Translating Extensive Form Games to Open Games with Agency

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Introduction

Game theory is the mathematical study of interaction among independent, self-interested agents.

– Essentials of Game Theory, Leyton-Brown and Shoham 2008

Examples: economic and ecological games, machine learning, etc. $\rightsquigarrow$ *cybernetics.*

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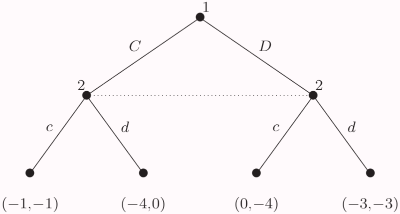
Classical game theory: **normal form** and **extensive form**

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Drawbacks: too little information (NF), too much information (EF), unclear causal relationships (both). Most importantly: non-compositional! ($\rightsquigarrow$ small scale)

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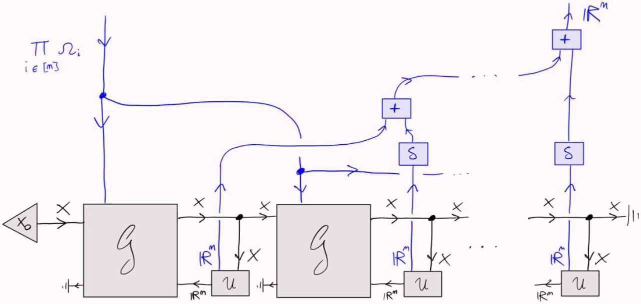
Open games as a framework for compositional game theory



Shows causality without being too unwieldy, compositional (⇝ large scale).

Also: string diagrams are nice to work with!

Introduction

Translating NF to open games is 'trivial' (there's only a utility function)...

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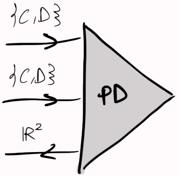

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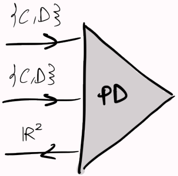

Translating EF to open games (mantaining a similar causal structure) is non-trivial:

1. EF is a **non-inductive definition**, and it's messy to map a whole tree in one go.
2. Open games don't have an **explicit notion of players**, necessary to model imperfect information and to correctly compute equilibria.

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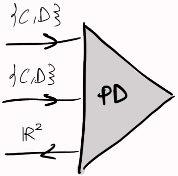

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To do so, we introduce:

1. Open games with **agency**, an improved compositional framework for games,
2. An operator calculus for games, in particular new **choice operators**,
3. **Inductive data types for EF trees** with (im)perfect information.

Open games with agency

What is a game?

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A game factors in two parts:

1. An **arena**, which models the **dynamics** of the game.
2. **Players**, which intervene in the arena by making **decisions** at different points.

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2. **Players**, which intervene in the arena by making **decisions** at different points.

A strategy $\omega \in \Omega_p$ for a player $1 \leq p \leq N$ is a policy p uses to make their decisions (e.g. a choice of move for each of p's rounds).

A strategy profile is a strategy for each player $\Omega = \Omega_1 \times \cdots \times \Omega_p$.

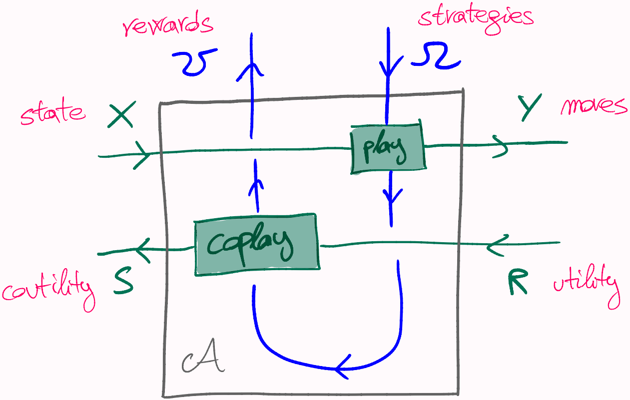
What is a Nash equilibrium?

A strategy profile is a **Nash equilibrium**
if no player has incentive to *unilaterally* change strategy.

