q_\theta = F\cos(\theta - \alpha) r; \qquad Q_r = F\sin(\theta - \alpha)$$

Equations of motion take the form

$$m\ddot{\theta}r^2 + 2m\dot{\theta}r\dot{r} + mgr\sin\theta = F\cos(\theta-\alpha)r$$
$$m\ddot{r} - m\dot{\theta}^2 r + k(r-r_0) - mg\cos\theta = F\sin(\theta-\alpha)$$

Example



Kinetic energy: $T = T_m + T_M = \frac{1}{2} m \dot{\vec{r}}_m \bullet \dot{\vec{r}}_m + \frac{1}{2} M \dot{\vec{x}} \bullet \dot{\vec{x}}$

$$\vec{r}_m = (x + l\sin\theta)\hat{i} + l\cos\theta\,\hat{j}, \quad \dot{\vec{r}}_m = (\dot{x} + l\dot{\theta}\cos\theta)\hat{i} - l\dot{\theta}\sin\theta\,\hat{j}$$

$$\Rightarrow T_m = \frac{1}{2} m\left[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\right] \qquad T_M = \frac{1}{2} M\dot{x}^2$$

$$= \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2\dot{x}l\dot{\theta}\cos\theta + \dot{\theta}^2 l^2\sin^2\theta\right)$$

$$= \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right)$$

Potential energy: $V = mgl\left(1-\cos\theta\right)+\frac{1}{2}kx^2$

Calculate generalized forces

$$dW = \sum_{i=1}^{n} \vec{F}_i \bullet d\vec{r}_i = \vec{F}_1 \bullet d\vec{r}_1 + \vec{F}_2 \bullet d\vec{r}_2$$

$$\vec{F}_1 = \left(-c\dot{x}\right)\hat{i}, \quad \vec{F}_2 = F\cos\theta\hat{i} - F\sin\theta\hat{j}$$

$$\vec{r}_1 = x\hat{i} \Rightarrow d\vec{r}_1 = dx\hat{i}$$

$$\vec{r}_2 = x\hat{i} + l\sin\theta\hat{i} + l\cos\theta\hat{j} \Rightarrow d\vec{r}_2 = dx\hat{i} + l\cos\theta\cdot d\theta\hat{i} - l\sin\theta\cdot d\theta\hat{j}$$

$$\therefore dW = \left(-c\dot{x}\right)\hat{i} \bullet dx\hat{i}$$

$$+\left(F\cos\theta\hat{i} - F\sin\theta\hat{j}\right)\bullet\left(dx\hat{i} + l\cos\theta\cdot d\theta\hat{i} - l\sin\theta\cdot d\theta\hat{j}\right)$$

$$dW = -c\dot{x}dx + F\cos\theta dx + Fl\cos^2\theta\cdot d\theta + Fl\sin^2\theta\cdot d\theta$$

$$dW = -c\dot{x}dx + F\cos\theta dx + Fld\theta = \left(-c\dot{x} + F\cos\theta\right)dx + Fld\theta$$

$$Q_x^* = -c\dot{x} + F\cos\theta$$

$$Q_\theta^* = Fl$$

The Lagrangian function:

$$L = T - V$$

$$= \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + \frac{1}{2}M\dot{x}^2 - mgl\left(1-\cos\theta\right) - \frac{1}{2}kx^2$$

Use Eq. (53)

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m\left(2l^2\dot{\theta} + 2l\dot{x}\cos\theta\right)$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\left(l^2\ddot{\theta} + l\ddot{x}\cos\theta - l\dot{x}\dot{\theta}\sin\theta\right)$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m\left(-2l\dot{x}\dot{\theta}\sin\theta\right) - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m\left(2\dot{x} + 2l\dot{\theta}\cos\theta\right) + M\dot{x}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) + M\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

