LESSON 43

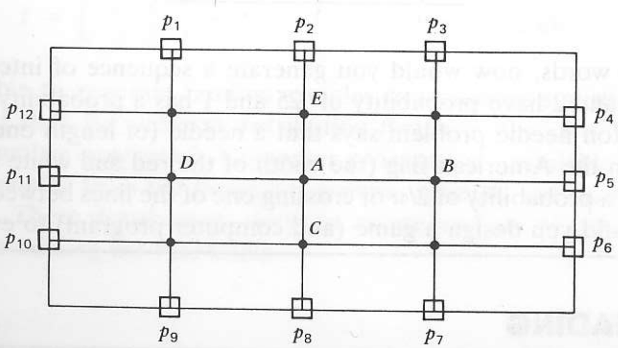
Monte Carlo Solution of Partial Differential Equations

PURPOSE OF LESSON: To show how random games (Monte Carlo methods) can be designed whose outcomes approximate solutions to differential equations. A specific game (tour du wino) is described whose outcome is the finite-difference approximation to a Dirichlet problem inside a square. The game is extended to include solutions to other problems as well.

In the previous lesson, we hinted at how games of chance might be designed whose expected outcomes were solutions or approximate solutions to problems in partial differential equations. This lesson shows how we can design such a game to approximate the solution of the Dirichlet problem

$$\text{PDE} \quad u_{xx} + u_{yy} = 0 \qquad 0 < x < 1 \qquad 0 < y < 1$$

$$\text{BC} \quad u(x,y) = g(x,y) = \begin{cases} 1 & \text{On the top of the square} \\ 0 & \text{On the sides and bottom of the square} \end{cases}$$



A = starting point for game ● = interior grid points

□ = end points p_i g_i = reward for ending at p_i

FIGURE 43.1 Board for tour du wino.

(After we show how the Monte Carlo method can solve this particular problem, we will then discuss more general problems.)

To illustrate the Monte Carlo method in this problem, we introduce a game called tour du wino. To play it, we need a board on which grid lines are drawn (Figure 43.1).

Now, for the rules of the game.

How Tour du Wino is Played

STEP 1 The wino starts from an arbitrary point (point A in our case).

STEP 2 At each stage of the game, the wino staggers off randomly to one of the four neighboring grid points. (In our case, the neighbors of A are B, C, D, and E, and the probability of going to each of these neighbors is .25).

STEP 3 After arriving at a neighboring point, the wino continues this process wandering from point to point until eventually hitting a *boundary point* p_i. He then stops, and we record that point p_i. This completes one *random walk*.

STEP 4 We repeat steps 1–3 until many random walks are completed. We now compute the fraction of times the wino had ended up at each of the boundary points p_i. Table 43.1. shows a typical result after 100 random walks.

STEP 5 Suppose the wino receives a reward g_i (g_i is the value of the BC at p_i) if he ends his walk at the boundary point p_i, and suppose the goal of the game

TABLE 43.1 Probability of Random Walk Ending at p_i (along with rewards g_i)

Boundary point p_i	$P_A(p_i)$ = fraction of times the wino ends at p_i	g_i = reward for ending at p_i
1	.04	1
2	.15	1
3	.03	1
4	.06	0
5	.17	0
6	.05	0
7	.06	0
8	.15	0
9	.03	0
10	.06	0
11	.16	0
12	.04	0

is to compute his average reward $R(A)$ for all the walks. The average reward is

$$R(A) = g_1 P_A(p_1) + g_2 P_A(p_2) + \ldots g_{12} P_A(p_{12})$$

The game is completed with the determination of $R(A)$. In our specific game with the values of g_i and $P_A(p_i)$ given in Table 43.1, we have

$$\begin{aligned} R(A) &= 1(.04) + 1(.15) + 1(.03) + 0(.06) + \ldots + 0(.04) \\ &= .22 \end{aligned}$$

We now tell the reason for playing this game.

