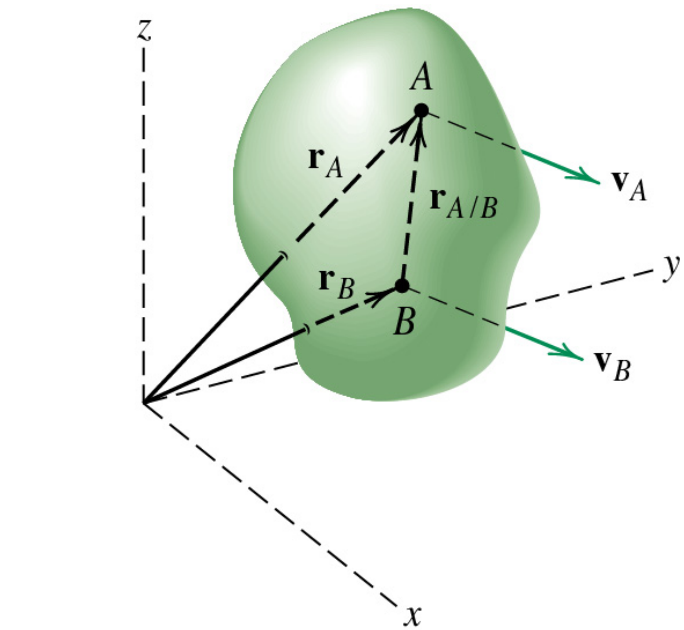
# 3. Relative Motion

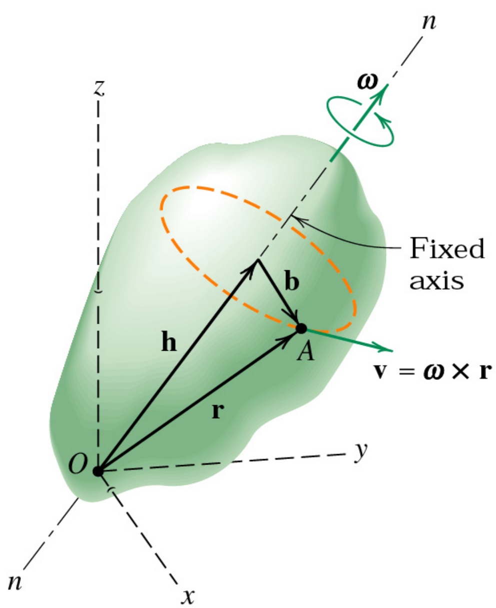
# Outline

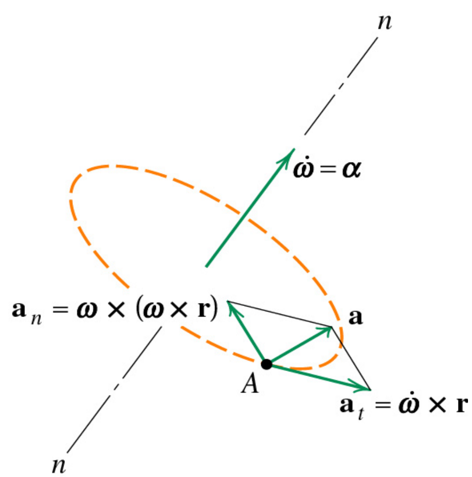
- Rotation transformations
- Finite rotations
  - Body-fixed rotations
  - Space-fixed rotations
- Angular velocity
- Angular acceleration
- Velocity and acceleration




