4. Kinematics of Rigid Bodies

Outline

- General equations
- Eulerian angles
- Interconnections
  - Ball-socket joint
  - Pin connection
  - Collar connection
- Rolling

1. General Equations

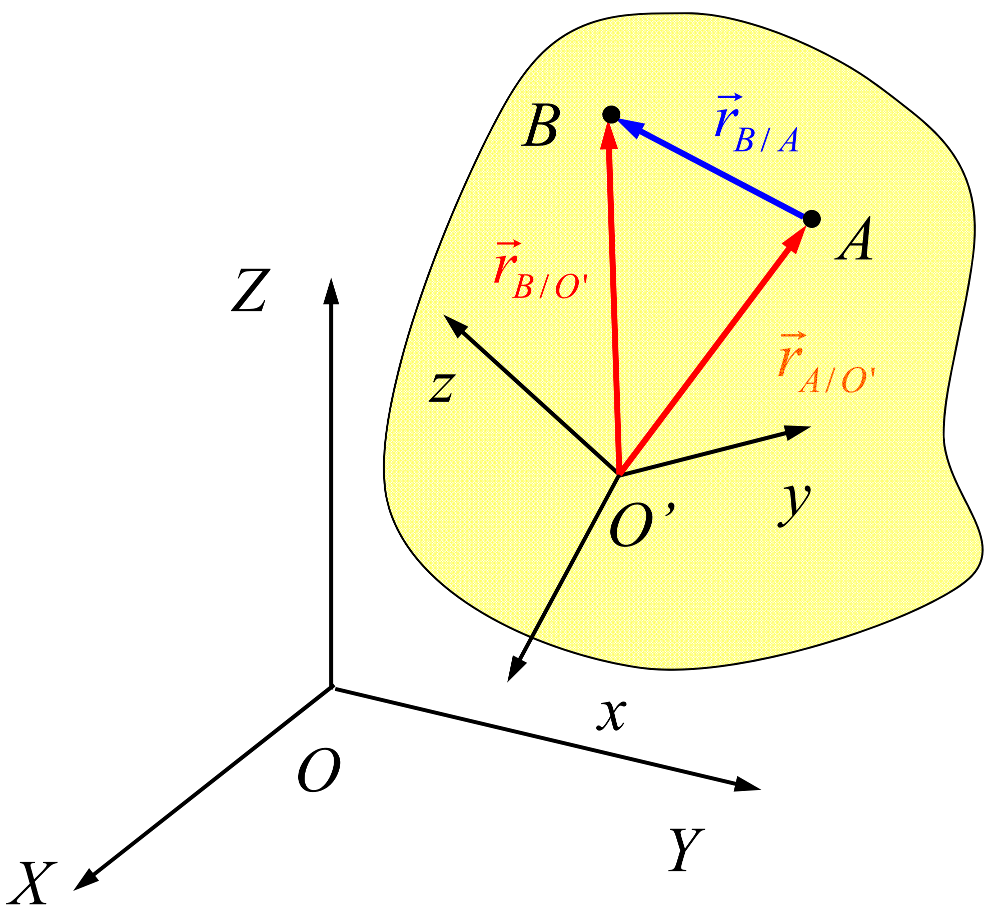
A rigid body is defined be to a collection of particles whose distance of separation is invariant. In this circumstance, any set of coordinate axes *xyz* that is scribed in the body will maintain its orientation relative to the body. Such a C.S. forms a body-fixed reference frame. For any point P

$$\left(\vec{v}_P\right)_{xyz} = 0, \ \left(\vec{a}_P\right)_{xyz} = 0$$

$$\vec{v}_P = \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{P/O'}$$

$$\vec{a}_P = \vec{a}_{O'} + \vec{\alpha} \times \vec{r}_{P/O'} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right)$$

1. General Equations



Point *A*:

$$\vec{v}_A = \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{A/O'}$$

$$\vec{a}_A = \vec{a}_{O'} + \vec{\alpha} \times \vec{r}_{A/O'} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{A/O'} \right)$$

Point *B*:

$$\vec{v}_B = \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{B/O'}$$

$$\vec{a}_B = \vec{a}_{O'} + \vec{\alpha} \times \vec{r}_{B/O'} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/O'} \right)$$

1. General Equations

Subtract the motion of *A* from that of *B*:

$$\vec{v}_B - \vec{v}_A = \vec{\omega} \times \left(\vec{r}_{B/O'} - \vec{r}_{A/O'} \right)$$

$$\vec{a}_B - \vec{a}_A = \vec{\alpha} \times \left(\vec{r}_{B/O'} - \vec{r}_{A/O'} \right) + \vec{\omega} \times \left[\vec{\omega} \times \left(\vec{r}_{B/O'} - \vec{r}_{A/O'} \right) \right]$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \\ \vec{a}_B &= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A} \right) \end{aligned}}$$

1. General Equations

- Conclusions:
  - For a rigid body, the selection of the origin of the body-fixed C.S. (*O*’) is arbitrary.
  - Given a set of n points in a rigid body, there are $n – 1$ independent equations between their velocities or accelerations. These equations may be obtained by relating one point to each of the other $n – 1$.

1. General Equations

- Chasle's theorem:

The general motion of a rigid body is a *superposition of a pure translation and a pure rotation*. In the translation, all points follow the movement of an arbitrary point A in the body and the orientation remains constant. The pure rotation of the motion is such that the arbitrary point A remains at rest.

1. General Equations

- Such a point A is called the instantaneous center of velocity, or *instant center*.
- However, although instant center method (centrod method) is useful for velocity analysis, it does not lead directly to acceleration analysis.
- So it is not the best method in real application.

Example 4.1 At an instant, point A has a velocity parallel to the diagonal AE. And observations:

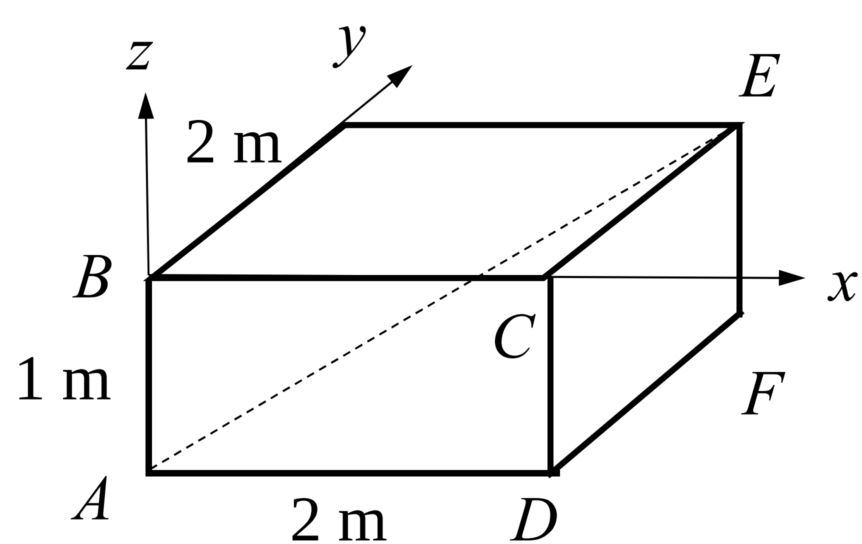
$$(v_B)_x = 10\ \text{m/s}$$

$$(v_C)_z = 20\ \text{m/s}$$

$$(v_D)_x = 10\ \text{m/s}$$

$$(v_E)_y = 5\ \text{m/s}$$

Determine whether these values are possible. If so, evaluate the velocity of corner F.



Given: $\vec{v}_A = v_A \hat{e}_{E/A}$ unknowns: $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$\vec{v}_B \bullet \hat{i} = 10 \text{ m/s}$ v_A

$\vec{v}_C \bullet \hat{k} = 20 \text{ m/s}$

$\vec{v}_D \bullet \hat{i} = 10 \text{ m/s}$

$\vec{v}_E \bullet \hat{j} = 5 \text{ m/s}$

$$\vec{r}_E = 2\hat{i} + 2\hat{j},\ \vec{r}_A = -1\hat{k} \Rightarrow \vec{r}_{E/A} = 2\hat{i} + 2\hat{j} + 1\hat{k}$$

$$\hat{e}_{E/A} = \frac{2\hat{i} + 2\hat{j} + 1\hat{k}}{\sqrt{2^2 + 2^2 + 1}} = \frac{1}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\Rightarrow \vec{v}_A = \frac{v_A}{3}\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$\Rightarrow \vec{v}_A \bullet \hat{i} = \frac{2v_A}{3},\ \vec{v}_A \bullet \hat{j} = \frac{2v_A}{3},\ \vec{v}_A \bullet \hat{k} = \frac{v_A}{3}$$

$$
\begin{cases}
\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} = \vec{v}_A + \left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) \times \hat{k} \\
\vec{v}_C = \vec{v}_A + \vec{\omega} \times \vec{r}_{C/A} = \vec{v}_A + \left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) \times \left(2\hat{i} + \hat{k}\right) \\
\vec{v}_D = \vec{v}_A + \vec{\omega} \times \vec{r}_{D/A} = \vec{v}_A + \left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) \times 2\hat{i} \\
\vec{v}_E = \vec{v}_A + \vec{\omega} \times \vec{r}_{E/A} = \vec{v}_A + \left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right) \times \left(2\hat{i} + 2\hat{j} + 1\hat{k}\right)
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\vec{v}_B = \vec{v}_A + \omega_y \hat{i} - \omega_x \hat{j} \\
\vec{v}_C = \vec{v}_A + \omega_y \hat{i} + \left(2\omega_z - \omega_x\right) \hat{j} - 2\omega_y \hat{k} \\
\vec{v}_D = \vec{v}_A + 2\omega_z \hat{j} - 2\omega_y \hat{k} \\
\vec{v}_E = \vec{v}_A + \left(2\omega_y - \omega_z\right) \hat{i} + \left(2\omega_z - \omega_x\right) \hat{j} + 2\left(\omega_x - \omega_y\right) \hat{k}
\end{cases}
$$

$$\vec{v}_B \bullet \hat{i} = 10 \text{ m/s} \Rightarrow \frac{2v_A}{3} + \omega_y = 10$$

$$\vec{v}_C \bullet \hat{k} = 20 \text{ m/s} \Rightarrow \frac{v_A}{3} - 2\omega_y = 20$$

$$\vec{v}_D \bullet \hat{i} = 10 \text{ m/s} \Rightarrow \frac{2v_A}{3} = 10$$

$$\vec{v}_E \bullet \hat{j} = 5 \text{ m/s} \Rightarrow \frac{2v_A}{3} + \left(2\omega_z - \omega_x\right) = 5$$

2. Eulerian Angles

- (from Wikipedia) The Euler angles were developed by Leonhard Euler to describe the orientation of a rigid body in 3-dimensional Euclidean space.
- To give an object a specific orientation it may be subjected to a sequence of three rotations described by the Euler angles.
- This is equivalent to saying that a rotation matrix can be decomposed as a product of three elemental rotations.

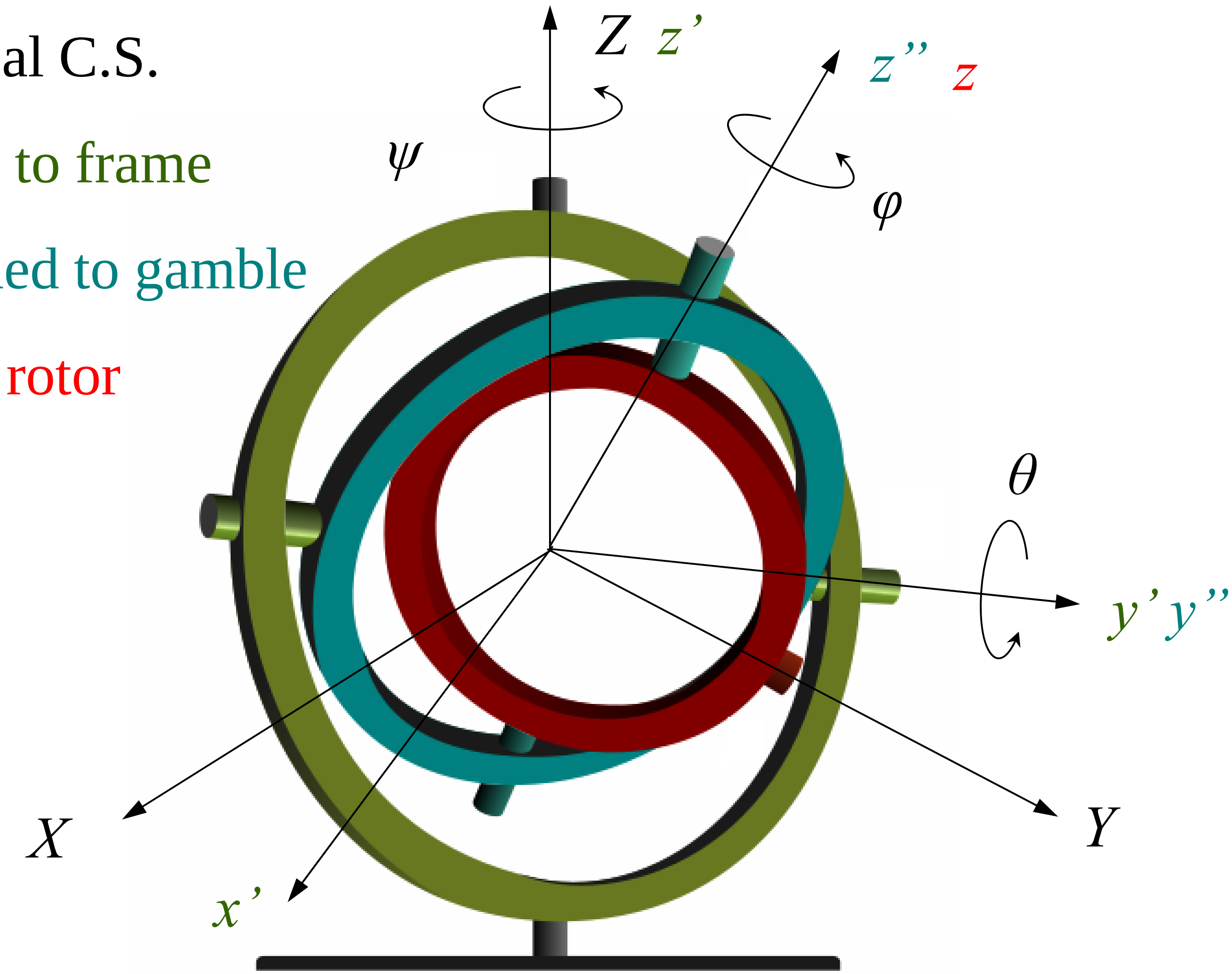
Leonhard Paul Euler
(1707 – 1783)

