Translating Extensive Form Games to Open Games with Agency

ACT2021

Matteo Capucci, Neil Ghani,
Jérémy Ledent, Fredrik Nordvall Forsberg

Mathematically Structured Programming group,
Department of Computer and Information Sciences,
University of Strathclyde

July 14th, 2021

Introduction

> *Game theory is the mathematical study of interaction among independent, self-interested agents.*
>
> *– Essentials of Game Theory [LS08]*

Open games as a framework for compositional game theory (string diagram)

Classical game theory: normal form and extensive form (PD in both forms)

Introduction

In this work, we give a canonical translation from extensive form games to open games

To do so, we introduce:

1. Open games with *agency*, an improved compositional framework for games,
2. An operator calculus for games, in particular new *choice operators*,
3. *Inductive data types for extensive form games* of (im)perfect information.

What is a game?

Game theory is the mathematical study of interaction among independent, self-interested agents.
– Essentials of Game Theory [LS08]

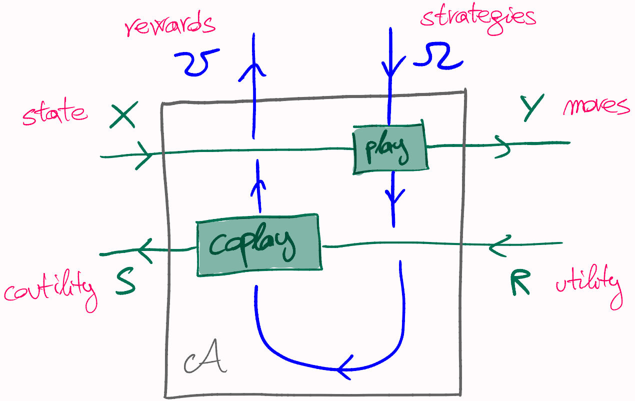
Examples: economic games, proof theory, machine learning, control theory, etc.

A game factors in two parts:

1. An **arena**, which models the interaction patterns in the game
2. **Players**, which intervene in the arena by making **decisions** at different points

What is a game?

An arena is an open system with three boundaries:



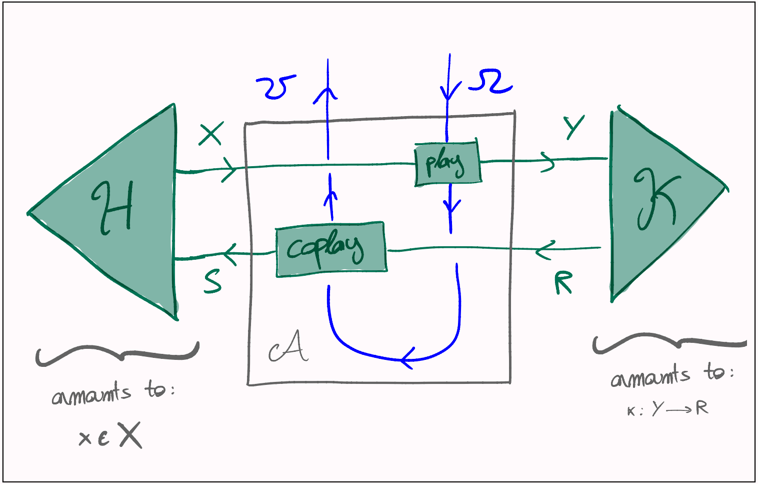
This is a parametrised lens $\mathcal{A} : (X, S) \xrightarrow{(\Omega,\mho)} (Y, R)$ specified by two maps

$$\text{play} : \Omega \times X \to Y,$$

$$\text{coplay} : \Omega \times X \times R \to \mho \times S$$

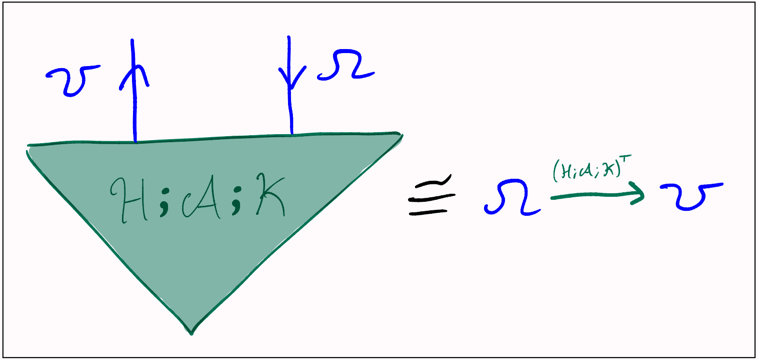
What is a game?