## Translation

$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$$



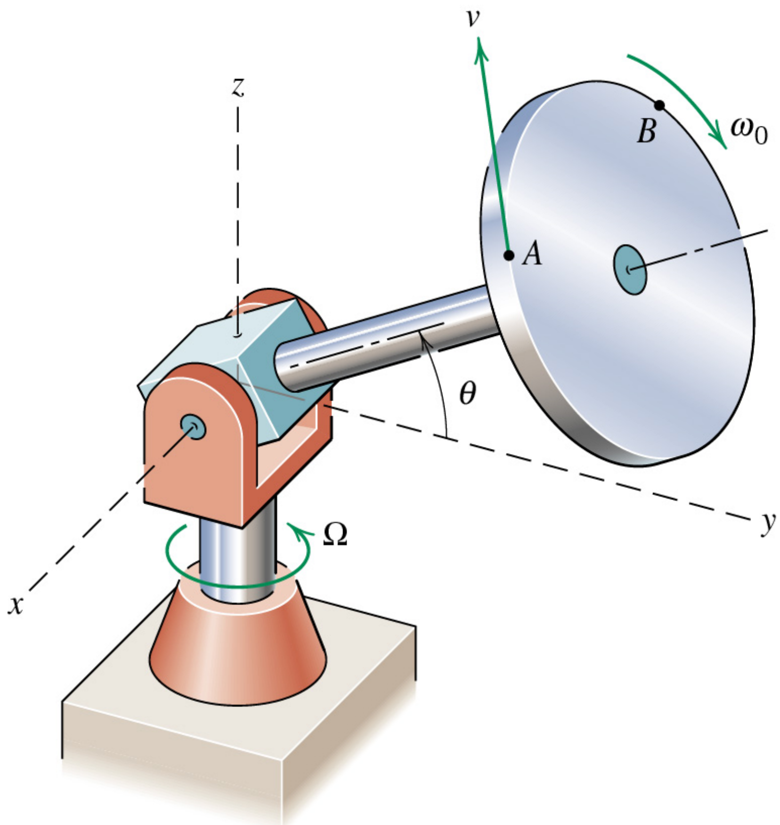




## Fixed-axis rotation

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \left( \vec{\omega} \times \vec{r} \right)$$

# 1. Rotation transformations



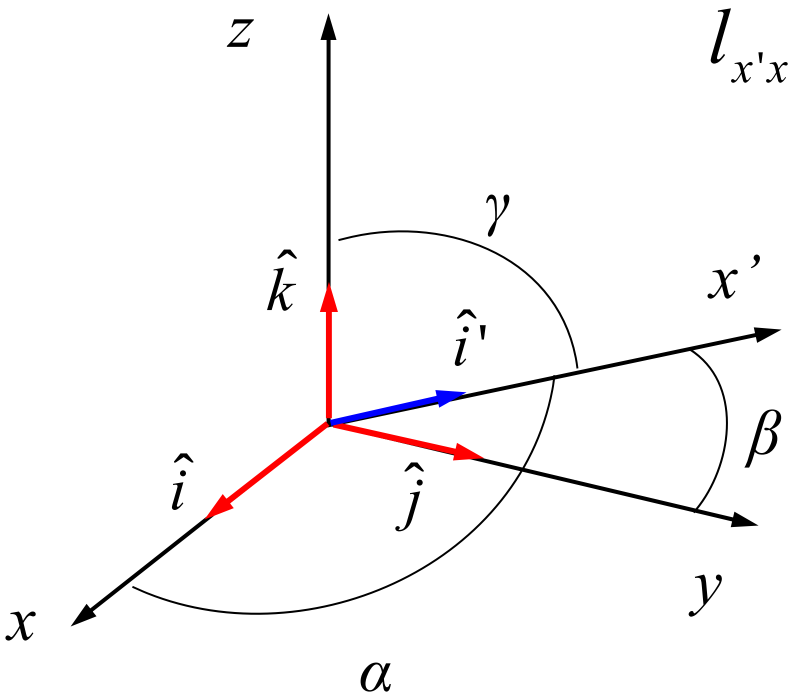

When a body rotates about a fixed point instead of a fixed axis, the angular velocity vector no longer remain fixed in direction. This change calls for a more general concept of rotation.

# 1. Rotation transformations

Assume: the origins of the coordinate systems coincide.

