The Unsteady Force Due to Vorticity Creation

$$
\begin{aligned}
\frac{\partial \mathbf{u}'}{\partial t} + U\frac{\partial \mathbf{u}'}{\partial x} + v\frac{\partial \mathbf{U}}{\partial r} &= -\nabla p' \\
\nabla \cdot \mathbf{u}' &= 0
\end{aligned}
\tag{1}
$$

Taking the divergence of the momentum equation and using the continuity equation we get an equation for the pressure

$$
\nabla^2 p = -2U'\frac{\partial v}{\partial x}. \tag{2}
$$

In component form Eq. **??** is

$$
\begin{aligned}
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial r} \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial r} + \frac{\partial u}{\partial x} &= 0
\end{aligned}
\tag{3}
$$

Taking the derivative with respect to r of Eq. 2 and using the first of Eqs. 3 we get an equation for the radial velocity v.

$$
\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla^2 - \frac{1}{r^2}\right) - \left(U'' - \frac{U'}{r}\right)\frac{\partial}{\partial x}\right] v = 0 \tag{4}
$$

Let's define the Fourier transform with respect to x

$$
\tilde{v}(r,\alpha,t) = \int_0^\infty v(r,x,t)e^{-i\alpha x}dx \qquad\qquad v(r,x,t) = \frac{1}{2\pi}\int_{C_\alpha} \tilde{v}(r,\alpha,t)e^{i\alpha x}d\alpha, \tag{5}
$$

where the bottom of the contour C_α lies below all singularities of the integrand and is closed by semi circle that encloses the singularities and goes off to infinity. Substituting this expression into 6,

$$
\left[\frac{\partial}{\partial t}\left(D^2 + \frac{D}{r} - \frac{1}{r^2} + \frac{\partial^2}{\partial x^2}\right) + U\frac{\partial}{\partial x}\left(D^2 + \frac{D}{r} - \frac{1}{r^2} + \frac{\partial^2}{\partial x^2}\right) - \left(U'' - \frac{U'}{r}\right)\frac{\partial}{\partial x}\right] v = 0 \tag{6}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(D^2\tilde{v} + \frac{D}{r}\tilde{v} - \frac{1}{r^2}\tilde{v} - \alpha^2\tilde{v}) + \frac{\partial}{\partial t}(-i\alpha v(r,0,t) - \frac{\partial v(r,0,t)}{\partial x}) \\
&\qquad + U(D^2(i\alpha\tilde{v} - v(r,0,t)) + \frac{D}{r}(i\alpha\tilde{v} - v(r,0,t)) - \frac{1}{r^2}(i\alpha\tilde{v} - v(r,0,t) + \\
&\qquad (-i\alpha^3\tilde{v} + \alpha^2 v(r,0,t) - i\alpha\frac{\partial v(r,0,t)}{\partial x} - \frac{\partial^2 v(r,0,t)}{\partial x^2})) - (i\alpha\tilde{v} - v(r,0,t))\left(U'' - \frac{U'}{r}\right) = 0,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}(D^2\tilde{v} + \frac{D}{r}\tilde{v} - \frac{1}{r^2}\tilde{v} - \alpha^2\tilde{v}) + i\alpha U(D^2\tilde{v} + \frac{D}{r}\tilde{v} - \frac{1}{r^2}\tilde{v} - \alpha^2\tilde{v}) - i\alpha\tilde{v}\left(U'' - \frac{U'}{r}\right) \\
&+ \frac{\partial}{\partial t}(-i\alpha v(r,0,t) - \frac{\partial v(r,0,t)}{\partial x}) + U(-D^2 - \frac{D}{r} + \frac{1}{r^2} + \alpha^2 - i\alpha\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2})v(r,0,t) + v(r,0,t))\left(U'' - \frac{U'}{r}\right) = 0,
\end{aligned}
\tag{8}
$$

$$
\left[\left(\frac{\partial}{\partial t} + i\alpha U\right)\left(D^2 + \frac{D}{r} - \alpha^2 - \frac{1}{r^2}\right) - i\alpha\left(U'' - \frac{U'}{r}\right)\right]\tilde{v}(r,\alpha,t) =
$$

