Math Problem Set 1

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Problem 1.3. There are three things to check:

- G_1 is neither an algebra nor a sigma-algebra
- G_2 is only an algebra, but it is not a sigma-algebra as it is not closed under countable unions.
- G_3 is both an algebra and a sigma-algebra.

Problem 1.7. We already showed that both of these sets are sigma-algebras. Obviously no sigma-algebra can be larger than $\mathcal{P}(X)$, since this is the largest collection of subsets of X. And since any sigma-algebra S must contain $\emptyset$, $X \in S$, so $\{\emptyset, X\} \subset S$.

Problem 1.10. *Proof.* I'll check the three axioms of sigma-algebras on $S = \bigcap_\alpha S_\alpha$:

- $\emptyset \in S_\alpha$ for each α, so $\emptyset \in S$.
- Let $E \in S$. Then $E \in S_\alpha$ for each α. Since each S_α is a sigma-algebra, $E^c \in S_\alpha$ for each α, which means that $E^c \in S$.
- Very similar to the above. Let $\{E_i\}_{n==1}^\infty \in S$. Then $E_i \in S_\alpha$ for each α and each i, and $\cup_{i=1}^\infty E_i \in S_\alpha$ for each α, so $\cup_{i=1}^\infty E_i \in S$.

□

Problem 1.17. *Proof.* There are two parts.

- *Monotonicity:*
  Let $A, B \in S, A \subset B$. Then we can decompose B as follows

$$\begin{aligned} B &= (B \cap A) \cup (B \cap A^c) \\ &= A \cup (B \cap A^c) \end{aligned}$$

  And since now we have written B as a union of disjoint sets, we can say that $\mu(B) = \mu(A) + \mu(B \cap A^c)$ and, by the nonnegativaty of μ, $\mu(B) \geq \mu(A)$.

- *Subadditivity:*
  Let $\{A_i\}_{i=1}^{\infty} \in S$. I will decompose $\cup_{i=1}^{\infty} A_i$ as follows:

$$\cup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap A_1^c \cap A_2^c) \cup ... \cup (A_i \cap A_1^c \cap ... \cap A_{i-1}^c) \cup ...$$

  So that the $\cup_{i=1}^{\infty} A_i$ is written as a union of disjoint sets. Moreover, we see, for instance, that $\mu(A_2 \cap A_1^c) \leq \mu(A_2)$ by the monotonicty of μ. Combining these two facts, we see that:

$$\begin{aligned} \mu(\cup_{i=1}^{\infty} A_i) &= \mu(A_1) + \mu(A_2 \cap A_1^c) + ... + \mu(A_i \cap A_1^c \cap ... \cap A_{i-1}^c) + ... \\ &\leq \mu(A_1) + \mu(A_2) + ... + \mu(A_n) + ... \end{aligned}$$

  Which is what we wanted to show.

□

Problem 1.20. *Proof.* . This proof is adapted from Dr. Richard Timoney's lectures for a measure theory course at Trinity College.
Let $\{A_i\}_i = 1^{\infty}$ be a decreasing sequence of measurable sets, as in the statement. Define $B_n = B_1 \setminus B_n$ for each $n \in \mathbb{N}$. See that for each n, $A_1 = B_n \cup A_n$ Because this is a disjoint union, we have $\mu(A_1) = \mu(B_n) + \mu(A_n)$ and thus:

$$\mu(A_n) = \mu(A_1) - \mu(B_n) \tag{1}$$

Now, $\{B_i\}_{i=1}^{\infty}$ is an increasing sequence of functions. Define $B = \cup_{i=1}^{\infty} B_i$, and from part (i) of the theorem,

$$\mu(B) = lim_{n\to\infty} \mu(B_n) \tag{2}$$

We'll use these two facts.
Now, note that

$$A_1 \setminus \bigcap_{i}^{\infty} A_i = A_1 \cap \big(\bigcap_{i}^{\infty} A_i\big)^c = \bigcup_{i=1}^{\infty} \big(A_1 \cap A_i^c\big) = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = B$$

Therefore,

$$\mu(A_1) = \mu\big(A_1 \setminus \bigcap_{i=1}^{\infty} A_i i\big) + \mu\big(\bigcap_{i=1}^{\infty} A_i\big) = \mu(B) + \mu\big(\bigcap_{i=1}^{\infty} A_i\big)$$

Therefore, as in equation (1),

$$\mu\big(\bigcap_{i=1}^{\infty} A_i\big) = \mu(A_1) - \mu(B)$$

and by equation (2) we see that

$$\mu\big(\bigcap_{i=1}^{\infty} A_i\big) = \mu(A_1) - lim_{n\to\infty} \mu(B_n)$$

And we can pass this limit outside and use equation (1), so that

$$\mu(\bigcap_{i=1}^{\infty} A_i) = lim_{n\to\infty}\big(\mu(A_1) - \mu(B_n)\big) = lim_{n\to\infty}\mu(A_n) \tag{3}$$

And (3) was what we wanted to show. □

Problem 2.10. We know from the countable subadditivity of the outer measure that

$$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

So the $\geq$ in (*) will never actually be $>$, and it is equivalent to replace it with equality.