XYZ – fixed global C.S.

x'y'z' – attached to frame

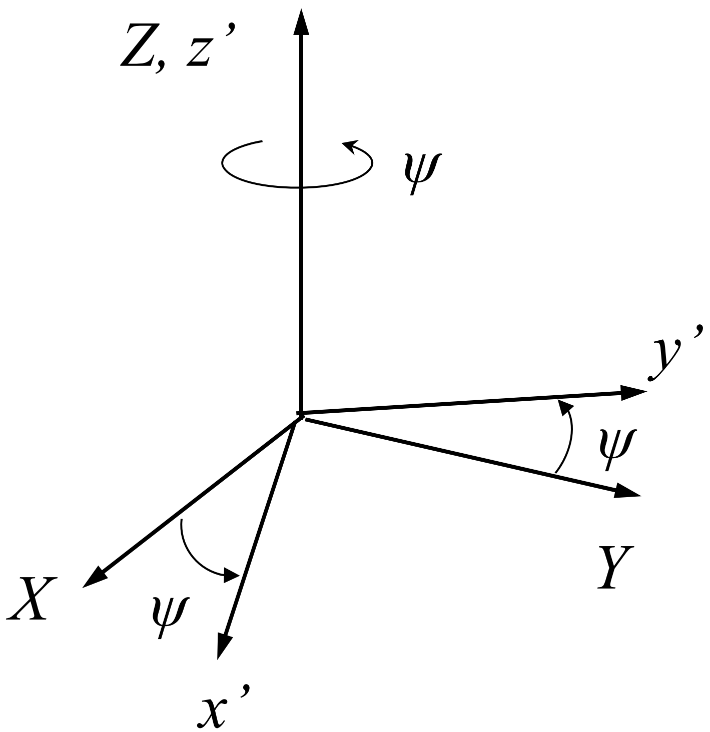
x''y''z'' – attached to gamble

xyz – attached to rotor



Precession

The first rotation is precession which is about the fixed Z axis. The angle of rotation is denoted ψ. The frame after transformation is $x'y'z'$.

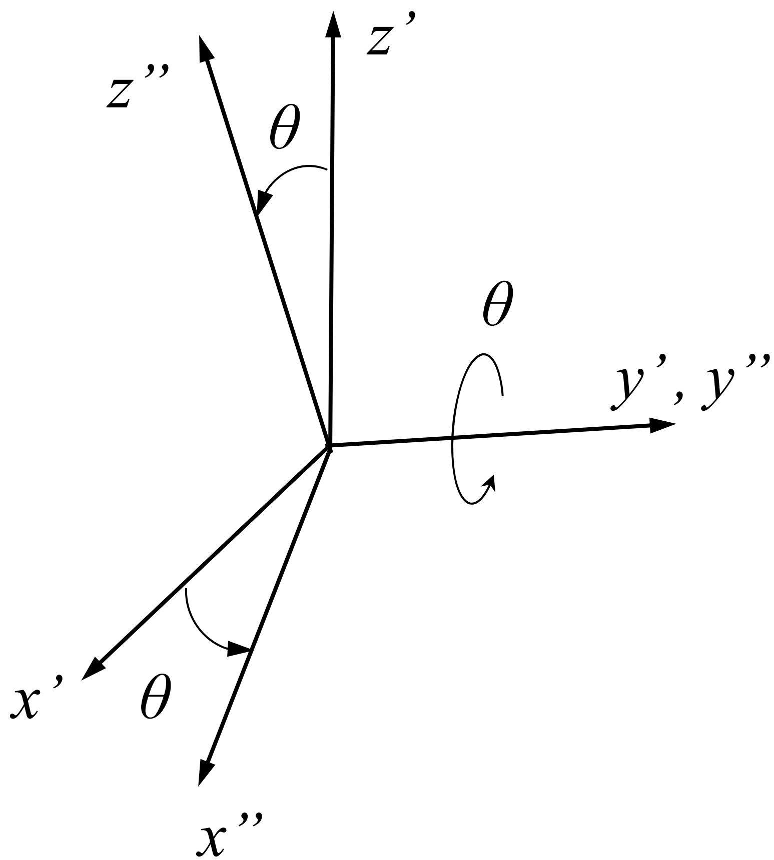


$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [R_\psi] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

$$[R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

nutation

The second rotation is nutation which is about the *y*' axis. The angle of rotation is denoted θ. The frame after transformation is *x*''*y*''*z*''.

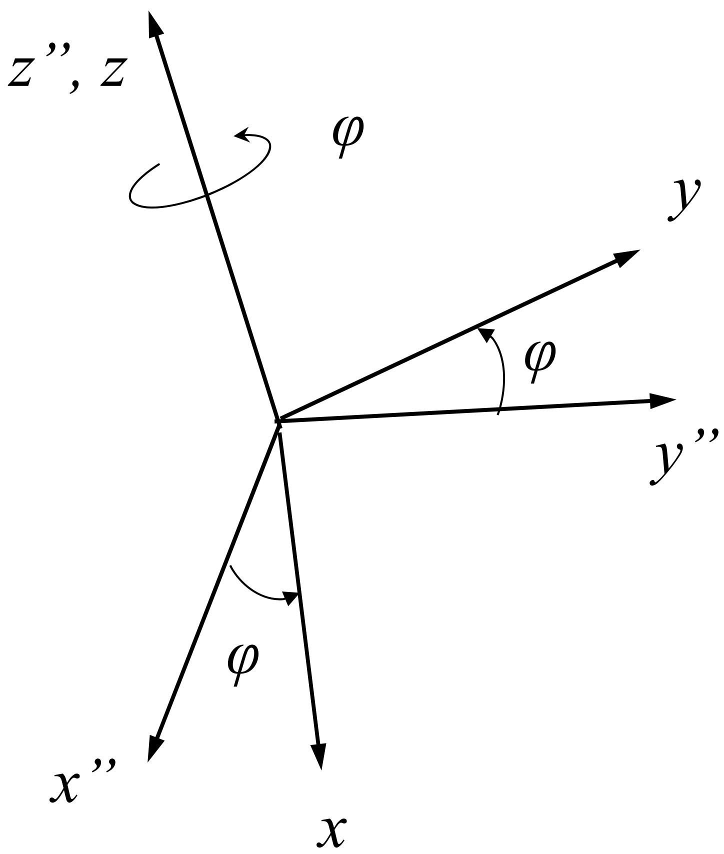


$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = [R_\theta] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}$$

$$[R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

spin

The last rotation is spin in which the *x''y''z''* frame moves to its final orientation. The *z''* axis is the spin axis and the angle of rotation is denoted φ.



$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left[R_\phi\right] \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix}$$

$$\left[R_\phi\right] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Eulerian Angles

The overall transformation

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left[R_\phi\right]\left[R_\theta\right]\left[R_\psi\right]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{4.11}$$

$$\vec{\omega} = \dot{\psi}\hat{K} + \dot{\theta}\hat{j}' + \dot{\phi}\hat{k}$$

$$\begin{aligned}\vec{\alpha} &= \ddot{\psi}\,\hat{K} + \ddot{\theta}\,\hat{j}' + \dot{\theta}\,\dot{\hat{j}}' + \ddot{\phi}\,\hat{k} + \dot{\phi}\,\dot{\hat{k}} \\ &= \ddot{\psi}\,\hat{K} + \ddot{\theta}\,\hat{j}' + \dot{\theta}\left(\dot{\psi}\,\hat{K} \times \hat{j}'\right) + \ddot{\phi}\,\hat{k} + \dot{\phi}\left(\vec{\omega} \times \hat{k}\right)\end{aligned}$$

$$\hat{K} = \sin\theta\left[-(\cos\phi)\hat{i} + (\sin\phi)\hat{j}\right] + (\cos\theta)\hat{k}$$

$$\hat{j}' = (\sin\phi)\hat{i} + (\cos\phi)\hat{j}$$

$$\begin{aligned}\vec{\omega} &= \left(-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi\right)\hat{i} \\ &\quad + \left(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)\hat{j} + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\hat{k}\end{aligned} \tag{4.16}$$

$$\begin{aligned}\vec{\alpha} = &\left(-\ddot{\psi}\sin\theta\cos\phi + \ddot{\theta}\sin\phi - \dot{\psi}\dot{\theta}\cos\theta\cos\phi + \dot{\psi}\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\dot{\phi}\cos\phi\right)\hat{i} \\ &+ \left(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi\right)\hat{j} \\ &+ \left(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta\right)\hat{k}\end{aligned} \tag{4.17}$$

Notes:

1) Utilization of Eulerian angles requires recognition of the appropriate axes of rotation. This involves identifying a fixed axis of rotation as the precession axis (may not be Z). Then the nutation axis precesses orthogonally to the precession axis. Finally, the spin axis precesses and nutates, while it remains perpendicular to the nutation axis.

2) There is no accepted as to how the coordinate axes should be assigned to the rotation axes. You need to replace the precession angle ψ, nutation angle θ, spin angle φ, and the direction unit vectors by the actual used quantities.

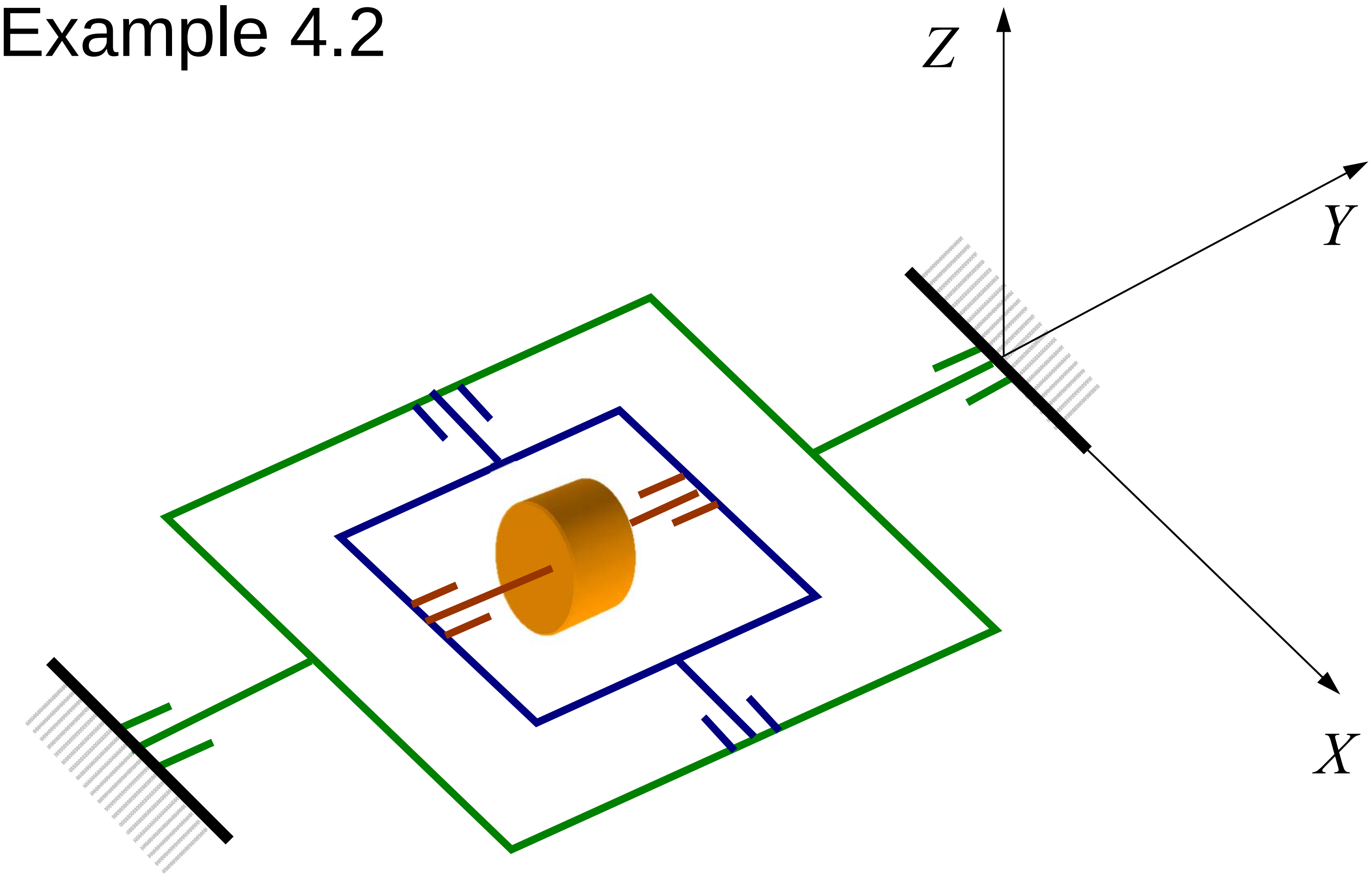
3) In the meantime, the form $R = R_\varphi R_\theta R_\psi$ does not change either. You only need to adjust the order of x, y, z in

