Math Problem Set 2

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Problem 3.1. There are two parts:

- (i)

$$\begin{aligned}||x+y||^2 - ||x-y||^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\ &= \langle x+y, x\rangle + \langle x+y, y\rangle - (\langle x-y, x\rangle + \langle x-y, -y\rangle) \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle - (\langle x, x-y\rangle + \langle -y, x-y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle \\ &= 4\langle x, y\rangle \text{ after some easy manipulations}\end{aligned}$$

  So it is clear that $\frac{1}{4}(||x+y||^2 - ||x-y||^2) = \langle x, y\rangle$

- (ii) As above,

$$\begin{aligned}||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x+y, x\rangle + \langle x+y, y\rangle + (\langle x-y, x\rangle + \langle x-y, -y\rangle) \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle + (\langle x, x-y\rangle + \langle -y, x-y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y, \rangle \text{ after some easy manipulations} \\ &= 2(||x||^2 + ||y||^2)\end{aligned}$$

  So it is clear that $\frac{1}{2}(||x+y||^2 + ||x-y||^2) = ||x||^2 + ||y||^2$

Problem 3.2. I proceed similarly: just start from the left side, expand, and simplify.

Problem 3.3. Let θ be the angle in question. Recall that

$$cos(\theta) - \frac{\langle f, g\rangle}{||f|| \cdot ||g||}$$

- (i)

$$\langle f,g\rangle = \int_0^1 fg dx = \int_0^1 x^6 dx = \left(\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{7}$$
$$||f||^2 = \int_0^1 f^2 dx = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$
$$||g||^2 = \int_0^1 g^2 dx = \int_0^1 x^1 0 dx = \left(\frac{x^{11}}{11}\right)\Big|_0^1 = \frac{1}{11}$$

Thus, we see that

$$cos(\theta) = \frac{\frac{1}{7}}{(\frac{1}{3}\cdot\frac{1}{11})^{\frac{1}{2}}} = \frac{33^{\frac{1}{2}}}{7} \tag{1}$$

And (1) implies that $\theta \approx 35°$.

- (ii)

$$\langle f,g\rangle = \int_0^1 fg dx = \int_0^1 x^6 dx = \left(\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{7}$$
$$||f||^2 = \int_0^1 f^2 dx = \int_0^1 x^4 dx = \left(\frac{x^5}{8}\right)\Big|_0^1 = \frac{1}{5}$$
$$||g||^2 = \int_0^1 g^2 dx = \int_0^1 x^8 dx = \left(\frac{x^9}{9}\right)\Big|_0^1 = \frac{1}{9}$$

Thus, we see that

$$cos(\theta) = \frac{\frac{1}{7}}{(\frac{1}{5}\cdot\frac{1}{9})^{\frac{1}{2}}} = \frac{45^{\frac{1}{2}}}{7} \tag{2}$$

And (1) implies that $\theta \approx 17°$

Problem 3.8. There are four parts

- (i)

Proof. This is just a matter of checking all the relevant details.
Norms = 1:

$$||cos(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt = \frac{1}{\pi}\frac{cos(x)sin(x)+x}{2}\Big|_{-\pi}^{\pi} = 1$$
$$||sin(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(t)dt = \frac{1}{\pi}\frac{-sin(2x)+2x}{4}\Big|_{-\pi}^{\pi} = 1$$
$$||cos(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(2t)dt = \frac{1}{\pi}\frac{sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$$
$$||sin(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(2t)dt = \frac{1}{\pi}\frac{-sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$$

Inner Products = 1:

$$\langle cos(t), sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(t)dt = \frac{sin^2(x)}{2}\Big|_{-\pi}^{\pi} = sin^2(\pi) - sin^2(-\pi) = 0$$

The proof is completed by checking all the other inner products similarly. □

- (ii)

$$\begin{aligned}||t||^2 &= \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} = \frac{2\pi^3}{3}\\ ||t|| &= \left(\frac{2\pi^3}{3}\right)^{\frac{1}{2}}\end{aligned}$$

- (iii)

$$\begin{aligned}proj_X(cos(3t)) &= \langle sin(t), cos(3t)\rangle sin(t) + \langle cos(t), cos(3t)\rangle cos(t)\\ &+ \langle sin(2t), cos(3t)\rangle sin(2t) + \langle cos(2t), cos(3t)\rangle cos(2t)\\ &= 0 + 0 + 0 + 0 = 0\end{aligned}$$