Problem 2.14. Most of the work has been done for us elsewhere in the notes. I will show a brief intermediate step:

Lemma 0.1. The sigma algebra generated by $\mathcal{A}$ from Example 2.2 is the borel sigma algebra.

Proof. I know that there is an open set in (A), $\sigma(\mathcal{A})$ at least contains $\sigma(\mathcal{O})$. To show the other inclusion, recognize that intervals of the form $(a, b]$ and $(-\infty, a]$ are in $\sigma(\mathcal{O})$ since we can get the closed edge of the interval by taking a complement of an open interval (Eventually. - there are certainly missing details here, my apologies...) □

The pre-measure ν as defined in example 2.2 is still a pre-measure on $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$. This means that we can apply the Caratheodory Extension Theorem 2.12 to see that $\sigma(\mathcal{O}) = \mathcal{M}$.

Problem 3.1. *Proof.* Let X be a countable set in $\mathbb{R}$. Then I can ennumerate the elements of X so that $\{x_1, ...\} = X$. Now, construct intervals I_ϵ^i for a given small ϵ and for each i so that $I_\epsilon^i = (x_i - \frac{\epsilon}{2^i}, x_i + \frac{\epsilon}{2^i})$. Now for

$$\mu(\bigcup_{i=1}^{\infty} I_\epsilon^i) \leq \sum_{i=1}^{\infty} \frac{2\epsilon}{2^i} = 2\epsilon$$

We see that even if we make ϵ arbitrarily small, $\bigcup_{i=1}^{\infty} I_\epsilon^i$ covers X, so the infimum of the measures of these covers is zero, which is the measure of X. □

Problem 3.4. This follows from the fact that the set of measurable sets is a sigma-algebra, and all these statements are equivalent.

Proof. I'll show that the sets being measurable are all equivalent statements.
$f^{-1}((-\infty, a))$ is measurable $\iff$ $f^{-1}([a, \infty))$ is measurable (they are complements). I will now show that:

$$f^{-1}((-\infty, a)) \in \mathcal{M} \iff f^{-1}((-\infty, a]) \in \mathcal{M}$$

$\Rightarrow$: Suppose sets of the form $f^{-1}((-\infty, a]) \in \mathcal{M}$. Construct a sequence of sets $E_i n = f^{-1}((-\infty, a - frac1n]) \in \mathcal{M}$. This countable union $\cup_{n=1}^{\infty} = f^{-1}((-\infty, a])$ is in $\mathcal{M}$
$\Leftarrow$: Suppose sets of the form $f^{-1}((-\infty, a)) \in \mathcal{M}$. Then their complements, sets of the form $f^{-1}([a, \infty))$ are also $in\mathcal{M}$. We can use a similar argument, employing sets of the form $f^{-1}([a + \frac{1}{n}, \infty))$ to show that sets of the form $f^{-1}((a, \infty)) \in \mathcal{M}$. This shows that the complements of these sets, $f^{-1}((-\infty, a])$, are also elements of $\mathcal{M}$.
To conclude, see that $f^{-1}((a, \infty] \in \mathcal{M} \iff f^{-1}((-\infty, a]) \in \mathcal{M}$ because the sets are complements. □

Problem 3.7. *Proof.* I'll go item by item.

- $f+g$: $f+g : X \to \mathcal{M}$ can be written as $F(f,g) : X \to \mathcal{M}$ where $F(x,y) = x+y$. This f is continuous, so it is measurable.
- fg: I can write fg similarly as $F(f,g) = fg$ and use the same argument.
- min: Define $\{f_n\}_{n=1}^{\infty}$ so that $f_1 = f, f_2 = f+2, ...f_i = f+i$. Similarly define $\{g_n\}_{n=1}^{\infty}$ so that $g_1 = g, g_2 = g+2, ...g_i = g+i$. f and g are the smallest elements of their respective sequences by construction. Now make a sequence $\{h_n\}_{n=1}^{\infty}$ where the odd terms are the fs and the even terms the gs. $inf_{n\in\mathbb{N}}\{h_n\}_{n=1}^{\infty} = min(f,g)$, and all the terms h_i are lebesgue measurable so by (2) $min(f,g)$ is lebesgue measurable.
- max: Just modify the above by making the sequences always be smaller that f and g, e.g, $f_1 = f, f_2 = f-2, ...$ and show that $max(f,g) = sup_{n\in\mathbb{N}}\{h_n\}_{n=1}^{\infty}$
- *Absolute value*: See that $|f| = max(f,-f)$, and therefore by what we've just shown it is lebesgue measurable.

