Notes on Hamiltonian Mechanics

Luigi Angelini, Matteo Capucci,
Francesco Monzani, Davide Dal Martello

March 12, 2020

Contents

1 Preliminary notions **2**

1.1 Smooth manifolds . . . 2

1.2 Vector bundles . . . 3

1.2.1 Pullback and pushforward . . . 5

1.2.2 Distributions . . . 6

1.3 Exterior derivative . . . 8

1.3.1 Singular Homology . . . 10

1.3.2 de Rham's Theorem . . . 11

1.4 Interior product . . . 14

1.4.1 Integration . . . 14

2 Symplectic geometry **15**

2.1 Symplectic forms on vector spaces . . . 15

2.1.1 Skew-symmetric bilinear maps . . . 15

2.1.2 Symplectic vector spaces . . . 17

2.2 Symplectic manifolds . . . 19

2.2.1 The canonical symplectic structure on the cotangent bundle . . . 21

2.3 Symplectomorphisms and Lagrangian manifolds . . . 24

2.4 Darboux's Theorem . . . 27

2.4.1 Tubular neighbourhoods . . . 27

2.4.2 Moser's theorems . . . 28

3 Hamiltonian mechanics **31**

3.1 Hamiltonian vector fields . . . 31

3.2 Physical meaning of a symplectic structure . . . 34

3.3 Poisson brackets . . . 35

3.4 Integrable systems . . . 38

3.5 Arnold–Liouville Theorem . . . 39

3.5.1 Applications . . . 46

4 Poisson manifolds **55**
4.1 The adjoint action of a Lie group . . . 55
4.2 Poisson manifolds . . . 61
4.2.1 Poisson tensor . . . 61
4.2.2 The Schouten–Nijenhuis bracket . . . 63
4.2.3 Poisson maps and Poisson submanifolds . . . 64
4.3 The symplectic foliation of a Poisson manifold . . . 65
4.4 Poisson cohomology . . . 69
4.5 Poisson vector fields . . . 69
4.5.1 Meaning of the Poisson cohomologies . . . 72

5 Integrability **75**
5.1 Classical R-matrix . . . 75
5.2 Lax Representations . . . 78
5.3 Bihamiltonian structures . . . 81
5.3.1 The Hamiltonian recursion in the constant rank case . . . 83
5.3.2 The Hamiltonian recursion in the symplectic case . . . 91
5.3.3 Nijenhuis structures . . . 91
5.3.4 The recursion . . . 94

6 Deformation quantization **96**
6.1 Deformation theory for commutative rings . . . 98
6.1.1 Star products . . . 100
6.1.2 Relation to Poisson manifolds . . . 103
6.1.3 Equivalence of star products . . . 104
6.2 Formal deformations of Poisson structures . . . 108
6.2.1 Kontsevich's Theorem . . . 109
6.2.1.1 Admissible graphs . . . 110
6.2.1.2 The weights . . . 113
6.2.1.3 The formula . . . 114
6.3 Retrospect . . . 116

Chapter 1

Preliminary notions

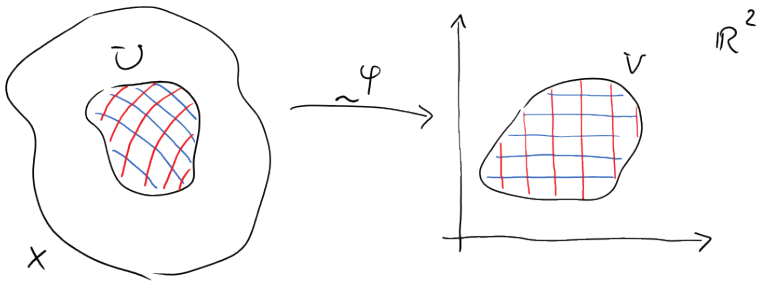
For this chapter we loosely refer to [**abate2011geometria**]. All manifolds in these notes are assumed to be Hausdorff and second-countable.

1.1 Smooth manifolds

A smooth manifold is a topological space which locally looks like an Euclidean space $\mathbb{R}^n$. To fix notation for the rest of the notes, we work out the precise definition behind this intuition.

Definition 1.1. A **chart** on a topological space M is a pair (U, ϕ) where U is an open subset of M and ϕ is an homeomorphism[1] from U to an open set $V \subseteq \mathbb{R}^n$, for suitable n.

A chart then brings the coordinates of $\mathbb{R}^n$ onto M:



Indeed, more often we'll not address directly the map ϕ but its components $\phi = (x^1, \dots, x^n)$. We'll usually say something like: let $(x^1, \dots, x^n)$ be coordinates around[2]

[1] An **homeomorphism** is an isomorphism of topological spaces, hence a continuous map whose inverse is again continuous.

[2] Usually this will imply, additionally, that $\phi(p) = 0$, so that we are taken $\operatorname{im}\phi$ to be a neighbourhood of the origin.

$p \in M$. This means we do not give much importance to U either, as long as we know it contains our point of interest.

Finally, notice that when two charts (U_1, ϕ_1) and (U_2, ϕ_2) overlap, then they are automatically "compatible" on their intersection, since $\phi^1 \circ \phi^2 - 1 : \phi_1(U_1 \cap U_2) \tilde{\to} \phi_2(U_1 \cap U_2)$ defines an homeomorphism on them. These maps are called **transition maps** and sew together the different local descriptions of the space M.

Definition 1.2. A **smooth manifold** is a topological space M together with a **smooth atlas**, i.e. a collection $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ of charts such that all the transition maps are $\mathcal{C}^\infty$-diffeomorphisms[3].

Remark 1.2.1. We stress the last word of the definition: as any two overlapping charts are "topologically comaptible", the smoothness of the manifold is ensued by the differentiability of the transition maps.

We conclude this brief introduction by noting that we rarely give an explicti atlas along with our manifolds. The reason is twofold:

1. for most spaces (e.g. spheres, tori, projective spaces), geometers worked out their construction in terms of an atlas once, and then it is understood we are using that;

2. we always assume the atlas to be **maximal**, that is, to contain every possible compatible chart (with a given starting set): as such a collection is clearly very large, it is impossible to give one explicitly.

1.2 Vector bundles

On a smooth manifold we are interested in studying so-called differential objects: vectors and forms, as well as smooth functions supported on the manifold.

Definition 1.3. A **map of smooth manifolds** $F : M \to N$ is a map from the manifold M to the manifold N which is locally represented by a differentiable function, i.e. for every $p \in M$, there exist a chart (U, ϕ) around p and a chart (V, ψ) around $F(p)$ such that $\psi \circ F \circ \phi^{-1}$ is a differentiable function in the usual sense.

The space of differentiable maps from M to $\mathbb{R}$ is denoted by $\mathcal{C}^\infty(M)$. It clearly is an $\mathbb{R}$-algebra, i.e. a ring whose addition makes it into an $\mathbb{R}$-module.

Definition 1.4. A **vector bundle** on a smooth manifold M is a map $\pi : E \to M$ from a smooth manifold E called the **total space** to the manifold M called the **base space** such that

[3] Hence homeomorphisms which are differentiable infinitely many times.

1. $\pi^{-1}(p)$ is a vector space for any p,
2. for each p, there is an open neighbourhood U of p and a diffeomorphism $\chi : \pi^{-1}(U) \xrightarrow{\sim} U \times \pi^{-1}(p)$ which is fiberwise an isomorphism of vector spaces and makes the following commute:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\chi} & U \times \pi^{-1}(p) \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array} \tag{1.2.1}$$

Intuitively, a vector bundle is a smooth assignment of vector spaces to each point of a manifold. A consequence of this is that every fiber $\pi^{-1}(p)$ has the same dimension r, called the **rank** of the bundle.

Definition 1.5. A **section** of a vector bundle $\pi : E \to M$ is any map $s : M \supseteq U \to E$ which is a left inverse of π on U, i.e.

$$\pi \circ s = \mathrm{id}_U. \tag{1.2.2}$$

If $U = M$, the section is said to be **global**.

A section of a vector bundle can be more easily depicted as a smooth assignment of vectors of E, where the vector at $p \in M$ is taken from the fiber there.

The most important vector bundle in differential geometry is the **tangent bundle** TM. Its construction is somewhat technical so we refer to [**abate2011geometria**], but intuitively it is the vector bundle whose fiber at $p \in M$ is the space of tangent vectors at p, suitably defined. Global sections of TM are called **vector fields** and the totality of them is denoted by $\mathfrak{X}(M)$.

Along with TM, we get the **cotangent bundle** T^*M, whose fibers are dual to those of TM. Global sections of T^*M are called **differential 1-forms** and the totality of them is denoted by $\Omega^1(M)$.

Finally, through fiberwise tensoring we get a **tensor bundle** $T^p_q M$ for each $p, q \in \mathbb{N}$. Global sections of T^h_k are called **h-covariant, k-contravariant tensors**, or simply **tensors of type** (h, k). Clearly $T^1M = TM$ and $T_1M = T^*M$. When after tensoring, we look at the skew-symmetric part of the tensors, we get an algebra called the **external algebra** of M. The product is denoted by $\wedge$, and is defined as

$$\xi_1 \wedge \ldots \wedge \xi_q = \sum_{\sigma \in S_q} (-1)^{sgn\sigma} \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(q)}, \quad \forall \xi_1, \ldots, \xi_q \in T^*M. \tag{1.2.3}$$

Skew-symmetric k-contravariant tensors are more commonly referred to as **differential k-forms**, and their collection is denoted by $\Omega^q(M)$. Skew-symmetric h-covariant tensors are more commonly referred to as **h-vectors** (or, more generally, **multivectors**) and their collection is denoted by $\mathfrak{X}^q(M)$.

1.2.1 Pullback and pushforward

Any smooth map $F : M \to N$ induces morphisms between the respective tangent and cotangent bundles. In this section, let p will designate a point of M and q a point of N.

Construction 1.6. For **vectors**, at each point p of M there's a well-defined map of fibers

$$dF_p : T_pM \to T_{F(p)}N \tag{1.2.4}$$

called the **differential of F at p**, whose intrinsic definition depends on the specific way the tangent bundle is defined. Anyway, once coordinates $(x^1, \dots, x^m)$ are fixed around p and coordinates $(y^1, \dots, y^n)$ are fixed around q, its expression becomes

$$dF_p \left(X^h|_p \partial_h|_p \right) = X^h|_p \, \frac{\partial F^k}{\partial x^h}|_{F(p)} \, \partial_k|_{F(p)} \tag{1.2.5}$$

where $\partial F^k / \partial x^h|_{F(p)}$ is the Jacobian of F at $F(p)$ computed with respect to the chosen coordinates[4].

Construction 1.7. The dual of dF_p, in the sense of linear algebra, is a corresponding fiberwise morphism between the cotangent bundles, but in the opposite direction. Recall that T_q^*N is the space of $\mathbb{R}$-valued functionals on T_qN. When $q = F(p)$, it's then natural to precompose any such functional with the differential of F at p to get a functional on T_pM. Diagrammatically, it is very intuitive:

$$\begin{array}{ccc} T_pM & \xrightarrow{dF_p} & T_{F(p)}N \\ & \searrow^{dF_p^*\xi} & \downarrow \xi \\ & & \mathbb{R} \end{array} \tag{1.2.6}$$

Hence

$$\begin{aligned} dF_p^* : T_{F(p)}^*N &\longrightarrow T_p^*M \\ \xi &\longmapsto \xi \circ dF_p \end{aligned} \tag{1.2.7}$$

Construction 1.8. When we want to bundle up all these fiberwise morphisms, we see that there's no problem for vectors:

$$\begin{aligned} dF : TM &\longrightarrow TN \\ (p, X_p) &\longmapsto (F(p), dF_p(X_p)) \end{aligned} \tag{1.2.8}$$

[4]This means that we should actually refer to the Jacobian of the local representation $\tilde{F}$ at the point $(x^1(p), \dots, x^m(p))$.

Conversely, dF_p^* is generally ill-defined when we replace p with $F^{-1}(q)$, so that to eliminate such concerns **we need to assume F to be a diffeomorphism**. Only in this case we can define

$$\begin{aligned} dF^* : T^*N &\longrightarrow T^*M \\ (q, \xi) &\longmapsto (F^{-1}(q), \xi \circ dF_{F^{-1}(q)}) \end{aligned} \tag{1.2.9}$$

However, we usually use vectors and covectors in the form of vector fields and differential forms. Thus we'd like to define a way to move sections along F.

Construction 1.9. For vector fields, this operation is called **pushforward**

$$\begin{aligned} F_* : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(N) \\ X &\longmapsto q \mapsto dF_{F^{-1}(q)}(X_{F^{-1}(q)}) \end{aligned} \tag{1.2.10}$$

Like dF^*, this map is well defined if and only if F is a diffeomorphism.

Construction 1.10. Conversely, the analogous map for differential forms, called the **pullback** is now well defined everytime:

$$\begin{aligned} F^* : \Omega^1(N) &\longrightarrow \Omega^1(M) \\ \omega &\longmapsto (p, X_p) \mapsto \omega_{F(p)}(dF_p(X_p)) \end{aligned} \tag{1.2.11}$$

Construction 1.11. We can extend both kinds of maps to *homogeneous* tensors by having them act componentwise: for example, the last becomes

$$\begin{aligned} F^* : \Omega^k(N) &\longrightarrow \Omega^k(M) \\ \omega_1 \wedge \ldots \wedge \omega_k &\longmapsto F^*\omega_1 \wedge \ldots \wedge F^*\omega_k. \end{aligned} \tag{1.2.12}$$

Notice the same restrictions as before apply. We stress it doesn't make sense to speak about pullback/pushforward of *mixed* tensors since mixed tensors also have mixed variancy of the induced maps (you'd have to pullback one part of the tensor and pushforward another); unless $M = N$.

1.2.2 Distributions

Definition 1.12. A k-dimensional distribution D on a smooth manifold M is a subset of TM such that $D_p = D \cap T_pM$ is a key dimensional subspace of T_pM for each $p \in M$.

The most common way to define a distribution on a manifold its through a set of generators.

Definition 1.13. The distribution generated by the vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$ is given pointwise by

$$D_p = \langle X_1|_p, \ldots, X_k|_p \rangle. \tag{1.2.13}$$

Definition 1.14. We say a k-dimensional distribution D is **smooth** if, for any point $p \in M$, there exists an open neighbourhood U of p such that D restricted to U can be generated by k vector fields of $\mathfrak{X}(M)$. Such a set of vector fields is called a **local frame** for D on U.

Evidently, a smooth distribution is again a vector bundle on M. Hence we can consider "D-valued vector fields", which are just section of D, i.e. vector fields with values in D_p at each point $p \in M$.

Definition 1.15. A distribution is called **involutive** if $\mathfrak{X}_D(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, i.e. if $\mathfrak{X}_D(M)$ is the Lie bracket of two sections of D is again a section of D.

The most useful kind of distributions are those giving or coming from **foliations**, i.e. partitions of M into smooth submanifolds[5] called **leaves**. Then any foliation clearly induce a distribution given by the tangent spaces of the leaves. It's then natural to ask when the converse is true.

Definition 1.16. An **integral submanifold** of a smooth distribution D is a submanifold $S \subseteq M$ such that $TS = D$ on S.

Definition 1.17. A smooth distribution D is said to be **integrable** if every point of M is contained in an integral submanifold of D.

Definition 1.18. Let D be a k-dimensional smooth distribution on an n-dimensional manifold M. A chart (U, ϕ) is **flat** for D if

1. $\phi(U) = V \times W$ where V is open in $\mathbb{R}^k$ and W is open in $\mathbb{R}^{n-k}$,
2. $(\partial_1, \ldots, \partial_k)$ is a local frame for D on U.

A distribution is called **completely integrable** if M admits an atlas of flat charts for D.

To say a chart on U is flat for D then means its integral submanifolds cut U in directions parallel to the coordinates on U, i.e. every set of the form $\{x \in U \mid x^{k+1} = c^1, \ldots, x^n = c^{n-k}\}$ for a point $(c_1, \ldots, c_{n-k}) \in V$ (defined as above) is an integral submanifold of D on U. The first bullet point in the definition says that the coordinates on U can be naturally partitioned in "parallel" directions (contained and generating the integral submanifolds) and "transverse" directions ("orthogonal" to the integral submanifolds).

The concepts of involutive, integrable and completely integrable smooth distribution are related by the following theorems:

Proposition 1.19. *Any smooth, completely integrable distribution is integrable.*

[5] In a **regular** folations, leaves have all the same dimension, otherwise the foliation is called **singular**.

Proposition 1.20. *Any smooth integrable distribution is involutive.*

Theorem 1.21 (Frobenius, [**abate2011geometria**])**.** *Any smooth involutive distribution is completely integrable.*

While the first two are relatively straightforward to prove, the third one is more involuted and technical.

$$\text{completely integrable} \xrightarrow{\textit{Prop. 1.20}} \text{integrable} \xrightarrow{\textit{Prop. 1.19}} \text{involutive} \qquad (1.2.14)$$
$$\text{involutive} \xrightarrow{\textit{Th. 1.21}} \text{completely integrable}$$

In the end, integrable, completely integrable and involutive mean the same thing.

1.3 Exterior derivative

Definition 1.22. The **exterior derivative** is the unique family of *graded* $\mathbb{R}$*-antiderivations* $d : \Omega^p(M) \to \Omega^{p+1}$ extending the differential defined by Construction 1.8, i.e. satisfying the following axioms:

1. It is $\mathbb{R}$-linear,
2. For any smooth function $f \in \mathbb{C}^\infty(M) = \Omega^0(M)$, d is the global differential of f,
3. It is a boundary map:
$$d^2 = 0, \qquad (1.3.1)$$
4. It is a graded antiderivation:
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta. \qquad (1.3.2)$$

Existence and uniqueness can be easily proved by building up from the first axiom using the other two. In particular, if $\omega = \omega_{i_1,\dots,i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$, then

$$d\omega = \frac{\partial \omega_{i_1,\dots,i_p}}{\partial x^{i_0}} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \qquad (1.3.3)$$

where indices ranges over the dimension of the ambient manifold, and summation is implied.

Definition 1.23. A form ω is said to be **closed** if $d\omega = 0$. It's said to be **exact** if $\omega = d\alpha$ for some form α.

Clearly, since $d^2 = 0$, **any exact form is closed**. The converse is not always true and this is the starting point of de Rham cohomology theory, which we'll see soon.

Lemma 1.24. *For any $p \in \mathbb{N}$ and $\alpha \in \Omega^p(M)$, for every k,*

$$d(d\alpha)^{\wedge k} = d(\underbrace{d\alpha \wedge \ldots \wedge d\alpha}_{k \text{ times}}) = 0 \tag{1.3.4}$$

that is, exterior powers of exact forms are closed.

Proof. We proceed by induction on k. Indeed, for $k = 1$ the statement is true by definition of d. Assume it is true for $k-1$, then

$$\begin{aligned} d\left((d\alpha)^{\wedge k}\right) &= d\left(d\alpha \wedge (d\alpha)^{\wedge k-1}\right) \\ &= \cancel{d^2\alpha} \wedge (d\alpha)^{k-1} + d\alpha \wedge \cancel{d\left(d\alpha^{\wedge k-1}\right)} = 0 \end{aligned} \tag{1.3.5}$$

□

Corollary 1.24.1. *Exterior powers of exact forms are again exact.*

Proof.

$$(d\alpha)^{\wedge k} = d\alpha \wedge (d\alpha)^{\wedge k-1} - \alpha \wedge d(\underbrace{(d\alpha)^{\wedge k-1}}_{=0}) = d\left(\alpha \wedge (d\alpha)^{\wedge k-1}\right). \tag{1.3.6}$$

□

So as we noted above, any exact form is closed, meaning the ds form the coboundary maps for a cochain complex of differential forms:

$$\ldots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \ldots \tag{1.3.7}$$

This is called the **de Rham complex** of M, and we can naturally associate a cohomology theory to it by taking homologies: let $\ker d_p = Z^p(M)$ be the susbpace of closed p-forms (p-**cocycles**) and $\operatorname{im} d_{p-1} = B^p(M)$ the subspace of exact p-forms (p-**coboundaries**). Then the p**-th de Rham cohomology** is defined as

$$H^p_{dR}(M) = Z^p(M)/B^p(M). \tag{1.3.8}$$

So de Rham cohomology encodes the failure of closed p forms to be exact. Surprisingly, although its definition is intrinsecally dependent on the smooth structure of M, **de Rham cohomology is a topological invariant**, meaning

$$M \underset{\mathsf{Top}}{\cong} N \implies H^\bullet_{dR}(M) \underset{\mathsf{Vec}}{H}{}^\bullet_{dR}(N) \tag{1.3.9}$$

This is proven by proving the equivalence of the cohomology to the (purely topological) singular homology.

1.3.1 Singular Homology

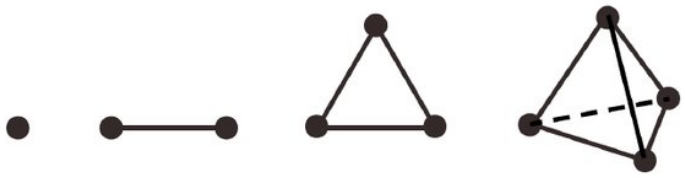
We will now describe an homology theory which is entirely topological. Indeed, in this construction there's no need to assume M to be a smooth manifold, any topological space will suffice.

Definition 1.25. A **singular r-simplex** is any set of the form

$$\Delta^r = [P_0, \dots, P_r] \tag{1.3.10}$$

where $P_0, \dots, P_r \in \mathbb{R}^r$ are affinely independent points and $[-]$ denotes the convex closure of its argument.

Basically simplices are a lower and higher dimensional generalization of triangles: Δ^0 is a point, Δ^1 a segment, Δ^2 a triangle, Δ^3 a tetrahedron, and so on:



Definition 1.26. A **singular r-simplex in a topological space** M is a continuous map $\sigma^r : U \to M$ where U is an open neighbourhood of a $\Delta^r \subseteq \mathbb{R}^r$.

By abuse of terminology, often we'll call r-simplex in M the image of σ^r.

Construction 1.27. Using the family of singular r-simplices in a topological space M as generators, we can build a free R-module $C_r(M; R)$ for all commutative rings R (usually R is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.). An element of such module is called an **r-chain** and it is a formal sum of r-simplices in M with coefficients taken from R.

Definition 1.28. The **boundary** of an r-simplex σ^r of M is the $(r-1)$-chain $\partial\sigma^r$ defined as

$$\partial\sigma^r := \sum_{k=0}^{r} (-1)^k [P_0, \dots, \hat{P}_k, \dots, P_r] \tag{1.3.11}$$

where $\hat{-}$ means we are skipping the term underneath.

The terms, without sign, appearing in the summation defining $\partial\sigma^r$ are called **faces** of the simplex. Thus the boundary of a simplex is the alternating sum of its faces.

By R-linear extension, the boundary construction induces a (family of) map(s) $\partial : C_r(M; R) \to C_{r-1}(M; R)$, called the **boundary map(s)**.

Lemma 1.29. *The boundary maps are boundary maps in the sense of homological algebra, i.e.*

$$\partial_{r-1}\partial_r = 0 \tag{1.3.12}$$

Proof. For conciseness sake, define

$$[\hat{h}, \hat{k}] = [p_0, \dots, \hat{p_h}, \dots, \hat{p_k}, \dots, p_r]. \tag{1.3.13}$$

Notice to show the claim it suffices to prove it for a single, arbitrary r-simplex, as they generate $C_r(M; R)$. Then it is just a matter of computation. We just need to be careful substituting the second ∂ as indices for it have to be properly shifted:

$$\begin{aligned}\partial(\partial[p_0, \dots, p_r]) &= \sum_{k=0}^{r}(-1)^k \partial[\hat{k}] \\ &= \sum_{k=0}^{r}(-1)^k \left(\sum_{h=0}^{k-1}(-1)^h[\hat{h}, \hat{k}] + \sum_{h=k+1}^{r}(-1)^{h-1}[\hat{k}, \hat{h}]\right) \\ &= \sum_{0\le h<k\le r}(-1)^{k+h}[\hat{h}, \hat{k}] - \sum_{0\le k<h\le r}(-1)^{k+h}[\hat{h}, \hat{k}] = 0,\end{aligned} \tag{1.3.14}$$

The very last equality is proven by swapping the indices of one summation (both will do) – as its apparent this operation doesn't add nor remove any term; and simply noticing we then get two equal things subtracted, hence zero. □

Chains $\sigma^r \in C_r$ such that $\partial\sigma^r = 0$ are called **cycles** ($\sigma^r \in Z^r(M; R) = \ker \partial_r$ in symbols), while when $\partial\delta = \sigma$ for some $\delta \in C_{r+1}$ we speak of **boundaries** (in symbols $\sigma^r \in B^r(M; R) = \operatorname{im} \partial_{r+1}$).

Definition 1.30. The **r-th singular homology of M with coefficients in R** is

$$H_r(M; R) := \ker \partial_r / \operatorname{im} \partial_{r+1} = Z^r(M; R)/B^r(M; R) \tag{1.3.15}$$

1.3.2 de Rham's Theorem

Unlike de Rham cohomology, singular homology is a topological construction. On the other hand, on a smooth manifold M we could have defined a "smooth version" by asking the maps $\sigma^r : U \to M$ to be not just continuous but smooth. Eventually, the two constructions turn out to be equivalent so we don't distinguish them in the following notation, but it's important to point out this is where the bridge between topological and *differential* structure is crossed.

The aim of this section is then to prove smooth singular homology can be computed as well using de Rham cohomology, thus an easier and more tame machinery. To link the

two, we'll use the natural pairing of differential forms and smooth simplicial chains given by integration:

$$\begin{aligned}(-,-) : C^r(M;\mathbb{R}) \times \Omega^r(M) &\longrightarrow \mathbb{R} \\ (\sum_{i=1}^{n} \alpha_n \sigma_n^r, \omega) &\longmapsto \int_C \omega := \sum_{i=1}^{n} \alpha_n \int_{\sigma_n^r} \omega\end{aligned} \tag{1.3.16}$$

Notice we fixed $R = \mathbb{R}$ for the coefficients of the homology as to get reals out of the integral.

A classical result links homological properties of differential forms to those of smooth simplicial chains:

Theorem 1.31 (Stokes)**.** *Let $\omega \in \Omega^r(M)$ and C be a smooth $(r+1)$-chain. Then*

$$\int_{\partial C} \omega = \int_C d\omega \tag{1.3.17}$$

Lemma 1.32. *The integral pairing passes to homology and cohomology, i.e. the pairing*

$$[-,-] : H_r(M;\mathbb{R}) \times H^r_{dR}(M) \longrightarrow \mathbb{R} \tag{1.3.18}$$

defined by

$$[\overline{C}, \overline{\omega}] = (C, \omega) \tag{1.3.19}$$

is a well-defined bilinear form.

Proof. Choose $C' \in C^{r+1}(M;\mathbb{R})$ and $\omega' \in \Omega^{r-1}$, then

$$C + \partial C' \in \overline{C}, \quad \omega + d\omega' \in \overline{\omega} \tag{1.3.20}$$

so that to show well-definedness we simply need to show the result does not depend on C' and ω'. Indeed, applying the extended definition of the integral, Stokes' Theorem and recalling $\partial C = 0$ and $d\omega = 0$ by definition of (co)homology, we get:

$$\begin{aligned}[\overline{C}, \overline{\omega}] &= (C + \partial C', \omega + d\omega') \\ &= \int_C \omega + \int_{\partial C'} \omega + \int_C d\omega' + \int_{\partial C'} d\omega' \\ &= \int_C \omega + \int_{C'} d\omega + \int_{\partial C} \omega' + \int_{\partial^2 C'} \omega' = (C, \omega).\end{aligned} \tag{1.3.21}$$

Bilinearity is then evident. □

Theorem 1.33 (de Rham)**.** *The bilinear form $[-,-]$ defined above is non-degenerate, hence it induces an isomorphism*

$$H_r(M;\mathbb{R}) \underset{\mathsf{Vec}}{\cong} H^r_{dR}(M) \tag{1.3.22}$$

Theorem 1.34 (Poincarè's duality)**.** *Let M be a* compact *orientable manifold of dimension m. The map*

$$\begin{aligned} \langle -,- \rangle : H^r_{dR}(M) \times H^{m-r}_{dR}(M) &\longrightarrow \mathbb{R} \\ (\overline{\omega}, \overline{\eta}) &\longmapsto \int_M \omega \wedge \eta \end{aligned} \tag{1.3.23}$$

is a well-defined, non-degenerate bilinear form, implying $H^r_{dR}(M) \underset{\mathsf{Vec}}{\cong} H^r_{dR}(M)$.

Proof sketch. For a compact orientable manifold we can produce a triangulation of M, that is, an m-chain homeomorphic to M itself. By compactness of M, the chain must be a cycle since $\partial M = 0$. However it can't be the boundary of anyhting bigger since it is of maximal dimension. Then by de Rham's again,

$$\int_{\Delta_m} \omega \wedge \eta \neq 0 \tag{1.3.24}$$

proving the theorem. □

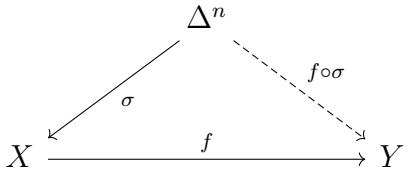
Definition 1.35. The **Euler characteristic** of M is the alternating sum of the dimensions of the homology groups of M:

$$\chi_M = \sum_{n=0}^{\dim M} (-1)^n \dim H_n(M) \tag{1.3.25}$$

Corollary 1.35.1. *Any odd-dimensional compact orientable smooth manifold has $\chi_M = 0$.*

Proof. By de Rham's Theorem and Poincarè's duality. □

Induced maps. Singular homology and de Rham cohomology are actually a family of functors, in the sense that maps between manifolds induce also maps between their cohomology groups in a way that is well-behaved with respect to composition. Indeed, if $f : X \to Y$ is a smooth map, it also brings singular simplices on X to singular simplices on Y:



(1.3.26)

thereby producing a morphism $f^* : H_n(X; R) \to H_n(Y; R)$. The same thing happens for cohomology groups, although with a different variancy, by using the pullback of forms (Construction 1.10), so we get a morphism $f^\sharp : H^n_{dR}(Y) \to H^n_{dR}(X)$.

