Reasoning under uncertainty

Universidad Carlos III de Madrid

AI

Outline

1. Introduction
2. Probabilistic reasoning
3. Bayesian networks
4. Markov Models
5. Fuzzy logic

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   - Hidden Markov Models
   - Markov Decision Processes (MDP)
   - Partially Observable MDPs (POMDPs)

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Summary of Probability and BN

- Probabilities can represent uncertainty in AI
- Inference by enumeration solves questions such as P(event|observation)
- Conditional independence allows for a more compact representation of probability distribution
- Bayesian networks is a graph representation based on conditional independence
- We can use exact or stochastic methods for inference

Motivation: Probability + time + actions

- Probabilistic reasoning:
  - How can we add the notion of time?
    - what is the probability of a user cliking in one of our web pages if she is now visiting one of our web pages?
    - Idea: We can create random variables for each time step: $\text{Web}_t, \text{Web}_{t+1}, \ldots$

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    - we have used search algorithms to select a sequence of actions to solve a problem, but what happens when actions are not deterministic?
    - how can we select actions such that we have the highest probability of reaching the goal state?

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- Tasks
  - Markov chains
  - Hidden Markov Models (HMMs)
  - Markov Decision Processes (MDPs)
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Markov Chains (Markov Processes)

- A sequence of random variables $X_0, X_1, \ldots, X_t$
- They represent the value of a random variable X over time
- Full observability
- Can be modeled as a chain-structured Bayesian network

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots$$

  - each node is identically distributed
  - (belief) state: probability distribution of X at a given time
  - initial state (probability distribution): $P(X_0)$
  - transition probabilities: how state evolves over time

$$P(X_t | X_{t-1})$$

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$$P(X_t | X_{t-1})$$

- Markov property: the past and the future are independent given the present

Example. User visiting our/other web pages

- States:
  - $X_0 = X_1 = \ldots =$ {other, our}

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- Transitions:

X_{t-1}	$P(X_t$=our)	$P(X_t$=other)
our	0.9	0.1
other	0.1	0.9

Example. User visiting our/other web pages

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X_{t-1}	$P(X_t$=our)	$P(X_t$=other)
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- Initial state: $P(X_0 =$ our$) = 1.0$, $P(X_0 =$ other$) = 0.0$

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- Transitions:

X_{t-1}	$P(X_t\text{=our})$	$P(X_t\text{=other})$
our	0.9	0.1
other	0.1	0.9

- Initial state: $P(X_0 = \text{our}) = 1.0$, $P(X_0 = \text{other}) = 0.0$
- What is the probability distribution after one step?

Example. User visiting our/other web pages

- States:
  - $X_0 = X_1 = \ldots =$\{other, our\}
  - $X_{t-1} = X_t =$\{other, our\}
- Transitions:

X_{t-1}	$P(X_t{=}\text{our})$	$P(X_t{=}\text{other})$
our	0.9	0.1
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- Initial state: $P(X_0 = \text{our}) = 1.0$, $P(X_0 = \text{other}) = 0.0$
- What is the probability distribution after one step?

$$
\begin{aligned}
P(X_1 = \text{our}) = \textstyle\sum_{i \in \{\text{our,other}\}} & P(X_1 = \text{our} | X_0 = i) P(X_0 = i) = \\
& P(X_1 = \text{our} \mid X_0 = \text{our}) P(X_0 = \text{our}) + \\
& P(X_1 = \text{our} \mid X_0 = \text{other}) P(X_0 = \text{other}) = \\
& 0.9 \times 1.0 + 0.1 \times 0.0 = 0.9
\end{aligned}
$$

$$P(X_1 = \text{other}) = 1 - P(X_1 = \text{our}) = 0.1$$

How about predicting the future?

- From initial observation of our

$$\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix} \quad \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} \quad \begin{pmatrix} 0.82 \\ 0.18 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$P(X_0) \quad P(X_1) \quad P(X_2) \quad \ldots \quad P(X_\infty)$$

- From initial observation of other

$$\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} \quad \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \quad \begin{pmatrix} 0.18 \\ 0.82 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$P(X_0) \quad P(X_1) \quad P(X_2) \quad \ldots \quad P(X_\infty)$$

Stationary Distributions

- If we simulate the chain long enough
  - uncertainty accumulates
  - eventually, we have no idea what the state is!
- Stationary distributions
  - for most chains, the distribution we end up in is independent of the initial distribution
  - usually, we can only predict a short time out