It can be closed by specifying an **initial state** and a **utility function**:



What is a game?

At the end of the day, the arena amounts to an evaluation of strategies with rewards:



What is a game?

Agents can then choose which strategies to play using a **selection function**:

$$\varepsilon : (\Omega \to \mho) \to \mathcal{P}\Omega$$

Typically, Ω is finite, $\mho = \mathbb{R}$ and ε is argmax.

Interlude: selection functions

Let $\mathbb{S}(X, S) := (X \to S) \to \mathscr{P}X$. This is a functor on lenses:

$$\begin{aligned} \mathbb{S}(\alpha) : \mathbb{S}(X, S) &\longrightarrow \mathbb{S}(Y, R) \\ \varepsilon &\longmapsto \lambda k \,.\, \{x \fatsemi \alpha \mid x \in \varepsilon(\alpha \fatsemi k)\} \end{aligned}$$

where $\alpha : (X, S) \to (Y, R)$ and we implicitly used $\mathrm{Lens}((1, 1), (Y, R)) \cong Y \to R$ and $\mathrm{Lens}((X, S), (1, 1)) \cong X$:

(drawing)

This functor is lax monoidal (**Nash product**):

$$\begin{aligned} - \boxtimes - : \mathbb{S}(X, S) \times \mathbb{S}(Y, R) &\longrightarrow \mathbb{S}(X \times Y, S \times R) \\ (\varepsilon, \eta) &\longmapsto \lambda k \,.\, \{(x, y) \mid x \in \varepsilon(k_1(x, y)) \text{ and } y \in \eta(k_2(x, y))\} \end{aligned}$$

(drawing)

This yields a moncat $\mathrm{Lens}_{\mathbb{S}} = \int_{mon} \mathbb{S}$.

The definition

Definition

An **open game with agency** G is a pair

$$\mathcal{A} : (X, S) \xrightarrow{(\Omega, \mho)} (Y, R), \quad \varepsilon : \mathbb{S}(\Omega, \mho)$$

More succintly: a $\mathbf{Lens}_{\mathbb{S}}$-parametrised lens.

Agency $\neq$ players: players arise from the presentation; if $\Omega = \prod_{p \in P} \Omega_p$, $\mho = \prod_{p \in P} \mho_p$ and $\varepsilon = \boxtimes_{p \in P} \varepsilon_p$ we speak of **open game with** (set of) **players** P.

Equilibria for $\mathrm{G} = (\mathcal{A}, \varepsilon)$ are given by:

$$\mathrm{eq}_{\mathrm{G}}(x, k) = \varepsilon(x \fatsemi \mathcal{A} \fatsemi k).$$

Composing arenas

Arenas are the compositional heart of open games with agency:

Composing arenas

Notice: as of now, these operators can't be extended to selection functions in a canonical way!

Reparametrisation

Most importantly arenas form a (locally fibred) bicategory: one can **reparametrise** along a lens $\alpha : (\Omega', \mho') \to (\Omega, \mho)$.

This is crucial for introducing agency!

Regrouping

As an example, if $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along

$$\mathrm{regroup}_r : \Big(\prod_{q\in Q} (r^*\Omega)_q, \prod_{q\in Q} (r^*\mho)_q\Big) \longrightarrow \Big(\prod_{p\in P} \Omega_p, \prod_{p\in P} \mho_p\Big)$$

which only 'regroups' the factors according to r.

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players P, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. So we regroup along $P + P \to P$ to get again an arena with players P.

Tying

The second most important application for reparametrisations is tying strategies at different point of the game. This is done by reparametrising along $(\Delta, \mathrm{combine})$ (where $\mathrm{combine} = +, \pi_2, \max, \ldots$).

Notice: $(\Delta, \mathrm{combine})^*(\mathcal{A}_1 \mathbin{⨟} \mathcal{A}_2)$ lies outside the image of $- \mathbin{⨟} -$, hence introduces 'non-compositional' effects. Indeed: agency is non-local.