When $f_0, f_1 : M_1 \to M_2$ are smooth maps, to say they're **smoothly homotopic** means there is a "smooth map interpolating between the two", i.e. a smooth map

$$h : M_1 \times [0,1] \to M_2, \ (p,t) \mapsto h_t(p) \tag{1.3.27}$$

such that

$$h_0 = f_0, \quad h_1 = f_1. \tag{1.3.28}$$

Then it can be shown that **the whenever f_0 and f_1 are homotopic, they induce the same homeomorphisms $f_0^\sharp$ and $f_1^\sharp$ on the de Rham cohomology groups**.

1.4 Interior product

Definition 1.36 (Interior product)**.** Let $X \in \mathfrak{X}(M)$. The **interior product** with X of a $(p+1)$-differential form ω is the unique p-differential form defined by

$$\iota_X\omega(X_1, \ldots, X_p) = \omega(X, X_1, \ldots, X_p), \quad \forall X_1, \ldots, X_p \in \mathfrak{X}(M). \tag{1.4.1}$$

Property 1.37. *Let $X \in \mathfrak{X}(M)$, $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$. Then the following hold:*

1. *ι_X is a boundary map:*

$$\iota_X^2 = 0. \tag{1.4.2}$$

2. *ι_X is an **graded antiderivation**:*

$$\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^p \omega \wedge (\iota_X\eta). \tag{1.4.3}$$

3. *ι_X commutes with the Lie derivative along X:*

$$\mathcal{L}_X\iota_X = \iota_X\mathcal{L}_X. \tag{1.4.4}$$

4. *Cartan's magic formula:*

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega). \tag{1.4.5}$$

1.4.1 Integration

Definition 1.38. A **volume form** on the n-dimensional manifold M is given by an n-differential form

Chapter 2

Symplectic geometry

Symplectic and Poisson structures came from the structures observed in analytical mechanics (originated from the need of studying complex dynamical systems in exact fashion, like motions of planets, oscillations of many particles systems, etc.).

2.1 Symplectic forms on vector spaces

2.1.1 Skew-symmetric bilinear maps

Definition 2.1. Let V be an $\mathbb{R}$-vector space with $\dim V = m$. Let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. We say Ω is **skew-symmetric** iff

$$\Omega(v, w) = -\Omega(w, v), \qquad \forall v, w \in V.$$

Theorem 2.2. *Let V be a finite-dimensional $\mathbb{R}$-vector space. Let Ω be a skew-symmetric, bilinear map on V. Then there exists a basis*

$$B = \{u_1, \ldots, u_k, e_1, \ldots, e_n, f_1, \ldots, f_n\} \tag{2.1.1}$$

of V such that

1. $\Omega(u_i, v) = 0 \quad \forall v \in V, \forall i = 1, \ldots, k,$
2. $\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \quad \forall i, j = 1, \ldots, n,$
3. $\Omega(e_i, f_j) = \delta_{ij} \quad \forall i, j = 1, \ldots, n,$

Proof. Let

$$U := \{u \in V \,|\, \Omega(u, v) = 0 \ \forall v \in V\}. \tag{2.1.2}$$

This is a non-empty set since $0 \in U$ and it is the kernel of $\tilde{\Omega}$ hence a linear subspace of V. Therefore if we let $u_1, \ldots, u_k$ be a basis for U (letting $k = \dim U$), the first item is proved.

We'll now build by induction a decomposition

$$V = U \oplus \langle e_1, f_1 \rangle \oplus \dots \langle e_n, f_n \rangle. \tag{2.1.3}$$

Let W be a complement of U (i.e. a subspace of V such that $U \oplus W = V$). Unless $k = \dim V$ (in which case the theorem is vacuously proved), $W \neq 0$. Then we can select a non-zero vector e_1 from W, for which there has to be at least another $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Otherwise $\Omega(e_1, w) = 0$ for every $w \in W$ implying $\Omega(e_1, v) = 0$ for any $v \in V$, since when $v \in U$ we already know $\Omega(e_1, v) = -\Omega(v, e_1) = 0$ by definition of U. But $U \cap V = 0$ and $e_1 \neq 0$ exclude this possibility.

Thus we choose $e_1, f_1 \in W$ such that, up to rescaling, $\Omega(e_1, f_1) = 1$. Notice, moreover, that the two vectors are linearly independent, else Ω would be zero on them by skew-symmetry. Thus we can add e_1 and f_1 to our basis, let $W_1 = \langle e_1, f_1 \rangle$, and if $W_1 = W$ we are done. Conversely, define

$$W_1^\Omega = \{w \in W \,|\, \Omega(w, w_1) = 0 \ \forall w_1 \in W_1\} \tag{2.1.4}$$

which is a sort of "Ω-orthogonal space" of W_1 inside W. We can easily prove that $W_1 \cap W_1^\Omega = \{0\}$: let $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$, then $0 = \Omega(v, e_1) = -b$ and $0 = \Omega(v, f_1) = a$, therefore $v = 0$. Now we will prove that $W = W_1 \oplus W_1^\Omega$. Suppose $v \in W$ has $\Omega(v, e_1) = c$, $\Omega(v, f_1) = d$; then $v = (-cf_1 + de_1) + (v + cf_1 - de_1)$ where $(-cf_1 + de_1) \in W_1$ and $(v + cf_1 - de_1) \in W_1^\Omega$.

Let's proceed further. As before, we can select a non-zero e_2 from W_1^Ω for which there must exist a $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) = 1$ up to rescaling, or e_2 would fall in U since $U \oplus W_1 \oplus W_1^\Omega = V$. We understand then our process can go on as long as $W_n \neq W$, giving us a basis which satisfies (2) because each e_i (resp. f_i) lives in the "Ω-orthogonal space" of the other e_i (resp. f_i), and (3) for the same reason plus the fact we make a suitable rescaling at each step. □

Corollary 2.2.1. *The form ω, with respect to the basis B, is represented as the following block matrix*

$$\begin{pmatrix} 0_k & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}. \tag{2.1.5}$$

Remark 2.2.1.

1. The above basis B is not unique.

2. The subspace U (and in particular its dimension) does not depend on the choice of the basis. It's an invariant of (V, Ω).

3. Since $k + 2n = m$, n is also a numerical invariant of (V, Ω).

2.1.2 Symplectic vector spaces

Let V be an n-dimensional $\mathbb{R}$-vector space. Let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. There is an induced linear map $\tilde{\Omega} : V \to V^*$ defined by

$$\tilde{\Omega}(v)(w) := \Omega(v, w). \tag{2.1.6}$$

In other words, we can say that Ω is an element of $V^* \otimes V^* \cong \mathrm{Hom}(V \otimes V, \mathbb{R}) \cong \mathrm{Hom}(V, V^*)$. We have

$$\otimes \ker \tilde{\Omega} = \{u \in V \,|\, \Omega(u, v) = 0 \,\forall v \in V\} = U \tag{2.1.7}$$

Definition 2.3. A **symplectic form** is a non-degenerate, skew-symmetric bilinear form. A vector space V equipped with a symplectic form on V is called a **symplectic vector field**.

Remark 2.3.1. The above theorem actually describes the Jordan decomposition for the map $\tilde{\Omega} : V \to V^*$. Obviously, if we considered the complex field this map could be diagonalized. In addition, the definition of symplectic form requires that the Jordan decomposition of $\tilde{\Omega}$ has no zero-eigenvalue blocks.

Remark 2.3.2. Notice the analogy to an inner-product space, where skew-symmetry of the form is replaced by simple symmetry. Indeed, some results generalize as they depend only on the non-degeneracy of the bilinear form, for instance partial application of the form Ω fixes a canonical isomorphism of V and its dual:

$$\begin{aligned} \tilde{\Omega} : V &\longrightarrow V^* \\ v &\longmapsto \Omega(v, -); \end{aligned} \tag{2.1.8}$$

On the other hand, the theory of inner product spaces and that of symplectic spaces start to diverge very early on.

While the decomposition of V we found in the previous proof is highly non-canonical (we made a lot of arbitrary choices!), the dimensions of the decomposition depend solely on Ω. In particular, notice that Ω is non-degenerate iff $k = 0$, hence we see immediately that, in light of Theorem 2.2, **a symplectic vector space is always even-dimensional**.

Equation (2.1.5) prescribes we can always find a "nice" basis for V in which Ω can be represented by a **symplectic matrix**:

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.1.9}$$

Even if, as noted before, the choice of such a basis is far from unique, we'll refer to it as a *canonical* basis.

Definition 2.4. A subspace W of a symplectic vector space (V, Ω) is said to be **symplectic** as well if $(W, \Omega|_W)$ is a symplectic vector space. If instead $\Omega|_W = 0$, the subspace is called **isotropic** and its Ω-orthogonal is then called **coisotropic**. When a subspace is both, it's named **Lagrangian**. and its Ω-orthogonal is then called **coisotropic**. When a subspace is both, it's named **Lagrangian**.

Example 2.1. Let (V, Ω) be a symplectic vector space with $V = \langle e_1, \dots, e_n, f_1, \dots, f_n \rangle$ where $e_1, \dots, e_n, f_1, \dots, f_n$ is a canonical basis. Then

1. $W = \langle e_{i_1}, \dots, e_{i_k}, f_{i_1}, \dots, f_{i_k} \rangle$ where $k \leq n$, $1 \leq i_s \leq n$ and $i_s \neq i_{s'}$ if $s \neq s'$, is a symplectic subspace of V.

2. $W = \langle e_{i_1}, \dots, e_{i_k} \rangle$ where $k \leq n$, is a isotropic subspace of V

Thus the decomposition (2.1.3) (when $U = 0$, so Ω is non-degenerate) is a decomposition of V in minimal symplectic subspaces.

Definition 2.5. A **symplectomorphism** is a linear map $\phi : (V, \Omega) \to (V', \Omega')$ between symplectic vector spaces (V, Ω) and (V', Ω') such that

$$\phi^* \Omega' = \Omega \tag{2.1.10}$$

where $\phi*$ denotes the pull-back (recall $(\phi^*\Omega')(u, v) := \Omega'(\phi(u), \phi(v))$). If such a ϕ exists, (V, Ω) and (V', Ω') are called **symplectomorphic**.

This is equivalent to requiring that the following diagram commutes:

$$\begin{array}{ccc} V \times V & \xrightarrow{\Omega} & \mathbb{R} \\ \downarrow{\scriptstyle \phi} \quad \downarrow{\scriptstyle \phi} & & \downarrow \\ V' \times V' & \xrightarrow{\Omega} & \mathbb{R} \end{array} \tag{2.1.11}$$

Remark 2.5.1. Notice that an homomorphism $\phi : V \xrightarrow{\sim} V'$ between symplectic spaces is forced to be an isomorphism by the compatibility condition with the symplectic forms.

Remark 2.5.2. In the case of symplectic vector spaces, Theorem 2.2 says that every symplectic vector space of dimension $2n$ is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$:

$$(V, \Omega) \underset{\mathsf{Symp}}{\cong} (\mathbb{R}^{2n}, \Omega_0) \tag{2.1.12}$$

where Ω_0 is the symplectic matrix when seen from the canonical basis of $\mathbb{R}^{2n}$. Notice that for vector spaces equipped with a non-degenerate *symmetric* form (**inner-product spaces**) the situation is different, since on $\mathbb{R}^n$ we can define at least n different metrics

differing by signature, i.e. any real inner-product space (V, g) is isomorphic to a "canonical" space $(\mathbb{R}^n, g_0)$ of the form

$$g_0 = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} \tag{2.1.13}$$

for some value of $k \neq n$.

2.2 Symplectic manifolds

Endowing the tangent bundle of a manifold with a symplectic structure, we get the differential analogue of a symplectic vector space.

Definition 2.6. Let M be a smooth, m-dimensional manifold. The 2-form $\omega \in \Omega^2(M)$ is a **symplectic form** if $d\omega = 0$ (it's closed) and $\omega|_p$ is symplectic (on T^*M) $\forall p \in M$. A **symplectic manifold** is a pair (M, ω) of a smooth manifold M and a symplectic form ω on it.

Therefore at each point $p \in M$ the tangent space T_pM is a symplectic vector space with symplectic form ω_p.

Remark 2.6.1. If $\omega \in \Omega^2(M)$ is symplectic then $\dim M = 2n$.

Example 2.2.

1. Any Euclidean space $\mathbb{R}^{2n}$ can be equipped with the following form

$$\omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i \tag{2.2.1}$$

where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are linear coordinates on $\mathbb{R}^{2n}$. With this definition ω_0 is, pointwise, the form Ω_0 discussed previously. Moreover, it is closed as

$$\begin{aligned} d\omega_0 &= \sum_{i=1}^{n} d(dx^i \wedge dy^i) \\ &= \sum_{i=1}^{n} d^2x^i \wedge y^i - dx^i \wedge d^2y^i = 0 \end{aligned} \tag{2.2.2}$$

2. The 2-sphere is a symplectic manifold. Indeed, consider its immersion in $\mathbb{R}^3$ defined as $S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = \langle p, p \rangle = 1\}$. Then

$$T_pS^2 = \{v \in \mathbb{R}^3 \mid \langle v, p \rangle = 0\} = \langle p \rangle^{\perp} \tag{2.2.3}$$

can be made into a symplectic vector space by means of

$$\omega|_p(u, v) := \langle p, u \times v \rangle, \quad \forall u, v \in T_pS^2. \tag{2.2.4}$$

which trivially defines a differential form $\omega \in \Omega^2(S^2)$. ω is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and, for instance, $v = u \times p$. Finally, since $\dim S^2 = 2$, the form ω is necessarily closed.

Since $\dim T_pM = \dim M$, M inherits the condition on the dimension of T_pM: M must be even-dimensional. However this time, even-dimensionality is not a sufficient condition for a manifold to admit symplectic forms. A further necessary condition is given by the following proposition.

Proposition 2.7. *Let (M, ω) be a symplectic manifold, with $\dim M = 2n$. Then the n-th exterior power of ω is a volume form, i.e. $\omega^{\wedge n} \not\equiv 0$ everywhere on M.*

Proof. The rank of $\omega^{\wedge n} \in \Omega^{2n}(M)$ is $2n$, so the claim makes sense. Choose $p \in M$ now, and fix a local frame $E_1, \dots, E_n, F_1, \dots, F_n$ around p in such a way that it is in p a canonical basis as per Theorem 2.2. Then:

$$\omega_p^{\wedge n}(E_1|_p, \dots, E_n|_p, F_1|_p, \dots, F_n|_p) = \omega_p(E_1|_p, F_1|_p) \cdot \dots \cdot \omega_p(E_n|_p, F_n|_p) = 1 \tag{2.2.5}$$

thus the form is everywhere positive. □

Corollary 2.7.1. *If M is symplectic, it is orientable.*

The obvious remark is then that non-orientable manifolds can't be symplectic even in the even-dimensional case; e.g. Möbius strip and the projective plane do not admit a symplectic structure.

Other obstructions are given by topology. For example

Theorem 2.8. *The 2-sphere is the only sphere admitting a symplectic structure.*

Proof. In Example 2.2 we exhibited a symplectic structure for S^2, so only we need to prove any S^{2n} with $n \geq 2$ does not admit one. Let $\omega \in \Omega^2(S^{2n})$ be a closed form. Recall $H_d^2 r(S^{2n}) = \{0\}$, as the only non-vanishing homologies (hence cohomologies by de Rham's Theorem) are the bottom and the top ones and we assumed $n > 1$. By definition of cohomology, this means all closed 2-forms are automatically exact, implying there exists an $\eta \in \Omega^1(S^{2n})$ such that $d\eta = \omega$. Hence, using Corollary 1.24.1 and Stokes' Theorem:

$$\int_{S^{2n}} \omega^{\wedge n} = \int_{S^{2n}} d(\eta \wedge \omega^{\wedge(n-1)}) = \int_{\partial S^{2n}} \eta \wedge \omega^{\wedge(n-1)} = 0 \tag{2.2.6}$$

Then, by Proposition 2.7, it must be $\omega = 0$ thus not symplectic. □

Remark 2.8.1. Notice **the proof can be generalized to any compact manifold with vanishing second cohomology**. Moreover, it shows a symplectic form cannot be exact on a compact manifold.

Definition 2.9. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds of the same dimension. A diffeomorphism $\phi : M_1 \to M_2$ is called a **symplectomorphism** if

$$\phi^* \omega_2 = \omega_1 \tag{2.2.7}$$

Remark 2.9.1. If N is a submanifold of a symplectic manifold (M, ω) and $(N, \omega|_N)$ is symplectic then N is a **symplectic submanifold** of M. In particular a symplectomorphism $\phi : M \to N$ is an embedding.

Remark 2.9.2. As with symplectic vector spaces, a symplectic form induces a natural isomorphism of $\mathfrak{X}(M)$ with $\Omega^1(M)$, which is the extension of the pointwise isomorphism $\tilde{\omega}_p : T_pM \to T_p^*M$, defined as before by partial application:

$$\begin{aligned} \omega : \mathfrak{X}(M) &\longrightarrow \Omega^1(M) \\ X &\longmapsto \iota_X \omega \end{aligned} \tag{2.2.8}$$

The inverse map is

$$\omega^{-1} : \Omega^1(M) \longrightarrow \mathfrak{X}(M) \tag{2.2.9}$$

and it is defined as to satisfy

$$\omega^{-1}(\iota_X \omega) = X. \tag{2.2.10}$$

2.2.1 The canonical symplectic structure on the cotangent bundle

The chief example of symplectic manifolds are cotangent bundles of smooth manifolds. In fact, there's a natural/canonical way to equip $M = T^*X$ with a symplectic structure. Notice that $\dim T^*X = 2 \dim X$ so that, in principle, we can expect to assign a symplectic structure to arbitrary cotangent bundles. Clearly, as we've seen, dimensionality isn't the only obstruction so we need to exhibit a symplectic form.

The construction of the symplectic structure exploits a particular, naturally defined, 1-form on T^*X. The symplectic form will then be it's exterior derivative.

Construction 2.10. Let $\pi : T^*X \to X$ be the bundle projection. Notice that a differential 1-form on T^*X is a function $T^*X \to T^*T^*X$ (with the additional condition of being compatible with the fibers, more on this later) which is exactly the type of function $d\pi^*$ is. If we unpack its definition from Construction 1.8, we get

$$\begin{aligned} d\pi^* : T^*X &\longrightarrow T^*T^*X \\ (x, \xi) &\longmapsto (\pi^{-1}(x),\ \xi \circ d\pi_{\pi^{-1}(x)}) \end{aligned} \tag{2.2.11}$$

So the definition fails. We already knew this: we remarked that dF^* is well-defined if and only if F is a diffeomorphism, which is certainly not the case for π. On the other

hand, we can try to fix this pathology by finding another way to choose a "canonical" element from $\pi^{-1}(x)$ in the mapping expression of $d\pi^*(x,\xi)$. Indeed, since π is surjective, that's the only obstruction! And we clearly see that those $\pi^{-1}(x)$ can be unambiguously resolved by simply using the argument of the sought function. So we define the mapping

$$\begin{aligned} \alpha : T^*X &\longrightarrow T^*T^*X \\ (x,\xi) &\longmapsto ((x,\xi), \xi \circ d\pi_{(x,\xi)}) \end{aligned} \tag{2.2.12}$$

or, using the fiberwise duals (Construction 1.7):

$$\alpha(x,\xi) = d\pi^*_{(x,\xi)}\xi. \tag{2.2.13}$$

This is clearly well-defined as the domain of $d\pi^*_{(x,\xi)}$ is precisely $T^*_{(x,\xi)}T^*X$. The 1-form we just defined is called **Liouville form** or **tautological one-form**.

The name *tautological* comes from the following property: for every $\mu \in \Omega^1(X)$,

$$\mu^*\alpha = \mu \tag{2.2.14}$$

where μ^* is the pullback of μ when seen as smooth map $X \to T^*X$.

Let's compute what this form looks like, fixing coordinates on the involved spaces. Notice, first, that coordinates on X suffice to fix relevant coordinates on both T^*X and T^*T^*X, by using the induced reference frames as bases for the fibers:

Space	Local frame	Coordinates
X	(does not apply)	$(x^1,\dots,x^n)$
T^*X	$(dx^1,\dots,dx^n)$	$(x^1,\dots,x^n,\xi^1,\dots,\xi^n)$
T^*T^*X	$(dx^1,\dots,dx^n,d\xi^1,\dots,d\xi^n)$	$(x^1,\dots,x^n,\xi^1,\dots,\xi^n,\chi^1,\dots,\chi^{2n})$

Basically, for T^*X and T^*T^*X, the second half of the coordinates are just the components with respect to the local frame on their left[1].

When we focus on a single fiber **half of the coordinates become superfluous since are fixed by the choice of fiber**. Therefore a covector $\xi \in T^*_xX$ will be written as

$$\xi = \xi_j(x)dx^j|_x, \tag{2.2.15}$$

while $\chi \in T^*_{(x,\xi)}T^*X$ (a double covector?) as

$$\chi = \chi_j(x,\xi)dx^j|_{(x,\xi)} + \chi_{j+n}(x,\xi)d\xi^j|_{(x,\xi)}. \tag{2.2.16}$$

[1]This is doable for any locally trivial vector bundle: any point $y \in E$ of a vector bundle $\pi : E \to M$ can be thought as a pair $(\pi(y), v)$ so that we use the coordinates of M for the first component, and components with respect to the local frame induced by the coordinates of M on $E_{\pi(y)}$ for the second.

In this setting, $\pi : T^*X \to X$ is the following map

$$\pi(x,\xi) = \pi(x^1,\dots,x^n,\xi^1,\dots,\xi^n) = (x^1,\dots,x^n), \quad \forall (x,\xi) \in T^*X \tag{2.2.17}$$

whose differential, expressed as its Jacobian matrix, is then

$$d\pi_{(x,\xi)} = \left(\partial_i \pi^j\right)|_{(x,\xi)} = \begin{pmatrix} \dfrac{\partial \pi^j}{\partial x^i} & \dfrac{\partial \pi^j}{\partial \xi^i} \end{pmatrix}|_{(x,\xi)} = \begin{pmatrix} I_n & 0 \end{pmatrix}. \tag{2.2.18}$$

Then

$$\alpha(x,\xi) = d\pi^*_{(x,\xi)}\xi = \xi \circ d\pi_{(x,\xi)} = \xi_j \frac{\partial \pi^j}{\partial x^k} dx^k|_{(x,\xi)} + \xi_j \frac{\partial \pi^j}{\partial \xi^k} d\xi^k|_{(x,\xi)} = \xi_k dx^k|_{(x,\xi)} \tag{2.2.19}$$

The "canonicity" of α is also witnessed by the following fact:

Theorem 2.11. *Let $F : X_1 \xrightarrow{\sim} X_2$ be a diffeomorphism and α, β the Liouville forms respectively on X_1 and X_2. Call $\mathcal{F}$ the dual morphism induced by F^{-1}. Then $\mathcal{F}^*\beta = \alpha$.*

$$\begin{array}{ccc} T^*X_1 & \xrightarrow{\mathcal{F}} & T^*X_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{F} & X_2 \end{array} \tag{2.2.20}$$

Proof. For each $(x,\xi) \in T^*X_2$,

$$\mathcal{F}(x,\xi) = (F(x), d(F^{-1})^*_{(x,\xi)}\xi), \tag{2.2.21}$$

so that

$$\begin{aligned} (\mathcal{F}^*\beta)_{(x,\xi)} &= \beta_{\mathcal{F}(x,\xi)} \circ d\mathcal{F}_{(x,\xi)} \\ &= d(F^{-1})^*_{(x,\xi)}\xi \circ d(\pi_2)_{\mathcal{F}(x,\xi)} \circ d(d(F^{-1})^*)_{(x,\xi)} \\ &= \xi \circ d(F^{-1})_{(x,\xi)} \circ d(\pi_2 \circ d(F^{-1})^*)_{(x,\xi)} \qquad \text{by chain rule} \\ &= \xi \circ d(F^{-1})_{(x,\xi)} \circ d(F \circ \pi_1)_{(x,\xi)} \qquad \text{by Diagram 2.2.20} \\ &= \xi \circ d(\pi_1)_{(x,\xi)} \\ &= d(\pi_1)^*_{(x,\xi)}\xi = \alpha_{(x,\xi)}; \end{aligned} \tag{2.2.22}$$

which amounts to the thesis. □

Definition 2.12. The **canonical symplectic form** on T^*X is

$$\omega = -d\alpha \tag{2.2.23}$$

Notice the minus sign, that's unavoidable and rigs pretty much all symplectic geometry. With the same coordinates we used before, ω is

$$\begin{aligned}\omega &= -d\alpha \\ &= -d(\xi_j dx^j) \\ &= -d\xi_j \wedge dx^j - \xi_j \wedge \cancelto{0}{d^2 x^j} = \sum_{j=1}^{n} d\xi^j \wedge dx^j\end{aligned} \tag{2.2.24}$$

2.3 Symplectomorphisms and Lagrangian manifolds

Theorem 2.11 says the pullback of $(f^{-1})^* : T^*X_1 \to T^*X_2$ is a symplectomorphism. In particular, when $X = X_1 = X_2$, there's a natural morphism of groups,

$$\mathrm{Diff}(X) \lhook\joinrel\longrightarrow \mathrm{Symp}(T^*X, \omega), \tag{2.3.1}$$

whose domain is the group of diffeomorphisms of X and whose codomain is the group of symplectomorphisms of (T^*X, ω). Recall that X lives inside T^*X as its zero section, thus the morphism is actually an inclusion, justifying its depiction by an hooked arrow. However, this map is not onto, since not all symplectomorphism arise in this way: for instance, translation along fibers of T^*X is not in the image.

This leaves us with the problem of finding these additional symplectomorphisms. If diffeomorphisms of the base space are too few to induce all of them, conversely diffeomorphisms of the total space are too many and we need to weed out all the ones which aren't preserving the symplectic structure.

Luckily, a new geometric concept comes to help:

Definition 2.13. A submanifold Y of a symplectic manifold (M, ω) is called **Lagrangian** when

1. $\dim Y = \frac{1}{2} \dim M$ and
2. Y is **isotropic**, i.e. if $\iota : Y \hookrightarrow M$ is the embedding of Y, then $\iota^*\omega = 0$

These submanifolds are closely related to symplectomorphisms. A first hint comes from the following result.

Exercise 2.3. Let $M = T^*X$ for a smooth manifold X, along with its canonical symplectic structure. Prove the graph of $\alpha \in \Omega^1(X)$ defines a Lagrangian submanifold of M if and only if α is closed, that is $d\alpha = 0$.

Proof. Indeed, let Γ be the graph of α. Then Γ is trivally an embedded submanifold of M of dimension $\frac{1}{2}\dim M$. Fix now coordinates as in the construction of Liouville's form. For the isotropy condition, a quick computation gives

$$d\iota = \begin{pmatrix} \delta^i_j & 0 \\ \partial_j \alpha_i & 0 \end{pmatrix} \tag{2.3.2}$$

from which

$$\iota^*\omega = \sum_{j=1}^{n} \iota^* dx^j \wedge \iota^* d\xi^j = \sum_{j=1}^{n} \delta^i_j dx^j \wedge (\partial_j \alpha_i) dx^j = -d\alpha \tag{2.3.3}$$

giving the claim. □

Consider now two symplectic manifolds (M_1, ω_1) and (M_2, ω_2), with the same dimension $\dim M_1 = \dim M_2$. We can endow $M_1 \times M_2$ with two different symplectic structures

$$\omega_\pm = \mathrm{pr}_1^*\omega_1 \pm \mathrm{pr}_2^*\omega_2 \tag{2.3.4}$$

where $\mathrm{pr}_\cdot$ denote a projection map.

Theorem 2.14. *A diffeomorphism $\phi : M_1 \to M_2$ is a symplectomorphism if and only if its graph submanifold*

$$\Gamma_\phi = \{(p, \phi(p)) \,|\, p \in M_1\} \xhookrightarrow{\ \iota\ } M_1 \times M_2 \tag{2.3.5}$$

is Lagrangian with respect to ω_-.

Proof. As $\dim M_1 = \dim M_2$ (because they're diffeomorphic through ϕ),

$$\dim \Gamma_\phi = \dim M_1 = \frac{1}{2}\dim(M_1 \times M_2). \tag{2.3.6}$$

Moreover, We have

$$\iota^*\omega_- = \iota^*\mathrm{pr}_1^*\omega_1 - \iota^*\mathrm{pr}_2^*\omega_2 = (\mathrm{pr}_1 \circ \iota)^*\omega_1 - (\mathrm{pr}_2 \circ \iota)^*\omega_2 \tag{2.3.7}$$

Since ι sends $p \mapsto (p, \phi(p))$, composing with the projection on the first component yields id_{M_1}, and composing with the other projection yields ϕ. Thus

$$\iota^*\omega_- = \mathrm{id}^*\omega_1 - \phi^*\omega_2 \tag{2.3.8}$$

Implying

$$\phi^*\omega_2 = \omega_1 \iff \iota^*\omega_- = 0 \tag{2.3.9}$$

which is the claim we wanted to prove. □

Remark 2.14.1. The theorem *does not* state any Lagrangian submanifold is the graph of a symplectomorphism: it is assumed, a fortiori, that ϕ is a diffeomorphism.

This result, combined with Exercise 2.3, gives us a systematic way to produce symplectomorphisms of a cotangent space.

