Games with players

Towards categorical foundations of cybernetics

An MSP101 talk

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April 29th, 2021 (Day 488 of the COVID Era)

What is a game?

Informal definition:

> *Game theory is the mathematical study of interaction among independent, self-interested agents.*
>
> *– Essentials of Game Theory [LS08]*

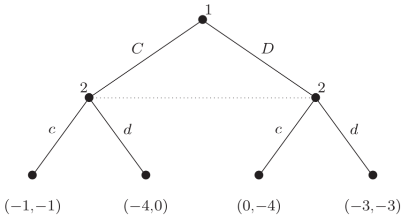
Examples:

1. Tic-tac-toe, chess, Monopoly, etc.
2. Economic games (includes/are included in: ecological games)
3. Social dilemmas (PD, 'tragedy of the commons', etc.)
4. Proof theory, model theory, etc.
5. Machine learning
6. **etc.**

Representing games

1. **Normal form**: A set of **players** P, an indexed set of **actions** $A : P \to$ **Set**, a **utility function** $u : \prod_{p \in P} A\ p \to (P \to R)$
2. **Extensive form**: A set of **players** P, a **tree** representing the unfolding of the game. Nodes are assigned to players and grouped in **information sets**. Branches are called **moves**. A **utility vector** assigned to each leaf.

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Extensive $\to$ normal

One can always convert an extensive form game into normal form:

1. Define
$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$
2. Define
$$\begin{aligned} u(\text{action profile } a_1, \ldots, a_n) &= u(\text{path } a_1, \ldots, a_n) \\ &= \text{payoff at the end of the path.} \end{aligned}$$

The converse is not always possible since normal-form games have too little structural information.

Solving games

Pre-formal definition: A **solution concept** is a notion of 'optimality' for ways to play a game.

A 'way to play' for a player $p \in P$ is called **strategy**:

$$\Omega\ p = \prod_{x \in p\text{'s nodes}} \text{moves at } x$$

Compare it with

$$A\ p = \sum_{x \in p\text{'s nodes}} \text{moves at } x$$

Key difference: strategies are a **comprehensive plan of action**: for each **state** of the game, no matter how unlikely, we plan an **action**.

A choice of strategy for each player is a **strategy profile**:

$$S = \prod_{p \in P} \Omega\ p$$

Nash equilibrium

The most important (and general) solution concept is **Nash equilibrium**:

Definition

A strategy profile $s \in S$ is a Nash equilibrium if no player has interest in unilaterally deviating its strategy.

e.g. for utility-maximizing players:

$$\forall p \in P, \forall s'_p \in \Omega\ p \quad u_i(s[s_p/s'_p]) \leq u_p(s)$$

It's not the only one: SGP, ESS, ϵ-Nash, trembling hand, etc.
Afaik, all are **refinements** of Nash.

Nash equilibrium: example

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Pros and cons

Problems with classical game theory:

1. Games are treated **monolithically**
2. Stuck in **early 20th century mathematical language**
3. Denotations are quite disappointing: normal form is too opaque, extensive form is too... extended

Open games are a proposed improvement:

1. Games are defined **compositionally**, including **equilibria**
2. Mathematically more sophisticated (grounded in **category theory**)
3. Denoted by **string diagrams**: halfway between normal and extensive form

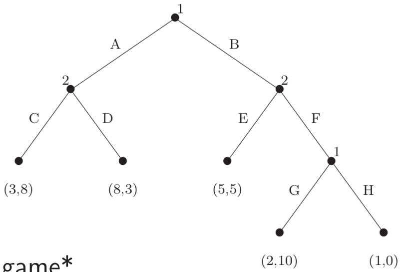
It follows the ACT tradition of 'opening up' systems: *always consider a system as part of an environment it interacts non-trivially with*

Open games

Warning: Open games are compositional structures, hence the single building blocks do not make much sense from a classical standpoint – you have to put them together to get something meaningful!

