Example Problems for Chapter 5

The material in the following pages is for today's class discussion.

Please review and let me know if you have any questions.

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I am available from 8:00-10:00 each morning for any questions or discussion that you would like to have.

First, a review of the most important concepts:

2. Angular momentum and inertia properties

2.3 Rate of change of angular momentum

$$\dot{\vec{H}}_A = \frac{\delta \vec{H}_A}{\delta t} + \vec{\omega} \times \vec{H}_A$$

where

$$\frac{\delta \vec{H}_A}{\delta t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z \right)\hat{i} + \left(I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z \right)\hat{j}$$
$$+ \left(I_{zz}\alpha_z - I_{xz}\alpha_z - I_{yz}\alpha_y \right)\hat{k}$$

3. Equation of motion

$$\sum \vec{F} = \frac{d^2}{dt^2}\left(m\vec{r}_{G/A}\right) = m\vec{a}_G \qquad \sum M_A = \frac{\delta \vec{H}_A}{\delta t} + \vec{\omega} \times \vec{H}_A$$

$$\vec{H}_A = \left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right)\hat{i} + \left(I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z\right)\hat{j}$$
$$+\left(I_{zz}\omega_z - I_{xz}\omega_z - I_{yz}\omega_y\right)\hat{k} = \left[I\right]_{xyz}\left\{\omega\right\}_{xyz}$$

$$\frac{\delta \vec{H}_A}{\delta t} = \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z\right)\hat{i} + \left(I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z\right)\hat{j}$$
$$+\left(I_{zz}\alpha_z - I_{xz}\alpha_z - I_{yz}\alpha_y\right)\hat{k} = \left[I\right]_{xyz}\left\{\alpha\right\}_{xyz}$$

1) point A is center of gravity G, or 2) point A has no acceleration

3. Equation of motion

Matrix form
$$\begin{Bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{Bmatrix} = m \begin{Bmatrix} a_{Gx} \\ a_{Gy} \\ a_{Gz} \end{Bmatrix}$$

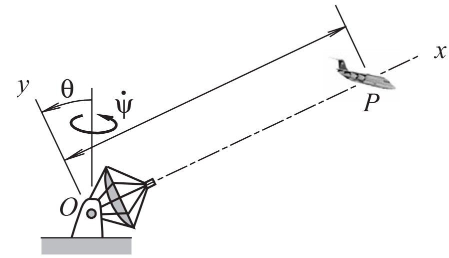
$$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = [I] \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} [I] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Coordinates attached to the body

The special case where *xyz* are principal axes leads to *Euler's equation*:

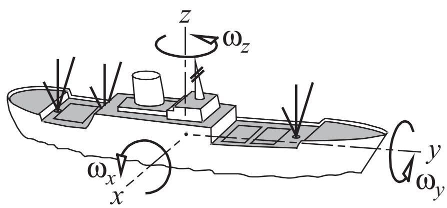
$$\begin{Bmatrix} \sum M_{Ax} \\ \sum M_{Ay} \\ \sum M_{Az} \end{Bmatrix} = \begin{Bmatrix} I_{xx}\alpha_x - \left(I_{yy} - I_{zz}\right)\omega_y\omega_z \\ I_{yy}\alpha_y - \left(I_{zz} - I_{xx}\right)\omega_x\omega_z \\ I_{zz}\alpha_z - \left(I_{xx} - I_{yy}\right)\omega_x\omega_y \end{Bmatrix}$$

EXERCISE 5.33 The radar antenna tracks airplane P by rotating about the vertical axis at $\dot{\psi}$ while the elevation angle θ is adjusted. Assuming that the body-fixed xyz axes are principal, what is the force–couple system acting at the stationary pivot O that must be applied to overcome the inertial resistance when ψ and θ are arbitrary time functions?



