Games with players

Towards categorical foundations of cybernetics

An MSP101 talk

Matteo Capucci

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What is a game?

Informal definition:

> *Game theory is the mathematical study of interaction among independent, self-interested agents.*
>
> *– Essentials of Game Theory [LS08]*

Examples:

1. Tic-tac-toe, chess, Monopoly, etc.
2. Economic games (includes/are included in: ecological games)
3. Social dilemmas (PD, 'tragedy of the commons', etc.)
4. Proof theory, model theory, etc.
5. Machine learning
6. **etc.**

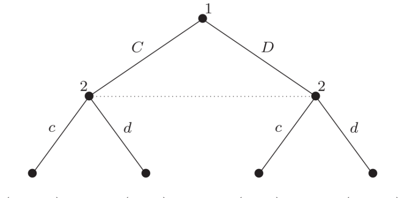
Representing games

1. **Normal form**: A set of players P, and indexed set of **actions** $A : P \to$ **Set**, and a **utility function**

$$u : \prod_{p \in P} A\ p \to (P \to R)$$

2. **Extensive form**: A set of players P, and a tree representing the unfolding of the game. Nodes are assigned to players and grouped in **information sets**. Branches are called **moves**. A **tility vectors** is assigned to each leaf.

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Extensive $\rightarrow$ normal

One can always convert an extensive form game into normal form:

1. Define
$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$
2. Define
$$\begin{aligned} u(\text{action profile } a_1, \ldots, a_n) &= u(\text{path } a_1, \ldots, a_n) \\ &= \text{payoff at the end of the path.} \end{aligned}$$

The converse is not always possible since normal-form games have too little structural information.

Solving games

Pre-formal definition: A **solution concept** is a notion of 'optimality' for ways to play a game.

A 'way to play' for a player $p \in P$ is called **strategy**:

$$\Omega\ p = \prod_{x \in p\text{'s nodes}} \text{moves at } x$$

Compare it with

$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$

Key difference: strategies are a **comprehensive plan of action**: for each **state** of the game, no matter how unlikely, we plan an **action.**

A choice of strategy for each player is a **strategy profile**:

$$S = \prod_{p \in P} \Omega\ p$$

Nash equilibrium

The most important (and general) solution concept is **Nash equilibrium**:

Definition

A strategy profile $s \in S$ is a Nash equilibrium if no player has interest in unilaterally deviating its strategy.

e.g. for utility-maximizing players:

$$\forall p \in P, \forall s'_p \in \Omega\ p \quad u_i(s[s_p/s'_p]) \leq u_p(s)$$

It's not the only one: SGP, ESS, ϵ-Nash, trembling hand, etc.
Afaik, all are **refinements** of Nash.

Nash equilibrium: example

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Pros and cons

Problems with classical game theory:

1. Games are treated **monolithically**
2. Stuck in **early 20th century mathematical language**
3. Denotations are quite disappointing: normal form is too opaque, extensive form is too... extended

Open games are a proposed improvement:

1. Games are defined **compositionally**, including **equilibria**
2. Mathematically more sophisticated (grounded in **category theory**)
3. Denoted by **string diagrams**: halfway between normal and extensive form

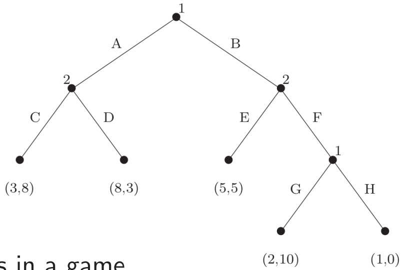
It follows the ACT tradition of 'opening up' systems: *always consider a system as part of an environment it interacts non-trivially with*

Open games

Warning: Open games are compositional structures, hence the single building blocks do not make much sense from a classical standpoint – you have to put them together to get something meaningful!

Intuitively, an 'atomic' open game is a forest of bushes:

Back and forth



There's two phases in a game

1. The 'forward phase': players take turns and make their own decisions until a leaf is reached
2. The 'backward phase': payoffs propagate back to players along the tree

The backward phase is done for analysis purposes: we observe how different decisions affect a player's payoff

$\longrightarrow$ **backward induction**

Lenses and bidirectional information flows

A **lens** models exactly this bidirectional information flow:

(pic of a lens)

Definition

Let **C** be a cartesian category (think: sets & functions).

A **lens** $(X, S) \to (Y, R)$ is a pair of maps

$$\text{view} : X \to Y$$

$$\text{update} : X \times R \to S$$

They can be generalized greatly, see **optics** [**reilly**]

Games as lenses

So a game can be represented naively as a lens $(X, S) \to (Y, R)$:

What's coutility?