Then the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta^*$$

$$m\left(l^2\ddot{\theta} + l\ddot{x}\cos\theta - \cancel{l\dot{x}\dot{\theta}\sin\theta}\right) + \cancel{ml\dot{x}\dot{\theta}\sin\theta} + mgl\sin\theta = Fl$$

$$ml^2\ddot{\theta} + ml\cos\theta\cdot\ddot{x} + mgl\sin\theta = Fl$$

and $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=Q_x^*$$

$$m\left(\ddot{x}+l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta\right)+M\ddot{x}+kx=-c\dot{x}+F\cos\theta$$

$$\left(M+m\right)\ddot{x}+c\dot{x}+ml\cos\theta\cdot\ddot{\theta}+kx=F\cos\theta$$

equations of motion:

$$\begin{cases} ml^2\ddot{\theta}+ml\cos\theta\cdot\ddot{x}+mgl\sin\theta=Fl \\ \left(M+m\right)\ddot{x}+c\dot{x}+ml\cos\theta\cdot\ddot{\theta}+kx=F\cos\theta \end{cases}$$

Linearized equations of motion:

$$\begin{cases} ml^2\ddot{\theta}+ml\ddot{x}+mgl\theta=Fl \\ \left(M+m\right)\ddot{x}+c\dot{x}+ml\ddot{\theta}+kx=F \end{cases}$$

In order to obtain the liearized equations, alternatively, we can approximate the Lagrangian function L up to **quadratic terms** and linearize each generalized force Q^*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \approx 1 - \frac{x^2}{2}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \approx x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \approx 1 + \frac{x^2}{2}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \approx x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \approx 1 + x + \frac{x^2}{2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \approx 1 + x + x^2$$

$$(1+x)^\mu = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \cdots + \frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}x^n + \cdots$$

$$L = \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + \frac{1}{2}M\dot{x}^2 - mgl\left(1-\cos\theta\right) - \frac{1}{2}kx^2$$

$$L \approx \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\right) + \frac{1}{2}M\dot{x}^2 - mgl\left[1-\left(1-\theta^2\right)\right] - \frac{1}{2}kx^2$$

$$= \frac{1}{2} m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\right) + \frac{1}{2}M\dot{x}^2 - mgl\theta^2 - \frac{1}{2}kx^2$$

and

$$Q_x^* = -c\dot{x} + F$$

$$Q_\theta^* = Fl$$

We should get the same linearized equations as before.

Equations of Motion for Small Perturbation about a Constant Solution (Linearization)

Recall: The equations of motion are given by Eq. (55)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^* \tag{55}$$

These are general nonlinear ordinary differential equations of the form:

$$\left[M\left(\underset{\sim}{q},t\right)\right]\ddot{\underset{\sim}{q}} + \underset{\sim}{g}\left(\underset{\sim}{q},\dot{\underset{\sim}{q}},t\right) = \underset{\sim}{Q}^*\left(\underset{\sim}{q},\dot{\underset{\sim}{q}},t\right) \tag{57}$$

For an autonomous system the equations do not contain *t* explicitly, i.e.

$$\left[M\left(\underset{\sim}{q}\right)\right]\ddot{\underset{\sim}{q}} + \underset{\sim}{g}\left(\underset{\sim}{q},\dot{\underset{\sim}{q}}\right) = \underset{\sim}{Q}^*\left(\underset{\sim}{q},\dot{\underset{\sim}{q}}\right) \tag{58}$$

A known solution associated with Eq. (57) is represented by $\underset{\sim}{\Phi}(t)$. We may then linearize (57) about $\underset{\sim}{\Phi}(t)$.

For the autonomous system given by equation (58) the constant solution are given by

$$\underset{\sim}{g}\left(\underset{\sim}{\Phi}\right)=\underset{\sim}{Q}^{*}\left(\underset{\sim}{\Phi}\right) \tag{59}$$

This is equivalent to set $-\dfrac{\partial L}{\partial q_j}=Q_j^{*}$

to find the static equilibrium

Note that $\underset{\sim}{\Phi}$ may or may not be an explicit function of time.

$\underset{\sim}{\Phi}$ may represent an equilibrium position or a reference motion.