Define $l_{p'q} = l_{qp'}$ to be the direction cosine

$$l_{x'x} = \cos\alpha, l_{x'y} = \cos\beta, l_{x'z} = \cos\gamma$$




$$\hat{i}' = l_{x'x}\hat{i} + l_{x'y}\hat{j} + l_{x'z}\hat{k}$$

$$\hat{j}' = l_{y'x}\hat{i} + l_{y'y}\hat{j} + l_{y'z}\hat{k}$$

$$\hat{k}' = l_{z'x}\hat{i} + l_{z'y}\hat{j} + l_{z'z}\hat{k}$$

$$\begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = [R] \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

where $[R] = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix}$ is called the rotation transformation matrix

An important property: [*R*] is an orthonormal matrix

$$\text{proof}: \begin{bmatrix} \hat{i}' & \hat{j}' & \hat{k}' \end{bmatrix} \begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \end{bmatrix} [R]^T [R] \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = 1$$

$$\Rightarrow [R]^T [R] = I \Rightarrow [R]^T = [R]^{-1}$$

Therefore, $$\begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = [R]^{-1} \begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = [R]^{T} \begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix}$$

An arbitrary point *A* in the space

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = A_{x'} \hat{i}' + A_{y'} \hat{j}' + A_{z'} \hat{k}'$$

$$\begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = \begin{bmatrix} A_{x'} & A_{y'} & A_{z'} \end{bmatrix} \begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = \begin{bmatrix} A_{x'} & A_{y'} & A_{z'} \end{bmatrix} [R] \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

$$\begin{bmatrix} A_{x'} & A_{y'} & A_{z'} \end{bmatrix} [R] = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix} \Rightarrow [R]^{T} \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = [R] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}$$

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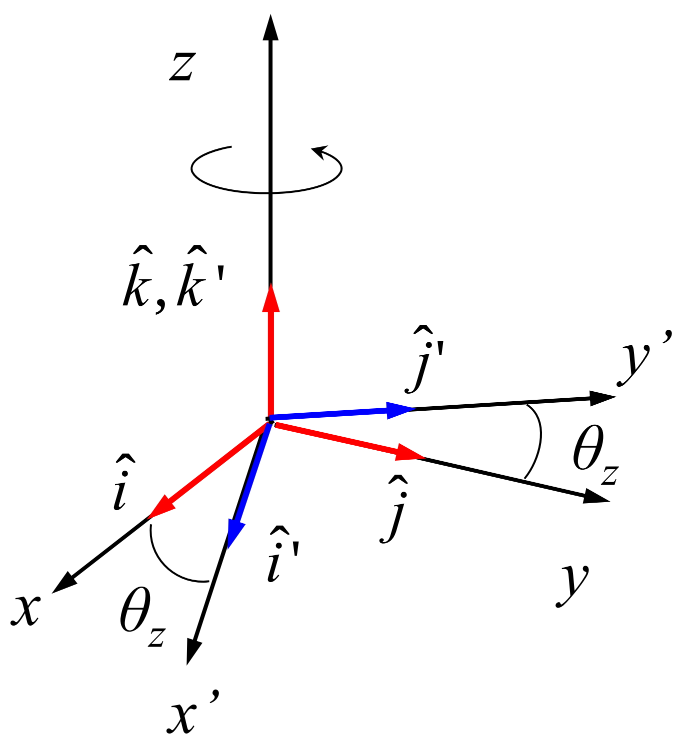
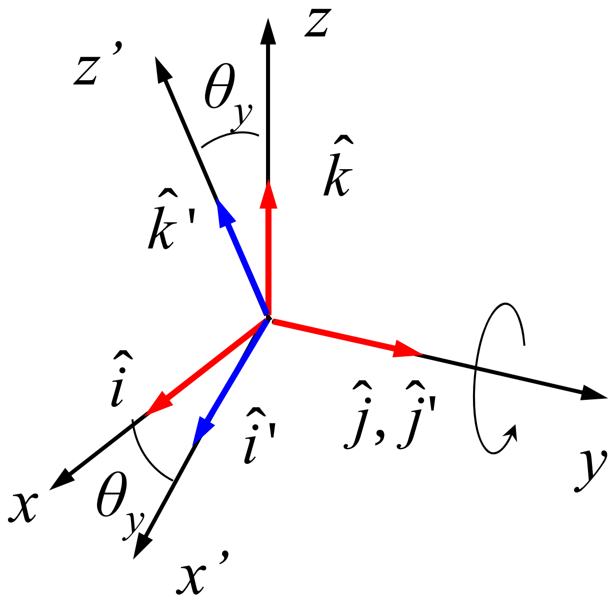