Intuitively, an 'atomic' open game is a forest of bushes:

Back and forth



There's two phases in a game*

1. The '**forward phase**'

   Players take turns and make their own decisions until a leaf is reached

2. The '**backward phase**'

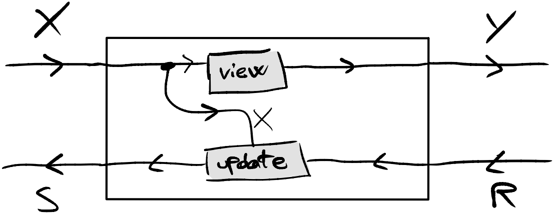
   Payoffs propagate back to players along the tree

   $\longrightarrow$ **backward induction**

*handwaving important philosophical point here

Lenses and bidirectional information flows

A **lens** models exactly this bidirectional information flow:



Definition

Let **C** be a cartesian category (think: sets & functions).

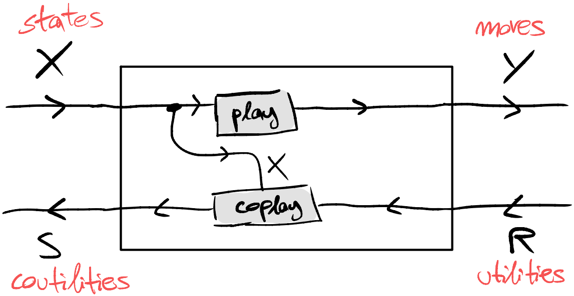
A **lens** $(X, S) \to (Y, R)$ is a pair of maps

$$\text{view} : X \to Y, \qquad \text{update} : X \times R \to S$$

They can be generalized greatly, see **optics** [Ril18]

Games as lenses

A game can be represented naively as a lens



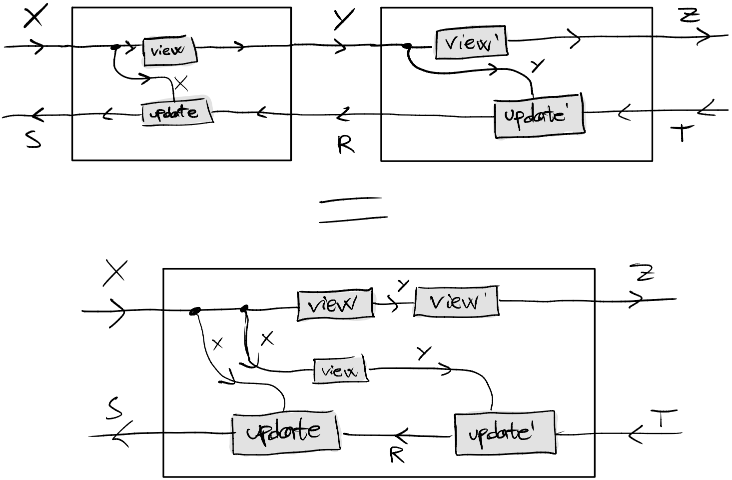
What's coutility?

> *For a given node x in a game, a player's continuation value (also called* **continuation payoff***) is the payoff that this* **player will eventually get** *contingent on the path of play* **passing through node** x*. – Strategy [Wat02]*

The coplay function takes on this job. In classical game theory it's hidden in backward induction.

It doesn't *have to* be trivial, but it often is.