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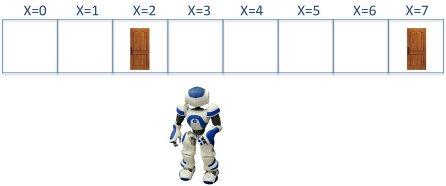
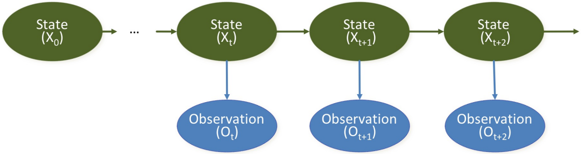
Hidden Markov Models

- Observations (partial)
- As a BN (localize a robot):





Hidden Markov Models

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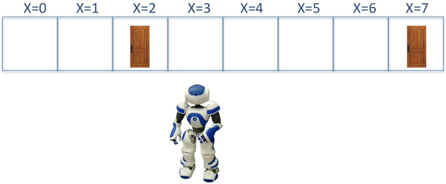
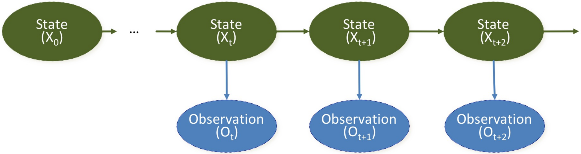
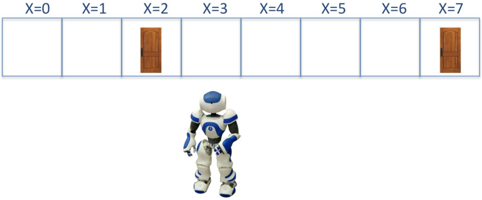
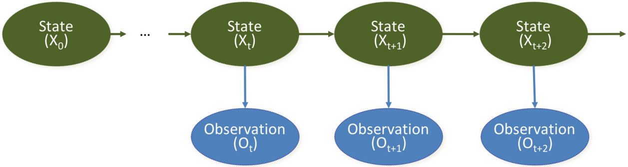




- Independence properties:
  - Markov property (influences the values of X_t)
  - the current observation is independent of everything else given current state (influences the values of O_t)

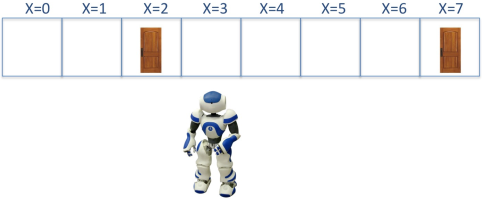
Example of HMM

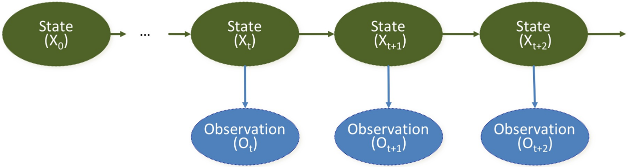




- State: probability distribution of the position of robot $P(X_t)$
- Initial state: $X_0 = 3 \rightsquigarrow$
  $P(X_0 = 0) = 0.0 \ldots P(X_0 = 3) = 1.0 \ldots P(X_0 = 7) = 0.0$
- Observation: it sees a door or not $P(Door)$

Simple example of HMM

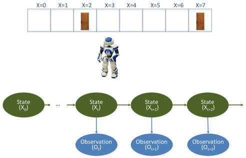




An HMM is defined by:

- Initial distribution: $P(X_0)$
- Transitions: $P(X_t|X_{t-1})$
- Observations (emissions): $P(O_t|X_t) = P(O|X)$

Ejemplo de HMM



- Distribución inicial: $P(X_0) = \langle 0, 0, 0, 1, 0, 0, 0, 0 \rangle$
- Transiciones: $P(X_t | X_{t-1})$

X_{t-1}	$P(X_t = 0)$	$P(X_t = 1)$	$\dots$	$P(X_t = 7)$
0	0.1	0.9	$\dots$	0.0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
7	0.0	0.0	$\dots$	0.1

- Observaciones: $P(O_t | X_t) = P(O | X)$

X_t	$P(\text{Puerta}_t = yes)$	$P(\text{Puerta}_t = no)$
0	0.0	1.0
$\dots$	$\dots$	$\dots$
7	1.0	0.0

