Artificial Intelligence

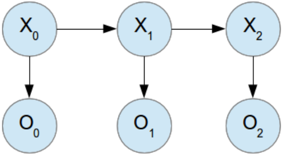
SCALAB
Grupo de Inteligencia Artificial

Universidad Carlos III de Madrid

2017-2018

Suppose we want to build a system for automatic speech recognition. Specifically we want to recognize words from acoustic signals. To simplify we will restrict to words of three letters, involving the letters a, r and t. The process can be modelled with a HMM in which states have just one variable representing the letter and observations correspond to the letter phoneme, extracted from the speech signal

- States X_t represent the letter in $\{a, r, t\}$
- Observations O_t represent the phoneme in $\{a, r, t\}$



Suppose the following prior probability and CPTs are known

- $P(X_0) = (0.3, 0.3, 0.4)$
- $P(X_{t+1}/X_t)$

X_t	$P(X_{t+1} = a/X_t)$	$P(X_{t+1} = r/X_t)$	$P(X_{t+1} = t/X_t)$
a	0.0	0.7	0.3
r	0.5	0	0.5
t	1.0	0.0	0.0

- $P(O_t/X_t)$

X_t	$P(O_t = a/X_t)$	$P(O_t = r/X_t)$	$P(O_t = t/X_t)$
a	0.9	0.1	0.0
r	0.1	0.8	0.1
t	0.0	0.2	0.8

In this case we will assume that CPTs are the same for all time steps.

1. What is the probability that the letter is *a* when we hear the phoneme *a*, $P(X_0 = a/O_0 = a)$?
2. What is the probability to hear the phoneme *a*, $P(O_0 = a)$?
3. If we observe *arr*, what is the most probable sequence of states?

Exercise 1 – solution

1. $P(X_0 = a|O_0 = a)$
   - $P(X_0 = a|O_0 = a) = \alpha P(O_0 = a|X_0 = a)P(X_0 = a) = \alpha 0.9 \times 0.3 = \alpha 0.27 = 0.9$
     To compute α we can compute the same probability for the rest of values for X_0:
   - $P(X_0 = r|O_0 = a) = \alpha P(O_0 = a|X_0 = r)P(X_0 = r) = \alpha 0.1 \times 0.3 = \alpha 0.03 = 0.1$
   - $P(X_0 = t|O_0 = a) = \alpha P(O_0 = a|X_0 = t)P(X_0 = t) = \alpha 0.0 \times 0.4 = 0$
     Then, $\alpha = \frac{1}{0.27+0.03} = 1/0.3$
2. This is exactly $1|\alpha$ of previous exercise.:

$$P(O_0 = a) = \sum_i P(O_0 = a|X_0 = i)P(X_0 = i) = P(O_0 = a|X_0 = a)P(X_0 = a)+$$
$$P(O_0 = a|X_0 = r)P(X_0 = r) + P(O_0 = a|X_0 = t)P(X_0 = t) = 0.9 \times 0.3 + 0.1 \times 0.3 + 0.0 \times 0.4 = 0.3$$

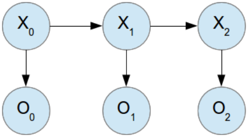
3. $P(X_0, X_1, X_2|O_0 = a, O_1 = r, O_2 = r)$?

$$P(X_0, X_1, X_2|O_0 = a, O_1 = r, O_2 = r) = \alpha P(X_0, X_1, X_2, O_0 = a, O_1 = r, O_2 = r) =$$
$$= P(O_0 = a|X_0)P(X_0)P(O_1 = r|X_1)P(X_1|X_0)P(O_2 = r|X_2)P(X_2|X_1)$$

We have to compute this probability for all the cases of X_0, X_1 and X_2 and take the most probable. For example, for $X_0 = a, X_1 = r, X_2 = t$:

$$P(X_0 = a, X_1 = r, X_2 = t|O_0 = a, O_1 = r, O_2 = r) = \alpha P(X_0 = a, X_1 = r, X_2 = t, O_0 = a, O_1 = r, O_2 = r) =$$
$$= P(O_0 = a|X_0 = a)P(X_0 = a)P(O_1 = r|X_1 = r)P(X_1 = r|X_0 = a)P(O_2 = r|X_2 = t)P(X_2 = t|X_1 = r) =$$
$$= \alpha 0.9 \times 0.3 \times 0.8 \times 0.7 \times 0.2 \times 0.5$$

- You sometimes get cold (C), which makes you sneeze (S). You also get allergies (A), which make you sneeze. Sometimes you are well (W), which does not make you sneeze. You decide to model this as an HMM where:
  - States X_t represent the state of the person that can be $\{well(w), allergy(a), cold(c)\}$
  - Observations O_t represent the observable symptom that can be sneeze(s) or not sneeze ($\neg s$)



  - $P(X_0) = (1, 0, 0)$ (i.e. $P(X_o = w) = 1$)
  - $P(X_{t+1}|X_t)$

X_t	$P(X_{t+1} = w\|X_t)$	$P(X_{t+1} = a\|X_t)$	$P(X_{t+1} = c\|X_t)$
w	0.7	0.2	0.1
a	0.6	0.3	0.1
c	0.2	0.2	0.6

  - $P(O_t|X_t)$

X_t	$P(O_t = s\|X_t)$
w	0.1
a	0.8
c	0.7

1. What is the probability that you are Well tomorrow (day 1)?
2. What is the probability of the sequence W,C,C,W on days 1 to 4?
3. What is the probability that on day 0 you observe *not sneeze* and on day 1 you observe *sneeze*?