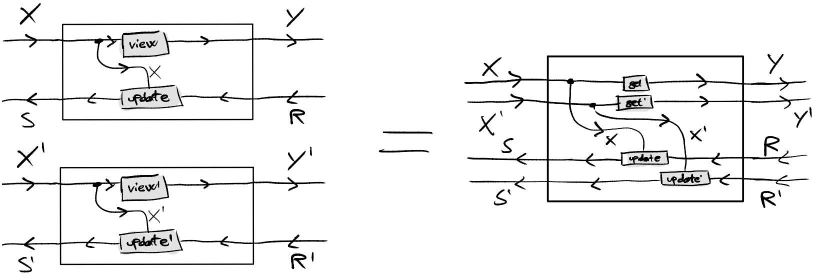
Sequential composition



e.g. chess

Parallel composition

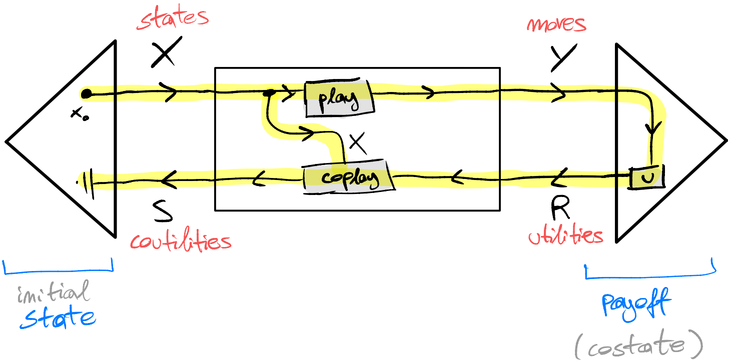
Play simultaneously



e.g. PD

Closing a game

We can use sequential composition to give a lens a **context**, i.e. an initial **state** and a **payoff function**



Slogan: '**Time flows clockwise**'

Open games

Recall:

> *Game theory is the mathematical study of* **interaction** *among independent, self-interested* **agents**.
>
> *– Essentials of Game Theory [LS08]*

A game factors in two parts

1. An **arena**, which models the interaction patterns in the game
2. A set of **agents**, i.e. the players, which make **decisions** at different points of a game

Without (2) a game would be only a **dynamical system**, whose dynamic is fixed. Instead, in a game **agents can vary the dynamics in response to the observed unfolding of the interaction**.

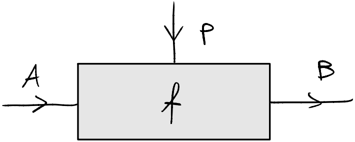
Interlude I: the Para construction

Originally from [FST19], but expanded greatly in the last year

Definition

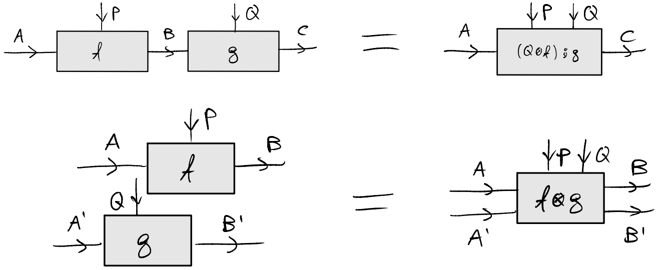
When **C** is symmetric monoidal, **Para**(**C**) is the category of parametrized morphisms of **C**:

1. objects are the same,
2. a morphism $A \to B$ is given by a choice of parameter $P : \mathbf{C}$ and a choice of morphism $f : P \otimes A \to B$ in **C**:



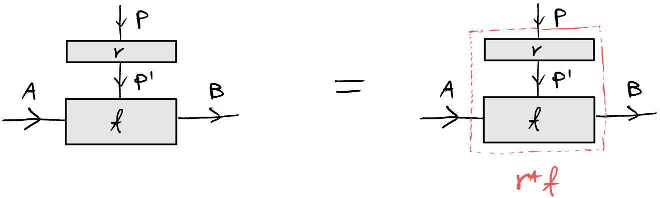
Interlude I: the Para construction

Para(C) is again symmetric monoidal:



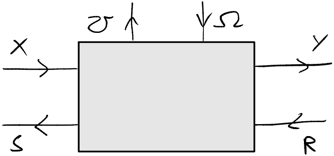
Interlude I: the Para construction

Most importantly, it's a **bicategory**:

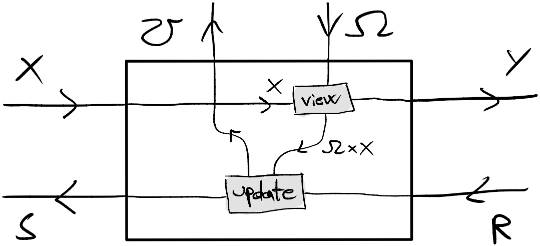


Interlude I: the Para construction

If our morphisms are 'bidirectional', we get an even more interesting picture:

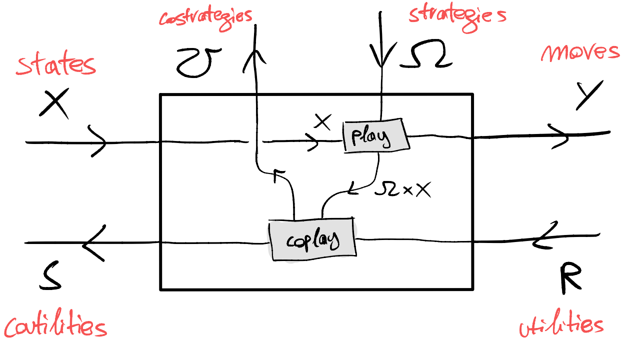


If we peek inside, we can see the new information flow:



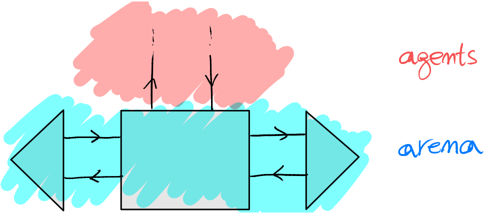
Putting players in a game

Let's go back to games...

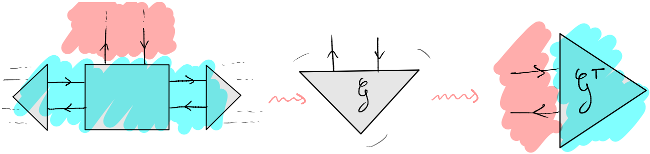


Open games

Slogan: **agents live in the parametrization direction**



The **arena** plays the role of a costate to them:

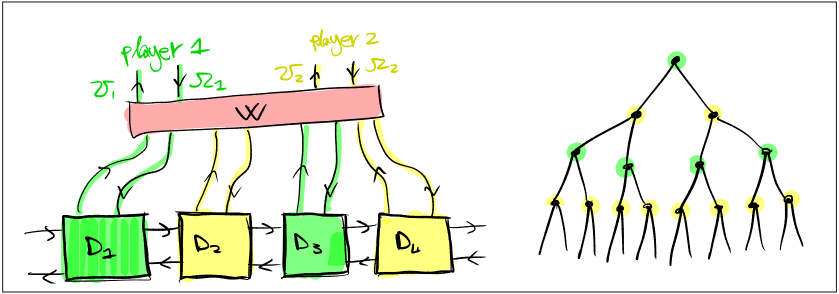


$-^\top$ (**transposition**) is the **arena lifting** (dynamics $\to$ agents) operation.

Open games

Arenas can be defined '**locally**': it's just information plumbing. Agents can't: an agent might observe and interact with the arena at multiple, causally 'distant' points.

We take advantage of the 2-cells in **Para**(**Lens**) to handle this:



Agency in open games

> *What does it mean to say that agents are self-interested? [...] It means that each agent has* **their own description** *of which* **states of the world they like***—which can include good things happening to other agents—and that* **they act** *in an attempt to bring about these states of the world.*
>
> *– Essentials of Game Theory [LS08]*

Agents have:

1. a way to **act in the world**
2. a way to **observe the world**
3. a way to **evaluate the world**

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2. a way to **observe the world** $\rightsquigarrow$ **costrategies**
3. a way to **evaluate the world** $\rightsquigarrow$ **selection function**

Interlude II: selection functions

Definition

A **continuation** on a object X with scalars an object R is a map

$$K_R(X) = (X \to R) \to R$$

It's a 'generalized quantifier': max, min, $\exists$, $\forall$

If the ambient category is cartesian closed, K_R defines a monad.

Interlude II: selection functions

Selection functions 'realize' quantifiers:

Definition

Def. A **selection function** on a object X with scalars an object R is a map

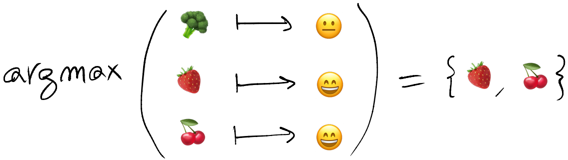
$$J_R(X) = (X \to R) \to X$$

Examples: argmax, argmin, Hilbert's ε

If the ambient category is cartesian closed, J_R defines a monad.