Exercise 5.33

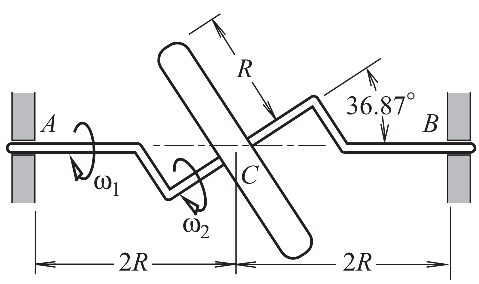
EXERCISE 5.34 The rotation rates of the ship with respect to the body-fixed centroidal xyx coordinate system in the sketch are the pitch ω_x, roll ω_y, and yaw ω_z. Consider the case in which these rates simultaneously attain their maximum magnitudes, with $\omega_x = 0.5$ rad/s, $\omega_y = -1.1$ rad/s, $\omega_z = 0.2$ rad/s. The accelerations of the center of mass at this instant are $a_x = 5\ \text{m/s}^2$, $a_y = -12\ \text{m/s}^2$, $a_z = 15\ \text{m/s}^2$. The mass of the ship is $40\left(10^6\right)$ kg, and the radii of gyration are $\kappa_x = 90$ m, $\kappa_y = 10$ m, $\kappa_z = 15$ m; it may be assumed that xyz are principal axes. Determine the force–couple system acting at the origin of xyz that is equivalent to the forces exerted on the ship by the ocean.



Exercise 5.34

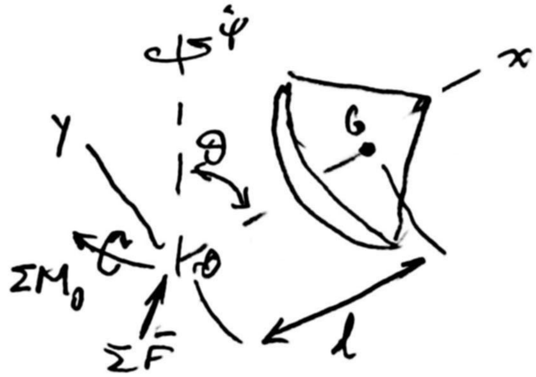
EXERCISE 5.35 A very thin circular disk rolls without slipping relative to the ground such that its plane is oriented vertically throughout the motion. Consider the situation in which the center C of the disk follows a circular path of radius ρ. Determine $\bar{H}_C$ and $d\bar{H}_C/dt$. From those results explain why the condition that the plane of the disk be vertical can be satisfied only if the center follows a straight path, unless forces other than gravity and the contact force are exerted on the disk.

EXERCISE 5.36 The device shown is a wobble plate, in which the precession rate ω_1 of shaft AB and the spin rate ω_2 of the disk relative to the shaft are constant. The mass of the shaft is negligible. Let λ denote the ratio of the angular speeds, such that $\omega_2 = \lambda\omega_1$. (a) In terms of ω_1 and λ, derive expressions for the angular velocity, angular momentum H_C relative to the center of the disk, and $d\bar{H}_C/dt$. (b) Evaluate the results in Part (a) for the case in which $\lambda = 3$, and draw a sketch depicting these quantities. Determine the magnitude of each and the angle between each quantity and the bearing axis AB. (c) Determine whether there is any value of λ for which no



Exercise 5.36

Exercise 5.33



Given arbitrary $\dot{\psi}$ & θ, xyz are principal axes.

Find $\Sigma \bar{F}$ & $\Sigma \bar{M}_0$.

Solution; Assume center of mass is on x axis $\Rightarrow \bar{r}_{G/O} = l\bar{i}$

Let $I_{xx} = I_1$, $I_{yy} = I_2$, $I_{zz} = I_3$

$$\bar{\omega} = \dot{\psi}\bar{e}_1 + \dot{\theta}\bar{e}_2, \quad \bar{\alpha} = \ddot{\psi}\bar{e}_1 + \ddot{\theta}\bar{e}_2 + \dot{\theta}(\dot{\psi}\bar{e}_1 \times \bar{e}_2)$$

Set $\bar{e}_1 = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}$, $\bar{e}_2 = -\bar{k}$

so $\bar{\omega} = \dot{\psi}\cos\theta\,\bar{i} + \dot{\psi}\sin\theta\,\bar{j} - \dot{\theta}\,\bar{k}$

$$\bar{\alpha} = (\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\bar{i} + (\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta)\bar{j} - \ddot{\theta}\bar{k}$$

