Games with players

Towards categorical foundations of cybernetics

An MSP101 talk

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Informal definition:

> *Game theory is the mathematical study of interaction among independent, self-interested agents.*
>
> – *Essentials of Game Theory [**eogt**]*

Examples:

1. Tic-tac-toe, chess, Monopoly, etc.
2. Economic games (includes/are included in: ecological games)
3. Social dilemmas (PD, 'tragedy of the commons', etc.)
4. Proof theory, model theory, etc.
5. Machine learning
6. **etc.**

Classical game theory

Two ways of representing a game:

1. **Normal form** There is a set of players P, and indexed set of **actions** $A : P \to \mathbf{Set}$, and a **utility function**

   $$u : \Pi A \to P \to R$$

   (PD)

2. **Extensive form** There is a set of players P, and a tree representing the unfolding of the game. Nodes are assigned to players and grouped in **information sets**. Branches are called **moves**. **Utility vectors** are assigned to each leaf. (PD)

Classical game theory

One can always convert an extensive form game into normal form:

1. Define
$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$
2. Define
$$\begin{aligned} u(\text{action profile } a_1, \ldots, a_n) &= u(\text{path } a_1, \ldots, a_n) \\ &= \text{payoff at the end of the path.} \end{aligned}$$

The converse is not always possible since normal-form games have too little structural information.

Classical game theory

Pre-formal definition: A **solution concept** is a notion of 'optimality' for ways to play a game.

A 'way to play' for a player $p \in P$ is called **strategy**:

$$\Omega\ p = \prod_{x \in p\text{'s nodes}} \text{moves at } x$$

Compare it with

$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$

Key difference: strategies are a **comprehensive plan of action**: for each **state** of the game, no matter how unlikely, we plan an **action**.

A choice of strategy for each player is a **strategy profile**:

$$S = \prod_{p \in P} \Omega\ p$$

Classical game theory

The most important (and general) solution concept is **Nash equilibrium**:

Definition

A strategy profile $s \in S$ is a Nash equilibrium if no player has interest in unilaterally deviating its strategy.

e.g. for utility-maximizing players:

$$\forall p \in P, \forall s'_p \in \Omega\ p \quad u_i(s[s_p/s'_p]) \leq u_p(s)$$

It's not the only one: SGP, ESS, ε-Nash, trembling hand, etc.
Afaik, all are **refinements** of Nash.

Classical game theory

Problems with classical game theory:

1. Games are treated **monolithically**: one defines a game all at once, and treats reuse/composition only informally.
2. Stuck in early **20th century mathematical language**
3. Denotations are quite disappointing: normal form is too opaque, extensive form is too... extended

Open games are a proposed improvement:

1. Defined **compositionally**. **This includes, most importantly, equilibria**
2. Mathematically more sophisticated (grounded in **category theory**)
3. Denoted by **string diagrams**: halfway between normal and extensive form

Follows the ACT tradition of 'opening up' systems: always consider a system as part of an environment it interacts non-trivially with

Open games

Warning: Open games are compositional structures, hence the single building blocks do not make much sense from a classical standpoint – you have to put them together to get something meaningful

Informally: an 'atomic' open game is a forest of bushes (pic)

Open games

First of all, let's model the information flow of a game. (pic of a tree)
There's two phases in a game

1. The 'play' or 'forward phase': players takes turn and make their own decisions until a leaf is reached
2. The 'coplay' or 'backward phase': payoffs propagate back to player along the tree

The backward phase is done for analysis purposes: we observe how different decisions would bring about different payoffs (**backward induction**)

Open games

A **lens** models exactly this bidirectional information flow:

(pic of a lens)

Slogan: '**Time flows clockwise**'

Definition

Let $\mathbf{C}$ be a cartesian category (think: sets & functions). A **lens** $(X, S) \to (Y, R)$ is a pair of maps

$$\text{view} : X \to Y$$
$$\text{update} : X \times R \to S$$

Intuition: **view** corresponds to the forward step, **update** to the backward step.