Construction 2.15. In the case $M_1 = T^*X_1$ and $M_2 = T^*X_2$, along with their canonical symplectic structure, then

$$T^*X_1 \times T^*X_2 = T^*(X_1 \times X_2) \tag{2.3.10}$$

and the space is symplectic with symplectic form

$$\omega_- = -d(\mathrm{pr}_1^*\alpha_1 - \mathrm{pr}_2^*\alpha_2). \tag{2.3.11}$$

On the other hand, the canonical symplectic form on $T^*(X_1 \times X_2)$ is

$$\omega_+ = -d(\mathrm{pr}_1^*\alpha_1 + \mathrm{pr}_2^*\alpha_2) \tag{2.3.12}$$

By Exercise 2.3, we know how to construct Lagrangian manifolds on $T^*(X_1 \times X_2)$ with respect to its canonical symplectic structure as cotangent bundle. However, Theorem 2.14 refers to the other symplectic structure of $T^*(X_1 \times X_2)$, obtained from its product structure. Thus we leverage the following procedure:

1. Produce a Lagrangian submanifold of $T^*(X_1 \times X_2)$ with respect to ω_+: choose any function $f \in \mathcal{C}^\infty(X_1 \times X_2)$ and consider $\Gamma_{df} = \{(x, y, d_x f, d_y f) \,|\, (x, y) \in X_1 \times X_2\}$.

2. Twist the graph to $\tilde{\Gamma}_{df} = \{(x, y, d_x f, -d_y f) \,|\, (x, y) \in X_1 \times X_2\}$ (because the only difference between ω_+ and ω_- is exactly this twist on the second factor) to get a Lagrangian submanifold with respect to ω_-.

3. For $\tilde{\Gamma}_{df}$ to be the graph of a diffeomorphism $\phi : T^*X_1 \xrightarrow{\sim} T^*X_2$, it needs to satisfy a local condition: it must not be "vertical" at any point on the T^*X_1-T^*X_2 plane, that is, the projection $\mathrm{pr}_1 : \tilde{\Gamma}_{df} \to T^*X_1$ must have no critical points. This means

$$\det\left(\frac{\partial^2 f}{\partial y^j \partial x^i}\right) \neq 0 \tag{2.3.13}$$

If this is satisfied, then ϕ sends the first and third component of a point in $\tilde{\Gamma}_{df}$ to the second and fourth components:

$$\phi : (x, \xi) \mapsto (y, \eta) \tag{2.3.14}$$

where

$$\begin{cases} \xi_i = \dfrac{\partial f}{\partial x^i}(x, y) \\ \eta_i = -\dfrac{\partial f}{\partial y^i}(x, y) \end{cases} \tag{2.3.15}$$

Eventually, in the case $X_1 = X_2 = X$, we just explained geometrically how **canonical transformations** of a phase-space manifold arise from **generating functions** $f \in \mathcal{C}^\infty(X \times X)$.

2.4 Darboux's Theorem

We'll now pursue a local description of symplectic manifolds. For this section, refer to [**cannas2005symplectic**] and [**cannas2005symplectic**].

2.4.1 Tubular neighbourhoods

Definition 2.16. Given a closed embedding $\iota : X \hookrightarrow M$ of a k-dimensional submanifold X into its n-dimensional ambient manifold M, its **normal bundle** is

$$NX := TM/TX \tag{2.4.1}$$

defined pointwise on X.

Definition 2.17. A **tubular neighbourhood** of X is an open neighbourhood U of $\iota(X)$ in M diffeomorphic to a convex[2] neighbourhood $U_0 \subseteq NX$ of the zero section of the normal bundle.

The convexity condition means that $U_0 \cap N_x X$ has to be convex as a subset of the vector space $N_x X$.

Hence the situation is the following:

$$\begin{array}{ccccccc} NX & \supseteq & U_0 & \xrightarrow{\quad\phi\quad} & U & \subseteq & M \\ & & & {\scriptstyle \pi_0} \searrow \quad \nearrow {\scriptstyle \iota} & & & \\ & & & {\scriptstyle \iota_0} \nwarrow \quad \swarrow {\scriptstyle \pi} & & & \\ & & & X & & & \end{array} \tag{2.4.2}$$

where π_0 is the bundle morphism, ϕ is the diffeomorphism we ask for in the definition and π arises from commutativity of the diagram.

Topological properties of tubular neighbourhoods. The choice of properties of a tubular neighbourhood is tuned as to make it a very useful tool to probe the de Rham cohomology of X. From 1.3.2 we know two spaces have the same cohomology when they are homotopically equivalent. Indeed, notice $\pi\iota = \mathrm{id}_X$ already. On the other hand we clearly don't have $\iota\pi = \mathrm{id}_U$. However, $\iota\pi$ is still homotopic to the identity, thanks to the convexity

[2] In a vector space, a subset is convex if any convex combination of elements of the subset is still in the subset.

condition on U_0. In fact U_0 is just a fattening of X and diffeomorphic to U, so that U contracts to X as well. In this way $\mathrm{id} \simeq \iota\pi$. Then $(\iota\pi)^* = \pi^*\iota^* = \mathrm{id}_U^* : H^k_{dR}(U) \to H^k_{dR}(U)$. Hence, π^* **is an isomorphism of de Rham cohomology groups**, which means

$$H^k_{dR}(U) \cong H^k_{dR}(X). \tag{2.4.3}$$

2.4.2 Moser's theorems

Theorem 2.18 (Moser's First Theorem)**.** *Let $(M, \omega_t = (1-t)\omega_0 + t\omega_1)$ be a symplectic* ***compact*** *manifold for each $t \in [0,1]$, and $\overline{\omega_0} = \overline{\omega_1} \in H^2_{dR}(M)$. Then there exists $\rho : M \times \mathbb{R} \to M$ such that $\rho_t : (M, \omega_t) \xrightarrow{\sim} (M, \omega_0)$ is a symplectomorphism for all $t \in [0,1]$.*

The idea is that if you have a "flowing" symplectic structure, you can compensate by a "flowing" change of coordinates. This is given by a one-parameter group of diffeomorphisms, hence by the flow of a vector field on M. In this vein, the theorem is basically equivalent to the following:

Lemma 2.19 (Moser's trick)**.** *Under the assumptions of Moser's First Theorem, there exists $V_t \in \mathfrak{X}(M)$ depending smoothly on t and such that*

$$\mathcal{L}_{V_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t} = 0. \tag{2.4.4}$$

Proof. Notice that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \omega_1 - \omega_0$, and since ω_1 and ω_0 are cohomologous, their difference is cohomologous to $\overline{0} \in H^2_{dR}(M)$. This means that $\omega_1 - \omega_0 = d\mu$ for some $\mu \in \Omega^1(M)$. Moreover, by Cartan's magic formula

$$\mathcal{L}_{V_t}\omega_t = d\iota_{V_t}\omega_t + \cancel{\iota_{V_t} d\omega_t}. \tag{2.4.5}$$

Hence the equation we want to prove has been reduced to

$$d(\iota_{V_t}\omega_t + \mu) = 0 \tag{2.4.6}$$

which of course holds in particular if $\iota_{V_t}\omega_t = -\mu$ (**Moser's equation**), i.e. when

$$V_t = -\omega_t^{-1}(\mu). \tag{2.4.7}$$

□

Proof of Moser's Theorem (2.18). Let ρ_t be a flow as claimed. Then if V_t is the vector field associated to it (i.e. its derivative[3]), we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^*\omega_t) = \rho_t^*\left(\mathcal{L}_{V_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right). \tag{2.4.8}$$

Then we see that, as M is compact, we could reverse the argument and get all of ρ_t by integration of V_t, which is given to us by Moser's trick. □

Remark 2.19.1. Moser's Theorem can be proved also in the more general case where ω_t is a continuous curve inside a single cohomology class, that is

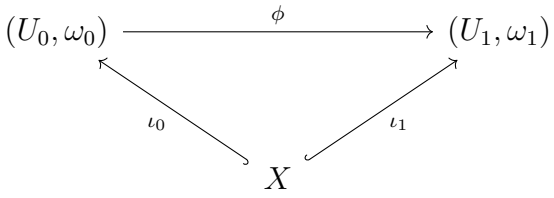
$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\omega_t} = \overline{0} \in H^2_{dR}(M). \tag{2.4.11}$$

A simple application of this is that if (M, ω) is symplectic and $\eta \in \Omega^1(M)$, then $(M, \omega + d\eta)$ is symplectic as well and symplectomorphic to the original manifold.

Theorem 2.20 (Moser's Relative Theorem)**.** *Let (M, ω_0) and (M, ω_1) be two symplectic structures on M, $\iota : X \hookrightarrow M$ a closed embedding of a compact submanifold such that*

$$\omega_0|_p = \omega_1|_p, \quad \forall p \in X \tag{2.4.12}$$

Then there are neighbourhoods U_0 and U_1 of $\iota(X)$ and a symplectomorphism $\phi : U_0 \xrightarrow{\sim} U_1$, which on X restricts to the identity, such that



(2.4.13)

Proof. Choose a tubular neighbourhood U_0 of X. Again, under the given assumptions on ω_0 and ω_1, $\omega_1|_{U_0} - \omega_0|_{U_0} = d\mu$ for $\mu \in \Omega^1(U_0)$. Let $\omega_t := \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td\mu$. This is symplectic for generic $t \in [0, 1]$, hence shrinking U_0 if necessary we can assume ω_t is symplectic on U_0.

To see the shrunk neighbourhood does not degenerate, notice that

$$\{(p, t) \in M \times [0, 1] \mid \det \omega_t(p) = 0\} \tag{2.4.14}$$

is closed. Same for $\iota(X) \times [0, 1]$. Then as the two do not intersect, they can be separated by an open neighbourhood around $\iota(X) \times [0, 1]$, whose projection on the first factor contains a tubular neighbourhood.

Solve now Moser's equation in this neighbourhood. We get then a flow $\rho_t : U_0 \times [0, 1] \to M$ which carries ω_t to ω_0 for each $t \in [0, 1]$. To see we proved the theorem, set $\phi = \rho_1$

[3]We have to be careful about the meaning of the parameter t! There are two different "times" to consider in a similar flow, indeed, ρ_t satisfies the following ODE:

$$V_t = \frac{\mathrm{d}}{\mathrm{d}s}(\rho_s \rho_t^{-1})|_{s=t} \tag{2.4.9}$$

It's like an usual flow in which vectors, however, change with time (*non-autonomous flow*), so at time t the curve has tangent vector V_t. For comparison, this is the equation in the *autonomous* (i.e. the field doesn't vary with time) case:

$$V = \frac{\mathrm{d}}{\mathrm{d}s}\rho_s|_{s=0}. \tag{2.4.10}$$

and $U_1 = \rho_1(U_0)$. Since $\omega_1|_{\iota(X)} = \omega_0|_{\iota(X)}$ we have $\rho_t|_X = \mathrm{id}_X \ \forall t$, hence in particular for $\rho_1 = \phi$, concluding the proof. □

Theorem 2.21 (Darboux)**.** *Let (M, ω) be a symplectic manifold and $p \in M$. Then there exist a chart $(U, (x^1, \dots, x^n, y^1, \dots, y^n))$ around p such that, on U*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{2.4.15}$$

Proof. Fix a symplectic basis for (T_pM, ω_p) (following the procedure given in Theorem 2.2) so that $\omega_p = \sum_{i=1}^n dx^{i'}|_p \wedge dy^{i'}|_p$ in those coordinates. By triviality of the tangent bundle, these coordinates are actually valid in a whole open neighbourhood U' of p, so that around p we get to define a symplectic form $\omega' = \sum_{i=1}^n dx^{i'} \wedge dy^{i'}$. Then by Moser's Relative Theorem applied to $X = \{p\}$, ω is equivalent to ω' on a neighbourhood $U_1 \cap U'$ of p, thanks to a diffeomorphism $\phi : U_0 \to U_1$. Hence setting $x^i = x^{i'} \circ \phi$, $y^i = y^{i'} \circ \phi$, we are done. □

Remark 2.21.1. The three theorems exposed in this section form a hierarchy of results about symplectomorphisms of (M, ω) fixing an immersed symplectic compact submanifold (X, ω'). Moser's Theorem addresses the case $X = M$, Darboux's Theorem addresses the case $X = \{\bullet\}$ and Moser's Relative Theorem addresses any other case in-between (be careful about the little differences between the three theorems, though).

Remark 2.21.2. What can we say about the different symplectic structures a manifold admits? Are they all symplectomorphic? A different yet related question is: **are there some invariants which can discern between non-equivalent symplectic structures?** Gromov proved a result stating that $\mathrm{Symp}(M, \omega)$ (the group of symplectomorphisms of (M, ω)) is either dense or closed in $\mathrm{Diff}(M)$ (*Gromov's alternative*). In the first case, it would have meant that symplectic topology is undistinguishable from differential topology, yet it was later proved that the second case holds, so that there is interest in studying the symplectic structures of a manifold. However, this no easy task. In fact, Darboux's theorem tells us **symplectic structures have no local invariants**: to distinguish them you have to look at global data, which is harder to work with. For example, we'll immediately see in the next chapter that a very intuitive cohomology theory fails to classify symplectic structures (while its later extension to Poisson structures partially succeed to do this). An algebraic invariant which accomplish this is called **Floer cohomology**, and requires more machinery than what we'll see here.

Chapter 3

Hamiltonian mechanics

In this chapter, we'll study how a symplectic structure may be used to produce a dynamics on the manifold. This is accomplished by using so-called Hamiltonian vector fields. The physical interpretation of the whole construction will be given, and we'll conclude by proving a result about the general properties of the dynamical systems we get in this way.

3.1 Hamiltonian vector fields

Definition 3.1. Given $h \in \mathcal{C}^\infty(M)$ on a symplectic manifold (M, ω), the **Hamiltonian vector field associated to** h is the vector field $X_h \in \mathfrak{X}(M)$ such that

$$\iota_{X_h}\omega = dh \tag{3.1.1}$$

or equivalently

$$X_h = \omega^{-1}(dh). \tag{3.1.2}$$

Definition 3.2. A vector field $X \in \mathfrak{X}(M)$ is **Hamiltonian** iff there exists a function $h \in \mathcal{C}^\infty(M)$ such that

$$\iota_X\omega = dh. \tag{3.1.3}$$

The subset of Hamiltonian fields is denoted by $\mathfrak{X}^{\mathrm{H}}(M)$.

Notice **the Hamiltonian condition is equivalent to require $\iota_X\omega$ to be exact**. By analogy then we could conceive the following definition:

Definition 3.3. A vector field $X \in \mathfrak{X}(M)$ is **symplectic** iff

$$d(\iota_X\omega) = 0. \tag{3.1.4}$$

The subset of symplectic fields is denoted by $\mathfrak{X}^{\mathrm{S}}(M)$.

Remark 3.3.1. These two kinds of fields are related in the same way exact and closed form are related:

$$\begin{array}{ccccc} B^1(M) & \subseteq & Z^1(M) & \subseteq & \Omega^1(M) \\ \downarrow{\scriptstyle \omega^{-1}} & & \downarrow{\scriptstyle \omega^{-1}} & & \downarrow{\scriptstyle \omega^{-1}} \\ \mathfrak{X}^{\mathrm{H}}(M) & \subseteq & \mathfrak{X}^{\mathrm{S}}(M) & \subseteq & \mathfrak{X}(M) \end{array} \tag{3.1.5}$$

Moreover, ω^{-1} carries the $\mathbb{R}$-module structure downstairs: indeed, the linear combination of Hamiltonian (resp. symplectic) fields is still Hamiltonian (resp. symplectic). **This means the obstructions for a symplectic vector field to be an Hamiltonian one are encoded by the same cohomology data for differential forms**, i.e. the module $H^1_{dR}(M)$. If one wonders wheter this could be extended to a whole cohomology theory, the answer is *yes*, although we'll need to wait the more general context of Poisson manifolds. The "symplectic cohomology" we'll get, however, will be canonically isomorphic to de Rham cohomology, as the first cohomology witnesses. As remarked in the previous chapter, this means that this flavor of cohomology is not enough to discern symplectic structures on M.

Example 3.1. Let $M = T^2$ with symplectic form $\omega = d\theta_1 \wedge d\theta_2$. The chosen coordinates represent angle variables around two closed circles around the torus, (for simplicity, we can think of them to be around the "principal" directions of the torus). Consider the fields $X_1 = \partial_{\theta^1}$ and $X_2 = -\partial_{\theta^2}$: they are symplectic but not Hamiltonian. Indeed, $\mathcal{L}_{X_i}\omega = 0$ for both $i = 1, 2$ but

$$\iota_{X_i}\omega = d\theta^{3-i} \tag{3.1.6}$$

which is **not** exact since θ^i can't be defined globally: they are only defined locally (at most on the whole torus with two independent[1] circles removed). Anyway, $d\theta^i$ does define a differential 1-form on the whole space (intuitively, since differentiating eliminates the 2π jump discontinuity which prevents a global definition). Actually this is the whole point of the first cohomology group: $\overline{d\theta_1}, \overline{d\theta_2}$ generate $H^1_{dR}(T^2)$ which is, indeed, $\mathbb{R} \times \mathbb{R}$.

Remark 3.3.2. An equivalent condition for a vector field to be symplectic can be obtained through Cartan's formula. In fact,

$$d\iota_X\omega = d\iota_X\omega + \iota_X \cancel{d\omega} = \mathcal{L}_X\omega, \tag{3.1.7}$$

so that a vector field is symplectic if and only if

$$\mathcal{L}_X\omega = 0. \tag{3.1.8}$$

[1] Non-homotopic, non-contractible loops which wind once.

Theorem 3.4. *The 1-parameter group of diffeomorphisms $\phi_t : M \tilde{\to} M$ associated to a symplectic vector field X preserves the symplectic structure:*

$$\phi_t^* \omega = \omega. \tag{3.1.9}$$

This means that ***the group is actually a group of symplectomorphisms****.*

Proof. Follows from Equation 2.4.8 and Remark 3.3.2. □

Example 3.2. Let $M = S^2$ with the symplectic structure given by the mixed product. Consider the immersion given by cylindrical coordinates

$$\begin{aligned} \iota : S^2 &\longrightarrow \mathbb{R}^3 \\ (\theta, h) &\longmapsto (\sqrt{1-h^2}\cos\theta,\ \sqrt{1-h^2}\sin\theta,\ h). \end{aligned} \tag{3.1.10}$$

Then

$$\begin{aligned} \iota_*(\partial_\theta) &= \partial_\theta x \partial_x + \partial_\theta y \partial_y + \partial_\theta z \partial_z = -y\partial_x + x\partial_y \\ \iota_*(\partial_h) &= \partial_h x \partial_x + \partial_h y \partial_y \partial_h z \partial_z = \frac{z}{1-z^2} \end{aligned} \tag{3.1.11}$$

where in the second computation we made the substitution $\cos\theta = x/\sqrt{1-z^2}$ and $\sin\theta = y/\sqrt{1-z^2}$. We use these to compute the symplectic form in coordinates. In truth, we already know

$$\omega(\partial_\theta, \partial_\theta) = \omega(\partial_h, \partial_h) = 0 \tag{3.1.12}$$

by antisimmetry, so we just go for the mixed one:

$$\begin{aligned} \omega(\partial_\theta, \partial_h)(x,y,z) &= \langle (x,y,z), \iota_*(\partial_\theta) \times \iota_*(\partial_h) \rangle \\ &= \langle (x,y,z), \frac{y^2 z}{1-z^2}\partial_z + y\partial_y + \frac{x^2 z}{1-z^2}\partial_z + x\partial_x \rangle \\ &= \langle (x,y,z), x\partial_x + y\partial_y + \frac{(x^2+y^2)z}{1-z^2}\partial_z \rangle \\ &= 1. \end{aligned} \tag{3.1.13}$$

So that $\omega = d\theta \wedge dh$.

Now consider $h \in \mathcal{C}^\infty(M)$ given by the "height function", so — by an happy coincidence of notations — literally the coordinate h on S^2. Its Hamiltonian field is defined by $\iota_{X_h}\omega = dh$ so that $X_h = \partial_\theta$, whose 1-parameter group of diffeomorphisms is $\phi_t(\theta, h) = (\theta + t, h)$ (a rotation of angle t), which evidently preserves the form ω.

Theorem 3.5. *Let $h \in \mathcal{C}^\infty(M)$. Then h is constant along the flow of X_h.*

Proof. We check $X_h(h)$, the derivative of h along the field X_h:

$$X_h(h) = dh(X_h) = \iota_{X_h} dh = \iota_{X_h}\iota_{X_h}\omega = 0. \tag{3.1.14}$$

□

Corollary 3.5.1. *Integral curves of X_h are contained in the level sets of h.*

Indeed, in Example 3.2 we took the height function on the sphere as Hamiltonian an indeed the group we got was rotations at a fixed height, i.e. the flow lines were parallels on the sphere.

3.2 Physical meaning of a symplectic structure

We'd like now to justify our mathematical definitions in terms of the properties we want an Hamiltonian physical model to have.

Construction 3.6. The starting point of any such model is the space of possible physical states of a chosen physical system, i.e. the set of its *positions* and its *momenta*. This is assumed to form a smooth manifold M, called the *phase space*. **We model physics as some rule associating, to a given energy function $h \in \mathcal{C}^\infty(M)$, an evolution of the physical state on the phase space**. We can think of such an evolution as the flow of a vector field X_h associated to h, although it should only depend on energy *differences* between states, hence on dh.

To produce such a vector field we necessarily need to make a choice of equations: this is where physical experience enters in the model. In the simplest case, we adopt Newton equations:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} = m\ddot{x} \tag{3.2.1}$$

which can be rewritten in terms of

$$h = \frac{(m\dot{x})^2}{2m} + U(x) = \underbrace{K(\dot{x})}_{\text{kinetic energy}} + \underbrace{U(x)}_{\text{potential energy}} \tag{3.2.2}$$

as

$$\begin{cases} \dot{x} = \dfrac{1}{m}\dfrac{\partial h}{\partial \dot{x}} \\ \ddot{x} = -\dfrac{1}{m}\dfrac{\partial h}{\partial x} \end{cases} \tag{3.2.3}$$

The dependence of X_h on dh is then *linear*, meaning it can be given at each point $p \in M$ by a linear map $T_p^*M \to T_pM$. Geometrically, such a smooth choice of linear maps is encoded by a tensor field $\eta \in \Gamma(TM \otimes TM)$, so that $X_h = \eta(dh)$. In this sense, η **is the mathematical incarnation of the laws of physics, because it associates evolution to energy**. If we *assume*[2] all energy gradients produce a non-zero dynamics then η is non-degenerate; hence we can work with its "inverse" (pointwise as a map) instead, call it $\omega \in \Gamma(T^*M \otimes T^*M)$.

[2]This is not always true! We'll see some instances of this and we'll introduce a formalism to treat those cases as well, the one of *Poisson manifolds*.

Now properties of ω, which is our "wannabe" symplectic form, will come from two additional modelling assumption. **The first is conservation of energy**, which is the condition $X_h(h) = 0$. Expressed in terms of ω, it becomes $\omega(X_h, X_h) = 0$. Since this has to be satisfied by *all* choices $h \in \mathcal{C}^\infty(M)$, including — locally — the coordinate functions x^i (thus on a local frame for TM), hence, locally, $\omega(X, X) = 0$ for any vector field X, which means ω is skew-symmetric ($\omega \in \Omega^2(M)$).

The second assumption is to have non-changing laws of physics (now embodied in ω), which translates mathematically to:

$$\phi_t^* \omega = \omega \tag{3.2.4}$$

where $\phi_t : M \to M$ is the flow of X_h. Notice, again, this must be true for any choice of h, so that by the same computation we did to prove Theorem 3.4, it implies $d\omega = 0$.

Now let $M = T^*X$ for some smooth manifold X, which we think as to represent the configuration space of a mechanical system, and let $\omega = -d\alpha = dx^j \wedge d\xi_j$ be the canonical symplectic structure on M. Fix a function $h \in \mathcal{C}^\infty(T^*X)$. Then its Hamiltonian vector field is obtained by applying the graded Leibniz rule for the interior product:

$$X_h = \frac{\partial h}{\partial \xi_j} \partial_{x^j} - \frac{\partial h}{\partial x^j} \partial_{\xi^j} \tag{3.2.5}$$

which corresponds very naturally to the vector field we got from Newton's equations. Indeed, they're the same up to rescaling by a factor $1/m$. Somehow, this "trick" to lower Newton's equations to the first-order, when taken seriously and globally on the configuration space, gives us the symplectic structure of T^*X.

3.3 Poisson brackets

We've seen in the previous sections that the dynamics on a symplectic manifold comes, in the first place, from a function $h \in \mathcal{C}^\infty(M)$. Moreover, in the physical interpretation of a symplectic manifold we can also imagine the smooth functions on M to represent observables of the system, i.e. quantities we can measure at any given state $p \in M$. Then we might try to find a way to study not only the dynamics of the system, but also the dynamics of the observables themselves.

To do this, we introduce a formalism which relies solely on the functions $\mathcal{C}^\infty(M)$. It will be so powerful that, as we'll see later in these notes, we will able to dispense with the fields X_h altogether.

Definition 3.7. The **Poisson brackets** of two functions $f, g \in \mathcal{C}^\infty(M)$ is defined as

$$\{f, g\} = \omega(X_f, X_g). \tag{3.3.1}$$

Theorem 3.8. *For all $f, g, h \in \mathcal{C}^\infty(M)$, the Poisson brackets satisfy the following properties:*

1. ***Bilinearity**:*
$$\{\alpha f + \beta g, h\} = \alpha\{f, g\} + \beta\{g, h\} \tag{3.3.2}$$
for any $\alpha, \beta \in \mathbb{R}$;

2. ***Skew-symmetry**:*
$$\{f, g\} = -\{g, f\}; \tag{3.3.3}$$

3. ***Leibniz rule**:*
$$\{f, gh\} = \{f, g\}h + g\{f, h\} \tag{3.3.4}$$
for all $f, g, h \in \mathcal{C}^\infty(M)$;

4. ***Jacobi identity**:*
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{3.3.5}$$
for all $f, g, h \in \mathcal{C}^\infty(M)$.

Theorem 3.9. *Let $U, V \in \mathfrak{X}^{\mathrm{S}}(M)$ be symplectic vector fields, then*
$$[U, V] = -X_{\omega(U,V)} \tag{3.3.6}$$

Proof. Fix an atlas, we'll verify $\iota_{[U,V]}\omega = -d(\omega(U, V))$ using coordinates computation. We'll use of the following facts (the order reflects where they are used in the computation):

1. The Lie derivative of a tensor is found by successively saturating the involved indices (one for the derivative + all the indices of the tensor) with the vector field we are differentiating along, so in particular for a 1-form, it is
$$\mathcal{L}_U\eta = [U^\mu(\partial_\mu\eta_\beta) + (\partial_\beta U^\mu)\eta_\mu]\, dx^\beta; \tag{3.3.7}$$
while for a 2-form, it is
$$\mathcal{L}_U\omega = [U^\mu(\partial_\mu\omega_{\alpha\beta}) + (\partial_\alpha U^\mu)\omega_{\mu\beta} + (\partial_\beta U^\mu)\omega_{\alpha\mu}]\, dx^\alpha \wedge dx^\beta. \tag{3.3.8}$$

2. Cartan's Magic Formula (1.4.5).

3. The fields U and V are symplectic, hence
$$d\iota_U\omega = d\iota_V\omega = 0. \tag{3.3.9}$$

Together with the above facts, two algebraic tricks are used:

1. In the second line, we use the "inverse" Leibniz rule:

$$(\partial_\mu V^\alpha)\omega_{\alpha\beta} = \partial_\mu(V^\alpha\omega_\alpha\beta) - V^\alpha(\partial_\mu\omega_{\alpha\beta}) \tag{3.3.10}$$

2. In the third line, we add and substract the identical terms marked in blue.

The computation goes like this:

$$\begin{aligned}
\iota_{[U,V]}\omega &= \left(U^\mu(\partial_\mu V^\alpha) - V^\mu(\partial_\mu U^\alpha)\right)\omega_{\alpha\beta}dx^\beta \\
&= U^\mu\partial_\mu(V^\alpha\omega_{\alpha\beta})dx^\beta - U^\mu V^\alpha(\partial_\mu\omega_{\alpha\beta})dx^\beta - V^\mu(\partial_\mu U^\alpha)\omega_{\alpha\beta}dx^\beta \\
&= U^\mu(\partial_\mu V^\alpha\omega_{\alpha\beta})dx^\beta + (\partial_\beta U^\mu)V^\alpha\omega_{\alpha\mu}dx^\beta \\
&\quad - U^\mu V^\alpha(\partial_\mu\omega_{\alpha\beta})dx^\beta - V^\alpha(\partial_\alpha U^\mu)\omega_{\mu\beta}dx^\beta - V^\alpha(\partial_\beta U^\mu)\omega_{\alpha\mu}dx^\beta \\
&= \mathcal{L}_U(\iota_V\omega) - \iota_V(\mathcal{L}_U\omega) \\
&= d\iota_U\iota_V\omega - \iota_U d\iota_V\omega - \iota_V d\iota_U\omega + \iota_V\iota_U d\omega \\
&= -d(\omega(U,V)).
\end{aligned} \tag{3.3.11}$$

Then the theorem is proved by definition of Hamiltonian vector. □

Remark 3.9.1. A fruitful reading of the last theorem is **the set of symplectic fields is closed under the operation of taking commutators**. The same applies to Hamiltonian fields since the result of the commutators is Hamiltonian and Hamiltonian vectors are symplectic. Therefore, the downstairs inclusions of Diagram 3.1.5 get promoted to inclusions of Lie algebras:

$$\mathfrak{X}^{\mathrm{H}}(M) \underset{\mathsf{LieAlg}}{\subseteq} \mathfrak{X}^{\mathrm{S}}(M) \underset{\mathsf{LieAlg}}{\subseteq} \mathfrak{X}(M). \tag{3.3.12}$$

Corollary 3.9.1. *The Poisson bracket puts a Lie algebra structure on $\mathcal{C}^\infty(M)$, and the assignment of the Hamiltonian field*

$$\begin{aligned}
(\mathcal{C}^\infty(M), \{-,-\}) &\longrightarrow (\mathfrak{X}^{Ham}(M), [-,-]) \\
f &\longmapsto X_f
\end{aligned} \tag{3.3.13}$$

*is an **anti**morphism of Lie algebras, surjective (by the very definition of Hamiltonian vector field), but not injective (as f and f + cost. have clearly the same image under this map).*

Definition 3.10. A **Poisson algebra** $(\mathcal{P}, \{-,-\})$ is a commutative associative algebra $\mathcal{P}$ equipped with an additional Lie algebra structure $\{-,-\}$ which satisfy a Leibniz rule relative to the algebra product:

$$\{f, gh\} = \{f,g\}h + g\{f,h\}, \quad \forall f,g,h \in \mathcal{P}. \tag{3.3.14}$$

The flow ϕ_t of X_h encodes the evolution of the state of the system under the Hamiltonian function $h \in \mathcal{C}^\infty(M)$. Evaluating f on ϕ_t makes f a function of time, thus providing a function $f(t)$ which spits out the value of f on the state reached by the system at t. Using the Poisson brackets, we can skip a step and get the evolution of f directly:

Theorem 3.11. *Let $f, h \in \mathcal{C}^\infty(M)$. Then the evolution of f on the flow of X_h is given by the bracket with h:*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, h\} \tag{3.3.15}$$

Proof.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = X_h(f) = \langle X_h, \iota_{X_f}\omega\rangle = \iota_{X_h}\iota_{X_f}\omega = \omega(X_f, X_h) = \{f, h\}. \tag{3.3.16}$$

□

3.4 Integrable systems

Definition 3.12. We call **Hamiltonian system** the data of a symplectic manifold together with a fixed choice of Hamiltonian $h \in \mathcal{C}^\infty(M)$.