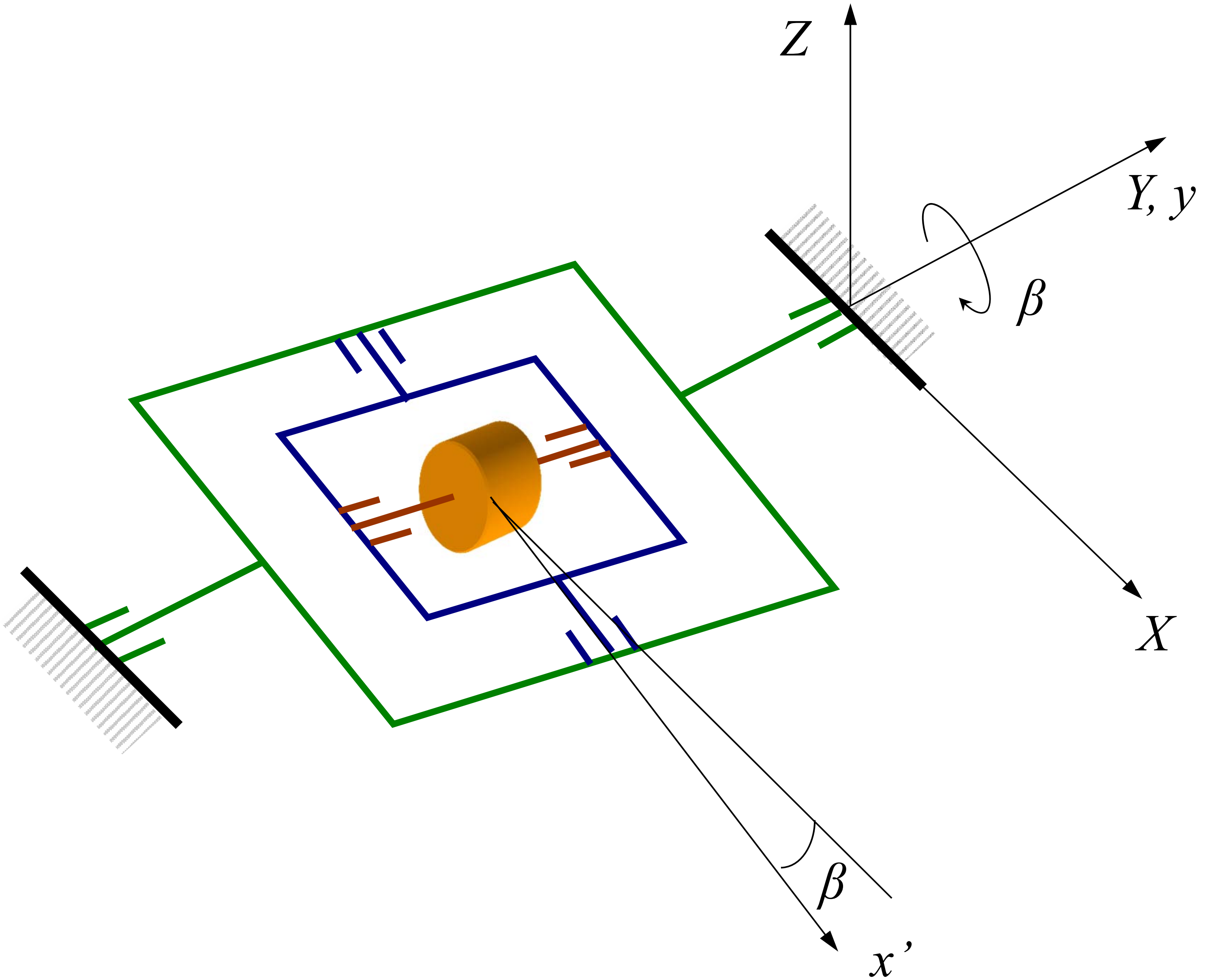
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left[R_\phi\right]\left[R_\theta\right]\left[R_\psi\right]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \qquad (4.11)$$

4) Furthermore,

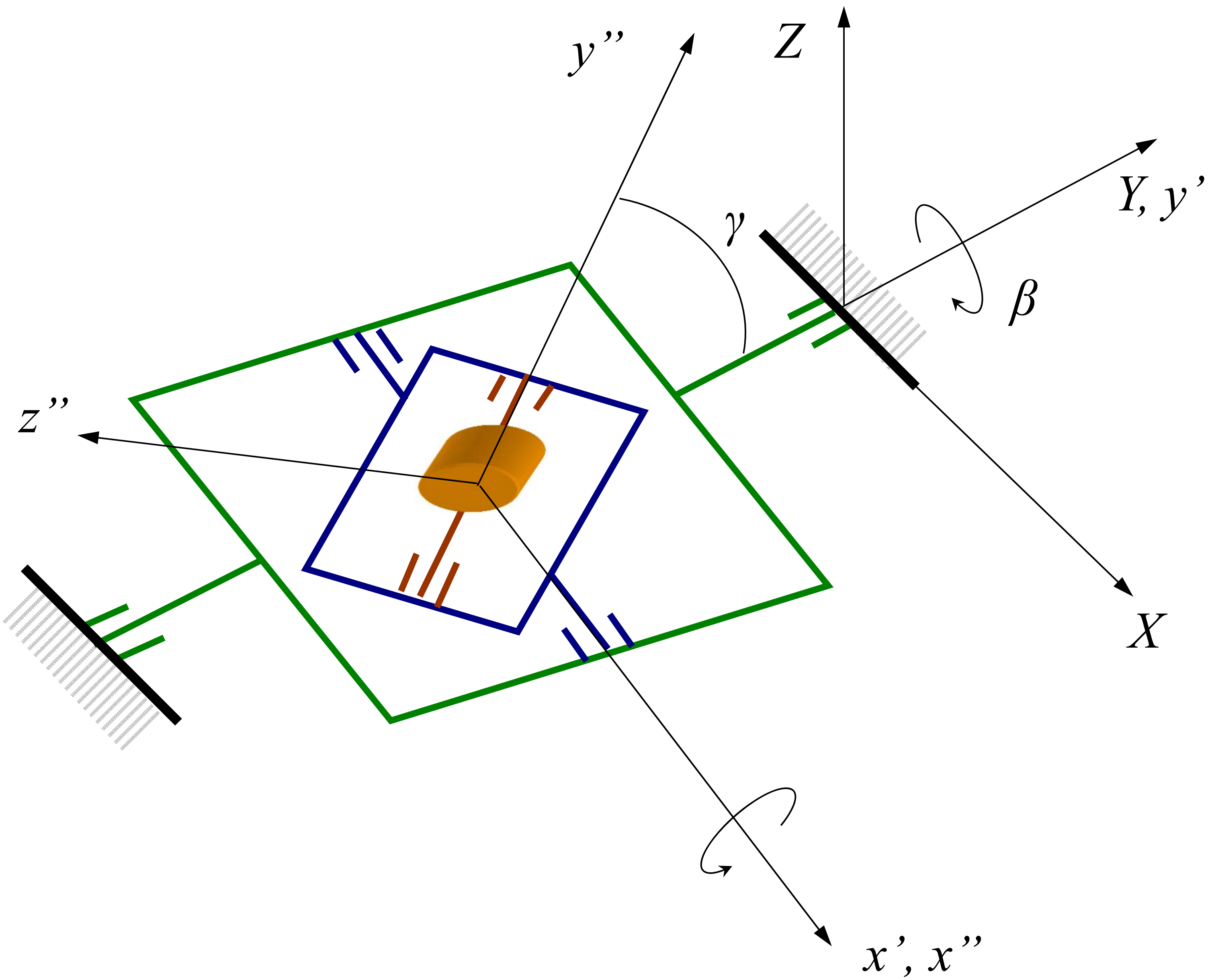
$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \left[R_\phi\right]\left[R_\theta\right]\left[R_\psi\right]\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} \qquad \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = \left[R_\phi\right]\left[R_\theta\right]\left[R_\psi\right]\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix}$$

Example 4.2

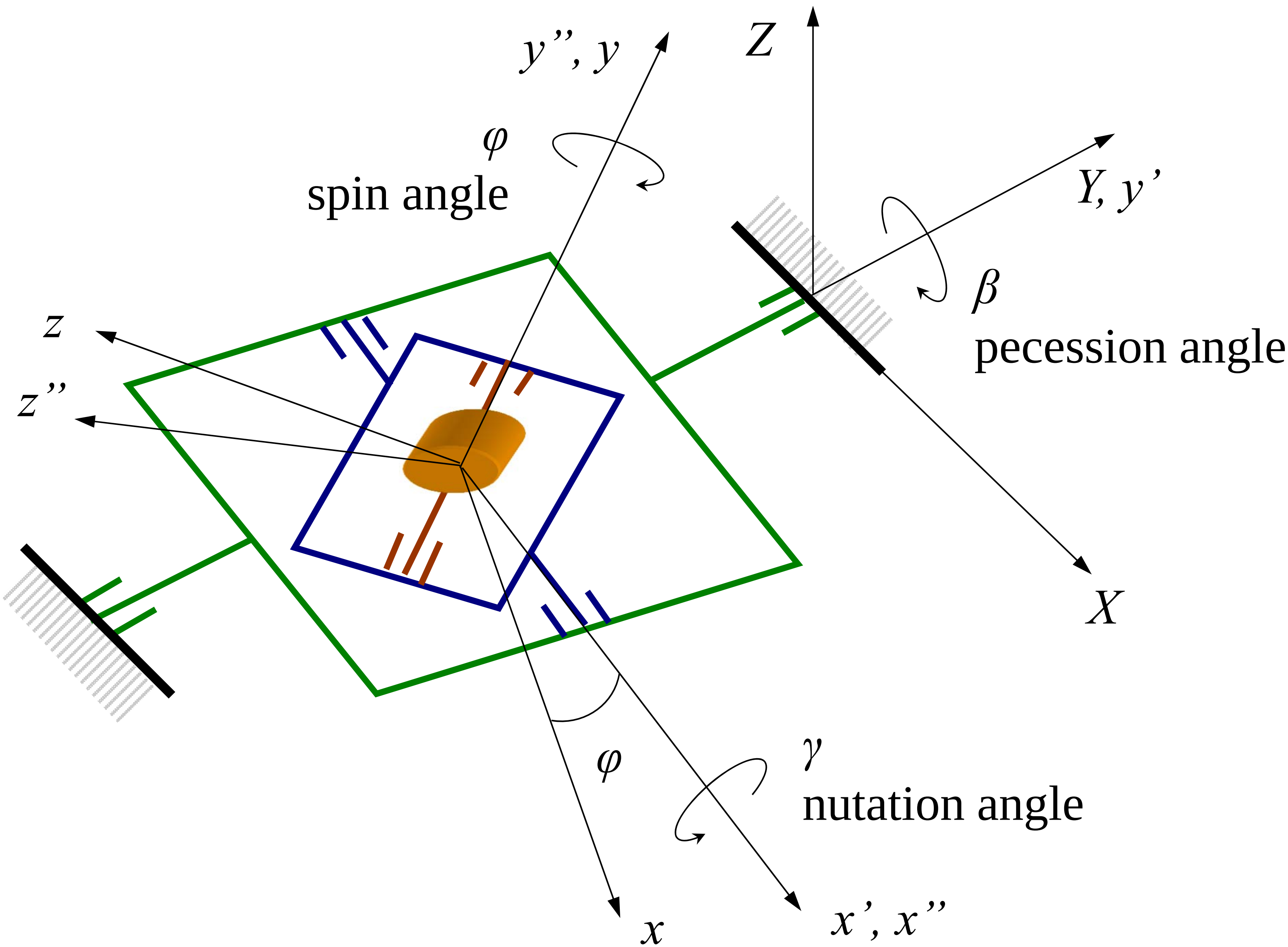




Z
Y, y'
β
X
β
x'



y''
Z
Y, y'
γ
β
z''
X
x', x''



y'', y
Z
φ
spin angle
Y, y'
β
pecession angle
z
z''
X
φ
γ
nutation angle
x
x', x''

$$\vec{\omega} = \dot{\beta}\hat{J} + \dot{\gamma}\hat{i}\,' + \dot{\phi}\hat{j} \tag{1}$$

$$\vec{\alpha} = \ddot{\beta}\hat{J} + \ddot{\gamma}\hat{i}\,' + \dot{\gamma}\left(\dot{\beta}\hat{J}\times\hat{i}\,'\right) + \ddot{\phi}\hat{j} + \dot{\phi}\left(\vec{\omega}\times\hat{j}\right) \tag{2}$$

