The Unsteady Force Due to Vorticity Creation

The force that must be applied to a fluid of volume V to generate vorticity $\boldsymbol{\omega}(t)$ inside the volume is

$$\mathbf{F}_{\omega}(t) = \frac{\rho}{2}\frac{\partial}{\partial t}\int_V \mathbf{r}\times\boldsymbol{\omega}(t)dV. \tag{1}$$

As a simple model of vortex creation we consider the axisymmetric jet emerging from the vocal folds of radius r_f surrounded by a vortical boundary layer of radius δ. As a model for this jet profile we consider the piecewise linear profile shown in Eq. 2. This profile possesses three regions. Inside the boundary layer the velocity is constant and equal to U_0. The vortical boundary layer has radius δ. Here the velocity rapidly drops from U_0 to zero. Outside the boundary the velocity is zero.

$$\mathbf{U} = \begin{cases} U_0\hat{x} & 0 \le r < r_f \\ U_0\frac{r_f+\delta-r}{\delta}\hat{x} & r_f \le r < r_f+\delta \\ 0\hat{x} & r_f+\delta \le r < r_0 \end{cases} \tag{2}$$

From this assumption it can be seen that the mean flow vorticity is $\boldsymbol{\Omega} = \frac{U_0}{\delta}\hat{\phi}$ inside the jet boundary layer and 0 outside of it. To calculate the unsteady portion of the vorticity let us separate the total flow into the mean flow and a small disturbance $\mathbf{u} = \mathbf{U} + \epsilon\mathbf{u}'$.

$$\mathbf{u}' = u'_x(x,r,\phi,t)\hat{x} + u'_r(x,r,\phi,t)\hat{r} + u'_\phi(x,r,\phi,t)\hat{\phi}. \tag{3}$$

Substituting this assumption into the Euler and continuity equations and gathering terms of order ϵ we get the linearized Euler and continuity equation for the disturbances

$$\begin{gathered}\frac{\partial\mathbf{u}'}{\partial t} + U\frac{\partial\mathbf{u}'}{\partial x} + u'_r\frac{\partial\mathbf{U}}{\partial r} = -\frac{\nabla p'}{\rho} \\ \nabla\cdot\mathbf{u}' = 0\end{gathered} \tag{4}$$

Let's define the Fourier transforms

$$\begin{aligned}\mathscr{F}_x[f(r,x,\theta,t)] &= \textstyle\int_0^\infty dx f(r,x,\theta,t)e^{-i\alpha x} \\ \mathscr{F}_\theta[f(r,x,\theta,t)] &= \textstyle\int_0^{2\pi} d\theta f(r,x,\theta,t)e^{-in\theta} \\ \mathscr{F}_t[f(r,x,\theta,t)] &= \textstyle\int_{-\infty}^{\infty} dt f(r,x,\theta,t)e^{-i\omega t}\end{aligned} \tag{5}$$

Let's define the multi dimensional Fourier transform

$$\hat{f}(r,\omega,\alpha,n) = \mathscr{F}[f(r,x,\theta,t)] = \int_{-\infty}^{\infty} dt \int_0^{2\pi} d\theta \int_0^{\infty} dx f(r,x,\theta,t)e^{-i(n\theta+\alpha x+\omega t)} \tag{6}$$

$$\mathscr{F}[\mathbf{u}'] = \begin{pmatrix}\hat{u}_x(r)\\ i\hat{u}_r(r)\\ \hat{u}_\theta(r)\end{pmatrix}\ \mathscr{F}[\tfrac{\partial\mathbf{u}'}{\partial t}] = \begin{pmatrix}i\omega\hat{u}_x(r)\\ -\omega\hat{u}_r(r)\\ i\omega\hat{u}_\theta(r)\end{pmatrix}\ \mathscr{F}[\tfrac{\partial\mathbf{u}'}{\partial x}] = \begin{pmatrix}i\alpha\hat{u}_x(r) - \mathscr{F}_t[\mathscr{F}_\theta[u_0(t)]]\\ -\alpha\hat{u}_r(r)\\ i\alpha\hat{u}_\theta(r)\end{pmatrix}\ \mathscr{F}[\nabla p'] = \begin{pmatrix}i\alpha\hat{p}(r)\\ \hat{p}'(r)\\ \frac{in}{r}\hat{p}(r)\end{pmatrix}\ \mathscr{F}[\nabla\cdot\mathbf{u}'] \tag{7}$$

$$\omega\begin{pmatrix}\hat{u}_x\\ -\hat{u}_r\end{pmatrix} + U\alpha\begin{pmatrix}\hat{u}_x\\ -\hat{u}_r\end{pmatrix} + \hat{u}_r\begin{pmatrix}\frac{\partial U}{\partial r}\\ 0\end{pmatrix} + \begin{pmatrix}i\alpha\hat{p}(r)\\ \frac{\partial\hat{p}}{\partial r}\end{pmatrix} = U\begin{pmatrix}\hat{u}_0\\ 0\end{pmatrix} \tag{8}$$