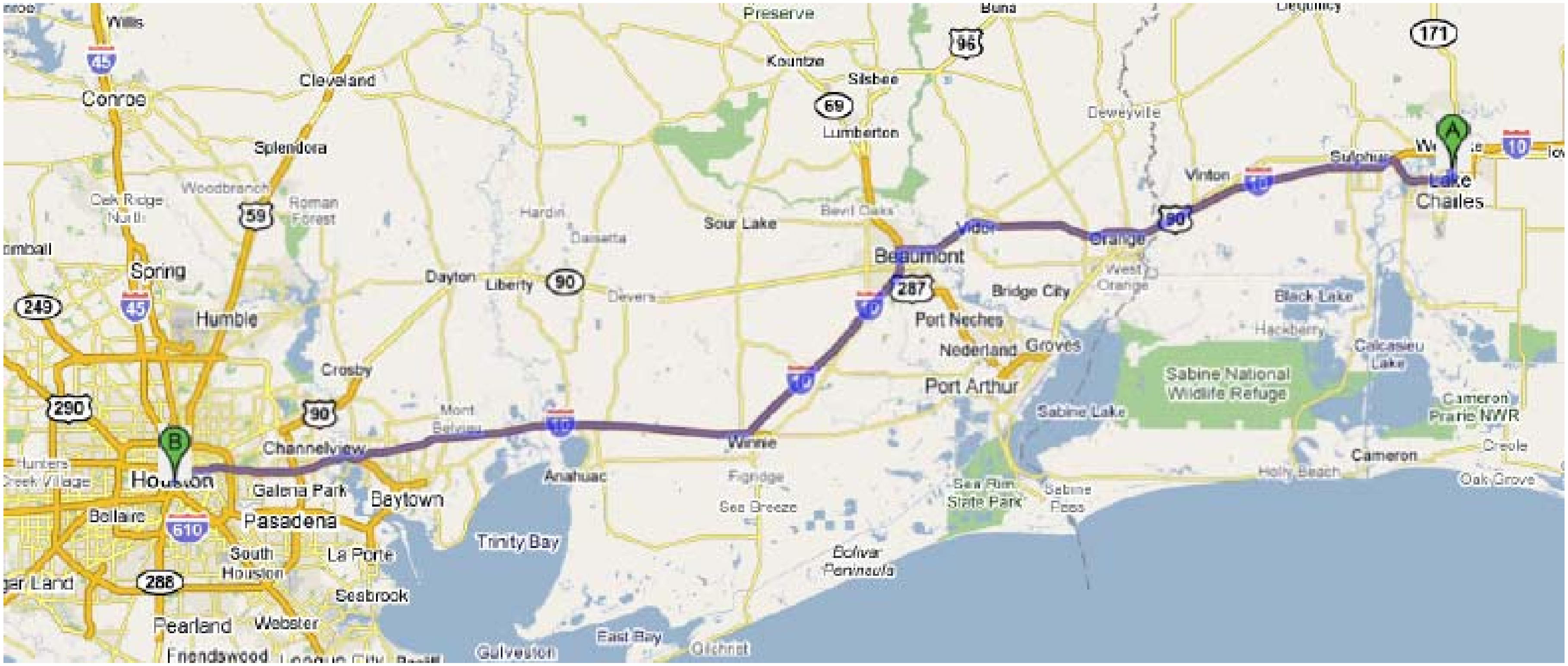
2. Particle Kinematics

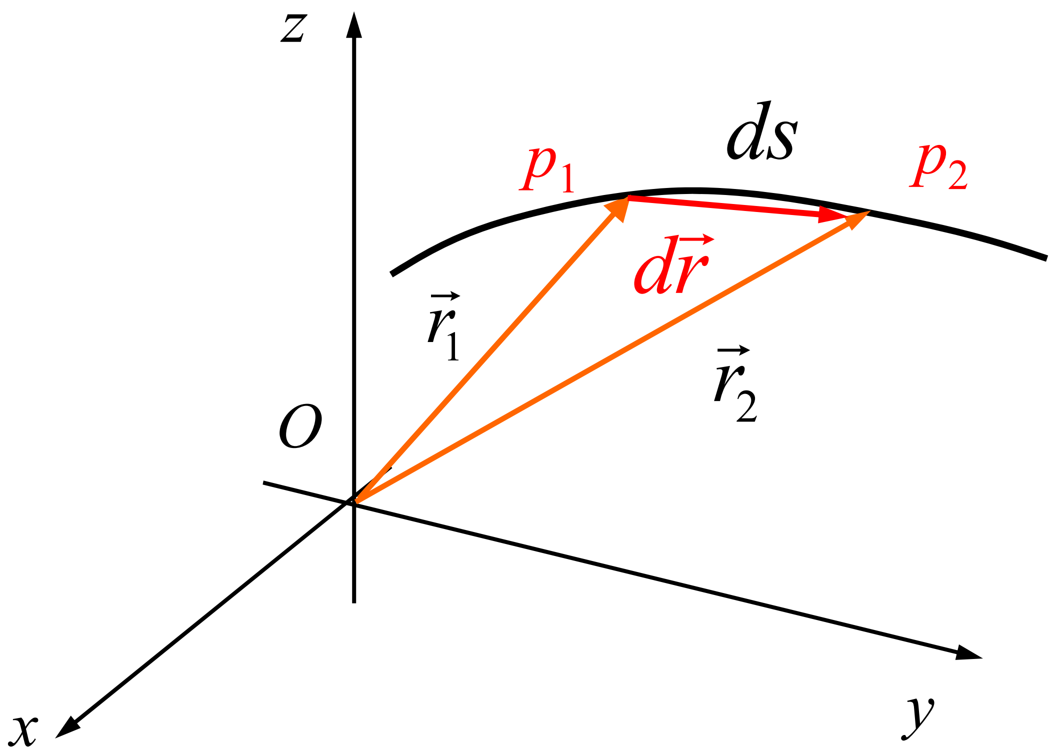
Outline

- Path variables – normal and tangential
- Cartesian coordinates
- Orthogonal curvilinear coordinates
- Joint kinematical descriptions