Use geometry

$$\hat{i}\,' = \hat{i}\,'' = \cos\phi\hat{i} + \sin\phi\hat{k}$$

$$\hat{k}\,'' = \cos\phi\hat{k} - \sin\phi\hat{i}$$

$$\hat{k}\,' = \sin\gamma\hat{j}\,'' + \cos\gamma\hat{k}\,'' = \sin\gamma\hat{j} + \cos\gamma\left(\cos\phi\hat{k} - \sin\phi\hat{i}\right)$$

$$= -\cos\gamma\sin\phi\hat{i} + \sin\gamma\hat{j} + \cos\gamma\cos\phi\hat{k}$$

$$\hat{J} = \hat{j}\,' = \cos\gamma\hat{j}\,'' - \sin\gamma\hat{k}\,'' = \cos\gamma\hat{j} - \sin\gamma\left(\cos\phi\hat{k} - \sin\phi\hat{i}\right)$$

$$= \sin\gamma\sin\phi\hat{i} + \cos\gamma\hat{j} - \sin\gamma\cos\phi\hat{k}$$

And plug them back in (1) and (2), they become

$$\vec{\omega} = \dot{\beta}\left(\sin\gamma\sin\phi\hat{i} + \cos\gamma\,\hat{j} - \sin\gamma\cos\phi\hat{k}\right)$$

$$+\dot{\gamma}\left(\cos\phi\hat{i} + \sin\phi\hat{k}\right) + \dot{\phi}\,\hat{j}$$

$$\Rightarrow \vec{\omega} = \left(\dot{\beta}\sin\gamma\sin\phi + \dot{\gamma}\cos\phi\right)\hat{i}$$

$$+\left(\dot{\beta}\cos\gamma + \dot{\phi}\right)\hat{j} + \left(-\dot{\beta}\sin\gamma\cos\phi + \dot{\gamma}\sin\phi\right)\hat{k} \qquad (3)$$

$$\vec{\alpha} = \underbrace{\ddot{\beta}\hat{J}}_{(A)} + \underbrace{\ddot{\gamma}\hat{i}\,'}_{(B)} + \underbrace{\dot{\gamma}\left(\dot{\beta}\hat{J}\times\hat{i}\,'\right)}_{(C)} + \underbrace{\ddot{\phi}\,\hat{j}}_{(D)} + \underbrace{\dot{\phi}\left(\vec{\omega}\times\hat{j}\right)}_{(E)}$$

$$(A) = \ddot{\beta}\sin\gamma\sin\phi\hat{i} + \ddot{\beta}\cos\gamma\,\hat{j} - \ddot{\beta}\sin\gamma\cos\phi\hat{k}$$

$$(B) = \ddot{\gamma}\cos\phi\hat{i} + \ddot{\gamma}\sin\phi\hat{k}$$

$$\left(C\right)=\dot{\gamma}\dot{\beta}\left(\sin\gamma\sin\phi\hat{i}+\cos\gamma\hat{j}-\sin\gamma\cos\phi\hat{k}\right)\times\left(\cos\phi\hat{i}+\sin\phi\hat{k}\right)$$

$$=\dot{\gamma}\dot{\beta}\left(-\sin\gamma\sin^2\phi\hat{j}-\cos\gamma\cos\phi\hat{k}+\cos\gamma\sin\phi\hat{i}-\sin\gamma\cos^2\phi\hat{j}\right)$$

$$=\dot{\gamma}\dot{\beta}\cos\gamma\sin\phi\hat{i}-\dot{\gamma}\dot{\beta}\sin\gamma\hat{j}-\dot{\gamma}\dot{\beta}\cos\gamma\cos\phi\hat{k}$$

$$\left(E\right)=\dot{\phi}\left(\vec{\omega}\times\hat{j}\right)$$

$$=\dot{\phi}\left[\left(\dot{\beta}\sin\gamma\sin\phi+\dot{\gamma}\cos\phi\right)\hat{i}+\left(-\dot{\beta}\sin\gamma\cos\phi+\dot{\gamma}\sin\phi\right)\hat{k}\right]\times\hat{j}$$

$$=\dot{\phi}\left(\dot{\beta}\sin\gamma\sin\phi+\dot{\gamma}\cos\phi\right)\hat{k}+\dot{\phi}\left(\dot{\beta}\sin\gamma\cos\phi-\dot{\gamma}\sin\phi\right)\hat{i}$$

$$\vec{\alpha}=\left(\ddot{\beta}\sin\gamma\sin\phi+\ddot{\gamma}\cos\phi+\dot{\gamma}\dot{\beta}\cos\gamma\sin\phi+\dot{\phi}\dot{\beta}\sin\gamma\cos\phi-\dot{\phi}\dot{\gamma}\sin\phi\right)\hat{i}$$

$$=\left(\ddot{\beta}\cos\gamma-\dot{\gamma}\dot{\beta}\sin\gamma+\ddot{\phi}\right)\hat{j} \tag{4}$$

$$=\left(-\ddot{\beta}\sin\gamma\cos\phi+\ddot{\gamma}\sin\phi+-\dot{\gamma}\dot{\beta}\cos\gamma\cos\phi+\dot{\phi}\dot{\beta}\sin\gamma\sin\phi+\dot{\phi}\dot{\gamma}\cos\phi\right)\hat{k}$$

The precession, nutation and spin axes are Y, x', and y, instead of Z, y', and z in the previous derivation. If you use the x, y, z order in (4.11), the transformation matrices should be about their corresponding rotation axes, e.g.

Precession: $$\left[R_\beta\right]=\begin{bmatrix}\cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta\end{bmatrix}$$

nutation: $$\left[R_\gamma\right]=\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}$$

spin: $$\left[R_\phi\right]=\begin{bmatrix}\cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi\end{bmatrix}$$

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \left[R_\phi\right]\left[R_\theta\right]\left[R_\psi\right]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

where

$$\left[R\right] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\gamma & -\sin\phi\cos\gamma \\ 0 & \cos\gamma & \sin\gamma \\ \sin\phi & -\cos\phi\sin\gamma & \cos\phi\cos\gamma \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\beta - \sin\phi\cos\gamma\sin\beta & \sin\phi\sin\gamma & -\cos\phi\sin\beta - \sin\phi\cos\gamma\cos\beta \\ \sin\gamma\sin\beta & \cos\gamma & \sin\gamma\cos\beta \\ \sin\phi\cos\beta + \cos\phi\cos\gamma\sin\beta & -\cos\phi\sin\gamma & -\sin\phi\sin\beta + \cos\phi\cos\gamma\cos\beta \end{bmatrix}$$

Calculation is easier using Eulerian angle equations

$$\vec{\omega} = \dot{\beta}\hat{J} + \dot{\gamma}\hat{i}' + \dot{\phi}\hat{j}$$

In Eqs. (4.16) (4.17), replace ψ by β, θ by γ. Also notice that the spin axis is y instead of z, we need also rotate the the coefficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$ as coefficients of $\hat{k}$, $\hat{i}$, and $\hat{j}$.

$$\begin{Bmatrix} z \\ x \\ y \end{Bmatrix} = \left[R_\phi\right]\left[R_\gamma\right]\left[R_\beta\right]\begin{Bmatrix} Z \\ X \\ Y \end{Bmatrix}$$

You only need rotate the axis order, but still write metrices $\left[R_\phi\right], \left[R_\gamma\right], \left[R_\beta\right]$ as they were rotating about Z, y' and z axes.

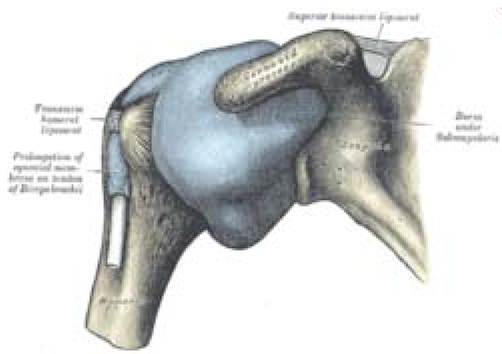
$$
\begin{aligned}
[R] &= [R_\phi][R_\gamma][R_\beta] \\
&= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi\cos\gamma & \sin\phi & -\cos\phi\sin\gamma \\ -\sin\phi\cos\gamma & \cos\phi & \sin\phi\sin\gamma \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi\cos\gamma\cos\beta - \sin\phi\sin\beta & \cos\phi\cos\gamma\sin\beta + \sin\phi\cos\beta & -\cos\phi\sin\gamma \\ -\sin\phi\cos\gamma\cos\beta - \cos\phi\sin\beta & -\sin\phi\cos\gamma\sin\beta + \cos\phi\cos\beta & \sin\phi\sin\gamma \\ \sin\gamma\cos\beta & \sin\gamma\sin\beta & \cos\gamma \end{bmatrix}
\end{aligned}
$$

3. Interconnections

- Joint (kinematic pair) – is a connection between two or more links at their nodes, which allows some motion (or potential motion) between the connected links

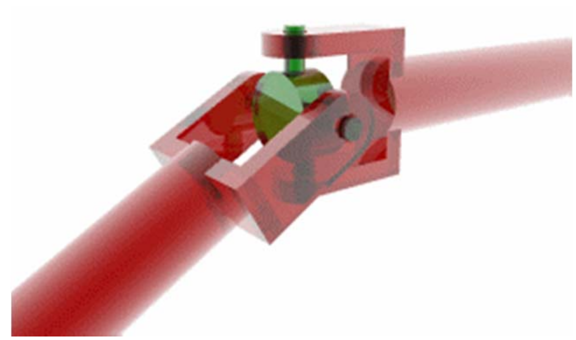
Hinge

(? DOF)



Shoulder (ball and socket joint)

(? DOF)

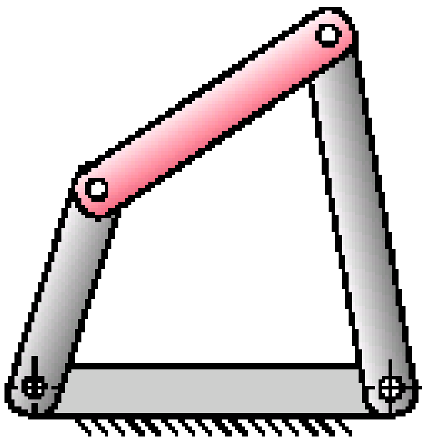


Universal joint

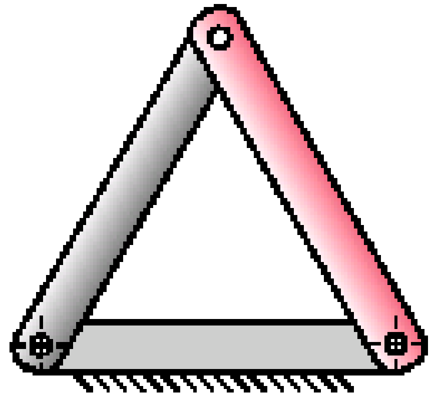
(? DOF)

3. Interconnections

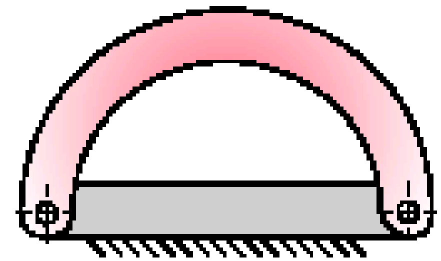
- DOFs and constraints
  - dialectical concepts
  - Joints reduce system DOF
  - More DOF means less constraints, and vise versa. When we study the motion of a system, it depends on the perspective you look at the problem.
  - We have seen how to classify the joints into lower and higher pairs. There are other methods to classify the joints. A commonly used one is by the DOFs or constraints.



Mechanism – DOF>0
(links have relative motion)



Structure – DOF =0
(no motion possible)



Preload structure
DOF <0

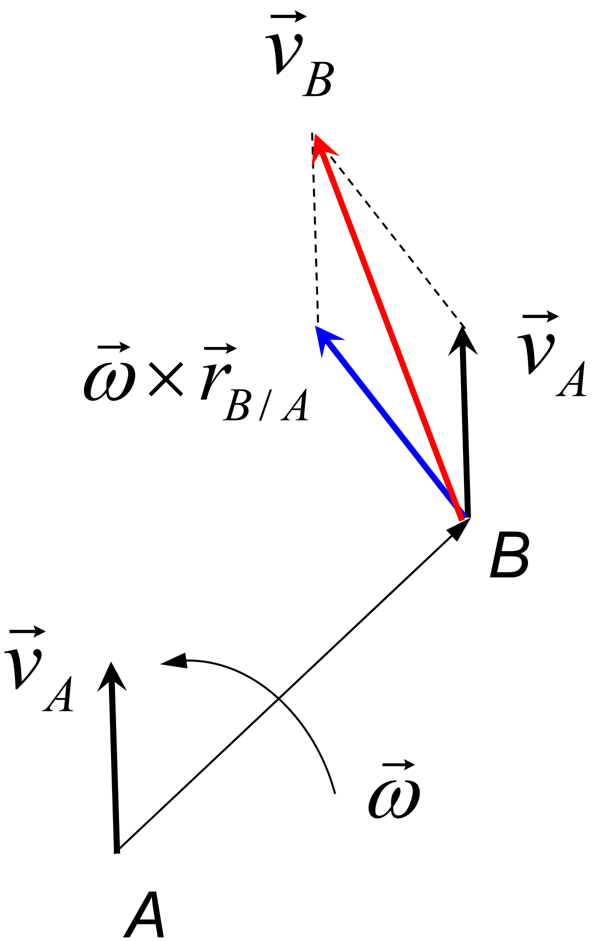
- Fully constrained: the system of equations is solvable such that the motion of each body (velocity, acceleration, and angular motion) can be evaluated
- Partially constrained: the number of unknowns are more than that of the equations.
- Rigid: no motion