Let us define $\underset{\sim}{y}(t)$ as the small perturbation in the neighborood of $\underset{\sim}{\Phi}(t)$ by

$$\underset{\sim}{q}(t) = \underset{\sim}{\Phi} + \underset{\sim}{y}(t) \tag{60}$$

In order to obtain the equations of small motions in terms of $\underset{\sim}{y}(t)$, we expected the quantities in Taylor series

$$\left[M\left(\underset{\sim}{q},t\right)\right] \approx \left[M_0\left(t\right)\right]\ddot{\underset{\sim}{q}} + \cdots$$

$$\underset{\sim}{g}\left(\underset{\sim}{q},\dot{\underset{\sim}{q}},t\right) \approx \underset{\sim}{g}_0\left(t\right) + \left.\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{q}}\right|_{\Phi} \underset{\sim}{y}(t) + \left.\frac{\partial \underset{\sim}{g}}{\partial \dot{\underset{\sim}{q}}}\right|_{\Phi} \dot{\underset{\sim}{y}}(t) + \cdots$$

$$\underset{\sim}{Q}^*\left(\underset{\sim}{q},\dot{\underset{\sim}{q}},t\right) \approx \underset{\sim}{Q}_0^*\left(t\right) + \left.\frac{\partial \underset{\sim}{Q}^*}{\partial \underset{\sim}{q}}\right|_{\Phi} \underset{\sim}{y}(t) + \left.\frac{\partial \underset{\sim}{Q}^*}{\partial \dot{\underset{\sim}{q}}}\right|_{\Phi} \dot{\underset{\sim}{y}}(t) + \cdots \tag{61}$$

If we only keep the linear terms and let

$$\left.\begin{aligned}
&\left[M(t)\right] \equiv \left[M_0(t)\right] && \text{(coeff. of } \ddot{\underset{\sim}{y}}\text{)}\\
&\left[P(t)\right] \equiv \left.\frac{\partial \underset{\sim}{g}}{\partial \dot{\underset{\sim}{q}}}\right|_{\Phi} - \left.\frac{\partial \underset{\sim}{Q}^{*}}{\partial \dot{\underset{\sim}{q}}}\right|_{\Phi} && \text{(coeff. of } \dot{\underset{\sim}{y}}\text{)}\\
&\left[S(t)\right] \equiv \left.\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{q}}\right|_{\Phi} - \left.\frac{\partial \underset{\sim}{Q}^{*}}{\partial \underset{\sim}{q}}\right|_{\Phi} && \text{(coeff. of } \underset{\sim}{y}\text{)}\\
&\underset{\sim}{h}(t) = \underset{\sim}{Q}_0^{*}(t) - \underset{\sim}{g}_0(t)
\end{aligned}\right\} \tag{62}$$

Then Eq. (57)

$$\left[M(t)\right]\ddot{\underset{\sim}{y}}(t) + \left[P(t)\right]\dot{\underset{\sim}{y}}(t) + \left[S(t)\right]\underset{\sim}{y}(t) = \underset{\sim}{h}(t) \tag{63}$$

If parameters $\left[M(t)\right]$, $\left[P(t)\right]$ and $\left[P(t)\right]$ are time invariant

$$[M]\ddot{\underset{\sim}{y}}(t)+[C]\dot{\underset{\sim}{y}}(t)+[S]\underset{\sim}{y}(t)=\underset{\sim}{h}(t) \tag{64}$$

Matrix $[M]$ is called the ***inertia matrix*** and is symmetric, i.e. $[M] = [M]^T$. Matrices $[C]$ and $[S]$ may be decomposed in to symmetric and skewsymmetric parts as

$$\begin{aligned}[C] &= [D] + [G] \\ [S] &= [K] + [N]\end{aligned} \tag{65}$$

where $[D]$ and $[K]$ are symmetric matrices while $[G]$ and $[N]$ are skewsymmetric.

Matrix $[D]$ represents the ***damping matrix***, derivable from the Rayleigh dissipation function;

$[G]$ arises due to the presence of gyration forces;

Matrix $[K]$ represents the ***conservative elastic forces***, and $[N]$ describes the non-conservative position forces.