Exercise 2: Solution

1. What is the probability that on day 2 you are Well?
   $P(X_1 = w) =$
   $P(X_1 = w | X_0 = w) P(X_0 = w)$
   = (.7)(1) = 0.7
   (Note that $P(X_0 = a) = P(X_0 = c) = 0$, so previous equation is simplified)
2. What is the probability of the sequence W, C, C, W on days 0 to 3?
   $P(X_0 = w, X_1 = c, X_2 = c, X_3 = w)$
   = $P(X_3 = w | X_2 = c) P(X_2 = c | X_1 = c) P(X_1 = c | X_0 = w) P(X_0 = w)$
   = (.2)(.6)(.1) (1) = 0.012
   (Terms with probabilities containing observations are simplified given that they are factors equal to 1)
3. What is the probability that on day 0 you observe *not sneeze* and on day 1 you observe *sneeze*?
   $P(O_0 = \neg s, O_1 = s)$
   $= P(O_0 = \neg s, O_1 = s | X_0 = w, X_1 = w) P(X_0 = w, X_1 = w)$
   $+ P(O_0 = \neg s, O_1 = s | X_0 = w, X_1 = a) P(X_0 = w, X_1 = a)$
   $+ P(O_0 = \neg s, O_1 = s | X_0 = w, X_1 = c) P(X_0 = w, X_1 = c)$
   $= P(O_0 = \neg s | X_0 = w) P(O_1 = s | X_1 = w) P(X_1 = w | X_0 = w) P(X_0 = w) + \ldots$
   $= (.9)(.1)(.7)(1) + (.9)(.8)(.2)(1) + (.9)(.7)(.1)(1)$
   $= 0.27$
   (Terms with $X_0 \neq w$ are simplified)

- We have a radar which every few seconds obtains a set of attributes about position and speed of a F-16 Fighting Falcon. We denote these observations as O_t, for an arbitrary time t. We want to build a system to determine if the F-16 Fighting Falcon is an imminent threat or not.
- Which of the models seen in class is adequate to construct this system? and how would you perform the reasoning to obtain the answer?

Exercise 3: Solution

- The problem can be modeled as a Hidden Markov Model.
- Observations: O_t.
- Hidden State: X_t: represents whether the F-16 is a threat or not at time t.
- The distribution $P(X_0)$ models the initial probability of the F16 being a threat.
- The distribution $P(X_{t+1}|X_t)$ allows to model the evolution of the threat, i.e., the probability of having a threat at time t depending on the previous state.
- The distribution $P(O_t|X_t)$ allows to model the observations obtained by the radar according to whether the F16 is a threat or not. The task of inference to solve is filtering: $P(X_t = t|O_0, \ldots, O_t)$

Exercise 4

- We have an agent with three states A, B and C where C is the goal state. In state A the agent can take two possible actions p and q, and in state B the agent can take action q.
  - The execution of action p in state A moves the agent to state B with probability 0.8, and stays in state A with probability 0.2.
  - The execution of action q in state A moves the agent to state C with probability 0.1, and stays in state A with probability 0.9.
  - The execution of action q in state B moves the agent to state A with probability 0.1, and moves the agent to state C the rest of the time. Each action has a cost of 1.
- Model formally the MDP.
- Specify the Bellman's equations that updates the values of states V(A) and V(B).
- Compute the expected value $V(s)$ for each of the states. After each iteration truncate values to only one decimal.
- Compute the optimal policy.

Model formally the MDP

- Defined as a tuple: $< S, A, P, R >$
  - S: States: $s_t \in \{A, B, C\}$
  - A: Actions: $\{p, q\}$
  - P: Transition function
  - $P_p\left(s_{t+1} \mid s_t\right)$:

	A	B	C
A	.2	.8	0

  - $P_q\left(s_{t+1} \mid s_t\right)$:

	A	B	C
A	0.9	0	.1
B	.1	0	.9

  - R: Reward. We define costs instead rewards
    c(p)=1
    c(q)=1

Exercise 4: Solution

Specify the Bellman's equations that updates the values of states V(A) and V(B)

- Update of state V(A):

$$V_{i+1}(A) = \min\left[c(p) + P_p(A \mid A)\, V_i(A) + P_p(B \mid A)\, V_i(B)\right.$$
$$\left. c(q) + P_q(C \mid A)\, V_i(C) + P_q(A \mid A)\, V_i(A)\right]$$