$$\begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$




$$\begin{Bmatrix}\hat{i}'\\ \hat{j}'\\ \hat{k}'\end{Bmatrix} = \begin{bmatrix}\cos\theta_y & 0 & -\sin\theta_y\\ 0 & 1 & 0\\ \sin\theta_y & 0 & \cos\theta_y\end{bmatrix}\begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix}$$

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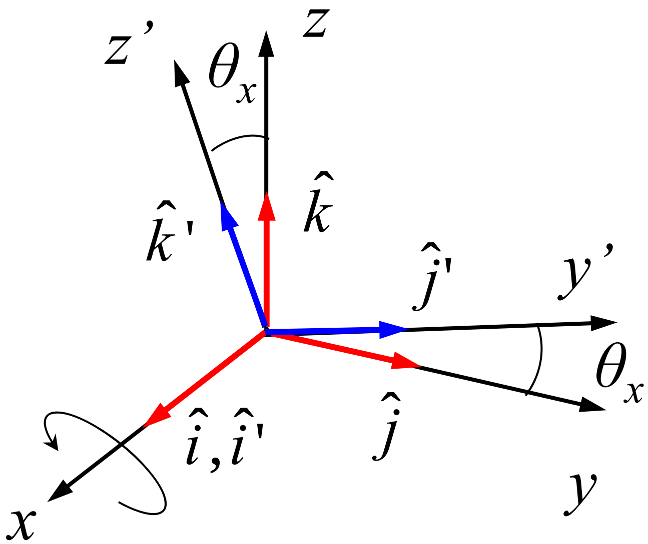


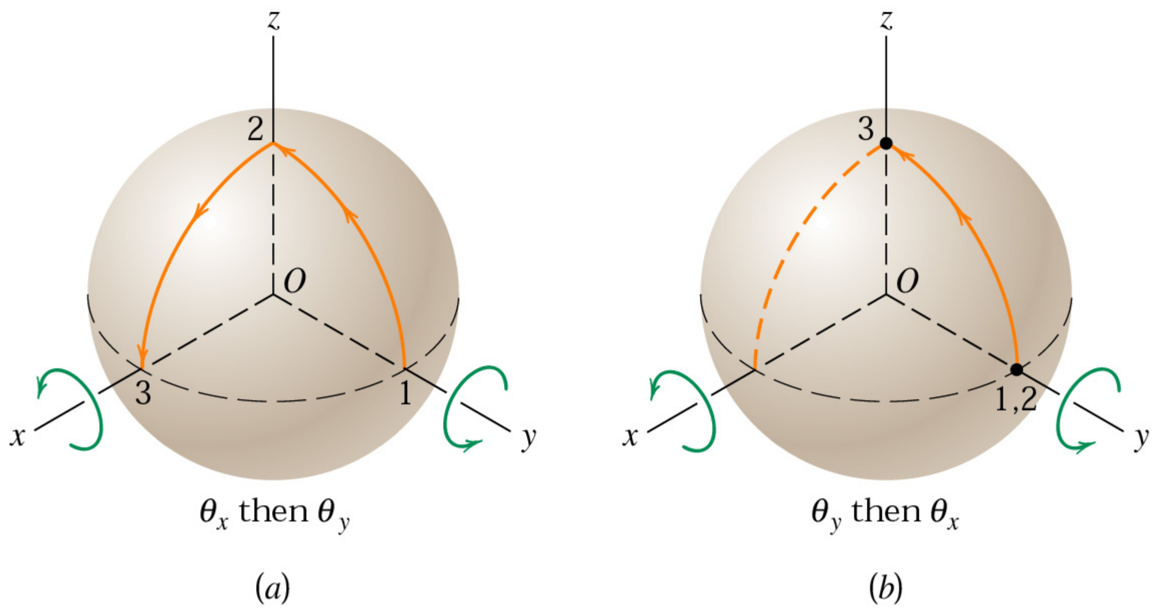

$$\begin{Bmatrix}\hat{i}'\\ \hat{j}'\\ \hat{k}'\end{Bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_x & \sin\theta_x\\ 0 & -\sin\theta_x & \cos\theta_x\end{bmatrix}\begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix}$$