Traditionally 'the goal' of game theory is determining Nash equilibria of games, though this is not necessarily the case anymore (see: Fudenberg and Levine 1998).

What is an arena?

An arena is an open system with three boundaries:



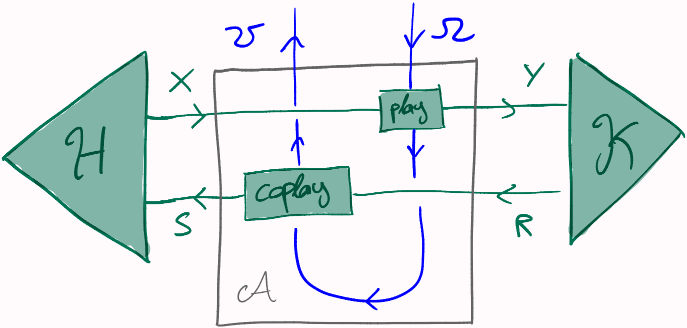
This is a parametrised lens $\mathcal{A} : (X, S) \xrightarrow{(\Omega,\mho)} (Y, R)$ specified by two maps

$$\text{play} : \Omega \times X \to Y,$$

$$\text{coplay} : \Omega \times X \times R \to \mho \times S$$

What is an arena?

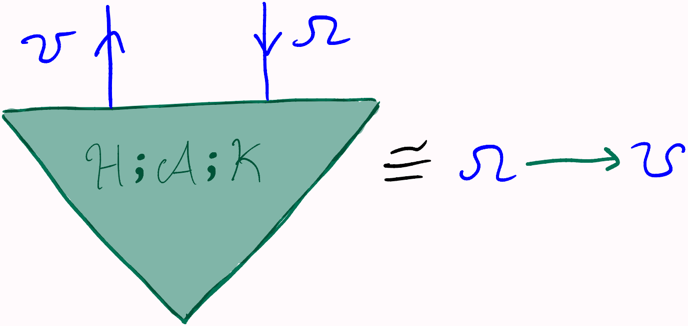
It can be closed by specifying an **initial state** and a **utility function**:



Observe that $\mathrm{Lens}(1,1)(X,S) \cong X$ and $\mathrm{Lens}(Y,R)(1,1) \cong Y \to R$.

What is an arena?

A **closed** arena amounts to an evaluation of strategies with rewards:



What is a selection function?

Agents express their preferences by means of a **selection function**:

$$\varepsilon : (\Omega \to \mho) \longrightarrow \mathsf{P}\Omega$$

Typically, Ω is 'finite', $\mho = \mathbb{R}$ and ε is argmax:

$$\mathrm{argmax}(u : \Omega \to \mathbb{R}) = \{\omega \in \Omega \,|\, \omega \text{ maximises } u\} \subseteq \Omega.$$

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The assignment $\mathbb{S}(\Omega, \mho) := (\Omega \to \mho) \longrightarrow \mathrm{P}\Omega$ is functorial on Lens.

Crucially, it admits a lax monoidal structure (**Nash product**):

$$- \boxtimes - : \mathbb{S}(\Omega_1, \mho_1) \times \mathbb{S}(\mho_1, \mho_2) \longrightarrow \mathbb{S}(\Omega_1 \times \Omega_2, \mho_1 \times \mho_2)$$

$$\begin{aligned}(\varepsilon \boxtimes \eta)(u) &= \{(\omega_1, \omega_2) \mid \omega_1 \in \varepsilon(u_1(-, \omega_2)) \text{ and } \omega_2 \in \eta(u_2(\omega_1, -))\} \\ &= \text{strategies s.t. 'no agent wants to unilaterally deviate'}\end{aligned}$$

where $u = (u_1, u_2) : \Omega_1 \times \Omega_2 \to \mho_1 \times \mho_2$.

