Math Problem Set 5

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MASSIVE CREDIT to Reiko Laski for creating the templates for the dictionaries. Indeed some of these simplex writeups are. And apologies for the weird numbering formatting.

Problem 1. 8.1 Jupyter notebook

Problem 2. 8.2 Jupyter Notebook

Problem 3. 8.3 Let x_1 denote the number of Barb soldiers sold and x_2 the number of Joey dolls. The optimization problem then is:

$$\begin{aligned} &\max_{x_1,x_2}(4x_1+3x_2) \\ &\text{subject to } 2x_1+2x_2 \leq 300 \\ &\qquad 15x_1+10x_2 \leq 1800 \\ &\qquad x_2 \leq 200 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Problem 4. 8.4 Let x_{ij} be the units that flow between node i and j. The optimization problem is:

$$\begin{aligned} \min_{x_{ij}, i\neq j} & 2x_{AB}+5x_{AD}+4x_{DE}+3x_{EF}+2x_{CF}+5x_{BC}+2x_{BD}+7x_{BE}+9x_{BF} \\ \text{subject to } & x_{AB}+x_{AD}=10 \\ & x_{BD}+x_{BE}+x_{BF}+x_{BC}-x_{AB}=1 \\ & x_{CF}-x_{BC}=-2 \\ & -(x_{EF}+x_{BF}+x_{CF})=-10 \\ & x_{EF}-(x_{DE}+x_{BE})=4 \\ & x_{DE}-(x_{AD}+x_{BD})=-3 \\ & 0 \leq x_{ij} \leq 6 \end{aligned}$$

Problem 5. 8.5

(i)

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + x_2 \\
\text{subject to} \quad & x_1 + 3x_2 + w_1 = 15 \\
& 2x_1 + 3x_2 + w_2 = 18 \\
& x_1 - x_2 + w_3 = 4 \\
& x_1, x_2, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{ccccccc}
\zeta & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2 \\
\hline\hline
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3 \\
\hline\hline
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3 \\
\hline
\end{array}
$$

Optimizer: $(6, 2)$
Optimum value: 20

(ii)

$$
\begin{aligned}
\text{maximize} \quad & 4x + 6y \\
\text{subject to} \quad & -x + 3x_2 + w_1 = 11 \\
& x + y + w_2 = 27 \\
& 2x + 5y + w_3 = 90 \\
& x, y, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{ccccccc}
\zeta & = & & & 4x & + & 6y \\
\hline
w_1 & = & 11 & + & x & - & y \\
w_2 & = & 27 & - & x & - & y \\
w_3 & = & 90 & - & 2x & - & 5y \\
\hline\hline
\zeta & = & 66 & + & 10x & - & 6w_1 \\
\hline
y & = & 11 & + & x & - & w_1 \\
w_2 & = & 16 & - & 2x & + & w_1 \\
w_3 & = & 35 & - & 7x & + & 5w_1 \\
\hline\hline
\zeta & = & 116 & + & \frac{8}{7}w_1 & - & \frac{10}{7}w_3 \\
\hline
y & = & 16 & - & \frac{2}{7}w_1 & - & \frac{1}{7}w_3 \\
w_2 & = & 6 & - & \frac{3}{7}w_1 & + & \frac{2}{7}w_3 \\
x & = & 5 & + & \frac{5}{7}w_1 & - & \frac{1}{7}w_3 \\
\hline\hline
\zeta & = & 132 & - & \frac{8}{3}w_2 & - & \frac{2}{7}w_3 \\
\hline
y & = & 12 & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \\
w_1 & = & 14 & - & \frac{7}{3}w_2 & + & \frac{2}{3}w_3 \\
x & = & 15 & - & \frac{5}{3}w_2 & + & \frac{1}{3}w_3 \\
\hline
\end{array}
$$

