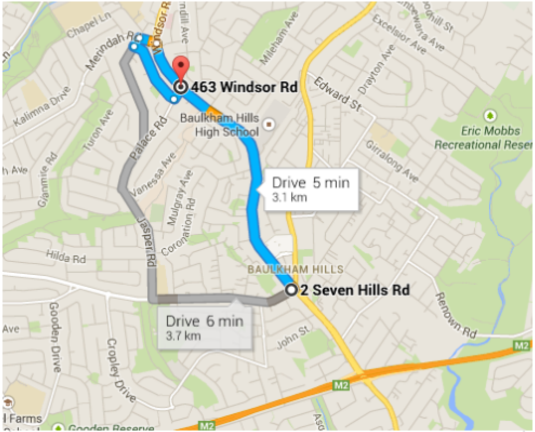
Travel / traffic assignment

- Supposing demand is fixed. How will people travel? What mode (car, bus, train, bike, walk)? If by car, then by what route? The split over these choices is called the **assignment**.
- Factors:
  - Travel time
  - Money (petrol, tickets etc.)
  - Convenience / quality
  - Ethical / environmental principles
- Simplest approach — roll all the factors up in one **(virtual) cost**.
- Assume **rational actors** - each traveller tries to minimise their personal cost.

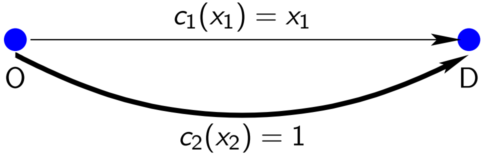


Long road vs the short road

- Rational actors will (surely?) choose the shortest road
- But it becomes congested (expensive) so over time - some people will use the "country road"
- The **(route) assignment** is the amount of traffic taking each route (to be determined).
- **Wardrop principle** — or **User Equilibrium** — the costs of the used routes will be equal and less than or equal to the costs of the unused routes

Congestion games: Pigou's example

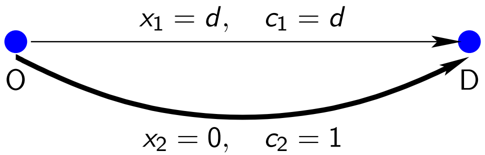
- <u>Basic idea</u>: the cost of a choice goes up if more people choose it.
- Suppose (demand) $d > 0$ is the number of 'users' making the choice (per unit time).
- **Assignment** is the number of users making each choice (per unit time) — to be determined.
- Simplest setting is a binary choice between two options — and for now ignore real-world transport things like routing, multiple origin-destination pairs etc.



- Pigou's example, costs per user (known as **link cost** functions) are $c_1(x_1)$ and $c_2(x_2)$, where x_1 and x_2 are the number of users taking the respective choice (per unit time).
- Link 1 is (kind of) a very short but easily congested road.
- Link 2 is (kind of) a long but very high capacity road.

Pigou's example: low demand and high demand

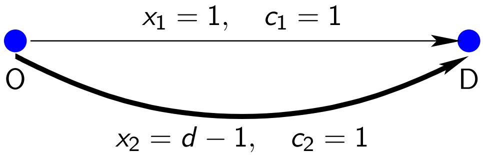
Low demand $d \leq 1$



- Everyone goes the 'shortest' way (link 1).
- Link 2 is more expensive and remains unused.

- Recall: $c_1(x_1) = x_1$ and $c_2(x_2) = 1$.

High demand $d > 1$



- Both links are used and have equal cost (per user) of one.
- As d increases further, all additional traffic goes to the 'high capacity' road (link 2).
- Throughout, we clearly require

$$x_1, x_2 \geq 0 \quad \text{and} \quad x_1 + x_2 = d.$$

Pigou's example: User Equilibrium and System Optimal

- Total (system) cost (per unit time)
$$f = x_1 c_1(x_1) + x_2 c_2(x_2)$$
which gives for the **User Equilibrium** assignment
$$f = \begin{cases} d^2 & \text{for } d \leq 1, \\ d & \text{for } d > 1. \end{cases}$$

- (Sometimes we express system cost as an average per user — here by dividing by demand d.)

- Is this the lowest system cost that can be achieved?

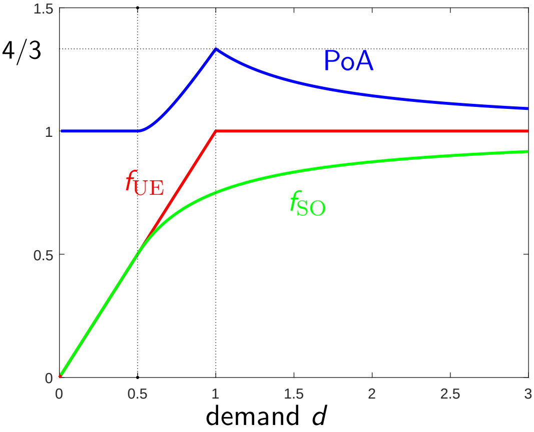
- **System Optimal** assignment minimises system cost f.

- Minimise
$$x_1^2 + x_2$$
subject to
$$x_1 + x_2 = d, \quad \text{and} \quad x_1, x_2 \geq 0.$$

- The optimal solution is
$$f = \begin{cases} d^2 & \text{for } d \leq 1/2, \\ d - 1/4 & \text{for } d > 1/2, \end{cases}$$
and this total cost is <u>lower</u> than for User Equilibrium for $d > 1/2$.

Pigou's example: the Price of Anarchy (PoA)

- Per user system costs f_{UE} and f_{SO} and their ratio $\mathrm{PoA} := f_{\mathrm{UE}}/f_{\mathrm{SO}}$.