$$\left[\left(i\alpha\frac{\partial}{\partial t}+i\alpha U\frac{\partial}{\partial x}+\frac{\partial^2}{\partial x\partial t}+U\frac{\partial^2}{\partial x^2}\right)+U(D^2+\frac{D}{r})-\left(U''-\frac{U'}{r}+U\left(\frac{1}{r^2}+\alpha^2\right)\right)\right]v(r,0,t) \quad (9)$$

where we have used the fact that

$$\begin{aligned}
\int_0^\infty \frac{\partial v(r,x,t)}{\partial x}e^{-i\alpha x}dx &= i\alpha\tilde{v}(r,\alpha,t)-v(r,0,t)\\
\int_0^\infty \frac{\partial^2 v(r,x,t)}{\partial x^2}e^{-i\alpha x}dx &= -\alpha^2\tilde{v}(r,\alpha,t)-i\alpha v(r,0,t)-\frac{\partial v(r,0,t)}{\partial x}\\
\int_0^\infty \frac{\partial^3 v(r,x,t)}{\partial x^3}e^{-i\alpha x}dx &= -i\alpha^3\tilde{v}(r,\alpha,t)+\alpha^2 v(r,0,t)-i\alpha\frac{\partial v(r,0,t)}{\partial x}-\frac{\partial^2 v(r,0,t)}{\partial x^2}
\end{aligned} \quad (10)$$

Now defining the Fourier transform with respect to t

$$\hat{v}(r,\alpha,\omega)=\int_{-\infty}^{\infty}\tilde{v}(r,\alpha,t)e^{i\omega t}dt \qquad \tilde{v}(r,\alpha,t)=\frac{1}{2\pi}\int_{C_\omega}\hat{v}(r,\alpha,\omega)e^{-i\omega t}d\omega, \quad (11)$$

where here the top of the contour C_ω lies above all singularities in the ω plane and is closed by a semi circle that travels downward. Substituting this into Eq. 9,

$$\begin{aligned}
&\left[(\omega-\alpha U)\left(D^2+\frac{D}{r}-\frac{1}{r^2}-\alpha^2\right)+\alpha\left(U''-\frac{U'}{r}\right)\right]\hat{v}(r,\alpha,\omega)=\\
&\quad i\left[\left(\alpha\omega+i\alpha U\frac{\partial}{\partial x}-i\omega\frac{\partial}{\partial x}+U\frac{\partial^2}{\partial x^2}\right)+U(D^2+\frac{D}{r})-\left(U''-\frac{U'}{r}+U\left(\frac{1}{r^2}+\alpha^2\right)\right)\right]\hat{v}_0(r,\omega)=\hat{f}_0(r,\omega)
\end{aligned} \quad (12)$$

where

$$\hat{v}_0(r,\omega)=\int_{-\infty}^{\infty}v(r,0,t)e^{i\omega t}dt. \quad (13)$$

Let's define the operator on the left as

$$\mathbf{L}(\alpha,\omega)=(\omega-\alpha U)\left(D^2+\frac{D}{r}-\frac{1}{r^2}-\alpha^2\right)+\alpha\left(U''-\frac{U'}{r}\right). \quad (14)$$

We have thus converted our problem to solving the inhomogeneous boundary value problem

$$\begin{aligned}
\mathbf{L}(\alpha,\omega)\hat{v}(r,\alpha,\omega)&=\hat{f}_0(r,\omega)\\
\hat{v}(0,\alpha,\omega)&=0\\
\hat{v}(2,\alpha,\omega)&=0
\end{aligned} \quad (15)$$

To find the dispersion relation we have to solve the homogeneous problem

$$\mathbf{L}(\alpha,\omega)\hat{v_h}(r,\alpha,\omega)=0. \quad (16)$$

For the spatial problem we select a real frequency ω and solve the polynomial eigenvalue problem,

$$\left[\mathbf{A}_0+\alpha\mathbf{A}_1+\alpha^2\mathbf{A}_2+\alpha^3\mathbf{A}_3\right]\hat{v_h}(r,\alpha,\omega)=0, \quad (17)$$

for eigenvalues α and eigenfunctions $\hat{v_h}(r,\alpha,\omega)$. The eigenvalues α are the solutions of the dispersion relation $D(\alpha,\omega)=0$ for a given ω.