2.1 Path Variables



2.1.1 Frenet's Formulas



$$\frac{ds}{dt} = v$$

$$\hat{e}_t = \frac{d\vec{r}}{ds}$$

$$\hat{e}_t \bullet \hat{e}_t = 1$$

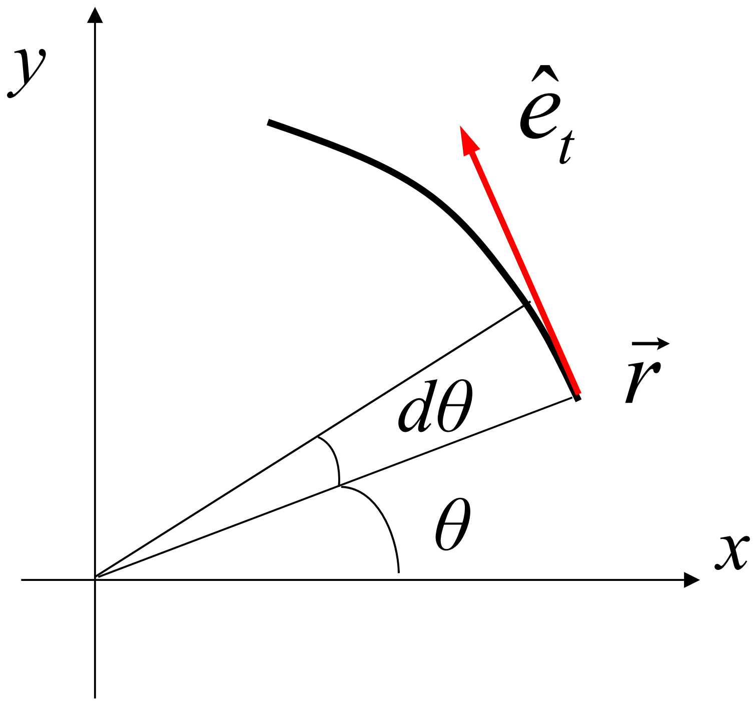
Differentiate $\rightarrow$ $$2\hat{e}_t \bullet \frac{d\hat{e}_t}{ds} = 0$$