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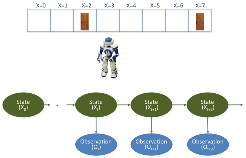

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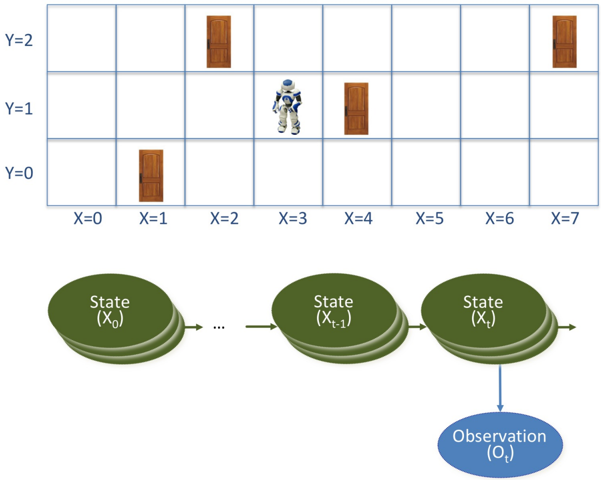
X_{t-1}	$P(X_t = 0)$	$P(X_t = 1)$	$\ldots$	$P(X_t = 7)$
0	0.1	0.9	$\ldots$	0.0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
7	0.0	0.0	$\ldots$	0.1

- Observations: $P(O_t | X_t) = P(O | X)$

X_t	$P(\mathrm{Door}_t = yes)$	$P(\mathrm{Door}_t = no)$
0	0.0	1.0
$\ldots$	$\ldots$	$\ldots$
7	1.0	0.0

More complex HMM



More complex HMM

- States: $S = (X, Y)$
- Probability distributions of states:
  - $P(X) = \langle P(X = 0), P(X = 1), \ldots, P(X = 7) \rangle$
  - $P(Y) = \langle P(Y = 0), P(Y = 1), P(Y = 2) \rangle$
  - $P(S) = P(X, Y) = \langle P(X = 0, Y = 0), \ldots, P(X = 7, Y = 2) \rangle$
- Initial distribution:
  - $P(X_0) = \langle 0, 0, 0, 1, 0, 0, 0, 0 \rangle$
  - $P(Y_0) = \langle 0, 1, 0 \rangle$
  - $P(S_0) = \langle 0, 0, \ldots, 1, \ldots, 0 \rangle$
- Observations: robot sees door (yes, no)

More complex HMM

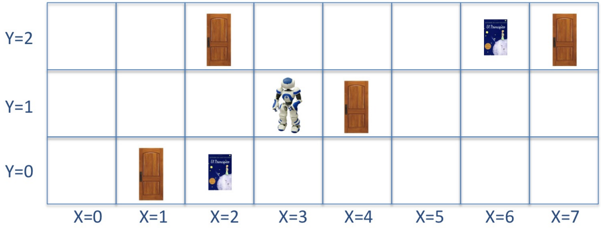
- Transitions: $P(S_t|S_{t-1})$

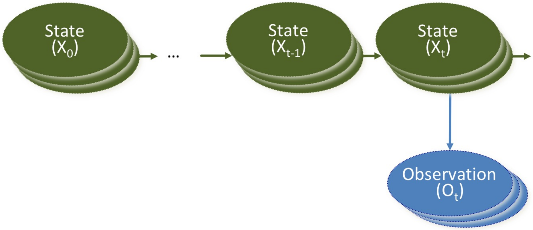
X_{t-1}	Y_{t-1}	$P(X_t = 0)$	$\dots$	$P(X_t = 7)$	$P(Y_t = 0)$	$P(Y_t = 1)$	$P(Y_t = 2)$
0	0	0.3	$\dots$	0.0	0.4	0.6	0.0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	
7	2	0.0	$\dots$	0.6	0.0	0.4	0.6

- Observations: $P(O_t|S_t) = P(O|S) = P(O|X, Y)$

X_t	Y_t	$P(O_t = yes)$	$P(O_t = no)$
0	0	0.0	0.0
1	0	1.0	0.0
$\dots$	$\dots$	$\dots$	$\dots$
7	2	1.0	0.0