Definition 3.13. Let (M, ω, h) be an Hamiltonian system. A function $f \in \mathcal{C}^\infty(M)$ such that $\{f, h\} = 0$ is called an **integral of motion** of the system.

The great insight of Hamiltonian formalism is that **integrals of motion are the key to describe explicitly the evolution of a system.** This intuition can be justified by the geometrical treatment we have given so far. Hence this section will be devoted to this purpose, and will culminate with a structural result about Hamiltonian systems, the Arnold–Liouville Theorem.

Definition 3.14. A set of functions $f_1, \ldots, f_k \in \mathcal{C}^\infty(M)$ is said to be **in involution** or to **commute** if

$$\{f_i, f_j\} = 0, \quad \forall i, j \leq k. \tag{3.4.1}$$

Construction 3.15. Let (M, ω) be a $2n$-dimensional symplectic manifold, let $h_1, \ldots, h_n$ be functions in involution and let finally $c \in \mathbb{R}^n$ be a regular value of

$$\begin{aligned} H : M &\longrightarrow \mathbb{R}^n \\ p &\longmapsto (h_1(p), \ldots, h_n(p)) \end{aligned} \tag{3.4.2}$$

meaning that $dh_1|_p, \ldots dh_n|_p$ are linearly independent at each $p \in H^{-1}(c)$.

By the implicit function theorem, $H^{-1}(c)$ **is a submanifold of** M **and its tangent spaces are generated by the Hamiltonian vector fields associated to the functions** $h_1, \dots, h_n$: indeed,

$$T_p H^{-1}(c) = \{X_p \in T_p M \mid X_p(h_i) = 0 \ \forall i \leq n\} \tag{3.4.3}$$

and

$$X_{h_i}(h_j) = \omega(X_{h_i}, X_{h_j}) = \{h_i, h_j\} = 0 \tag{3.4.4}$$

and the X_{h_i} are independent. Moreover, **this computation also shows** $H^{-1}(c)$ **is a Lagrangian submanifold of** M, since it has dimension n and ω vanishes on its generators. Then H is said to be a **Lagrangian fibration**.

Definition 3.16. A set of functions $f_1, \dots, f_k \in \mathcal{C}^\infty(M)$ is said to be **independent** if

$$\forall p \in M, df_1|_p, \dots, df_n|_p \text{ are linearly independent over } T^*M. \tag{3.4.5}$$

Construction 3.15 then shows that:

Lemma 3.17. *A set $f_1, \dots, f_k$ of functions independent and in involution generates an* ***isotropic distribution****, which means at every point $x \in M$ they span a subspace of T_xM depending smoothly on x*

$$\langle X_{f_1}|_x, \dots, X_{f_k}|_x \rangle \underset{\mathsf{Vec}}{\subseteq} T_x M \tag{3.4.6}$$

such that

$$\omega(X, Y) = 0, \quad \forall X, Y \in \langle X_{f_1}, \dots, X_{f_k} \rangle. \tag{3.4.7}$$

In particular, when $k = \frac{1}{2} \dim M$, the distribution is Lagrangian.

Definition 3.18. An Hamiltonian system (M, ω, h) is called **completely integrable** if it possesses $n = \frac{1}{2} \dim M$ integrals of motion which are independent and in involution.

3.5 Arnold–Liouville Theorem

Example 3.3 (Pendulum)**.** When $\dim M = 2$, every Hamiltonian system whose energy function h is not closed ($dh \neq 0$) is an integrable system (because we need just one function, and we have h already). The configuration space of a simple pendulum is S^1, and thus $M = T^*S^1 \cong S^1 \times \mathbb{R}$ is such a 2-dimensional Hamiltonian system, with Hamiltonian given by the energy function:

$$h(\theta, \xi) = \frac{\xi^2}{2m\ell^2} + m\ell(1 - \cos\theta) \tag{3.5.1}$$

where m is the mass of the pendulum and ℓ the length of the rod. The parameter θ will refer to the anticlockwise angle measured from the down direction. We choose units as to have $m = \ell = 1$. Then

$$h(\theta, \xi) = \frac{\xi^2}{2} + 1 - \cos\theta. \tag{3.5.2}$$

We plot the level sets of h:

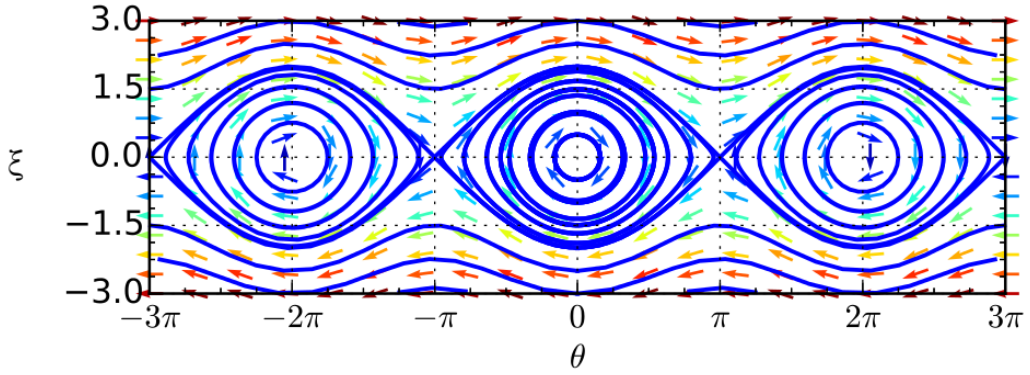


Figure 3.1

The critical points of h can be read from the expression of dh: $\xi d\xi + \sin\theta d\theta$, and are

$$\xi = 0, \quad \theta = 0, \pm\pi \tag{3.5.3}$$

In these points, the system is not integrable anymore (as h is not regular, see Construction 3.15), so we remove them from the manifold M. In this new manifold, which is again symplectic and smooth, the system is everywhere integrable and its integral submanifolds are either circles (for closed orbits) or copies of $\mathbb{R}$ (for the extremal orbit which is a circle with two points removed, as well as for larger orbits which are unaffected).

It is somewhat surprising that the fibration phenomenon we have seen at the end of Example 3.3 is not peculiar to that example: **integral submanifolds happen to be always tori or Euclidean spaces**. This is the subject of Arnold–Liouville Theorem.

We start with some vocabulary:

Definition 3.19. A **locally trivial surjective submersion** is a surjective submersion $f : M \to N$ such that there exists a manifold F and for every point $p \in N$ there exist an open neighbourhood U and a diffeomorphism $f^{-1}(U) \xrightarrow{\sim} U \times F$ which makes the following commute:

$$\begin{array}{ccccc} M & \supset & f^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times F \\ & & & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & & & U & \end{array} \tag{3.5.4}$$

Definition 3.20. A **proper map** $f : X \to Y$, where X and Y are topological spaces, is a continuous map such that f^{-1} maps compact sets to compact sets.

Lemma 3.21 ([**mo2019properness**])**.** *Any surjective submersion with compact and connected fibers is proper.*

Lemma 3.22 (Ehresmann)**.** *Any proper surjective submersion is locally trivial.*

Theorem 3.23 (Arnold–Liouville)**.** *Let (M, ω) be a symplectic manifold equipped with $h_1, \dots, h_n \in \mathcal{C}^\infty(M)$ functions in involution and define $H = (h_1, \dots, h_n) : M \to \mathbb{R}^n$. Suppose H is a **Lagrangian fibration**; i.e. its fibers are Lagrangian submanifolds of constant dimension. Let $p \in M$ such that the flows of $X_{h_1}, \dots, X_{h_n}$, starting at p, are complete*[3]*, and call $F^{(p)}$ the connected component of $H^{-1}(H(p))$ containing p. Then*

1. *$F^{(p)}$ is diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{T}^k$ through a diffeomorphism that maps the flows of $X_{h_1}, \dots, X_{h_n}$ to linear flows on $\mathbb{R}^{n-k} \times \mathbb{T}^k$; i.e. a linear $\mathbb{R}^n$-action on $\mathbb{R}^{n-k} \times \mathbb{T}^k$;*
2. *in a neighbourhood of $F^{(p)}$ the functions $h_1, \dots, h_n$ can be completed to a system of Darboux coordinates;*
3. *the set M_C of points whose fiber is compact is open, and $H|_{M_C}$ is a locally trivial Lagrangian fibration with fiber $\mathbb{T}^n$;*
4. *in a neighbourhood of each fiber of $H|_{M_C}$ there exists a system of Darboux coordinates formed by n angle variables on $\mathbb{T}^n$ and n integrals of motion called **action variables** (where the term* action *comes from their physical meaning).*

Remark 3.23.1. The statement of **the theorem is agnostic about the choice of Hamiltonian function**. Indeed, it is not about Hamiltonian manifolds but about symplectic ones with n functions in involution. Basically, this comes from the fact that the choice of h is (from the mathematical point of view) arbitrary, since it comes from the physical situation: the mathematical content of the Hamiltonian formalism is given solely by the set of functions in involution.

Remark 3.23.2. The condition on H is artificial since, following Example 3.3, we can always remove from M the critical points of H and apply the theorem to $M \setminus \operatorname{crit} H$ (such a subset is always open, because it is the preimage of the open set of non-critical values). On this manifold H is a fibration and by Construction 3.15 its fibers are Lagrangian.

[3]Which means that the flow ϕ_t is defined for all $t \in \mathbb{R}$, or — equivalently — that it is infinitely prolongable. This is always the case when M is compact.

Remark 3.23.3. To state the theorem globally (since any local statement about a symplectic manifold is trivial) we must ask for completeness of the flow. Indeed, to get fibers diffeomorphic to $\mathbb{R}$ and $\mathbb{T}^1$ we need complete flow lines.

Remark 3.23.4. The fourth statement only makes sense because of the previous one, otherwise there wouldn't be a way to extend the angle variables on a fiber to the neighboring ones. Moreover, in general the action variables will be a "straightened-up" version of the integrals of motions, hence not the same.

Proof.

1. We use the 1-parameter diffeomorphism groups $\phi_t^{X_{h_1}}, \ldots, \phi_t^{X_{h_n}}$ of the given vector fields $X_{h_1}, \ldots, X_{h_n}$ to build an action[4] of $\mathbb{R}^n$ on $F^{(p)}$ by defining

$$\phi_{(t^1,\ldots,t^n)}(x) = \phi_{t^1}^{X_{h_1}} \circ \ldots \circ \phi_{t^n}^{X_{h_n}}, \quad \forall x \in F^{(p)} \tag{3.5.5}$$

It is well-defined (as a group action) by virtue of the commutativity of the flows, which they inherit from their vector fields:

$$\begin{aligned} \phi_{(t^1+s^1,\ldots,t^n+s^n)}(x) &= \phi_{t^1+s^1}^{X_{h_1}} \circ \ldots \circ \phi_{t^n+s^n}^{X_{h_n}} \\ &= \phi_{t^1}^{X_{h_1}} \circ \phi_{s^1}^{X_{h_1}} \circ \ldots \circ \phi_{t^n}^{X_{h_n}} \circ \phi_{s^n}^{X_{h_n}} \\ &= \phi_{t^1}^{X_{h_1}} \circ \ldots \circ \phi_{t^n}^{X_{h_n}} \circ \phi_{s^1}^{X_{h_1}} \circ \ldots \circ \phi_{s^n}^{X_{h_n}}. \end{aligned} \tag{3.5.6}$$

Moreover, it is a transitive[5] action by the following argument. It's easy to see each point $x \in F^{(p)}$ has an open neighbourhood on which the flows, starting from x itself, define a chart. This is true because the fields are independent, so we can appeal to Frobenius' Theorem (Theorem 1.21). To show transitivity, we need to show any two $x, y \in F^{(p)}$ can be linked by flow lines. So pick a curve connecting x and y (it exists since $F^{(p)}$ is connected by assumption). By compactness of the curve, a finite number of the charts we defined using the flows suffice to cover the whole path from x and y, and we can assume the chart centered at x and the one centered at y to be in the covering. Now starting at x, follow the flow lines until they meet a point in the next chart, and then pick up another flow line of the second chart from there, and so on until we reach y. Clearly this defines a single segment of a flow line linking x to y, implying ϕ is transitive.

We then notice that, locally around any point, the action is free[6] (indeed, as we already observed, each point is the center of a chart given by the flows) meaning the

[4] An action of a group G on the set M is an homomorphism of groups $f : G \to \operatorname{Aut} M$, i.e. an assignment of permutations of M to elements of G, such that $f(gh) = f(g) \circ f(h)$ for every $g, h \in G$.

[5] An action of a group G on the set M is **transitive** iff for every pair of points $p, q \in M$, there is a $g \in G$ whose action brings p to q.

[6] An action of a group G on the set M is **free** if the identity of G is the only element inducing the identity on M.

stabilizer of any point is a discrete subgroup of $\mathbb{R}^n$. Such a subgroup, we'll call it Γ, does not depend on the point we chose on $F^{(p)}$ (we can show this using transitivity of ϕ and abelianity of $\mathbb{R}^n$). Moreover, a discrete subgroup for $\mathbb{R}^n$ can only be a lattice of points generated by integer multiples of a family of k independent vectors of $\mathbb{R}^n$. This suggests $\mathbb{R}^n/\Gamma \cong \mathbb{R}^{n-k} \times \mathbb{T}^k$. Moreover, this isomorphism of groups is also a *diffeo*morphism as Γ is a closed Lie subgroup of $\mathbb{R}^n$. The action ϕ then prolongs this diffeomorphism to $F^{(p)}$, proving the claim.

2. So now we know that $F^{(p)} \cong \mathbb{R}^{n-k} \times \mathbb{T}^k$ and by definition $h_1, \dots, h_n$ are constant on $F^{(p)}$. Since their differentials are linearly independent, they define a system of coordinates on the cotangent space of each fiber. Then we can use the $h_1, \dots, h_n$ to move across fibers, and we will show that together with $t^1, \dots, t^n$ (the flow times) they form a system of Darboux coordinates for a suitable choice of starting points for the flows on each fiber $H^{-1}(c)$ in a neighbourhood of $F^{(p)}$.

   Let us restrict to a tubular neighbourhood of $F^{(p)}$ where, since $F^{(p)}$ is Lagrangian (by Lemma 3.17), we have $\omega = d\eta$ for some 1-form η defined on such a neighbourhood (since tubular neighbourhoods inherit cohomology).

   Start by making an arbitrary choice of starting points for the flows. They span an n-dimensional submanifold of the tubular neighbourhoods, which is transverse (i.e., the tangent spaces are never parallel) to each fiber $H^{-1}(c)$. The $t^1, \dots, t^n, h_1, \dots, h_n$ form a system of coordinates for the tubular neighbourhood. Let us compute the components of ω in such a system:

$$\begin{aligned} \omega(\partial_{t^i}, \partial_{t^j}) &= \omega(X_{h_i}, X_{h_j}) = \{h_i, h_j\} = 0 \\ \omega(\partial_{t^i}, \partial_{h_j}) &= \iota_{X_{h_i}} \omega(\partial_{h_j}) = dh_i(\partial_{h_j}) = \delta_i^j \\ \omega(\partial_{h_i}, \partial_{h_j}) &=: V^{ij} \in \mathcal{C}^\infty(U). \end{aligned} \tag{3.5.7}$$

   Hence we can write

$$\omega = dt^i \wedge dh_i + V^{ij} dh_i \wedge dh_j. \tag{3.5.8}$$

   To get Darboux coordinates we want to eliminate the second term. Let's use the freedom of starting points to make the V^{ij} vanish at p. Now, since $t^1, \dots, t^n$ commute, $\mathcal{L}_{\partial_{t^i}} dt^j = 0$, and likewise for $h_1, \dots, h_n$. Also $\mathcal{L}_{\partial_{t^i}} \omega = 0$ by Theorem 3.4. Hence, for each $k = 1, \dots, n$,

$$\mathcal{L}_{\partial_{t^k}} V^{ij} = \mathcal{L}_{\partial_{t^k}} \omega(\partial_{h_i}, \partial_{h_j}) = 0 \tag{3.5.9}$$

   meaning $V^{ij} = V^{ij}(h_1, \dots, h_n)$, that is, **they only depend on the transverse coordinates**. On the other hand, in the tubular neighbourhood we are working

in, ω is an exact form. In fact, so is its first term (from (3.5.8)):

$$dt^i \wedge dh_i = d(t^i dh_i). \tag{3.5.10}$$

We then impose $V^{ij} dh_i \wedge dh_j$ to be exact, which means the second term has to be the differential of some 1-form W depending only on $h_1, \dots, h_n$, which implies that

$$V^{ij} = \partial_{h_j} W^i - \partial_{h_i} W^j. \tag{3.5.11}$$

Then

$$\omega = d(t^i dh_i + W^i dh_i) = d(t^i + W^i) \wedge dh_i. \tag{3.5.12}$$

By defining new coordinates $\tilde{t}^i = t^i + W^i(h_1, \dots, h_n)$, it's clear we get a Darboux system. Since the W^is depend only on the integral of motions, these new coordinates are still "times" for the flows but considered with a different starting point, as stated initially.

3. Can we say something more when fibers are compact? Are they "common" or do they appear isolated? We'll show they form open sets in M. Let M_C be the union of all compact connected components of any fiber. We claim M_C is open in M.

   Suppose M_C is not open. Then $M \setminus M_C$ is not closed, so that not all sequences of points $\{p_i\}_{i \in \mathbb{N}} \subseteq M \setminus M_C$ and converging in M converge in $M \setminus M_C$. This means we can select one of those, such that $p_i \longrightarrow p \in M_C$ while $p_i \in M \setminus M_C$ for all $i \in \mathbb{N}$. Let U be an open neighbourhood of $K^{(p)}$ (the compact connected component of $H^{-1}(H(p))$ containing p) such that $\overline{U}$ is compact and small enough as to not intersect other connected components of the same fiber $H^{-1}(H(p))$. For i large enough, $(\overline{U} \setminus U) \cap F^{(p_i)} = \varnothing$: indeed, if it were otherwise, we could choose points p_i' in each intersection $(\overline{U} \setminus U) \cap F^{(p_i)}$ and from those it can be selected a converging subsequence $\left\{p'_{i_k}\right\}_{k \in \mathbb{N}}$ by virtue of compactness of $\overline{U} \setminus U$ (it is closed in a compact). Let $p' \in \overline{U} \setminus U$ be the limit point of the subsequence. Since the two subsequences come from the same fibers, $H(p'_{i_k}) = H(p_{i_k})\ \forall k$. Using continuity of H, we obtain $H(p') = H(p)$, implying $p' \in K^{(p)}$, contradicting the assumption that $K^{(p)}$ doesn't intersect the boundary of U. So $F^{(p_i)} \cap (\overline{U} \setminus U) = \varnothing$ which implies (since the p_i accumulate on p)

$$F^{(p_i)} \subseteq U \quad \forall i \text{ large enough.} \tag{3.5.13}$$

Since $F^{(p_i)}$ is closed (since it's the preimage of $H(p_i)$, closed) and contained in $\overline{U}$, meaning it is compact itself. Moreover, $p_i \in M_C$, a contradiction with their definition. Thus M_C is open, hence a (sub)manifold (of M). Apply now Lemma 3.21 and Ehresmann's Lemma (3.22) in succession to the map H on M_C to conclude the proof.

4. Since $H : M_C \to \mathbb{R}^n$ is locally trivial, we can identify a neighbourhood of each fiber with $U_0 \times \mathbb{T}^n$ and $U_0 \subseteq \mathbb{R}^n$ is open (H is a fibration in tori). By point (2), $\omega = d(t^i dh_i) = -d(h_i dt^i)$ where $h_1, \dots, h_n$ are linear coordinates on $\mathbb{R}^n$ and $t^1, \dots, t^n$ are linear functions on $\mathbb{T}^n$. What we want to do now is a change of coordinates to get angle coordinates on the torus. Since we'd like this change to be structure-preserving, we'll try to get it as a canonical transformation, thus a symplectomorphism found through a generating function.

   Start by choosing a basis $\gamma_1, \dots, \gamma_n \in H_1(\mathbb{T}^n)$, hence smooth circles around each factor of $\mathbb{T}^n$. Now define

$$I_\alpha(h_1, \dots, h_n) := -\int_{\gamma_\alpha} h_i \, dt^i, \tag{3.5.14}$$

   Since H is constant on each γ_α, the h_i in the integral are (non-zero) constants, so that the assignment is a linear function:

$$(I_\alpha) = \underbrace{(-\int_{\gamma_\alpha} dt^i)}_{A}(h^i). \tag{3.5.15}$$

   Since we choose $\gamma_1, \dots, \gamma_n$ to be linearly independent, the matrix A can be easily shown to be invertible. Then we can write:

$$h_\alpha = h_\alpha(I_1, \dots, I_n). \tag{3.5.16}$$

   Now define

$$S(I_1, \dots, I_n, t^1, \dots, t^n) := -\int_{t_0}^{t} h_i(I_1, \dots, I_n) \, dt^i. \tag{3.5.17}$$

   Then by definition $h_\alpha = -\partial S / \partial t_\alpha$, thus calling $\phi^\alpha = \partial S / I_\alpha$ we get a canonical transformation whose generating function is S. Moreover, these second set of coordinates are the angle coordinates we were looking for. Indeed, fix a β:

$$\begin{aligned} \phi^\alpha(p + \gamma_\beta) &= \frac{\partial}{\partial I_\alpha} \left(\int_{t_0}^{t(p)+\gamma_\beta} h_i(I_1, \dots, I_n) \, dt^i \right) \\ &= \frac{\partial}{\partial I_\alpha} (S(I_1, \dots, I_n, t^1(p), \dots, t^n(p)) - I_\beta) \\ &= \phi^\alpha(p) - \delta^\alpha_\beta. \end{aligned} \tag{3.5.18}$$

   That shows ϕ^α is a periodic function in the α-th direction.

□

3.5.1 Applications

Example 3.4 (Simple pendulum)**.** Let's analyze Example 3.3 in light of Arnold–Liouville Theorem. Figure 3.1 shows the fibration induced by h on T^*S^1. We need to take in consideration that the diagram should actually be wrapped around a cylinder. Hence not only the closed curves near the singular point $(0,0)$ are compact fibers, but also the ones further away, which look open on the plane but close up on the cylinder. The open fibers are the two halves of the "doubly punctured" circle which constitutes the separatrix between the two families of compact fibers.

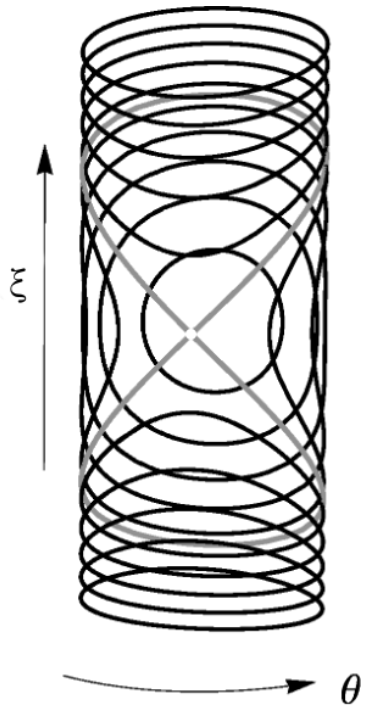


Figure 3.2: *The manifold T^*S^1, with compact fibers as solid black lines and open ones in grey.*

Example 3.5 (Spherical pendulum)**.** This time $M = T^*S^2$, we choose θ (longitude, $0 < \theta < 2\pi$) and ϕ (latitude, $0 < \phi < \pi$) as angle coordinates for S^2 and ξ, η for the remaining ones. The Hamiltonian of the system is

$$h(\theta, \phi, \xi, \eta) = \frac{1}{2}\left(\eta^2 + \frac{\xi^2}{\sin^2\phi}\right) + \cos\phi \tag{3.5.19}$$

whose differential is

$$dh = \eta d\eta + \frac{\xi}{\sin^2\phi} d\xi - \left(\frac{\xi^2 \cos\phi}{\sin^3\phi} + \sin\phi\right) d\phi \tag{3.5.20}$$

which is singular iff $\eta = \xi = 0$, $\phi = 0, \pi$. Since $\dim S^2 = 2$, we need another integral of motion to apply Arnold–Liouville. Indeed, notice h doesn't really depend on θ! So we put $h_2 = \xi$ "$(= \dot{\theta})$" (check this commutes!).

Let's check the points where dh_1 is not linearly indipendent from dh_2 (as with the singular points check for dh, this is in order to satisfy the hypothesis of Arnold–Liouville). As $dh_2 = d\xi$, this happens only when dh_1 has just the $d\xi$ term, i.e. the other two vanish:

$$\eta = 0, \quad \xi = \pm\sqrt{-\frac{\sin^4\phi}{\cos\phi}}. \tag{3.5.21}$$

Notice the condition on ξ can be satisfied iff $\phi > \pi/2$ (since it is already less than π by definition).

On the singular points of dh_1, $X_{h_1} = X_{h_2} = 0$ (where the second equality is extended to the points outside the chosen chart by a simple continuity argument) while in the second set of points (where the differentials are not independent)

$$\begin{aligned} X_{h_1} &= \pm\frac{1}{\sqrt{-\cos\phi}}\partial_\theta \\ X_{h_2} &= \partial_\theta \end{aligned} \tag{3.5.22}$$

The first case corresponds to the configurations where the pendulum is vertical, either upwards or downwards. The fact that the two Hamiltonian vector fields are zero tells us it has to stay still to maintain the balance. The second case represents circle orbits below the equator, those where gravity is balanced by centrifugal force.

Exercise 3.6. Apply Arnold–Liouville Theorem to the "updated" phase space of the spherical pendulum, in particular describe the fibers of the Lagrangian fibration and the locus M_C of the compact ones.

Example 3.7 (Kepler's problem on the plane)**.** Consider $M = T^*(\mathbb{R}^2 \setminus \{0\})$ with polar coordinates r, θ for the points and p_r, p_θ for covectors. The symplectic form on this manifold is

$$\omega = dr \wedge dp_r + d\theta \wedge dp_\theta \tag{3.5.23}$$

along with Hamiltonian[7]

$$h = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}, \quad k, m > 0 \tag{3.5.24}$$

The two integrals of motion are then $h_1 = h$ and $h_2 = p_\theta$.

As before, we now study the linear independence of dh_1, dh_2. First of all, they are

$$\begin{aligned} dh_1 &= \frac{p_r}{m}dp_r + \frac{p_\theta}{mr^2}dp_\theta + \left(\frac{k}{r^2} - \frac{p_\theta^2}{mr^3}\right)dr \\ dh_2 &= dp_\theta \end{aligned} \tag{3.5.25}$$

[7]We get this form from a suitable choice of description of the two-body system, from the reference frame of the center of mass.

thus they're indipendent unless

$$p_r = 0, \quad r = \frac{p_\theta^2}{mk} \tag{3.5.26}$$

which correspond to particular circular orbits.

To find the fibers, choose an arbitrary point $c = (c_1, c_2) \in \mathbb{R}^2$. Hence the equations of a level set of $H = (h_1, h_2)$ are

$$\begin{cases} c_1 = \dfrac{p_r^2}{2m} + V_{c_2}(r), \quad \text{where } V_{c_2}(r) = \dfrac{c_2^2}{2mr^2} - \dfrac{k}{r} \\ c_2 = p_\theta \end{cases} \tag{3.5.27}$$

The term $V_{c_2}(r)$ is called the *effective potential*. We then consider two cases.

