Translating Extensive Form Games to Open Games with Agency

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Introduction

Game theory is the mathematical study of interaction among independent, self-interested agents.

– Essentials of Game Theory [LS08]

Examples: economic and ecological games, machine learning, etc. $\rightsquigarrow$ *cybernetics.*

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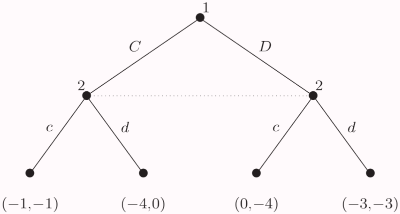
Classical game theory: **normal form** and **extensive form**

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Drawbacks: too little information (NF), too much information (EF), unclear causal relationships (both). Most importantly: non-compositional! ($\rightsquigarrow$ small scale)

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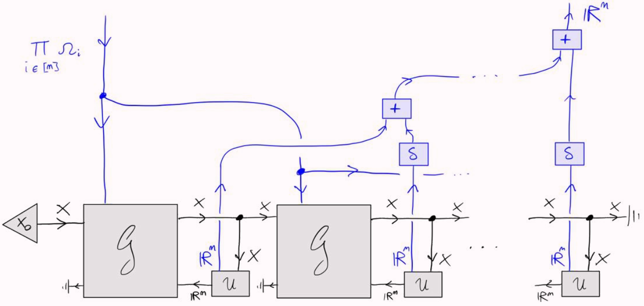
Open games as a framework for compositional game theory



Shows causality without being too unwieldy, compositional ($\rightsquigarrow$ large scale).

Also: string diagrams are nice to work with!

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Translating EF to open games (mantaining a similar causal structure) is non-trivial:

1. EF is a **non-inductive definition**, and it's messy to map a whole tree in one go.
2. Open games don't have an **explicit notion of players**, necessary to model imperfect information and to correctly compute equilibria.

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To do so, we introduce:

1. Open games with **agency**, an improved compositional framework for games,
2. An operator calculus for games, in particular new **choice operators**,
3. **Inductive data types for EF trees** with (im)perfect information.

Open games with agency

What is a game?

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2. **Players**, which intervene in the arena by making **decisions** at different points.

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2. **Players**, which intervene in the arena by making **decisions** at different points.

A strategy $\omega \in \Omega_p$ for a player $1 \leq p \leq N$ is a policy p uses to make their decisions (e.g. a choice of move for each of p's rounds).

A strategy profile is a strategy for each player $\Omega = \Omega_1 \times \cdots \times \Omega_p$.

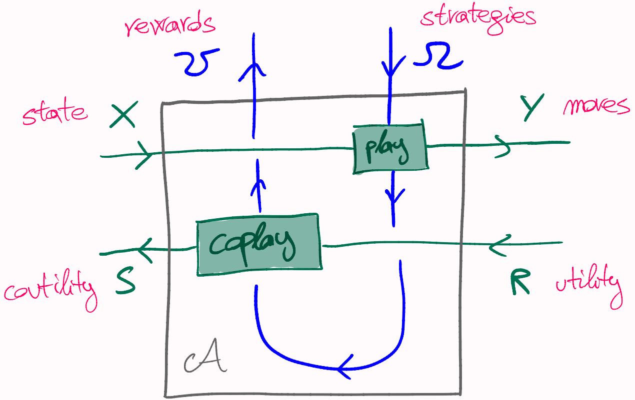
What is a Nash equilibrium?

A strategy profile is a **Nash equilibrium**
if no player has incentive to *unilaterally* change strategy.

Traditionally 'the goal' of game theory is determining Nash equilibria of games, though this is not necessarily the case anymore (see: [Fud+98]).

What is an arena?

An arena is an open system with three boundaries:



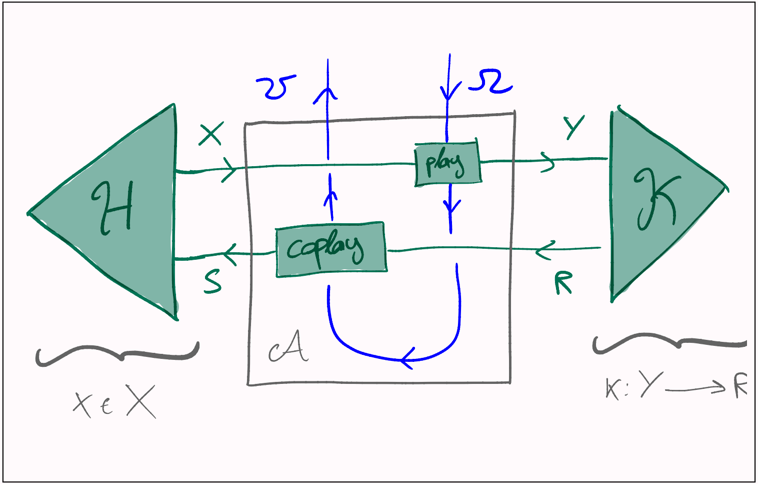
This is a parametrised lens $\mathcal{A} : (X, S) \xrightarrow{(\Omega, \mho)} (Y, R)$ specified by two maps

