MECH7710: Optimal Estimation and Control: Homework #1

Due on February 14, 2020

Matthew Boler

Problem 1

Use the MATLAB®convolve function to produce discrete PDFs for throws of six dice as follows:

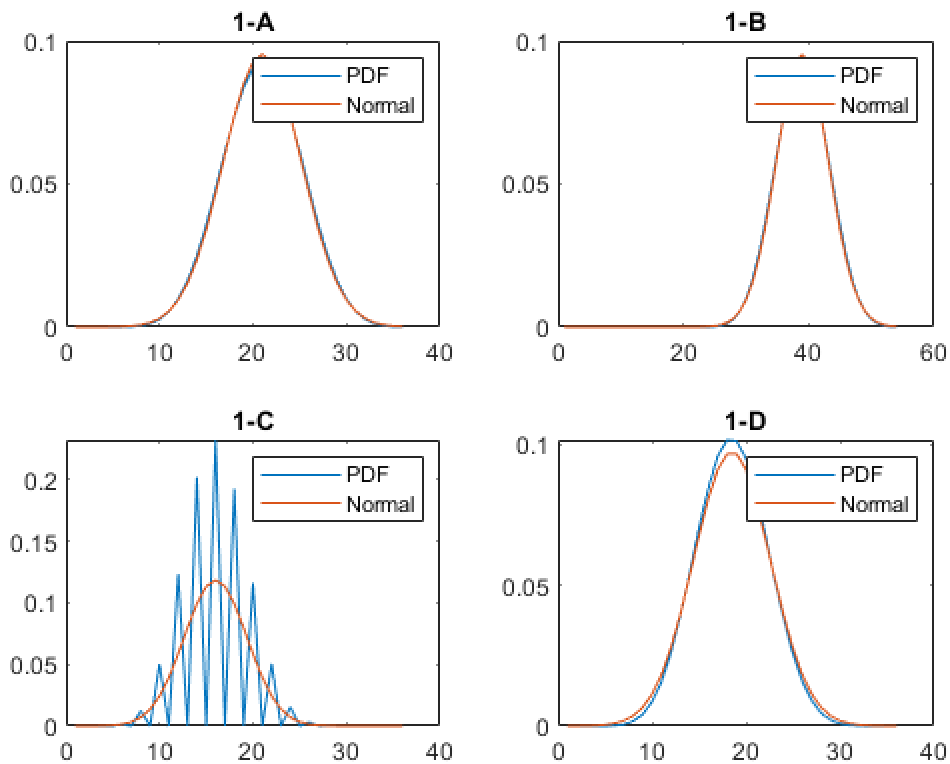
a 6 numbered 1, 2, 3, 4, 5, 6

b 6 numbered 4, 5, 6, 7, 8, 9

c 6 numbered 1, 1, 3, 3, 3, 5

d 3 numbered 1, 2, 3, 4, 5, 6 and 3 numbered 1, 1, 3, 3, 3, 5

Solution
The PDFs of each are shown below:



Problem 2

a What is the PDF for 2 fair dice?

b What are $E(X_1)$, $E(X_1 - E(X_1))$, $E(X_1^2)$, $E((X_1 - E(X_1))^2)$, and $E(((X_1 - E(X_1)) * (X_2 - E(X_2))))$?

c Form the covariance matrix for x_1 and x_2

d Find the PDF matrix for $V_1 = x_1$ and $v_2 = x_1 + x_2$

e What is the mean, $E(v_1 - E(v_1))$, rms, and variance of v_1?

f What is the mean, $E(v_2 - E(v_2))$, rms, and variance of v_2?

g What is the new covariance matrix?

