Assignment 2 STA457H1F

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October 11, 2024

1 Exercise - Canadian/U.S. dollar exchange rates

1.1 Augmented Dickey-Fuller test

Conducting the ADF test for various lags - k=1, 2, 5, 10 - we have no evidence to reject the null hypothesis, with p-values hovering around 0.5. This would imply that the data is non-stationary or, alternatively, that we have no evidence to think that the data is stationary.

1.2 Ljung-Box and Bartlett tests

A white noise process represents random, uncorrelated data with constant mean and variance.

Computing the Ljung-Box test we want to check if autocorrelation exists in the two time series models. The test is defined as:
· H_0: residuals are independently distributed with no autocorrelation present
· H_1: residuals exhibit autocorrelation, hence they are not independently distributed

For first differences, we obtain a p-value of 0.01047 - lower than the critical value of 0.05 - indicating that we can reject the null hypothesis and say that the first differences are not independently distributed. This implies that the time series of first differences is not a white noise process.

For absolute first differences, we obtain a p-value of 1.631e-09 - lower than the critical value of 0.05 - indicating that we can confidently reject the null hypothesis and say that the absolute first differences are autocorrelated. This implies that the time series of absolute first differences is not a white noise process.

Computing the Bartlett test we want to check whether our samples have equal variances. The test is defined as:
· H_0: samples have equal variance
· H_1: at least two samples have different variance

For the first differences, we obtain a p-value of 0.04076001 - barely lower than the critical value of 0.05 - hence we can reject the hypothesis that all samples have equal variance. This discards automatically the assumption of a white process model for the first differences, even though we have to mention that it might loosely resemble it. Moreover, the cumulative periodogram in Figure 1 shows that the time series points lie not that far off from the straight line with slope 1 and intercept 0. This again confirms the fact that the time series does loosely resemble white noise.

For the absolute values of the first differences, we obtain a p-value of 0.0007286529 - lower than the critical value of 0.05 - hence we can reject the hypothesis that all samples have equal variance. This discards automatically the assumption of a white process model for the absolute first differences. Moreover, the cumulative periodogram in Figure 2 shows that the time series points lie far from the straight line with slope 1 and intercept 0, confirming that the white process assumption doesn't fit the series.

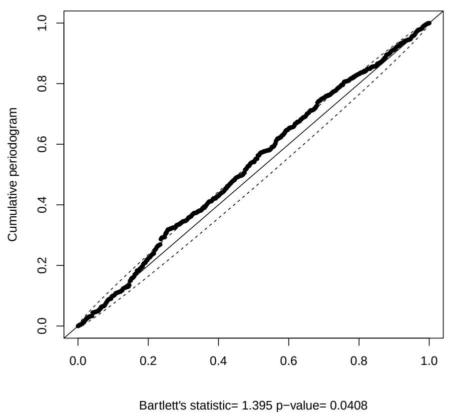

Figure 1:

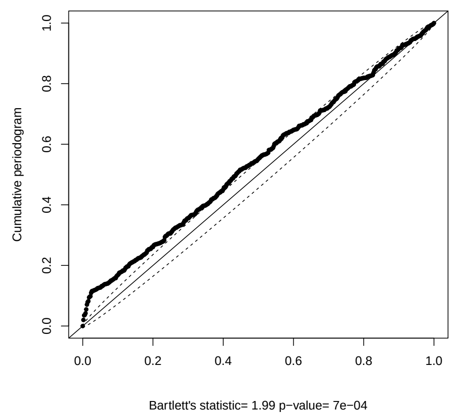

Figure 2:

1.3 AR(1) and ARMA(1,1) models

We fit AR(1) and ARMA(1,1) models to the absolute first differences. Using the AIC metrics - of respectively -9367.118 and -9402.841 - we can say that the ARMA(1,1) fits better the data.

2 Tide heights at Sooke Basin

2.1 AR model using AIC

We fit an AR model to the time series using AIC to choose the model order. Then we plot the estimated spectral density function.