- There is often a kind of 'system' penalty for selfish free will — known as the **Price of Anarchy**

- For Pigou's network, the PoA is $4/3$ at demand $d = 1$, which is the theoretical maximum for affine cost functions.

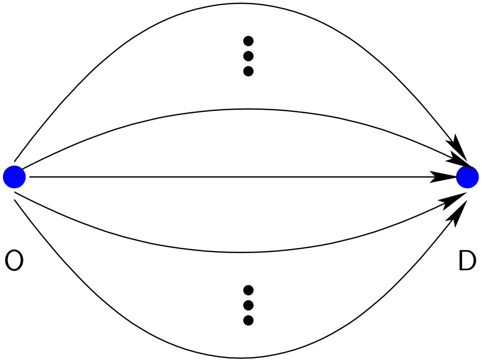
- Although the System Optimal assignment achieves lower system costs for $d > 1/2$, it is unfair.

- For $d > 1/2$, SO has $x_1 = 1/2$ (thus $c_1 = 1/2$) and $x_2 = d - 1/2$ ($c_2 = 1$), so the users on link 1 get a lucky deal.

Pigou's example: Beckmann formulation

- To solve for UE — we in effect solved the following **complementarity** problem
  - $c_1(x_1) = c_2(x_2)$ with $x_1 + x_2 = d$ and $x_1, x_2 \geq 0$; OR
  - $x_1 = 0$, with $c_1(0) \geq c_2(x_2 = d)$ (doesn't happen in our example); OR
  - $x_2 = 0$, with $c_2(0) \geq c_1(x_1 = d)$ (happens for $d < 1$)
- Handling the various cases is a bit of a nightmare!!!
- So solve the equivalent **Beckmann** formulation:
  Minimise
  $$\hat{f} := \int_0^{x_1} c_1(\tilde{x})\,\mathrm{d}\tilde{x} \quad + \quad \int_0^{x_2} c_2(\tilde{x})\,\mathrm{d}\tilde{x},$$
  subject to $x_1 + x_2 = d$ and $x_1, x_2 \geq 0$.
- For the Pigou example, $\hat{f} = \frac{1}{2}x_1^2 + x_2$, cf total system cost $f = x_1^2 + x_2$.
  So use same solution methods as SO problem.
  Cases distinguished by internal vs boundary minima.

General choice problem (no routing)

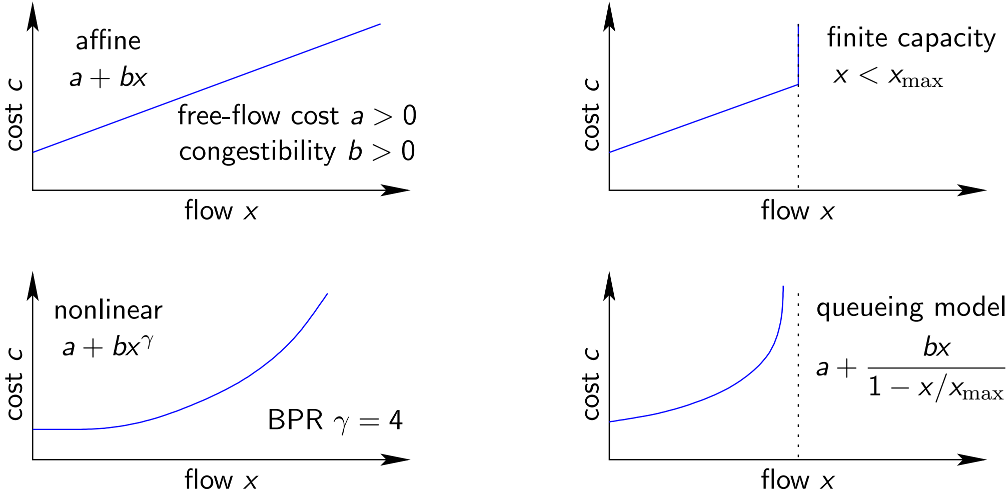


- Suppose n parallel links with flows x_i and cost functions c_i.
- Constraints $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = d$ must always hold.

- SO problem. Minimise
$$\sum_{i=1}^{n} x_i c_i(x_i)$$
- UE problem (Beckmann). Minimise
$$\sum_{i=1}^{n} \int_0^{x_i} c_i(\tilde{x}) \,\mathrm{d}\tilde{x}$$
- UE problem (Complementarity).

For each subset $\mathcal{I} := \{i_1, i_2, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$, solve
$c_{i_1}(x_{i_1}) = c_{i_2}(x_{i_2}) = \ldots = c_{i_m}(x_{i_m}) =: \mathcal{C}$,
and check $c_i(0) \geq \mathcal{C}$ for all $i \notin \mathcal{I}$.

Extensions — more interesting (realistic?) link costs



Further reading / exercises

- Tim Roughgarden's notes https://theory.stanford.edu/~tim/f13/l/l11.pdf
- Ben Heydecker's popular article https://ima.org.uk/3410/network-models-route-choice/
- Yosef Sheffi's book http://web.mit.edu/sheffi/www/selectedMedia/sheffi_urban_trans_networks.pdf Start with part 1 chapter 1
- General background search (e.g. Wikipedia) **Braess's paradox**, **John Glen Wardrop**, **Convex optimization**, **Congestion game**, **Price of anarchy**, **Nash equilibrium**, **BPR (bureau of public roads) functions**, **Route assignment**
- Calculations?

1. Two parallel links with other link costs, e.g. $1 + 2x$ and $2 + x$
2. Investigate price of anarchy for two parallel links and nonlinear costs.
3. Re-work Braess network calculations for alternative link costs — see internet