Math Problem Set 2

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Problem 3.1. There are two parts:

- (i)

$$\begin{aligned}
||x+y||^2 - ||x-y||^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\
&= \langle x+y, x\rangle + \langle x+y, y\rangle - (\langle x-y, x\rangle + \langle x-y, -y\rangle) \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle - (\langle x, x-y\rangle + \langle -y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - \langle x, x\rangle - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle \\
&= 4\langle x, y\rangle \text{ after some easy manipulations}
\end{aligned}$$

  So it is clear that $\frac{1}{4}(||x+y||^2 - ||x-y||^2) = \langle x, y\rangle$

- (ii) As above,

$$\begin{aligned}
||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x+y, x\rangle + \langle x+y, y\rangle + (\langle x-y, x\rangle + \langle x-y, -y\rangle) \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle + (\langle x, x-y\rangle + \langle -y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y, \rangle \text{ after some easy manipulations} \\
&= 2(||x||^2 + ||y||^2)
\end{aligned}$$

  So it is clear that $\frac{1}{2}(||x+y||^2 + ||x-y||^2) = ||x||^2 + ||y||^2$

Problem 3.2. I proceed similarly: just start from the left side, expand, and simplify.

Problem 3.3. Let θ be the angle in question. Recall that

$$cos(\theta) - \frac{\langle f, g\rangle}{||f|| \cdot ||g||}$$

- (i)

$$\langle f, g\rangle = \int_0^1 fg dx = \int_0^1 x^6 dx = \left(\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{7}$$
$$||f||^2 = \int_0^1 f^2 dx = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$
$$||g||^2 = \int_0^1 g^2 dx = \int_0^1 x^1 0 dx = \left(\frac{x^{11}}{11}\right)\Big|_0^1 = \frac{1}{11}$$

Thus, we see that

$$cos(\theta) = \frac{\frac{1}{7}}{(\frac{1}{3}\cdot\frac{1}{11})^{\frac{1}{2}}} = \frac{33^{\frac{1}{2}}}{7} \tag{1}$$

And (1) implies that $\theta \approx 35°$.

- (ii)

$$\langle f, g\rangle = \int_0^1 fg dx = \int_0^1 x^6 dx = \left(\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{7}$$
$$||f||^2 = \int_0^1 f^2 dx = \int_0^1 x^4 dx = \left(\frac{x^5}{8}\right)\Big|_0^1 = \frac{1}{5}$$
$$||g||^2 = \int_0^1 g^2 dx = \int_0^1 x^8 dx = \left(\frac{x^9}{9}\right)\Big|_0^1 = \frac{1}{9}$$

Thus, we see that

$$cos(\theta) = \frac{\frac{1}{7}}{(\frac{1}{5}\cdot\frac{1}{9})^{\frac{1}{2}}} = \frac{45^{\frac{1}{2}}}{7} \tag{2}$$

And (1) implies that $\theta \approx 17°$

Problem 3.8. There are four parts

- (i)

Proof. This is just a matter of checking all the relevant details.
Norms = 1:

$$||cos(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt = \frac{1}{\pi}\frac{cos(x)sin(x)+x}{2}\Big|_{-\pi}^{\pi} = 1$$
$$||sin(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(t)dt = \frac{1}{\pi}\frac{-sin(2x)+2x}{4}\Big|_{-\pi}^{\pi} = 1$$
$$||cos(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(2t)dt = \frac{1}{\pi}\frac{sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$$
$$||sin(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(2t)dt = \frac{1}{\pi}\frac{-sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$$

Inner Products = 1:

$$\langle cos(t), sin(t) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(t)dt = \frac{sin^2(x)}{2}\Big|_{-\pi}^{\pi} = sin^2(\pi) - sin^2(-\pi) = 0$$

The proof is completed by checking all the other inner products similarly. □

- (ii)

$$\begin{aligned} ||t||^2 &= \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} = \frac{2\pi^3}{3} \\ ||t|| &= \left(\frac{2\pi^3}{3}\right)^{\frac{1}{2}} \end{aligned}$$