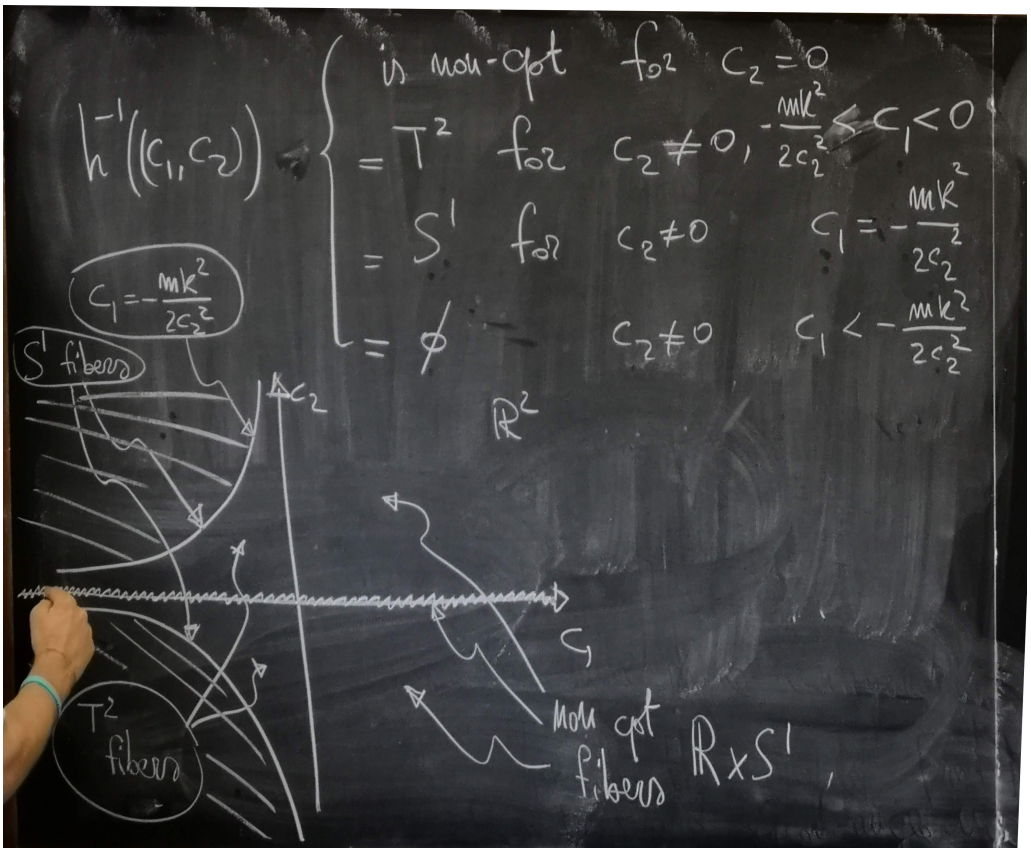
1. $c_2 = 0$, yielding
$$c_1 = \frac{p_r^2}{2m} - \frac{k}{r} \implies r = \frac{1}{k}\frac{1}{\frac{p_r^2}{2m} - c_1} \tag{3.5.28}$$
We observe that r is free to be as large as we please, hence these fibers correspond to unbounded orbits of the system and, geometrically, to non-compact fibers.

2. $c_2 \neq 0$, then inspecting the first equation we see that $c_1 < 0$ implies V_{c_2} negative, thus when we draw its graph we notice this is only the case when $r < 0$. In this case, we get compact level sets. Clearly, the level set is empty whenever $c_1 < \min V_{c_2}(r)$.

Therefore

$$H^{-1}(c_1, c_2) \begin{cases} \text{is non-compact} = \mathbb{R} \times S^1 & \text{for } c_2 = 0,\ c_1 \geq 0 \\ = \mathbb{T}^2 & \text{when } c_2 \neq 0,\ -\dfrac{mk^2}{2c_2^2} < c_1 < 0 \\ = S^1 & \text{when } c_2 \neq 0,\ c_1 = -\dfrac{mk^2}{2c_2^2} \\ = \varnothing & \text{when } c_2 \neq 0,\ c_1 < -\dfrac{mk^2}{2c_2^2} \end{cases} \tag{3.5.29}$$

Notice that the third case happens when $r = c_2^2/mk$, which we excluded previously for reasons of linear independence. Indeed, it represent a separatrix between compact and non-compact trajectories more or less like in the example of the pendulum. We can represent the fibration on the plane as such:



Thus the region of the plane giving compact fibers (i.e. tori) is

$$B = \left\{(h, p_\theta) \in \mathbb{R}^2 \,|\, h < 0,\ 0 < |p_\theta| < k\sqrt{-m/2h}\right\} \tag{3.5.30}$$

We can now move on to find the action-angle variables of the system: we start choosing γ_1, γ_2 basis for $H_1(H^{-1}(c_1, c_2))$. As the first one, we choose the circle in the θ direction parametrized by

$$\begin{aligned} &\gamma_1 : S^1 \longrightarrow H^{-1}(c_1, c_2) \\ &t \overset{\theta}{\longmapsto} \begin{cases} \theta = t \\ r = \dfrac{-k + \sqrt{k^2 + \frac{c_1 c_2^2}{m}}}{2c_1} \\ p_r = 0 \\ p_\theta = c_2 \end{cases} \end{aligned} \tag{3.5.31}$$

The expression for r is computed by imposing the circle to stay inside the fiber $H^{-1}(c_1, c_2)$, once we fixed $p_r = 0$ (which is an arbitrary choice).

The other loop γ_2, independent from γ_1, is naturally sought in the complementary part (r, p_r) of the fibers: as seen in Figure 3.3, an S^1 is actually present (the disentanglement of the r-dynamics from the θ-dynamics is due to the θ-independence of h).

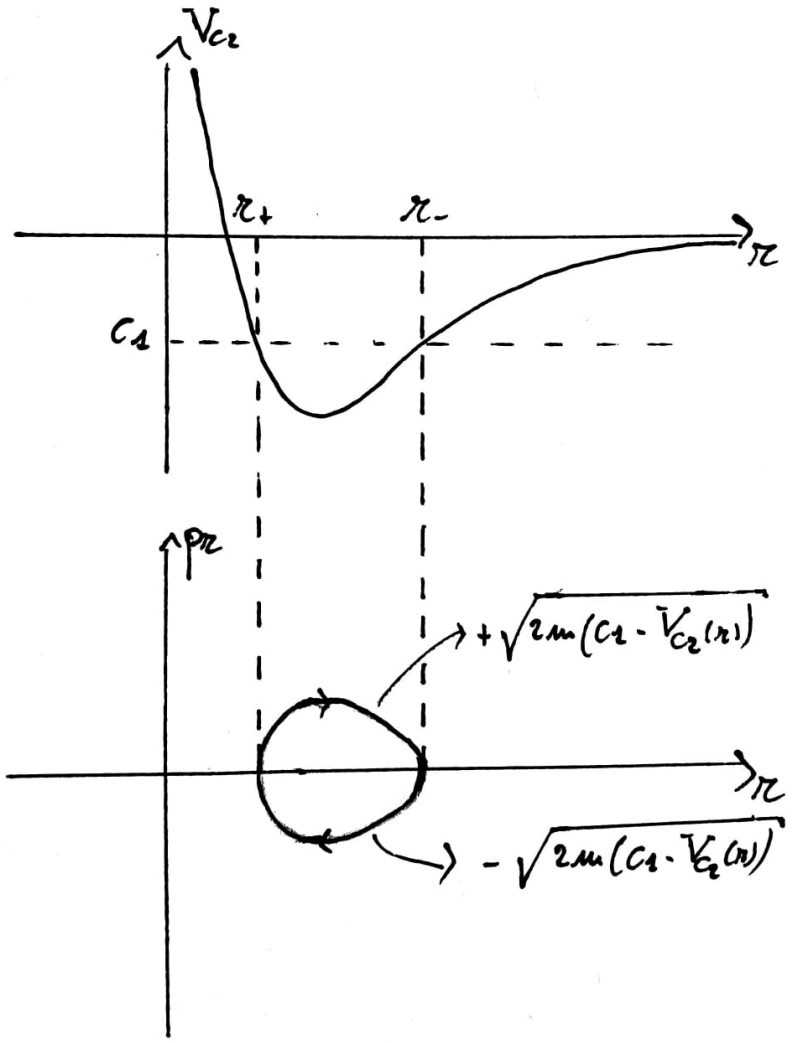


Figure 3.3

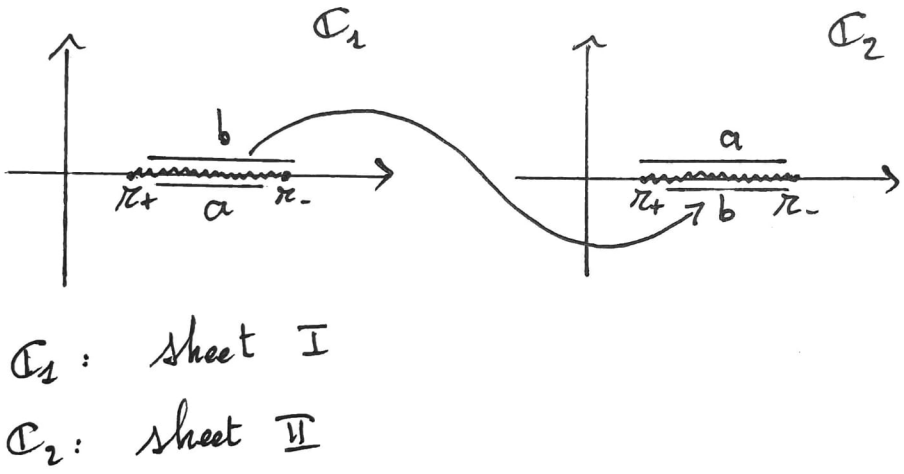


Figure 3.4

Although harder to visualize, this should make clear why γ_2 can be chosen to be formed by two branches $\gamma_2^{\pm}$ stitched together at the ends:

$$\gamma_2^{\pm} : (r_+, r_-) \longrightarrow H^{-1}(c_1, c_2)$$
$$t \longmapsto \begin{cases} \theta = 0 \qquad \text{(arbitrary choice)} \\ r = \pm t \\ p_r = \pm\sqrt{2m\left(c_1 - V_{c_2}(\pm t)\right)} \\ p_\theta = c_2 \end{cases} \tag{3.5.32}$$

The expression for p_r is computed by imposing the circle to stay inside the fiber $H^{-1}(c_1, c_2)$, once we fixed $p_\theta = c_2$. The values r_+ and r_- are given by the zeroes of the discriminant of $h_1 = c_1$:

$$r_\pm = \frac{-k \pm \sqrt{k^2 + \frac{c_1 c_2^2}{m}}}{2c_1} \qquad (r_+ < r_-). \tag{3.5.33}$$

Put together, γ_2 goes go back and forth between r_+ and r_- using the p_r dimension.

Then recall $\omega = d(-p_\theta d\theta - p_r dr)$, so, since $p_r = 0$ on γ_1.

$$I_1(c_1, c_2) = -\int_{\gamma_1} (p_\theta d\theta + p_r dr) = -2\pi c_2 \tag{3.5.34}$$

As for computing I_2, γ_2 is obtained by spanning r from r_+ to r_- setting $p_r(r) = +g(r)$, and then spanning r from r_- to r_+ setting $p_r(r) = -g(r)$, where $g(r) = \sqrt{2m(c_1 - V_{c_2}(r))}$, therefore

$$\begin{aligned} I_2(c_1, c_2) &= -\int_{\gamma_2} p_r dr \\ &= -\left(\int_{r_+}^{r_-} g(t)\, dt + \int_{r_-}^{r_+} (-g(t))\, dt\right). \end{aligned} \tag{3.5.35}$$

To compute it, we use residues by interpreting the integral in the complex t plane.

In order to do that, we aim to rewrite I_2 as a circuit integral and then to deform the integration path using the properties of integrals of holomorphic functions. Let us start by defining the following function

$$\pi_r(t) = \frac{\sqrt{-2mc_1}}{t}\sqrt{(t - r_-)(t - r_+)}. \tag{3.5.36}$$

This is clearly a multivalued function on two sheets, I and II, with two branches. It displays only two ramification points in $r_\pm$, therefore we take the cut connecting them as shown in Figure 3.4. Finally, it features a simple pole in zero, while being holomorphic elsewhere.

Next, consider its restrictions to the real axis above and below the cut (respectively b and a sides): one easily gets

$$\begin{aligned} \pi_r(t)_{|_b} &= ig(t) \\ \pi_r(t)_{|_a} &= -ig(t) \end{aligned} \tag{3.5.37}$$

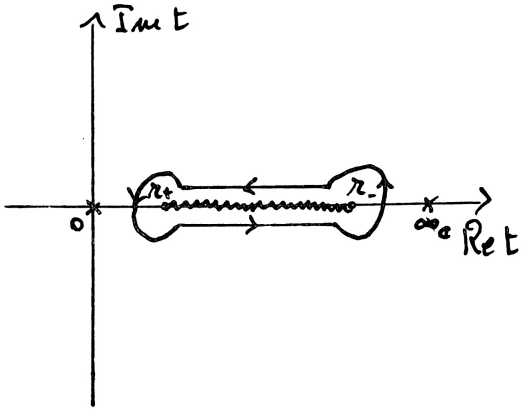


Figure 3.5

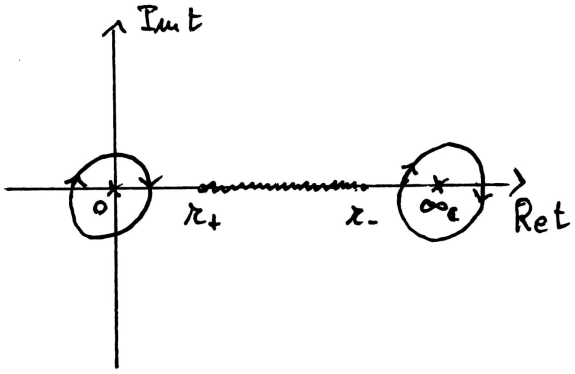


Figure 3.6

therefore one can consider the line integral

$$I = \int_\gamma \pi_r(t)\, dt \tag{3.5.38}$$

where γ is the path shown in Figure 3.5.

One can continuously deform the integration path γ without changing the result (since π_r is holomorphic outside zero), so let us shrink γ to the segment $[r_+, r_-]$, getting

$$\begin{aligned} I &= \int_{r_-}^{r_+} ig(x)\, dx + \int_{r_+}^{r_-} (-ig(x))\, dx \\ &= i\, I_2 \end{aligned} \tag{3.5.39}$$

so that I_2 can actually be rewritten as

$$I_2(c_1, c_2) = -i \int_\gamma \pi_r(t)\, dt. \tag{3.5.40}$$

Now we can use the residue theorem and trade γ for the union of the circles C_0 and C_∞ around the pole in zero and around $\infty_{\mathbb{C}}$ as integration path (see Figure (3.6)). Keeping track of the inherited orientations of the circles, one gets

$$\begin{aligned} I_2(c_1, c_2) &= \left(\int_{C_0} \pi_r(t)\, dt + \int_{C_\infty} \pi_r(t)\, dt \right) \\ &= 2\pi i \left(-\operatorname*{Res}_{t=0}(\pi_r(t)) - \operatorname*{Res}_{t=\infty_{\mathbb{C}}}(\pi_r(t)) \right) \end{aligned} \tag{3.5.41}$$

Let us carry out the residue computation:

1. **In $t = 0$.** Let us expand π_r in its Laurent series around $t = 0$: one finds

$$\pi_r(t) = \frac{\sqrt{-2mc_1}}{t} \sqrt{r_+ r_-} + \mathcal{O}(1), \tag{3.5.42}$$

so that

$$\begin{aligned} \operatorname*{Res}_{t=0}(\pi_r(t)) &= \sqrt{-2mc_1 r_+ r_-} \\ &= c_2 \end{aligned} \tag{3.5.43}$$

where we took $c_2 > 0$ and we used $r_+ r_- = -\frac{c_2^2}{2mc_1}$.

2. **In $t = \infty_{\mathbb{C}}$.** Let us expand π_r in its Laurent series around $t = \infty_{\mathbb{C}}$: one finds

$$\pi_r(t) = \sqrt{-2mc_1} - \frac{\sqrt{-2mc_1}(r_+ + r_-)}{2t} + \mathcal{O}(t^{-2}), \tag{3.5.44}$$

therefore

$$\begin{aligned} \operatorname*{Res}_{t=\infty_{\mathbb{C}}}(\pi_r(t)) &= \frac{\sqrt{-2mc_1}}{2}(r_+ + r_-) \\ &= \frac{mk}{\sqrt{-2mc_1}}. \end{aligned} \tag{3.5.45}$$

Finally, one gets

$$\begin{aligned} I_2(c_1, c_2) &= -i\left[-2\pi i\left(c_2 + \frac{mk}{\sqrt{-2mc_1}}\right)\right] \\ &= -2\pi\left(c_2 + \frac{mk}{\sqrt{-2mc_1}}\right). \end{aligned} \tag{3.5.46}$$

Reinserting back the functions for c_1 and c_2, one gets the following results:

$$\begin{aligned} I_1 &= -2\pi p_\theta \\ I_2 &= -2\pi p_\theta - 2\pi\sqrt{\frac{mk^2}{-2\left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}\right)}} \end{aligned} \tag{3.5.47}$$

from which we can obtain the generalized momenta:

$$\begin{aligned} p_\theta &= -\frac{1}{2\pi}I_1 \\ p_r &= \sqrt{2m\left(-2\pi^2\frac{mk^2}{(I_2 - I_1)^2} - \frac{1}{4\pi^2}\frac{I_1^2}{2mr^2} + \frac{k}{r}\right)} \end{aligned} \tag{3.5.48}$$

Call ϕ^α the angle variables. Then

$$S(\theta, r, I_1, I_2) = \int_{(\phi_0, r_0)}^{(\theta, r)} (p_\theta d\theta + p_r dr) \tag{3.5.49}$$

where the integration path is not relevant as long as it remains inside $H^{-1}(c_1, c_2)$. Then $\phi^\alpha = \partial S/\partial I_\alpha$, but **it's not possible to compute S analytically**. However, we can still find the canonical transformation to get the action variables. We start by writing h in terms of I_1 and I_2, an expression not depending on any other variable since H is costant along the fibers. From the relative computation, we get

$$h = h(I_1, I_2) = 2\pi^2\frac{mk^2}{(I_2 - I_1)^2} \tag{3.5.50}$$

We know the equations of motion imply

$$\begin{cases} \dot{I}_\alpha = 0 \qquad \text{since it is an integral of motion} \\ \dot{\phi}^\alpha = \dfrac{\partial h}{\partial I_\alpha} = \pm 4\pi^2\dfrac{mk^2}{(I_2 - I_1)^3} \end{cases} \tag{3.5.51}$$

Since ϕ^α comes from the linear action on the torus, we can now get the angle coordinates:

$$\phi^\alpha = \dot{\phi}^\alpha t + \phi_0^\alpha \tag{3.5.52}$$

The information we extracted from the system allows us, for example, to analyze the question of the periodicity of the motion. By computing the slope of the line described by the ϕ^α on the toric fiber, we can answer positively that question:

$$\frac{\dot{\phi}^1}{\dot{\phi}^2} = -1 \in \mathbb{Q} \tag{3.5.53}$$

Remark 3.23.5. The last part of the previous example shows an approach which works generally – once we have $I_1, \ldots, I_n$, we can express h in terms of them and then the angle variables as

$$\phi^\alpha = \frac{\partial h(I_1, \ldots, I_n)}{\partial I_\alpha} t + \phi_0^\alpha \tag{3.5.54}$$

Chapter 4

Poisson manifolds

4.1 The adjoint action of a Lie group

A Lie group is a group supported on a smooth manifold and whose product and inverse maps are smooth. This is a very fertile combination of structures from which a canonical symplectic structure of certain associated manifolds (the **coadjoint orbits**) will emerge naturally. As Lie groups are a quite large class of manifolds which includes groups of matrices, such a treatment is relevant to many physical situations.

It is interesting to study the (smooth) actions of such a group, i.e. group maps

$$\rho : G \times M \longrightarrow M \tag{4.1.1}$$

where M is a smooth manifold. Any such action, by virtue of the smooth structures involved, lifts by pushforward to an action on the fields

$$\begin{aligned} \rho_* : G \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (g, X) &\longmapsto (\rho_g)_* X \end{aligned} \tag{4.1.2}$$

Even when we do not have at our disposal another manifold M for G to act on, G acts on itself already in interesting ways. The easiest way a group acts on itself is with the **left action**:

$$\begin{aligned} L : G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned} \tag{4.1.3}$$

By noting that $L_e = \mathrm{id}_G$, meaning vectors at the identity are fixed by the action, we can guess the following identification

$$T_eG \underset{\mathsf{LieAlg}}{\cong} \mathfrak{X}^L(G) = \{X \in \mathfrak{X}(G) \,|\, (L_g)_* X = X \ \forall g \in G\} \tag{4.1.4}$$

In fact, any vector in T_eG (which is left-invariant) can be extended to a left-invariant vector field by prolonging it using the action's flow; transitivity of L takes care of injectivity

issues. Evaluation at e is the inverse map. Both T_eG and $\mathfrak{X}^L(G)$, along with the usual brackets $[-,-]$, are called the **Lie algebra** of G and denoted by $\mathfrak{g}$. Incidentally, $\mathfrak{g}$ is also *a* Lie algebra with respect to the usual bracket of vector fields, thus $\mathfrak{g} \underset{\mathsf{LieAlg}}{\subseteq} \mathfrak{X}(G)$.

Of course, these construction can be carried out for the ***right*** **action**, too, defined obviosuly as multplication on the right. The fields invariant for this action are denoted by $\mathfrak{X}^R(G)$ and for the same reasons as before, they form a Lie algebra isomorphic to T_eG, thus to $\mathfrak{g}$. However, this doesn't mean right-invariant fields are automatically left-invariant — at all.

Since $\mathfrak{g}$ can be considered a tangent space of G, to any action $\rho : G \times G \to G$ fixing the identity we can associate an action on the Lie algebra, simply by taking differentials at e:

$$\begin{aligned} \mathfrak{p} : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, X_e) &\longmapsto d\rho_g(X_e) \end{aligned} \tag{4.1.5}$$

Proposition 4.1. *When $\rho : G \times G \to G$ is an action fixing the identity, then the action $\mathfrak{p}$ is isomorphic (in the representation-theoretic sense) to ρ_* restricted to $\mathfrak{X}^L(G)$, that is, the following commutes for every $g \in G$:*

$$\begin{array}{ccc} \mathfrak{X}^L(G) & \xrightarrow{(\rho_g)_*} & \mathfrak{X}^L(G) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi} \\ T_eG & \xrightarrow{d\rho_g} & T_eG \end{array} \tag{4.1.6}$$

where ϕ is the isomorphism given by evaluation at e discussed above.

Proof. Let $X \in \mathfrak{X}^L(G)$, then

$$\begin{aligned} \phi((\rho_g)_*(X)) &= (\rho_g)_*(X)_e \\ &= d(\rho_g)_{\rho_g^{-1}e}(X_{\rho_g^{-1}e}) \\ &= d(\rho_g)_e(X_e) \qquad \text{because } \rho \text{ fixes } e \in G \\ &= d\rho_g(\phi(X)). \end{aligned} \tag{4.1.7}$$

□

The simplest of such actions is the **conjugation action** (since the left action and the right action do not fix the identity):

$$\begin{aligned} I : G \times G &\longrightarrow G \\ (g, x) &\longmapsto gxg^{-1} \end{aligned} \tag{4.1.8}$$

The identity $e \in G$ is a fixed point for this action.

Definition 4.2. The **adjoint action** of a Lie group G is the action

$$\begin{aligned} \mathrm{Ad} : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, X) &\longmapsto (I_g)_* X \end{aligned} \tag{4.1.9}$$

induced on its Lie algebra $\mathfrak{g}$ by the conjugate action

Construction 4.3. Fix $X \in \mathfrak{g}$ and call $g(t)$ its flow through the identity. If we are given a smooth action $\rho : G \times M \to M$ on a smooth manifold M, $g(t)$ induces a 1-parameter group of diffeomorphisms of M:

$$\phi_t := \rho_{g(t)}. \tag{4.1.10}$$

Its derivative then defines a field $X_M \in \mathfrak{X}(M)$. This procedure can be embodied by a map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{X}(M) \\ X &\longmapsto X_M \end{aligned} \tag{4.1.11}$$

called the **infinitesimal action** (even if it's not an action on its own right) of X on M. It can be proven to be a Lie algebra homomorphism.

Definition 4.4. The infinitesimal action associated to the adjoint action of G unto $\mathfrak{g}$ is denoted by

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathfrak{X}(\mathfrak{g}) \\ X &\longmapsto X_{\mathfrak{g}} \end{aligned} \tag{4.1.12}$$

Since ad_X is a vector field on $\mathfrak{g}$, it yields a vector for any left-invariant vector field Y we pass to it. Prove:

$$\mathrm{ad}_X |_Y = [X, Y] \quad \forall X \in \mathfrak{g}. \tag{4.1.13}$$

Remark 4.4.1. Notice that $\mathrm{ad}_X |_Y \in T_X\mathfrak{g}$, but **since T_eG is a vector space its tangent space is naturally isomorphic to itself**. This is why we can consider $[X, Y]$ (which is an element of $\mathfrak{g}$, since both X and Y are left-invariant and pushforward respects the bracket) both as an element of $\mathfrak{g}$ and as a vector.

Definition 4.5. The **coadjoint action** Ad^* is the dual map to Ad, hence the unique map

$$\mathrm{Ad}^* : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \tag{4.1.14}$$

such that

$$\langle \mathrm{Ad}^*_g \alpha, X \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}} X \rangle, \quad \forall \alpha \in \mathfrak{g}^*, X \in \mathfrak{g}. \tag{4.1.15}$$

Its infinitesimal action is the map $\mathrm{ad}^*_X : \mathfrak{g} \to \mathfrak{X}(\mathfrak{g}^*)$ satisfying

$$\langle \mathrm{ad}^*_X |_\alpha, Y \rangle = \langle \alpha, \mathrm{ad}_{-X} |_Y \rangle = -\langle \alpha, [X, Y] \rangle. \tag{4.1.16}$$

Exercise 4.1. Prove the last claim.

Definition 4.6. The **coadjoint orbit** through $\alpha \in \mathfrak{g}^*$ is

$$\Omega_\alpha = \{\mathrm{Ad}^*_g \alpha \,|\, g \in G\} \subseteq \mathfrak{g}^*. \tag{4.1.17}$$

Theorem 4.7. *There is an identification* $T_\alpha\Omega_\alpha \cong \mathfrak{g}/\ker(\mathrm{ad}^*|_\alpha)$.

Proof. Let's start by trying to understand intuitively what is the kernel of $\mathrm{ad}^*|_\alpha$. Remember ad^* represents the infinitesimal action of G on $\mathfrak{g}^*$, thus we may think of its kernel at a given point $\alpha \in \mathfrak{g}^*$ as the stabilizer of this action, i.e. those vector fields which do not move α. It's easy then to picture the claimed isomorphism: the tangent space of the coadjoint orbit is made by the vectors along which the action can move α, thus it makes sense to say these are exactly all the vectors of the infinitesimal action modulo the ones that do not move α, namely the ones in the infinitesimal stabilizer.

Consider the submersion $g \longmapsto \mathrm{Ad}^*_g(\alpha)$. We have $T_gG \cong T_eG = \mathfrak{g}$ through the left action of G on itself and $T_{\mathrm{Ad}^*_g(\alpha)}\Omega_\alpha \cong T_\alpha\Omega_\alpha$ for analogous reasons. Its differential at the identity of e then is

$$\begin{aligned} \mathrm{ad}^* : \mathfrak{g} &\longrightarrow T_\alpha\Omega_\alpha \\ X &\longmapsto \mathrm{ad}^*_X \end{aligned} \tag{4.1.18}$$

and by surjectivity, the claim follows from the First Isomorphism Theorem. □

Theorem 4.8. *There 2-form* $\omega \in \Omega^2(\Omega_\alpha)$ *defined by*

$$\omega(X,Y)_\xi = \omega(\mathrm{ad}^*_X, \mathrm{ad}^*_Y)_\xi = \langle \xi, [X,Y] \rangle, \quad \forall X, Y \in \mathfrak{X}(\Omega_\alpha),\ \forall \xi \in \Omega_\alpha, \tag{4.1.19}$$

is symplectic.

Proof. It is indeed well-defined as 2-form: it is trivially bilinear and skew-symmetric.

It is non-degenerate because

$$\begin{aligned} \langle \alpha, [X,Y] \rangle &= \langle \alpha, \mathrm{ad}_X|_Y \rangle \\ &= -\langle \mathrm{ad}^*_X|_\alpha, Y \rangle = 0 \qquad \forall Y \in \mathfrak{X}(\Omega_\alpha) \\ &\iff X \in \ker \mathrm{ad}^*_\bullet|_\alpha, \end{aligned} \tag{4.1.20}$$

that is, since $\mathrm{ad}^*_\bullet|_\alpha$ is surjective $\forall \alpha \in \mathfrak{g}^*$,

$$\omega(\mathrm{ad}^*_X, \bullet) = 0 \iff X \in \ker \mathrm{ad}^*_\bullet|_\alpha. \tag{4.1.21}$$

On the other hand, by the previous theorem,

$$X \in T_\alpha\Omega_\alpha \cong \mathfrak{g}/\ker \mathrm{ad}^*_\bullet|_\alpha, \tag{4.1.22}$$

which means exactly that $[X] = 0 \in T_\alpha\Omega_\alpha$, so ω is non-degenerate on its domain.