Reason for Playing Tour du Wino

It turns out that the average reward we just obtained is the approximate solution to our Dirichlet problem at point A. This interesting observation is based on two facts.

1. Suppose the wino started at a point A that was on the *boundary* of the square. Each resulting random walk ends immediately at that point, and the wino collects the amount g_i. Thus, his average reward for starting from a boundary point is also g_i.
2. Now suppose the wino starts from an interior point. Then, the average reward $R(A)$ is clearly the average of the four average rewards of the four neighbors

$$R(A) = \frac{1}{4}[R(B) + R(C) + R(D) + R(E)]$$

Again, we ask why the wino's average reward $R(A)$ approximates the solution of the Dirichlet problem at A. We have seen that $R(A)$ satisfies the two equations

$$\begin{aligned} R(A) &= \frac{1}{4}[R(B) + R(C) + R(D) + R(E)] \qquad (A \text{ an interior point}) \\ R(A) &= g_i \qquad (A \text{ a boundary point}) \end{aligned}$$

If we let g_i be the value of the boundary function $g(x,y)$ at the boundary point p_i, then our two equations are exactly the two equations we arrived at when we solved the Dirichlet problem by the finite-difference method. That is, $R(A)$ corresponds to $u_{i,j}$ in the finite-difference equations

$$\begin{aligned} u_{i,j} &= \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) \qquad (i,j) \text{ an interior point} \\ u_{i,j} &= g_{i,j} \qquad g_{i,j} \text{ the solution at a boundary point } (i,j) \end{aligned}$$

Hence, $R(A)$ will approximate the true solution of the PDE at A. The tour du wino game can be summarized in three steps

Solution of Laplace's Equation by the Monte Carlo Method

These rules give the solution at one point inside the square.

STEP 1 Generate several random walks starting at some specific point A and ending once you hit a boundary point. Keep track of how many times you hit each boundary point.

STEP 2 After completing the walks, compute the fraction of times you have ended at each point p_i. Call these fractions $P_A(p_i)$.

STEP 3 Compute the *approximate solution* $u(A)$ from the formula

$$u(A) = g_1 P_A(p_1) + g_2 P_A(p_2) + \ldots g_N P_A(p_N)$$

where g_i is the value of the function at p_i and N is the number of boundary points.

The game tour du wino can be modified to solve more complicated problems, as in the following example.

Solution to a Dirichlet Problem with Variable Coefficients

Consider the following elliptic boundary-value problem inside a square:

$$\begin{array}{lll} \text{PDE} & u_{xx} + (\sin x)u_{yy} = 0 & 0 < x < \pi \qquad 0 < y < \pi \\ \text{BC} & u(x,y) = g(x,y) & \text{On the boundary of the square} \end{array}$$

To solve this problem, we replace u_{xx}, u_{yy}, and $\sin x$ by

$$\begin{aligned} u_{xx} &= [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]/h^2 \\ u_{yy} &= [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]/k^2 \qquad \text{(Central-difference approximation)} \\ \sin x &= \sin x_j \end{aligned}$$

and plug them into the PDE. The grid points can be seen in Figure 43.2.

Making these substitutions and solving for $u_{i,j}$ gives

$$u_{i,j} = \frac{u_{i,j+1} + u_{i,j-1} + \sin x_j(u_{i+1,j} + u_{i-1,j})}{2(1 + \sin x_j)}$$

Look very carefully at this last equation. The coefficients of $u_{i+1,j}$, $u_{i-1,j}$, $u_{i,j+1}$, and $u_{i,j-1}$ are positive and sum to one. In other words, the solution

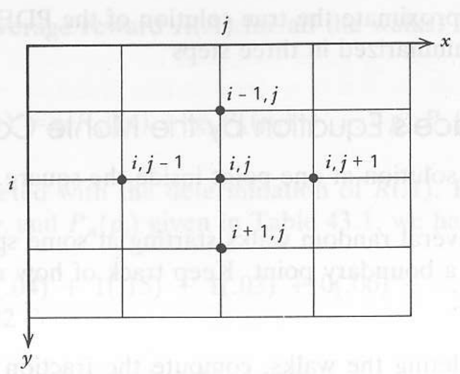


FIGURE 43.2 Grid points of the random walk.