# 1. Rotation transformations

- Example 3.1 (pp. 60)

# 2. Finite rotations

- Finite rotations are not proper vectors since they do not obey the parallel law of vector addition and are not commutative.

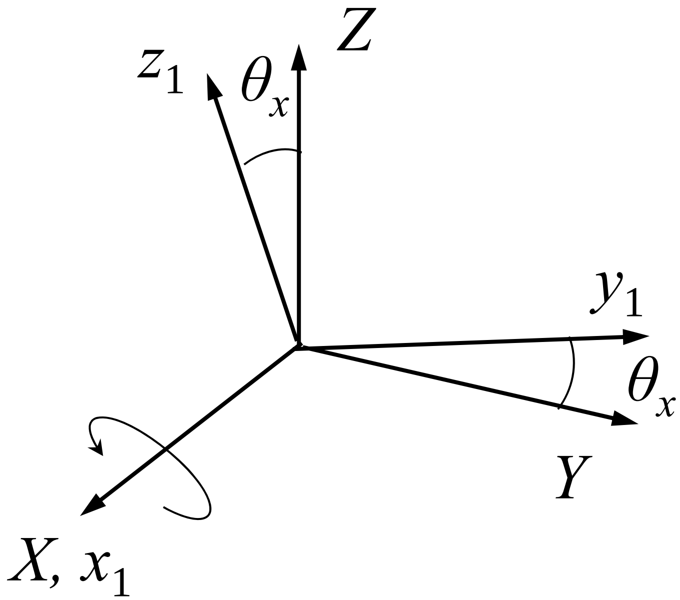



- The finite orientation of a coordinate system depends on the sequence in which rotation occur, as well as the magnitude of the individual rotations and the orientation of their respective axes. (pp. 67)

# 2.1 Body-fixed rotations

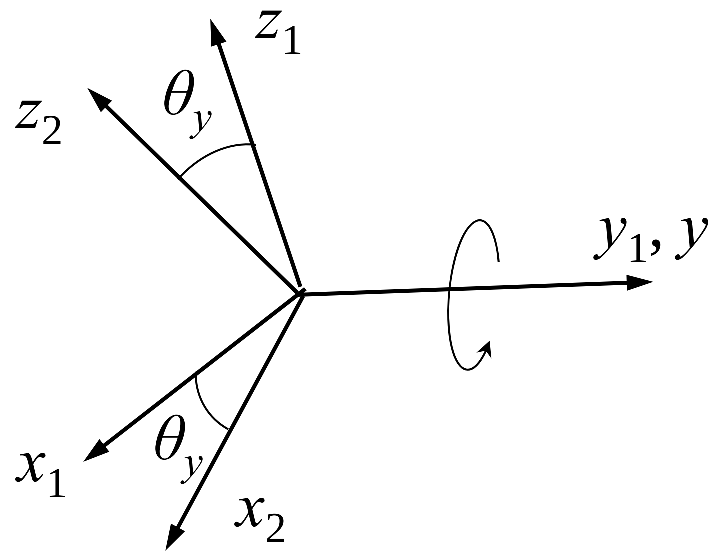
- In a body-fixed rotation sequence, each rotation is about one of the axes of the coordinate system at the preceding step in the sequence.
- On the other hand, in space-fixed rotation, the rotations at all the steps in a sequence is about an axis of the fixed coordinate system.