(We see that $cos(3t)$ is orthogonal to X)

- (iv)

$$proj_X(t) = 0 + 2sin(t) + 0 - sin(2t)$$

Problem 3.9. *Proof.* In $\mathbb{R}^2$, a rotation about the origin by arbitrary angle θ can be described by the matrix

$$M = \begin{bmatrix} cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{bmatrix}$$

So that $M\left(\begin{pmatrix} x\\ y\end{pmatrix}\right) = \left(\begin{smallmatrix} xcos(\theta) - ysin(\theta)\\ xsin(\theta)+ycos(\theta)\end{smallmatrix}\right)$. Let $a = \left(\begin{smallmatrix} a_1\\ a_2\end{smallmatrix}\right), b = \left(\begin{smallmatrix} b_1\\ b_2\end{smallmatrix}\right) \in \mathbb{R}^2$. Then

$$\begin{aligned}\langle M(a), M(b)\rangle &= \left\langle \begin{pmatrix} a_1cos(\theta) - a_2sin(\theta)\\ a_1sin(\theta) + a_2cos(\theta)\end{pmatrix}, \begin{pmatrix} b_1cos(\theta) - b_2sin(\theta)\\ b_1sin(\theta) + b_2cos(\theta)\end{pmatrix} \right\rangle\\ &= a_1b_1cos^2(\theta) + a_2b_2sin^2(\theta) + a_1b_1sin^2(\theta) + a_2b_2cos^2(\theta\\ &= a_1b_1 + a_2b_2 = \langle a, b\rangle\end{aligned}$$

So we see that M is an orthonormal operator. (N.b that this is a terribly inefficient way to prove this - I should have just shown that the columns of M were orthonormal! □

Problem 3.10. Recall that taking the Hermitian is flipping rows and columns and taking the conjugate.

- (i)

Proof. We need to show both directions.
$\Rightarrow$: Let Q be an orthonormal matrix. Then $\langle m, n\rangle = \langle Qm, Qn\rangle \implies m^H n = (Qm)^H Qn = m^H Q^H Qn$. And because m and n were arbitrarily chosen, the only way that this equality holds is if $Q^H Q = I$, and this gives us that $QQ^H = Q$ since left inverse $\implies$ right inverse.
$\Leftarrow$: Let Q be a matrix so that $Q^H Q = I$. Then consider $\langle Qm, Qn\rangle = (Qm)^H Qn = m^H Q^H Qn = m^H n = \langle m, n\rangle$. $\square$

- (ii)

Proof. This is pretty easy:

$$||x|| = \sqrt[2]{\langle x, x\rangle} = \sqrt[2]{\langle x, x\rangle} = ||Qx||$$

$\square$

- (iii)

Proof. Assume Q is orthonormal. Then $QQ^H = Q^H Q = I \implies Q^H = Q^{-1}$. I'll prove the following short lemma.

Lemma 0.1. For Q orthonormal, Q^H is orthonormal.

Recall that $(Q^H)^H = Q$, and see that

$$(Q^H)^H Q^H = Q^H (Q^H)^H = I$$

which proves the lemma.
And since $Q^{-1} = Q^H$, Q^{-1} is orthonormal. $\square$

- (iv)

Proof. We'll examine the elements of the identity matrix element by element. First note that:

$$I_{ij} = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Then we'll compare this to what we get when we multiply QQ^H, which we know is equal to I in all its coordinates. First, though, for any matrix A, define A^i to be the "ith row" of A and A_j to be the jth column. Then

$$\delta_{ij} = (Q^H Q)_{ij} = (Q^H)^i Q_j =$$

Recall now that $(Q^H)^i = \bar{Q}_i$, by definition of the Hermitian. But now we see that

$$\langle \bar{Q}_i, q_j\rangle = \delta_{ij}$$

and the columns of Q are orthonormal. $\square$

- (v) A counterexample would be the matrix $M = \begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}$. $\det B = 1$, but $Be_1 = 2e_1$ and $||e_1|| \neq ||2e_1|| = ||Be_1||$ which violates what we proved in (ii).
- (vi)

  Proof. This is also quite short:

  $$(Q_1Q_2)(Q_1Q_2)^H = Q_1Q_2Q_2^HQ_1^H = Q_1Q_1^H = I$$

  And we also get the left inverse by properties of inverses. □

Problem 3.11. Suppose that $x_1, ..., x_n$ is as et of linearly *dependent* vectors. Let's apply Gram-Schmidt. Eventually, we will arrive at a vector x_k which is linearly dependent upon $x_1, ..., x_{k-1}$. But then, if $X = span(x_1, ..., x_{k-1})$, then $x_k \in X$ and $p_{k-1} = proj_X(x_k) = x_k$, which forces $q_k = 0$. In the end, if we throw out all these zeroes, we still get an orthonormal basis $q_1, ...q_m$ of X where $m = \dim X$.