This means $\hat{e}_t$ and $\frac{d\hat{e}_t}{ds}$ are perpendicular

Define the unit vector of the normal direction:

$$\hat{e}_n = \rho \frac{d\hat{e}_t}{ds}$$

where ρ is the radius of curvature



$$\hat{e}_t = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\frac{d\hat{e}_t}{d\theta} = -\cos\theta\hat{i} - \sin\theta\hat{j}$$

$$ds = r\,d\theta$$

$$\frac{d\hat{e}_t}{ds} = \frac{d\hat{e}_t / d\theta}{ds / d\theta} = \frac{1}{r}\left[-\cos\theta\hat{i} - \sin\theta\hat{j}\right]$$

$$\Rightarrow \left|\frac{d\hat{e}_t}{ds}\right| = \frac{1}{r}$$

The plane constructed by $\hat{e}_t$ and $\hat{e}_n$ is called the osculating plane. And the direction perpendicular to the osculating plane is called *binormal*.

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n$$

These three unit vectors form an orthogonal space. Therefore, any vector in the space can be written as the linear combination of these three unit vectors. And the coefficients are simply the projections of the vector onto these three directions.

$$\frac{d\hat{e}_n}{ds} = \left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_t\right)\hat{e}_t + \left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_n\right)\hat{e}_n + \left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_b\right)\hat{e}_b$$

$$\frac{d\hat{e}_n}{ds} = \underbrace{\left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_t\right)}_{-\frac{1}{\rho}} \hat{e}_t + \cancel{\left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_n\right)} \hat{e}_n + \underbrace{\left(\frac{d\hat{e}_n}{ds} \bullet \hat{e}_b\right)}_{\frac{1}{\tau}} \hat{e}_b$$

Define torsion τ: $$\frac{1}{\tau} = \frac{d\hat{e}_n}{ds} \bullet \hat{e}_b$$

$$\boxed{\frac{d\hat{e}_n}{ds} = -\frac{1}{\rho}\hat{e}_t + \frac{1}{\tau}\hat{e}_b}$$

Note: $\hat{e}_t \bullet \hat{e}_n = 0 \Rightarrow \hat{e}_t \bullet \frac{d\hat{e}_n}{ds} + \frac{d\hat{e}_t}{ds} \bullet \hat{e}_n = 0 \Rightarrow \hat{e}_t \bullet \frac{d\hat{e}_n}{ds} = -\frac{d\hat{e}_t}{ds} \bullet \hat{e}_n = -\frac{1}{\rho}$

$\hat{e}_n \bullet \hat{e}_n = 1 \Rightarrow \frac{d\hat{e}_n}{ds} \bullet \hat{e}_n = 0$

$$\frac{d\hat{e}_b}{ds} = \left(\cancel{\frac{d\hat{e}_b}{ds} \bullet \hat{e}_t}\right)\hat{e}_t + \underbrace{\left(\frac{d\hat{e}_b}{ds} \bullet \hat{e}_n\right)}_{-\frac{1}{\tau}}\hat{e}_n + \left(\cancel{\frac{d\hat{e}_b}{ds} \bullet \hat{e}_b}\right)\hat{e}_b$$

Note:

$$\hat{e}_t \bullet \hat{e}_b = 0 \Rightarrow \frac{d\hat{e}_t}{ds} \bullet \hat{e}_b + \hat{e}_t \bullet \frac{d\hat{e}_b}{ds} = 0 \Rightarrow \hat{e}_t \bullet \frac{d\hat{e}_b}{ds} = -\frac{d\hat{e}_t}{ds} \bullet \hat{e}_b = -\frac{1}{\rho}\hat{e}_n \bullet \hat{e}_b = 0$$

$$\hat{e}_b \bullet \hat{e}_n = 0 \Rightarrow \frac{d\hat{e}_b}{ds} \bullet \hat{e}_n + \hat{e}_b \bullet \frac{d\hat{e}_n}{ds} = 0 \Rightarrow \frac{d\hat{e}_b}{ds} \bullet \hat{e}_n = -\hat{e}_b \bullet \frac{d\hat{e}_n}{ds} = -\hat{e}_b \bullet \left(\frac{1}{\tau}\hat{e}_b\right) = -\frac{1}{\tau}$$

$$\boxed{\frac{d\hat{e}_b}{ds} = -\frac{1}{\tau}\hat{e}_n} \qquad \frac{1}{\tau} = \left|\frac{d\hat{e}_b}{ds}\right|$$