The definition

Definition

An **open game with agency** is a pair

$$\mathsf{G} \quad = \quad (\ \mathcal{A} : (X, S) \xrightarrow{(\Omega,\mho)} (Y, R), \quad \varepsilon : \mathbb{S}(\Omega, \mho)\ )$$

whose equilibria are given by

$$\mathsf{eq}_{\mathsf{G}}(x, k) = \varepsilon(x \mathbin{⨟} \mathcal{A} \mathbin{⨟} k).$$

In this way we recover the equilibrium predicate of open games.

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Definition

G **has set of players** P when

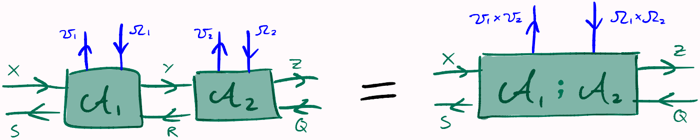
$$\Omega = \prod_{p \in P} \Omega_p, \quad \mho = \prod_{p \in P} \mho_p, \quad \varepsilon = \boxtimes_{p \in P} \varepsilon_p$$

in which case

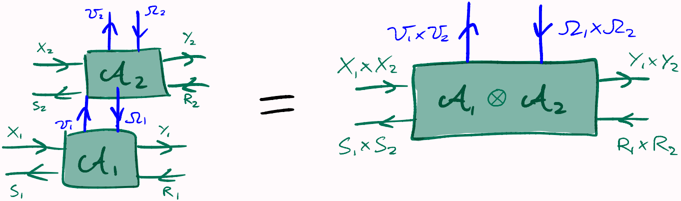
$$\mathrm{eq}_{\mathsf{G}}(x, k) = (\boxtimes_{p \in P} \varepsilon_p)(x \mathbin{⨟} \mathcal{A} \mathbin{⨟} k).$$

Composing games

Sequential composition



Parallel composition

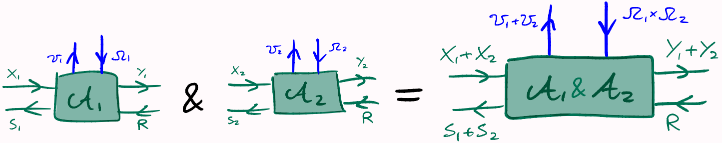


Notice: these operators <u>can</u> be extended to games:

$$(\mathcal{A}_1, \varepsilon) \fatsemi (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \fatsemi \mathcal{A}_2, \varepsilon \boxtimes \eta), \quad (\mathcal{A}_1, \varepsilon) \otimes (\mathcal{A}_2, \eta) := (\mathcal{A}_1 \otimes \mathcal{A}_2, \varepsilon \boxtimes \eta)$$

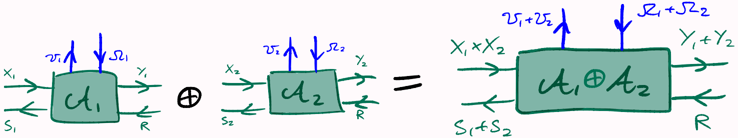
Composing arenas

External choice



The 'environment' chooses which game to play, agents are prepared to play both.

Internal choice



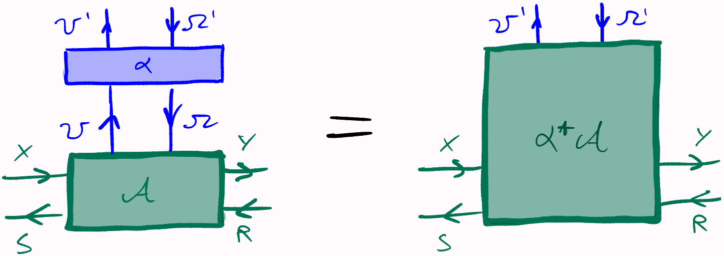
The 'environment' can play either game, agents choose which one.

Notice: these operators can't be extended to selection functions in a canonical way!

(Actually $\oplus$ can if we refine our typing judgments)

Reparametrisation

Most importantly arenas form a (locally fibred) bicategory: one can **reparametrise** along a lens $\alpha : (\Omega', \mho') \to (\Omega, \mho)$.



This is crucial for introducing agency!