Interlude II: selection functions

Notice: often quantifiers are realized by multiple elements...



So a better type for selection functions is

$$(X \to R) \to \mathsf{P}X$$

where P is the powerset monad.

Interlude II: selection functions

Also notice:

$$\begin{array}{ccc} (X \to R) & \xrightarrow{\hspace{6em}} & \mathrm{P}X \\ \text{costates of } (X, R) & & \mathrm{P}(\text{states of } (X, R)) \end{array}$$

so we arrive to a general definition:

Definition

Let **C** be a monoidal category. Then the selection functions functor is given by

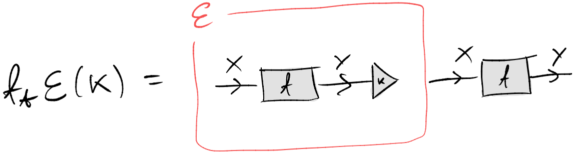
$$\mathbf{Sel} : \mathbf{C} \longrightarrow \mathbf{Cat}$$
$$X \longmapsto \mathbf{C}(X, I) \to \mathrm{P}\mathbf{C}(I, X)$$

The codomain is **Cat** since this set is ordered by pointwise inclusion:

$$\varepsilon \leq \varepsilon' \quad \underline{\text{iff}} \quad \forall k \in \mathbf{C}(X, I),\ \varepsilon(k) \subseteq \varepsilon'(k)$$

Interlude II: selection functions

Sel is a functor because it also acts on morphism by **pushforward**:



Idea: selection functions are a relation between states and costates.

(**Probably better idea**: selection functions are predicates on *contexts*)

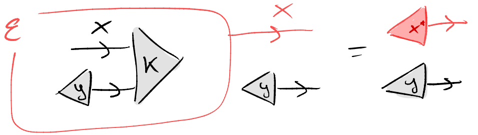
Interlude II: selection functions

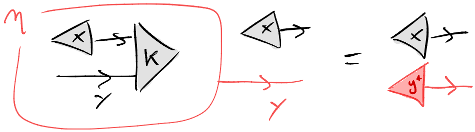
Finally, **Sel** is **lax monoidal** with **Nash product**:

$$- \boxtimes - : \mathbf{Sel}(X) \times \mathbf{Sel}(Y) \to \mathbf{Sel}(X \otimes Y)$$

$$(\varepsilon \boxtimes \eta)(k) = \{x \otimes y \in (X \otimes y)_* \varepsilon(k) \cap (x \otimes Y)_* \eta(k)\}$$

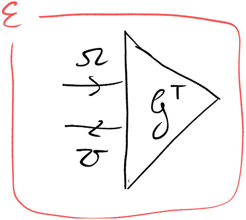
Best 'unilateral deviations':



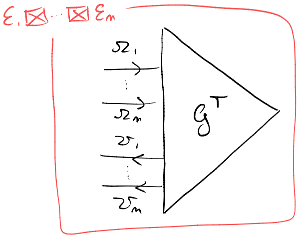


Agency in open games

Idea: An agent's interest is embodied by a selection function on their arenas:



Multiple agents ⊠ together their selection to select common arenas:



Open games

So an open game is

Definition

1. A **parametrized lens**:

$$\mathcal{G} : (X, S) \xrightarrow{(\Sigma,\Sigma')} (Y, R)$$

2. A set of **selection functions** indexed by player P:

$$\forall p \in P,\ \varepsilon_p \in \mathbf{Sel}(\Omega_p, \mho_p)$$

3. A **wiring 2-cell**:

$$w : \prod_{p \in P} (\Omega_p, \mho_p) \longrightarrow (\Sigma, \Sigma')$$

Open games

As anticipated, equilibria are given by

$$\mathrm{eq}_{\mathcal{G}}(x, u) = (\boxtimes_{p \in P}\, \varepsilon_p)(w \fatsemi (x \fatsemi \mathcal{G} \fatsemi u)^\top) \qquad \subseteq \prod_{p \in P} \Omega_p$$