2.1.1 Frenet's Formulas (summary)

$$\boxed{\frac{d\hat{e}_t}{ds} = \frac{1}{\rho}\hat{e}_n}$$

$$\boxed{\frac{d\hat{e}_n}{ds} = -\frac{1}{\rho}\hat{e}_t + \frac{1}{\tau}\hat{e}_b}$$

$$\boxed{\frac{d\hat{e}_b}{ds} = -\frac{1}{\tau}\hat{e}_n}$$

2.1.2 Parametric form

$$\vec{r} = x(\alpha)\hat{i} + y(\alpha)\hat{j} + z(\alpha)\hat{k}$$

$$\hat{e}_t = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{d\alpha}\frac{d\alpha}{ds} = \frac{\vec{r}\,'}{s\,'}$$

where $$\vec{r}\,' = x'(\alpha)\hat{i} + y'(\alpha)\hat{j} + z'(\alpha)\hat{k}$$

$$\hat{e}_t \bullet \hat{e}_t = 1 \Rightarrow s\,' = \sqrt{\vec{r}\,' \bullet \vec{r}\,'} = \sqrt{\left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2}$$

$$\Rightarrow s = \int_{\alpha_0}^{\alpha} \left[\left(x'\right)^2 + \left(y'\right)^2 + \left(z'\right)^2 \right] d\alpha$$

$$\hat{e}_n = \rho \frac{d\hat{e}_t}{ds} = \rho \frac{d\hat{e}_t}{d\alpha}\frac{d\alpha}{ds} = \frac{\rho}{s'}\left[\frac{\vec{r}\,''}{s'} - \frac{\vec{r}\,'s''}{\left(s'\right)^2}\right]$$

$$= \frac{\rho}{\left(s'\right)^3}\left(\vec{r}\,''s' - \vec{r}\,'s''\right)$$

$$s' = \sqrt{\vec{r}\,'\bullet\vec{r}\,'} \Rightarrow s'' = \frac{1}{2}\frac{\vec{r}\,''\bullet\vec{r}\,' + \vec{r}\,'\bullet\vec{r}\,''}{\sqrt{\vec{r}\,'\bullet\vec{r}\,'}} = \frac{\vec{r}\,'\bullet\vec{r}\,''}{s'}$$

$$\hat{e}_n = \frac{\rho}{\left(s'\right)^4}\left[\vec{r}\,''\left(s'\right)^2 - \vec{r}\,'\left(\vec{r}\,'\bullet\vec{r}\,''\right)\right]$$

$$\frac{1}{\rho} = \frac{1}{\left(s'\right)^3}\left[\left(\vec{r}\,''\bullet\vec{r}\,''\right)\left(s'\right)^2 - \left(\vec{r}\,'\bullet\vec{r}\,''\right)^2\right]^{1/2}$$

In the case of planar motion in which $z = 0$, the radius of curvature can be simplified as

$$\frac{1}{\rho} = \frac{\left[1+\left(y'\right)^2\right]^{3/2}}{\left|y''\right|} = \frac{\left[1+\left(x'\right)^2\right]^{3/2}}{\left|x''\right|}$$

$$\hat{e}_b = \hat{e}_t \times \hat{e}_n = \frac{\vec{r}\,'}{s'} \times \frac{\rho}{\left(s'\right)^4}\left[\vec{r}\,''\left(s'\right)^2 - \vec{r}\,'\left(\vec{r}\,'\bullet\vec{r}\,''\right)\right]$$

$$= \frac{\rho}{\left(s'\right)^3}\vec{r}\,'\times\vec{r}\,''$$

$$\frac{1}{\tau} = -\frac{\rho^2}{\left(s'\right)^6}\left[\vec{r}\,''\bullet\left(\vec{r}\,'\times\vec{r}\,'''\right)\right]$$