Optimizer: $(15, 12)$
Optimum value: 132

Problem 6. 8.6

$$
\begin{array}{ll}
\text{maximize} & 4b + 3j \\
\text{subject to} & 15b + 10j + w_1 = 1800 \\
& 2b + 2j + w_2 = 300 \\
& j + w_3 = 200 \\
& b, j, w_1, w_2, w_3 \geq 0
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4b & + & 3j \\
\hline
w_1 & = & 1800 & - & 15b & - & 10j \\
w_2 & = & 300 & - & 2b & - & 2j \\
w_3 & = & 200 & - & j & & \\
\hline\hline
\zeta & = & 450 & + & b & - & \frac{3}{2}w_2 \\
\hline
w_1 & = & 300 & - & 5b & + & 5w_2 \\
j & = & 150 & - & b & - & \frac{1}{2}w_2 \\
w_3 & = & 50 & + & b & + & \frac{1}{2}w_2 \\
\hline\hline
\zeta & = & 510 & - & \frac{1}{5}w_1 & - & \frac{1}{2}w_2 \\
\hline
b & = & 60 & - & \frac{1}{5}w_1 & + & w_2 \\
j & = & 90 & + & \frac{1}{5}w_1 & - & \frac{3}{2}w_2 \\
w_3 & = & 110 & - & \frac{1}{5}w_1 & + & \frac{3}{2}w_2 \\
\hline
\end{array}
$$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls
Maximal profit: $510

Problem 7. 8.7

- (i) COME BACK TO
- (ii)

$$
\begin{array}{ll}
\text{maximize} & 5x_1 + 2x_2 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 = 15 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& 4x_1 - 3x_2 + x_5 = -12 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Auxiliary problem:

$$
\begin{array}{ll}
\text{maximize} & -x_0 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 - x_0 = 15 \\
& 3x_1 + 5x_2 + x_4 - x_0 = 15 \\
& 4x_1 - 3x_2 + x_5 - x_0 = -12 \\
& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
x_4 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
x_5 & = & -12 & - & 4x_1 & + & 3x_2 & + & x_0 \\
\hline\hline
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & x_5 \\
\hline
x_3 & = & 27 & - & x_1 & - & 6x_2 & + & x_5 \\
x_4 & = & 27 & + & x_1 & - & 8x_2 & + & x_5 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & x_5 \\
\hline\hline
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}x_4 & - & \frac{5}{8}x_5 \\
\hline
x_3 & = & \frac{27}{4} & - & \frac{7}{4}x_1 & + & \frac{3}{4}x_4 & + & \frac{1}{4}x_5 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}x_4 & + & \frac{1}{8}x_5 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}x_4 & + & \frac{5}{8}x_5 \\
\hline
\end{array}
$$

We see that the optimum for the auxillary problem is nonzero, so there is no way to make x_0 become 0, and therefore we say that the original problem has no feasible points.

- (iii) For this one there's actually no need to set up an auxillary problem - which is *really* nice.

$$
\begin{array}{ccccc}
\hline\hline
\zeta & = & & -3x_1 & +x_2 \\
\hline
w_1 & = & 4 & & -x_2 \\
w_2 & = & 6 & +2x_1 & -3x_2 \\
\hline\hline
\zeta & = & 2 & -\frac{7}{3}x_1 & -\frac{1}{3}w_2 \\
\hline
w_1 & = & 2 & -2x_1 & +w_2 \\
x_2 & = & 2 & +\frac{2}{3}x_1 & -\frac{1}{3}w_2
\end{array}
$$

The optimum value is 2, which is attained at (0, 2).

Problem 8. 8.8 I'll go very simple for these example questions...

$$
\begin{aligned}
\text{maximize} \quad & -x-y-z \\
\text{subject to} \quad & -x+y \leq 1 \\
& x-y \leq 1 \\
& x-z \leq 1 \\
& -x+z \leq 1 \\
& x,y,z \geq 0
\end{aligned}
$$

gives an unbounded closed feasible region (it's like this rectangular prism corridor which shoots out from the origin), but the optimum is attained at (0, 0).