Therefore, in general the equations of motion for small vibration about a constant solution $\Phi(t)$ is given by

$$[M]\,\ddot{\underset{\sim}{y}}(t)+[D+G]\,\dot{\underset{\sim}{y}}(t)+[K+N]\,\underset{\sim}{y}(t)=\underset{\sim}{h}(t) \tag{66}$$

For free vibration $\underset{\sim}{h}(t)\equiv\underset{\sim}{0}$

Note: the derivative of a vector with respect to another vector

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

Instead of using the approach described above, we can also obtain linear equations using ***state space method*** as in the following.

Note that Eq. (57) may be rewritten as a set of first order equations as

$$\dot{\underset{\sim}{z}}(t) = \underset{\sim}{f}(\underset{\sim}{z}, t) \tag{67}$$

where $\underset{\sim}{z}$ and $\underset{\sim}{f}$ are $2N$ vectors given by

$$\underset{\sim}{z}(t) \equiv \begin{Bmatrix} \underset{\sim}{z}_1(t) \\ \underset{\sim}{z}_2(t) \end{Bmatrix};$$

$$\underset{\sim}{f}(\underset{\sim}{z}, t) \equiv \begin{Bmatrix} \left[M\left(\underset{\sim}{z}_2, t\right)\right]^{-1} \left\{\underset{\sim}{Q}^*\left(\underset{\sim}{z}_2, \underset{\sim}{z}_1, t\right) - \underset{\sim}{g}\left(\underset{\sim}{z}_2, \underset{\sim}{z}_1, t\right)\right\} \\ \underset{\sim}{z}_1(t) \end{Bmatrix} \tag{68}$$

Note we defined: $\underset{\sim}{q} = \underset{\sim}{z}_2; \dot{\underset{\sim}{q}} = \dot{\underset{\sim}{z}}_2 = \underset{\sim}{z}_1$ (69)

In many situations, the modeling of a physical phenomenon naturally results in a set of nonlinear differential equations of the form represented by Eq. (67)

The constant solutions $\Psi(t)$ of Eq. (67) can be obtained from

$$\dot{\underset{\sim}{\Psi}}(t) = \underset{\sim}{f}(\underset{\sim}{\Psi}, t) \tag{70a}$$

or, if the system is autonomous, then from

$$\underset{\sim}{f}(\underset{\sim}{\Psi}, t) = 0 \tag{70b}$$

As before, we define the perturbation $\underset{\sim}{x}(t)$ by

$$\underset{\sim}{z}(t) = \underset{\sim}{\Psi}(t) + \underset{\sim}{x}(t) \tag{71}$$

and substitute back in Eq. (67)

$$\dot{\underset{\sim}{\Psi}}(t) + \dot{\underset{\sim}{x}}(t) = \underset{\sim}{f}(\underset{\sim}{\Psi}, t) + \left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{z}}\right|_{\underset{\sim}{\Psi}} \underset{\sim}{x}(t) + \cdots \tag{72}$$

If we denote $$\left.\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{z}}\right|_{\underset{\sim}{\Psi}} \equiv \left[A(t)\right]_{2N\times 2N} \tag{73}$$

and use Eq. (70), then Eq. (72) becomes

$$\dot{\underset{\sim}{x}}(t) = \left[A(t)\right]\underset{\sim}{x}(t) \tag{74}$$

However, in many instances an input vector $\underset{\sim}{u}(t)$ appears in the equation of motion (67), i.e.

$$\dot{\underset{\sim}{z}}(t) = \underset{\sim}{f}(\underset{\sim}{z},\underset{\sim}{u},t) \tag{75}$$

In these situations, the linearization leads to a set of equations of the form

$$\dot{\underset{\sim}{x}}(t) = \left[A(t)\right]\underset{\sim}{x}(t) + \left[B(t)\right]\underset{\sim}{u}(t) \tag{76}$$

where $\underset{\sim}{x}$ is a $2N$ vector, $\left[A(t)\right]$ is a $2N\times 2N$ matrix, $\left[B(t)\right]$ is a $2N\times k$ matrix, and $\underset{\sim}{u}(t)$ is a k vector.