- Update of state V(B):

$$V_{i+1}(B) = c(q) + P_q(A \mid B)\, V_i(A) + P_q(C \mid B)\, V_i(C)$$

Exercise 4: Solution

Compute the expected value $V(s)$ for each of the states

- Iteration 0:

$$V_0(A) = 0$$

$$V_0(B) = 0$$

$$V_i(C) = 0$$

- Updates:

$$\begin{aligned} V_{i+1}(A) = \min [&c(p) + P_p(A \mid A)\, V_i(A) + P_p(B \mid A)\, V_i(B) \\ &c(q) + P_q(C \mid A)\, V_i(C) + P_q(A \mid A)\, V_i(A)] \end{aligned}$$

$$V_{i+1}(B) = c(q) + P_q(A \mid B)\, V_i(A) + P_q(C \mid B)\, V_i(C)$$

- You reach a fix point after 6 iterations:

i	0	1	2	3	4	5	6
$V_i(A)$	0	1	1.9	2.2	2.3	2.4	2.4
$V_i(B)$	0	1	1.1	1.1	1.2	1.2	1.2

Compute the optimal policy

- We eventually obtain the values:
  - $V(A) = 2.4$
  - $V(B) = 1.2$
  - $V(C) = 0$
- What is then the optimal policy π^*?
  - $\pi^*(A)$,
    - For **p** we have:
      $c(p) + P_p(A|A)V(A) + P_p(B|A)V(B) = 1 + (0.2)(2.4) + (0.8)(1.2) = 2.44$
    - For **q** we have $c(q) + P_q(C|A)V(C) + P_q(A|A)V(A) = 1 + (0.1)(0) + (0.9)(2.4) = 3.16$

    Therefore, $\pi^*(A) =$p
  - $\pi^*(B)$ =q, only applicable action.
  - $\pi^*(C)$ not defined. There are no actions to be applied at the goal.

Exercise 5: MDP

- To treat a certain type of tumor a doctor can execute three actions: surgery, chemotherapy or radiotherapy.
  - For the option of surgery, the likelihood of Cure is 0.5, with probability 0.4 the tumor will be regenerated exactly as it was, and with probability 0.1 there will be metastasis.
  - For the radiation therapy, the probability of Cure is 0.3, with probability 0.6 there will be metastasis with probability 0.1 the tumor regenerates.
  - For chemotherapy, the probability of Cure is 0.3, with tumor regeneration in all other cases.
- To treat metastasis there are two options: radiotherapy or chemotherapy.
  - For radiation the probability of Cure is 0.3, and for chemotherapy the probability of Cure is 0.6 (all other cases the patient continues with metastasis).
- The cost of radiotherapy is 6, of chemotherapy 10 and the cost of surgery is 100.
- Model the MDP and execute the value iteration algorithm

Model formally the MDP

- Defined as a tuple: $< S, A, P, R >$
  - S: States: $s_t \in \{t, m, c\}$ (tumor, metastasis, cure). Cure is a goal state.
  - A: Actions: $\{q, r, s\}$. q: chemotherapy, r: radiotherapy, s : surgery.
  - P: Transition table:
    $P_q\left(s_{t+1} \mid s_t\right)$:

	t	m	c
t	0.7	0	0.3
m	0	0.4	0.6

  - $P_r\left(s_{t+1} \mid s_t\right)$

	t	m	c
t	0.1	0.6	0.3
m	0	0.7	0.3

  - $P_s\left(s_{t+1} \mid s_t\right)$

	t	m	c
t	0.4	0.1	0.5

  - R: Rewards: c(s)=100, c(q)=10, c(r)=6

Bellman Equations:

- Update of state V(m):

$$V_{i+1}(m) = \min \left[c(q) + P_q(c \mid m)\, V_i(c) + P_q(m \mid m)\, V_i(m) \right., \\ \left. c(r) + P_r(c \mid m)\, V_i(c) + P_r(m \mid m)\, V_i(m) \right]$$

- Update of state V(t):

$$V_{i+1}(t) = \min \left[c(q) + P_q(c \mid t)\, V_i(c) + P_q(t \mid t)\, V_i(t) \right., \\ c(r) + P_r(t \mid t)\, V(t) + P_r(m \mid t)\, V(m) + P_r(c \mid t)\, V(c), \\ \left. c(s) + P_s(t \mid t)\, V(t) + P_s(m \mid t)\, V(m) + P_s(c \mid t)\, V(c) \right]$$

- - Iteration 0:

$$V_i(c) = 0$$

$$V_0(m) = 0$$

$$V_0(t) = 0$$

- Iterations:

i	0	1	2	3	4	5	6	7	8	9	10
$V_i(T)$	0	6	10.2	13.1	15.1	16.5	17.2	17.5	17.6	17.7	17.7
$V_i(M)$	0	6	10.2	13.1	15.1	16	16.4	16.5	16.6	16.6	16.6