Solution The PDF for two dice can be represented as the following matrix:

$$\frac{1}{6^2} * ones(6,6)$$

$$\begin{aligned}
E(X_1) &= \sum x f(x) = 3.5 \\
E(X_1 - E(X_1)) &= E(X_1) - E(X_1) = 0 \\
E(X_1^2) &= \sum x_1^2 f(x_1) = 15.167 \\
E[(X_1 - E(X_1))^2] &= \sum (x_1 - E(X_1))^2 f(x) = 2.916 \\
E[(X_1 - E(X_1)) * (X_2 - E(X_2))] &= \sum \sum (x_1 - E(X_1)(x_2 - E(X_2))) f(x_1, x_2) = 2.916
\end{aligned}$$

$$Cov(X_1, X_2) = \begin{bmatrix} \sigma_{x_1}^2 = 2.916 & E[X_1X_2]] - E[X_1]E[X_2] = 0 \\ E[X_1X_2]] - E[X_1]E[X_2] = 0 & \sigma_{x_2}^2 = 2.916 \end{bmatrix}$$

The PDF for $V_1 = X_1$ and $V_2 = X_1 + X_2$ is ($V_1 \in [1,6]$ on rows, $V_2 \in [2,12]$ on columns)

$$\frac{1}{6^2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
E[v_1] &= E[x_1] = 3.5 \\
E[v_1 - E(v_1)] &= E[x_1] - E[x_1] = 0 \\
rms_{v_1} &= \sqrt{\sum v_1^2 f(v_1)} = 3.89 \\
\sigma_{v_1}^2 &= E[(v_1)^2] - E[v_1]^2 = 2.916
\end{aligned}$$

$$\begin{aligned}
E[v_2] &= E[x_1 + x_2] = E[x_1] + E[x_2] = 7 \\
E[v_2 - E(v_2)] &= 0 \\
rms_{v_2} &= \sqrt{\sum v_2^2 f(v_2)} = \sqrt{(E[v_2]^2 + \sigma_{v_2}^2)} = 9.11 \\
\sigma_{v_2}^2 &= \sigma_{x_1} + \sigma_{x_2} = 5.832
\end{aligned}$$

Problem 3

Two random vectors X_1 and X_2 are uncorrelated if

$$E[(X_1 - \overline{X_1})(X_2 - \overline{X_2})] = 0$$

Show that:

Problem 3 continued on next page...

1. Independent random vectors are uncorrelated
2. Uncorrelated Gaussian random vectors are independent

Solution If two random vectors are independent, then $f(x_1, x_2) = f(x_1)f(x_2)$. From this we get

$$\begin{aligned} E[X_1 X_2] &= \sum\sum x_1 x_2 f(x_1) f(x_2) \\ &= \sum x_1 f(x_1) \sum x_2 f(x_2) \\ &= E[X_1]E[X_2] \end{aligned}$$

Since $Cov(X_1, X_2) = E[X_1X_2] - E[X_1]EX_2 = 0$, the vectors are uncorrelated.

If two gaussian random vectors are uncorrelated, then $\rho_{1,2} = 0$. From this, the exponential term in the PDF takes the form $\exp(\alpha_1 X_1^2 + \alpha_2 X_2^2)$, which can clearly be written as a separable product of exponentials, and the vectors are thus independent.

Problem 4

Consider a sequence created by throwing a pair of dice and summing the numbers, which are $[-2.5, -1.5, -0.5, 0.5, 1.5, 2.5]$. Call this $V_0(k)$.

a What is the PDF?

b What are the mean and variance?

If we generate a new random sequence

$$V_N(k+1) = (1-r)V_N(k) + rV_0(k),$$

$V_N(k)$ is serially correlated.

c What are the steady-state mean and variance of this new sequence?

d What is the covariance function $R(k) = E[V_N(k)V_N(k-L)]$?

e Are there any practical constrains on r?

Solution

$$\begin{aligned} f(v_0) &= \frac{1}{6^2}\begin{bmatrix}1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1\end{bmatrix} \in [-5, 5] \\ E[v_0] &= 0 \\ \sigma_{v_0}^2 &= 5.832 \end{aligned}$$

If we expand the sequence definition backwards in time a few steps, we find that it is described by the summation

$$V_n(k) = (1-r)^k V_0(0) + \sum_{i=0}^{k-1} r(1-r)^i V_0(i)$$

Our $V_0(0)$ term goes to 0 as $k \to \inf$, and

$$\begin{aligned} E[\sum_{i=0}^{k-1} r(1-r)^i V_0(i)] &= rE[V_0]\sum_{i=0}^{k-1}(1-r)^i \\ &= 0 \end{aligned}$$

Problem 4 continued on next page. . .