$$\text{play} : \Omega \times X \to Y,$$

$$\text{coplay} : \Omega \times X \times R \to \mho \times S$$

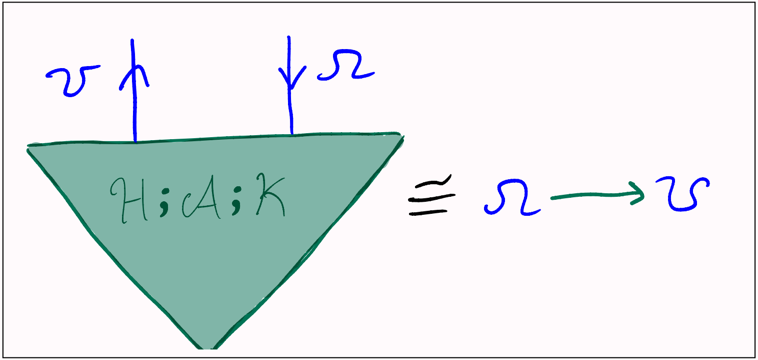
What is an arena?

It can be closed by specifying an **initial state** and a **utility function**:



What is an arena?

At the end of the day, the arena amounts to an evaluation of strategies with rewards:



What is a selection function?

Agents express their preferences by means of a **selection function**:

$$\varepsilon : (\Omega \to \mho) \longrightarrow \mathsf{P}\Omega$$

Typically, Ω is 'finite', $\mho = \mathbb{R}$ and ε is argmax:

$$\text{argmax}(u : \Omega \to \mathbb{R}) = \{\omega \in \Omega \,|\, \omega \text{ maximises } u\} \subseteq \Omega.$$

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The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, it admits a lax monoidal structure (**Nash product**):

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\mho_1, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$\begin{aligned}(\varepsilon \boxtimes \eta)(u) &= \{(\omega_1, \omega_2) \,|\, \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\} \\ &= \text{strategies s.t. `no agent wants to unilaterally deviate'}\end{aligned}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

The definition

Definition

An **open game with agency** is a pair

$$\mathrm{G} \quad = \quad (\ \mathcal{A} : (X, S) \xrightarrow{(\Omega,\mho)} (Y, R), \quad \varepsilon : \mathbb{S}(\Omega, \mho)\ )$$

whose equilibria are given by

$$\mathrm{eq}_{\mathrm{G}}(x, k) = \varepsilon(x \fatsemi \mathcal{A} \fatsemi k).$$

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Definition

G **has set of players** P when

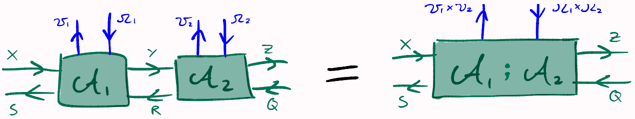
$$\Omega = \prod_{p \in P} \Omega_p, \quad \mho = \prod_{p \in P} \mho_p, \quad \varepsilon = \bigboxtimes_{p \in P} \varepsilon_p$$

in which case

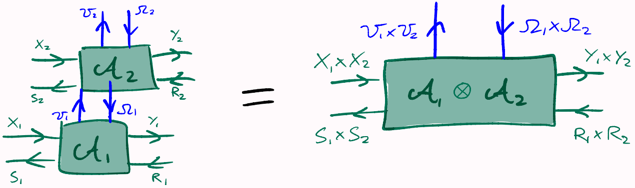
$$\mathrm{eq}_{\mathsf{G}}(x, k) = (\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_n)(x \mathbin{⨟} \mathcal{A} \mathbin{⨟} k).$$