□

Problem 3.14. Recall the definition of uniform convergence: We want to show that

$$\forall\epsilon > 0, \exists N = N(\epsilon) \text{ such that } n \geq N \implies |f(x) - s_n(x)| < \epsilon, \forall x \in X$$

Proof. Suppose that $f(x) < M$. Fix $\epsilon > 0$. We construct intervals and simple functions just as we did in the proof of 3.13. Let $N_1 > M, N_1 \in \mathbb{N}$. Then $f(x) < N_1$ for all x and $x \notin E_{\infty}^{N_1}$. We also see that there exists N_2 such that

$$N_2 > N_1 \text{ and } \frac{1}{2^{N_2}} < \epsilon$$

Now, it follows that for $n > N_2$,

$$\forall x \in X, x \in E_i^n \text{ for some index } 0 \leq i \leq N_2, i \in \mathbb{N}$$

Then $f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$ and our simple function in this interval is $s_n(x) = \frac{i-1}{2^n}$ Recall now that we've chosen this same N_2 for ALL $x \in X$, and so $|f(x) - s_n(x)| < \frac{1}{2^n} < \frac{1}{2^{N2}} < \epsilon$ implies uniform convergence. □

Problem 4.13. To show that $f \in \mathcal{L}^1(\mu, E)$, it suffices to show that $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are both finite. The fact that $||f|| < M$ on E implies that $\int_E |f| d\mu < M$. Thus, $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are both finite, and we are done. (???)

Problem 4.14. I'll argue that the contrapositive is true.

Proof. Suppose that there exists a set A such that $\mu(A) > 0$ and $f(x) = \pm\infty$ Now, since $\mu(A) > 0$, either B the set on which $f(x) = \infty$ or C the set on which $f(x) = -\infty$ must have nonzero measure. So the proof splits into two cases.
Case 1: $\mu(B) \neq 0$. Then

$$\int_B f d\mu = \infty \implies \int_B f^+ d\mu = \infty$$

And we see then that

$$\int_B f^+ d\mu = \infty \implies \int_E f^+ d\mu = \infty$$

because $B \subset E$ and f^+ is a positive function. And this means that $f \notin \mathcal{L}(E, \mu)$, which is what we wanted to show.
Case 2: $\mu(C) \neq 0$. This case is exactly similar... We have

$$\int_C f d\mu = -\infty \implies \int_C f^- d\mu = \infty$$

And we see then that

$$\int_C f^- d\mu = \infty \implies \int_E f^- d\mu = \infty$$

because $C \subset E$ and f^- is a positive function. And this means that $f \notin \mathcal{L}(E, \mu)$, which is what we wanted to show. $\square$

Problem 4.15. *Proof.* Define $h = g - f$. on E Because $g \geq f, h(x) \geq 0$ and $\int_E h d\mu \geq 0$. Consider now the following lemma.

Lemma 0.2. (Linearity of Integral)
For any $f, g \in \mathcal{L}^1(E, \nu)$, $\alpha \in X$

$$\int_E \alpha f f \mu = \alpha \int_E d\mu$$
$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu$$

I state this lemma without proof, since the property is somewhat intuitive (if this didn't hold, we would have quite a bad definition for an integral!). If a proof is desired, it can be found in the lecture notes to Richard Timoney's Lebesgue Integral course at Trinity College.

Using this fact, we see that

$$\int_E g d\mu = \int_E f + h d\mu \geq \int_E f d\mu$$

Which is what we wanted to show $\square$

Problem 4.16. *Proof.* Suppose that $f \in \mathcal{L}(E, \mu)$. Then by definition, $\int_E f^+ d\mu < \infty, \int_E f^- d\mu < \infty$. Since $A \subset E$ and we're taking the integral of a positive function over a more restrictive domain, we can now say that $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$. And this means that $f \in \mathcal{L}(A, \mu)$. $\square$

Problem 4.21. *Proof.* By the above theorem, $\lambda(\cdot) = \int_{\cdot} f d\mu$ is a measure on $\mathcal{M}$. So we know that it is countably additive, i.e,

$$\lambda(A) = \lambda(A \setminus B) + \lambda(A \cap B) = \lambda(A \setminus B) + \lambda(B)$$

and thus

$$\int_A f d\mu = \int_{A \setminus B} f d\mu + \int_B f d\mu = \int_B f d\mu$$

(which is even more than what we wanted to show!) $\square$