Performing the same expansion for $V_n(k)^2$, we get

$$V_n(k)^2 = (1-r)^{2k} V_n(0)^k + \sum_{i=0}^{k-1} r^2 (1-r) V_0(i)^2$$

Again, our $V_0(0)$ term goes to 0, and

$$\begin{aligned}
E[\sum_{i=0}^{k-1} r^2 (1-r)^{2i} V_0(i)^2] &= r^2 \sum_{i=0}^{k-1} (1-r)^{2i} E[V_0(i)^2] \\
&= [r^2 \sum_{i=0}^{k-1} (1-r)^{2i}](\sigma_{v0}^2 - 0)
\end{aligned}$$

From these,

$$\begin{aligned}
\sigma_{vn}^2 &= E[V_n^2] - E[V_n]^2 \\
&= \sigma_{v0}^2 [r^2 \sum_{i=0}^{k-1} (1-r)^{2i}]
\end{aligned}$$

To find $R(L)$ we'll start with a small example where $L = 1$.

$$\begin{aligned}
E[V_N(k)V_N(k-1)] &= E[((1-r)V_N(k-1) + rV_0(k-1)) * V_N(k-1)] \\
&= E[(1-r)V_N(k-1)^2] + E[rV_0(k-1)V_N(k-1)] \\
&= (1-r)E[V_N(k-1)^2]
\end{aligned}$$

From this, we expand the general solution to $R(L) = (1-r)^L E[V_N(k-L)^2]$. There's still some work to be done in exactly solving back to $V_N(1)$, but this gets the general idea that the sequence is less correlated with itself over larger time steps. A practical constraint on r is that $|r| \in [0, 1]$, otherwise the sequence goes to infinity.

Problem 5

A random variable x has a PDF given by

$$f_X = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

a What is the mean of x?

b What is the variance of x?

Solution

$$\begin{aligned}
\mu_x &= \int_0^2 x \frac{x}{2} dx \\
&= \frac{x^3}{6} |_0^2 \\
&= \frac{4}{3}
\end{aligned}$$

Problem 5 continued on next page...

$$\begin{aligned}
\sigma_x^2 &= E[X^2] - E[X]^2 \\
&= \int_0^2 x^2 \frac{x}{2} dx - \frac{16}{9} \\
&= \frac{x^4}{8}|_0^2 - \frac{16}{9} \\
&= 2 - \frac{16}{9}
\end{aligned}$$

Problem 6

Consider a normally-distributed 2D vector X, with mean 0 and

$$P_X = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

a Find the eigenvalues of P_X

b What are the principal axes?

c Plot the likelihood ellipses for $c = 0.25, 1, 1.5$

d What is the probability of finding X inside each of these ellipses?