Open games

So a game can be represented naively as a lens $(X, S) \to (Y, R)$ where

... X are **states of the game**

... Y are **moves**

... R are **utilities**

... S are **coutilities**

In open games we call the forward part of a lens '**play**' and backward part '**coplay**'

Open games

What's coutility?

> *For a given node x in a game, a player's continuation value (also called continuation payoff) is the payoff that this player will eventually get contingent on the path of play passing through node x.*
>
> – *Strategy* [**strategy**]

The coplay function takes on this job. In classical game theory it's hidden in backward induction. It doesn't **have to** be trivial, but it often is.

Open games

Lenses, hence games, can then be composed in at least two ways:

(sequential)

(parallel)

Open games

We can use sequential composition to give a lens a **context**, i.e. an initial **state** and a **payoff function**

(pic)

Remember: '**time flows clockwise**'

Open games

What's missing?

> *What does it mean to say that agents are self-interested? [...] [I]t means that each agent has* **[their] own description** *of which* **states of the world [they like]***—which can include good things happening to other agents—and that* **[they act]** *in an attempt to bring about these states of the world.*
>
> *– Essentials of Game Theory [***eogt***]*

Three things:

1. A way for agents to **act in the world**
2. A way for agents to **represent the world**
3. A way for agents to **evaluate the world**

At the moment, agents do not intervene in the unfolding of the game: plays and coplays are fixed

We need to give a 'steering wheel' to agents

Interlude I: the Para construction

Para originates in [**backpropasafunctor**] an has been adopted by this group (Bruno) to represent all sort of stuff. Also Toby Smithe has come up with a Para-like construction, Proxy.

Interlude I: the Para construction

Definition

When **C** is symmetric monoidal, **Para**(**C**) is the category of parametrized morphisms of **C**:

1. objects are the same,
2. a morphism $A \to B$ is given by a choice of parameter $P : \mathbf{C}$ and a choice of morphism $P \otimes A \to B$ in **C**:

(pic of para morphism)

Interlude I: the Para construction

Para(C) is again symmetric monoidal:

(pic of seq comp)

(pic of par comp)

and, most importantly, it's a **bicategory**

(pic of 2-cell)

Interlude I: the Para construction

Now, if our morphisms are ‘bidirectional’, we get an even more interesting picture:

(pic of para optic)

If we peek inside, we can see the new information flow:

(pic of para lens, opened up)

Open games

Idea: if 'games without agency' are lenses, 'games with agency' are *parametrized* lenses:

> parameters are **strategies**: **the ways an agent acts** in the world
>
> coparameters are **'costrategies'**: **the ways an agent represents** the world

The only piece still missing is ways for agents to **evaluate** the world.

Interlude II: selection functions

Definition

A **continuation** on a object X with scalars an object R is a map

$$K_R(X) = (X \to R) \to R$$

It's a 'generalized quantifier': max, min, $\exists$, $\forall$

If the ambient category is cartesian closed, K_R defines a monad.

Interlude II: selection functions

Selection functions ‘realize’ quantifiers:

Definition

Def. A **selection function** on a object X with scalars an object R is a map

$$J_R(X) = (X \to R) \to X$$

Examples: argmax, argmin, Hilbert's ε

If the ambient category is cartesian closed, J_R defines a monad.

Interlude II: selection functions

Notice: often quantifiers are realized by multiple elements...

(ambiguous max function)

So a better type for selection functions is

$$(X \to R) \to \mathsf{P}X$$

where P is the powerset monad.