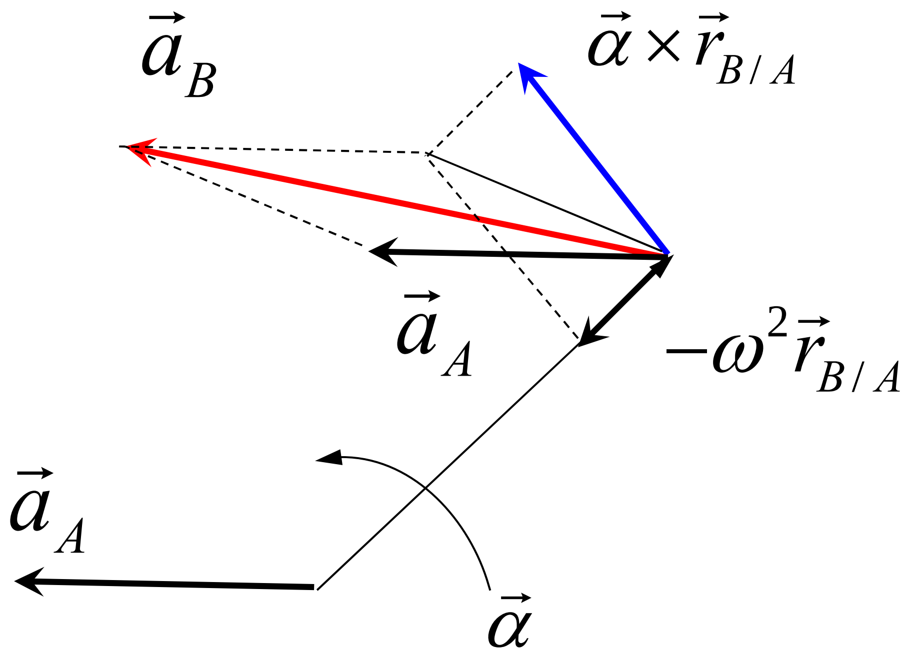
Planar motion

$$\vec{\omega} = \omega \hat{k}, \;\; \vec{\alpha} = \dot{\omega} \hat{k} = \alpha \hat{k},$$

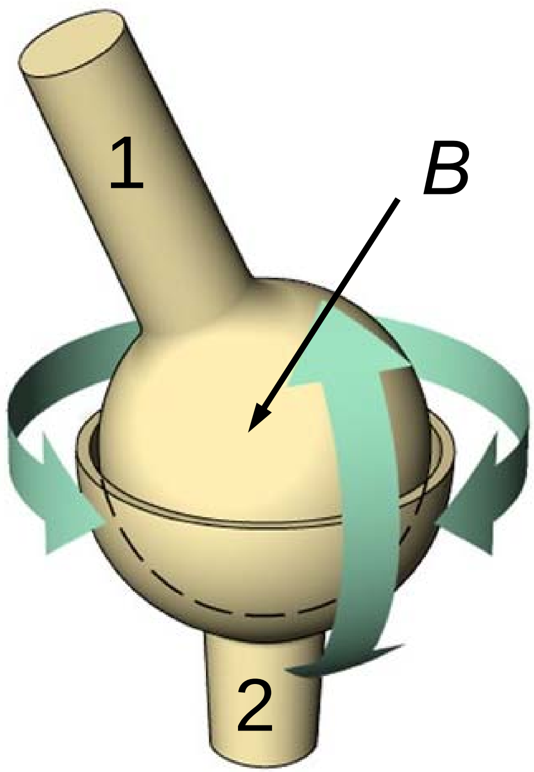
$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$$





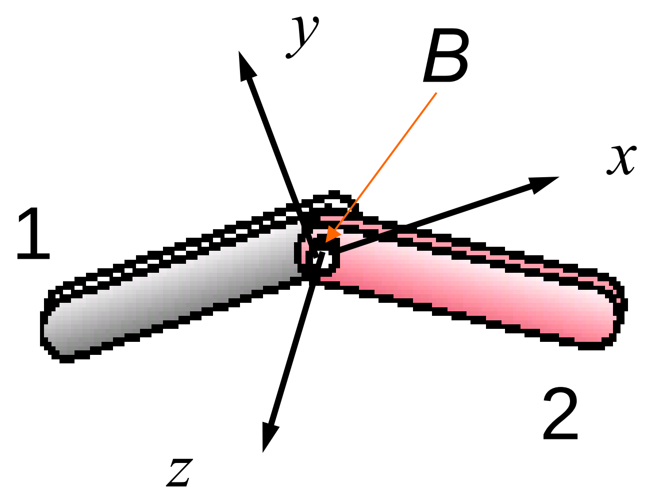
Ball-socket joint



$$\vec{v}_{B1} = \vec{v}_{B2}$$

$$\vec{a}_{B1} = \vec{a}_{B2}$$

Pin connection



$$\vec{v}_{B1} = \vec{v}_{B2} \qquad \vec{a}_{B1} = \vec{a}_{B2}$$

$$\vec{\omega}_2 = \vec{\omega}_1 + \dot{\phi}\hat{k}$$

$$\vec{\alpha}_2 = \vec{\alpha}_1 + \ddot{\phi}\hat{k} + \dot{\phi}\left(\vec{\omega}_1 \times \hat{k}\right)$$

collar connection: Assume bar AB straight

$$\left(\vec{v}_{C2}\right)_{AB} = u\hat{e}_{B/A} \qquad \left(\vec{a}_{C2}\right)_{AB} = \dot{u}\hat{e}_{B/A}$$

Consider:

$$\vec{v}_P = \vec{v}_{O'} + \left(\vec{v}_P\right)_{xyz} + \vec{\omega}\times\vec{r}_{P/O'}$$

$$\vec{a}_P = \vec{a}_{O'} + \left(\vec{a}_P\right)_{xyz} + \vec{\alpha}\times\vec{r}_{P/O'} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{P/O'}\right) + 2\vec{\omega}\times\left(\vec{v}_P\right)_{xyz}$$

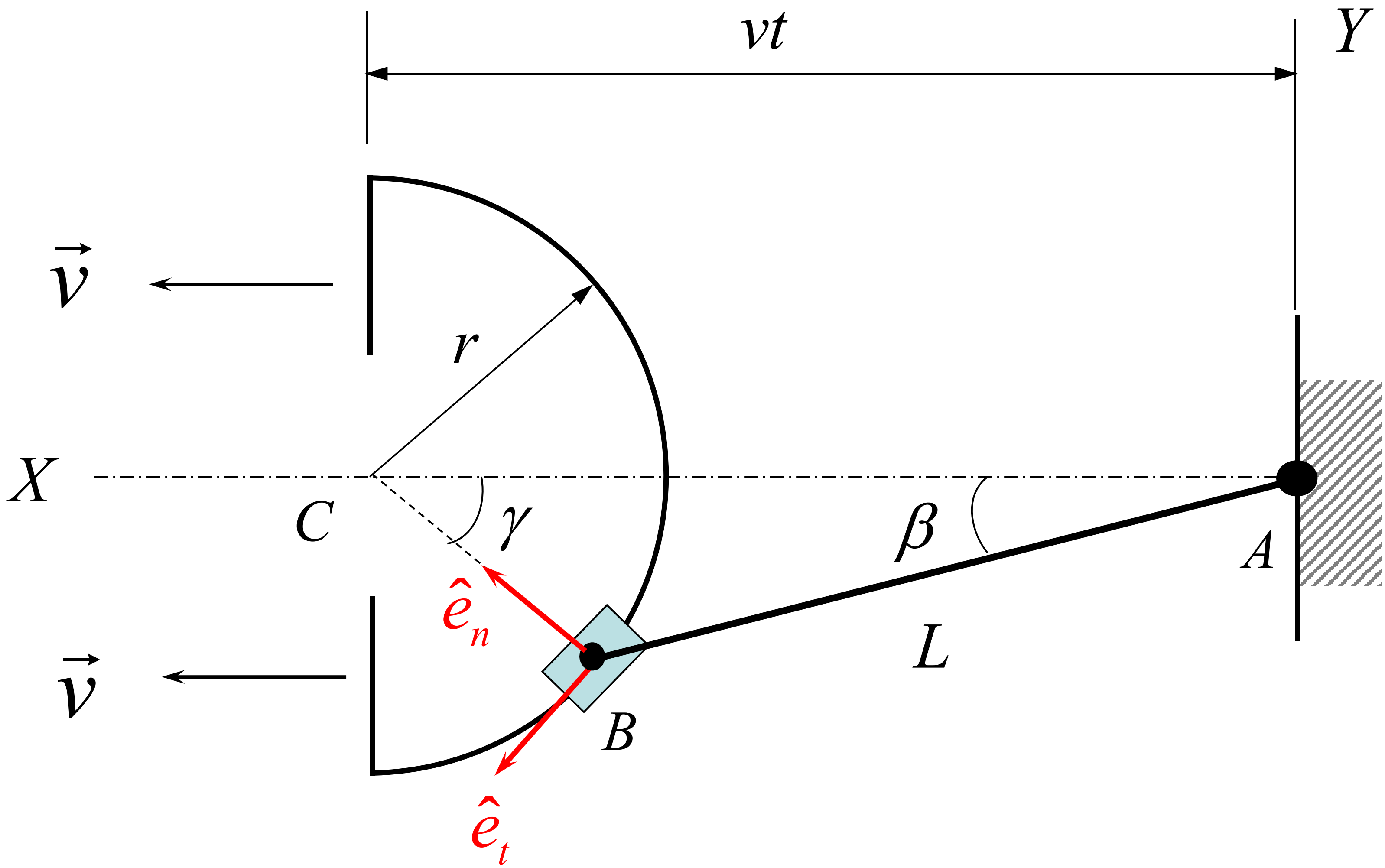
$$\Rightarrow \boxed{\begin{aligned} \vec{v}_{C2} &= \vec{v}_{C1} + u\hat{e}_{B/A} \\ \vec{a}_{C2} &= \vec{a}_{C1} + \dot{u}\hat{e}_{B/A} + 2u\vec{\omega}\times\hat{e}_{B/A} \end{aligned}}$$

collar with pin connection

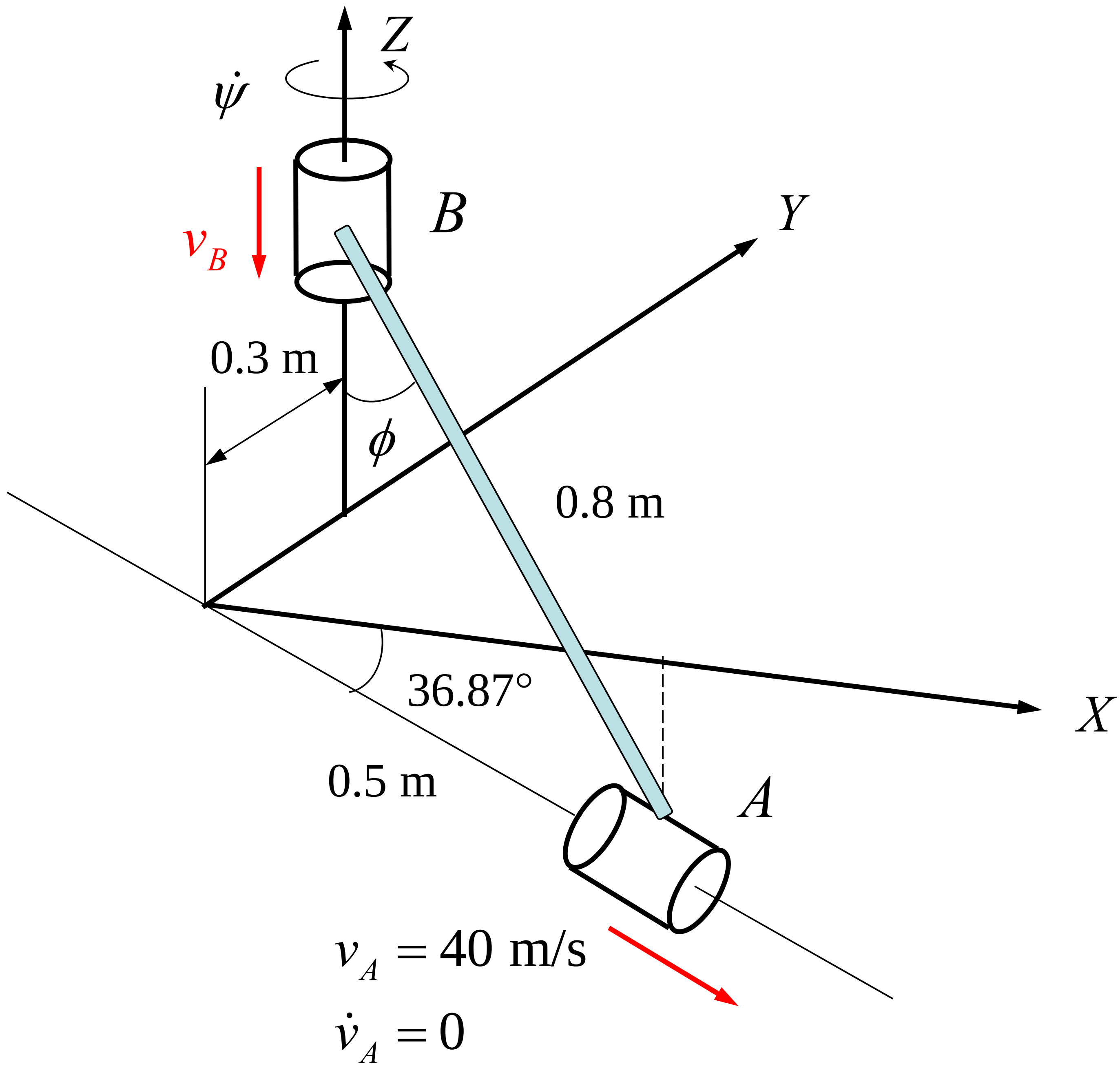
$$\vec{\omega}_2 = \vec{\omega}_1 + \dot{\psi}\hat{i}_1 + \dot{\phi}\hat{k}_2$$

$$\vec{\alpha}_2 = \vec{\alpha}_1 + \ddot{\psi}\hat{i}_1 + \dot{\psi}\left(\vec{\omega}_1 \times \hat{i}_1\right) + \ddot{\phi}\hat{k}_2 + \dot{\phi}\left(\vec{\omega}_1 \times \hat{k}_2\right)$$

