Pesky irrational people!!!

Classical (Wardrop) user equilibrium (UE) supposes users

- are all the same (in how they weight the components of virtual cost)
- have perfect information (of the state of the network)
- are perfectly rational

But none of these assumptions are actually true!

New modelling idea: **Stochastic User Equilibrium (SUE)**

- use randomness to model heterogeneity / imperfect information / irrationality
- unlike (Wardrop) UE, used choices (i.e., routes) do not have to cost the same

Intro to **discrete choice models**

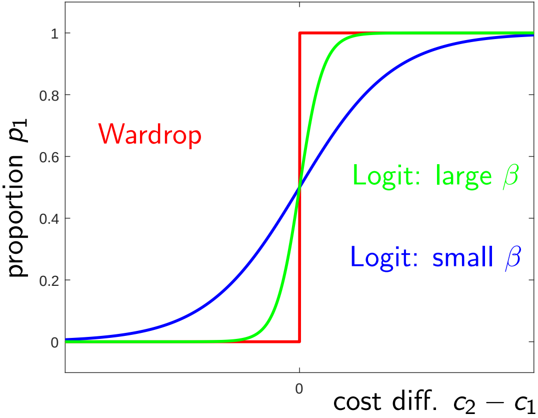
- Simplest setting: binary choice between two alternatives with costs c_1 and c_2
- Basic **logit** model:

$$p_1 = \frac{\exp(-\beta c_1)}{\exp(-\beta c_1) + \exp(-\beta c_2)}$$

and

$$p_2 = \frac{\exp(-\beta c_2)}{\exp(-\beta c_1) + \exp(-\beta c_2)}$$

- $p_1, p_2 > 0$ are the proportions of users taking the two choices. Note $p_1 + p_2 = 1$.
- $\beta > 0$ is the sensitivity to cost



- Logit also called the **softmax** function
- Higher sensitivity β means a harder max, approaching Wardrop setting
- NB a (very small but nonzero) number of people will make each choice, no matter how bad it is

Discrete choice models: a bit more generally

- For n choices, so called **multinomial** choice,

$$p_i = \frac{f(c_i)}{\sum_{j=1}^{n} f(c_j)}$$

will do the 'right' thing, provided $f > 0$ is a decreasing function of c.

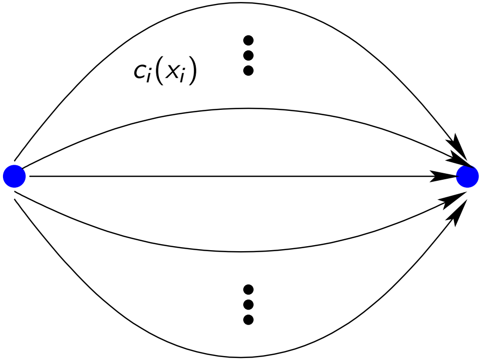
- For Logit $f(c) = \exp(-\beta c)$.
- NB automatically each $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$
- Partial derivatives:

$$\frac{\partial p_i}{\partial c_i} < 0 \quad \text{and} \quad \frac{\partial p_i}{\partial c_j} > 0 \ (i \neq j)$$

(natural interpretation).

- Sometimes — choice models expressed in terms of **utility** (usefulness) u_i for each choice — the opposite of cost, e.g. $u_i = -c_i$.

Parallel networks: SUE model



- Prescribe link cost functions $c_i(x_i)$ for links $i = 1, 2, \ldots, n$.
- Prescribe total demand d.

- Proportions for each choice are $p_i = x_i/d$.
- Solve (using Logit with sensitivity $\beta > 0$)

$$\frac{x_i}{d} = \frac{\exp[-\beta c_i(x_i)]}{\sum_{j=1}^{n} \exp[-\beta c_j(x_j)]}$$

for $i = 1, 2, \ldots, n$

- Note $\sum_{i=1}^{n} x_i = d$ automatically from the construction of the system
- In short n (usually, nasty nonlinear) equations in n unknowns — give it to a computer!

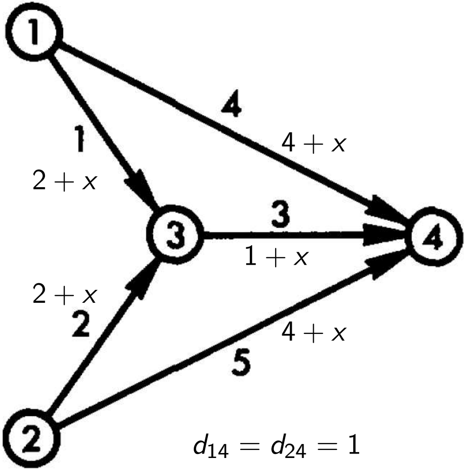
More general networks and SUE

- Unlike Wardrop UE problems, there is no ‘Beckmann trick’, and progress can only be made in the route variables.

- This means:
  - You have to compute (all or representatively some of) the routes
  - You have to compute the link-route incidence matrix
  - You must express link costs in terms of route variables.
  - You then compute route costs in terms of route variables.
  - Route costs are input into the choice function.
  - You get large systems of nonlinear equations to solve for the route variables.

- In general — massive computation, and no easy examples that can be worked out by hand.

SUE solution — cf complementarity example from before



$d_{14} = d_{24} = 1$

R1: $\{4\}$. R2: $\{1,3\}$. R3: $\{5\}$. R4: $\{2,3\}$

- Compute link costs in route variables — formally $\mathbf{c}(\mathbf{Ay})$ — then compute route costs in route variables, is $\mathbf{A}^{\mathrm{T}}\mathbf{c}(\mathbf{Ay})$:

R1: $4 + y_1$

R2: $(2 + y_2) + (1 + y_2 + y_4)$
$= 3 + 2y_2 + y_4$

R3: $4 + y_3$

R4: $3 + y_2 + 2y_4$

- (Instead of complementarity) solve:

$$y_1/d_{14} = f(c_{R_1})/[f(c_{R_1}) + f(c_{R_2})]$$
$$y_2/d_{14} = f(c_{R_2})/[f(c_{R_1}) + f(c_{R_2})]$$
$$y_3/d_{24} = f(c_{R_3})/[f(c_{R_3}) + f(c_{R_4})]$$
$$y_4/d_{24} = f(c_{R_4})/[f(c_{R_3}) + f(c_{R_4})]$$

- Automatically $y_1 + y_2 = d_{14}$ and $y_3 + y_4 = d_{24}$. Unlike Wardop UE, no need to check for unused routes.

Further reading and next steps

- Wikipedia: **Discrete choice**
- Sheffi chapters 10–12
- Warning (1): discrete choice theory should be based formally on **random utility theory** — it gives rise more naturally to **Probit** choice functions
- Warning (2): random utility theory takes careful account of choices which are not truly independent — e.g., routes with some links in common. This is way too hard to cover in a course like this!!!