Composing games

Sequential composition



Parallel composition

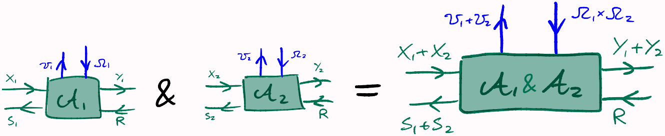


Notice: these operators can be extended to games:

$$(\mathcal{A}_1, \varepsilon) \mathbin{;} (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \mathbin{;} \mathcal{A}_2, \varepsilon \boxtimes \eta), \quad (\mathcal{A}_1, \varepsilon) \otimes (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \otimes \mathcal{A}_2, \varepsilon \boxtimes \eta)$$

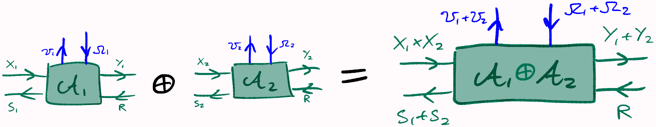
Composing arenas

External choice



The ‘environment’ chooses which game to play, agents are prepared to play both.

Internal choice



The ‘environment’ can play either game, agents choose which one.

Notice: these operators <u>can’t</u> be extended to selection functions in a canonical way!

(Actually $\oplus$ <u>can</u> if we refine our typing judgments)

Reparametrisation

Most importantly arenas form a (locally fibred) bicategory: one can **reparametrise** along a lens $\alpha : (\Omega', \mho') \to (\Omega, \mho)$.

This is crucial for introducing agency!

Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along

$$\mathsf{regroup}_r : (\prod_{q \in Q} (r^*\Omega)_q, \prod_{q \in Q} (r^*\mho)_q) \longrightarrow (\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p)$$

which only permutes & reindex the factors according to r.

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players P, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. So we regroup along $P + P \to P$ to get again an arena with players P.

(drawing)

Tying

The second most important application for reparametrisations is tying strategies at different point of the game. This is done by reparametrising along $(\Delta, \mathsf{combine})$ (where $\mathsf{combine} = +, \pi_2, \mathsf{max}, \ldots$).

Notice: $(\Delta, \mathsf{combine})^*(\mathcal{A}_1 \fatsemi \mathcal{A}_2)$ lies outside the image of $- \fatsemi -$, hence introduces 'non-compositional' effects. Indeed: **agency is non-local.**

Extensive form games & their translation

Extensive form

(pic)

Extensive form

(pic)

A **(perfect information) extensive-form tree** is a term of

$$\text{data PETree} = \text{Leaf} \;\; R^{P} \quad | \quad \text{Node} \;\; P \;\; (n : \mathbb{N}^{+}) \;\; ([n] \to \text{PETree})$$

where P = set of **players**, R = type of **rewards** (usually $\mathbb{R}$).

Extensive form

Strategies can be inductively defined...

$$\mathsf{strat}_{\mathsf{PET}} : \mathsf{PETree} \to P \to \mathsf{Set}$$

$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Leaf}\ v)\ p = 1$$

$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Node}\ q\ n\ f)\ p$$

$$= (\mathsf{if}\ p \equiv q\ \mathsf{then}\ [n]\ \mathsf{else}\ 1) \times (\Pi\, m \in [n])\, \mathsf{strat}_{\mathsf{PET}}\ (f\ \ m)\ p$$

$$\mathsf{prof}_{\mathsf{PET}} : \mathsf{PETree} \to \mathsf{Set}$$

$$\mathsf{prof}_{\mathsf{PET}}\ \ T = (\Pi\, p \in P)\ \mathsf{strat}_{\mathsf{PET}}\ \ T\ \ p$$

Extensive form

As well as **moves** (= paths in tree) and payoff function:

$$\mathrm{path}_{\mathrm{PET}} : \mathrm{PETree} \to \mathrm{Set}$$

$$\mathrm{path}_{\mathrm{PET}}\ (\mathrm{Leaf}\ v) = 1$$

$$\mathrm{path}_{\mathrm{PET}}\ (\mathrm{Node}\ p\ n\ f) = (\Sigma\, m \in [n])\ \mathrm{path}_{\mathrm{PET}}\ (f\ \ m)$$

$$\mathrm{payoff}_{\mathrm{PET}} : (T : \mathrm{PETree}) \to (\mathrm{path}_{\mathrm{PET}}\ \ T) \to R^P$$

$$\mathrm{payoff}_{\mathrm{PET}}\ (\mathrm{Leaf}\ v)\ \bullet = v$$

$$\mathrm{payoff}_{\mathrm{PET}}\ (\mathrm{Node}\ p\ n\ f)\ (m, \pi) = \mathrm{payoff}_{\mathrm{PET}}\ (f\ \ m)\ \pi$$

Translation

Finally, we can translate any PETree to an open game with agency:

(pic)

where

1. $\mathrm{Dec}([n], R^P)$ is the 'trivial' game where moves $=$ strategies $= [n]$ and coplay is just Δ.
2. $\mathrm{regroup}_p$ is regrouping along $\{p\} + P \to P$.

and we equip this arena with $\boxtimes_{p \in P} \mathrm{argmax}(- \mathbin{⨟} \pi_p)$.

