It is then convenient to use an inner product on $\mathfrak{h}$ to identity linear functionals on $\mathfrak{h}$ with elements of $\mathfrak{h}$ itself. We define the inner product of H and H' in $\mathfrak{h}$ by

$$\left\langle H, H' \right\rangle = \operatorname{trace}(H^* H'), \tag{6.13}$$

or, explicitly,

$$\langle \operatorname{diag}(a, b, c), \operatorname{diag}(d, e, f) \rangle = \bar{a}d + \bar{b}e + \bar{c}f,$$

where $\operatorname{diag}(\cdot,\cdot,\cdot)$ is the diagonal matrix with the indicated diagonal entries. If ϕ is a linear functional on $\mathfrak{h}$, there is (Proposition A.11) a unique vector λ in $\mathfrak{h}$ such that ϕ may be represented as $\phi(H) = \langle \lambda, H \rangle$ for all $H \in \mathfrak{h}$. If we represent the linear functional in the previous paragraph in this way, we arrive at a new, basis-independent notion of a weight.

Definition 6.21. Let $\mathfrak{h}$ be the subspace of $\mathsf{sl}(3;\mathbb{C})$ spanned by H_1 and H_2 and let (π, V) be a representation of $\mathsf{sl}(3;\mathbb{C})$. An element λ of $\mathfrak{h}$ is called a **weight** for π if there exists a nonzero vector v in V such that

$$\pi(H)v = \langle \lambda, H \rangle\, v$$

for all H in $\mathfrak{h}$. Such a vector v is called a **weight vector** with weight λ.

If λ is a weight in our new sense, the ordered pair (m_1, m_2) in Definition 6.1 is given by

$$m_1 = \langle \lambda, H_1 \rangle ; \quad m_2 = \langle \lambda, H_2 \rangle .$$

It is easy to check that for all $U \in N$, the adjoint action of U on $\mathfrak{h}$ preserves the inner product in (6.13). Thus, the action of the Weyl group on $\mathfrak{h}$ is unitary: $\langle w \cdot H, w \cdot H' \rangle = \langle H, H' \rangle$. Since the roots are just the nonzero weights of the adjoint representation, we now also think of the roots as elements of $\mathfrak{h}$.

Theorem 6.22. *Suppose that (Π, V) is a finite-dimensional representation of $\mathsf{SU}(3)$ with associated representation (π, V) of $\mathsf{sl}(3;\mathbb{C})$. If $\lambda \in \mathfrak{h}$ is a weight for V then $w \cdot \lambda$ is also a weight of V with the same multiplicity. In particular, the roots are invariant under the action of the Weyl group.*

Proof. Suppose that λ is a weight for V with weight vector v. Then for all $U \in N$ and $H \in \mathfrak{h}$, we have

$$\begin{aligned}
\pi(H)\Pi(U)v &= \Pi(U)(\Pi(U)^{-1}\pi(H)\Pi(U))v \\
&= \Pi(U)\pi(U^{-1}HU)v \\
&= \left\langle \lambda, U^{-1}HU \right\rangle \Pi(U)v.
\end{aligned}$$

Here, we have used that U is in N, which guarantees that $U^{-1}HU$ is, again, in $\mathfrak{h}$. Thus, if w is the Weyl group element represented by U, we have