To solve the inhomogeneous problem we will find the eigenfunctions and eigenvalues of the operator $\mathbf{L}(\alpha,\omega)$ and express the inhomogeneous solution as a sum of the eigenfunctions. That is we have to solve the eigenvalue problem

$$\mathbf{L}(\alpha,\omega)\hat{v}_n(\alpha,\omega)=\lambda_n(\alpha,\omega)\hat{v}_n(\alpha,\omega) \quad (18)$$

Expanding the solutions and driving term of Eq. 12 on to the eigenfunctions,

$$\begin{aligned}
\hat{v}&=\sum_n a_n\hat{v}_n\\
\hat{f}_0&=\sum_n f_n\hat{v}_n
\end{aligned} \quad (19)$$

Plugging these back into the original problem

$$\sum_n a_n \lambda_n \hat{v}_n = \sum_n f_n \hat{v}_n. \tag{20}$$

Since the eigenvectors $\{\hat{v}_1, \hat{v}_2, ...\}$ are linearly independent we have,

$$a_n = \frac{f_n}{\lambda_n} \tag{21}$$

and,

$$\hat{v}(r,\alpha,\omega) = \sum_n \frac{(\hat{f}_0(r,\alpha,\omega), \hat{v}_n(r,\alpha,\omega))}{\lambda_n(\alpha,\omega)} \hat{v}_n(r,\alpha,\omega) \tag{22}$$

Thus, for a given forcing $\hat{f}_0$ we have solved the problem in (α,ω) space. Now we must invert back to (x,t) space. First performing the spatial inversion,

$$v(r,x,t) = \frac{1}{4\pi^2} \int_{C_\omega} d\omega \int_{C_\alpha} d\alpha \left(\sum_n \frac{(\hat{f}_0(r,\alpha,\omega), \hat{v}_n(r,\alpha,\omega))}{\lambda_n(\alpha,\omega)} \hat{v}_n(r,\alpha,\omega) \right) e^{i(\alpha x - \omega t)} \tag{23}$$

Now we use the fact that $\lambda_n(\alpha,\omega)$ is an analytic function expect at its poles, which are given by the solutions of $D(\alpha,\omega) = 0$ and a branch cut which is given by the solutions of $(\omega - \alpha U) = 0$. The branch cut does not pose a problem, since we can deform our contour around. For a given ω let us denote the discrete solutions of the dispersion relation in the α plane as α_m. We can then use the residue theorem to evaluate Eq. 23.

$$v(r,x,t) = \frac{i}{2\pi} \sum_m \int_{C_\omega} d\omega (\hat{f}_0(r,\alpha_m,\omega), \hat{v}_h(r,\alpha_m,\omega)) \hat{v}_h(r,\alpha_m,\omega) e^{i(\alpha_m x - \omega t)}. \tag{24}$$

As a simple example consider the driving flow

$$v(r,0,t) = \frac{2-r}{2} e^{-i\omega_d t}, \tag{25}$$

and mean flow

$$\begin{aligned} U(r) &= \frac{2-r^2}{2} \\ U'(r) &= -r \\ U''(r) &= -1. \end{aligned} \tag{26}$$

Physically this corresponds to fully developed Pousille pipe flow with a sinusoidal driving velocity at $x = 0$. The Fourier transform of the driving velocity with respect to time is

$$\hat{v}_0(r,\omega) = \frac{2-r}{2} \frac{i}{\omega - \omega_d}. \tag{27}$$

Inserting this into the definition of $\hat{f}_(r,\omega)$ we get,

$$\hat{f}_0(r,\omega) = i \left[\alpha\omega - \frac{2-r^2}{4r} - \frac{2-r^2}{2r^2}\frac{2-r}{2} - \frac{2-r^2}{2}\frac{2-r}{2}\alpha^2 \right] \left(\frac{i}{\sqrt{2\pi}(\omega-\omega_d)} + \sqrt{\frac{\pi}{2}}\delta(\omega-\omega_d) \right). \tag{28}$$

$$\hat{f}_0(r,\omega) = -\left[-\frac{r^3\alpha^2}{2} + r^2\alpha^2 - \left(\frac{1}{2} - \alpha^2\right) r - \left(2\alpha^2 - \alpha\omega - \frac{1}{2}\right) + \frac{1}{2r} - \frac{1}{2r^2} \right] \frac{1}{\omega - \omega_d}. \tag{29}$$