It is important to observe that for an autonomous system, the linearized equations in the neighborhood of the constant solution $\underset{\sim}{\Phi}$ or $\underset{\sim}{\Psi}$ are always homogeneous and have the forms given by equation (66) with $\underset{\sim}{h}(t) \equiv \underset{\sim}{0}$ or Eq. (74)

More on linearization…

Lagrangian Formulation for Small Oscillation (free)

Let the coordinates for the equilibrium position of oscillation be

$$q_1 = q_2 = \cdots q_N = 0 \qquad (77)$$

Further the arbitrary constant associated with the potential energy function $V(q_1,q_2,\ldots,q_N)$ can be chosen such that $V \equiv 0$ for $q_i = 0$. We can expand V in Taylor series about the equilibrium position ($q_i = 0$) and retain terms up to the second order. We have

$$V \cong \sum_{i=1}^{N} \left.\frac{\partial V}{\partial q_i}\right|_{q_i=0} \cdot q_i + \frac{1}{2}\sum_{i,j=1}^{N} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\substack{q_i=0\\ q_j=0}} \cdot \left(q_i q_j\right) + \cdots \qquad (78)$$

Note: $$\frac{\partial^2 V}{\partial q_i \partial q_j} = \frac{\partial^2 V}{\partial q_j \partial q_i} \qquad (79)$$

Since the origin is the equilibrium position of the motion, the potential energy in this configuration must have a stationary value. In another word, V has its either maximum or minimum value at the equilibrium position. So the first derivatives are zero:

$$\left.\frac{\partial V}{\partial q_i}\right|_{q_i=0} = 0, \quad i = 1, 2, \cdots N \qquad (80)$$

(78) becomes $$V = \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 V}{\partial q_i \partial q_j} \cdot \left(q_i q_j \right) \qquad (81)$$

Let $$K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} \tag{82}$$

From (79) we know that $K_{ij} = K_{ji}$, i.e, matrix $[K]$ is symmetric. Eq. (81) becomes

$$V = \frac{1}{2} \sum_{i,j=1}^{N} K_{ij} q_i q_j \equiv \frac{1}{2} \{q\}^T [K] \{q\} \tag{83}$$

We observe that the generalized conservative forces are obtained by

$$Q_{ci} = \frac{\partial V}{\partial q_i} = -\sum_{j=1}^{N} K_{ij} q_j, \quad i = 1, 2, \cdots N \tag{84}$$

Therefore, K_{ij} are the generalized stiffness or spring constants of the system.

We shall now consider the kinetic energy T of the system. Let us assume that the system is scleronomic and thus the displacement vector $\underset{\sim}{r}$ does not depend on t and can be expressed as

$$r_k = r_k(q_1, q_2, \cdots, q_N), \quad k = 1, 2, \cdots, n \tag{85}$$

Therefore,

$$T \equiv \sum_{k=1}^{n} \frac{1}{2} m_k v_k^2 = \sum_{k=1}^{n} \frac{1}{2} m_k \left(\sum_{i=1}^{N} \frac{\partial r_k}{\partial q_i} \dot{q}_i \right)^2 \tag{86}$$

If we define

$$m_{ij} \equiv \sum_{k=1}^{n} m_k \frac{\partial r_k}{\partial q_i} \frac{\partial r_k}{\partial q_j} \tag{87}$$

then

$$T = \sum_{k=1}^{n} \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \{\dot{q}\}^T [M] \{\dot{q}\} \tag{88}$$

From (87) and (88) we observe that $M_{ij} = M_{ji}$, i.e, matrix $[M]$ is symmetric.