Yet a more complex HMM





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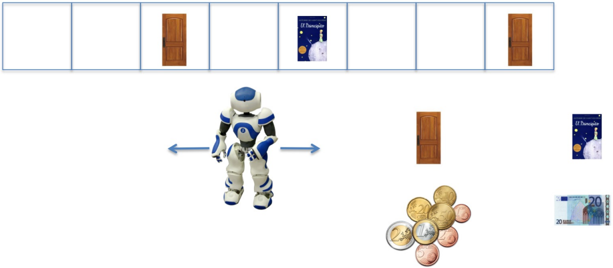
Markov Decision Processes (MDP)



- In HMMs we were following the robot: inferring where it was (from partial observations)
- In MDP, we want to control the robot:

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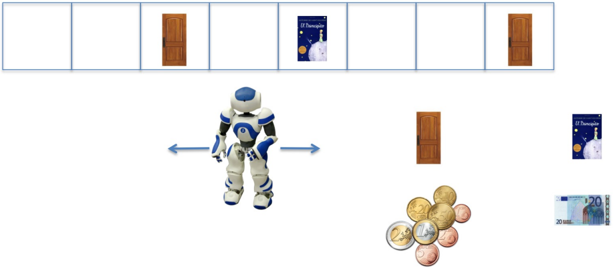

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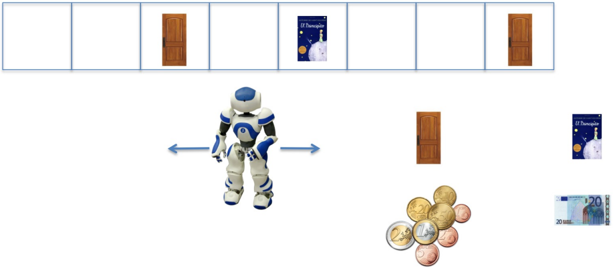

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  - such that they maximize the reward

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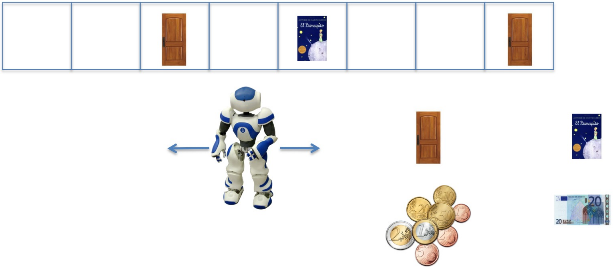

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  - select actions
  - such that they maximize the reward
  - over time

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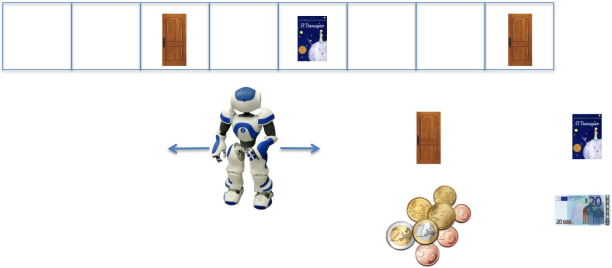
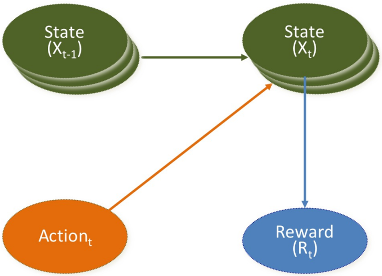


- In HMMs we were following the robot: inferring where it was (from partial observations)
- In MDP, we want to control the robot:
  - select actions
  - such that they maximize the reward
  - over time
  - based on full observations: complete state

Markov Decision Processes (MDP). BN's view



Markov Decision Process (MDP)

- Markov process where we make a decision at every round
- MDP: $< S, A, T, R >$
  - S: set of states (including initial and goal state)
  - A: set of actions
  - T: transition function

$$T(s, a, s') = P(S_{t+1} = s' | S_t = s, A_t = a)$$

  - R: reward function

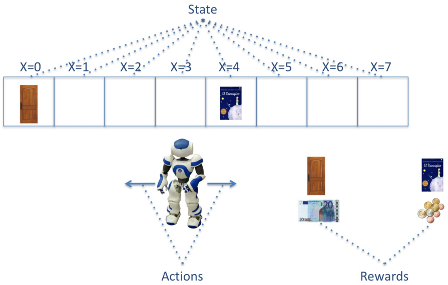
$$R(S_t = s, A_t = a) = R(s, a)$$

    - sometimes just $R(s)$

Example

MDP Tuple: $< S, A, T, R >$

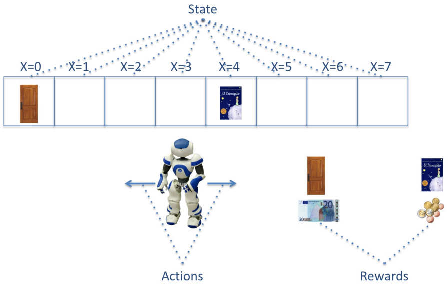
- S: position of robot
  - initial state: $X_0 = 3$
  - goal state: $X_n = 0$



