Assignment practice problems: background, definitions, parallel networks and the Braess paradox

R.E. Wilson

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1 Definitions etc.

1. In transport — what are the modes? Which of them are classed as sustainable modes? Which of them are classed as active modes?

2. Describe the various components of a virtual cost, in so far as they relate to travel choice.

3. (In words) what is route assignment?

4. Give a concise statement of Wardrop's (user equilibrium) principle, in so far as it relates to route assignment.

5. Describe (in words) the complementarity method for solving user equilibrium (UE) problems.

6. Define (in words) the system optimal (SO) assignment.

7. Define (in words) the price of anarchy.

8. Research the BPR cost functions and describe typical parameters that are associated to them.

9. Describe in general terms (without discussing particular network details) the Braess Paradox effect.

10. Describe (in words) the Beckmann formulation of UE problems.

11. Compare and contrast the complementarity and Beckmann formulations for solving the UE problem with a single OD pair.

2 Parallel networks

These questions relate to the parallel network shown in Fig. 1, with n links. The simple case where $n = 2$ and the cost functions are $c_1(x_1) = x_1$ and $c_2(x_2) = 1$ recovers the Pigou example used in the lecture.

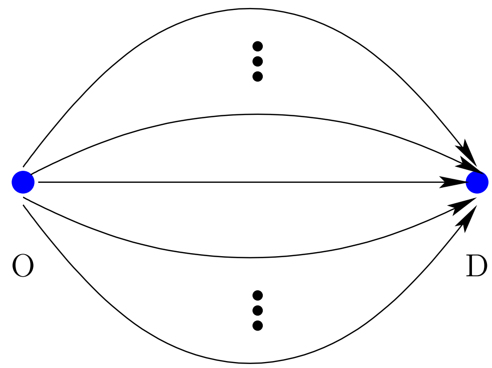

Figure 1: Parallel network with n links and a single OD pair with demand $d > 0$. Denote the link flows $x_1, x_2, \ldots, x_n \geq 0$ with costs $c_1(x_1), c_2(x_2), \ldots, c_n(x_n) \geq 0$.

12. Extend the Pigou example by computing the UE and SO flows for a network with two parallel links and cost functions $c_1(x_1) = 1/2 + x_1$ and $c_2(x_2) = 1 + x_2/2$. Your answer should consider general behaviour over a range of demand d from 0 to ∞. You may wish to begin by considering a few special values: e.g., $d = 1/2$, $d = 1$, $d = 2$ etc. You should:

    - Compute the UE solution by both complementarity and Beckmann (minimisation) methods.
    - Compute the SO solution by a minimisation method.
    - Compute the Price of Anarchy and find its maximum value and the value of d at which this maximum is achieved.

13. Extend the previous question to consider more general cost functions $c_1(x_1) = a + x$ and $c_2(x_2) = 1 + bx$, where $a, b \geq 0$ are parameters. Verify (*) that the PoA is bounded above by 4/3 in all cases.

14. (*) Consider a parallel network with $n = 3$ links and cost functions $c_1(x_1) = 1$, $c_2(x_2) = x$ and $c_3 = a + bx$, where $a, b > 0$ are parameters. Compute the UE solutions for all possible $a, b, d > 0$ considering the complementarity problem and the various possible patterns of used and unused links. Investigate SO solutions (hard: requires minimisation in two variables) and thus the PoA. Extension: replace $c_1(x_1)$ and $c_2(x_2)$ with the forms in Q12.

15. (*) By considering a pair $n = 2$ of parallel links, and by designing your own nonlinear cost functions, investigate whether the PoA is still bounded above by 4/3.

3 Braess paradox

This section relates to the Braess network with nodes, links and routes as described in Fig. 2.

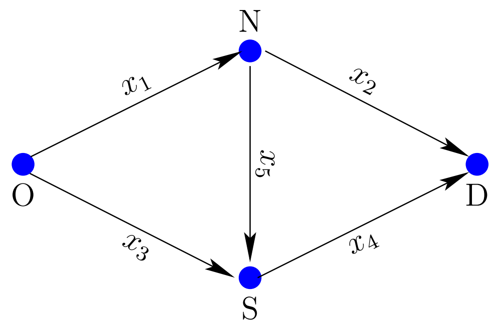

Figure 2: Braess network with labelling of nodes and link flows. The northern route is links 1,2. The southern routes is links 3,4. The mid-town route is links 1,5,4.

16. Work through the lecture example carefully so that you understand it. (*) Will the network display the Braess paradox if we change the value of demand d? (*) Will the network still display the paradox if we change the link costs over so that c_1 and c_4 are constant and c_2 and c_3 are proportional to flow? (*) Investigate what happens if the mid-town link is replaced by one of a constant cost $a > 0$, for various values of a?

17. Work through the UE and SO solutions of the Braess network for the parameters given in Sheffi p76 (p92 in the pdf). Here $c_1 = 10x_1$, $c_4 = 10x_4$, $c_2 = 50 + x_2$, $c_3 = 50 + x_3$ and $c_5 = 10 + x_5$, and demand $d = 6$ units. Hence demonstrate the Braess paradox. (Note Sheffi draws the network the other way up from me, with the mid-town link going up the page.)

18. Consider the Braess network with $c_1(x_1) = a + x_1$, $c_4(x_4) = a + x_4$, $c_2(x_2) = 1 + bx_2$, $c_3(x_3) = 1 + bx_3$ and $c_5(x_5) = 0$. Derive conditions on the parameters a, b and $d > 0$ for the Braess paradox to be displayed.