Interlude II: selection functions

Also notice:

$$\begin{array}{ccc} (X \to R) & \longrightarrow & \mathrm{P}X \\ \text{costate of } (X, R) & & \text{state of } (X, R) \end{array}$$

so we arrive to a general definition:

Definition

Let $\mathbf{C}$ be a monoidal category. Then the selection functions functor is given by

$$\begin{aligned} \mathbf{Sel} : \mathbf{C} &\longrightarrow \mathbf{Cat} \\ X &\longmapsto \mathbf{C}(X, I) \to \mathrm{P}\mathbf{C}(I, X) \end{aligned}$$

The codomain is **Cat** since this set is ordered by pointwise inclusion:

$$\varepsilon \leq \varepsilon' \quad \underline{\text{iff}} \quad \forall k \in \mathbf{C}(X, I),\ \varepsilon(k) \subseteq \varepsilon'(k)$$

Interlude II: selection functions

Sel is a functor because it also acts on morphism by **pushforward**:

(pic from wiki)

Idea: selection functions are a relation between states and costates

(Probably better: selection functions are predicates on *contexts*)

Interlude II: selection functions

Finally, **Sel** is **lax monoidal** with **Nash product**:

$$\boxtimes : \mathbf{Sel}(X) \times \mathbf{Sel}(Y) \to \mathbf{Sel}(X \otimes Y)$$

$$(\varepsilon \boxtimes \eta)(k) = \{x \otimes y \in (X \otimes y)_* \varepsilon(k) \cap (x \otimes Y)_* \eta(k)\}$$

(pic)

Open games

To see why we call this the Nash product, let's go back to games...

(pic of para optic)

At this point, we only miss one piece of data:

3. A way for agents to **evaluate the world**

Idea: for each player, we pick a selection function

$$\varepsilon \in \textbf{Sel}(\Omega, \mho)$$

to model their preferences

Open games

Now, careful. Recall:

> *Game theory is the mathematical study of* **interaction** *among independent, self-interested* **agents**.
>
> – *Essentials of Game Theory [**eogt**]*

A game factors in two parts

1. An **arena**, which models the interaction patterns in the game
2. A set of **agents**, i.e. the players, which make **decisions** at different points of a game

Without (2) a game would be only a **dynamical system**, whose dynamic is fixed. Instead, in a game **agents can vary the dynamics in response to the observed unfolding of the interaction**.

Open games

Therefore: **agents live in the parametrization direction**

(pic: para optic with arena and agents areas highlighted)

The arena plays the role of a costate to them:

(pic of $\top$ action)

Open games

Finally, if arenas can be defined '**locally**' because they are information plumbing, agents' interests can't since they are defined '**globally**': an agent might observe and interact with the arena at multiple, causally 'far' points.

We take advantage of the 2-cells in Para(Lens) to handle this:

(pic)

Open games

Great! So an open game is

Definition

1. A **parametrized lens**

$$\mathcal{G} : (X, S) \xrightarrow{(P,P')} (Y, R)$$

2. A set of players $\{1, ..., n\}$, each represented by their own **selection function**

$$\varepsilon_i \in \mathbf{Sel}(\Omega_i, \mho_i)$$

3. A **wiring 2-cell**:

$$w : \prod_{i \in I} (\Omega_i, \mho_i) \longrightarrow (P, P')$$

Open games

Equilibria are given by

$$\mathrm{eq}(x, u) = (\varepsilon_1 \boxtimes \cdots \boxtimes \varepsilon_n)(w; (x; G; u)^\top)$$

It can be shown that this definition is compositional in the natural way, and recovers Nash equilibria as a solution concept (which justifies calling $\boxtimes$ **Nash product**)

Open games

This solves long-standing problems with open games:

1. We finally compute ther **right set of Nash equilibria**
2. We can handle situations of **imperfect recall**
3. We can define equilibria for **internal choice**
4. We can even model coalitional games (future work)

Still a lot to explore.

Cybernetics

1. Lenses model dynamical systems (see also: DJM, Spivak, Schultz, Vasilakopoulou, etc.)
2. Parametrized lenses model cybernetic systems
3. Parametrized $\cdots$ parametrized lenses model ...?

   Hierarchical agency? Higher-order cybernetic systems?

Thanks for your attention!

Questions?

References