

MONTE CARLO SOLUTION TO LAPLACE'S EQ ON A RECTANGULAR DOMAIN

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ABSTRACT. We use the Tour Du Wino Method, an algorithm based on Monte-Carlo simulation, to approximate a solution to Laplace's Eq on a rectangular domain with Dirichlet boundary conditions

1. INTRODUCTION

We seek to solve Laplace's Equation

$$(1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on a rectangular domain $0 \leq x \leq 7$, $0 \leq y \leq 9$ with boundary conditions $u(0, y) = u(7, y) = u(x, 0) = 0$ and $u(x, 9) = 12$

2. BACKGROUND

Write about application (electric) and how this equation comes about

3. IMPLEMENTATION

We programmed a Monte-Carlo simulation using the Tour Du Wino method that utilizes a 2D random walker on a uniform Cartesian grid that records the boundary point on which it lands. It then repeats this process m times starting at this same point. It then averages the value of the boundary condition at each point recorded and assigns this to the value of the solution at the given point.[1]

The proof of this is very straightforward. The expected value for some point a where $R(a)$ is the value it will take is

$$E[R(a)] = \frac{1}{4}(E[R(b)] + E[R(c)] + E[R(d)] + E[R(e)])$$

where b, c, d , and e are the neighboring points.

This can be rewritten as

$$(2) \quad u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

which can be rearranged as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0$$

The centered finite difference approximations for the second partial derivatives are

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$$u_{xx} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \quad u_{yy} = \frac{u_{i,j+1} + u_{i,j-1} - u_{i,j}}{h^2}$$

It is now clear the above formulation can be expressed as $h^2 u_{xx} + h^2 u_{yy} = 0$ and dividing away h^2 leaves us $u_{xx} + u_{yy} = 0$ which is then Laplace's Eq.

Our initial algorithm was

Algorithm 1 Original Tour Du Wino

```

1: Initialize  $pos1$  and  $pos2$  as the first interior coordinate pair
2: Generate random number  $r \sim U(0, 1)$ 
3: if  $r \leq 0.25$  then
4:    $pos1 \leftarrow pos1 + 1$ 
5: else if  $r \leq 0.5$  then
6:    $pos1 \leftarrow pos1 - 1$ 
7: else if  $r \leq 0.75$  then
8:    $pos2 \leftarrow pos2 + 1$ 
9: else
10:   $pos2 \leftarrow pos2 - 1$ 
11: end if
12: if  $pos1$  or  $pos2 == 1$  or  $n$  then
13:   Save and continue to next starting point
14: else
15:   Repeat
16: end if
17: Repeat for all starting points

```

This was incredibly slow so we parallelized the algorithm and have the following

Algorithm 2 Improved Tour Du Wino

```

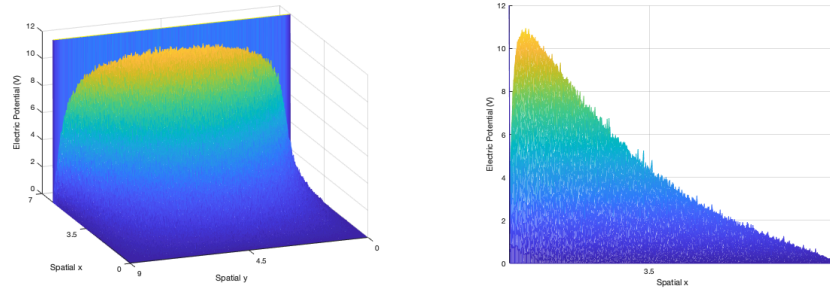
1: Initialize  $pos1$  and  $pos2$  as all the coordinate pairs in  $2 : n - 1$ , repeated  $m$  times
2: Create  $2 \times 1 \times n^2 m$  arrays,  $r$ , to store coordinates for each walker
3: Generate  $1 \times n^2 m$  array  $\sim U(0, 1)$ 
4: if  $r_i \leq 0.25$  then
5:    $pos1_i \leftarrow pos1_i + 1$ 
6: else if  $r_i \leq 0.5$  then
7:    $pos1_i \leftarrow pos1_i - 1$ 
8: else if  $r_i \leq 0.75$  then
9:    $pos2_i \leftarrow pos2_i + 1$ 
10: else
11:   $pos2_i \leftarrow pos2_i - 1$ 
12: end if
13: if  $pos1_i$  or  $pos2_i == (1 \text{ or } n)$  then
14:   Remove  $pos1_i$  and  $pos2_i$  from list
15:   Decrease length of random number array
16: end if
17: Repeat until the array is empty