- List all the constraints
- Figure out the DOF
- Evaluate velocities first
- Then accelerations

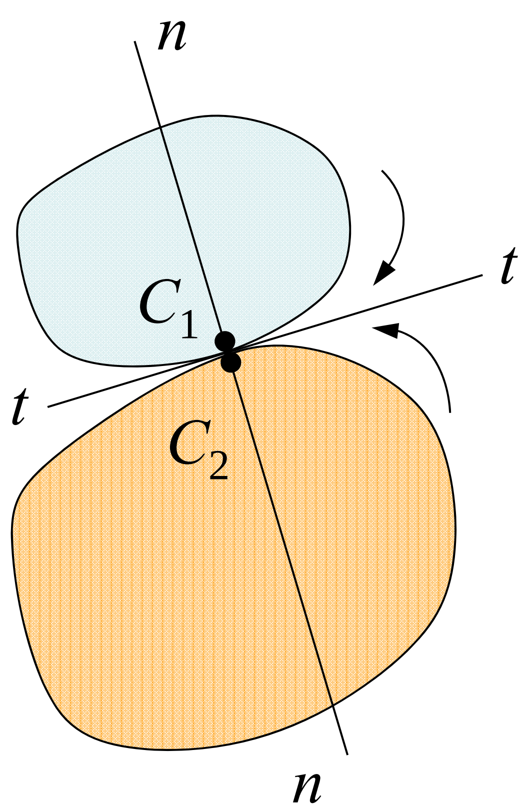


vt
Y
$\vec{v}$
r
X
C
γ
β
A
$\hat{e}_n$
L
$\vec{v}$
B
$\hat{e}_t$



Z
$\dot{\psi}$
B
v_B
Y
0.3 m
ϕ
0.8 m
36.87°
X
0.5 m
A
$v_A = 40$ m/s
$\dot{v}_A = 0$

4. Rolling



- Rolling means the perpendicular velocities of two cylinders are the same, otherwise they will out of contact.

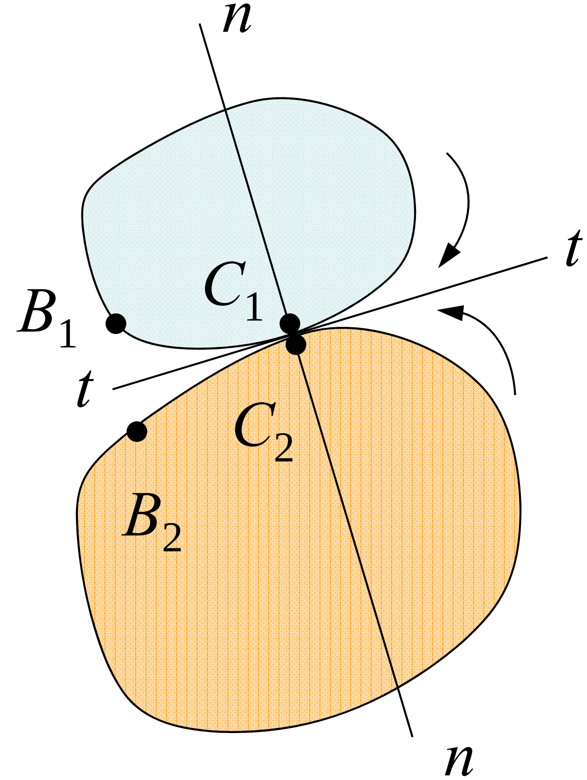
$$\vec{v}_{C1} \bullet \hat{n} = \vec{v}_{C2} \bullet \hat{n}$$

- Rolling without slipping:

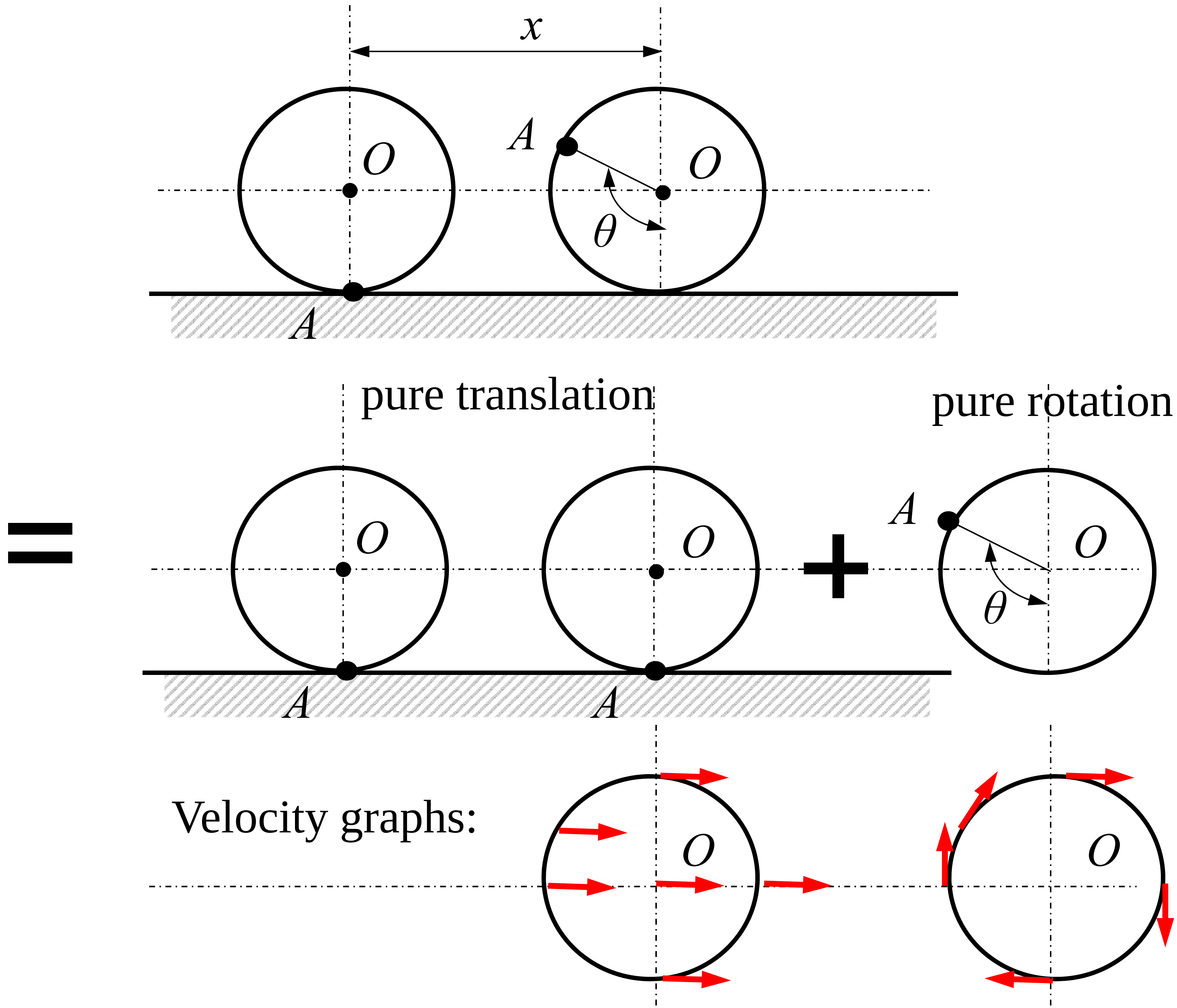
$$\text{arclength } B_1C_1 = B_2C_2$$

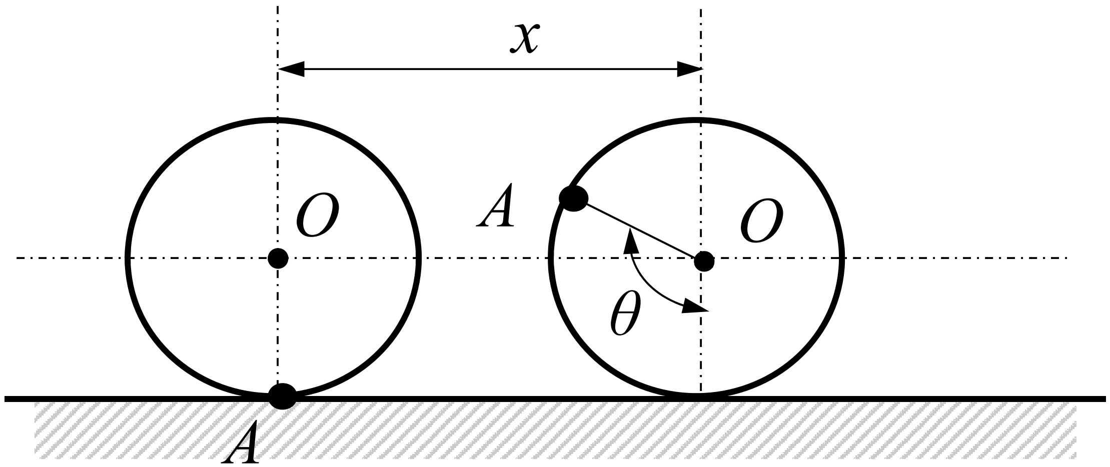
- The tangential velocities are also the same.

$$\vec{v}_{C1} = \vec{v}_{C2}$$



- Note: the accelerations are not the same!


x
A
O
θ
A
pure translation
pure rotation
=
+
Velocity graphs:



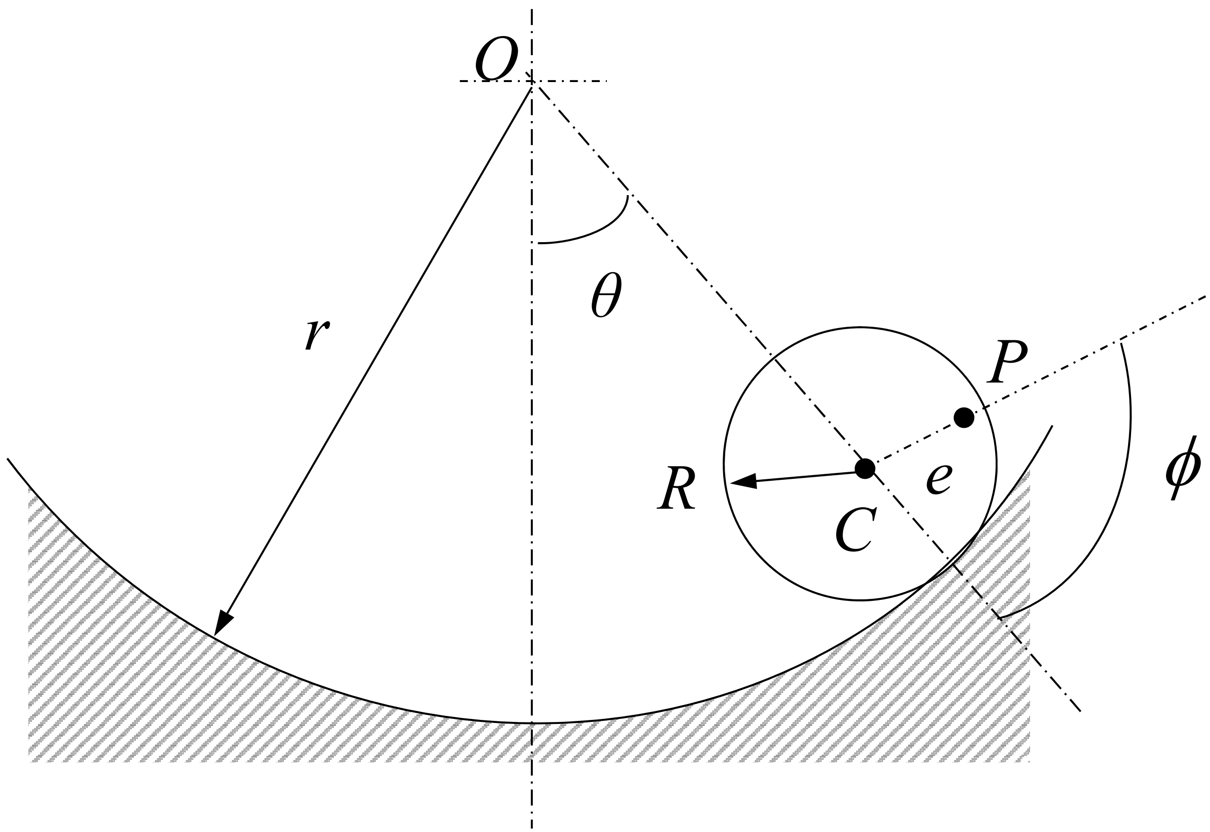
$$x_O = R\theta \Rightarrow \dot{x}_O = v = R\dot{\theta} \Rightarrow \ddot{x}_O = \dot{v} = R\ddot{\theta}$$

