

# Math Problem Set 2

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**Problem 3.1.** There are two parts:

- (i)

$$\begin{aligned} ||x + y||^2 - ||x - y||^2 &= \langle x + y, x + y \rangle - \langle x - y, x - y \rangle \\ &= \langle x + y, x \rangle + \langle x + y, y \rangle - (\langle x - y, x \rangle + \langle x - y, -y \rangle) \\ &= \langle x, x + y \rangle + \langle y, x + y \rangle - (\langle x, x - y \rangle + \langle -y, x - y \rangle) \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle - \langle x, x \rangle - \langle x, -y \rangle - \langle -y, x \rangle - \langle -y, -y \rangle \\ &= 4\langle x, y \rangle \text{ after some easy manipulations} \end{aligned}$$

So it is clear that  $\frac{1}{4}(|x + y|^2 - |x - y|^2) = \langle x, y \rangle$

- (ii) As above,

$$\begin{aligned} ||x + y||^2 + ||x - y||^2 &= \langle x + y, x + y \rangle + \langle x - y, x - y \rangle \\ &= \langle x + y, x \rangle + \langle x + y, y \rangle + (\langle x - y, x \rangle + \langle x - y, -y \rangle) \\ &= \langle x, x + y \rangle + \langle y, x + y \rangle + (\langle x, x - y \rangle + \langle -y, x - y \rangle) \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle + \langle x, x \rangle + \langle x, -y \rangle + \langle -y, x \rangle + \langle -y, -y \rangle \\ &= 2\langle x, x \rangle + 2\langle y, y \rangle \text{ after some easy manipulations} \\ &= 2(||x||^2 + ||y||^2) \end{aligned}$$

So it is clear that  $\frac{1}{2}(|x + y|^2 + |x - y|^2) = ||x||^2 + ||y||^2$

**Problem 3.2.** I proceed similarly: just start from the left side, expand, and simplify.

**Problem 3.3.** Let  $\theta$  be the angle in question. Recall that

$$\cos(\theta) = \frac{\langle f, g \rangle}{||f|| \cdot ||g||}$$

- (i)

$$\begin{aligned}\langle f, g \rangle &= \int_0^1 f g dx = \int_0^1 x^6 dx = \left( \frac{x^7}{7} \right) \Big|_0^1 = \frac{1}{7} \\ \|f\|^2 &= \int_0^1 f^2 dx = \int_0^1 x^2 dx = \left( \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{3} \\ \|g\|^2 &= \int_0^1 g^2 dx = \int_0^1 x^{10} dx = \left( \frac{x^{11}}{11} \right) \Big|_0^1 = \frac{1}{11}\end{aligned}$$

Thus, we see that

$$\cos(\theta) = \frac{\frac{1}{7}}{\left(\frac{1}{3} \cdot \frac{1}{11}\right)^{\frac{1}{2}}} = \frac{33^{\frac{1}{2}}}{7} \quad (1)$$

And (1) implies that  $\theta \approx 35^\circ$ .

- (ii)

$$\begin{aligned}\langle f, g \rangle &= \int_0^1 f g dx = \int_0^1 x^4 dx = \left( \frac{x^5}{5} \right) \Big|_0^1 = \frac{1}{5} \\ \|f\|^2 &= \int_0^1 f^2 dx = \int_0^1 x^2 dx = \left( \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{3} \\ \|g\|^2 &= \int_0^1 g^2 dx = \int_0^1 x^8 dx = \left( \frac{x^9}{9} \right) \Big|_0^1 = \frac{1}{9}\end{aligned}$$

Thus, we see that

$$\cos(\theta) = \frac{\frac{1}{5}}{\left(\frac{1}{3} \cdot \frac{1}{9}\right)^{\frac{1}{2}}} = \frac{45^{\frac{1}{2}}}{7} \quad (2)$$

And (1) implies that  $\theta \approx 17^\circ$

**Problem 3.8.** There are four parts

- (i)

*Proof.* This is just a matter of checking all the relevant details.

Norms = 1:

$$\begin{aligned}\|\cos(t)\|^2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(t) dt = \frac{1}{\pi} \frac{\cos(x)\sin(x) + x}{2} \Big|_{-\pi}^{\pi} = 1 \\ \|\sin(t)\|^2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(t) dt = \frac{1}{\pi} \frac{-\sin(2x) + 2x}{4} \Big|_{-\pi}^{\pi} = 1 \\ \|\cos(2t)\|^2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(2t) dt = \frac{1}{\pi} \frac{\sin(4x) + 4x}{8} \Big|_{-\pi}^{\pi} = 1 \\ \|\sin(2t)\|^2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(2t) dt = \frac{1}{\pi} \frac{-\sin(4x) + 4x}{8} \Big|_{-\pi}^{\pi} = 1\end{aligned}$$

Inner Products = 1:

$$\langle \cos(t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \sin(t) dt = \frac{\sin^2(x)}{2} \Big|_{-\pi}^{\pi} = \sin^2(\pi) - \sin^2(-\pi) = 0$$

The proof is completed by checking all the other inner products similarly.  $\square$

- (ii)

$$\begin{aligned} \|t\|^2 &= \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3} \Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} = \frac{2\pi^3}{3} \\ \|t\| &= \left( \frac{2\pi^3}{3} \right)^{\frac{1}{2}} \end{aligned}$$

- (iii)

$$\begin{aligned} \text{proj}_X(\cos(3t)) &= \langle \sin(t), \cos(3t) \rangle \sin(t) + \langle \cos(t), \cos(3t) \rangle \cos(t) \\ &\quad + \langle \sin(2t), \cos(3t) \rangle \sin(2t) + \langle \cos(2t), \cos(3t) \rangle \cos(2t) \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned}$$

(We see that  $\cos(3t)$  is orthogonal to  $X$ )

- (iv)

$$\text{proj}_X(t) = 0 + 2\sin(t) + 0 - \sin(2x)$$

**Problem 3.9.** *Proof.* In  $\mathbb{R}^2$ , a rotation about the origin by arbitrary angle  $\theta$  can be described by the matrix

$$M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So that  $M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{pmatrix}$ . Let  $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ . Then

$$\begin{aligned} \langle M(a), M(b) \rangle &= \left\langle \begin{pmatrix} a_1\cos(\theta) - a_2\sin(\theta) \\ a_1\sin(\theta) + a_2\cos(\theta) \end{pmatrix}, \begin{pmatrix} b_1\cos(\theta) - b_2\sin(\theta) \\ b_1\sin(\theta) + b_2\cos(\theta) \end{pmatrix} \right\rangle \\ &= a_1b_1\cos^2(\theta) + a_2b_2\sin^2(\theta) + a_1b_1\sin^2(\theta) + a_2b_2\cos^2(\theta) \\ &= a_1b_1 + a_2b_2 = \langle a, b \rangle \end{aligned}$$

So we see that  $M$  is an orthonormal operator. (N.b that this is a terribly inefficient way to prove this - I should have just shown that the columns of  $M$  were orthonormal!  $\square$ )

**Problem 3.10.** Recall that taking the Hermitian is flipping rows and columns and taking the conjugate.

- (i)

*Proof.*  $\square$