Still compositional wrt sequential and parallel composition of arenas:

$$\mathrm{eq}_{\mathcal{G} \fatsemi \mathcal{H}}(x, u) = \{\omega \otimes \xi \mid \omega \in \mathrm{eq}_{\mathcal{G}}(x, \mathcal{H}(\xi \fatsemi w') \fatsemi u) \wedge \xi \in \mathrm{eq}_{\mathcal{H}}(x \fatsemi \mathcal{G}(\omega \fatsemi w), u)\}$$

$$\mathrm{eq}_{\mathcal{G} \otimes 1}((x, \_), u) = \mathrm{eq}_{\mathcal{G}}(x, u)^*$$

It recovers Nash equilibria as a solution concept, which justifies calling $\boxtimes$ **Nash product**.

*harder to generalize to monoidal categories / optics

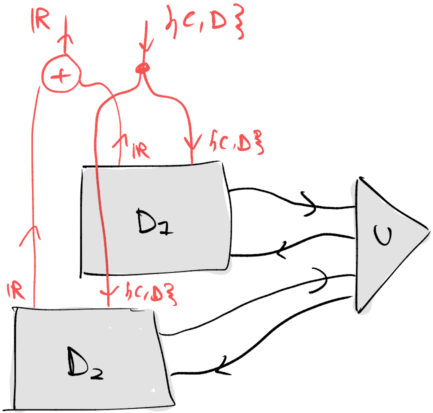
Open games

This solves long-standing problems with open games:

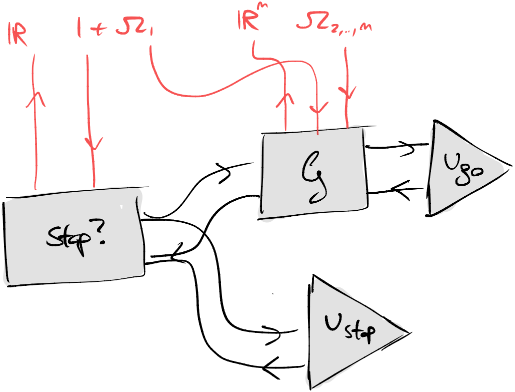
1. We finally compute the **right set of Nash equilibria** (instead of one-shot deviations)
2. We can better handle situations of **imperfect information**
3. We can define **internal choice** (perhaps: approach to **cooperative game theory**)

A lot to explore!

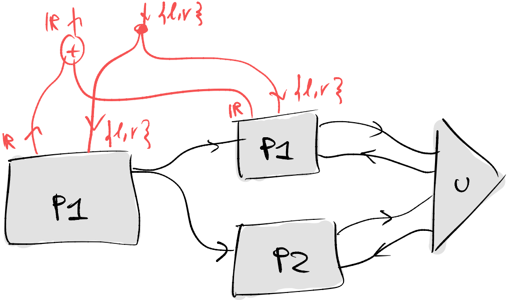
Example I: cooperative Prisoner's Dilemma



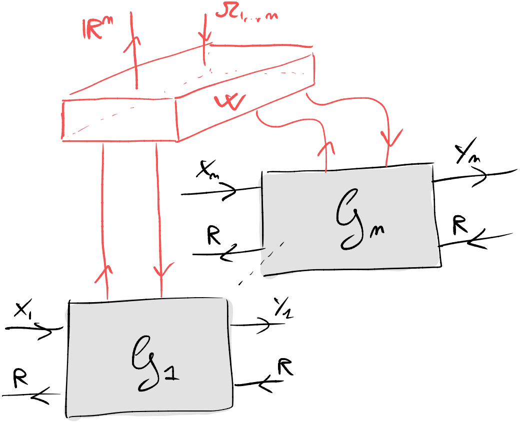
Example II: stopped game



Example III: imperfect recall



Example IV: internal choice



Thanks for your attention!

Questions?