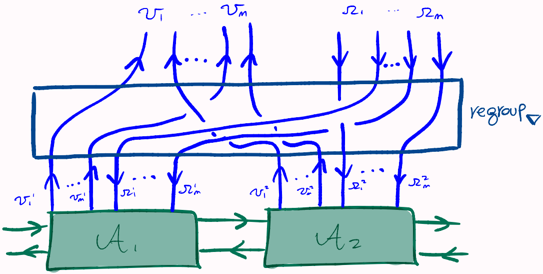
Regrouping

If $\mathcal{A}$ has set of players P and $r : P \to Q$ is a function, we can turn $\mathcal{A}$ into an arena with players Q by reparametrising along the permutation induced by r:

$$\mathrm{regroup}_r : \left(\textstyle\prod_{q \in Q} (r^* \Omega)_q, \prod_{q \in Q} (r^* \mho)_q\right) \longrightarrow \left(\textstyle\prod_{p \in P} \Omega_p, \prod_{p \in P} \mho_p\right)$$

Example

If $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same players P, $\mathcal{A}_1 \fatsemi \mathcal{A}_2$ has players $P + P$. Regrouping along $\nabla : P + P \to P$ restores the correct set of players.



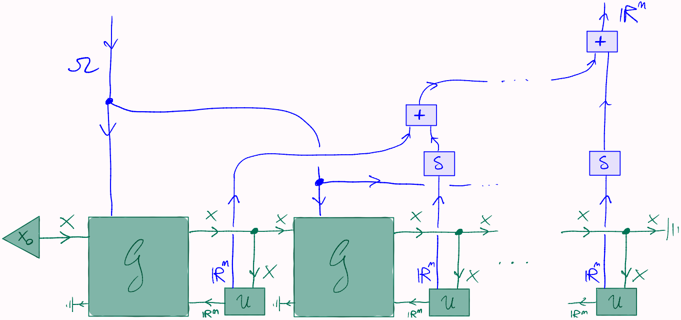
Here $\Omega_i = (\nabla^* \Omega)_i = \Omega_i^1 \times \Omega_i^2$, and likewise for $\mho$.

Tying

By reparametrising along $(\Delta, \text{combine})$ (where $\text{combine} = +, \pi_2, \max, \dots$), we can enforce the same strategies to be played at different points of a game.

Example

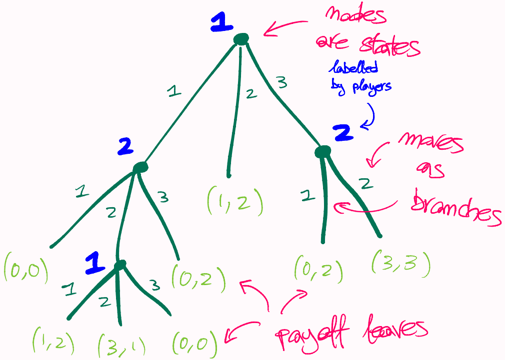
Play the same strategy every round:



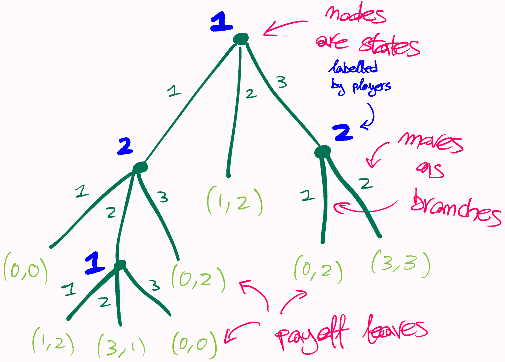
Notice: $(\Delta, \text{combine})^*(\mathcal{A}_1 \fatsemi \mathcal{A}_2)$ lies outside the image of $- \fatsemi -$, hence introduces 'non-compositional' effects. Indeed: **agency is non-local.**

Extensive form games
& their translation

Extensive form



Extensive form



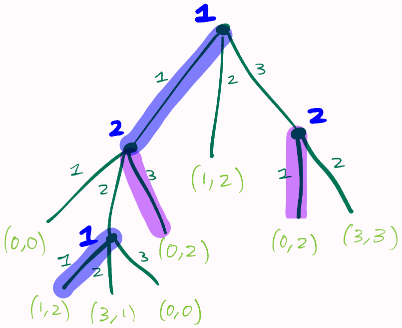
Definition

A **(perfect information) extensive-form tree** is a term of

$$\text{data PETree} = \text{Leaf} \;\; R^P \quad | \quad \text{Node} \;\; P \;\; (n : \mathbb{N}^+) \;\; ([n] \to \text{PETree})$$

where P = set of **players**, R = type of **rewards** (usually $\mathbb{R}$).