2.1.3 Kinematrical relations

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\hat{e}_t, \text{ where } v = \frac{ds}{dt}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{dt} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{ds}\dot{s} = \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{ds}$$

$$\Rightarrow \vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

where $\dot{v} = v\dfrac{dv}{ds}$ is the tangential acceleration and

$\dfrac{v^2}{\rho}$ is the centripetal acceleration

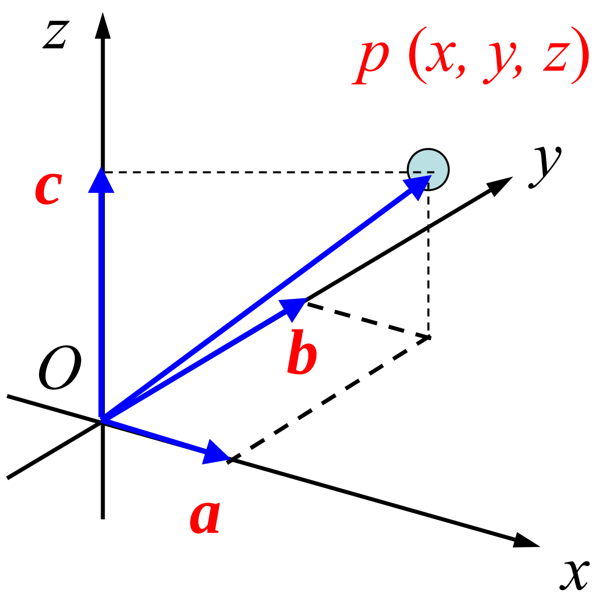
2.2 Rectangular Cartesian Coordinates

- The simplest set of extrinsic coordinates
- *xyz* axes are orthogonal and right-handed
- Motions in the *x*-, *y*- and *z*-directions are uncoupled. That means none of the motion parameters for one direction appears in another direction.
- The *xyz* axes may be functions of time

$$\vec{r}_{P/O} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}; \quad v_x = \dot{x},\ v_y = \dot{y},\ v_z = \dot{z}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}; \quad a_x = \dot{v}_x,\ a_y = \dot{v}_y,\ a_z = \dot{v}_z$$



2.2 Rectangular Cartesian Coordinates

- Limitations
  - Projectile motion near earth's surface
  - When motion covers a long range
- The use of curvilinear coordinates is more suitable for this type of problem.

2.3 Orthogonal curvilinear coordinates

- Polar coordinates (2D)

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \dot{r}\hat{e}_r + \dot{\theta}r\hat{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

2.3.1 Coordinates and Unit Vectors

$$x = x(\alpha,\beta,\gamma),\ y = y(\alpha,\beta,\gamma),\ z = z(\alpha,\beta,\gamma)$$