> *For a given node x in a game, a player's continuation value (also called continuation payoff) is the payoff that this player will eventually get contingent on the path of play passing through node x.*
>
> – *Strategy* [**strategy**]

The coplay function takes on this job. In classical game theory it's hidden in backward induction. It doesn't *have to* be trivial, but it often is.

Sequential composition

Parallel composition

Closing a game

We can use sequential composition to give a lens a **context**, i.e. an initial **state** and a **payoff function**

(pic)

Slogan: '**Time flows clockwise**'

What's missing?

> *What does it mean to say that agents are self-interested? [...] It means that each agent has* **their own description** *of which* **states of the world they like***—which can include good things happening to other agents—and that* **they act** *in an attempt to bring about these states of the world.*
>
> *– Essentials of Game Theory [LS08]*

At the moment, agents do not intervene: plays and coplays are fixed

We need to give agents:

1. a way to **act in the world**
2. a way to **observe the world**
3. a way to **evaluate the world**

Interlude I: the Para construction

Originally from [FST19], but expanded greatly in the last year

Definition

When $\mathbf{C}$ is symmetric monoidal, $\mathbf{Para}(\mathbf{C})$ is the category of parametrized morphisms of $\mathbf{C}$:

1. objects are the same,
2. a morphism $A \to B$ is given by a choice of parameter $P : \mathbf{C}$ and a choice of morphism $f : P \otimes A \to B$ in $\mathbf{C}$:

(pic of para morphism)

Interlude I: the Para construction

Para(C) is again symmetric monoidal:

(pic of seq comp)

(pic of par comp)

and, most importantly, it's a **bicategory**

(pic of 2-cell)

Interlude I: the Para construction

Now, if our morphisms are ‘bidirectional’, we get an even more interesting picture:

(pic of para optic)

If we peek inside, we can see the new information flow:

(pic of para lens, opened up)

Open games

Idea: if 'games without agency' are lenses, 'games with agency' are *parametrized* lenses:

parameters are **strategies**:

the ways an agent acts in the world

coparameters are **'costrategies'**:

the ways an agent observes the world

The only piece still missing is ways for agents to **evaluate** the world.

Interlude II: selection functions

Definition

A **continuation** on a object X with scalars an object R is a map

$$K_R(X) = (X \to R) \to R$$

It's a 'generalized quantifier': max, min, $\exists$, $\forall$

If the ambient category is cartesian closed, K_R defines a monad.

Interlude II: selection functions

Selection functions ‘realize’ quantifiers:

Definition

Def. A **selection function** on a object X with scalars an object R is a map

$$J_R(X) = (X \to R) \to X$$

Examples: argmax, argmin, Hilbert's ε

If the ambient category is cartesian closed, J_R defines a monad.

Interlude II: selection functions

Notice: often quantifiers are realized by multiple elements...

So a better type for selection functions is

$$(X \to R) \to \mathsf{P}X$$

where P is the powerset monad.

Interlude II: selection functions

Also notice:

$$\underset{\text{costate of } (X,R)}{(X \to R)} \xrightarrow{\qquad\qquad\qquad} \underset{\text{state of } (X,R)}{\mathrm{P}X}$$

so we arrive to a general definition:

Definition

Let $\mathbf{C}$ be a monoidal category. Then the selection functions functor is given by

$$\begin{aligned} \mathbf{Sel} : \mathbf{C} &\longrightarrow \mathbf{Cat} \\ X &\longmapsto \mathbf{C}(X, I) \to \mathrm{P}\mathbf{C}(I, X) \end{aligned}$$

The codomain is **Cat** since this set is ordered by pointwise inclusion:

$$\varepsilon \leq \varepsilon' \quad \underline{\text{iff}} \quad \forall k \in \mathbf{C}(X, I),\ \varepsilon(k) \subseteq \varepsilon'(k)$$

Interlude II: selection functions

Sel is a functor because it also acts on morphism by **pushforward**:

(pic from wiki)

Idea: selection functions are a relation between states and costates

(Probably better: selection functions are predicates on *contexts*)

Interlude II: selection functions

Finally, **Sel** is **lax monoidal** with **Nash product**:

$$\boxtimes : \mathbf{Sel}(X) \times \mathbf{Sel}(Y) \to \mathbf{Sel}(X \otimes Y)$$

$$(\varepsilon \boxtimes \eta)(k) = \{x \otimes y \in (X \otimes y)_* \varepsilon(k) \cap (x \otimes Y)_* \eta(k)\}$$

(pic)

Open games

To see why we call this the Nash product, let's go back to games...