Finally, let us prove ω is closd. One can show, using the Lie algebra axioms (after long computations we are skipping) that $\mathcal{L}_{\mathrm{ad}^*_Z}\omega = 0$ for all $Z \in T_\alpha\Omega_\alpha$. On the other hand, notice that

$$\iota_{\mathrm{ad}^*_Z}\omega = \omega(\mathrm{ad}^*_Z, \bullet) = -\langle \mathrm{ad}^*_Z|_\alpha, \bullet\rangle, \tag{4.1.23}$$

and thanks to the identification

$$\begin{aligned} \mathrm{ad}^*_Z|_\alpha \in T_\alpha\Omega_\alpha &\cong \mathfrak{g}/\ker \mathrm{ad}^*_\bullet|_\alpha \\ \mathrm{ad}^*_Z|_\alpha &\sim Z \end{aligned} \tag{4.1.24}$$

one can write $\langle \mathrm{ad}^*_Z|_\alpha, \bullet\rangle = dZ$, therefore

$$\iota_{\mathrm{ad}^*_Z}\omega|_\alpha = -dZ, \tag{4.1.25}$$

and using Cartan's formula

$$0 = \mathcal{L}_{\mathrm{ad}^*_Z}\omega = d\iota_{\mathrm{ad}^*_Z}\omega + \iota_{\mathrm{ad}^*_Z}d\omega = \iota_{\mathrm{ad}^*_Z}d\omega \qquad \forall\, \mathrm{ad}^*_Z, \tag{4.1.26}$$

which implies that $d\omega = 0$ since ad^*_Z generates $T_\alpha\Omega_\alpha$. □

Definition 4.9. A symmetric non-degenerate bilinear form $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is called Ad-**invariant** if

$$(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = (X, Y), \quad \forall g \in G,\, \forall X, Y \in \mathfrak{g}. \tag{4.1.27}$$

Exercise 4.2. Prove that the identification $\mathfrak{g} \cong \mathfrak{g}^*$ induced by an Ad-invariant form commutes with both Ad and Ad^*:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g}^* \\ \downarrow{\scriptstyle \mathrm{Ad}_g} & & \downarrow{\scriptstyle \mathrm{Ad}^*_g} \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}^*. \end{array} \tag{4.1.28}$$

Example 4.3. Let $G = SO(3)$ (the group of orientation-preserving rotations of the 2-dimensional sphere), identified with the manifold of order 3 square, orthogonal matrices with positive determinant.

We can obtain an explicit description of its Lie algebra as $T_{I_3}SO(3)$, by differentiating the relation $X^TX = I_3$:

$$\begin{aligned} \mathfrak{so}(3) &= \{\frac{\mathrm{d}}{\mathrm{d}t}|_{I_3} X \mid X \in \mathbb{R}^{3\times 3},\ X^TX = I_3\} \\ &= \{X \in \mathbb{R}^{3\times 3} \mid X + X^T = 0\}. \end{aligned} \tag{4.1.29}$$

The Lie bracket on $\mathfrak{so}(3)$ is the usual commutator of matrices (in the ring theoretic sense). There is an isomorphism

$$\begin{aligned} \phi : \mathbb{R}^3 &\xrightarrow{\sim} \mathfrak{so}(3) \\ (x,y,z) &\longmapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \end{aligned} \tag{4.1.30}$$

sending the vector product to the commutator:

$$\phi(v \times w) = [\phi(v), \phi(w)] \tag{4.1.31}$$

So

$$(\mathfrak{so}(3), [-,-]) \underset{\mathsf{LieAlg}}{\cong} (\mathbb{R}^3, \times). \tag{4.1.32}$$

The action Ad_g applied to v results in $g\phi(v)g^{-1}$. It produces a rotation of v by g: $= \phi(gv)$. We have an Ad-invariant structure:

$$(\phi(v), \phi(w)) := v \cdot w, \quad \forall v, w \in \mathbb{R}^3. \tag{4.1.33}$$

The coadjoint action is then

$$\mathrm{Ad}^*_g(\phi(v)) = \phi(g^{-1}v)\cdot \tag{4.1.34}$$

Consider

$$\begin{aligned} \Omega_{\phi(v)} &= \{\mathrm{Ad}^*_g(\phi(v)) \,|\, g \in SO(3)\} \\ &= \phi\{g^{-1}v \,|\, g \in SO(3)\} \end{aligned} \tag{4.1.35}$$

The symplectic form is then

$$\omega(\mathrm{ad}^*_{\phi(v)}, \mathrm{ad}^*_{\phi(w)})_{\phi(u)} = u \cdot (v \times w) \tag{4.1.36}$$

which is the usual symplectic structure on the sphere!

We've said $T_v\Omega_v$ describes the infinitesimal action – thus the infinitesimal rotations in this case — departing from v. Thus we can understand the quotient structure as this

$$\begin{aligned} \mathfrak{so}(3) &\rightsquigarrow \text{all infinitesimal rotations of the sphere} \\ &/ \\ \ker \mathrm{ad}^*|_v &\rightsquigarrow \text{infinitesimal rotations around } v \\ &= \\ T_v\Omega_v &\rightsquigarrow \text{infinitesimal rotations away from } v \\ &= \text{infinitesimal movements on the sphere starting at } v \end{aligned} \tag{4.1.37}$$

4.2 Poisson manifolds

Definition 4.10. A **Poisson manifold** $(M, \{-,-\})$ is a smooth manifold M together with a Poisson algebra structure $\{-,-\}$ on $\mathcal{C}^\infty(M)$.

Example 4.4. Any symplectic manifold is a Poisson manifold with Poisson brackets given by

$$\{f, g\} = \omega(X_f, X_g) = \omega^{-1}(df, dg). \tag{4.2.1}$$

Example 4.5 ([**fernandes2014lectures**])**.** Let $\mathfrak{g}$ be any finite dimensional Lie algebra. If $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$, then the differential of f at ξ can be viewed naturally[1] as an element of $\mathfrak{g}$. Hence we can define a Poisson bracket on $\mathfrak{g}^*$ (called the **Lie–Poisson bracket**) by simply reusing the Lie structure:

$$\{f, g\}(\xi) := \langle [df, dg]_{\mathfrak{g}}, \xi \rangle, \quad \forall \xi \in \mathfrak{g}^*. \tag{4.2.2}$$

The fact it satisfies the required properties is straightforward, if tedious, to check.

Definition 4.11. On a Poisson manifold $(M, \{-,-\})$ a **Casimir function** is a function $f \in \mathcal{C}^\infty(M)$ such that

$$\{f, -\} = 0. \tag{4.2.3}$$

Remark 4.11.1. Since a symplectic form is non-degenerate, **there are no Casimir functions on a symplectic manifold**.

4.2.1 Poisson tensor

Reference: [**fernandes2014lectures**]. On a symplectic manifold we can prove the value of $\{f, g\}$ at a point $p \in M$ only depends on df_p and dg_p: let indeed $h \in \mathcal{C}^\infty(M)$ be such that $dh_p = 0$, then

$$\{f, gh\}(p) = \{f, g\}(p)\, \cancel{h(p)} + g(p)\{f, h\}(p) = g(p)\, \underbrace{\cancel{\omega_p^{-1}(df_p, dh_p)}}_{\text{by non-degeneracy}} = 0. \tag{4.2.4}$$

Moreover, notice this dependence is linear and skew-symmetric. Hence we see that a Poisson structure can be encoded by a map $\Pi_p : T_p^*M \wedge T_p^*M \longrightarrow \mathbb{R}$ at each point $p \in M$. In other words, a tensor field $\Pi \in \Gamma(TM \wedge TM)$ on M:

Definition 4.12. The **Poisson tensor** of a Poisson manifold M is the unique tensor field such that

$$\{f, g\} = \Pi(df, dg). \tag{4.2.5}$$

[1] Because $\mathfrak{g}$ is assumed to be finite dimensional; otherwise $df \in \mathfrak{g}^{**}$ which might not be naturally isomorphic to $\mathfrak{g}$.

Example 4.6. For a symplectic manifold (M, ω) the Poisson tensor is simply ω^{-1}.

The Poisson tensor brings some of the amenities of a symplectic structure to a Poisson manifold – in fact, anything definable by means of ω^{-1}. For instance, this is the case for Hamiltonian vector fields:

$$X_f = \Pi^\sharp(df). \tag{4.2.6}$$

Here, $\Pi^\sharp$ is the map $\Omega^1(M) \to \mathfrak{X}^1(M)$ induced by partial application of Π on the left.

On the other hand, we can start with a manifold equipped with a Poisson tensor and recover the bracket as a byproduct. In fact, in local coordinates, such a tensor can be written as

$$\Pi = \frac{1}{2}\Pi^{ij}\,\partial_i \wedge \partial_j \tag{4.2.7}$$

and its associated bracket[2] as

$$\{f, g\} = \frac{1}{2}\Pi^{ij}\,(\partial_i f\,\partial_j g - \partial_j f\,\partial_i g). \tag{4.2.8}$$

This also shows that $\Pi^{ij} = \{x_i, x_j\}$, so that

$$\{x_i, \{x_j, x_k\}\} = \frac{1}{2}\Pi^{i\ell}\,\partial_\ell \Pi^{jk} \tag{4.2.9}$$

which allows us to convert the Jacobi identity of the brackets to the following condition on the components of Π:

$$\Pi^{i\ell}\,\partial_\ell \Pi^{jk} + \Pi^{k\ell}\,\partial_\ell \Pi^{ij} + \Pi^{j\ell}\,\partial_\ell \Pi^{ki} = 0. \tag{4.2.10}$$

Therefore **we call a Poisson tensor** (or Poisson structure) **any skew-symmetric tensor of type** $(2, 0)$ (also called a *bivector field*) **which satisfies Equation 4.2.10**.

Remark 4.12.1. Notice **the Jacobi condition is quadratic on the components so it is a non-linear PDE of the second order**. The immediate consequence is that linear combinations of Poisson tensors are not Poisson tensors themselves.

Exercise 4.7. Show that, for a symplectic form ω, Equation (4.2.10) translates to the condition $d\omega = 0$.

Example 4.8. Consider $\mathbb{R}^{2n}$ with linear coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$. Equip it with the tensor

$$\Pi = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \tag{4.2.11}$$

which is called the **canonical Poisson structure on** $\mathbb{R}^{2n}$. Indeed, this is simply the inverse of the canonical symplectic structure on $\mathbb{R}^{2n}$!

[2] Recall $X \wedge Y = \frac{1}{2}(X \otimes Y - Y \otimes X)$.

Example 4.9. On $\mathbb{R}^n$, consider a skew-symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. We can define a **quadratic Poisson bracket** on this space by the formula:

$$\Pi = \frac{1}{2}\sum_{i,j\leq n} a_{ij}x^i x^j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{4.2.12}$$

Example 4.10. Consider $\mathbb{R}^n_+$ (the positive quadrant of the Euclidean n-space) equipped with a quadratic Poisson structure. A popular biological model, called the **Lotka–Volterra equations**, models population dynamics using the following Hamiltonian:

$$h = \eta_j x^j + k_j \log x^j \tag{4.2.13}$$

for a certain choice of the parameters $\eta^i, k_i \in \mathbb{R}$. To find the dynamics of the system, we apply a quadratic Poisson tensor to $dh = \sum_{j=1}^n \eta_j dx^j + k_j (x^j)^{-1} dx^j$ to get

$$X_h = \Pi^\sharp(dh) = \sum_{j=1}^n (\underbrace{a_{ij}\eta_j}_{\gamma_{ij}} x^i x^j + \underbrace{a_{ij}k_j}_{\varepsilon_i} x^i)\partial_i, \tag{4.2.14}$$

whose components are then the Lotka–Volterra equations:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \gamma_{ij}x^i x^j + \varepsilon_i x^i. \tag{4.2.15}$$

Example 4.11. For a Lie algebra dual $\mathfrak{g}^*$, equipped with the natural Poisson structure of Example 4.5, the Poisson tensor has the contraction of the **structure constants** of the Lie algebra (the numbers c_k^{ij} below) as components:

$$\Pi^{ij} = \{x^i, x^j\} = [dx^i, dx^j]_{\mathfrak{g}} = c_k^{ij} x^k. \tag{4.2.16}$$

4.2.2 The Schouten–Nijenhuis bracket

The Jacobi condition has a more concise expression in terms of the **Schouten–Nijenhuis bracket**, which is a graded extension of the customary Lie bracket to (skew-symmetric) multivector fields:

$$[-,-] : \mathfrak{X}^m(M) \times \mathfrak{X}^n(M) \longrightarrow \mathfrak{X}^{n+m-1}(M), \qquad \text{where } \mathfrak{X}^k(M) = \Gamma(TM^{\wedge k}). \tag{4.2.17}$$

It is defined inductively starting from the Lie bracket (taken for the case $n = m = 1$) and the derivative of functions along vector fields (taken for the case $n = 1$, $m = 0$) by imposing the rules to make it into a graded anti-derivation on $\bigwedge^\bullet TM$:

1. **Graded skew-symmetry**:

$$[A, B] = (-1)^{(|A|-1)(|B|-1)}[B, A]. \tag{4.2.18}$$

2. **Graded Leibniz rule**:

$$[A, B \wedge C] := [A, B] \wedge C + (-1)^{|B|(|A|-1)} B \wedge [A, C]. \tag{4.2.19}$$

In this formalism, the Jacobi identity for the Poisson tensor can be expressed intrinsecally as:

$$[\Pi, \Pi] = 0. \tag{4.2.20}$$

Exercise 4.12. Show the Schouten–Nijenhuis bracket inherits a graded version of the Jacobi identity for the Lie bracket:

$$\begin{aligned}(-1)^{(|A|-1)(|C|-1)} [A, [B, C]] \\ + (-1)^{(|B|-1)(|A|-1)} [B, [C, A]] \\ + (-1)^{(|C|-1)(|B|-1)} [C, [A, B]] = 0.\end{aligned} \tag{4.2.21}$$

4.2.3 Poisson maps and Poisson submanifolds

Definition 4.13. A **Poisson map** is a smooth map $\phi : (N, \Pi_N) \to (M, \Pi_M)$ between Poisson manifolds such that

$$\phi_* \Pi_N = \Pi_M. \tag{4.2.22}$$

Using the brackets to express the structure of a Poisson manifold, the condition becomes

$$\{f \circ \phi, g \circ \phi\}_N = \{f, g\}_M \circ \phi, \quad \forall f, g \in \mathcal{C}^\infty(M). \tag{4.2.23}$$

Exercise 4.13 ([**fernandes2014lectures**])**.** Show the map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^n$, where the domain and codomain are equipped with the canonical and quadratic Poisson structures respectively (Example 4.8 and Example 4.9), defined by

$$(p^i, q^j) \mapsto x^i = \mathrm{e}^{p^i - \frac{1}{2} a_{ij} q^j} \tag{4.2.24}$$

is Poisson.

Example 4.14 ([**fernandes2014lectures**])**.** We've seen in Example 4.5 that the dual of any Lie algebra carries a natural Poisson structure. In fact, for any map $\Psi : \mathfrak{h} \to \mathfrak{g}$, the dual map $\Psi^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map for the natural structures

Definition 4.14. A **Poisson submanifold** of a Poisson manifold (M, Π_M) is a Poisson manifold (N, Π_N) together with a Poisson injective immersion $\iota : N \hookrightarrow M$.

4.3 The symplectic foliation of a Poisson manifold

Notice that when the rank of Π is constant throughout M we obtain a *distribution* on M, given by $\operatorname{im}\Pi^\sharp$. It is smooth since it is generated by the vector fields $X_f = \Pi^\sharp(df)$.

Lemma 4.15 ([**fernandes2014lectures**])**.** *A submanifold N of a Poisson manifold (M,Π) is Poisson if only if the distribution induced by $\Pi^\sharp$ on M is a distribution on N too:*

$$\operatorname{im}\Pi|_p^\sharp \subseteq d\iota_p(T_pN), \quad \forall p \in N. \tag{4.3.1}$$

Moreover, in this case the Poisson substructure on N is unique as such.

Proof. Suppose N is a Poisson submanifold. We want to prove (4.3.1) holds. Notice first that Π_N is ι-*related* to Π (because, more strongly, it is its pushforward), that is

$$d\iota_p(\Pi_N|_p) = \Pi|_{\iota(p)}, \quad \forall p \in N, \tag{4.3.2}$$

which basically means $\Pi|_N = \Pi_N$. Then, sharping both tensors and precomposing $\Pi_N^\sharp$ with the pullback of covectors, we clearly have:

$$d\iota_p \circ \Pi_N^\sharp|_p \circ d\iota_p^* = \Pi|_{\iota(p)}^\sharp, \quad \forall p \in N. \tag{4.3.3}$$

This means the following commutes

$$\begin{array}{ccc} T^*_{\iota(p)}M & \xrightarrow{\Pi|^\sharp_{\iota(p)}} & T_{\iota(p)}M \\ \Big\downarrow{\scriptstyle d\iota_p^*} & & \Big\uparrow{\scriptstyle d\iota_p} \\ T^*_pN & \xrightarrow{\Pi_N|_p} & T_pN \end{array} \tag{4.3.4}$$

and it's now very apparent both (4.3.1) (by commutativity in the upper right corner) and uniqueness of Π_N (by injectivity of $d\iota_p$) hold.

Conversely, assume now (4.3.1) holds. Then the previous considerations about the maps $d\iota_p$ and $d\iota_p^*$ allow us to define

$$\Pi_N^\sharp|_p = (d\iota_p)^{-1} \circ \Pi^\sharp|_{\iota(p)} \circ (d\iota_p^*)^{-1}, \tag{4.3.5}$$

where the inverses are just left and right sided respectively, and meant to be set-theoretical inverses. This means $d\iota_p^{-1}$ is defined by sending any $v \in T_{\iota(p)}M \setminus d\iota_p(T_pN)$ to an arbitrarly chosen $v \in T_pN$, whereas $(d\iota_p^*)^{-1}$ choses an arbitrary lift $\hat{\alpha}$ from the $d\iota_p$-preimage of any $\alpha \in T_p^*M$. Then we want to show the values of Π_N do not actually depend on any of these choices. As for the first case, the assumption (4.3.1) guarantees no point outside $d\iota(TN)$

is ever passed to $d\iota_p^{-1}$, hence the choice of v doesn't matter and doesn't ruin smoothness and linearity. As for the second, let $\hat{\alpha}, \hat{\alpha}'$ be both different lifts of α, i.e.

$$d\iota_p^* \hat{\alpha} = d\iota_p^* \hat{\alpha}' \implies \hat{\alpha} - \hat{\alpha}' = \beta \in \ker d\iota_p^*. \tag{4.3.6}$$

Recall $\ker d\iota_p^* = \{\psi \mid \psi(\operatorname{im} d\iota_p) = 0\}$, so that, in particular, β vanishes on $\operatorname{im} \Pi^\sharp|_{\iota(p)}$. Then, however chosen $\gamma \in T^*_{\iota(p)}M$,

$$\begin{aligned} \langle \Pi^\sharp|_{\iota(p)}(\hat{\alpha}) - \Pi^\sharp|_{\iota(p)}(\hat{\alpha}'), \gamma \rangle &= \langle \Pi^\sharp|_{\iota(p)}(\beta), \gamma \rangle \\ &= \Pi|_{\iota(p)}(\beta, \gamma) \qquad \text{by undoing the } \sharp \\ &= -\Pi|_{\iota(p)}(\gamma, \beta) \qquad \text{by skew-symmetry of } \Pi \\ &= -\langle \Pi|_{\iota(p)}(\gamma), \beta \rangle \\ &= \beta(\Pi|_{\iota(p)}(\gamma)) = 0. \end{aligned} \tag{4.3.7}$$

Thus $\Pi^\sharp|_{\iota(p)}(\hat{\alpha}) = \Pi^\sharp|_{\iota(p)}(\hat{\alpha}')$, as desired.

Therefore, it's only left to show Π_N is a Poisson tensor. Bilinearity and skew-symmetry easily permeate from Π. To show Jacobi's identity, we recall that pushforwards preserve the Lie bracket hence, by induction on the grading, the SN bracket too. Meanwhile, the previous argument showed $\iota_* \Pi_N = \Pi$. Putting all together, we get

$$\iota_*[\Pi_N, \Pi_N] = [\iota_* \Pi_N, \iota_* \Pi_N] = [\Pi, \Pi] = 0, \tag{4.3.8}$$

and since ι is an immersion, this implies $[\Pi_N, \Pi_N] = 0$, as wished. □

We've seen a symplectic structure always gives rise to a Poisson structure, yet the opposite is not true. In fact, Π isn't always non-degenerate, and this prevents Π^{-1} to be a well-defined symplectic form. However, we know from (multi)linear algebra (by Theorem 2.2 for instance) that, at least pointwise, Π has $2n$ dimensions on which is degenerate and k on which is not, such that $\dim M = 2n + k$.

The aim of this section is then to show this pointwise argument has global and local refinements.

Definition 4.16. A point $p \in M$ is called **regular for the Poisson structure** Π if there exists a neighbourhood U of p on which the rank of Π is constant. Otherwise p is called **singular**.

Lemma 4.17. *The set of regular points $M_{reg} \subseteq M$ for a Poisson manifold (M, Π) is open and dense.*

Proof. Openness is obvious since non-singularity is a non-vanishing condition for some minors, hence an "open" condition. Let $p_0 \in M$ be singular. Take a neighbourhood V

of p_0 and consider the function $p \mapsto \operatorname{rk}(\Pi_p)$ on V. It takes a finite number of values (as it is finite and bounded by $\dim M$). Then it has a maximum value attained at a point $p' \in V$. In local coordinates, this means $\Pi^{ij}(p')$ has a minor or size $\operatorname{rk}(\Pi_{p'})$ whose determinant doesn't vanish. By a continuity argument, this actually happens in a whole open neighbourhood W of p' and since the rank in p' is maximal, this means in W the rank is constant implying p' is regular. As V was arbitrary, we proved that any open neighbourhood of a point M contains a regular point, hence regular points are dense in M as claimed. □

Theorem 4.18. *Let (M, Π) be a Poisson manifold with Π of constant rank. Then $\operatorname{im}\Pi^\sharp$ is an integrable distribution and on each leaf S there is a unique induced symplectic structure $\omega \in \Omega^r(S)$ such that*

$$\iota_*(\omega^{-1}) = \Pi|_{\iota(S)} \tag{4.3.9}$$

where ι is the inclusion map of S in M.

Proof. The distribution $\operatorname{im}\Pi^\sharp$ is generated by Hamiltonian vector fields (per (4.2.6)) and hence involutive by Theorem 3.9. Therefore, by Frobenius' Theorem (Theorem 1.21), it is an integrable distribution. Then M foliates into leaves (i.e., maximal integral submanifolds of $\operatorname{im}\Pi^\sharp$), denote S one of them. By definition $\operatorname{im}\Pi|_p^\sharp = d\iota_p(T_pS)$, so by Lemma 4.15 there exists a unique Poisson structure Π_S on S, which satisfies

$$\Pi_S^\sharp|_p = (d\iota_p)^{-1} \circ \Pi^\sharp|_p \circ (d\iota_p^*)^{-1} \tag{4.3.10}$$

so that

$$\begin{aligned} \operatorname{im}\Pi_S^\sharp &= \operatorname{im}(d\iota^{-1}\Pi^\sharp(d\iota^*)^{-1}) \\ &= \operatorname{im}(d\iota^{-1}\Pi^\sharp) \qquad \text{by surjectivity of } d\iota^* \\ &= d\iota^{-1}\operatorname{im}(\Pi^\sharp) \qquad \text{by surjectivity of } d\iota \\ &= d\iota^{-1}d\iota(TS) = TS. \end{aligned} \tag{4.3.11}$$

Hence $\Pi^\sharp$ is surjective. Since both S and M are finite-dimensional, this suffices to prove Π_S is non-degenerate, hence the inverse of a symplectic form ω. Then the claim is just the pullback condition (4.2.22). □

When (M, Π) is of constant rank (the **regular case**) then this theorem basically tells us **Poisson manifolds are a lasagna of symplectic manifolds**, or more formally, any Poisson manifold is the disjoint union of symplectic submanifolds of the same dimension. On the other hand, even in the general case this holds at least locally (morally, by a smoothness argument applied to the reasoning we did initially):

Theorem 4.19 (Darboux–Weinstein)**.** *Let* (M, Π) *be a Poisson manifold and* $p \in M$*. There exists a chart*

$$(U, (x^1, \dots, x^n, \xi_1, \dots, \xi_n, y^1, \dots, y^k)) \tag{4.3.12}$$

centered at p *such that*

$$\Pi = \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_i} + \frac{1}{2} c^{ij} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} \tag{4.3.13}$$

with $c^{ij}(p) = 0$*.*

Notice n depends on p, and — in general— isn't constant on M. The idea of the theorem is that Π^{-1} **is "almost symplectic": there are** k **degenerate directions given by the** y^i.

Remark 4.19.1. If (M, Π) is Poisson manifold and Π has constant rank $2n$ and corank k, then Theorem 4.18 tells M is locally foliated by symplectic leaves given by the level sets of k Casimir functions $y^1, \dots, y^k$ such that

$$\Pi(dy^1) = \dots = \Pi(dy^k) = 0. \tag{4.3.14}$$

We can define (everywhere but only locally) a family of Darboux charts $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ on the leaves, smoothly varying along the transverse direction, which together with the k Casimir functions form a Darboux–Weinstein chart such that

$$\Pi = \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_i}, \tag{4.3.15}$$

that is, such that $c^{ij} = 0 \ \forall i, j$.

Finally the following result, which we do not prove, shows that the intuition about Poisson manifolds being stacks of symplectic submanifolds is still true: we just need to account for singular leaves. Hence the constant rank condition just buys us regularity of the foliation.

Theorem 4.20 ([**fernandes2014lectures**])**.** *Let* (M, Π) *be a Poisson manifold. Then every point* $x_0 \in M$*, regular or not, is contained in a single symplectic leaf, i.e. a maximal integrable submanifold of* $\operatorname{im} \Pi^\sharp$*.*

Example 4.15 (Euler's equations for a rigid body)**.** On $\mathbb{R}^3$ the Poisson bracket is

$$\{f, g\}(x, y, z) = (df \times dg)(x, y, z) = \begin{vmatrix} \partial_x f & \partial_y f & \partial_z f \\ \partial_x g & \partial_y g & \partial_z g \\ x & y & z \end{vmatrix} \tag{4.3.16}$$

This Poisson structure is the same as the one on $\mathfrak{so}(3)$ – by the isomorphism we exhibited in Example 4.3 – and $\mathfrak{so}^*(3)$, since the inner product of $\mathbb{R}^3$ induces an identification. In more practical terms, we get the fundamental Poisson brackets

$$\{x, y\} = z, \quad \{x, z\} = -y, \quad \{y, z\} = x. \tag{4.3.17}$$

To get the symplectic foliation induced by the Poisson structure, we need to find a Casimir function for the bracket. Such is any solution to the PDE $\Pi^\sharp(df) = 0$, which in this case is

$$\begin{cases} x\partial_z f - z\partial_x f = 0 \\ y\partial_x f - x\partial_y f = 0 \\ z\partial_y f - y\partial_z f = 0 \end{cases} \tag{4.3.18}$$

One possible solution of this system is $f(x, y, z) = x^2 + y^2 + z^2$. The induced foliation is then, clearly, a foliation in concentric spheres, and it is easy to see they inherit a symplectic structure from the Poisson tensor. Moreover, we get regular leaves for $f > 0$ and a singular leaf at $f = 0$.

??????

4.4 Poisson cohomology

4.5 Poisson vector fields

Definition 4.21. A **Poisson vector field** $X \in \mathfrak{X}(M)$ on a Poisson manifold (M, Π) is a vector field such that

$$\mathcal{L}_X \Pi = 0 \tag{4.5.1}$$

or, equivalently (applying Leibniz rule):

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}. \tag{4.5.2}$$

Echoing analogous statements for Hamiltonian and symplectic fields, we have:

Theorem 4.22. *Let (M, Π) be a Poisson manifold. Then*

1. *the flow ϕ_t of a Poisson vector field $X \in \mathfrak{X}(M)$ preserves the Poisson structure:*

$$(\phi_t)_* \Pi = \Pi, \tag{4.5.3}$$

2. *Hamiltonian vector fields are Poisson,*

3. *the bracket of Poisson fields is Poisson.*

Proof.

1. The same as in the symplectic case.

2. By definition of Hamiltonian vector field in a Poisson manifold and leveraging Jacobi's identity:

$$X_h(\{f,g\}) = -\{h,\{f,g\}\} = \{f,\{g,h\}\} + \{g,\{h,f\}\} = \{X_h(g),f\} + \{g,X_h(f)\}. \tag{4.5.4}$$

3. Suppose X and Y are Poisson fields, then

$$\mathcal{L}_{[X,Y]}\Pi = \mathcal{L}_X\mathcal{L}_Y\Pi - \mathcal{L}_Y\mathcal{L}_X\Pi = 0. \tag{4.5.5}$$

□

Remark 4.22.1. As in the symplectic case, not every Poisson vector field is Hamiltonian: consider any smooth manifold M equipped with the trivial Poisson structure $\Pi = 0$. Then any vector field is Poisson, yet only the zero section is an Hamiltonian vector field: $\Pi^\sharp(df) = 0 \in \mathfrak{X}(M)$ for any $f \in \mathcal{C}^\infty(M)$. Moreover, in the case Π is non-degenerate, the manifold is symplectic so the same considerations of Remark 3.3.1 apply, as we will see in the next section.

We clearly see a parallel between Poisson vs. Hamiltonian fields and symplectic vs. Hamiltonian fields. Indeed, the symplectic definitions are identical to the Poisson analogous. We've already analyzed the relationship between Hamiltonian and symplectic fields: we've seen they stand in the same relationship as closed and exact forms of the ambient manifold are (Remark 3.3.1), hence the first de Rham cohomology already gives which obstructions might prevent a symplectic field from being Hamiltonian. In the same spirit, we wish now to construct a cohomology theory which could encode the obstructions preventing a Poisson field from being Hamiltonian, that is, Poisson cohomology.