Example

MDP Tuple: $< S, A, T, R >$

- S: position of robot
  - initial state: $X_0 = 3$
  - goal state: $X_n = 0$
- A: left, right

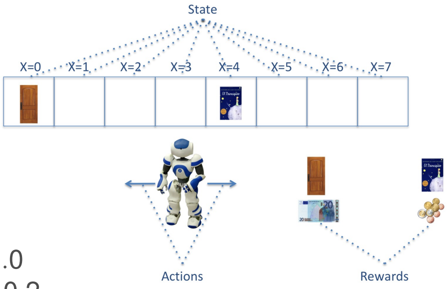


Example

MDP Tuple: $< S, A, T, R >$

- S: position of robot
  - initial state: $X_0 = 3$
  - goal state: $X_n = 0$
- A: left, right
- T: transition function
  - $P(X_t = 0 | X_{t-1} = 0, \text{left}) = 1.0$
  - $P(X_t = 0 | X_{t-1} = 0, \text{right}) = 0.2$
  - $P(X_t = 1 | X_{t-1} = 0, \text{left}) = 0.0$
  - $P(X_t = 1 | X_{t-1} = 0, \text{right}) = 0.8$
  - $\ldots$

  If action is not applicable, the robot stays in the same position



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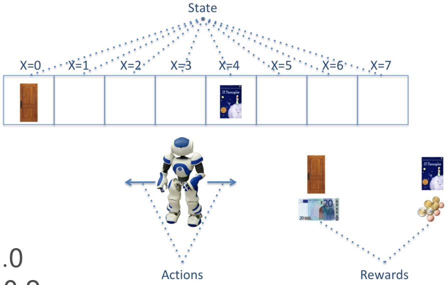

MDP Tuple: $< S, A, T, R >$

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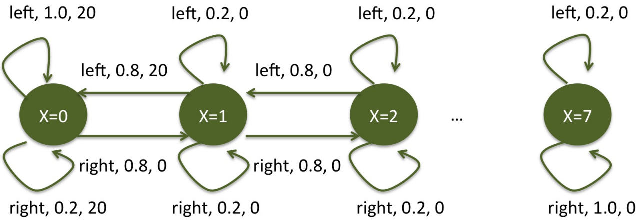
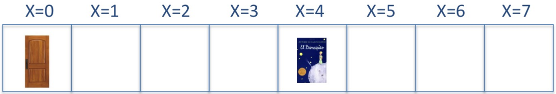
  - initial state: $X_0 = 3$
  - goal state: $X_n = 0$
- A: left, right
- T: transition function
  - $P(X_t = 0 | X_{t-1} = 0, \text{left}) = 1.0$
  - $P(X_t = 0 | X_{t-1} = 0, \text{right}) = 0.2$
  - $P(X_t = 1 | X_{t-1} = 0, \text{left}) = 0.0$
  - $P(X_t = 1 | X_{t-1} = 0, \text{right}) = 0.8$
  - $\ldots$
    If action is not applicable, the robot stays in the same position
- R: reward
  - $R(X = 0, \text{left}) = R(X = 0, \text{right}) = R(X = 0) = 20$
  - $R(X = 1, \text{left}) = R(X = 1, \text{right}) = R(X = 1) = 0$
  - $\ldots$
  - $R(X = 4, \text{left}) = R(X = 4, \text{right}) = R(X = 4) = 3.88$
  - $\ldots$

Example. MDP as a Probabilistic state machine





Markov Decision Process (MDP)

- MDP: $< S, A, T, R >$
- Task: choose a sequence of actions
    - not just one decision or one action
    - utility based on a sequence of decisions