Problem 9. 8.9 I'll re-use the same example function, but change the objective function.

$$\begin{aligned} &\text{maximize} \ \ x + y + z \\ &\text{subject to} \ \ -x + y \leq 1 \\ &\qquad\qquad\quad x - y \leq 1 \\ &\qquad\qquad\quad x - z \leq 1 \\ &\qquad\qquad\quad -x + z \leq 1 \\ &\qquad\qquad\quad x, y, z \geq 0 \end{aligned}$$

I have the same feasible region, but now the minimum of the objective function is $(0, 0)$, and there is no maximum. (To see this note that the triple (N, N, N) satisfies all constraints for any $N \in \mathbb{R}$)

Problem 10. 8.10 Boy, this example I've picked really is a workhorse:

$$\begin{aligned} &\text{maximize} \ \ x + y + z \\ &\text{subject to} \ \ -x + y \leq -1 \\ &\qquad\qquad\quad x - y \leq -1 \\ &\qquad\qquad\quad x - z \leq -1 \\ &\qquad\qquad\quad -x + z \leq -1 \\ &\qquad\qquad\quad x, y, z \geq 0 \end{aligned}$$

I've flipped all the constraints, and now I get an empty feasible region. This can be seen just by looking at the first two constraints - if $-x+y \leq -1$, then $x-y \geq 1 > -1$, so we see that there are no points which satisfy both of the first two constraints.

Problem 11. 8.11 I'll modify the workhorse by bounding the prism within two hyperplanes (two more constraints - the geometric intuiton is more clear if you think of $-x - y - z \leq -1$ as $-x - y - z \geq -1$)

$$\begin{aligned} &\text{maximize} \ \ x + y + z \\ &\text{subject to} \ \ -x + y \leq 1 \\ &\qquad\qquad\quad x - y \leq 1 \\ &\qquad\qquad\quad x - z \leq 1 \\ &\qquad\qquad\quad -x + z \leq 1 \\ &\qquad\qquad\quad -x - y - z \leq -1 \\ &\qquad\qquad\quad x + y + z \leq 5 \\ &\qquad\qquad\quad x, y, z \geq 0 \end{aligned}$$

The auxillary problem is :

$$\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{subject to} \quad & -x+y+x_0 \le 1 \\
& x-y+x_0 \le 1 \\
& x-z+x_0 \le 1 \\
& -x+z+x_0 \le 1 \\
& -x-y-z+x_0 \le -1 \\
& x+y+z+x_0 \le 5 \\
& x,y,z \ge 0
\end{aligned}$$

If this problem solves with an optimum $-x_0 = 0$ (which it should) then the point we get will be a feasible point for the original problem.

Problem 12. 8.12

$$\begin{aligned}
\text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{subject to} \quad & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\
& 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\
& x_1 + x_7 = 0 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0
\end{aligned}$$

$$\begin{array}{ccccccccccc}
\zeta & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
x_5 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_6 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
\hline\hline
\zeta & = & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_5 \\
\hline
x_1 & = & & & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_5 \\
x_6 & = & & & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_5 \\
\hline\hline
\zeta & = & 1 & - & 30x_2 & - & 42x_4 & - & 18x_5 & - & x_7 \\
\hline
x_1 & = & 1 & & & & & & & - & x_7 \\
x_6 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_5 & - & 2x_7 \\
x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_5 & - & x_7 \\
\hline
\end{array}$$

The optimum is 1, which is attained at $(1, 0, 1, 0)$

Problem 13. 8.15

Proof. Note that we have $x, y \succeq 0$. Thus we have the following implications:

$$\begin{aligned}
A^T y \succeq c &\implies (A^T y)^T x \ge c^T x \\
Ax \preceq b &\implies (Ax)^T y \le b^T y
\end{aligned}$$

And we see that

$$(Ax)^T y = x^T A^T y = (y^T A^T x)^T = y^T A^T x = (Ay)^T x^*$$

because all these are real numbers and thus equal to their transposes. We conclude by seeing

$$c^T x \leq (A^T y)^T x = (A^T y)^T x \leq b^T x$$

$\square$

Problem 14. 8.17 Let the primal problem be

$$\begin{aligned} \text{maximize}_x \quad & c^T x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

then the dual is:

$$\begin{aligned} \text{minimize}_y \quad & b^T y \\ \text{subject to} \quad & A^T y \succeq c \\ & y \succeq 0 \end{aligned}$$

Then by definition the dual of the dual is

$$\begin{aligned} \text{maximize}_z \quad & c^T z \\ \text{subject to} \quad & (A^T)^T z \succeq b \\ & z \succeq 0 \end{aligned}$$

and since $(A^T)^T = A$, this is just the primal problem.