- (iii)

$$\begin{aligned} proj_X(cos(3t)) &= \langle sin(t), cos(3t) \rangle sin(t) + \langle cos(t), cos(3t) \rangle cos(t) \\ &+ \langle sin(2t), cos(3t) \rangle sin(2t) + \langle cos(2t), cos(3t) \rangle cos(2t) \\ &= 0 + 0 + 0 + 0 = 0 \end{aligned}$$

(We see that $cos(3t)$ is orthogonal to X)

- (iv)

$$proj_X(t) = 0 + 2sin(t) + 0 - sin(2t)$$

Problem 3.9. *Proof.* In $\mathbb{R}^2$, a rotation about the origin by arbitrary angle θ can be described by the matrix

$$M = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

So that $M\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{smallmatrix} xcos(\theta) - ysin(\theta) \\ xsin(\theta) + ycos(\theta) \end{smallmatrix}\right)$. Let $a = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right) \in \mathbb{R}^2$. Then

$$\begin{aligned} \langle M(a), M(b) \rangle &= \left\langle \begin{pmatrix} a_1cos(\theta) - a_2sin(\theta) \\ a_1sin(\theta) + a_2cos(\theta) \end{pmatrix}, \begin{pmatrix} b_1cos(\theta) - b_2sin(\theta) \\ b_1sin(\theta) + b_2cos(\theta) \end{pmatrix} \right\rangle \\ &= a_1b_1cos^2(\theta) + a_2b_2sin^2(\theta) + a_1b_1sin^2(\theta) + a_2b_2cos^2(\theta \\ &= a_1b_1 + a_2b_2 = \langle a, b \rangle \end{aligned}$$

So we see that M is an orthonormal operator. (N.b that this is a terribly inefficient way to prove this - I should have just shown that the columns of M were orthonormal! □

Problem 3.10. Recall that taking the Hermitian is flipping rows and columns and taking the conjugate.

- (i)

Proof. We need to show both directions.
$\Rightarrow$: Let Q be an orthonormal matrix. Then $\langle m, n\rangle = \langle Qm, Qn\rangle \implies m^H n = (Qm)^H Qn = m^H Q^H Qn$. And because m and n were arbitrarily chosen, the only way that this equality holds is if $Q^H Q = I$, and this gives us that $QQ^H = Q$ since left inverse $\implies$ right inverse.
$\Leftarrow$: Let Q be a matrix so that $Q^H Q = I$. Then consider $\langle Qm, Qn\rangle = (Qm)^H Qn = m^H Q^H Qn = m^H n = \langle m, n\rangle$. $\square$

- (ii)

  Proof. This is pretty easy:

  $$||x|| = \sqrt[2]{\langle x, x\rangle} = \sqrt[2]{\langle x, x\rangle} = ||Qx||$$

  $\square$

- (iii)

  Proof. Assume Q is orthonormal. Then $QQ^H = Q^H Q = I \implies Q^H = Q^{-1}$. I'll prove the following short lemma.

  Lemma 0.1. For Q orthonormal, Q^H is orthonormal.

  Recall that $(Q^H)^H = Q$, and see that

  $$(Q^H)^H Q^H = Q^H (Q^H)^H = I$$

  which proves the lemma.
  And since $Q^{-1} = Q^H$, Q^{-1} is orthonormal. $\square$

- (iv)

  Proof. We'll examine the elements of the identity matrix element by element. First note that:

  $$I_{ij} = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

  Then we'll compare this to what we get when we multiply QQ^H, which we know is equal to I in all its coordinates. First, though, for any matrix A, define A^i to be the "ith row" of A and A_j to be the jth column. Then

  $$\delta_{ij} = (Q^H Q)_{ij} = (Q^H)^i Q_j =$$

  Recall now that $(Q^H)^i = \bar{Q}_i$, by definition of the Hermitian. But now we see that

  $$\langle \bar{Q}_i, q_j\rangle = \delta_{ij}$$

  and the columns of Q are orthonormal. $\square$

- (v) A counterexample would be the matrix $M = \begin{smallmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}$. $\det B = 1$, but $Be_1 = 2e_1$ and $||e_1|| \neq ||2e_1|| = ||Be_1||$ which violates what we proved in (ii).