(pic of para optic)

At this point, we only miss one piece of data:

3. A way for agents to **evaluate the world**

Idea: for each player, we pick a selection function

$$\varepsilon \in \mathbf{Sel}(\Omega, \mho)$$

to model their preferences

Open games

Now, careful. Recall:

> *Game theory is the mathematical study of* **interaction** *among independent, self-interested* **agents***.*
>
> *– Essentials of Game Theory [LS08]*

A game factors in two parts

1. An **arena**, which models the interaction patterns in the game
2. A set of **agents**, i.e. the players, which make **decisions** at different points of a game

Without (2) a game would be only a **dynamical system**, whose dynamic is fixed. Instead, in a game **agents can vary the dynamics in response to the observed unfolding of the interaction**.

Open games

Therefore: **agents live in the parametrization direction**

(pic: para optic with arena and agents areas highlighted)

The **arena** plays the role of a costate to them:

(pic of $\top$ action)

$-^\top$ (transposition) is an **arena lifting** operation.

Open games

Arenas can be defined '**locally**': it's just information plumbing. Agents can't: an agent might observe and interact with the arena at multiple, causally 'distant' points.

We take advantage of the 2-cells in **Para**(**Lens**) to handle this:

(pic)

Open games

Great! So an open game is

Definition

1. A **parametrized lens**:

$$\mathcal{G} : (X, S) \xrightarrow{(\Sigma, \Sigma')} (Y, R)$$

2. A set of **selection functions** indexed by player P:

$$\forall p \in P,\ \varepsilon_i \in \mathbf{Sel}(\Omega_i, \mho_i)$$

3. A **wiring 2-cell**:

$$w : \prod_{p \in P} (\Omega_p, \mho_p) \longrightarrow (\Sigma, \Sigma')$$

Open games

Equilibria are given by

$$\mathrm{eq}(x, u) = (\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_n)(w \mathbin{⨟} (x \mathbin{⨟} G \mathbin{⨟} u)^\top)$$

It can be shown that this definition is compositional in the natural way, and recovers Nash equilibria as a solution concept (which justifies calling $\boxtimes$ **Nash product**)

Open games

This solves long-standing problems with open games:

1. We finally compute ther **right set of Nash equilibria** (instead of one-shot deviations)
2. We can better handle situations of **imperfect information**
3. We can define **internal choice**
4. Speculative: we might have a direct line of approach to **cooperative game theory**

A lot to explore!

Thanks for your attention!

Questions?

...or, let's skip to the bonus section $\rightarrow$

Cybernetics

From $\kappa\upsilon\beta\varepsilon\rho\nu\alpha\omega$ (*to govern, to steer*):

> *Science concerned with the study of systems of any nature which are capable of receiving, storing and processing information so as to use it for control.* – *Kolmogorov*

Lenses model dynamical systems (see: [Mye20])

Parametrized lenses (+ decorations) model cybernetic systems!

The missing bits are **storage** and **feedback**.

$\underbrace{\text{Parametrized} \cdots \text{parametrized}}_{n \text{ times}}$ lenses model **...?**

Hierarchical agency?

Higher-order cybernetics

Agents

1. act in the arena,
2. then observe the result of their behaviour,
3. then change their action accordingly,

until an equilibrium is reached.

Higher-order cybernetics

Agents

1. act in the arena,
2. then **observe** the result of their behaviour,
3. then change their action accordingly,

until an equilibrium is reached.

	1st	**2nd**	**3rd**	$\cdots$
non-trivial observation		yes	yes	$\cdots$
non-trivial analysis			yes	$\cdots$
$\vdots$				$\ddots$

Periodic table of cybernetic types

Reasoning negatively:

	-2nd	**-1st**	**0th**	**1st**	**2nd**	$\cdots$
non-trivial system		yes	yes	yes	yes	$\cdots$
non-trivial context			yes	yes	yes	$\cdots$
non-trivial interaction				yes	yes	$\cdots$
non-trivial observation					yes	$\cdots$
$\vdots$						$\ddots$

Learners as 2nd-order cybernetic systems

(pic learner)

Given a parameter $p \in P$, a learner can only observe $\mathcal{L}^\top$ on an infinitesimal neighbourhood of P

$\rightsquigarrow$ **2nd-order cybernetic systems**

We can consider **Para**(**Lens**(**Smooth**)) a 2nd-order cybernetic **doctrine**.

Cybernetics

This is actually a strength:

1. We can encode the selection in the parameter dynamics
2. We can analyze locally and iteratively
   (vs. games 'global and one-step')

A learner produces its own selection function as a fixpoint:

$$\mathrm{eq}(x, u) = \{p \in \mathrm{fixpoint}(p,\ GD \mathbin{;} w \mathbin{;} (x \mathbin{;} \mathcal{L} \mathbin{;} u)^{\top})\}$$

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