Problem 15. 8.18
Solving the standard problem:

$$\begin{array}{ccccccc}
\zeta & = & & & x_1 & + & x_2 \\ \hline
w_1 & = & 3 & - & 2x_1 & - & x_2 \\
x_2 & = & 5 & - & x_1 & - & 3x_2 \\
x_5 & = & 4 & - & 2x_1 & - & 3x_2 \\ \hline\hline
\zeta & = & \frac{3}{2} & + & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\ \hline
x_1 & = & \frac{3}{2} & - & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\
w_2 & = & \frac{7}{2} & - & \frac{5}{2}x_2 & + & \frac{1}{2}w_1 \\
w_3 & = & 1 & - & 2x_2 & + & w_1 \\ \hline\hline
\zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_1 \\ \hline
x_1 & = & \frac{5}{4} & - & \frac{3}{4}w_1 & + & \frac{1}{4}w_3 \\
w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3 \\ \hline
\end{array}$$

The function attains its optimum $\frac{7}{4}$ at the point $(\frac{5}{4}, \frac{1}{2})$

$$\begin{aligned} \text{minimize } & 3y_1 + 5y_2 + 4y_3 \\ \text{subject to } & 2y_1 + y_2 + 3y_3 \geq 1 \\ & y_1 + 3y_2 + 3y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Which in standard form is:

$$\begin{aligned} \text{maximize } & -3y_1 - 5y_2 - 4y_3 \\ \text{subject to } & -2y_1 - y_2 - 3y_3 \leq -1 \\ & -y_1 - 3y_2 - 3y_3 \leq -1 \\ & y_1, y_2, y_3 \leq 0 \end{aligned}$$

(0, 0) is not a feasible point, so I create the auxillary problem:

$$\begin{aligned} \text{maximize } & -y_0 \\ \text{subject to } & -2y_1 - y_2 - 3y_3 - y_0 \leq -1 \\ & -y_1 - 3y_2 - 3y_3 - y_0 \leq -1 \\ & y_1, y_2, y_3, y_0 \leq 0 \end{aligned}$$

$$\begin{array}{rcrcrcrcrcr} \zeta & = & & & & & & & & - & v_0 \\ \hline v_1 & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & + & v_0 \\ v_2 & = & -1 & + & y_1 & + & 3y_2 & + & 3y_3 & + & v_0 \\ \hline\hline \zeta & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & - & v_1 \\ \hline v_0 & = & 1 & - & 2y_1 & - & y_2 & - & 2y_3 & + & v_1 \\ v_2 & = & & - & y_1 & + & 2y_2 & + & y_3 & + & v_1 \\ \hline\hline \zeta & = & & & & & & & & - & v_0 \\ \hline y_2 & = & 1 & - & 2y_1 & - & 2y_3 & + & v_1 & - & v_0 \\ v_2 & = & 2 & - & 5y_1 & - & 3y_3 & + & 3v_1 & - & 2v_0 \\ \hline\hline \end{array}$$

Now I substitute into the objective function and solve the original problem.

$$\begin{array}{rcrcrcrcr} \zeta & = & -2 & + & y_1 & - & 3y_2 & - & 2v_1 \\ \hline y_3 & = & \frac{1}{2} & - & y_1 & - & \frac{1}{2}y_2 & + & \frac{1}{2}v_1 \\ v_2 & = & \frac{1}{2} & - & 2y_1 & + & \frac{3}{2}y_2 & + & \frac{3}{2}v_1 \\ \hline\hline \zeta & = & -\frac{7}{4} & - & \frac{3}{2}y_2 & - & \frac{5}{4}v_1 & - & \frac{1}{2}v_2 \\ \hline y_3 & = & \frac{1}{4} & - & 2\frac{3}{2}y_2 & - & \frac{1}{4}v_1 & + & \frac{1}{2}v_2 \\ y_1 & = & \frac{1}{4} & + & \frac{3}{2}y_2 & + & \frac{3}{4}v_1 & - & \frac{1}{2}v_2 \\ \hline \end{array}$$

The function attains its optimum $\frac{7}{4}$ at the point $(\frac{1}{4}, 0, \frac{1}{4})$
The two problems match up which is good!