...or, let's skip to the bonus section $\rightarrow$

Cybernetics

From $\kappa\upsilon\beta\varepsilon\rho\nu\alpha\omega$ (*to govern, to steer*):

> *Science concerned with the study of systems of any nature which are capable of receiving, storing and processing information so as to use it for control.* – *Kolmogorov*

Lenses model dynamical systems (see: [Mye20])

Parametrized lenses (+ decorations) model cybernetic systems!

The missing bits are **storage** and **feedback**.

$\underbrace{\text{Parametrized} \cdots \text{parametrized}}_{n \text{ times}}$ lenses model **...?**

Hierarchical agency?

Higher-order cybernetics

Agents

1. act in the arena,
2. then observe the result of their behaviour,
3. then change their action accordingly,

until an equilibrium is reached.

Higher-order cybernetics

Agents

1. act in the arena,
2. then **observe** the result of their behaviour,
3. then change their action accordingly,

until an equilibrium is reached.

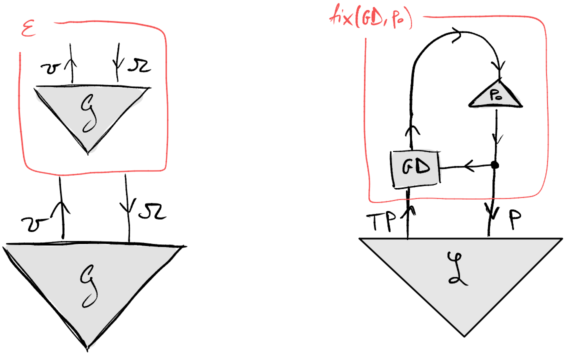
	1st	**2nd**	**3rd**	$\cdots$
non-trivial observation		yes	yes	$\cdots$
non-trivial analysis			yes	$\cdots$
$\vdots$				$\ddots$

Periodic table of cybernetic types

Reasoning negatively:

	-2nd	**-1st**	**0th**	**1st**	**2nd**	⋯
non-trivial system		yes	yes	yes	yes	⋯
non-trivial context			yes	yes	yes	⋯
non-trivial interaction				yes	yes	⋯
non-trivial observation					yes	⋯
⋮						⋱

Games vs. learners



A learner produces its own selection function as a fixpoint:

$$p \in \mathrm{eq}(x, u) \quad \underline{\text{iff}} \quad p = p \fatsemi GD \fatsemi w \fatsemi (x \fatsemi \mathcal{L} \fatsemi u)^{\top}$$

2nd order cybernetics

Given a parameter $p \in P$, a learner can only observe $\mathcal{L}^\top$ on an infinitesimal neighbourhood of p

$\rightsquigarrow$ **2nd-order cybernetic systems**

We can consider **Para**(**Lens**(**Smooth**)) a **2nd-order cybernetic doctrine** (terminology borrowed from [Mye])

This is actually a strength:

1. We can encode the selection in the parameter dynamics
2. We can analyze locally and iteratively
   (vs. games 'global and one-step')

References I

Kevin Leyton-Brown and Yoav Shoham. "Essentials of game theory: A concise multidisciplinary introduction". In: *Synthesis lectures on artificial intelligence and machine learning* 2.1 (2008), pp. 1–88.

Mitchell Riley. "Categories of Optics". In: *arXiv: Category Theory* (2018).

Joel Watson. *Strategy: an introduction to game theory*. Vol. 139. WW Norton New York, 2002.

Brendan Fong, David Spivak, and Rémy Tuyéras. "Backprop as Functor: A compositional perspective on supervised learning". In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE. 2019, pp. 1–13.

References II

David Jaz Myers. "Double Categories of Open Dynamical Systems". In: *arXiv preprint arXiv:2005.05956* (2020).

David Jaz Myers. *Categorical Systems Theory*. URL: https://github.com/DavidJaz/DynamicalSystemsBook/tree/master/book.

Neil Ghani et al. "Compositional game theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018, pp. 472–481.

G.S.H. Cruttwell et al. "Categorical Foundations of Gradient-Based Learning". In: *arXiv preprint arXiv:2103.01931* (2021).