To condense this expression we define $R(r,\alpha,\omega)$ such that

$$\hat{f}_0(r,\alpha,\omega) = \frac{R(r,\alpha,\omega)}{\omega - \omega_d}. \tag{30}$$

Substituting this into Eq. 24 and using the residue theorem

$$v(r,x,t) = -\sum_m (R(r,\alpha_m,\omega_d), \hat{v}_h(r,\alpha_m,\omega_d)) \hat{v}_h(r,\alpha_m,\omega_d) e^{i(\alpha_m x - \omega_d t)}. \tag{31}$$

CONNECTION TO ACOUSTIC FLOW

To derive an expression for $\hat{f}_0(r,\omega)$ we recall from the previous chapter that the axial velocity of the acoustic flow is

$$u(r,x,t) = cL\sum_m q_m(t)\Re\left[\frac{s_m}{c}\sinh(\frac{s_m}{c}x)\right] - \frac{(x-L)}{L}u_0(t) \tag{32}$$

From the incompressible continuity equation we can derive an expression for radial velocity

$$v(r,x,t) = -\frac{r-r_0}{2}\left(cL\sum_m q_m(t)\Re\left[\frac{s_m^2}{c^2}\cosh(\frac{s_m}{c}x)\right] - \frac{u_0(t)}{L}\right) \tag{33}$$

Rewriting this in the non dimensional coordinates of this chapter, $r^* = \frac{2r}{r_0}$, $v^* = \frac{v}{U_0}$, $t^* = \frac{2tU_0}{r_0}$,

$$v^*(r^*,x^*,t^*) = -\frac{r^*-2}{4}\left(\frac{r_0 L}{cU_o}\sum_m q_m(t^*)\Re\left[s_m^2\cosh\left(\frac{s_m r_0}{2c}x^*\right)\right] - \frac{r_0}{L}u_0^*(t^*)\right) \tag{34}$$

From now on we will drop the stars with the understanding that these quantities are non dimensional. Find the first and second derivatives and evaluating those functions at $x=0$

$$\begin{aligned}
v(r,0,t) &= -\frac{r-2}{4}\left(\frac{r_0 L}{U_0 c}\sum_m Re(s_m^2)q_m(t) - \frac{r_0}{L}u_0(t)\right)\\
\frac{\partial v(r,0,t)}{\partial x} &= 0\\
\frac{\partial^2 v(r,0,t)}{\partial x^2} &= -\frac{rLr_0^3}{16c^3U_0}\sum_m Re(s_m^4)q_m(t)
\end{aligned} \tag{35}$$

Taking the Fourier transforms with respect to t

$$\begin{aligned}
\hat{v}_0(r,\omega) &= \frac{r-2}{4}\left(\frac{r_0 L}{U_0 c}\sum_m Re(s_m^2)\hat{q}_m(\omega) + \frac{r_0}{L}\hat{u}_0(\omega)\right)\\
\frac{\partial \hat{v}_0(r,\omega)}{\partial x} &= 0\\
\frac{\partial^2 \hat{v}_0(r,\omega))}{\partial x^2} &= -\frac{rLr_0^3}{16c^3U_0}\sum_m \hat{q}_m(\omega)Re(s_m^4)
\end{aligned} \tag{36}$$

We can now simplify the expression for $\hat{f}_0$. Using the fact that first derivatives with respect to x and second derivatives with respect to r are 0,

$$\hat{f}_0(r,\omega) = i\left[\alpha\omega + U\frac{\partial^2}{\partial x^2} + U\frac{D}{r} - U'' + \frac{U'}{r} - U\left(\frac{1}{r^2}+\alpha^2\right)\right]\hat{v}_0(r,\omega) \tag{37}$$

Substituting Eqs. 35 into Eq. 36,

$$\hat{f}_0(r,\omega) = i\frac{r-2}{4}\left(\alpha\omega - U'' + \frac{U'}{r} - \frac{U}{r^2} + U\alpha^2 + \frac{U}{r(r-2)}\right)\left(\frac{r_0 L}{U_0 c}\sum_m Re(s_m^2)\hat{q}_m(\omega) + \frac{r_0}{L}\hat{u}_0(\omega)\right) - iU\frac{rLr_0^3}{16c^3U_0}\sum_m Re(s_m^4)\hat{q}_m(\omega) \tag{38}$$