$u_{i,j}$ is a *weighted average* of the solutions at the four neighboring points. Hence, we modify our game so that the wino doesn't stagger off to each neighbor with probability .25, but, rather, with a probability equal to the coefficient of the respective term. In other words, if the wino is at the point (i,j), he then goes to the point:

$$(i,j+1) \text{ with probability } \frac{1}{2(1+\sin x_j)}$$

$$(i,j-1) \text{ with probability } \frac{1}{2(1+\sin x_j)}$$

$$(i+1,j) \text{ with probability } \frac{\sin x_j}{2(1+\sin x_j)}$$

$$(i-1,j) \text{ with probability } \frac{\sin x_j}{2(1+\sin x_j)}$$

Other than this slight modification, the game is exactly the same as before. Modifications for other problems may be more subtle, but the ideas are similar. The reader may wish to devise his or her own games to solve other problems. The parabolic case is considered in the problems.

NOTES

1. Observe that once the fractions $P_A(p_i)$ (the fraction of times the wino ends at p_i) are computed, we can then find the solution $u(A)$ for any other boundary conditions g_i just by plugging the $P_A(p_i)$ into the formula

$$u(A) = g_1 P_A(p_1) + g_2 P_A(p_2) + \ldots + g_N P_A(p_N)$$

That is, we don't have to recompute new random walks.

2. In many cases, a researcher wants to find the solution of a PDE at only one point. If the boundary is fairly complicated and if the PDE involves three or four dimensions, then Monte Carlo methods may come to the rescue. In fact, Monte Carlo methods were originally developed to study difficult neutron-diffusion problems that were impossible to solve analytically.

PROBLEMS

1. Write a flow diagram for tour du wino to solve the problem

$$\begin{array}{llll} \text{PDE} & u_{xx} + u_{yy} = 0 & 0 < x < 1 & 0 < y < 1 \\ \text{BC} & u(x,y) = g(x,y) & \text{On the boundary} & \end{array}$$

at an interior point. Let the number of horizontal and vertical grid lines be arbitrary.

2. Write a computer program to carry out the flow diagram in problem 1.
3. How would the game tour du wino be modified to solve

$$\begin{array}{llll} \text{PDE} & u_{xx} + x^2 u_{yy} = 0 & 0 < x < 1 & 0 < y < 1 \\ \text{BC} & u(x,y) = g(x,y) & \text{On the boundary} & \end{array}$$

What is the nature of the random walk?

4. Can you devise a modified tour du wino game that will solve

$$\begin{array}{llll} \text{PDE} & u_{xx} + u_{yy} = 0 & 0 < x < 1 & 0 < y < 1 \\ \text{BCs} & \begin{cases} u(x,1) = 0 \\ u(x,0) = 0 \\ u(0,y) = 1 \\ \dfrac{\partial u}{\partial x}(1,y) = 0 \end{cases} & 0 < x < 1 & 0 < y < 1 \end{array}$$

5. Derive the Monte Carlo game for solving the following parabolic IBVP:

$$\begin{array}{lll} \text{PDE} & u_t = \alpha^2 u_{xx} - \beta u_x - \gamma u & 0 < x < 1 \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = f(t) \\ u(1,t) = g(t) \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = \phi(x) & 0 \leq x \leq 1 \end{array}$$

HINT Replace the PDE by the finite-difference approximation from the Crank-Nicolson method; solve for $u_{i+1,j}$ in terms of its five neighbors $u_{i+1,j-1}$, $u_{i+1,j+1}$, $u_{i,j-1}$, $u_{i,j}$, $u_{i,j+1}$, and go on from there.