## rotation 1




$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_1] = [R_x]$$

## rotation 2




$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [R_2] \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}, \quad [R_2] = [R_y]$$

$$= [R_2][R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

# 2.1 Body-fixed rotations

Let *xyz* be a reference frame that undergoes a sequence of rotations about its own axes, and let *XYZ* mark the initial orientation of *xyz*. The transformation from *XYZ* to the final *xyz* components is obtained by pre-multiplying (from right to left) the sequence of transformation matrices for the individual single-axis rotation. For *n* rotations:

$$[R] = [R_n] \cdots [R_2][R_1]$$

# 2.2 Space-fixed rotations

rotation 1
$$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = [R_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_1] = [R_X]$$

rotation 2
$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [R_2] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_2] = [R_Y]$$

Final:
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_1] \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = [R_1][R_2] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

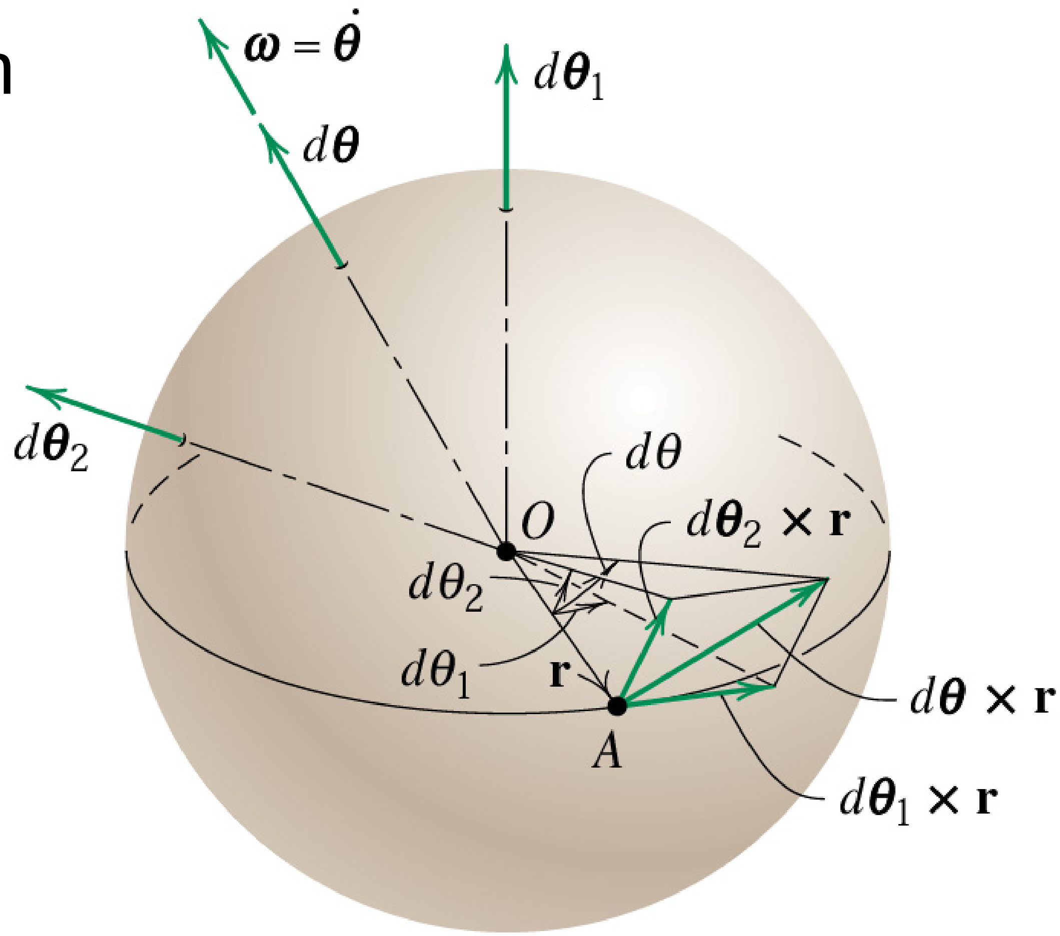
# 2.2 Space-fixed rotations

Let *xyz* be a reference frame that undergoes a sequence of rotations about the space-fixed axes *XYZ* with which it initially coincides. The transformation from *XYZ* to the final *xyz* components is obtained by post-multiplying (from left to right) the sequence of transformation matrices for the individual single-axis rotation. For *n* rotations:

$$[R] = [R_1][R_2]\cdots[R_n]$$

# 3. Angular velocity

- However, infinitesimal rotations do obey the parallel law of vector addition.
- The final orientation of a coordinate system is unaffected by the sequence of a set of infinitesimal rotations.