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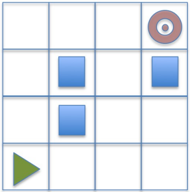
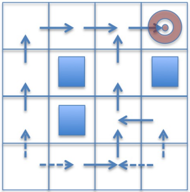
- MDP: $< S, A, T, R >$
- Task: choose a sequence of actions
  - not just one decision or one action
  - utility based on a sequence of decisions
- In order to choose a sequence of actions, a policy is computed

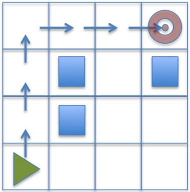
Path (plan) vs Policy (Deterministic)



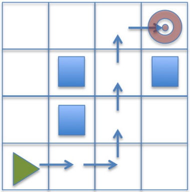
Task



Optimal policy



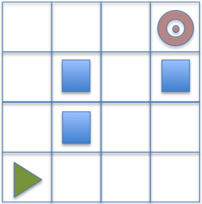
Optimal path



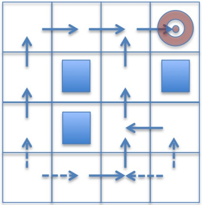
Optimal path

Path (plan) vs Policy (probabilistic)

- Stochastic Action: achieves the desired effect with 0.8 probability
- Example: Action $\leftarrow$ moves to the Left with 0.8 probability, Up with 0.1 probability and Down with 0.1 probability
- There is no optimal path
- Policy knows what to do regardless of the effect of any action at any time-step



Task



Optimal policy

Policies

- Policy: complete mapping from states to actions
- Policy is like a plan, but not quite
  - generated ahead of time, like a plan
- Unlike traditional plans, it is not a sequence of actions that an agent must execute
  - if there are failures in execution, the agent can continue executing a policy
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- Maximizes expected reward, rather than just reaching a goal state
- For every MDP there exists an optimal policy
  - for every possible start state, there is no better option than to follow the policy

Optimal policy

- We can reason about
  - maximizing expected reward, or
  - minimizing expected cost
- Optimal Policy π^*: lowest expected cost
- Cost models
  - deterministic: if action a always changes the state to s', the cost is

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How can we define an optimal policy?

Expected value of a state

V(*s*): expected cost to arrive to the goal from *s*

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- Search:
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How do we compute $V(S)$?

Computing $V(S)$. Deterministic

- Bellman Equation:
  - If s is a goal, $V(s) = 0$
  - Otherwise (s' is the new state after applying a at state s):

$$V(s) = \min_{a \in A(s)} [c(a) + V(s')]$$

$$\begin{aligned}
V(S_0) &= \min_{a \in A(s_0)} [c(a) + V(s')] \\
V(S_1) &= \min_{a \in A(s_1)} [c(a) + V(s')] \\
\ldots & \\
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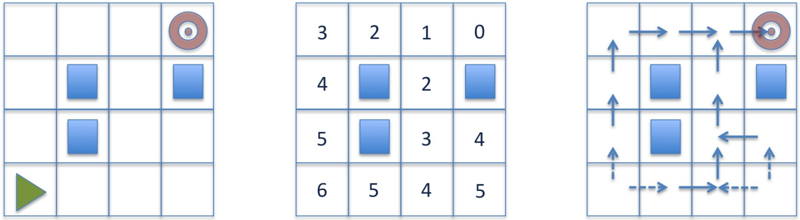
- Optimal Policy (s' is the new state after applying a at state s)

$$\pi^*(s) = \arg\min_a [c(a) + V(s')]$$

Path vs Policy (deterministic)



Task Values Optimal policy

Bellman Equation for MDPs

- Stochastic domains
  - the expected value of action *a* (and perfect execution from there):

$$c(a) + \sum_{s' \in S} P(s'|s, a)V(s')$$

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- Optimal Policy

$$\pi^*(s) = \arg\min_a [c(a) + \sum_{s' \in S} P(s'|s,a)V(s')]$$

Path vs Policy (probabilistic)

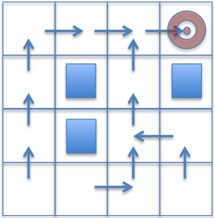
- Stochastic Action: achieves the desired effect with 0.8 probability
- Example: Action $\leftarrow$ moves to the Left with 0.8 probability, Up with 0.1 probability and Down with 0.1 probability