Also $\bar{a}_G = \bar{\alpha} \times \bar{r}_{G/O} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{G/O})$, $\bar{r}_{G/O} = l\,\bar{i}$

$$\bar{H}_o = I_1\dot{\psi}\cos\theta\,\bar{i} + I_2\dot{\psi}\sin\theta\,\bar{j} - I_3\dot{\theta}\,\bar{k}$$

$$\partial\bar{H}_o/\partial t = I_1(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta)\bar{i} + I_2(\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}\cos\theta)\bar{j} - I_3\ddot{\theta}\bar{k}$$

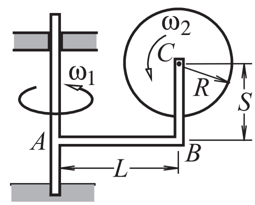
$$\bar{M}_o = \dot{\bar{H}}_o = \partial\bar{H}_o/\partial t + \bar{\omega} \times \bar{H}_o$$

$$= [I_1\ddot{\psi}\cos\theta - (I_1 - I_2 + I_3)\dot{\psi}\dot{\theta}\sin\theta]\bar{i} + [I_2\ddot{\psi}\sin\theta - (I_1 - I_2 - I_3)\dot{\psi}\dot{\theta}\cos\theta]\bar{j} - [I_3\ddot{\theta} + (I_1 - I_2)\dot{\psi}^2\sin\theta\cos\theta]\bar{k} \quad \triangleleft$$

$$\Sigma\bar{F} = -ml[\dot{\theta}^2 + \dot{\psi}^2(\sin\theta)^2]\bar{i} - ml[\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta]\bar{j} - ml[\ddot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}\cos\theta]\bar{k} \quad \triangleleft$$

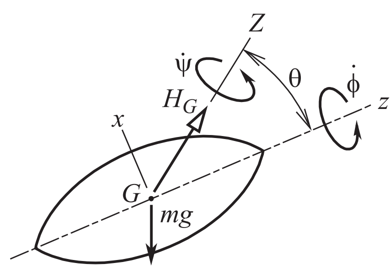
dynamic reactions are generated at bearings A and B. Explain your answer in terms of the properties of H_C.

EXERCISE 5.37 Arm ABC rotates about the vertical axis at constant rate ω_1, and the disk rotates relative to the arm at constant rate ω_2. The mass of the disk is m, and its centroidal moments of inertia are $I_1 = 0.5mR^2$ about its centerline and $I_2 = 0.25mR^2$ about a diameter. (a) Draw one or more sketches depicting the angular momentum $\bar{H}_C$ of the disk about its center. (b) Based on the sketch(es) in Part (a) and the manner in which the system rotates, evaluate $d\bar{H}_C/dt$. (c) Compare the result in Part (b) with what is obtained by evaluating $\partial \bar{H}_C/\partial t + \bar{\omega} \times \bar{H}_C$.



Exercise 5.37

EXERCISE 5.38 The topic of rotation of a body in free motion is treated extensively in Chapter 10. Some key aspects of that study are described in the sketch, which shows a body that is axisymmetric about the body-fixed z axis. The moment of inertia of the body about this axis is I_1, and the moment of inertia about any axis intersecting the center of mass and perpendicular to z is I_2. The body is in free flight and air resistance is negligible, so the only force acting on the body is its weight. Because this force acts at the center of mass G, $\Sigma \bar{M}_G = d\bar{H}_G/dt \equiv \bar{0}$, so the angular momentum $\bar{H}_G$ is constant. Let the fixed Z axis denote the constant direction of $\bar{H}_G$. Eulerian angles are used to describe the rotation of the body, with precession angle ψ being defined as the rotation about the fixed Z axis and the spin angle ϕ being the rotation about the z axis of symmetry. The nutation angle θ is the angle between these two axes, as shown in the sketch. (a) Describe the angular velocity of the body in terms of ψ, θ, and ϕ. Use this description of $\bar{\omega}$ to derive an expression for $\bar{H}_G$. (b) Derive a component description for $\bar{H}_G$ based on the fact that $\bar{H}_G$ is parallel to the Z axis. Match this description to the expression for $\bar{H}_G$ in Part (a). Show that the two descriptions are consistent only if the nutation angle is constant, that is, $\dot{\theta} = 0$. (c) From the fact that $\dot{\theta} = 0$, it follows that at any instant the angular velocity $\bar{\omega}$ must lie in the xz plane, so that $\bar{\omega} = \omega \sin\beta\, \bar{i} + \omega \cos\beta\, \bar{k}$, where β is the angle between $\bar{\omega}$ and the z axis, and the x axis lies in the plane formed by Z and z. Compare the descriptions of $\bar{\omega}$ and $\bar{H}_G$ in terms of ω and β with the corresponding expressions in terms of ψ, θ, and ϕ. From that comparison, derive the expression