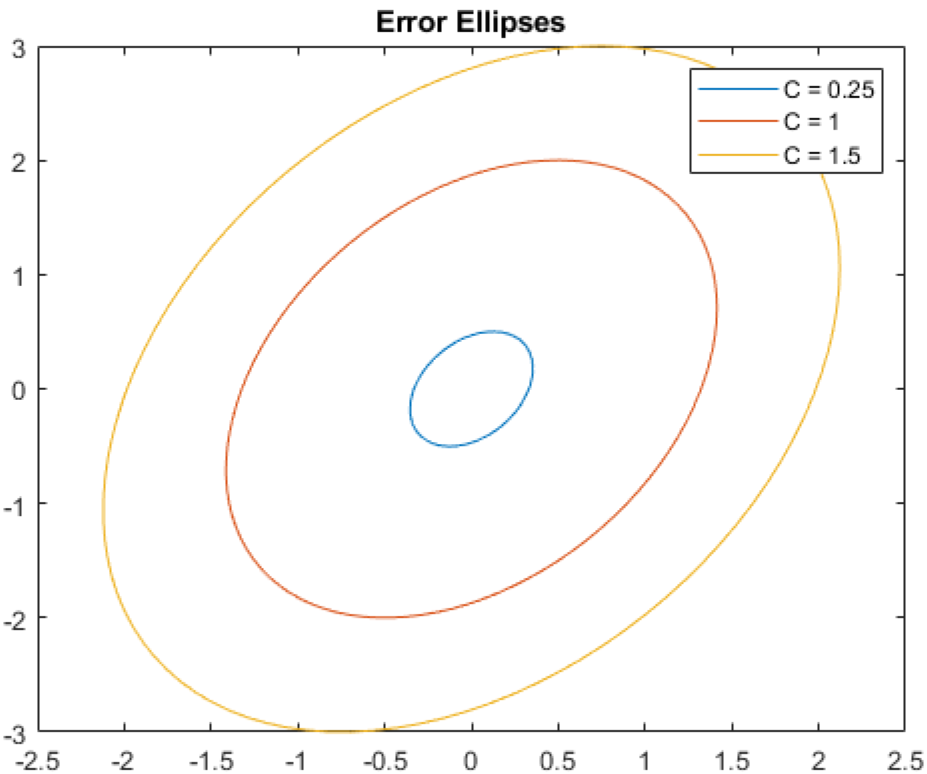
Solution

```
[V, D] = eig(P_x);
% Eigenvalues:
D = 1.5858      0
    0           4.4142
% Principal axes:
V = -0.9239     0.3827
    0.3827      0.9239
```

We find our error ellipses via:

```
function [c] = error_ellipse(covariance, k)
    theta = linspace(0, 2*pi, 100);
    a = k*[cos(theta); sin(theta)];
    [V, D] = eig(covariance);
    A = D^(-1/2) * V;
    A_inv = pin(A);
    c = A_inv * a;
end
```

Problem 6 continued on next page. . .



We expect these ellipses to contain X with the following probabilities:

```
c = [0.25; 1; 1.5];
alpha = c .^ 2
P = chi2cdf(alpha, 2);
```

$$C = 0.25 : P(X) = 0.0308$$
$$C = 1.00 : P(X) = 0.3935$$
$$C = 1.50 : P(X) = 0.6753$$

Problem 7

Given $x \sim N(0, \sigma_x^2)$ and $y = 2x^2$

a Find the PDF of y

b Draw the PDFs of x and y on the same plot for $\sigma_x^2 = 2.0$

c How is the density function changed by this transformation?

d Is y a normal random variable?

Solution For an RV X $N(0, \sigma_x^2)$ transformed as $Y = g(X) = 2X^2$, we slightly abuse the density transformation technique:

$$f_y(y) = |\frac{d}{dy} g^{-1}(y)| f_x(g^{-1}(y))$$

Problem 7 continued on next page. . .

To use this, g must be invertible. Instead we will handle it piecewise for $x < 0$ and $x > 0$.

$$
\begin{aligned}
g^{-1}(y) &= \pm sqrt(\frac{y}{2}) \\
|\frac{d}{dy}g^{-1}(y)| &= \frac{1}{2sqrt(2y)} \\
f_x(x) &= \frac{1}{sqrt(2\pi\sigma_x^2)}exp(\frac{-1}{2}(\frac{x-\mu}{\sigma})^2)
\end{aligned}
$$

To handle our uninvertible transformation, we only performt the calculation for $x > 0$ and double it to account for $x < 0$ mapping to the same value as their positive equivalent.

$$
\begin{aligned}
f_y(y) &= 2 * |\frac{1}{2sqrt(2y)}|f_x(sqrt(\frac{y}{2})) \\
&= \frac{1}{2\sigma_x sqrt(\pi y)}exp(\frac{-y}{4\sigma_x^2})
\end{aligned}
$$