$$\alpha = \alpha(x,y,z),\ \beta = \beta(x,y,z),\ \gamma = \gamma(x,y,z)$$

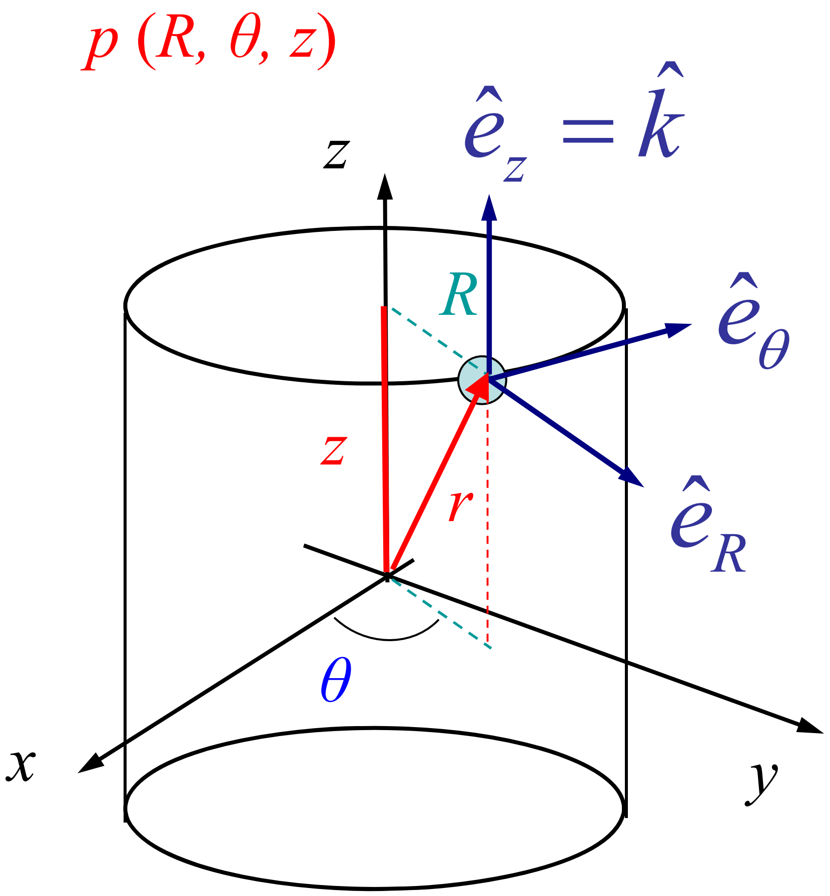
For any particle

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$ds_\lambda = h_\lambda d\lambda,\quad \lambda = \alpha,\ \beta,\ \text{or } \gamma$$

$$\hat{e}_\lambda = \frac{\partial\vec{r}}{\partial s_\lambda} = \frac{1}{h_\lambda}\left(\frac{\partial\vec{r}}{\partial\lambda}\right)$$

- Cylindrical coordinate system



transform

$$R = \sqrt{x^2 + y^2}$$

$$\theta = \arctan \frac{y}{x}$$

$$z = z$$

Inverse transform

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$z = z$$

- Cylindrical coordinate system

$$\vec{r} = R\cos\theta\hat{i} + R\sin\theta\,\hat{j} + z\hat{k}$$

$$\hat{e}_R = \frac{1}{h_R}\frac{\partial\vec{r}}{\partial R} = \frac{1}{h_R}\left(\cos\theta\hat{i} + \sin\theta\,\hat{j}\right)$$

since $\hat{e}_R \bullet \hat{e}_R = 1 \Rightarrow \underline{h_R = 1};$

Similarly, $\hat{e}_\theta = \dfrac{1}{h_\theta}\dfrac{\partial\vec{r}}{\partial\theta} = \dfrac{1}{h_\theta}\left(-R\sin\theta\hat{i} + R\cos\theta\,\hat{j}\right)$

$$\hat{e}_\theta \bullet \hat{e}_\theta = 1 \Rightarrow \underline{h_\theta = R};$$

and $\underline{h_z = 1}.$

2.3.2 Kinematical formulas

$$\vec{r} = \vec{r}(\alpha, \beta, \gamma)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}(\alpha, \beta, \gamma)$$

$$= \frac{\partial \vec{r}}{\partial \alpha}\dot{\alpha} + \frac{\partial \vec{r}}{\partial \beta}\dot{\beta} + \frac{\partial \vec{r}}{\partial \gamma}\dot{\gamma}$$

$$= h_\alpha \dot{\alpha} \hat{e}_\alpha + h_\beta \dot{\beta} \hat{e}_\beta + h_\gamma \dot{\gamma} \hat{e}_\gamma$$

$$= \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \dot{\lambda} \hat{e}_\lambda$$

2.3.2 Kinematical formulas

$$\vec{v} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \dot{\lambda} \hat{e}_\lambda$$

$$\vec{a} = \sum_{\lambda=\alpha,\beta,\gamma} \left[h_\lambda \ddot{\lambda} \hat{e}_\lambda + \sum_{\mu=\alpha,\beta,\gamma} \frac{\partial h_\lambda}{\partial \mu} \dot{\mu} \dot{\lambda} \hat{e}_\lambda + \sum_{\mu=\alpha,\beta,\gamma} h_\lambda \dot{\lambda} \frac{\partial \hat{e}_\lambda}{\partial \mu} \dot{\mu} \right]$$

$$\vec{a} = \sum_{\lambda=\alpha,\beta,\gamma} \left[h_\lambda \ddot{\lambda} \hat{e}_\lambda + \sum_{\mu=\alpha,\beta,\gamma} \left(\frac{\partial h_\lambda}{\partial \mu} \hat{e}_\lambda + h_\lambda \frac{\partial \hat{e}_\lambda}{\partial \mu} \right) \dot{\mu} \dot{\lambda} \right]$$