To start, we look for a coboundary map in the definitions of Poisson and Hamiltonian fields. A Poisson vector field is defined by the condition

$$\mathcal{L}_X\Pi = 0 \tag{4.5.6}$$

and an Hamiltonian vector field by

$$X = \Pi^\sharp(df) \text{ for some } f \in \mathcal{C}^\infty(M). \tag{4.5.7}$$

We leverage the inductive definition of the Schouten–Nijenhuis bracket to convert the Lie derivative along X in a bracket:

$$[\Pi, X] = 0 \tag{4.5.8}$$

and in the same vein, we can express $\Pi^\sharp(df)$ as a bracket:

$$\begin{aligned}\Pi^\sharp(df) &= \frac{1}{2}\Pi^{ij}\partial_j f\,\partial_i - \frac{1}{2}\Pi^{ij}\partial_i f\,\partial_j \\ &= \frac{1}{2}\Pi^{ij}\partial_i \wedge [f,\partial_j] - \frac{1}{2}[f,\Pi^{ij}\partial_i]\wedge\partial_j \\ &= [\frac{1}{2}\Pi^{ij}\partial_i\wedge\partial_j, f] \\ &= [\Pi, f].\end{aligned} \tag{4.5.9}$$

Hence we might guess our coboundary map to be $d_\Pi = [\Pi, -]$. To be such a map, it must square to zero:

Lemma 4.23. *Let $\Pi \in \mathfrak{X}^2(M)$ be a Poisson tensor. Then*

$$[\Pi,[\Pi,A]] = 0, \quad \forall A \in \mathfrak{X}^k(M). \tag{4.5.10}$$

Proof. Using the graded Jacobi identity (Exercise 4.12), and recalling $|\Pi| = 2$:

$$\begin{aligned}(-1)^{k-1}\,[\Pi,[\Pi,A]] - [\Pi,[A,\Pi]] + (-1)^{k-1}\,[A,\underbrace{[\Pi,\Pi]}_{=0}] &= 0 \\ 2(-1)^{k-1}\,[\Pi,[\Pi,A]] &= 0 \qquad \text{by graded skew-symmetry} \\ [\Pi,[\Pi,A]] &= 0.\end{aligned} \tag{4.5.11}$$

□

The cochain complex arising from these coboundaries is called the **Poisson complex**:

$$\ldots \xrightarrow{d_\Pi} \mathfrak{X}^p(M) \xrightarrow{d_\Pi} \mathfrak{X}^{p+1}(M) \xrightarrow{d_\Pi} \ldots \tag{4.5.12}$$

Definition 4.24. The r-th Poisson cohomology of (M,Π) is

$$H^r_\Pi(M) = \ker d^{(r)}_\Pi / \operatorname{im} d^{(r-1)}_\Pi \tag{4.5.13}$$

where $d^{(k)}_\Pi$ denotes $d_\Pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$.

Example 4.16. For the trivial Poisson tensor $\Pi = 0$, $H^r_\Pi \cong \mathfrak{X}^r(M)$.

Example 4.17. In the symplectic case (Π non-degenerate), then $\Pi^\sharp$ induces an identification of $\Omega^\bullet$ with $\mathfrak{X}^\bullet$, which commutes with the de Rham's and Poisson differentials:

$$\begin{array}{ccccccc} \ldots & \xrightarrow{d} & \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) & \xrightarrow{d} & \ldots \\ & & \downarrow{\scriptstyle \Pi^\sharp} & & \downarrow{\scriptstyle \Pi^\sharp} & & \\ \ldots & \xrightarrow{d_\Pi} & \mathfrak{X}^p(M) & \xrightarrow{d_\Pi} & \mathfrak{X}^{p+1}(M) & \xrightarrow{d_\Pi} & \ldots \end{array} \tag{4.5.14}$$

This is called, in homological algebra, a **(co)chain isomorphism** and implies that the cohomologies arising from the two complexes are naturally isomorphic. Thus, **in the symplectic case, Poisson cohomology coincides with de Rham cohomology**, as expected.

Example 4.18. Consider the Lie–Poisson structure on $\mathfrak{g}^*$. For this case, it turns out

$$H^k_\Pi(\mathfrak{g}^*) \cong H^k_L(\mathfrak{g}) \otimes \mathrm{Cas}(\mathfrak{g}^*) \tag{4.5.15}$$

where

1. $H^k_L(\mathfrak{g})$ is the k-th Lie cohomology[3] of $\mathfrak{g}$, and
2. $\mathrm{Cas}(\mathfrak{g}^*)$ is the space of Casimir functions on $\mathfrak{g}^*$.

4.5.1 Meaning of the Poisson cohomologies

We now set to interpret the Poisson cohomologies of a given manifold (M, Π). Let's start with the zeroth cohomology: it is the space of all $f \in \mathcal{C}^\infty(M)$ such that $[\Pi, f] = 0$. As we've shown in a previous calculation, this is the same as saying $\{f, -\} = 0$, meaning

$$H^0_\Pi(M) = \mathrm{Cas}(M) = \text{Casimir functions.} \tag{4.5.17}$$

Next, the interpretation of the first cohomology was our motivational example:

$$H^1_\Pi(M) = \frac{\text{Poisson vector fields}}{\text{Hamiltonian vector fields}} \tag{4.5.18}$$

Finally, both 2-cocycles and 2-boundaries assume meaning in the context of **Poisson infinitesimal deformations**. A *deformation* of Π is another Poisson tensor Π' obtained by adding a bivector $\Delta = \Pi' - \Pi$. An *infinitesimal deformation* is a deformation "up to the first-order", i.e. it is of the form $\Pi + \varepsilon\Delta$ for $\varepsilon > 0$, but we only require $[\Pi', \Pi'] = O(\varepsilon^2)$, hence

$$[\Pi + \varepsilon\Delta, \Pi + \varepsilon\Delta] = [\Pi,\Pi] + 2\varepsilon[\Pi, \Delta] + \varepsilon^2[\Delta, \Delta] = O(\varepsilon^2) \iff [\Pi, \Delta] = 0. \tag{4.5.19}$$

[3] The **Lie cohomology** of a (real) Lie algebra $\mathfrak{g}$ of finite dimension n is defined from the Chevalley–Eilenberg complex, whose objects are the k-th exterior powers of $\mathfrak{g}^*$ and the coboundary maps $d^{(k)}_L : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$ are given by:

$$(d_L\alpha)(X_0, \ldots, X_k) = \sum_{0\le i\le j\le n} (-1)^{i+j}\alpha([X_i, X_j], X_2, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \tag{4.5.16}$$

Alternatively, d_L can be defined axiomatically and inductively from its values on the 1-$\mathfrak{g}$-forms

From a cohomological point of view, this latter condition might be better expressed as $\Delta \in \ker d_\Pi^{(2)}$.

However, a deformation of Π can be also given through a local diffeomorphism, like the flow of a vector field X. In this way, the deformed Poisson tensor is[4]

$$(\phi_\varepsilon)_*\Pi = \Pi + \varepsilon \mathcal{L}_X \Pi = \Pi + \varepsilon(\underbrace{\mathcal{L}_X \Pi}_{\Delta}). \tag{4.5.20}$$

As we see in the above expression, this second case can be seen as a particular form of the first one, in fact a "trivial" one: diffeomorphisms are just changes of coordinates, and they don't really change the Poisson structure. Notice moreover that Δ is given above as the Taylor expansion of $[\Pi, X]$, hence $\Delta \in \operatorname{im} d_\Pi^{(1)}$.

Therefore we reach an interpretation for $H^2_\Pi(M)$: **it is the space of non-trivial infinitesimal deformations of** Π:

$$H^2_\Pi(M) = \frac{\text{infinitesimal deformations of } \Pi}{\text{trivial infinitesimal deformations of } \Pi} \tag{4.5.21}$$

Remark 4.24.1. The fact that first-order infinitesimal deformations exist does not imply a full-blown deformation exist! However, if it does, then it determines completely the coefficients of such deformation.

Higher cohomologies do not have, in general, an enlightening interpretation in terms of geometrical property of the manifold, but the general cohomological interpretation remains valid: they represent obstructions to "deformations" of higher orders.

Example 4.19. Let $M = S^2$ equipped with the usual symplectic structure (the area form pulled back from the volume form of $\mathbb{R}^3$). We know then $H^2_\Pi(S^2) \cong H^2_{dR}(S^2) \cong \mathbb{R}$ because, up to rescaling, there's only one volume closed form which is not exact.

Indeed, any infinitesimal deformation has the form $\Pi + \varepsilon\Pi = (1+\varepsilon)\Pi$, i.e. a rescaling. But the diffeomorphism between a sphere of radius 1 and one of radius $1+\varepsilon$ is not a symplectomorphism since it is not area preserving, hence they are all different Poisson structures.

Example 4.20 (Ginzburg, [**ciccoli2006lectures**])**.** Consider $(\mathbb{R}^2, \Pi_0 = (x^2+y^2)\,\partial_x \wedge \partial_y)$. The Poisson tensor has only one singular point, the origin. Therefore the symplectic foliation is composed of two leaves: the origin and the rest of $\mathbb{R}^2$. The Poisson cohomology is

$$\begin{aligned} H^0_\Pi(\mathbb{R}^2, \Pi_0) &= \mathbb{R} \qquad \text{(since constant functions on the symp. leaves are Casimir)} \\ H^1_\Pi(\mathbb{R}^2, \Pi_0) &= \operatorname{span}_{\mathbb{R}}\{x\,\partial_x + y\,\partial_y,\ y\,\partial_x - x\,\partial_y\} \\ H^2_\Pi(\mathbb{R}^2, \Pi_0) &= \operatorname{span}_{\mathbb{R}}\{\partial_x \wedge \partial_y,\ \Pi_0\}. \end{aligned} \tag{4.5.22}$$

[4]Notice higher order terms here vanish completely since $\mathcal{L}_X^2\Pi = [\Pi, [\Pi, X]] = 0$ by Lemma 4.23.

See the reference for explicit computations.

Chapter 5

Integrability

The first two sections are based on [**oevel1989r**].

5.1 Classical R-matrix

Definition 5.1. An **R-matrix** is a linear map $R : \mathfrak{g} \to \mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ having the property that the modified bracket

$$[X, Y]_R = [RX, Y] + [X, RY], \quad \forall X, Y \in \mathfrak{g} \tag{5.1.1}$$

defines a Lie product on $\mathfrak{g}$.

As customary, to the R-bracket on $\mathfrak{g}$ is associated a Poisson bracket on $\mathfrak{g}^*$:

$$\{f, g\}_R(\xi) = \langle [df, dg]_R, \xi \rangle, \quad \forall \xi \in \mathfrak{g}^*, \forall f, g \in \mathcal{C}^\infty(\mathfrak{g}^*). \tag{5.1.2}$$

Proposition 5.2. *A linear map $R : \mathfrak{g} \to \mathfrak{g}$ on a Lie algebra $(\mathfrak{g}, [-,-])$ is an R-matrix if and only if*

$$[B_R(X, Y), Z] + [B_R(Y, Z), X] + [B_R(Z, X), Y] = 0 \tag{5.1.3}$$

where

$$B_R(X, Y) = [RX, RY] - R[X, Y]_R \tag{5.1.4}$$

Proof. Exercise, it's just a computation. □

The operator B_R measures how far R is from being a Lie algebra homomorphism. It is quadratic on both terms. Notice that, while $B_R = 0$ certainly satisfies the hypotheses of the theorem, it doesn't have to be the case:

Corollary 5.2.1. *A* sufficient *condition for a linear map $R : \mathfrak{g} \to \mathfrak{g}$ to be an R-matrix is for R to satisfy the* ***modified Yang–Baxter equation*** *(mYBE):*

$$B_R(X, Y) = -c[X, Y], \tag{5.1.5}$$

for some $c \in \mathbb{R}$.

Clearly, up to rescaling of R, there are actually just two cases of the mYBE: the one with $c = 0$ (corresponding to the case where R is an automorphism of the algebra) and the one with $c = 1$. For this second case, there is a special class of solutions that arises in a very simple manner:

Construction 5.3. Let $(\mathfrak{g}, [-,-])$ be a Lie algebra which decomposes (as vector space!) as $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, for $\mathfrak{g}_\pm \leq \mathfrak{g}$. We do **not** require $[\mathfrak{g}_+, \mathfrak{g}_+] = 0$, i.e. the direct sum does not need to be one of Lie algebras. Let $\mathrm{pr}_\pm$ denote the projections onto $\mathfrak{g}_+$ and $\mathfrak{g}_+$ respectively and write $X_\pm = \mathrm{pr}_\pm X$ for any vector $X \in \mathfrak{g}$. Then we claim $R = \frac{1}{2}(\mathrm{pr}_+ - \mathrm{pr}_-)$ is an R-matrix. In fact observe that, by virtue of the bilinearity of $[-,-]$, the R-bracket follows the splitting perfectly:

$$\begin{aligned}
[X, Y]_R &= \frac{1}{2}[X_+ - X_-, Y] + \frac{1}{2}[X, Y_+ - Y_-] \\
&= \frac{1}{2}\left([X_+ - X_-, Y_+ + Y_-] + [X_+ + X_-, Y_+ - Y_-]\right) \\
&= \frac{1}{2}\left([X_+, Y_+] - [X_-, Y_-]\right)
\end{aligned} \tag{5.1.6}$$

Then its clear we can decompose the Jacobi identity of $[-,-]_R$ into two Jacobi identities for $[-,-]$, which hold by assumption. This class of examples is very ample as any decomposition of $\mathfrak{g}$ works.

Occasionally people find solutions of the mYBE which do not come from the decomposition we've treated. In fact, there are a lot of examples but the great majority of them is in the infinite-dimensional context. Indeed, this theory comes from that environment (in particular the study of PDEs), and only later it was realized there was some honest finite-dimensional geometry behind it.

Proposition 5.4. *If $h \in \mathrm{Cas}(\mathfrak{g}^*, \{-,-\})$ then its Hamiltonian vector field with respect to the Poisson R-bracket is*

$$X_h^R|_\alpha = \mathrm{ad}^*_{R(dh|_\alpha)}\,|_\alpha. \tag{5.1.7}$$

Proof. For all $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $\alpha \in \mathfrak{g}^*$,

$$\begin{aligned} X_h^R(f)|_\alpha &= \{f,h\}_R|_\alpha \\ &= \langle [df|_\alpha, dh|_\alpha]_R, \alpha\rangle \\ &= \langle [R(df|_\alpha), dh|_\alpha] + [df|_\alpha, R(dh|_\alpha)], \alpha\rangle \\ &= \langle [R(df|_\alpha), dh|_\alpha], \alpha\rangle + \langle [df|_\alpha, R(dh|_\alpha)], \alpha\rangle \\ &= \langle \operatorname{ad}_{R(df|_\alpha)}|_{dh|_\alpha}, \alpha\rangle - \langle \operatorname{ad}_{R(dh|_\alpha)}|_{df|_\alpha}, \alpha\rangle \\ &= -\operatorname{ad}^*_{R(df|_\alpha)}(h)|_\alpha + \operatorname{ad}^*_{R(dh|_\alpha)}(f)|_\alpha \end{aligned} \tag{5.1.8}$$

Since ad^*_X is tangent to the symplectic leaves (Theorem 4.7) and h is Casimir, the first term vanishes and the claim follows. □

Corollary 5.4.1. *If $f, g, \in \operatorname{Cas}(\mathfrak{g}^*, \{-,-\})$, then they commute with respect to the Poisson R-bracket.*

Proof. Since f is constant on coadjoint orbits:

$$\{f,g\}_R = \operatorname{ad}^*_{R(dg)}(f) = 0. \tag{5.1.9}$$

□

Remark 5.4.1. The flow of X_h^R for h Casimir lies in the intersection of the coadjoint orbits of $(\mathfrak{g}, [-,-])$ and $(\mathfrak{g}, [-,-]_R)$. So it is ad^* of something but also the ad^*_R of something.

Remark 5.4.2. Let $(-,-)$ be an Ad-invariant (bilinear, non-degenerate) form on $\mathfrak{g}$. Then this provides an identification $\mathfrak{g} \cong \mathfrak{g}^*$ sending ad_X to $-\operatorname{ad}^*_X$. In this case the equations of motion (Hamilton's) become (for the R-matrix)

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\operatorname{ad}_{R(dh)}|_X = [X, R(dh)]. \tag{5.1.10}$$

Example 5.1. In the case $\mathfrak{g} = \mathfrak{gl}(n)$ (the Lie algebra associated to square matrices with complex entries, but seen as a real manifold). Then $\operatorname{Ad}_g X = gXg^{-1}$. We consider the form $(X,Y) := \operatorname{Tr}(XY)$. It is Ad-invariant since

$$\begin{aligned} \operatorname{Tr}(gXg^{-1}gYg^{-1}) &= \operatorname{Tr}(gXYg^{-1}) \\ &= \operatorname{Tr}(g^{-1}gXY) \qquad \text{b.c. trace is cyclic-invariant} \\ &= \operatorname{Tr} XY. \end{aligned} \tag{5.1.11}$$

Some Casimir functions in $(\mathfrak{g}^*, \{-,-\})$ are given by $h_i = \operatorname{Tr}(X^i)$ (where the apex is an exponent), because any such function is Ad-invariant, hence constant on (co)adjoint orbits.

By the last remark, *any* Casimir function h on the Poisson algebra on $\mathfrak{g}^*$ let's you find the evolution given by X:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [X, R(dh)] \tag{5.1.12}$$

The h_i, for $i = 1, \ldots, n$, are in involution, so they represent symmetries of the system. These will turn out to be the most general spectral invariants of the matrix X.

These last ideas will be applied in the informative example of the Toda lattice (Example 5.2).

Given an Ad-invariant form $(-,-)$ and an R-matrix, then if $h \in \mathcal{C}^\infty(\mathfrak{g}^*)$ from the computation of Lemma 5.4 we get:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [X, R\nabla h] + R^*[X, \nabla h] \tag{5.1.13}$$

where R^* is the $(-,-)$-adjoint of R ($(RX, Y) = (X, R^*Y)$). If h is Casimir on $\mathfrak{g}^*$ the second term drops, as we've seen above. Alternatively, in terms of adjoint actions:

$$X_h^R(f)|_\alpha = \mathrm{ad}^*_{R(dh|_\alpha)}(f)|_\alpha - \mathrm{ad}^*_{R(df|_\alpha)}(h)|_\alpha. \tag{5.1.14}$$

In Equation (5.1.13), we could put any $A \in \mathfrak{g}$ in lieu of ∇h, interpreted as a 1-form ??.

Every other Casimir provides an integral of motion and they all commute. In the case of $\mathfrak{gl}(n)$, or its Lie subalgebras, a set of Casimirs is given by $h_i = \operatorname{Tr} X^i$.

5.2 Lax Representations

Definition 5.5. Given a vector field $X \in \mathfrak{X}(N)$ on a smooth manifold N, an open $U \subseteq N$, a **Lax representation for** (X, U) is a smooth map $(L, M) : N \to \mathfrak{gl}(n) \times \mathfrak{gl}(n)$ such that

$$L_* X(L(x)) = [L(x), M(x)] \tag{5.2.1}$$

or, equivalently,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M]. \tag{5.2.2}$$

It mimics the evolution of a quantum operator in the Heisenberg formalism.

Proposition 5.6. *The eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ of the Lax matrix L are integrals of motion of the system.*

Proof. Let $t_k = \operatorname{Tr} L^k$, then eigenvalues of L are roots of its characteristic polynomial:

$$\det(L - \lambda I) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + \ldots + c_n). \tag{5.2.3}$$

Its coefficients can be written in terms of the trace through **Newton's formula**:

$$c_m = -\frac{t_m}{m} + \frac{1}{2!}\sum_{i+j=m}\frac{t_i t_j}{ij} - \frac{1}{3!}\sum_{i+j+k=m}\frac{t_i t_j t_k}{ijk} + \ldots + (-1)^m\frac{t_1^m}{m!}. \tag{5.2.4}$$

Now, using the cyclic invariance of the trace to tackle the derivative, we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr} L^k &= k\operatorname{Tr}([L,M]L^{k-1}) \\ &= k\operatorname{Tr}(LML^{k-1} - ML^k) \\ &= 0.\end{aligned} \tag{5.2.5}$$

Then, as $c_1, \ldots, c_n$ depend only on the traces, which we've just shown to be constant of motions, the eigenvalues are constant of motions as well. □

Theorem 5.7. *Let (L, M) be a Lax representation for a vector field on some smooth manifold N. Let $G : N \to \mathrm{GL}(n)$ be a smooth map. Then*

$$L' = GLG^{-1}, \quad M' = GMG^{-1} - \frac{\mathrm{d}G}{\mathrm{d}t}G', \tag{5.2.6}$$

where t is the time of the Lax equation, is another Lax representation.

Remark 5.7.1. The transformation is in the same spirit of the gauge transformation of a fiber bundle connection!

Proof. Using the formula $\frac{\mathrm{d}G^{-1}}{\mathrm{d}t} = -G^{-1}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-1}$, we get

$$\begin{aligned}\frac{\mathrm{d}L'}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(GLG^{-1}) \\ &= \frac{\mathrm{d}G}{\mathrm{d}t}LG^{-1} + G[L,M]G^{-1} - GLG^{-1}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-1} \\ &= \frac{\mathrm{d}G}{\mathrm{d}t}G^{-1}GLG^{-1} + [GLG^{-1}, GMG^{-1}] - GLG^{-1}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-1} \\ &= \left[\frac{\mathrm{d}G}{\mathrm{d}t}G^{-1}, L'\right] + [L', GMG^{-1}] \\ &= [L', M'].\end{aligned} \tag{5.2.7}$$

□

Corollary 5.7.1. *Since L' is conjugate to L, hence they have the same eigenvalues, the Lax representation obtained in this way yields the same integrals of motion.*

Example 5.2 (Classical finite Toda lattice, from [**oevel1989r**])**.** Take $\mathfrak{g} = \mathfrak{gl}(n)$ as before and split it into $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ is the space of lower triangular matrices and $\mathfrak{g}_- = \mathfrak{so}(n)$ the space of skew-symmetric ones. We will use the duality given by the trace to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then

$$\mathfrak{g}_+^* = \mathfrak{g}_-^\perp = \{\text{symmetric matrices}\}, \tag{5.2.8}$$

and

$$\mathfrak{g}_-^* = \mathfrak{g}_+^\perp = \{\text{strictly lower triangular matrices}\}. \tag{5.2.9}$$

Consider the projection given by

$$\begin{aligned} \mathrm{pr}_+(L) &= \underline{\ell}(L) + \overline{u}(L)^T + d(L), \\ \mathrm{pr}_-(L) &= \overline{u}(L) - \overline{u}(L)^T, \end{aligned} \tag{5.2.10}$$

where $\underline{\ell}$ extracts the strictly lower triangular part, $\overline{u}$ the strictly upper triangular part and d the diagonal. Likewise, we define projection operators on the dual algebra:

$$\begin{aligned} \mathrm{pr}_+^*(L) &= d(L) + \overline{u}(L) + \overline{u}(L)^T, \\ \mathrm{pr}_-^*(L) &= \underline{\ell}(L) - \overline{u}(L)^T. \end{aligned} \tag{5.2.11}$$

Then let

$$\begin{aligned} R(L) &= \mathrm{pr}_+(L) - \mathrm{pr}_-(L) = \underline{\ell}(L) + 2\overline{u}(L)^T + d(L) - \overline{u}(L) \\ R^*(L) &= \mathrm{pr}_+^*(L) - \mathrm{pr}_-^*(L) = -\underline{\ell}(L) + 2\overline{u}(L)^T + d(L) + \overline{u}(L). \end{aligned} \tag{5.2.12}$$

We will build a hierarchy of commuting Hamiltonian equations on the space $\mathfrak{g}$. Now, using Equation 5.1.13 we can put a Poisson structure on $\mathfrak{g}$:

$$\Pi_R^\sharp(A)|_L = [L, RA] + R^*[L, A]. \tag{5.2.13}$$

We could now get the integrals of motion from the traces, but that approach would be uninformative for the purpose of finding the system corresponding to the Toda lattice. We take another route. Let

$$S = \{b + aT_- + T_-a\} \underset{\mathsf{LieAlg}}{\subseteq} \mathfrak{g} \tag{5.2.14}$$

where

$$a = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_{n-1} & \\ & & & & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 & & & & \\ & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & \\ & & & & b_n \end{pmatrix}$$

$$T_+ = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & & 1 & 0 \end{pmatrix}. \tag{5.2.15}$$

One could prove that, for any $L \in S$ and $A \in \mathfrak{g}$, $\Pi_R^\sharp(A)|_L \in S$, which means $\Pi_R^\sharp(A)$ is tangent to S if $L \in S$. This means $\operatorname{im}(\Pi_R(L)) \subseteq TS$, hence by Theorem 4.3.1 there is a unique Poisson submanifold structure on S, given by the restriction of Π_R to S. The function $c_k = \frac{1}{k} \operatorname{Tr} L^k$ is Casimir for the Poisson structure of the dual algebra, and

$$\frac{\mathrm{d}L}{\mathrm{d}t_k} = [L, R\nabla c_k] \quad \text{and} \quad \{c_i, c_j\}_R = 0. \tag{5.2.16}$$

For instance, take $k = 2$.

$$\frac{\mathrm{d}L}{\mathrm{d}t_2} = [L, R\nabla c_2] = [L, RL] \iff \begin{cases} \dot{b}_i = 4(a_i^2 - a_{i-1^2}) \\ \dot{a}_i = 2a_i(b_{i+1} - b_i) \end{cases} \tag{5.2.17}$$

These are **Toda equations** and describe a finite line of particles interacting (exponentially) with their neighbours. This is a good example of a system which is not integrable at first sight, but where its abstract study brings out the n integral of motions given by the Casimir functions we found, so we can completely integrate it.

5.3 Bihamiltonian structures

We can think of the space of Poisson structures as a manifold, the submanifold of the manifold of bivectors defined by the quadratic equation (4.2.10). It is not a vector space *per se*, but since it lives inside $\mathfrak{X}^2(M)$, which *is* a vector space, can contain subspaces.

Definition 5.8. A **bihamiltonian structure** on a smooth manifold M is a 2-dimensional linear subspace of the space of Poisson structures on M.

An ordered basis (Π_0, Π_1) of such a space is called a **pair of compatible Poisson tensors**. Since this means any linear combination of them is again a Poisson tensor, we've already seen we must have

$$[\Pi_0, \Pi_1] = 0. \tag{5.3.1}$$

Let us denote $\{-,-\}_i$ the Poisson bracket associated to the Poisson tensor Π_i.

Theorem 5.9 (Magri)**.** *Let M be a smooth manifold equipped with two different Poisson structures, Π_0 and Π_1. Consider a sequence of functions $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}^\infty(M)$ satisfying the following "Hamiltonian recursion":*

$$\{-, h_{p+1}\}_0 = \{-, h_p\}_1, \quad \forall p \in \mathbb{N}. \tag{5.3.2}$$

Then $\{h_p, h_q\}_0 = \{h_p, h_q\}_1 = 0$ for all $p, q \in \mathbb{N}$.

Remark 5.9.1. The relation is not really a recursion, since to really write h_{p+1} in terms of h_p one has to invert $\Pi_0^\sharp$ and then "integrate" which are both operations not possible in general (as Π_0 might be singular).

Proof. Suppose $q - p = 2m$, for $m \neq 0$. Then, by using recursion and skew-symmetry:

$$\begin{aligned} \{h_p, h_q\}_0 &= \{h_p, h_{q-1}\}_1 \\ &= -\{h_{q-1}, h_p\}_1 \\ &= -\{h_{q-1}, h_{p+1}\}_0 \\ &= \{h_{p+1}, h_{q-1}\}_0 \end{aligned} \tag{5.3.3}$$

We managed to dimish the difference of the indices of 2, so that by repeating this m times we'd arrive to $\{h_{p+m}, h_{q-m}\}_0 = 0$. Likewise, if $q - p = 2m + 1$ then we can arrive at

$$\{h_p, h_q\}_0 = \ldots = \{h_{p+m}, h_{q-m}\}_0 = \{h_{p+m}, h_{q-m-1}\}_1 = 0. \tag{5.3.4}$$

Hence all the functions are in involution for the 0-brackets, and by the recursion relation, it is a trivial exercise to check they are in involution for the 1-bracket too. □

Notice **we actually didn't use the compatibility of Π_0 and Π_1 to get this result**. Indeed, it is not necessary. However a bihamiltonian structure gives us a canonical way to produce a sequence of functions in recursion in two particular cases:

1. when the rank of any $\Pi \in \operatorname{span}_{\mathbb{R}}\{\Pi_0, \Pi_1\}$ is constant (and non-maximal), or
2. when one of the Poisson structures is non-degenerate, hence we have at our disposal a symplectic structure.

Let's start with the first one.

5.3.1 The Hamiltonian recursion in the constant rank case

Definition 5.10. A marked line $\Pi_0 + \varepsilon\Pi_1$ in a bihamiltonian structure $\{\Pi_0, \Pi_1\}$ is called a **Poisson pencil**.

Lemma 5.11. *Let $\Pi_\varepsilon = \Pi_0 + \varepsilon\Pi_1$ be a Poisson pencil of constant rank for small ε and $k = \operatorname{cork}(\Pi_0 + \varepsilon\Pi_1) = \operatorname{cork}(\Pi_0)$. Then if $c_1, \dots, c_k \in \operatorname{Cas}(M, \Pi_0)$, then*

$$\{c_i, c_j\}_1 = 0, \quad \forall i, j \leq k. \tag{5.3.5}$$

Remark 5.11.1. This is remindful of the alike theorem for Lax representations (Corollary 5.4.1).