3	2	1	0
4		2	
5		3	4
6	5	4	5

Deterministic values

3.7	2.5	1.2	0.0
5.0		2.5	
6.2		3.9	5.3
7.4	6.7	5.4	6.4

Probabilistic values



Optimal policy

Solving the Bellman equation

Value Iteration Algorithm

$V(s) = 0$ for all states s
while not(Termination) do
    for each state $s \in S$
        $V(s) := \min_{a \in A(s)}[c(a) + \sum_{s' \in S} P(s'|s,a)V(s')]$
Return $[V(s_1), \ldots, V(s_n)]$

Example of Value Iteration

- States: s_1, s_2, s_3
- Goal: s_3
- Action: Right
  - transits from s_i to s_{i+1} with probability 0.8
  - with probability 0.2, it stays in s_i
  - cost: 1
- At start: $V(s_1) = V(s_2) = 0$
- $V(s_3) = 0$ always, because it is the goal

Example of Value Iteration

$$
\begin{aligned}
\text{Bellman eq: } V(s_1) &= c(\text{Right}) + (0.8V(s_2) + 0.2V(s_1) + 0V(s_3)) = 1 \\
V(s_2) &= c(\text{Right}) + (0.8V(s_3) + 0.2V(s_2) + 0V(s_1)) = 1 \\
\text{Iteration 1: } V(s_1) &= 1 + (0.8 \times 0 + 0.2 \times 0 + 0 \times 0) = 1 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 0 + 0 \times 0) = 1 \\
\text{Iteration 2: } V(s_1) &= 1 + (0.8 \times 1 + 0.2 \times 1) = 2 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 1) = 1.2 \\
\text{Iteration 3: } V(s_1) &= 1 + (0.8 \times 1.2 + 0.2 \times 2) = 2.36 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 1.2) = 1.24 \\
\text{Iteration 4: } V(s_1) &= 1 + (0.8 \times 1.24 + 0.2 \times 2.36) = 2.464 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 1.24) = 1.248 \\
\text{Iteration 5: } V(s_1) &= 1 + (0.8 \times 1.248 + 0.2 \times 2.464) = 2.4912 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 1.248) = 1.2496 \\
\text{Iteration 6: } V(s_1) &= 1 + (0.8 \times 1.2496 + 0.2 \times 2.4912) = 2.49792 \\
V(s_2) &= 1 + (0.8 \times 0 + 0.2 \times 1.2496) = 1.24992
\end{aligned}
$$

Termination on Value Iteration

- Value Iteration ends when it reaches a *fixed point*, i.e., when values do not change from one iteration to the next
- In practice, it stops when $\max_s \mid V(s)^t - V(s)^{t+1} \mid \leq \epsilon$
- For $\epsilon = 0.1$ the previous example stops after Iteration 5
- For $\epsilon = 0.01$ the previous example does not stop after Iteration 6

Outline

1 Introduction

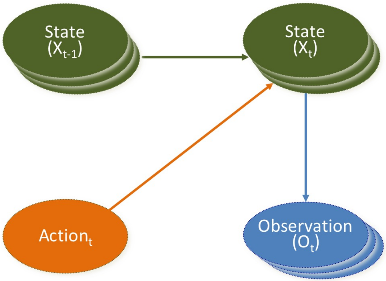
2 Probabilistic reasoning

3 Bayesian networks

4 Markov Models
 - Introduction
 - Markov chains
 - Hidden Markov Models
 - Markov Decision Processes (MDP)
 - Partially Observable MDPs (POMDPs)

5 Fuzzy logic

Partially Observable MDPs (POMDPs)



Markov tasks

- Markov chains
- Markov chains + partial observability = HMM
- Markov chains + actions = MDP
- Markov chains + partial observability + actions = HMM + actions = MDP + partial observability = POMDP

	Observability	
actions	**full**	**partial**
no	Markov chains	HMM
yes	MDP	POMDP

Summary

- To model the notion of probabilities + time we can use variables X_t
- Markov Assumption:
  - states that variables X_t depend solely of variables in t-1
  - allows us to represent the distribution compactly
- We can model non-deterministic actions with probabilities extending the search model
- Solutions are not longer paths but policies
- Bellman equation characterizes the expected value of being in a state
- Expected values allow us to compute optimal policies
- We can obtain these values by an iterative algorithm

Credits

- Material of previous years at UC3M
- Material by Héctor Geffner
- Book and teaching notes of *Artificial Intelligence: A Modern Approach*. Russell&Novig. 2nd edition
- Comparative table:
  `http://www.cassandra.org/pomdp/pomdp-faq.shtml`

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References