$$[R_x]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_x & \sin\theta_x\\ 0 & -\sin\theta_x & \cos\theta_x\end{bmatrix},[R_y]=\begin{bmatrix}\cos\theta_y & 0 & -\sin\theta_y\\ 0 & 1 & 0\\ \sin\theta_y & 0 & \cos\theta_y\end{bmatrix},[R_z]=\begin{bmatrix}\cos\theta_z & \sin\theta_z & 0\\ -\sin\theta_z & \cos\theta_z & 0\\ 0 & 0 & 1\end{bmatrix}$$

For infinitesimal rotations $d\theta_x, d\theta_y, d\theta_z$

$$[R_x]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & d\theta_x\\ 0 & -d\theta_x & 1\end{bmatrix}, [R_y]=\begin{bmatrix}1 & 0 & -d\theta_y\\ 0 & 1 & 0\\ d\theta_y & 0 & 1\end{bmatrix}, [R_z]=\begin{bmatrix}1 & d\theta_z & 0\\ -d\theta_z & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Neglect second order differentials, we can prove that

$$[R_x][R_y][R_z]=[R_z][R_y][R_x]$$

and $d\vec{\theta}=d\theta_x\hat{i}+d\theta_y\hat{j}+d\theta_z\hat{k}$

# 3. Angular velocity

In a moving reference frame with an angular velocity $\vec{\omega}$, for any vector $\vec{A}$ whose coordinates relative to that reference frame are constant, the rate of change of $\vec{A}$ is

$$\vec{\omega} \times \vec{A}$$

where $$\vec{\omega} = \sum_i \omega_i \hat{e}_i$$

(for example) $$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

The primary application of this theorem is to differentiate the unit vectors of a moving reference frame.

# 4. Angular acceleration

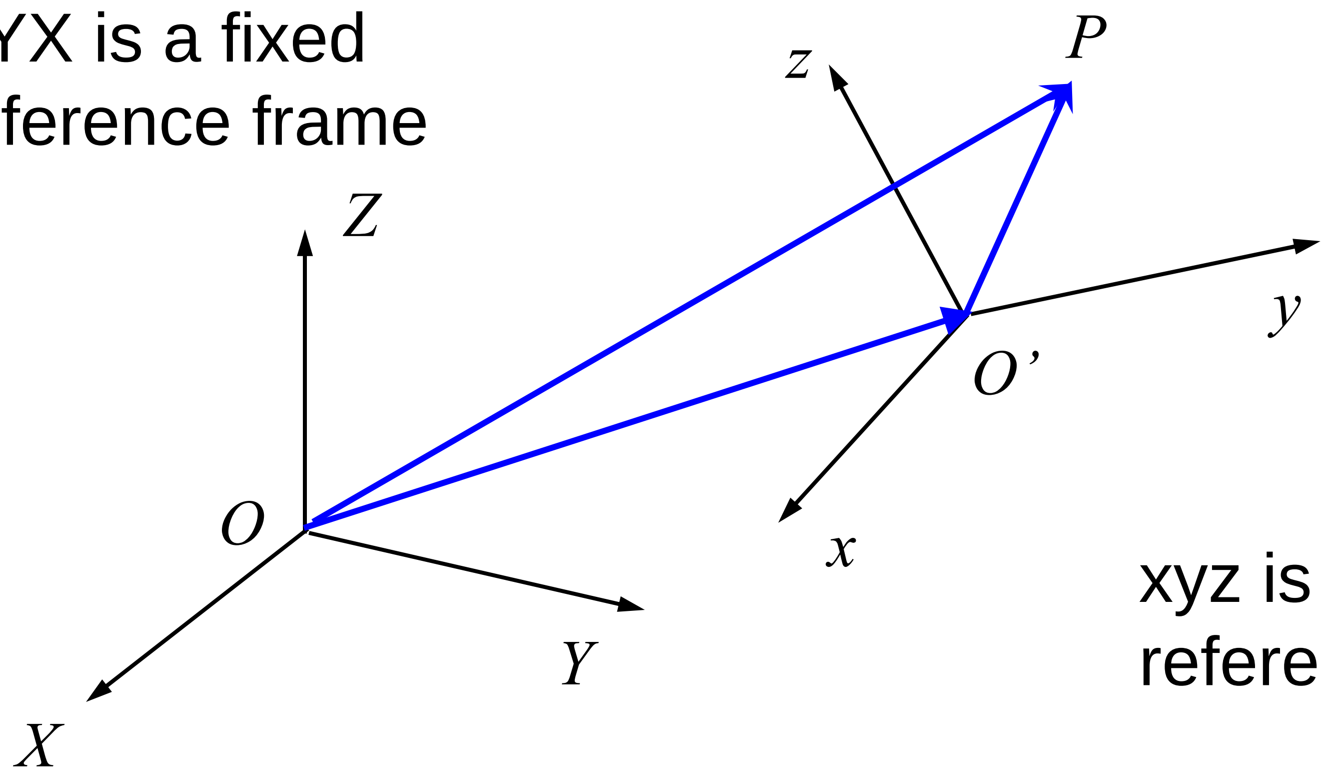
$$\vec{\omega} = \sum_i \omega_i \hat{e}_i$$