Hence the potential energy V and the kinetic energy T for small oscillation about the null equilibrium position can be expressed as

$$\boxed{\begin{aligned} V &= \frac{1}{2}\sum_{i,j=1}^{N} K_{ij} q_i q_j \equiv \frac{1}{2}\{q\}^T [K]\{q\} \\ T &= \sum_{k=1}^{n} \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2}\{\dot{q}\}^T [M]\{\dot{q}\} \end{aligned}}$$

$$(83)$$

$$(88)$$

We may define the Lagrangian function as $L = T - V$ and linearize the non-conservative generalized forces. Then the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^* \qquad (89)$$

would yield linear equations of the form

$$m_{ij}\ddot{q}_j + K_{ij}q_j = Q_{Li}^* \qquad (90a)$$

or

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q_L^*\} \qquad (90b)$$

4. Hamilton’s Principle

Consider $$\frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta\vec{r}_i\right) = \ddot{\vec{r}}_i \bullet \delta\vec{r}_i + \dot{\vec{r}}_i \bullet \delta\dot{\vec{r}}_i$$

$$= \ddot{\vec{r}}_i \bullet \delta\vec{r}_i + \delta\left(\frac{1}{2}\dot{\vec{r}}_i \bullet \dot{\vec{r}}_i\right)$$

$$\Rightarrow \ddot{\vec{r}}_i \bullet \delta\vec{r}_i = \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta\vec{r}_i\right) - \delta\left(\frac{1}{2}\dot{\vec{r}}_i \bullet \dot{\vec{r}}_i\right) \qquad (91)$$

Multiplying by m_i and summing over the entire system of particles, we obtain

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \bullet \delta\vec{r}_i = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta\vec{r}_i\right) - \delta\sum_{i=1}^{n}\frac{1}{2}\left(m_i \dot{\vec{r}}_i \bullet \dot{\vec{r}}_i\right)$$

$$= \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta\vec{r}_i\right) - \delta T \qquad (92)$$

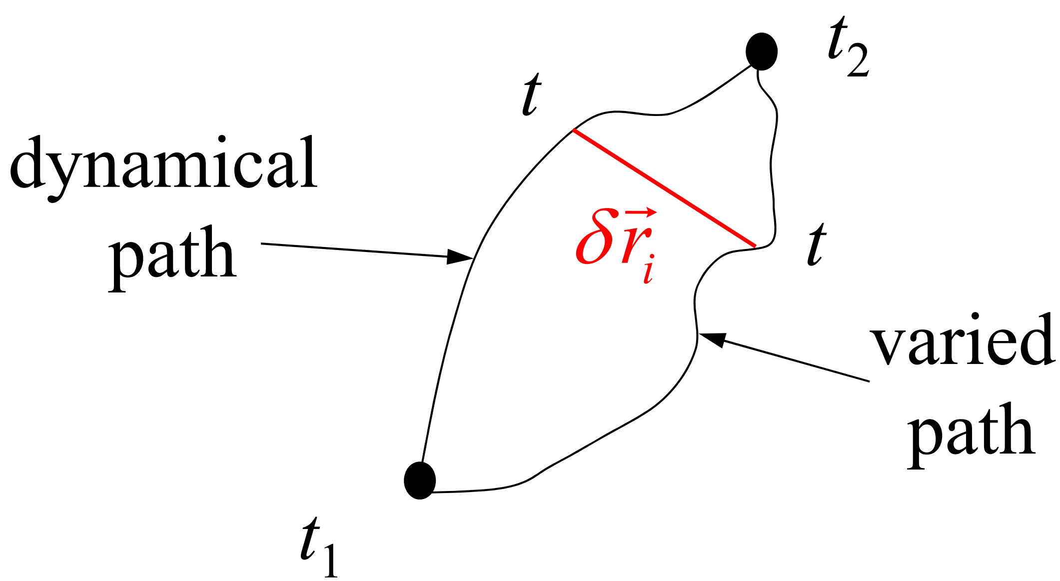
Recall Eq. (49): $$\sum_{i=1}^{n}\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right)\bullet\delta\vec{r}_i = 0$$

and $$\delta W = \sum_i \vec{F}_i\delta\vec{r}_i$$

Eq. (92) $$\Rightarrow \delta T + \delta W = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta\vec{r}_i\right) \tag{93}$$

The instantaneous configuration of a system is given by *n* generalized coordinates. These values correspond to a point in an *n*-dimensional space known as the configuration space. The configuration of the system changes with time, tracing a path known as the <u>dynamical path</u> (also called the <u>true path</u>) in the configuration space.