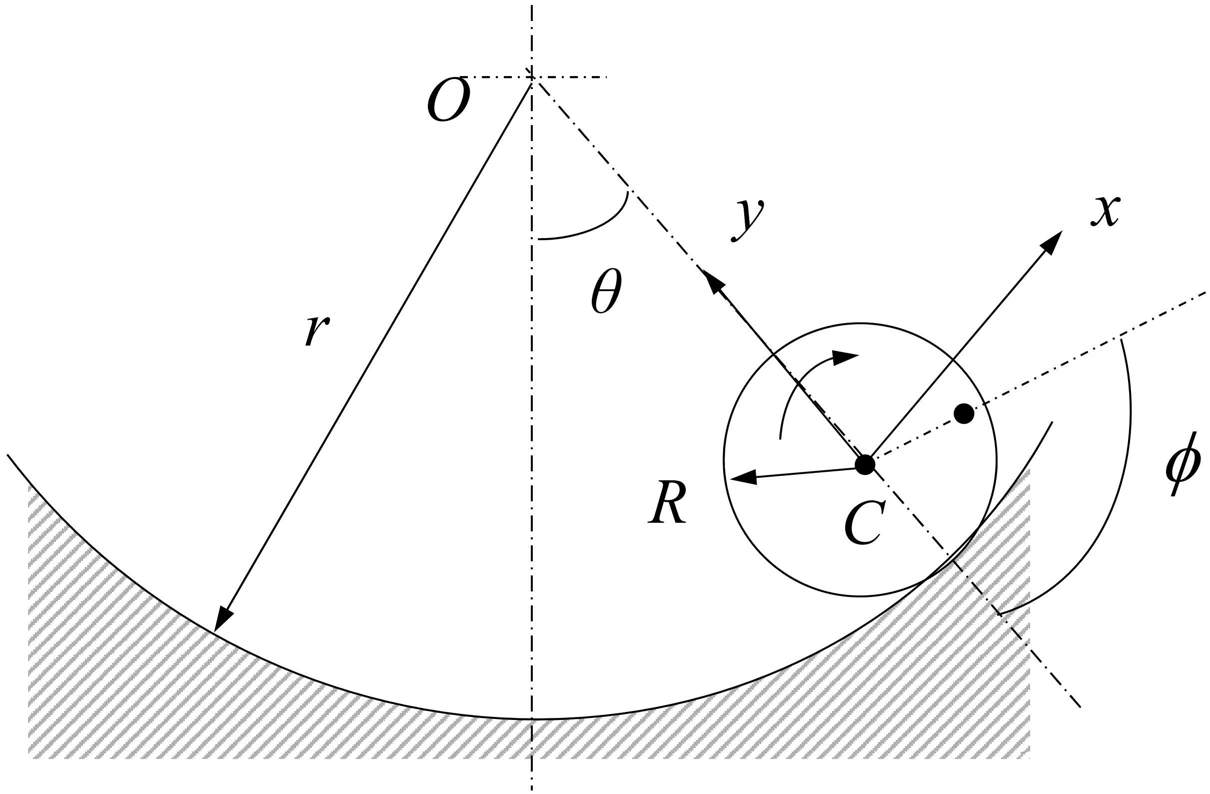
$$\vec{r}_{A/O} = R(\theta - \sin\theta)\hat{i} - R\cos\theta\,\hat{j}$$

$$\vec{v} = v(1-\cos\theta)\hat{i} + v\sin\theta\,\hat{j}$$

$$\vec{a} = \left[\dot{v}(1-\cos\theta) + \frac{v^2}{R}\sin\theta\right]\hat{i} + \left[\dot{v}\sin\theta + \frac{v^2}{R}\cos\theta\right]\hat{j}$$

Example 4.6 A cylinder of radius R rolls without slipping inside a semi-cylindrical cavity. Point P is collinear with the vertical ceterline when the vertical angle θ locating the cylinder's center C is zero. Derive expressions for the velocity and acceleration of point P in terms of φ and the speed v of center C. Then convert those expressions in terms of θ. Note: v is not constant.

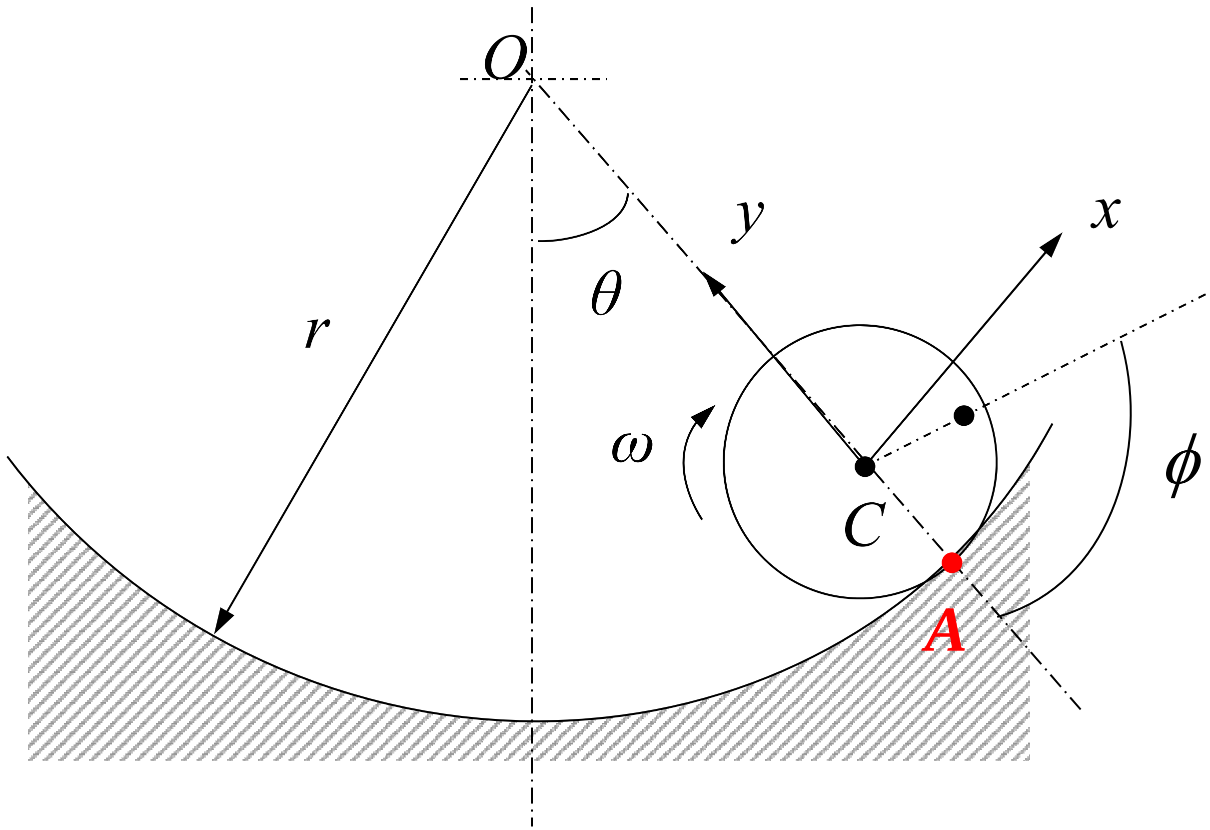




Attach the reference frame to the moving cylinder where x is tangent to the path of C. If the cylinder moves to the right, the angular velocity is negative. And since this is a planar motion problem, the direction of ω does not change. Therefore,

$$\vec{\omega} = -\omega\hat{k}, \quad \vec{\alpha} = -\dot{\omega}\hat{k}$$

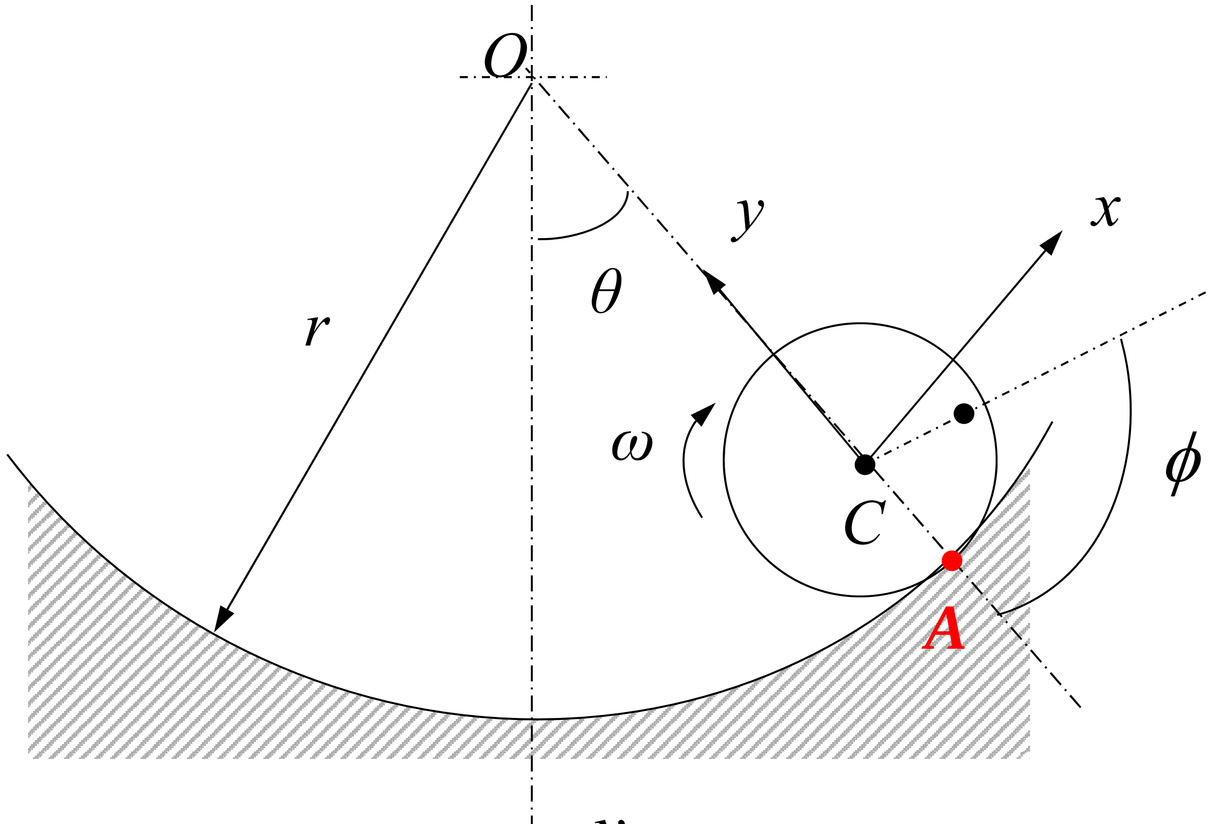
w.r.t. O $$\vec{v} = v\hat{i}, \quad \vec{a} = \dot{v}\hat{i} + \frac{v^2}{r-R}\hat{j}$$



$$\vec{v} = \cancel{\vec{v}_A} + \vec{\omega} \times \vec{r}_{C/A} = -\omega\hat{k} \times R\hat{j} = \omega R\hat{i}$$

$$\vec{v} = v\hat{i}$$

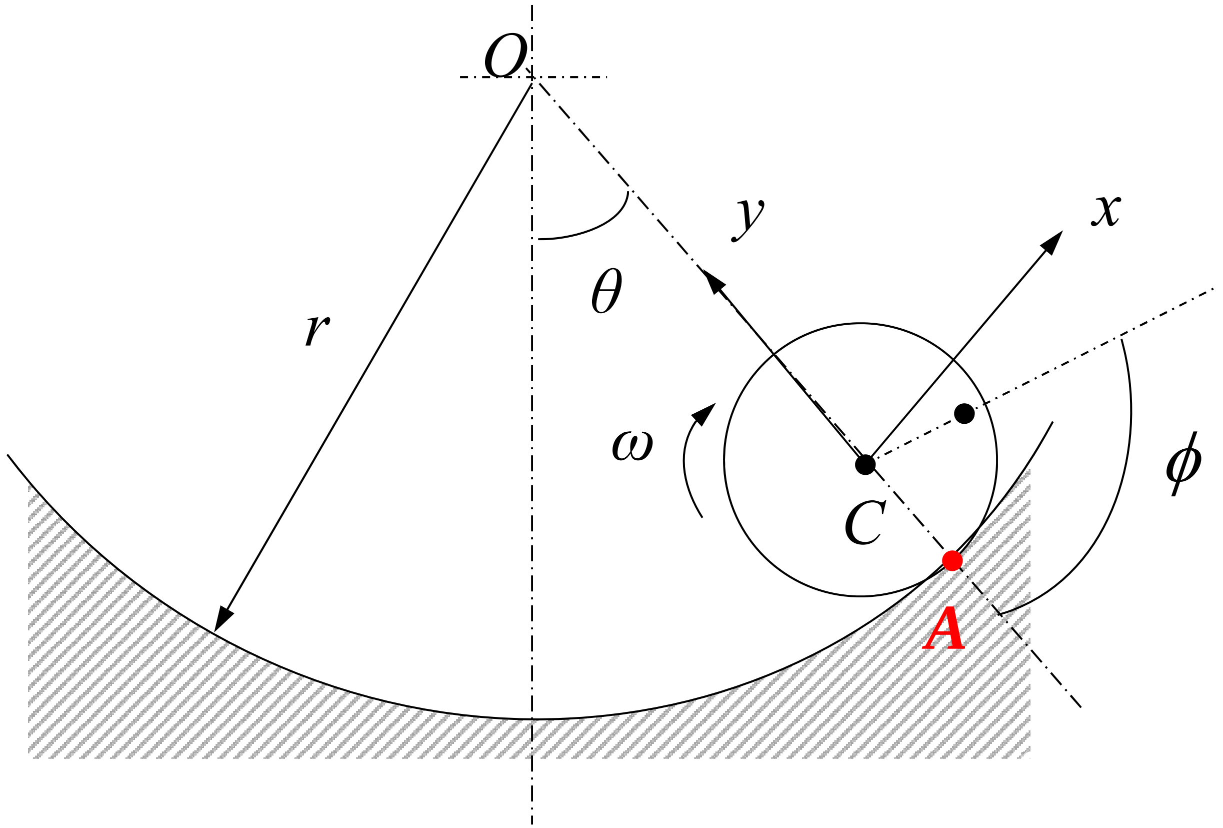
$$\Rightarrow \omega = \frac{v}{R} \Rightarrow \dot{\omega} = \frac{\dot{v}}{R}$$



$$\vec{v}_P = \vec{v}_C + \vec{\omega} \times \vec{r}_{P/C} = v\hat{i} - \frac{v}{R}\hat{k} \times e\left[(\sin\phi)\hat{i} - (\cos\phi)\hat{j}\right]$$

$$= v\hat{i} - \frac{ve}{R}\left[(\sin\phi)\hat{j} + (\cos\phi)\hat{i}\right]$$

$$= \left(v - \frac{ve}{R}\cos\phi\right)\hat{i} - \frac{ve}{R}(\sin\phi)\hat{j}$$