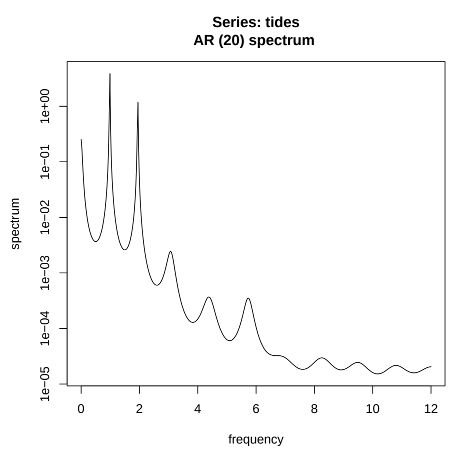

Figure 3:

As expected, we can see in Figure 2 that the underlying spectral density function of the process has peaks at the appropriate frequencies of multiples of 1.

2.2 White noise tests

We conduct the following graphical and formal tests to assess the validity of the assumption of the white noise model for the residuals:

· Plot the correlogram: in Figure 4, we can see that the values of autocorrelation are not close to zero for all lags. At certain lag values the residuals show significant autocorrelation which disputes the white noise assumption;

· Compute the Box-Ljung-Pierce tests: for both tests, the p-values obtained are less than 2.2e-16 - lower than the critical value 0.05 - which allows us to reject confidently the white noise assumption for the residuals;

· Conduct the Bartlett test and plot the cumulative periodogram: in Figure 5, we can see that the points of the time series do not lie closely to the straight line with slope 1 and intercept 0 which further proves the invalidity of the white noise assumption;

· Conduct the runs tests: we obtain a p-value of 0.03227 which is low enough for us to reject the white noise null hypothesis.

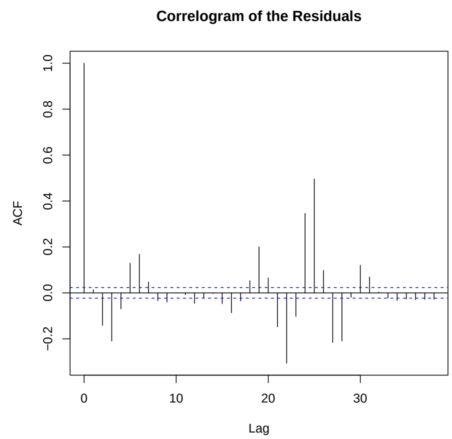

Figure 4:

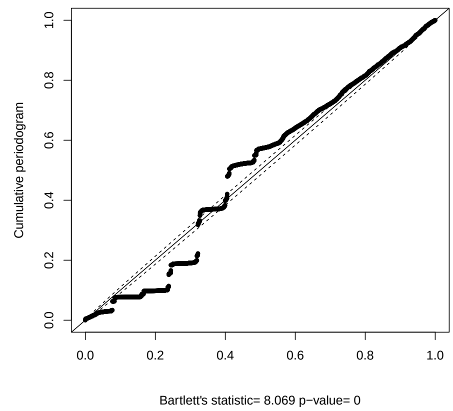

Figure 5:

If the residuals from the AR model are not well-modelled by a white noise process, it suggests that the model has not fully captured all of the structures present in the time series.

This can have the following implications on the spectral density function estimate:

· There will be unexplained variance left in the residuals that the model was not able to explain.

· The spectral density function is based on the assumption that the residuals are white noise. If this assumption is violated, the spectral density estimate may be biased or incomplete. This could lead to inaccuracies in identifying the true frequencies at which the time series exhibits periodic behaviour.

To assess the validity of the spectral density function, we can compare the variance of the residuals to the variance of the original time series. The variance of the residuals is 0.00302446 whereas the variance of the original time series is 0.3632181. The residuals account for only a small portion of the total variance, with the AR model explaining most of the variance. Hence the spectral density function estimate is still valid.

3 Stationary stochastic process

3.1 Recursive definition

We derive the recursive definition of Y_t in the following way:

$$
\begin{aligned}
Y_t = \alpha \sum_{u=0}^{+\infty} (1-\alpha)^u X_{t-u} = \alpha[(1-\alpha)^0 X_t + \sum_{u=1}^{+\infty} (1-\alpha)^u X_{t-u}] \\
= \alpha X_t + \alpha \sum_{v=0}^{+\infty} (1-\alpha)^{v+1} X_{(t-1)-v} \\
= \alpha X_t + (1-\alpha)[\alpha \sum_{v=0}^{+\infty} (1-\alpha)^v X_{(t-1)-v}] \\
= \alpha X_t + (1-\alpha) Y_{t-1}
\end{aligned} \tag{1}
$$

