**MECH 7710 Optimal Estimation and Control**

**Mid-Term Exam**

**April 6, 2020**

**General Instruction**

- This exam is open book, open notes, and open calculators. It is okay to use Matlab for calculation. Do not use built-in Matlab tools like "pinv"
- You have 24 Hours
- Solve each problem fully showing all your work. Clearly indication your answers
- Total credit is 100 points

**Name:** _____

**Date:** _____

I certify that I have performed this exam according to the Auburn University honor code and have not given or received any external information during the exam.

**Signature:** _____

Note:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1. (10 Points) Assume that *t* observations of a random variable are given:

$$\tilde{y}_k = y_k \qquad \forall\, k = 1, \dots, t$$

Find a recursive formula for calculating the mean at time *t*.

2. (20 Points) Consider the continuous system below
    a. Calculate the expected steady state Kalman filter estimation error variance
    b. Calculate the steady state Kalman gain
    c. Calculate the open loop and closed loop estimator eigenvalues
    d. Plot the location of all possible closed loop eigenvalues. What do you notice about the minimum Kalman gain and the slowest closed loop estimator the Kalman filter will use?

$$\dot{x} = x + w$$
$$y = x + v$$
$$E[w] = E[v] = 0$$
$$E[ww^T] = Q$$
$$E[vv^T] = R$$

3. (30 Points) Given the measurements shown in the table:
    a. Use least squares to estimate the slope and y-intercept of a line assuming the x values are inputs and the y values are measurements that are corrupted by zero mean Gaussian noise with a standard deviation of 0.2
    b. What is the expected error of the estimate?
    c. Used weighted least squares to calculate the estimates assuming the last measurement is twice as bad (1 $\sigma$) as the first two measurements

| k | x | y |
|---|---|---|
| 1 | 0 | 3 |
| 2 | 1 | 2 |
| 3 | 2 | 0 |

4. (20 Points) A random number generator generates integers from 1 to 9 (inclusive). All outcomes are equally likely, and each integer is generated independently of any previous integer. Let Σ denote the sum of two consecutively generated integers - $\Sigma = N_1 + N_2$.
   (*Hint: It may be helpful to write out the discrete joint PDF)*
   a. Given $N_1 > 8$, what is the conditional probability that $\Sigma$ is odd.
   b. Given $\Sigma > 10$, what is the conditional probability that at least one of the two integers is >7?
   c. Using Bayes Rule, find the conditional probably that $\Sigma = 7$ given $\Sigma$ is odd.

5. (20 Points) Consider the linearized pendulum driven by a motor torque with a force disturbance. The system is sampled at 100 Hz. The pendulum has an optical encoder with 360 counts/revolutions to measure angular position. The motor has a tachometer that measures angular velocity with an accuracy of 0.2 deg/s (1 σ). However, the tachometer measurement has a bias that should be modeled as a Markov process with a time constant of 10 seconds and a unit white noise input

$$J\ddot{\theta} + b\dot{\theta} + mgl\theta = \tau_m - F_d l \qquad where \quad E[F_d] = \mu_d \quad and \quad E[F_d F_d^T] = \sigma_d^2$$

   a. Find A, Bu, Bw, C, D, Q, and R
   b. Find the first order approximation (i.e. Taylor Series) of Ad
   c. Calculate Qd