- (vi)

  Proof. This is also quite short:

  $$(Q_1Q_2)(Q_1Q_2)^H = Q_1Q_2Q_2^HQ_1^H = Q_1Q_1^H = I$$

  And we also get the left inverse by properties of inverses. □

Problem 3.11. Suppose that $x_1, ..., x_n$ is as et of linearly *dependent* vectors. Let's apply Gram-Schmidt. Eventually, we will arrive at a vector x_k which is linearly dependent upon $x_1, ..., x_{k-1}$. But then, if $X = span(x_1, ..., x_{k-1})$, then $x_k \in X$ and $proj_X(x_k) = x_k$...

Problem 3.13. (Simply is procedural)

$$\begin{aligned} q_1 &= 1 \\ q_2 &= \frac{x - p_{k-1}}{||x - p_1||} \\ p_1 &= \langle 1, x \rangle \cdot 1 = 0 \\ ||x|| = \frac{\pi}{2} &\implies q_2 = \frac{2x}{\pi} \\ p_2 = \langle x^2, 1 \rangle \cdot 1 + \langle x^2, \frac{2x}{\pi} \rangle \cdot \frac{2x}{\pi} &= \frac{\pi}{2} \\ q_3 = \frac{x^2 - \frac{\pi}{2}}{||x^2 - \frac{\pi}{2}||} &= \frac{8}{\pi(2\pi^2 - 4\pi + 3)}(x^2 - \frac{\pi}{2}) \end{aligned}$$

Problem 3.16.

First I must show that topological equivalence is an equivalence relation.

Proof. I must show three things: (i) $x \sim x$. (ii) $x \sim y \implies y \sim x$, (iii) $x \sim y$ and $y \sim z \implies x \sim z$.
(i) $|| \cdot ||_1 \sim || \cdot ||_1$ trivially. Let $M \geq m$, then $m||x||_1 \leq ||x||_1 \leq M||x||_1$ for all x.
(ii) Also trivial: Suppose $|| \cdot ||_1 \sim || \cdot ||_2$. Then $m||x||_1 \leq ||x||_2 \leq M||x||_1$ for all x, which implies that $M^{-1}||x||_2 \leq ||x||_1 \leq m^{-1}||x||_2$.
(iii) Suppose $|| \cdot ||_1 \sim || \cdot ||_2$, and $|| \cdot ||_2 \sim || \cdot ||_3$. Then $m||x||_1 \leq ||x||_2 \leq M||x||_1$ and $n||x||_2 \leq ||x||_3 \leq N||x||_2$. But we get from this that $mn||x||_1 \leq ||x||_3 \leq MN||x||_1$. □

Now I'll show that the $1, 2$, and ∞ norms are topologically equivalent.

Proof. (i) $|| \cdot ||_1 \sim || \cdot ||_2$:
If we think about the inner product as the standard dot-product, then we have

$$(||x||_1)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i||x_j| \geq \sum_{i=1}^{n} x_i^2 = \langle x, x, \rangle = (||x||_2)^2$$

(the inequality comes because we simply threw out some positive terms on the left side). This implies that $||x||_1 \geq ||x||_2$. Moreover,

$$(\sqrt[2]{n}||x||_2)^2 = n\sum_{i=1}^{n} x_i$$

(ii) $||\cdot||_\infty \sim ||\cdot||_2$

$$||x||_\infty = max_{1\leq i\leq n}\{x_i\} = \sqrt[2]{(max_{1\leq i\leq n}\{x_i\})^2} \leq \sqrt[2]{\sum_{i=1}^{n} x_i} = ||x||_2$$

$\square$

Problem 3.26.