3.2 Evaluation of $\{|\Psi(\omega)|^2\}$

From the recursive relationship we have just derived

$$Y_t = \alpha X_t + (1-\alpha) Y_{t-1} \tag{2}$$

we can take the Fourier transform from both sides in the following way

$$\hat{Y}(\omega) = \alpha \hat{X}(\omega) + (1-\alpha) e^{-i\omega} \hat{Y}(\omega) \tag{3}$$

$$\hat{Y}(\omega)(1-(1-\alpha)e^{-i\omega}) = \alpha \hat{X}(\omega) \tag{4}$$

$$\hat{Y}(\omega) = \frac{\alpha}{1-(1-\alpha)e^{-i\omega}} \hat{X}(\omega) \tag{5}$$

Hence

$$\Psi(\omega) = \frac{\alpha}{1-(1-\alpha)e^{-i\omega}} \tag{6}$$

From this, we can derive

$$
\begin{aligned}
|\Psi(\omega)|^2 = |\frac{\alpha}{1-(1-\alpha)e^{-i\omega}}|^2 \\
= \frac{\alpha^2}{1-2(1-\alpha)\cos(\omega)+(1-\alpha)^2}
\end{aligned} \tag{7}
$$

In the equation for $|\Psi(\omega)^2|$, ω appears only in the $\cos(x)$ function which can take a range of values in [-1, 1].

Hence $|\Psi(\omega)^2|$ will have:

· maximum at ω = 2kπ with k$\in$ **N** which evaluates to

$$|\Psi(\omega)|^2 = \frac{\alpha^2}{\alpha^2} = 1 \tag{8}$$

· minimum at ω = (2k+1)π with k$\in$ **N** which evaluates to

$$|\Psi(\omega)|^2 = \frac{\alpha^2}{(2-\alpha)^2} \tag{9}$$

This implies that at the frequencies of:

· f = $\frac{\omega}{2\pi} = \frac{2k\pi}{2\pi} = k$ with k$\in$ **N** will obtain largest values, such as 1, 2, 3 etc.

· f = $\frac{\omega}{2\pi} = \frac{(2k+1)\pi}{2\pi} = k + \frac{1}{2}$ with k$\in$ **N** will obtain smallest values, such as $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc.

Values of α closer to 0 will make $|\Psi(\omega)^2|$ take values in a broader range from close to 0 to 1, whereas values of α closer to 1 will make $|\Psi(\omega)^2|$ take values in a smaller range close to 1.

A RScript for Exercise 1

Figure 6

B RScript for Exercise 2

Figure 7

```r
library(tseries)

dollar <- scan(file.choose())
dollar <- ts(log(dollar))

# a) Conduct the ADF test for various lags
adf.test(dollar, k=1)
adf.test(dollar, k=2)
adf.test(dollar, k=5)
adf.test(dollar, k=10)

# b) Conduct Ljung-Box and Bartlett tests on fist differences
#    and absolute first differences
returns <- diff(dollar)
abs_returns <- abs(returns)

# Ljung-Box test
Box.test(returns, lag=10, type="Ljung")
Box.test(abs_returns, lag=10, type="Ljung")

# Bartlett test
source(file.choose())
bartlett(returns, plot=T)$p.value
bartlett(abs_returns, plot=T)$p.value

# c) Fit AR(1) and ARMA(1,1)
r1 <- arima(abs_returns, order=c(1,0,0))
r2 <- arima(abs_returns, order=c(1,0,1))
r1$aic
r2$aic
```

Figure 6:

```r
tides <- scan(file.choose())
tides <- ts(tides, frequency=24)

# a) Fit an AR model using AIC
r1 <- ar.yw(tides, aic=T, order.max=20)
spec.ar(r1)

# b) Test if white noise model assumption for residuals is correct
resid <- r1$resid[(21):length(tides)]

# Correlogram
acf(resid, main="Correlogram of the Residuals")

# Box-Ljung-Pierce tests
Box.test(resid, lag=10, type="Box-Pierce")
Box.test(resid, lag=10, type="Ljung-Box")

# Bartlett test
source(file.choose())
bartlett(resid, plot=T)$p.value

# Runs tests
library(tseries)
y <- factor(ifelse(resid>0,-1,1))
runs.test(y)

# Comparing variances
var(resid)
var(tides)
```

Figure 7: