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HW 7

1.) Likelihood $= p(x_t|\lambda) = \begin{cases} (\lambda^{x_t} e^{-\lambda})/x_t! & x_t \geq 0 \\ 0 & o.w \end{cases}$

prior $p(\lambda) = \begin{cases} e^{-\lambda} & \text{for } \lambda > 0 \\ 0 & o.w \end{cases}$

a.) $E[\lambda] = \int_0^\infty \lambda e^{-\lambda} d\lambda$ $\quad u = \lambda \quad du = d\lambda$ $\quad \frac{dv}{d\lambda} = e^{-\lambda} \quad v = -e^{-\lambda}$

$= \lambda e^{-\lambda}\Big|_0^\infty + \int_0^\infty e^{-\lambda} d\lambda = -e^{-\lambda}\Big|_0^\infty = 1$

$\boxed{E[\lambda] = 1}$

b.) $p(x_1, x_2, \ldots, x_n | \lambda)$

$= \prod_i \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \boxed{\frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod_i x_i!}}$

=

$\frac{p(\text{Data}|\lambda)\, p(\lambda)}{\int p(\text{Data}|\lambda)\, p(\lambda)\, d\lambda}$

c.) posterior $p(\lambda | x_i, x_2, \ldots, x_n) \propto p(x_1, x_2, \ldots x_n | \lambda) \cdot p(\lambda)$

$\propto \boxed{\frac{\lambda^r e^{-\lambda(n+1)}}{\prod_{i=1}^n x_i!}} \propto \lambda^r e^{-\lambda(n+1)}$

d) const. of proportionality $= \frac{1}{\int_0^\infty P(x_1, x_2 \ldots x_n|\lambda)\, p(\lambda)\, d\lambda}$

$\frac{1}{c} = \int_0^\infty \frac{\lambda^r e^{-\lambda(n+1)}}{\prod_i x_i!} d\lambda = \frac{r!}{\prod_{i=1}^n x_i! (n+1)^{r+1}} \Rightarrow \boxed{c = \frac{(n+1)^{r+1}}{r!} \prod x_i!}$

then the posterior:

$$p(\lambda \mid x_1, x_2, \ldots x_n) = \frac{p(x_1, x_2, \ldots x_n \mid \lambda)\; p(\lambda)}{\int_0^\infty p(x_1, x_2, \ldots x_n \mid \lambda)\; p(\lambda)\, d\lambda}$$

$$\frac{\cancel{\prod x_i!}\,(n+1)^{r+1}}{r!} \frac{\lambda^r e^{-\lambda(n+1)}}{\cancel{\prod x_i!}}$$

we can immediately see this is a Gamma distribution

$$f(\lambda, \alpha, \beta) = \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}$$ where $\beta = n+1$

$\alpha = r+1$

$\Gamma(r+1) = r!$

$$\boxed{\text{so } p(\lambda \mid x_1, \ldots x_n) \sim \text{Gamma}(r+1, n+1)}$$

e.) $n = 10$, $r = \sum x_i = 35$

find $E(\lambda \mid x_1, \ldots x_n)$

$$= \int_0^\infty \frac{\lambda^{r+1} (n+1)^{r+1} e^{-\lambda(n+1)}}{\Gamma(r+1)} d\lambda = \boxed{3.27} = \frac{\Gamma(r+2)}{(n+1)\, r!}$$

via running R code

f.) see R code

$$\boxed{\lambda_l = 2.3 \;,\; \lambda_r = 4.4}$$

3.) Qu 8.3

Let $A = \{0.4 < X \leq 0.6\}$, Find $E(X), V(X), Pr(A)$

a.) Bernoulli

$$f(x) = \begin{cases} p^x (1-p)^{(1-x)} & \text{for } x = 0,1 \\ 0 & \text{o.w} \end{cases}$$

$$E(x) = \sum_{i \in X} x_i f(x_i) = 0(1-p) + 1p = \boxed{p} = \boxed{0.5}$$

$$E(X^2) = \sum_{i=0}^{1} x_i^2 f(x_i) = p, \quad E^2(X) = p^2$$

$$V(X) = E(X^2) - E^2(X) = p - p^2 = \boxed{p(1-p)} = \boxed{1/4}$$

$$\boxed{Pr(A) = 0}$$

b.) $X \sim N(\mu = 0.5, \sigma^2 = 0.25)$

we know for a normal distribution

$$\boxed{\begin{aligned} E(X) &= \mu = 0.5 \\ V(X) &= \sigma^2 = 0.25 \end{aligned}}$$

$$Pr(A) = \int_{0.4}^{0.6} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} = \boxed{0.158519}$$ using R

c.) Exponential $\lambda = 2$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0,\ \lambda > 0 \\ 0 & x \le 0 \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x) = \lambda \int_0^{\infty} x e^{-\lambda x}\, dx$$

$u = \lambda x \quad du = \lambda dx$

$\frac{dv}{dx} = e^{-\lambda x} \quad v = -\frac{e^{-\lambda x}}{\lambda}$

$$= x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$$

$$0 + \frac{1}{\lambda} \Rightarrow \boxed{E(x) = \frac{1}{\lambda}} = \boxed{\frac{1}{2}}$$

$$V(x) = E(x^2) - E^2(x)$$

$$= \lambda \int_0^{\infty} x^2 e^{-\lambda x}\, dx - \frac{1}{\lambda^2}$$

$u = x^2 \quad \frac{dv}{dx} = e^{-\lambda x}$

$\frac{du}{dx} = 2x \quad v = -\frac{1}{\lambda} e^{-\lambda x}$

$$= \lambda \left[-\frac{1}{\lambda} x^2 e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} \frac{2x e^{-\lambda x}}{\lambda}\, dx \right]$$

$$= 0 + 2 \int_0^{\infty} x e^{-\lambda x}\, dx = 2 \frac{E(x)}{\lambda} - E(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}} = \boxed{\frac{1}{4}}$$

$$P(A) = \int_{0.4}^{0.6} \lambda e^{-\lambda x} = -e^{-2x} \Big|_{0.4}^{0.6} = \boxed{0.14815}$$

4) 8.4

c.) $\bar{X} \sim \text{Exponential}(\lambda)$

$k = 2n = 20$

then $F(\bar{X} \le c) = F(k\lambda\bar{X} \le k\lambda c)$

$E(\bar{X}) = \mu = \frac{1}{\lambda} = \boxed{\frac{1}{2}}$

$V(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{n\lambda^2} = \boxed{\frac{1}{40}}$

$P(B) = P(k\lambda(0.6)) - P(k\lambda(0.4))$

$= P(\approx 24) - P(\approx 16) \sim \boxed{0.5}$ using the Table A.2 from Goldberger

(hard to get exact numbers from table)

Increasing the sample size lowers the variance of the sample as $1/n$. as the width of the distribution shrinks