Extensive form games
& their translation

Extensive form

Extensive form is a classical formalism for games:

(nice pic)

A game is presented as a (rooted) tree, which should reflect the possible unfoldings of the game. Each node is labelled with a *player*. Branches represent possible *moves* and *payoff vectors* await at the leaves.

Two kinds:

1. Perfect information: any player always knows exactly the state of the game (path from root to their nodes)
2. Imperfect information: otherwise

As customary in classical game theory, all players argmax their payoff.

PETs as inductive type

We can easily represent PETs with an inductive type:

(inductive type)

where P : Set are players, R : Set are rewards.

(tree and corresponding term)

PETs

Correspondigly, we can define what strategies/moves/states/utilities are with inductive definitions:

(def)

Translation

Finally, we can translate any PETree to an open game:

(pic)

where

1. $\mathrm{Dec}([n], R^P)$ is the 'trivial' game where moves = strategies = $[n]$ and coplay is just Δ.
2. $\mathrm{regroup}_p$ is regrouping along $\{p\} + P \to P$.

IETs

Imperfect information extensive form games are decorated with *information sets*:
(pic)
Nodes in the same information set cannot be distinguished by players. Players do not share information sets, each player has their own partition of their nodes in information sets. Nodes in the same information set have the same moves.
Limit case: all information sets are singleton = perfect information.

IETs as inductive type

We can also represent IETs with an inductive type:

(inductive type)

where P : Set are players, R : Set are rewards, I : Set is a set of information labels, belongs : $I \to P$ assigns information sets to players, $n : I \to \mathbb{N}^+$ assigns uniformly moves to nodes of the same information set.

(tree and corresponding term)

Conclusions & future work

In this work:

1. We have seen how games naturally decompose in *arenas* and *selection functions*, with *reparametrisations* playing a key role
2. This allows to handle long-range correlations in players' behaviour
3. Hence we can easily map extensive form trees to open games with agency

What's next?

1. Is the translation 'functorial'? Possible domain cat described in [Str21].
2. Once the diagram is drawn, can we simplify it using topological moves? e.g. mapping 'simultaneity' to $\otimes$.
3. Infinite trees?
4. Can we treat SPE?
5. What kind of assignment is $\mathrm{argmax} : \mathrm{Arenas} \to \mathrm{Games}$?
6. Can we make selections compositional?
7. Dependent types

Thanks for your attention!

Questions?

References I

Kevin Leyton-Brown and Yoav Shoham. "Essentials of game theory: A concise multidisciplinary introduction". In: *Synthesis lectures on artificial intelligence and machine learning* 2.1 (2008), pp. 1–88.

Peter A Streufert. "A Category for Extensive-Form Games". In: *arXiv preprint arXiv:2105.11398* (2021).

David Jaz Myers. "Double Categories of Open Dynamical Systems". In: *arXiv preprint arXiv:2005.05956* (2020).

Joel Watson. *Strategy: an introduction to game theory*. Vol. 139. WW Norton New York, 2002.

Brendan Fong, David Spivak, and Rémy Tuyéras. "Backprop as Functor: A compositional perspective on supervised learning". In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2019, pp. 1–13.

Neil Ghani et al. "Compositional game theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018, pp. 472–481.

References II

G.S.H. Cruttwell et al. "Categorical Foundations of Gradient-Based Learning". In: *arXiv preprint arXiv:2103.01931* (2021).

Mitchell Riley. "Categories of Optics". In: *arXiv: Category Theory* (2018).

David Jaz Myers. *Categorical Systems Theory*. URL: https://github.com/DavidJaz/DynamicalSystemsBook/tree/master/book.

Martin Escardó and Paulo Oliva. "Selection functions, bar recursion and backward induction". In: *Math. Struct. Comput. Sci.* 20.2 (2010), pp. 127–168.

Joe Bolt, Jules Hedges, and Philipp Zahn. "Bayesian open games". In: *arXiv preprint arXiv:1910.03656* (2019).

Matteo Capucci et al. "Towards foundations of categorical cybernetics". In: *arXiv preprint arXiv:2105.06332* (2021).