Math Problem Set 1

Matthew Brown
OSM Boot Camp 2018

June 20, 2018

Problem 1.3. There are three things to check:

- G_1 is neither an algebra nor a sigma-algebra
- G_2 is only an algebra, but it is not a sigma-algebra as it is not closed under countable unions.
- G_3 is both an algebra and a sigma-algebra.

Problem 1.7. We already showed that both of these sets are sigma-algebras. Obviously no sigma-algebra can be larger than $\mathcal{P}(X)$, since this is the largest collection of subsets of X. And since any sigma-algebra S must contain $\emptyset$, $X \in S$, so $\{\emptyset, X\} \subset S$.

Problem 1.10. *Proof.* I'll check the three axioms of sigma-algebras on $S = \bigcap_\alpha S_\alpha$:

- $\emptyset \in S_\alpha$ for each α, so $\emptyset \in S$.
- Let $E \in S$. Then $E \in S_\alpha$ for each α. Since each S_α is a sigma-algebra, $E^c \in S_\alpha$ for each α, which means that $E^c \in S$.
- Very similar to the above. Let $\{E_i\}_{n==1}^\infty \in S$. Then $E_i \in S_\alpha$ for each α and each i, and $\cup_{i=1}^\infty E_i \in S_\alpha$ for each α, so $\cup_{i=1}^\infty E_i \in S$.

$\square$

Problem 1.17. *Proof.* There are two parts.

- *Monotonicity:*
  Let $A, B \in S, A \subset B$. Then we can decompose B as follows

$$\begin{aligned} B &= (B \cap A) \cup (B \cap A^c) \\ &= A \cup (B \cap A^c) \end{aligned}$$

  And since now we have written B as a union of disjoint sets, we can say that $\mu(B) = \mu(A) + \mu(B \cap A^c)$ and, by the nonnegativaty of μ, $\mu(B) \geq \mu(A)$.

- *Subadditivity:*
  Let $\{A_i\}_{i=1}^{\infty} \in S$. I will decompose $\cup_{i=1}^{\infty} A_i$ as follows:

  $$\cup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap A_1^c \cap A_2^c) \cup ... \cup (A_i \cap A_1^c \cap ... \cap A_{i-1}^c) \cup ...$$

  So that the $\cup_{i=1}^{\infty} A_i$ is written as a union of disjoint sets. Moreover, we see, for instance, that $\mu(A_2 \cap A_1^c) \leq \mu(A_2)$ by the monotonicty of μ. Combining these two facts, we see that:

  $$\begin{aligned} \mu(\cup_{i=1}^{\infty} A_i) &= \mu(A_1) + \mu(A_2 \cap A_1^c) + ... + \mu(A_i \cap A_1^c \cap ... \cap A_{i-1}^c) + ... \\ &\leq \mu(A_1) + \mu(A_2) + ... + \mu(A_n) + ... \end{aligned}$$

  Which is what we wanted to show.

$\square$

Problem 1.20. *Proof.* . This proof is adapted from Dr. Richard Timoney's lectures for a measure theory course at Trinity College.
Let $\{A_i\}_i = 1^{\infty}$ be a decreasing sequence of measurable sets, as in the statement. Define $B_n = B_1/B_n$ for each $n \in \mathbb{N}$. See that for each n, $A_n = B_n \cup A_1$ Because this is a disjoint union, we have:

$$\mu(A_n) = \tag{1}$$

$\square$