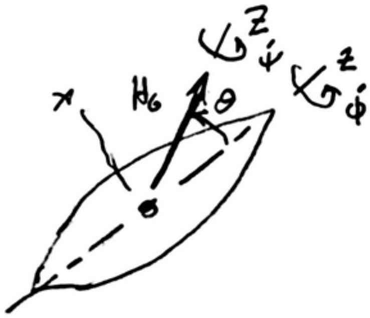


Exercise 5.38

$$\tan\beta = \frac{I_1}{I_2}\tan\theta.$$

(d) Derive an expression for $\dot{\psi}$ in terms of $\dot{\phi}$ and θ.

Exercise 5.38



Given $I_{xx} = I_{yy} = I_2$, $I_{zz} = I_1$, constant $\bar{H}_G$.
Find (a) $\bar{\omega}$ & $\bar{H}_G$ in terms of Eulerian angles, (b) show that $\dot{\theta} = 0$, (c) relation for angle β of $\bar{\omega}$ relative to xyz, (d) expression for $\dot{\psi}$

solution: $\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k} = \dot{\psi}\sin\theta\,\bar{i} + \dot{\theta}\,\bar{j} + (\dot{\phi} + \dot{\psi}\cos\theta)\bar{k}$

$\bar{H}_G = I_2\dot{\psi}\sin\theta\,\bar{i} + I_2\dot{\theta}\,\bar{j} + I_1(\dot{\phi} + \dot{\psi}\cos\theta)\bar{k}$ ◁

but $\bar{H}_G = H_G\bar{K} = H_G\sin\theta\,\bar{i} + H_G\cos\theta\,\bar{k}$

so $\bar{H}_G \cdot \bar{j} = 0 = I_2\dot{\theta} \Rightarrow \theta$ is constant ◁

set $\bar{\omega} = \omega\sin\beta\,\bar{i} + \omega\cos\beta\,\bar{k} \Rightarrow \bar{H}_G = I_2\omega\sin\beta\,\bar{i} + I_1\omega\cos\beta\,\bar{k}$

so $\bar{H}_G \cdot \bar{i} = I_2\omega\sin\theta = I_2\dot{\psi}\sin\theta = H_G\sin\theta$

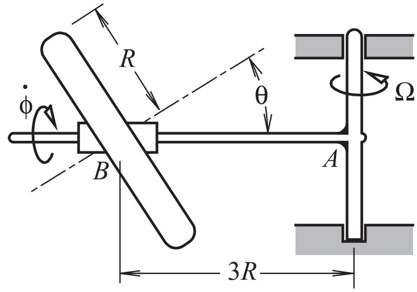
$\bar{H}_G \cdot \bar{k} = I_1\omega\cos\beta = I_1(\dot{\phi} + \dot{\psi}\cos\theta) = H_G\cos\theta$

$$\frac{\bar{H}_G \cdot \bar{i}}{\bar{H}_G \cdot \bar{k}} = \frac{I_2}{I_1}\tan\beta = \frac{I_2\dot{\psi}\sin\theta}{I_1(\dot{\phi} + \dot{\psi}\cos\theta)} = \tan\theta$$ ◁