A different path known as the <u>varied path</u> is obtained if at any given instant one allows a small variation in position δr_i with no associated change in time (i.e. $\delta t = 0$). However, it is required that at two instants t_1 and t_2, the dynamical path and the varied path coincide:



This implies $\quad \delta\vec{r}_i = 0$ at $t = t_1$ and t_2 (94)

We multiply (93) by dt and integrate between t_1 and t_2

$$\int_{t_1}^{t_2} \left(\delta T + \delta W\right) dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \delta \vec{r}_i\right) dt$$

$$= \sum_{i=1}^{n} \int_{t_1}^{t_2} m_i d\left(\dot{\vec{r}}_i \bullet \delta \vec{r}_i\right) = \sum_{i=1}^{n} m\left(\dot{\vec{r}}_i \bullet \delta \vec{r}_i\right)\Bigg|_{t_1}^{t_2}$$

from Eq. (94) $\Rightarrow \boxed{\int_{t_1}^{t_2} \delta\left(T + W\right) dt = 0}$ (95)

This above equation can be regarded as the generalized Hamilton's principle. Note that W consists of work due to conservative as well as non-conservative forces. If all forces are conservative, then

$$\delta W = -\delta V$$

the Eq. (95) $\delta \int_{t_1}^{t_2} L dt = 0 \quad$ where $L = T - V$ (96)

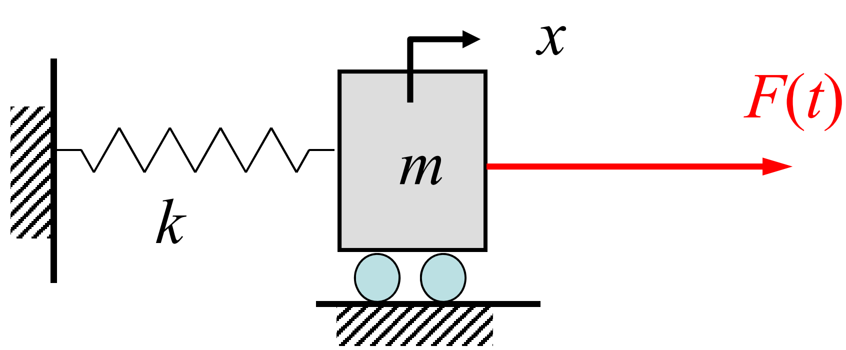
This is a mathematical statement of Hamilton's principle.

It states that the true path renders the value of the integral

$\int_{t_1}^{t_2} L dt$ stationary with respect to all possible paths between

two instant t_1 and t_2. This stationary value actually is a minimum.

Example



$$T=\frac{1}{2}m\dot{x}^2,\quad V=\frac{1}{2}kx^2,\quad W=F(t)x$$

$$\int_{t_1}^{t_2}\delta\left[\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2+F(t)x\right]dt=0$$

$$\Rightarrow\int_{t_1}^{t_2}m\dot{x}\delta\dot{x}dt-\int_{t_1}^{t_2}kx\delta xdt+\int_{t_1}^{t_2}F(t)\delta xdt=0$$

$$\Rightarrow m\int_{t_1}^{t_2}\dot{x}\frac{d}{dt}(\delta x)dt-k\int_{t_1}^{t_2}x\delta xdt+\int_{t_1}^{t_2}F(t)\delta xdt=0$$

integrate by part

$$\underbrace{\cancel{m\dot{x}\left(\delta x\right)\Big|_{t_1}^{t_2}}}_{=0} - m\int_{t_1}^{t_2} \ddot{x}\delta x dt - k\int_{t_1}^{t_2} x\delta x dt + \int_{t_1}^{t_2} F(t)\delta x dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left[m\ddot{x} + kx - F(t)\right]\delta x dt = 0$$

Since δx is arbitrary between t_1 and t_2, the EOM:

$$m\ddot{x} + kx = F(t)$$