Extensive form: strategies



Definition

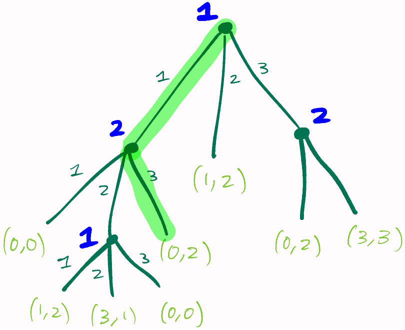
$$\mathsf{strat}_{\mathsf{PET}} : \mathsf{PETree} \to P \to \mathsf{Set}$$

$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Leaf}\ v)\ p = 1$$

$$\mathsf{strat}_{\mathsf{PET}}\ (\mathsf{Node}\ q\ n\ f)\ p$$

$$= (\mathsf{if}\ p \equiv q\ \mathsf{then}\ [n]\ \mathsf{else}\ 1) \times (\Pi\, m \in [n])\ \mathsf{strat}_{\mathsf{PET}}\ (f\ \ m)\ p$$

Extensive form: moves



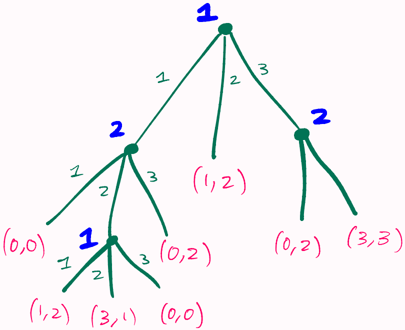
Definition

$$\mathsf{path}_{\mathsf{PET}} : \mathsf{PETree} \to \mathsf{Set}$$

$$\mathsf{path}_{\mathsf{PET}}\ (\mathsf{Leaf}\ v) = 1$$

$$\mathsf{path}_{\mathsf{PET}}\ (\mathsf{Node}\ p\ n\ f) = (\Sigma\, m \in [n])\ \mathsf{path}_{\mathsf{PET}}\ (f\ m)$$

Extensive form: payoff function



Definition

$$\mathsf{payoff}_{\mathsf{PET}} : (T : \mathsf{PETree}) \to (\mathsf{path}_{\mathsf{PET}}\ T) \to R^P$$

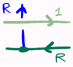
$$\mathsf{payoff}_{\mathsf{PET}}\ (\mathsf{Leaf}\ v)\ \bullet = v$$

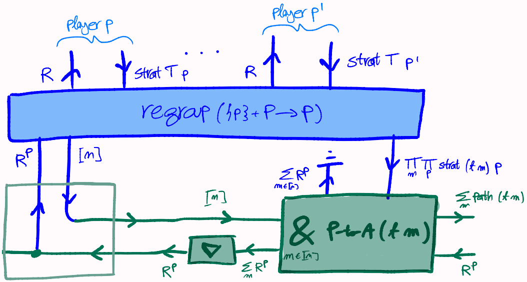
$$\mathsf{payoff}_{\mathsf{PET}}\ (\mathsf{Node}\ p\ n\ f)\ (m, \pi) = \mathsf{payoff}_{\mathsf{PET}}\ (f\ m)\ \pi$$

Translation

Finally, we can translate any PETree to an open game with agency:

$\text{PET-to-Arena} : (T : \text{PETree}) \longrightarrow \mathrm{Lens}_{(\prod_p \mathrm{strat}\ T\ p,\, R^P)}(1, R^P)(\mathrm{path}\ T, R^P)$