```

This algorithm has improved efficiency as it can take advantage of parallel computing while immediately deleting all "completed" walkers from the arrays so that it doesn't waste time.

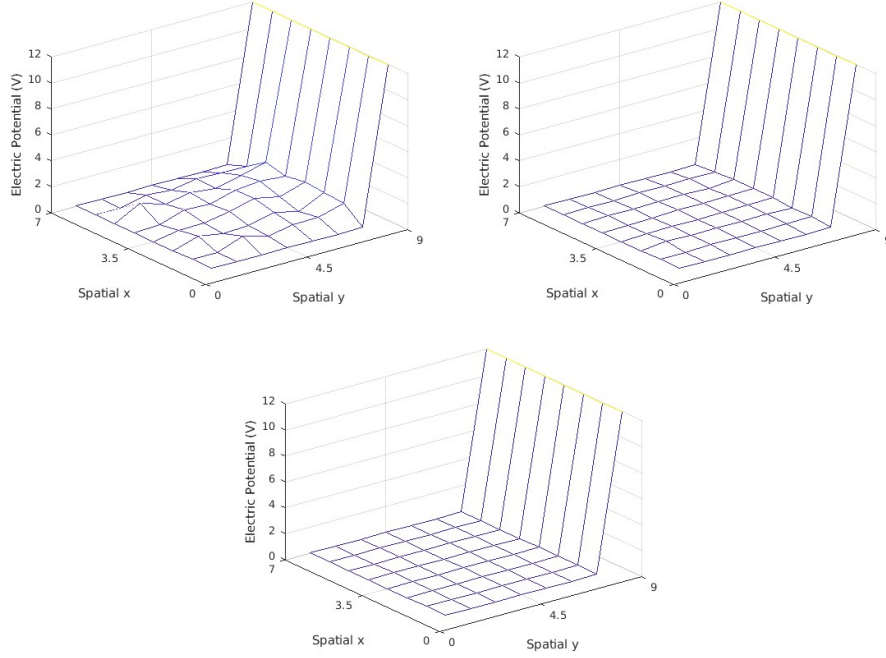
4. COMPUTATIONAL RESULTS

FIGURE 1. Numerical solution with a grid size of 350x350 and 350 realizations



4.1. Plots.

4.2. Convergence Towards Analytical Solution. For a small grid size n , the solution has significant error regardless of number of trials, m .



Clearly, a larger grid size is necessary to see results. By setting $n = m = 350$, we obtain Figure 1 above.

4.3. Computation Time. We tested various different grid sizes and number of trials as seen above. We have found the computation time to be $O(n^4m^2)$ where n is the size of the grid and m is the number of trials.

For $n = 8$ and $m = 10,000$, the computation time was 0.47 seconds. For the final solution plot above, we used $n = m = 350$ and found the computation time to be 553 minutes or approximately 9.2 hours.

5. DISCUSSION

We have found that the Tour Du Wino is effective in solving Laplace's Equation on a rectangular domain with Dirichlet boundary conditions which allows for a simulation of electric potential from a line charge diffusing through a rectangular domain.

As far as contribution to the project, Matt Cassini wrote the code to implement the Tour Du Wino algorithm and the error convergence, Moises Ramos derived and plotted the analytical solution and made the random walker animation, and Marissah McNeil wrote the code for the equipotential lines.

REFERENCES

- [1] S.J. Farlow. *Partial Differential Equations for Scientists and Engineers*. Dover Books on Mathematics. Dover Publications, 2012.

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