Also from $\bar{H}_G \cdot \bar{i}$; $I_2\dot{\psi} = H_G$ ◁

substitute into $\bar{H}_G \cdot \bar{k}$: $I_1(\dot{\phi} + \dot{\psi}\cos\theta) = I_2\dot{\psi}\cos\theta$

or $$\dot{\psi} = \frac{I_1\dot{\phi}}{(I_2 - I_1)\cos\theta}$$ ◁

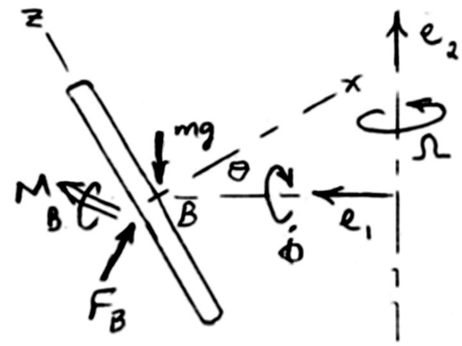


Exercise 5.39

EXERCISE 5.39 The disk is mounted obliquely on its hub, which spins at angular speed Ω about the horizontal arm AB of the T-bar. Consequently, the disk's center line forms a constant angle θ relative to arm AB. This rotation rate is $\dot{\phi}$, with $\phi = 0$ corresponding to the instant depicted in the sketch, where the disk's axis is situated in the vertical plane. Both this rotation rate and the precession rate Ω are constant. The disk's mass is m, whereas the hub and both shafts have negligible mass. Derive an expression for the force–couple system exerted on the disk by hub B when $\phi = 0$.

EXERCISE 5.40 Determine the force–couple systems in Exercise 5.39 as a function of the rotation angle ϕ.

Exercise 5.39



Given constant Ω and $\dot\phi$.

Find $\bar{F}_B$ & $\bar{M}_B$ at position shown.

Solution: $I_{xx} = \frac{1}{2}mR^2$, $I_{yy} = I_{zz} = \frac{1}{4}mR^2$

$\bar\omega = \dot\phi \bar{e}_1 + \Omega \bar{e}_2$, $\bar\alpha = \dot\phi \Omega \bar{e}_2 \times \bar{e}_1$

Set $\bar{e}_1 = -\cos\theta\,\bar{i} + \sin\theta\,\bar{k}$

$\bar{e}_2 = \sin\theta\,\bar{i} + \cos\theta\,\bar{k}$

$\bar\omega = (\Omega\sin\theta - \dot\phi\cos\theta)\bar{i} + (\Omega\cos\theta + \dot\phi\sin\theta)\bar{k}$, $\bar\alpha = -\dot\phi\Omega\bar{j}$

Circular motion $\Rightarrow$ $\bar{a}_B = -3R\Omega^2\bar{e}_1$

$\bar{H}_B = \frac{1}{2}mR^2(\Omega\sin\theta - \dot\phi\cos\theta)\bar{i} + \frac{1}{4}mR^2(\Omega\cos\theta + \dot\phi\sin\theta)\bar{k}$

$\frac{\partial \bar{H}_B}{\partial t} = -\frac{1}{4}mR^2\dot\phi\Omega\bar{j}$

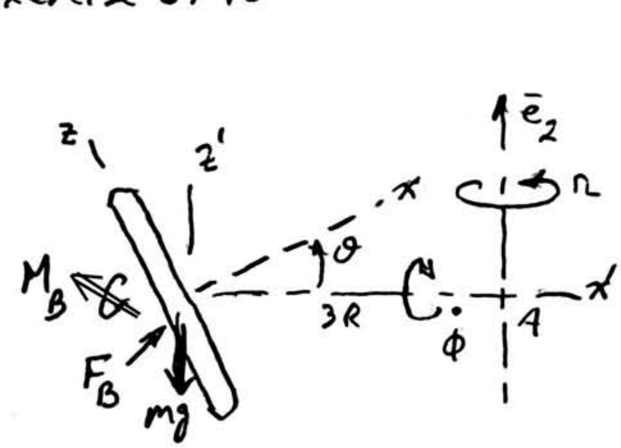
$\Sigma\bar{F} = \bar{F}_B - mg\bar{e}_2 = m\bar{a}_B \Rightarrow \bar{F}_B = -3mR\Omega^2\bar{e}_1 - mg\bar{e}_2$ ◁

