MONTE CARLO SOLUTION TO LAPLACE'S EQ ON A RECTANGULAR DOMAIN

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Abstract. We use the Tour Du Wino Method, an algorithm based on Monte-Carlo simulation, to approximate a solution to Laplace's Eq on a rectangular domain with Dirichlet boundary conditions

1. Introduction

We seek to solve Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

on a rectangular domain $0 \leq x \leq 7, \quad 0 \leq y \leq 9$ with boundary conditions $u(0, y) = u(7, y) = u(x, 0) = 0$ and $u(x, 9) = 12$

2. Background

Write about application (electric) and how this equation comes about

3. Implementation

We programmed a Monte-Carlo simulation using the Tour Du Wino method that utilizes a 2D random walker on a uniform Cartesian grid that records the boundary point on which it lands. It then repeats this process m times starting at this same point. It then averages the value of the boundary condition at each point recorded and assigns this to the value of the solution at the given point.[1]

The proof of this is very straightforward. The expected value for some point a where R(a) is the value it will take is

$E[R(a)] = \frac{1}{4}(E[R(b)] + E[R(c)] + E[R(d)] + E[R(e)])$

where b,c,d, and e are the neighboring points.

This can be rewritten as

$$u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) \tag{2}$$

which can be rearranged as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0$$

The centered finite difference approximations for the second partial derivatives are

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$$u_{xx} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \quad u_{yy} = \frac{u_{i,j+1} + u_{i,j-1} - u_{ij}}{h^2}$$

It is now clear the above formulation can be expressed as $h^2 u_{xx} + h^2 u_{yy} = 0$ and dividing away h^2 leaves us $u_{xx} + u_{yy} = 0$ which is then Laplace's Eq.

Our initial algorithm was

Algorithm 1 Original Tour Du Wino

Initialize $pos1$ and $pos2$ as the first interior coordinate pair
Generate random number $r =\sim U(0, 1)$
if $r \leq 0.25$ **then**
    $pos1 \leftarrow pos1 + 1$
else if $r \leq 0.5$ **then**
    $pos1 \leftarrow pos1 - 1$
else if $r \leq 0.75$ **then**
    $pos2 \leftarrow pos2 + 1$
else
    $pos2 \leftarrow pos2 - 1$
end if
if $pos1$ or $pos2 == 1$ or n **then**
    Save and continue to next starting point
else
    Repeat
end if
Repeat for all starting points

This was incredibly slow so we parallelized the algorithm and have the following

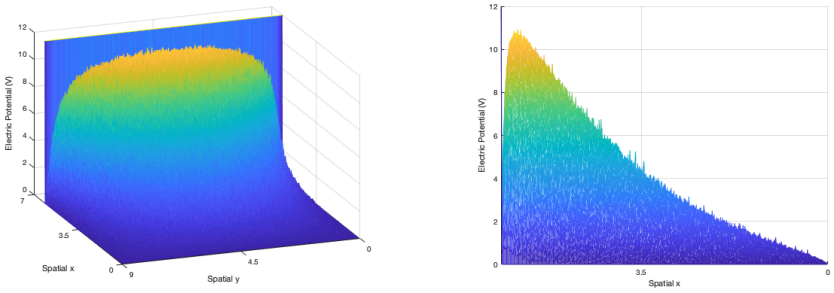
Algorithm 2 Improved Tour Du Wino

Initalize $pos1$ and $pos2$ as all the coordinate pairs in $2 : n-1$, repeated m times
Create 2 $1 \times n^2 m$ arrays, r, to store coordinates for each walker
Generate $1 \times n^2 m$ array $\sim U(0, 1)$
if $r_i \leq 0.25$ **then**
    $pos1_i \leftarrow pos1_i + 1$
else if $r_i \leq 0.5$ **then**
    $pos1_i \leftarrow pos1_i - 1$
else if $r_i \leq 0.75$ **then**
    $pos2_i \leftarrow pos2_i + 1$
else
    $pos2_i \leftarrow pos2_i - 1$
end if
if $pos1_i$ or $pos2_i == (1$ or $n)$ **then**
    Remove $pos1_i$ and $pos2_i$ from list
    Decrease length of random number array
end if
Repeat until the array is empty

This algorithm has improved efficiency as it can take advantage of parallel computing while immediately deleting all "completed" walkers from the arrays so that it doesn't waste time.

4. Computational Results

Figure 1. Numerical solution with a grid size of 350x350 and 350 realizations



4.1. **Plots.**

4.2. **Convergence Towards Analytical Solution.** Put in plots when error analysis code finishes

4.3. **Computation Time.** We tested various different grid sizes and number of trials as seen above. We have found the computation time to be $O(n^4m^2)$ where n is the size of the grid and m is the number of trials.

For $n = 8$ and $m = 10,000$, the computation time was 0.47 seconds. For the final solution plot above, we used $n = m = 350$ and found the computation time to be 553 minutes or approximately 9.2 hours.

References

[1] S.J. Farlow. *Partial Differential Equations for Scientists and Engineers*. Dover Books on Mathematics. Dover Publications, 2012.

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