Problem 3.16. Done with Albi

Problem 3.17. *Proof.*

$$\begin{aligned} \hat{R}x &= \hat{Q}^H b \\ \iff \hat{Q}\hat{R}x &= b \\ \iff Ax &= b \\ \iff A^HAx &= A^Hb \end{aligned}$$

Note that THIS IS NOT THE CORRECT PROOF AND I NEED TO FIX THIS □

Problem 3.23. doesn't look bad

Problem 3.24. Procedural (show things are norms)

Problem 3.26. First I must show that topological equivalence is an equivalence relation.

Proof. REST OF THIS DONE WITH ALBI SEE GIT I must show three things: (i) $x \sim x$. (ii) $x \sim y \implies y \sim x$, (iii) $x \sim y$ and $y \sim z \implies x \sim z$.
(i) $|| \cdot ||_1 \sim || \cdot ||_1$ trivially. Let $M \geq m$, then $m||x||_1 \leq ||x||_1 \leq M||x||_1$ for all x.
(ii) Also trivial: Suppose $|| \cdot ||_1 \sim || \cdot ||_2$. Then $m||x||_1 \leq ||x||_2 \leq M||x||_1$ for all x, which implies that $M^{-1}||x||_2 \leq ||x||_1 \leq m^{-1}||x||_2$.
(iii) Suppose $|| \cdot ||_1 \sim || \cdot ||_2$, and $|| \cdot ||_2 \sim || \cdot ||_3$. Then $m||x||_1 \leq ||x||_2 \leq M||x||_1$ and $n||x||_2 \leq ||x||_3 \leq N||x||_2$. But we get from this that $mn||x||_1 \leq ||x||_3 \leq MN||x||_1$. □

Now I'll show that the $1, 2,$ and ∞ norms are topologically equivalent.

Proof. (i) $||\cdot||_1 \sim ||\cdot||_2$:
If we think about the inner product as the standard dot-product, then we have

$$(||x||_1)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i||x_j| \geq \sum_{i=1}^{n} x_i^2 = \langle x, x, \rangle = (||x||_2)^2$$

(the inequality comes because we simply threw out some positive terms on the left side). This implies that $||x||_1 \geq ||x||_2$. Moreover,

$$(\sqrt[2]{n}||x||_2)^2 = n\sum_{i=1}^{n} x_i$$

(ii) $||\cdot||_\infty \sim ||\cdot||_2$

$$||x||_\infty = max_{1\leq i\leq n}\{x_i\} = \sqrt[2]{(max_{1\leq i\leq n}\{x_i\})^2} \leq \sqrt[2]{\sum_{i=1}^{n} x_i} = ||x||_2$$

□

Problem 3.28. Related to Previous

Problem 3.29. I will prove two statements.

The norm of an orthonormal matrix is 1:

Proof. Let Q be an orthonormal matrix. Then

$$||Qx|| = ||x|| \implies sup_{x\neq 0}\frac{||Qx||}{||x||} = ||Q|| = 1$$

□

If $R_x : M_n(\mathbb{F}) \to \mathbb{F}, R_x(A) = Ax$, then $||R_x|| = ||x||$:

Proof. The first step is to show $||R_x|| < ||x||$.

$$||R_x|| = sup_{A\neq 0}\frac{||R_x(A)||}{||A||} = sup_{A\neq 0}\frac{||Ax||}{||A||} = sup_{A\neq 0}\frac{||Ax||\cdot||x||}{||A||\cdot||x||}$$

By Remark 3.5.12, $||Ax|| \leq ||A||\cdot||x||\forall x \in \mathbb{F}^n$, so

$$||R_x|| = sup_{A\neq 0}\frac{||Ax||\cdot||x||}{||A||\cdot||x||} \leq sup_{A\neq 0}\frac{||Ax||\cdot||x||}{||Ax||} = ||x||$$

