University of Strathclyde Glasgow

Games with players

Towards categorical foundations of cybernetics

An MSP101 talk

Matteo Capucci

April 29th, 2021 (Day 488 of the COVID Era)

Informal definition:

> *Game theory is the mathematical study of interaction among independent, self-interested agents.*
>
> – *Essentials of Game Theory* [**eogt**]

Examples:

1. Tic-tac-toe, chess, Monopoly, etc.
2. Economic games (includes/are included in: ecological games)
3. Social dilemmas (PD, 'tragedy of the commons', etc.)
4. Proof theory, model theory, etc.
5. Machine learning
6. **etc.**

Classical game theory

Two ways of representing a game:

1. **Normal form** There is a set of players P, and indexed set of **actions** $A : P \to \textbf{Set}$, and a **utility function**

$$u : \Pi A \to P \to R$$

   (PD)

2. **Extensive form** There is a set of players P, and a tree representing the unfolding of the game. Nodes are assigned to players and grouped in **information sets**. Branches are called **moves**. **Utility vectors** are assigned to each leaf. (PD)

Classical game theory

One can always convert an extensive form game into normal form:

1. Define
$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$
2. Define
$$\begin{aligned} u(\text{action profile } a_1, \ldots, a_n) &= u(\text{path } a_1, \ldots, a_n) \\ &= \text{payoff at the end of the path.} \end{aligned}$$

The converse is not always possible since normal-form games have too little structural information.

Classical game theory

Pre-formal definition: A **solution concept** is a notion of 'optimality' for ways to play a game.

A 'way to play' for a player $p \in P$ is called **strategy**:

$$\Omega\, p = \prod_{x \in p\text{'s nodes}} \text{moves at } x$$

Compare it with

$$A\, p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$

Key difference: strategies are a **comprehensive plan of action**: for each **state** of the game, no matter how unlikely, we plan an **action**.

A choice of strategy for each player is a **strategy profile**:

$$S = \prod_{p \in P} \Omega\, p$$

Classical game theory

The most important (and general) solution concept is **Nash equilibrium**:

Definition

A strategy profile $s \in S$ is a Nash equilibrium if no player has interest in unilaterally deviating its strategy.

e.g. for utility-maximizing players:

$$\forall p \in P, \forall s'_p \in \Omega\ p \quad u_i(s[s_p/s'_p]) \leq u_p(s)$$

It's not the only one: SGP, ESS, ε-Nash, trembling hand, etc.
Afaik, all are **refinements** of Nash.

Classical game theory

Problems with classical game theory:

1. Games are treated **monolithically**: one defines a game all at once, and treats reuse/composition only informally.
2. Stuck in early **20th century mathematical language**
3. Denotations are quite disappointing: normal form is too opaque, extensive form is too... extended

Open games are a proposed improvement:

1. Defined **compositionally**. **This includes, most importantly, equilibria**
2. Mathematically more sophisticated (grounded in **category theory**)
3. Denoted by **string diagrams**: halfway between normal and extensive form

Follows the ACT tradition of 'opening up' systems: always consider a system as part of an environment it interacts non-trivially with

Thanks for your attention!

Questions?

References