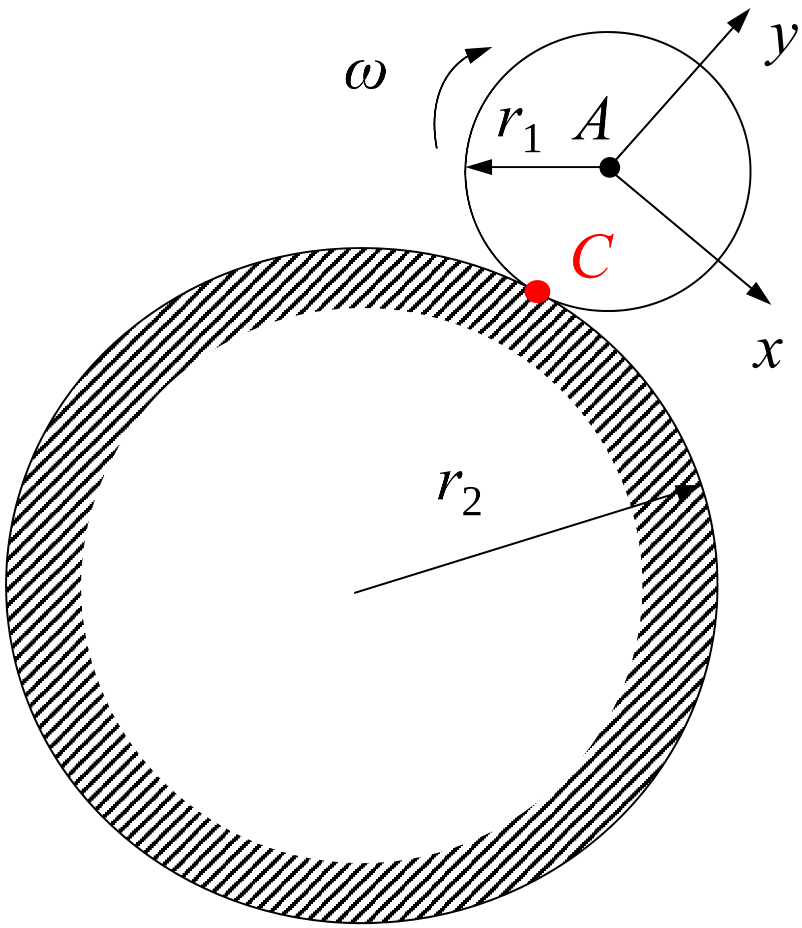


$$\vec{a}_P = \vec{a}_C + \vec{\alpha} \times \vec{r}_{P/C} - \omega^2 \vec{r}_{P/C}$$

$$= \dot{v}\hat{i} + \frac{v^2}{r-R}\hat{j} - \frac{\dot{v}}{R}\hat{k} \times e\left[(\sin\phi)\hat{i} - (\cos\phi)\hat{j}\right] - \left(\frac{v}{R}\right)^2 e\left[(\sin\phi)\hat{i} - (\cos\phi)\hat{j}\right]$$

$$= \left(\dot{v} - \frac{\dot{v}e}{R}\cos\phi - \frac{v^2 e}{R^2}\sin\phi\right)\hat{i} + \left[-\frac{\dot{v}e}{R}(\sin\phi) + v^2\left(\frac{1}{r-R} + \frac{e}{R^2}\cos\phi\right)\right]\hat{j}$$

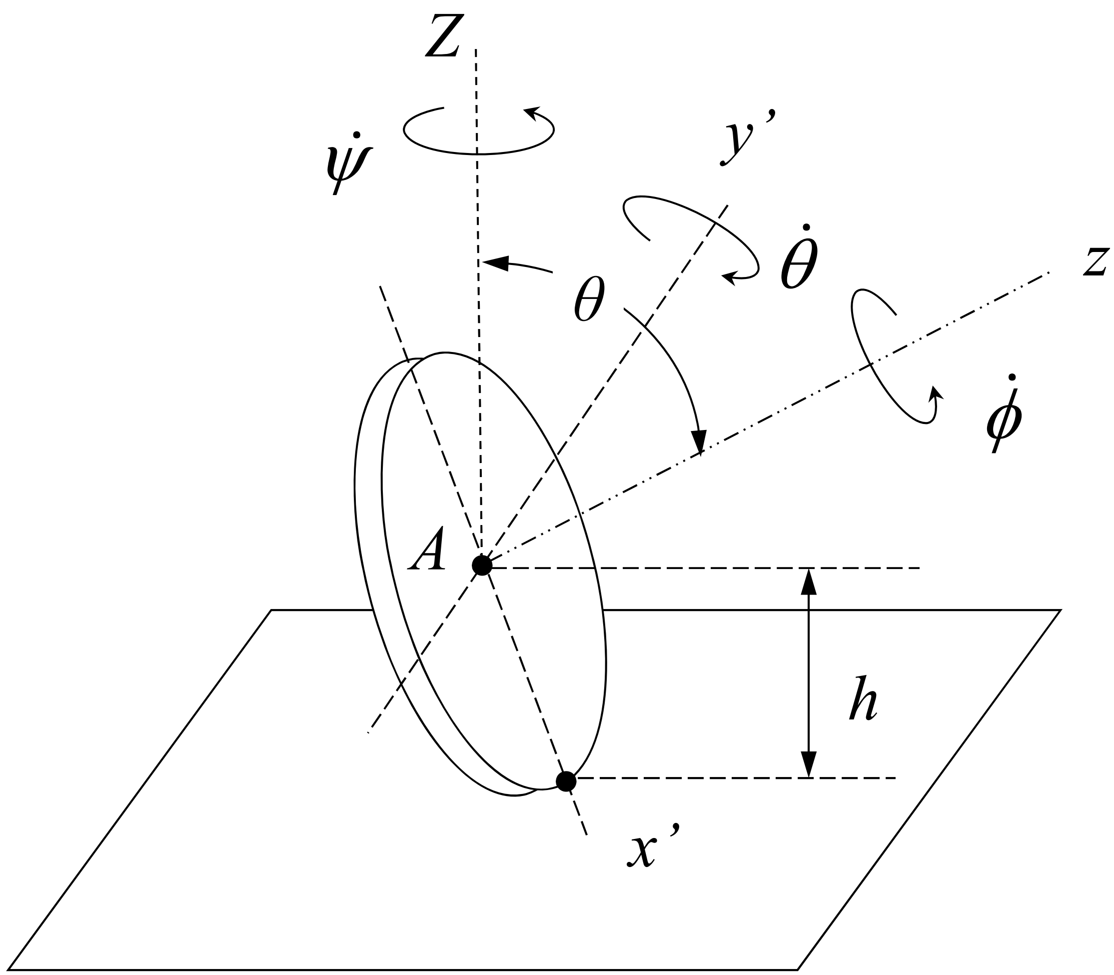
- Acceleration is more complicated.
- Intuitively, it looks like the contact points only accelerate relative to each other perpendicular to the contact plane due to the absence of slipping.
- However, the statement is not true in many cases of spatial motion. (see pp. 143)
- The lack of a direct constraint condition for acceleration represents a dilemma whose resolution lied in the existence of another constraint.
- Round objects care of primary concern. The reference point is the center and the constraint is the distance from a contact point to the center of the round object is constant.



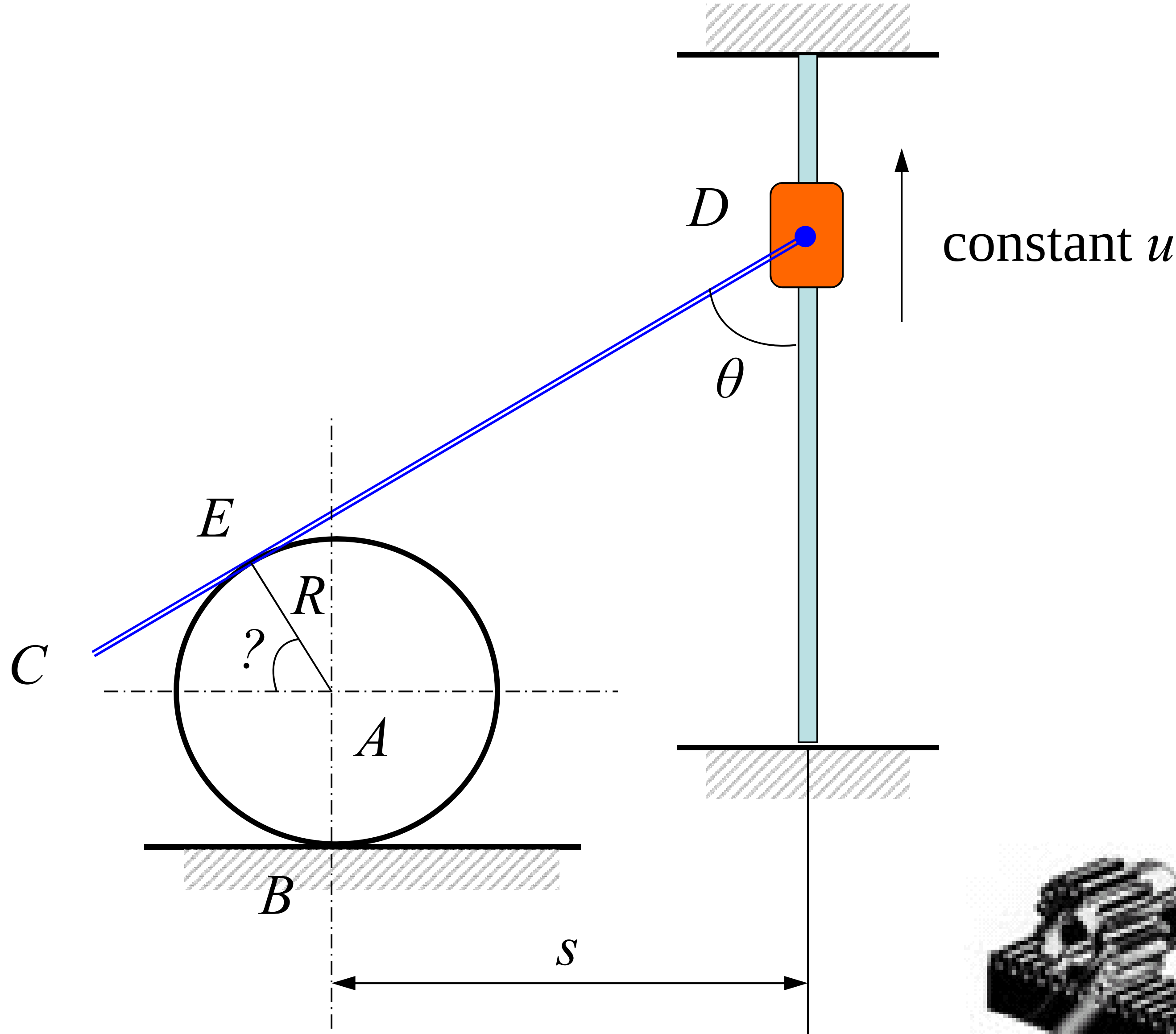
$$\vec{v}_A = v_A \hat{i}, \quad \vec{a}_A = \dot{v}_A \hat{i} - \frac{v_A^2}{r_1 + r_2} \hat{j}$$

$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{A/C}$$
$$= -\omega \hat{k} \times r_1 \hat{j} = \omega r_1 \hat{i}$$
$$\Rightarrow v_A = \omega r_1 \Rightarrow \dot{v}_A = \dot{\omega} r_1$$

Because of the round shape and the restriction to planar motion, the algebraic relation between the speed of each geometric center and the angular speeds of the contacting bodies will not change. Using this information we can completely solve v_A and a_A without consideration of a_C.



Z
$\dot{\psi}$
y'
$\dot{\theta}$
z
θ
$\dot{\phi}$
A
h
x'


D
constant u
θ
E
R
?
C
A
B
s