$\Sigma\bar{M}_B = \bar{M}_B = \frac{\partial \bar{H}_B}{\partial t} + \bar\omega \times \bar{H}_B$

$= mR^2[-\frac{1}{4}\Omega\dot\phi + \frac{1}{4}(\Omega\sin\phi - \dot\phi\cos\phi)(\Omega\cos\phi + \dot\phi\sin\phi)]$

$= mR^2[-\frac{1}{2}\Omega\dot\phi(\cos\theta)^2 + (\Omega^2 - \dot\phi)\sin\phi\cos\phi]$ ◁

Exercise 5.40



Given constant Ω & $\dot{\phi}$ with $\phi = 0$ in illustrated position. Find $\bar{F}_B$ & $\bar{M}_B$ at arbitrary ϕ.

Solution: Attach xyz to the disk and $x'y'z'$ to the T-bar.

$$\bar{\omega} = \Omega \bar{k}' - \dot{\phi}\bar{i}', \quad \bar{\alpha} = -\dot{\phi}(\Omega \bar{k}' \times \bar{i}') = -\Omega\dot{\phi}\bar{j}'$$

Transform to xyz: Starting from $x'y'z'$, rotate by $-\theta$ about y' axis, then by $-\phi$ about original x' axis. Body-fixed rotation sequence $\Rightarrow [x\ y\ z]^T = [R][x'\ y'\ z']^T$

$$[R] = [R_y(-\theta)][R_x(-\phi)] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Then
$$\begin{Bmatrix} \bar{\omega}\cdot\bar{i} \\ \bar{\omega}\cdot\bar{j} \\ \bar{\omega}\cdot\bar{k} \end{Bmatrix} = [R]\begin{Bmatrix} -\dot{\phi} \\ 0 \\ \Omega \end{Bmatrix}, \quad \begin{Bmatrix} \bar{\alpha}\cdot\bar{i} \\ \bar{\alpha}\cdot\bar{j} \\ \bar{\alpha}\cdot\bar{k} \end{Bmatrix} = [R]\begin{Bmatrix} 0 \\ -\Omega\dot{\phi} \\ 0 \end{Bmatrix}$$

Also $I_{xx} = \frac{1}{2}mR^2$, $I_{yy} = I_{zz} = \frac{1}{4}mR^2$, principal axes

$$\{\dot{H}_G\} = [I]\begin{Bmatrix} \bar{\alpha}\cdot\bar{i} \\ \bar{\alpha}\cdot\bar{j} \\ \bar{\alpha}\cdot\bar{k} \end{Bmatrix} + \begin{Bmatrix} \bar{\omega}\cdot\bar{i} \\ \bar{\omega}\cdot\bar{j} \\ \bar{\omega}\cdot\bar{k} \end{Bmatrix} \otimes [I]\begin{Bmatrix} \bar{\omega}\cdot\bar{i} \\ \bar{\omega}\cdot\bar{j} \\ \bar{\omega}\cdot\bar{k} \end{Bmatrix}$$

Also $\bar{a}_B = 3R\Omega^2\bar{i}'$

so $\Sigma\bar{F} = \bar{F}_B - mg\bar{k}' = m\bar{a}_B \Rightarrow \bar{F}_B = 3mR\Omega^2\bar{i}' + mg\bar{k}'$ ◁

$$\Sigma\bar{M}_B = \bar{M}_B = \dot{\bar{H}}_B = \frac{1}{2}mR^2\Omega\dot{\phi}\sin\phi\sin\theta\,\bar{i}$$
$$+ mR^2\left[\frac{1}{4}\left((\Omega\cos\phi)^2 - \dot{\phi}^2\right)\sin\theta\cos\theta - \frac{1}{2}\Omega\dot{\phi}\cos\phi(\cos\theta)^2\right]\bar{j} + mR^2\left[\frac{1}{4}\Omega^2\sin\phi\cos\phi\sin\theta - \frac{1}{2}\Omega\dot{\phi}\sin\phi\cos\theta\right]\bar{k}$$ ◁