- Cylindrical coordinate system

recall $$\vec{r} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j} + z\hat{k}$$

$$h_R = 1,\ h_\theta = R,\ h_z = 1$$

$$\Longrightarrow \quad \vec{v} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \dot{\lambda}\hat{e}_\lambda = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

$$\vec{a} = \underbrace{\ddot{R}\hat{e}_R + \left(\frac{\partial h_R}{\partial R}\hat{e}_R + h_R\frac{\partial \hat{e}_R}{\partial R}\right)\dot{R}^2 + \left(\frac{\partial h_R}{\partial \theta}\hat{e}_R + h_R\frac{\partial \hat{e}_R}{\partial \theta}\right)\dot{R}\dot{\theta} + \left(\frac{\partial h_R}{\partial z}\hat{e}_R + h_R\frac{\partial \hat{e}_R}{\partial z}\right)\dot{R}\dot{z}}_{\lambda=R}$$

$$\underbrace{+ R\ddot{\theta}\hat{e}_\theta + \left(\frac{\partial h_\theta}{\partial R}\hat{e}_\theta + h_\theta\frac{\partial \hat{e}_\theta}{\partial R}\right)\dot{\theta}\dot{R} + \left(\frac{\partial h_\theta}{\partial \theta}\hat{e}_\theta + h_\theta\frac{\partial \hat{e}_\theta}{\partial \theta}\right)\dot{\theta}^2 + \left(\frac{\partial h_\theta}{\partial z}\hat{e}_\theta + h_\theta\frac{\partial \hat{e}_\theta}{\partial z}\right)\dot{\theta}\dot{z}}_{\lambda=\theta}$$

$$\underbrace{+ \ddot{z}\hat{e}_z + \left(\frac{\partial h_z}{\partial R}\hat{e}_z + h_z\frac{\partial \hat{e}_z}{\partial R}\right)\dot{z}\dot{R} + \left(\frac{\partial h_z}{\partial \theta}\hat{e}_z + h_z\frac{\partial \hat{e}_z}{\partial \theta}\right)\dot{z}\dot{\theta} + \left(\frac{\partial h_z}{\partial z}\hat{e}_z + h_z\frac{\partial \hat{e}_z}{\partial z}\right)\dot{z}^2}_{\lambda=z}$$

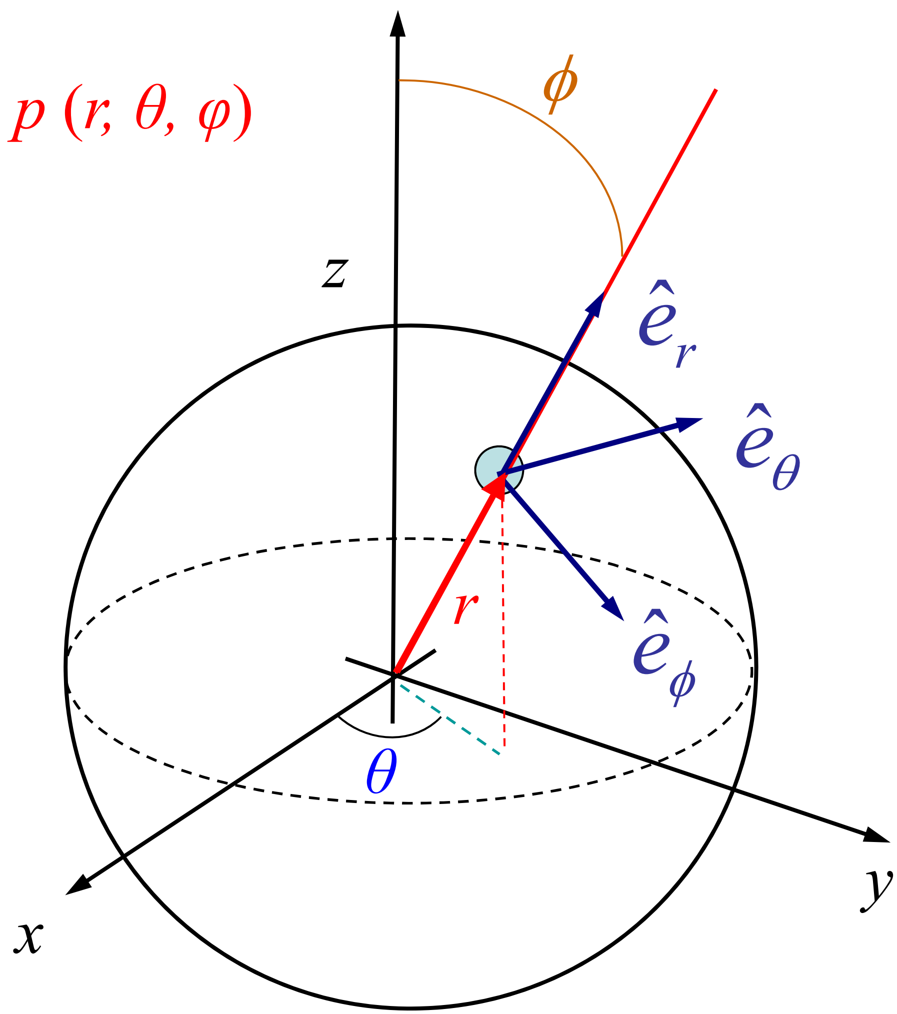
$$\vec{a} = \ddot{R}\hat{e}_R + \left(\cancel{\frac{\partial h_R}{\partial R}}\hat{e}_R + h_R \cancel{\frac{\partial \hat{e}_R}{\partial R}} \right)\dot{R}^2 + \left(\cancel{\frac{\partial h_R}{\partial \theta}}\hat{e}_R + h_R \underbrace{\frac{\partial \hat{e}_R}{\partial \theta}}_{\left(-\sin\theta\hat{i}+\cos\theta\hat{j}\right)=\hat{e}_\theta} \right)\dot{R}\dot{\theta} + \left(\cancel{\frac{\partial h_R}{\partial z}}\hat{e}_R + h_R \cancel{\frac{\partial \hat{e}_R}{\partial z}} \right)\dot{R}\dot{z}$$