PET-to-Arena(Leaf v) = 



PET-to-Arena(Node p n f) =

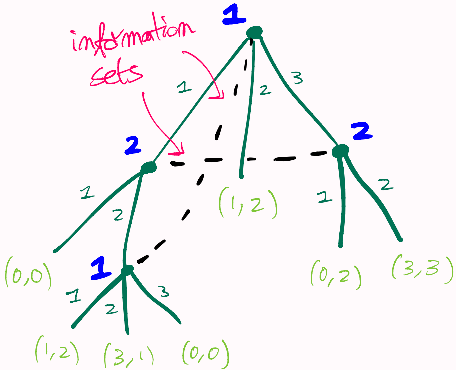
and we equip this arena with $\boxtimes_{p \in P} \mathrm{argmax}(- \fatsemi \pi_p)$.

Imperfect information

Sometimes players can't access the whole state of the game (i.e. history of play)

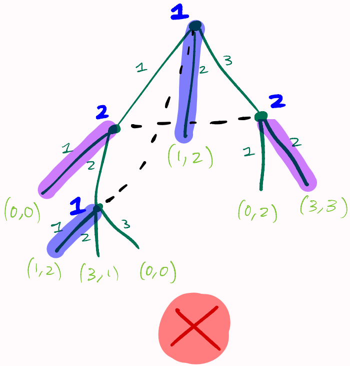
Imperfect information

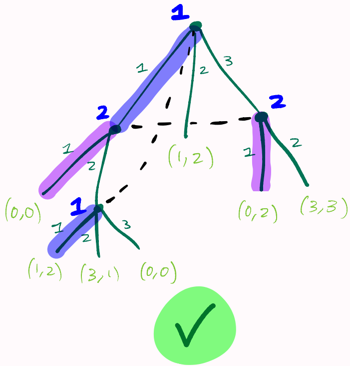
Information (or better, lack thereof) is represented by sets of 'indistinguishable states' called **information sets**.



Imperfect information

Strategies need to respect information sets: players can't distinguish between states in the same set:





It gets a bit more complicated with mixed strategies.

Adding imperfect information

Definition

An **imperfect information extensive-form tree** is a term of

$$\text{data IETree} = \text{Leaf } R^P \mid \text{Node } (i : I) \; ([n\ i] \to \text{IETree})$$

where

1. $I : \text{Set}$ is a set of **information labels**,
2. $n : I \to \mathbb{N}^+$ assigns moves to nodes of the same information set and
3. there is a (surjective) map $\text{belongs} : I \to P$.

Limit case: all information sets are singletons $\equiv$ perfect information.

Translation

Idea: information sets are instances of 'tying':

1. Forget about information sets and recover a perfect information game:

$$\text{IET-to-PET} : \text{IETree} \to \text{PETree}$$

$$\text{IET-to-PET}\ (\text{Leaf}\ v) = \text{Leaf}\ v$$

$$\text{IET-to-PET}\ (\text{Node}\ i\ f) = \text{Node}\ (\text{belongs}\, i)\ (n\ i)\ (\lambda m\,.\,\text{IET-to-PET}\ (f\ m))$$

2. Translate it using PET-to-Arena.
3. Reparametrise along clone which ties strategies in the same information sets.

$$\text{IET-to-Arena}(T) = \text{clone}^*(\text{PET-to-Arena}\ (\text{IET-to-PET}\ T))$$

Conclusions

In this work:

1. We have seen how games naturally decompose in *arenas* and *selection functions*, with *reparametrisations* playing a key role
2. This allows to handle long-range correlations in players' behaviour
3. Hence we can easily map extensive form trees to open games with agency

Future directions

1. Is the translation 'functorial'? Possible domain cat described in Streufert 2021.
2. Once the diagram is drawn, can we simplify it using topological moves? e.g. mapping 'simultaneity' to $\otimes$.
3. Infinite trees?
4. Can we treat SPE?
5. What kind of assignment is $\mathsf{argmax} : \mathsf{Arenas} \to \mathsf{Games}$?
6. Can we make selections compositional?
7. Dependent types

Thanks for your attention!

Questions?

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