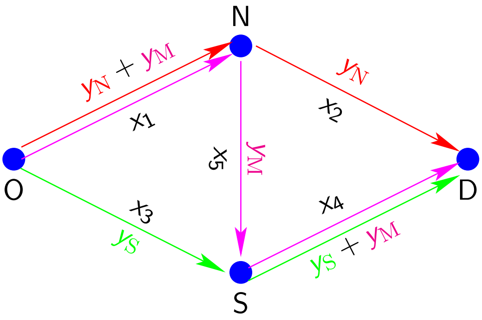
General concepts: links and routes



- Let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}}$ be the vector of **link flows** (alt. **link variables**).

- Let $\mathbf{y} = (y_{\mathrm{N}}, y_{\mathrm{S}}, y_{\mathrm{M}})^{\mathrm{T}}$ be the vector of **route flows** (alt. **route variables**).

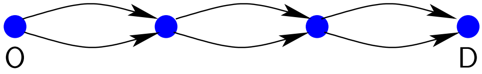
- Then $\mathbf{x} = \mathbf{A}\mathbf{y}$, where here

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is the **link-route incidence matrix**.

- Link flows can always be determined from route flows.
- Here (for Braess), route flows (3 of them) can also be determined from link flows (5 of them).
- More usually: cannot compute **y** from **x**.

More routes than links?



▶ Here: 6 links, but 8 routes

▶ You cannot (uniquely) determine route flows $\mathbf{y}$ from link flows $\mathbf{x}$.

▶ Traffic assignment problems (UE and SO) have a unique solution in link flows but not usually in route flows.

▶ For real-world networks — the number of all possible routes is very large indeed.

Possible coping strategy (!!!!):

▶ Compute only with some of the possible routes (n shortest / most plausible ones?).

▶ Compute the link-route incidence matrix $\mathbf{A}$ (where $\mathbf{x} = \mathbf{A}\mathbf{y}$).

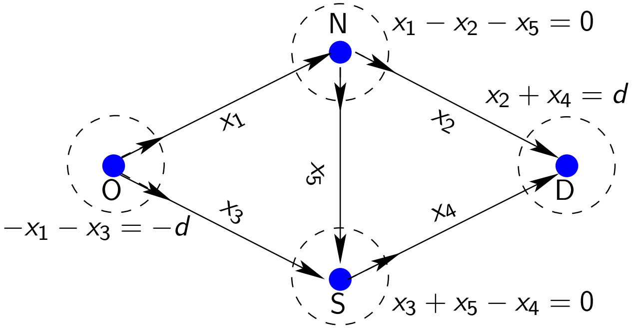
▶ Require throughout $y_1 + y_2 + \ldots + y_n = d$ (demand satisfied) and $y_i \geq 0$.

▶ System cost f and Beckmann function $\hat{f}$ are functions of link flows $\mathbf{x}$, so minimise either $f(\mathbf{A}\mathbf{y})$ or $\hat{f}(\mathbf{A}\mathbf{y})$.

▶ Get (typically) non-unique solution $\mathbf{y}$. Unique $\mathbf{x}$ recovered from $\mathbf{x} = \mathbf{A}\mathbf{y}$.

Solving assignment problems in link variables

- But it is also possible to solve for assignments without <u>ever</u> computing the routes or route flows!!!



- Problem becomes to minimise $f(\mathbf{x})$ or $\hat{f}(\mathbf{x})$ subject to

$$\mathbf{x} \geq \mathbf{0} \quad \text{and} \quad \mathbf{B}\mathbf{x} = \mathbf{s}$$

where $\mathbf{B}$ and $\mathbf{s}$ model conservation of flow or sinks / sources at the nodes.

Solving assignment in link variables — summing up

Let's suppose affine link cost functions $c_i = a_i + b_i x_i$.

Then minimise

$$g(\mathbf{x}) = \mathbf{a}^{\mathrm{T}}\mathbf{x} + \alpha\mathbf{x}^{\mathrm{T}}\mathrm{diag}\{\mathbf{b}\}\mathbf{x},$$

subject to

$$\mathbf{x} \geq \mathbf{0} \quad \text{and} \quad \mathbf{B}\mathbf{x} = \mathbf{s}$$

where $\mathbf{B}$ and $\mathbf{s}$ model conservation of flow or sinks / sources at the nodes.

- $\alpha = 1$ gives $g \equiv f$ and thus SO problem.
- $\alpha = 1/2$ gives $g \equiv \hat{f}$ and thus (Beckmann) UE problem.

Note $\mathbf{a} = (a_1, a_2, \ldots, a_m)^{\mathrm{T}}$ and $\mathrm{diag}\{\mathbf{b}\}$ is the matrix with $b_1, b_2, \ldots, b_m$ on the diagonal and zeroes elsewhere.

These are *quadratic programmes with linear constraints* — and can be solved with the Matlab command quadprog. Note the route variables $\mathbf{y}$ are not involved — in fact, routes and route variables need not even be computed at all.

Doing it in route variables — not usually recommended

Find some routes (approximate, cheap) or all routes (exact, expensive) and find link-route incident matrix so that $\mathbf{x} = \mathbf{A}\mathbf{y}$.

Then minimise

$$g(\mathbf{A}\mathbf{y}) = \left(\mathbf{A}^{\mathrm{T}}\mathbf{a}\right)^{\mathrm{T}}\mathbf{y} + \alpha\mathbf{y}^{\mathrm{T}}\left[\mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\mathbf{b}\}\mathbf{A}\right]\mathbf{y},$$

subject to

$$\mathbf{y} \geq \mathbf{0} \quad \text{and} \quad \mathbf{1}_m^{\mathrm{T}}\mathbf{y} = d.$$

Here the second condition expresses $y_1 + y_2 + \ldots + y_n = d$, that is, the route variables add up to the demand.

For multiple OD pairs, the second condition generalises to $\mathbf{M}\mathbf{y} = \mathbf{d}$, where $\mathbf{M}$ is the *OD-route* incidence matrix and $\mathbf{d}$ is a vector of OD demands.

As we shall see — multiple OD pairs may not be so easy for the link-based formulation.