$$+ R\ddot{\theta}\hat{e}_\theta + \left(\underbrace{\frac{\partial h_\theta}{\partial R}}_{1}\hat{e}_\theta + h_\theta \cancel{\frac{\partial \hat{e}_\theta}{\partial R}} \right)\dot{\theta}\dot{R} + \left(\cancel{\frac{\partial h_\theta}{\partial \theta}}\hat{e}_\theta + \underbrace{h_\theta}_{R} \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{-\hat{e}_R} \right)\dot{\theta}^2 + \left(\cancel{\frac{\partial h_\theta}{\partial z}}\hat{e}_\theta + h_\theta \cancel{\frac{\partial \hat{e}_\theta}{\partial z}} \right)\dot{\theta}\dot{z}$$

$$+ \ddot{z}\hat{e}_z + \cancel{\left(\frac{\partial h_z}{\partial R}\hat{e}_z + h_z \frac{\partial \hat{e}_z}{\partial R} \right)}\dot{z}\dot{R} + \cancel{\left(\frac{\partial h_z}{\partial \theta}\hat{e}_z + h_z \frac{\partial \hat{e}_z}{\partial \theta} \right)}\dot{z}\dot{\theta} + \cancel{\left(\frac{\partial h_z}{\partial z}\hat{e}_z + h_z \frac{\partial \hat{e}_z}{\partial z} \right)}\dot{z}^2$$

$$\Longrightarrow \quad \vec{a} = \left(\ddot{R} - R\dot{\theta}^2 \right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta} \right)\hat{e}_\theta + \ddot{z}\hat{k}$$

- Spherical coordinate system



transform

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \arctan \frac{\sqrt{x^2 + y^2}}{z}$$

$$\theta = \arctan \frac{y}{x}$$

Inverse transform

$$x = r \sin\phi \cos\theta$$

$$y = r \sin\phi \sin\theta$$

$$z = r \cos\phi$$

- Spherical coordinate system

$$h_r = 1,\ h_\phi = r,\ h_\theta = r\sin\phi$$

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi + r\dot{\theta}\sin\phi\hat{e}_\theta$$

$$\begin{aligned}\vec{a} = &\left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi\right)\hat{e}_r \\ &+\left(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi\right)\hat{e}_\phi \\ &+\left(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\phi}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi\right)\hat{e}_\theta\end{aligned}$$

2.4 Joint Kinematical Descriptions

- The motion is independent of the way how it is defined.
- We could consider the kinematical description that best matches the parameters of the actual system to be the “natural” one.
- We shall investigate situations in which no one formulation is entirely natural, although more than one has elements that are suitable.
- Express the unit vectors in one coordinate system by using those in another coordinate system. (See Fig. 2.9)