Now I'll show equality. For the $\leq$ above to be strict, we must have $||Ax|| < ||A||\cdot||A||$ for all operators A (because we're taking the supremum). $||x|| > 0$, so I can rearrange for the condition:

$$\frac{||Ax||}{||x||} < ||A||, \text{ for all operators } A, \text{ vectors } x$$

In other words, no x achieves the supremum which is encoded in the definition of $||A||$. I will use the previous result to show that this will never hold.
Let $q_1 = e_1$ (or some other vector with norm 1). I can use the gram-schmidt algorithm to construct an orthonormal basis $q_1, ...q_n$ for $\mathbb{F}^n$. Let Q be the matrix with these basis vectors as its columns. Then Q is an orthonormal matrix. Specifically, $||Q|| = 1$ and it achieves $\frac{||Qx||}{||x||} = ||Q|| = 1$ at all nonzero x.
This shows that the inequality can never be strict, so we have $||R_x|| = ||x||$ □

Problem 3.30. Show something is a norm (not bad)

Problem 3.37. The first thing is to define the standard basis, which is $\mathcal{B} = \{1, x, x^2\}$ Evaluate L on the basis vectors:

$$L(1) = 0, L(x) = 1, L(x^2) = 2$$

Now, for $p \in V$, p can be written as a linear combination of these basis vectors. So

$$L(p) = L(a_1 1 + a_2 x + a_3 x^2) = a_1 L(1) + a_2 L(x) + a_3 L(x^2) = \langle (L(1) \cdot 1, L(x), L(x^2)), (a_1, a_2, a_3) \rangle$$

which is the idea behind the Riesz Representation theorem. So we see that in this case, $q = (0, 1, 2)$... which squares with what we know about derivatives.

Problem 3.38. Let $\mathcal{B}$ as above.

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Odd - I want to apply the algorithm from 3.7.9 but I can't figure out why the fg term disappears!

Problem 3.39. There are 4 things to show.

- (i)

  Proof.

  $$\langle (S+T)v, w \rangle = \langle Sv, w \rangle + \langle Tv, w \rangle = \langle v, S^* w \rangle + \langle v, T^* w \rangle = \langle v, (S^* + T^*)w \rangle$$
  $$\langle \alpha T^* v, w \rangle = \alpha \langle Tv, w \rangle = \alpha \langle v, T^* w \rangle = \langle v, \overline{\alpha} T^* w \rangle$$

  □

- (ii)

  Proof.

  $$\langle S^* v, w \rangle = \overline{\langle w, S^* v \rangle} = \overline{\langle Sw, v \rangle} = \langle v, Sw \rangle$$

  □

- (iii)

Proof.

$$\langle STv, w\rangle = \langle Tv, S^*w\rangle = \langle v, T^*S^*w\rangle$$

□

- (iv)

  Proof. Consider the composition $T^*(T^{-1})^*$.

  $$\langle T^*(T^{-1})^*x, y\rangle = \langle (T^{-1})^*x, Ty\rangle = \langle x, (T^{-1})Ty\rangle = \langle x, y\rangle$$

  Since the above is true for all x, y, we must have $T^*(T^{-1})^* = I$ □

Problem 3.40.

- (i) (Considering A as the operator)

  $$\langle AB, C\rangle = \text{ tr } (AB)^H C = \text{ tr } B^H A^H C = \langle B, A^H C\rangle$$

- (ii) (seems easy!)

Problem 3.44. *Proof.* By the fundamental subspaces theorem, $\text{Ker}(A^H) = \text{Range}(A)$. So we can reformulate the second possibility to: there exists $y \in \text{ Range}(A)^\perp : \langle y, b\rangle \neq 0$. Consider now $p = proj_{\text{Range}(A)} b$. If $p = b$, then $b \in \text{Range}(A)$ and we have the first case. Otherwise, the procedure creates a residual vector r, $r = b - p$. $r \in \text{ Range}(A)^\perp$, and

$$\langle r, b\rangle = \overline{\langle p + r, r\rangle} = \overline{\langle p, r\rangle} + \overline{\langle r, r\rangle} = \langle p, b\rangle + \langle r, r\rangle = \langle r, r\rangle \neq 0$$

which is the second case. □

Problem 3.45.

Problem 3.47.

Problem 3.48.

Problem 3.50.