Adding imperfect information

Sometimes players can't access the whole state of the game (i.e. history of play)

Adding imperfect information

Information (or better, lack thereof) is represented by sets of 'indistinguishable states' called **information sets**

Adding imperfect information

Strategies need to respect information sets: players can't distinguish between states in the same set:

(pic)

It gets a bit more complicated with mixed strategies.

Adding imperfect information

Definition

An **imperfect information extensive-form tree** is a term of

$$\text{data IETree} = \text{Leaf}\ \ R^P \mid \text{Node}\ \ (i : I)\ \ ([n\ i] \to \text{IETree})$$

where I : Set is a set of **information labels** and $n : I \to \mathbb{N}^+$ assigns moves to nodes of the same information set. Additionally, there is an epi map belongs : $I \to P$.

Limit case: all information sets are singletons = perfect information.

Translation

Idea: information sets are instances of 'tying' reparametrisations.

1. Forget about information sets and recover a perfect information game:

$$\mathsf{IET\text{-}to\text{-}PET} : \mathsf{IETree} \to \mathsf{PETree}$$

$$\mathsf{IET\text{-}to\text{-}PET}\ (\mathsf{Leaf}\ v) = \mathsf{Leaf}\ v$$

$$\mathsf{IET\text{-}to\text{-}PET}\ (\mathsf{Node}\ i\ f) = \mathsf{Node}\ (\mathsf{belongs}\, i)\ (n\ i)\ (\lambda m\,.\,\mathsf{IET\text{-}to\text{-}PET}\ (f\ m))$$

2. Translate it using PET-to-arena.
3. Reparametrise along clone which ties strategies in the same information sets.

Conclusions & future work

In this work:

1. We have seen how games naturally decompose in *arenas* and *selection functions*, with *reparametrisations* playing a key role
2. This allows to handle long-range correlations in players' behaviour
3. Hence we can easily map extensive form trees to open games with agency

What's next?

1. Is the translation 'functorial'? Possible domain cat described in [Str21].
2. Once the diagram is drawn, can we simplify it using topological moves? e.g. mapping 'simultaneity' to $\otimes$.
3. Infinite trees?
4. Can we treat SPE?
5. What kind of assignment is $\mathrm{argmax} : \mathrm{Arenas} \to \mathrm{Games}$?
6. Can we make selections compositional?
7. Dependent types

Thanks for your attention!

Questions?

References I

Kevin Leyton-Brown and Yoav Shoham. "Essentials of game theory: A concise multidisciplinary introduction". In: *Synthesis lectures on artificial intelligence and machine learning* 2.1 (2008), pp. 1–88.

Drew Fudenberg et al. *The theory of learning in games*. Vol. 2. MIT press, 1998.

Peter A Streufert. "A Category for Extensive-Form Games". In: *arXiv preprint arXiv:2105.11398* (2021).

David Jaz Myers. "Double Categories of Open Dynamical Systems". In: *arXiv preprint arXiv:2005.05956* (2020).

Joel Watson. *Strategy: an introduction to game theory*. Vol. 139. WW Norton New York, 2002.

Brendan Fong, David Spivak, and Rémy Tuyéras. "Backprop as Functor: A compositional perspective on supervised learning". In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2019, pp. 1–13.

References II

Neil Ghani et al. "Compositional game theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018, pp. 472–481.

G.S.H. Cruttwell et al. "Categorical Foundations of Gradient-Based Learning". In: *arXiv preprint arXiv:2103.01931* (2021).

Mitchell Riley. "Categories of Optics". In: *arXiv: Category Theory* (2018).

David Jaz Myers. *Categorical Systems Theory*. URL: https://github.com/DavidJaz/DynamicalSystemsBook/tree/master/book.

Martin Escardó and Paulo Oliva. "Selection functions, bar recursion and backward induction". In: *Math. Struct. Comput. Sci.* 20.2 (2010), pp. 127–168.

Joe Bolt, Jules Hedges, and Philipp Zahn. "Bayesian open games". In: *arXiv preprint arXiv:1910.03656* (2019).

Matteo Capucci et al. "Towards foundations of categorical cybernetics". In: *arXiv preprint arXiv:2105.06332* (2021).