Self study--Complex roots

- Roots γ_i sometimes have a complex solution
- However, in a real system with $Q(\gamma)$ having real coefficients, such roots come in complex conjugate pairs
- Hence they can be combined to form a real 2nd order characteristic mode e.g.,
  - $$y_0[n] = c_1\gamma^n + c_2(\gamma^*)^n$$
    $$= c_1|\gamma|^n e^{j\beta n} + c_2|\gamma|^n e^{-j\beta n}$$

$\gamma = |\gamma| e^{j\beta}$

for a real system, c_1 and C_2 must be complex conjugates, thus leading to

$$= c|\gamma|^n \cos(\beta n + \theta)$$

$c_1 = c_2^*$

$c_1 = \frac{c}{2} e^{j\theta}$

$c_2 = \frac{c}{2} e^{-j\theta}$

$y_0[n] = c/2\, e^{j\theta} |\gamma|^n e^{j\beta n} + c/2\, e^{-j\theta} |\gamma|^n e^{-j\beta n}$

$= c/2 |\gamma|^n \left[e^{j[\beta n + \theta]} + e^{-j[\beta n + \theta]} \right]$

$= c|\gamma|^n \cos(\beta n + \theta)$

Example—(self study) Find the zero-input response of
$y[n] - 1.56y[n-1] + 0.81y[n-2] = x[n-1] + 3x[n-2]$ with IC: $y[-1] = 2$ and $y[-2] = 1$

Advance notation:

$$y[n+2] - 1.56y[n+1] + 0.81y[n] = x[n+1] + 3x[n]$$

$$Q(\gamma) = \gamma^2 - 1.56\gamma + 0.81$$

$|\gamma|$ β

$$\gamma = \frac{1.56 \pm \sqrt{2.4336 - 3.24}}{2} = 0.78 \pm j0.45 = 0.9e^{\pm j\frac{\pi}{6}}$$

$$y_c[n] = c_1[\gamma_1]^n + c_2[\gamma_2]^n = c_1[0.9e^{j\pi/6}]^n + c_2[0.9e^{-j\pi/6}]^n$$

Since all a_n and b_n are real, $c_2 = c_1^*$ and, thus $c_1 = Ke^{j\theta}$, $c_2 = Ke^{-j\theta}$

$$y_c[n] = 0.9^n\left(Ke^{j(\frac{\pi}{6}n+\theta)} + Ke^{-j(\frac{\pi}{6}n+\theta)}\right)$$

$$y_c[n] = K \cdot 0.9^n \cdot \cos\left(\frac{\pi}{6}n + \theta\right)$$

Applying the ICs:

$$y[-1] = \frac{K}{0.9}\cos\left(-\frac{\pi}{6} + \theta\right) = 2$$

$$y[-2] = \frac{K}{0.81}\cos\left(-\frac{\pi}{3} + \theta\right) = 1$$

$$y[-1] = \frac{K}{0.9}\cos\left(-\frac{\pi}{6}+\theta\right) = 2$$
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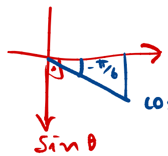
$\Rightarrow$

$\sin x = \cos(x - \frac{\pi}{2})$

Trigonometric identity:
$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\rightarrow$



$$\cos(\theta - \tfrac{\pi}{6}) = \underbrace{\cos(\tfrac{\pi}{6})}_{\sqrt{3}/2}\cos(\theta) + \underbrace{\sin(\tfrac{\pi}{6})}_{1/2}\sin(\theta)$$

$$\cos(\theta-\tfrac{\pi}{6}) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$$
$$\therefore\ \cos(\theta-\tfrac{\pi}{3}) = \cos\tfrac{\pi}{3}\cos\theta + \sin\tfrac{\pi}{3}\cos\theta$$
$$\cos(\theta-\tfrac{\pi}{3}) = \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta$$

$\rightarrow$

$$K\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = 1.8$$
$$K\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right] = 0.81$$

$$K\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = 1.8$$
$$+\ \left\{K\left[\frac{-\sqrt{3}}{2}\cos\theta + \frac{\overset{-3}{\cancel{\sqrt{3}}}}{2}\sin\theta\right] = 0.81\times(-\sqrt{3})\right\}\times(-\sqrt{3})$$
$$K\left[0 + \frac{1-3}{2}\sin\theta\right] = 1.8 - 0.81\cdot\sqrt{3}$$
$$K\left(\frac{-2}{2}\sin\theta\right) = 0.397$$
$$K\sin\theta = -0.397 \qquad (I)$$

$$\left\{K\left[\frac{\overset{-3}{\cancel{\sqrt{3}}}}{2}\cos\theta + \frac{-\sqrt{3}}{2}\sin\theta\right] = 1.8\times(-\sqrt{3})\right\}\times(-\sqrt{3})$$
$$+\ K\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right] = 0.81$$
$$K\left[\frac{-3+1}{2}\cos\theta + 0\right] = 0.81 - 1.8\cdot\sqrt{3}$$
$$K\left[-\cos\theta\right] = -2.308$$
$$K\cos\theta = 2.308 \qquad (II) \rightarrow K = \frac{2.308}{\cos\theta}$$

(II) in (I):
$$2.308\cdot\overbrace{\frac{\sin\theta}{\cos\theta}}^{\tan\theta} = -0.397$$
$$\tan\theta = \frac{-0.397}{2.308}$$
$$\theta = \tan^{-1}\left(\frac{-0.397}{2.308}\right) = -9.76^\circ$$

Back from (I):
$$K = \frac{-0.397}{\underbrace{\sin(-9.76^\circ)}_{-0.1695}} \cong 2.34$$

Finally:
$$y_c[n] = K\cdot 0.9^n\cdot\cos\left(\frac{\pi}{6}n + \theta\right)u[n]$$
$$y_c[n] = 2.34\,(0.9)^n\cos\left(\frac{\pi}{6}n - 9.76^\circ\right)u[n]$$