Differentiate it using the product rule

$$\vec{\alpha} = \sum_i \left[ \dot{\omega}_i \hat{e}_i + \omega_i \left( \vec{\Omega}_i \times \hat{e}_i \right) \right]$$

# 5. Velocity and acceleration

ZYX is a fixed reference frame




xyz is a moving reference frame

$$\vec{r}_P = \vec{r}_{O'} + \vec{r}_{P/O'}$$

$$\text{If } \vec{r}_{P/O'} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{d}{dt}\vec{r}_{P/O'} = \underbrace{\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}}_{\left(\vec{v}_P\right)_{xyz}} + \vec{\omega} \times \vec{r}_{P/O'}$$

Then differentate $\vec{r}_P = \vec{r}_{O'} + \vec{r}_{P/O'}$ w.r.t time

$$\underline{\vec{v}_P = \vec{v}_{O'} + \left(\vec{v}_P\right)_{xyz} + \vec{\omega} \times \vec{r}_{P/O'}}$$

Differentate w.r.t time one more time

$$\vec{a}_P = \vec{a}_{O'} + \left(\vec{a}_P\right)_{xyz} + \vec{\omega} \times \left(\vec{v}_P\right)_{xyz} + \vec{\alpha} \times \vec{r}_{P/O'} + \vec{\omega} \times \left[\left(\vec{v}_P\right)_{xyz} + \vec{\omega} \times \vec{r}_{P/O'}\right]$$

$$\Rightarrow \underline{\vec{a}_P = \vec{a}_{O'} + \left(\vec{a}_P\right)_{xyz} + \vec{\alpha} \times \vec{r}_{P/O'} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right) + 2\vec{\omega} \times \left(\vec{v}_P\right)_{xyz}}$$

# 5. Velocity and acceleration

Notes:

(1) $\left(\vec{v}_P\right)_{xyz}$ and $\left(\vec{a}_P\right)_{xyz}$ are the velocity and acceleration in the local C.S.

(2) special case: if point *P* is fixed in the moving C.S.

$$\left(\vec{v}_P\right)_{xyz} = 0, \ \left(\vec{a}_P\right)_{xyz} = 0$$

In this case the velocity and acceleration can be simplified

$$\vec{v}_P = \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{P/O'}$$

$$\vec{a}_P = \vec{a}_{O'} + \vec{\alpha} \times \vec{r}_{P/O'} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O'}\right)$$