Proof. Let $2m = \operatorname{rk}(\Pi_0)$, so that $2m + k = \dim M$. Let us choose Darboux–Weinstein coordinates for Π_0 centered at $p \in M$ ($c^{ij}(p) = 0$ in the notation of Theorem 4.19). Then

$$\Pi_0^{ab}|_p = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3.6}$$

Hence

$$(\Pi_0 + \varepsilon\Pi_1)^{ab} = \begin{pmatrix} O(\varepsilon) & I + O(\varepsilon) & O(\varepsilon) \\ -I + O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) & O(\varepsilon) \end{pmatrix}. \tag{5.3.7}$$

The fact that $\operatorname{rk}(\Pi_\varepsilon) = \operatorname{rk}(\Pi_0) = 2m$ implies the determinant of any minor of order $2m + 1$ vanishes. In particular, consider the minor M_{ij} we get from adjoining the i-th row and the j-th column ($2m < i, j \leq 2m + k$) to the upper left minor of size $2m$. Developing the determinant of such a minor with respect to the adjoined row, we obtain[1]

$$0 = \det M_{ij}(\varepsilon) = \underbrace{(-1)^{4m+2}(\Pi_0^{ij} + \varepsilon\Pi_1^{ij})}_{\text{rightmost element}} + \underbrace{O(\varepsilon)}_{\text{others}} = \varepsilon\Pi_1^{ij} + O(\varepsilon^2), \quad \forall i, j. \tag{5.3.8}$$

Since $\det M_{ij}$ is a polynomial in ε which vanishes on a whole neighbourood of $\varepsilon = 0$, thus it must be the zero polynomial, implying $\Pi_1^{ij} = 0$ for $2m < i, j \leq 2m + k$. Now recall that $\Pi^{ij} = \{x^i, x^j\}$, so this result about the components of Π_1 translates to the thesis, since the last k coordinates of a Darboux–Weinstein system are (a basis for) Casimirs. □

[1] An appreciable digression should be made about how did we obtain the stated equalities. Eventually, the employed results are the consequence of a careful and attentive application of the definition of determinant to the peculiar structure the involved matrices have, hence a challenging, if pedant, exercise. Explicitly, we are using the fact that:

$$\begin{vmatrix} O(\varepsilon) & I + O(\varepsilon) \\ -I + O(\varepsilon) & O(\varepsilon) \end{vmatrix} = 1 + O(\varepsilon), \tag{5.3.9}$$

while the same matrix with any line replaced by one whose entries are $O(\varepsilon)$ has determinant $O(\varepsilon)$. Applying Laplace's formula to the stated row leads to the expression of $\det M_{ij}$ we used.

Lemma 5.12. *Let Π be a Poisson structure of constant rank $2m$ on a $(2m+k)$-dimensional smooth manifold M. Locally, in a sufficiently small open ball (in the topological sense, i.e. homeomorphic to a ball of suitable dimension) $U \subseteq M$:*

1. *if $\overline{X} \in H^1_\Pi(U)$, then $\overline{X} = 0$ (i.e. the Poisson vector field X is Hamiltonian) if and only if $X = \Pi^\sharp \eta$ for some $\eta \in \Omega^1$ (hence not necessarily closed),*
2. *if $\overline{\Delta} \in H^2_\Pi(U)$, then $\overline{\Delta} = 0$ (i.e. the first-order deformation given by Δ is trivial) if and only if is zero on the Casimirs.*

Proof.

1. If $\overline{X} = 0$, then $X = \Pi^\sharp(df)$ for some $f \in \mathcal{C}^\infty(M)$, then of course $X \in \operatorname{im} \Pi^\sharp$. Conversely, suppose $X \in \operatorname{im} \Pi^\sharp$. Let's start by putting Darboux–Weinstein coordinates (x^i, ξ^i, c^j) on U of M. Use the Casimir coordinates to foliate the ball in symplectic leaves. We'll denote with U_c the symplectic leaf where Casimirs have a fixed value $c \in \mathbb{R}^k$. Then on any of these, $\Pi|_{U_c} = \partial_{x^i} \wedge \partial_{\xi^i}$. Consider now its transpose, which is simply Darboux 2-form $\omega_c = dx^i \wedge d\xi^i$. Extending ω_c on the whole ball U, we would get a closed form, even if not a symplectic one. Now, since $[\Pi, X] = 0$ then $\mathcal{L}_X \omega_c$ as well, thus:

$$\mathcal{L}_X \omega_c = d\iota_X \omega_c + \iota_X d\omega_c = d\iota_X \omega_c = 0. \tag{5.3.10}$$

   Then, locally on U_c, $\iota_X \omega_c = df_c$ where $f_c \in \mathcal{C}^\infty(U_c)$. We can write then

$$\iota_X \omega = df + \text{transverse}^2 \text{ part} \in \Omega^1(U), \tag{5.3.11}$$

   where $f(x, \xi, c) = f_c(x, \xi) \in \mathcal{C}^\infty(U)$. Remember, now, that $\iota_- \omega$ is the inverse of $\omega^{-1} = \Pi|^\sharp_{U_c}$. Thus, leafwise

$$\begin{aligned} X|_{U_c} &= \Pi|^\sharp_{U_c}(\iota_X \omega) \\ &= \Pi^\sharp|_{U_c}(df|_{U_c} + \text{transverse part}) \\ &= \Pi^\sharp|_{U_c}(df_c) + \underbrace{\Pi^\sharp|_{U_c}(\text{transverse part})}_{=0 \text{ by def.}} \end{aligned} \tag{5.3.12}$$

   Thus, since the symplectic foliation of U partitions U, we get $\Pi^\sharp(df) = [\Pi, f] = X$.

2. If $\overline{\Delta} = 0$, then $\Delta = \mathcal{L}_X \Pi = [X, \Pi]$ for $X \in \mathfrak{X}(U)$, so that for $c_1, c_2 \in \operatorname{Cas}(U)$

$$\Delta(dc_1, dc_2) = (\mathcal{L}_X \Pi)(dc_1, dc_2), \tag{5.3.13}$$

[2]Here and in the following "transverse" means along the k directions of degeneracy of the Poisson structure.

and to compute it we recall that

$$\begin{aligned} 0 = \mathcal{L}_X(\{c_1, c_2\}) = \mathcal{L}_X(\Pi(dc_1, dc_2)) = \\ = (\mathcal{L}_X)(dc_1, dc_2) + \Pi(\mathcal{L}_X dc_1, dc_2) + \Pi, (dc_1, \mathcal{L}_X dc_2), \end{aligned} \tag{5.3.14}$$

so that, since $\mathcal{L}_X df = d(X(f))$ for all $f \in \mathcal{C}^\infty(M)$,

$$\Delta(dc_1, dc_2) = -\{X(c_1), c_2\} - \{c_1, X(c_2)\} = 0, \tag{5.3.15}$$

as stated.

Conversely, choose Darboux–Weinstein coordinates $x^i = (q^j, p^h, c^\ell)$. Consider the k vectors given by the components of Δ in the transverse directions:

$$X_a = \Delta^{i\ 2m+a} \frac{\partial}{\partial x^i}, \quad 1 \leq a \leq k. \tag{5.3.16}$$

Since Δ is zero on Casimirs, this means $X_a^{i+2m} = 0$ for $i = 1, \ldots, k$, therefore $X_a \in \mathrm{im}(\Pi^\sharp)$ since it is tangent to the symplectic leaves.

Now, let us compute explicitly the components of $[\Pi, \Delta]$ and of $[\Pi, X_a]$. As for $[\Pi, \Delta]$, one finds by direct computation (using the S-N brackets definition and properties)

$$\begin{aligned} [\Pi, \Delta]^{ijh} = &- (\Delta^{hl}\partial_h \Pi^{ij} + \Delta^{ih}\partial_h \Pi^{jl}) \\ &- (\Pi^{hi}\partial_h \Delta^{jl} + \Pi^{hj}\partial_h \Delta^{li}), \end{aligned} \tag{5.3.17}$$

and since Π^{ij} are constant in D-W coordinates

$$[\Pi, \Delta]^{ij\,2m+a} = -(\Pi^{hi}\partial_h \Delta^{j\,2m+a} + \Pi^{hj}\partial_h \Delta^{2m+a\,i}). \tag{5.3.18}$$

As for $[\Pi, X_a]$, by direct computation one gets

$$[\Pi, X_a]^{ij} = -(\Pi^{hi}\partial_h \Delta^{j\,2m+a} + \Pi^{hj}\partial_h \Delta^{2m+a\,i}), \tag{5.3.19}$$

which proves that

$$[\Pi, X_a]^{ij} = [\Pi, \Delta]^{ija}, \quad 1 \leq a \leq k. \tag{5.3.20}$$

Now, by hypothesis $[\Pi, \Delta] = 0$, so we find

$$\mathcal{L}_{X_a} \Pi = 0, \tag{5.3.21}$$

that is X_a is a Poisson vector field for all $a = 1, \ldots, k$: $\overline{X_a} \in H^1_\Pi(U)$.

Therefore, we just built k vector fields X_a such that

(a) $\overline{X_a} \in H^1_\Pi(U)$;

(b) $X_a \in \mathrm{im}(\Pi)$,

so by the previous point, $X_a = [\Pi, f_a]$ for some $f_a \in \mathcal{C}^\infty(U)$. In local coordinates this reads $\Delta^{i\,a+2m} = 2\Pi^{ij}\partial_j f_a$ $(1 \leq a \leq k)$, which implies the constraint on the f_a

$$\Pi^{2m+a\,j}\partial_j f_b = 0. \tag{5.3.22}$$

Let us now change representative for this cohomology class, taking $\Delta' = \Delta - [\Pi, Z]$ as the new one, where we defined *ad hoc*

$$Z = \sum_{a=1}^{k} Z^{2m+a}\frac{\partial}{\partial c^a} = \sum_{a=1}^{k} 2f_a\frac{\partial}{\partial c^a}. \tag{5.3.23}$$

Let us show why this is convenient. By direct computation one finds

$$[\Pi, Z]^{i\,2m+a} = \Delta^{i\,2m+a}, \tag{5.3.24}$$

which implies that now the new representative is such that

$$\Delta'^{i\,2m+a} = \Delta'^{2m+a\,i} = 0, \tag{5.3.25}$$

while it keeps holding, just like for Δ,

$$\Delta'^{2ma+a\,2m+b} = 0. \tag{5.3.26}$$

This means that Δ' only has non-zero components along $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial p^i}$ directions, that is $\Delta'^\sharp$ can be really "restricted" to the symplectic leaves U_c, for all $c \in U$:

$$\Delta'_c : \Omega^1(U_c) \longrightarrow \mathfrak{X}(U_c). \tag{5.3.27}$$

This allows us to lower the indices of Δ and define, said $\omega = dq^i \wedge dp_i$,

$$\delta = \omega_{\bullet k}\Delta^{kl}\omega_{l\bullet} \in \Omega^2(U), \tag{5.3.28}$$

which means $\delta = \omega \circ \Delta \circ \omega$, and correspondingly we have the leaf-restricted version

$$\delta_c = \omega_c \circ \Delta_c \circ \omega_c \quad \forall c \in U. \tag{5.3.29}$$

Now, $\Pi_\epsilon = \Pi + \epsilon\Delta$ is Poisson at first order in ϵ if and only if $[\Pi, \Delta] = 0$, and this is true by hypothesis. Moreover, we know Π_c is symplectic and Δ_c is well defined, therefore $\Pi_\epsilon|_c = \Pi_c + \epsilon\Delta_c$ is Poisson if and only if $[\Pi|_c, \Delta_c] = 0$, which is true by hypothesis, and since

$$\Pi_\epsilon|_c^{-1} = \omega_\epsilon|_c = \omega_c - \epsilon\omega_{\bullet h}\Delta^{hl}\omega_{l\bullet} + \mathcal{O}(\epsilon^2) \tag{5.3.30}$$

then $\omega_\epsilon|_c$ is symplectic if and only if $d\omega_\epsilon|_c = 0$, that is iff $d\delta_c = 0$, which shows that δ_c is closed on the symplectic leaves U_c.

Therefore, analogously to what we did in point 1. of this proof, by Poincarè lemma $d\delta_c$ is locally exact:

$$\delta_c = d\phi_c \quad \exists \phi_c \in \Omega^1(U_c) \quad \forall c \in U \tag{5.3.31}$$

so that, going back outside the leaves, $\delta = d\phi + \tilde{\delta}$ for some $\tilde{\delta} \in \Omega^2(U)$, where $\phi(q,p,c) = \phi_c(q,p)$ and $\tilde{\delta}$ contains at least one dc^a in each summand. Now, to recover Δ' we compose the identity $\delta = d\phi + \tilde{\delta}$ with Π on both sides:

$$\underbrace{\Pi\delta\Pi}_{\Delta'} = \Pi d\phi\Pi + \underbrace{\Pi\tilde{\delta}\Pi}_{0} \tag{5.3.32}$$

so $\Delta' = \Pi d\phi\Pi$, and by direct computations one finds that

$$\Pi d\phi\Pi = -[\Pi, \Pi(\phi)] \tag{5.3.33}$$

that is $\Delta' = -[\Pi, \Pi(\phi)]$ (where we mean $\Pi(\phi) = \Pi^{ij}\phi_j\partial_i \in \mathfrak{X}(M)$). This proves that $\bar{\Delta}' = 0$, as we wanted.

□

Corollary 5.12.1. *Let $\Pi_0 + \varepsilon\Pi_1$ be a Poisson pencil of constant rank for small ε and on some ball $U \subseteq M$. Then*

$$\overline{\Pi}_1 = 0 \in H^2_{\Pi_0}(U). \tag{5.3.34}$$

Proof. From Lemma 5.11 we know Π_1 vanishes on the Casimirs of Π_0, hence the corollary follows from the second claim of Lemma 5.12. □

The meaning of the last result is that if (Π_0, Π_1) forms a Poisson pencil of constant rank, then we can locally reabsorb any deformation of Π_0 by Π_1 with a coordinate change.

Theorem 5.13. *Let $\Pi_\varepsilon = \Pi_0 - \varepsilon\Pi_1$ be a Poisson pencil of constant rank for small ε on a ball U on the manifold M. Let $c^1(\varepsilon), \dots, c^k(\varepsilon)$ be $k = \operatorname{cork}(\Pi_0) = \operatorname{cork}(\Pi_0 - \varepsilon\Pi_1)$ independent Casimirs for Π_ε and let*

$$c^i(\varepsilon) = c^i_{-1} + c^i_0\varepsilon + c^i_1\varepsilon^2 + \dots \tag{5.3.35}$$

*be their Taylor expansion in ε around $\varepsilon = 0$ (it's a power series of ε). Then the sequences $\left\{c^i_p\right\}_{p\in\mathbb{N}}$ are in Hamiltonian recursion, and moreover they **all** commute with each other with respect to both brackets:*

$$\{c^i_p, c^j_q\}_0 = \{c^i_p, c^j_q\}_1 = 0 \tag{5.3.36}$$

Proof. Since $c^i(\varepsilon)$ is a Casimir of Π_ε,

$$\begin{aligned} 0 &= \{-, c^i(\varepsilon)\}_\varepsilon \\ &= \{-, c^i(\varepsilon)\}_0 - \varepsilon\{-, c^i(\varepsilon)\}_1 \\ &= \left(\sum_{p=-1}^{\infty} \{-, c^i_p(\varepsilon)\}_0 \varepsilon^{p+1}\right) + \varepsilon\left(\sum_{p=-1}^{\infty} \{-, c^i_p(\varepsilon)\}_1 \varepsilon^{p+1}\right) \\ &= \sum_{p=-1}^{\infty} (\{-, c^i_p\}_0 - \{-, c^i_{p+1}\}_1)\varepsilon^{p+1} \end{aligned} \tag{5.3.37}$$

where we set $c^i_{-2} = 0$. Then the fact all the coefficients of the above power series vanish correspond to the wanted recursion relation. Theorem 5.9 takes care of the commuting relations among each of the k sequences $\left\{c^i_p\right\}_{p\in\mathbb{N}}$, so we just need to prove the functions commute also when in different sequences. To do this, we turn the trick we used for Magri's Theorem on its head: we spread the functions apart until one of them hits -1, at which point we can use the fact c^i_{-1} is Casimir at $\varepsilon = 0$ (because the rest of the terms of (5.3.35) vanish):

$$\begin{aligned} \{c^i_p, c^j_q\}_0 &= \{c^i_p, c^j_{q-1}\}_1 \\ &= -\{c^j_{q-1}, c^i_p\}_1 \\ &= -\{c^j_{q-1}, c^i_{p+1}\}_0 \\ &= \{c^i_{p+1}, c^j_{q-1}\}_0 \\ &= \dots \\ &= \{c^i_{p+q+1}, c^i_{-1}\}_0 = 0 \end{aligned} \tag{5.3.38}$$

□

Hence the "canonical way" to get an Hamiltonian recursion out of a bihamiltonian structure of constant rank is to find the Taylor expansion of the Casimirs. **How do we find it?**

Construction 5.14. One strategy goes like this. We have shown that if $\Pi_0 - \varepsilon\Pi_1$ is a Poisson pencil of constant rank, then $\overline{\Pi}_1 = 0 \in H^2_{\Pi_0}(U)$. This means $\Pi_1 = \mathcal{L}_Z\Pi_0$ for some $Z \in \mathfrak{X}^1(U)$. Let us make the additional assumption that $\mathcal{L}^2_Z\Pi_0 = 0$, i.e. the Poisson pencil is **exact**.

On a fixed coordinate patch, one can rectify $Z = \partial_1$ so that Π_0^{ij} will depend linearly on x^1 (since the second derivative vanishes), while Π_1^{ij} will be independent of it (since $\mathcal{L}_{\partial_1}\Pi_1 = \mathcal{L}_{\partial_1}\mathcal{L}_{\partial_1}\Pi_0 = 0$). This means that, if $\mathsf{shift}^1_\varepsilon$ is the change of coordinates sending x_1 to $x_1 - \varepsilon$, then

$$\mathsf{shift}^1_\varepsilon(\Pi_0)^{ij} = \Pi_0^{ij} - \varepsilon\Pi_1^{ij} = \Pi_\varepsilon^{ij}. \tag{5.3.39}$$

This shows that the Casimirs of Π_ε are $c^i(\varepsilon) = \mathsf{shift}^1_\varepsilon(c^i_{-1})$, hence we have a simple way to get the desired functions from the Casimirs of Π_0:

$$c^i(\varepsilon) = \exp(\varepsilon Z)c^i_{-1} = c^i_{-1} + \varepsilon Z(c^i_{-1}) + \frac{\varepsilon^2}{2}Z(Z(c^i_{-1})) + \dots \tag{5.3.40}$$

Example 5.3 (Toda lattice)**.** Remember Toda's equations from Example 5.2 (we rescaled them by a factor to get nicer numbers later on):

$$\begin{cases} \dot{a}_i = a_i(b_{i+1} - b_i) \\ \dot{b}_i = 2(a_i^2 - a_{i-1}^2) \end{cases} \tag{5.3.41}$$

Geometrically, we think of $a_1, \dots, a_{n-1}$ and $b_1, \dots, b_n$ as coordinates on a manifold. We equip it with a Poisson structure by setting

$$\{a_i, b_i\}_1 = -a_i, \quad \{a_i, b_{i+1}\}_1 = a_i \tag{5.3.42}$$

and zero in every other case.

Usually, one gets the a_is and the b_is from another set of (physical) coordinates (p_i, q_i) by the following change of variables:

$$\begin{cases} a_i = \dfrac{1}{2}\mathrm{e}^{\frac{1}{2}(q_i - q_{i+1})} \\ b_i = -\dfrac{1}{2}p_i \end{cases} \tag{5.3.43}$$

With respect to those coordinates, the bracket is just $\{q_i, p_j\}_1 = \delta_{ij}$. The Hamiltonian of the system is

$$H = \frac{1}{2}(b_1^2 + \dots + b_n^2) + a_1^2 + \dots + a_{n-1}^2. \tag{5.3.44}$$

The interpretation for this Hamiltonian is that the first term is a kinetic energy term while the second is a potential energy term for an exponential potential which is defined among neighbouring particles. The system has also a Lax representation as

$$\begin{aligned} L &= \begin{pmatrix} b_1 & a_1 & & & 0 \\ a_1 & & \ddots & & \\ & & \ddots & & \\ & & \ddots & & a_{n-1} \\ 0 & & & a_{n-1} & b_{n-1} \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & a_1 & & & 0 \\ -a_1 & & \ddots & & \\ & & \ddots & & \\ & & \ddots & & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix} = \mathrm{pr}_L = \overline{u}(L) - \overline{(u)}L^T \end{aligned} \tag{5.3.45}$$

In lieu of $B = RL$ we have $B = \mathrm{pr}_- L$, but it is the same thing as

$$R = \mathrm{pr}_+ - \mathrm{pr}_- = \mathrm{pr}_+\mathrm{pr}_- - 2\mathrm{pr}_- = \mathrm{id} - 2\mathrm{pr}_-. \tag{5.3.46}$$

The symmetries of the system are given by the trace of L, $b_1 + \ldots + b_n$. It is a Casimir function since

$$\begin{aligned} &\{\mathrm{Tr}\, L, b_i\}_1 = 0, \\ &\{\mathrm{Tr}\, L, a_i\}_1 = \{b_1, a_i\}_1 + \ldots + \{b_i, a_i\}_1 + \{b_{i+1}, a_i\}_1 + \ldots + \{b_n, a_i\}_1 \\ &\qquad = 0 + \ldots + 0 + \cancel{-a_i + a_i} + 0 + \ldots + 0 = 0. \end{aligned} \tag{5.3.47}$$

This, and the traces of higher powers of L, are the symmetries of the lattice (but only the first are Casimirs).

We now give a second Hamiltonian structure to the system, by providing new Poisson brackets. The pivotal observation is that the bracket we just defined is independent from the b_i coordinates, as we see from the definition (a b_i coordinate never appears on the right hand side). So we guess (there are actually good theoretical reason for this) there might be a second compatible Poisson bracket linearly dependent on the b_is, to which we could then apply the machinery we just derived for this case. So we start by adding a linear dependence on b_is:

$$\{a_i, b_i\}_2 = -a_i b_i, \quad \{a_i, b_{i+1}\}_2 = a_i b_{i+1} \tag{5.3.48}$$

and then we correct the rest of the terms in order to satisfy the Jacobi identity:

$$\{a_i, a_{i+1}\}_2 = \frac{1}{2} a_i a_{i+1}, \quad \{b_i, b_{i+1}\}_2 = 2a_i^2. \tag{5.3.49}$$

So the Poisson pencil is exact (because Π_1 is independent of the b_i and Π_2 depends only linearly on them), and

$$Z = \sum_{i=1}^{n} \partial_{b_i}. \tag{5.3.50}$$

One can show $\det L$ is a Casimir for $\{-,-\}_2$, and we get $\mathsf{shift}^1_\varepsilon(\det L) = \det(L - \varepsilon I)$. This latter object is the characteristic polynomial of L evaluated at ε, thus the coefficients of its Taylor expansion in ε coincides with its coefficient as a polynomial, are exactly n and given by the traces of successive powers of L.

The situation of this example is general: given an R-matrix for a Lie–Poisson structure, there's a canonical way to construct a second (and actually even a third) compatible quadratic Poisson structure. The spectral invariants we get from the Lax representation are then in Hamiltonian recursion for this pair of Poisson structures.

5.3.2 The Hamiltonian recursion in the symplectic case

In the case one of the Poisson is symplectic, we do not have enough Casimir functions[3] so we need a different approach.

5.3.3 Nijenhuis structures

For this section, we refer to [**magri1984geometrical**]. In this seminal paper, Magri and Morosi set out the theory of Poisson–Nijenhuis geometry, which deals with "compatible" tensor structures of Poisson, Nijenhuis and symplectic type. The article obtains hierarchies of symmetries out of this especially nice situation, through the method of Hamiltonian recursions.

Definition 5.15. The **Nijenhuis torsion** is the operator

$$T(N)(X,Y) = [NX,NY] - N([NX,Y]+[X,NY]) + N^2[X,Y], \quad \forall X,Y \in \mathfrak{X}(M), \tag{5.3.51}$$

where $N \in \mathfrak{X}_1^1(M)$.

Definition 5.16. A tensor field $N \in \mathfrak{X}_1^1(M)$ is called **Nijenhuis** if it satisfies $T(N) = 0$.

Definition 5.17. A **Nijenhuis manifold** is a smooth manifold M equipped with a Nijenhuis tensor N.

However, the most interesting situation comes when a Nijenhuis structure coexists (nicely) with a symplectic or Poisson structure:

Definition 5.18. A **symplectic–Nijenhuis manifold** M is a symplectic manifold (M,ω) equipped with a Nijenhuis tensor N such that

$$S(\omega,N) = 0, \quad \omega N - N^*\omega = 0. \tag{5.3.52}$$

where S is an operator defined as

$$S(\eta,Q)(X,Y) := \mathcal{L}_{QY}(\eta)X - \mathcal{L}_{QX}(\eta)Y + \eta Q[X,Y] + d\langle \eta QY, X\rangle, \quad X,Y \in \mathfrak{X}(M). \tag{5.3.53}$$

The first condition is a compatibility, *a fortiori* condition, which arises from computation regarding how Hamiltonian vector are preserved by the Nijenhuis tensor (it basically implies that the symmetries arising from Lemma 5.23 can be Hamiltonian if we start with Hamiltonian fields). The second condition expresses antisymmetry for ωN, where the

[3] This is especially true for the constant rank case: in fact, under this assumption the non-degeneracy of one Poisson structure forces non-degeneracy in the other, so we really do not have any Casimir.

spurious minus sign comes from the antisymmetry of ω itself. In the case $N = \Pi\omega$, then it is fullfilled and it's really an antisymmetry condition (since $\omega\Pi\omega$ is the composition of an odd number of antisymmetric tensors, thus antisymmetric itself).

Definition 5.19. A **Poisson–Nijenhuis manifold** is a Poisson manifold (M, Π) equipped with a Nijenhuis tensor N such that

$$R(\Pi, N) = 0, \quad N\Pi - \Pi N^* = 0 \tag{5.3.54}$$

where R is an operator defined as

$$R(P, Q)(\alpha, X) := \mathcal{L}_{P\alpha}(Q)X - P\mathcal{L}_X(Q^*\alpha) + P\mathcal{L}_{QX}\alpha, \quad \alpha \in \Omega^1(M), X \in \mathfrak{X}(M). \tag{5.3.55}$$

The first intriguing geometric result about these structures is that any "symplectic–Poisson" manifold can be given the structure of both a symplectic–Nijenhuis manifold, by defining $N = \Pi\omega$, and that of a Poisson–Nijenhuis manifold, by setting $\omega = \Pi^{-1}N$ (actually, in the case that Π is non-degenerate).

Lemma 5.20. *Let Π be a Poisson structure on M and ω a compatible symplectic form. Then*

$$d(\omega(\Pi\omega)^n) = 0, \quad \forall n \in \mathbb{N}. \tag{5.3.56}$$

Proof. Since ω is compatible with Π, (ω^{-1}, Π) is a bihamiltonian structure. In particular, $\omega^{-1} - \varepsilon\Pi$ is Poisson and non-degenerate if ε is sufficiently small, so that $(\omega^{-1} - \varepsilon\Pi)^{-1}$ is symplectic, and

$$(\omega^{-1}(1 - \varepsilon\omega\Pi))^{-1} = \left(\sum_{n=0}^{\infty}(\varepsilon\omega\Pi)^n\right)\omega = \omega + \varepsilon\omega\Pi\omega + \varepsilon^2\omega\Pi\omega\Pi\omega + \dots. \tag{5.3.57}$$

Because symplectic implies closed, and d it's a linear operator, the above expression gives the thesis for every value of $n \in \mathbb{N}$. □

Proposition 5.21. *Let Π be a Poisson tensor and ω a symplectic form on some manifold M, such that $d(\omega\Pi\omega) = 0$. Then $\Pi\omega$ is a Nijenhuis tensor.*

Proof. The proof rests on the following identity, whose lengthy proof can be found in [**magri1984geometrical**]: given $P \in \mathfrak{X}^2(M)$ and $\eta \in \Omega^2(M)$,

$$T(P\eta)(X, Y) = [P, P](\eta X, \eta Y) - PS(\eta, P\eta)(X, Y). \tag{5.3.58}$$

Now, when $P = \Pi$ and $\eta = \omega$, we have $[\Pi, \Pi] = 0$ as Π is Poisson, $d\omega = 0$ as ω is symplectic, and $d(\Pi\Omega) = 0$ by hypothesis, therefore all the terms of $T(\Pi\omega)$ vanish, as claimed. □

Indeed symplectic, Poisson and Nijenhuis structures dance around each other in an interesting way:

Symplectic	**Poisson**	**Nijenhuis**	**given that**
ω	Π	$\Pi\omega$	$d(\omega\Pi\omega) = 0$
ω	$N\omega^{-1}$	N	Conditions (5.3.52)
$\Pi^{-1}N$	Π	N	Conditions (5.3.54) + Π is non-degenerate.

But even more interesting is the fact that symplectic– and Poisson–Nijenhuis manifolds actually come in well-behaved hierarchies: starting from (ω, N) or (Π, N) we can define new structures of the same type by iterated composition:

$$P_{k+1} = N^k P, \quad \Omega_{k+1} = \Omega N^k. \tag{5.3.59}$$

This in turn originates from the fact that Nijenhuis structures come in hierarchies, given again by iterated composition (so that when (M, N) is Nijenhuis manifold, so is (M, N^k) for any $k \in \mathbb{N}$).

Lemma 5.22. *When N is a Nijenhuis tensor,*

$$\mathcal{L}_{NX}(N) = N\mathcal{L}_X(N), \quad \forall X \in \mathfrak{X}(M). \tag{5.3.60}$$

Proof.

$$\begin{aligned} T(N)(X,Y) &= [NX, NY] - N([X, NY] + [NX, Y]) + N^2[X, Y] \\ &= (\mathcal{L}_{NY}(NX) - N\mathcal{L}_{NY}(X)) - N(\mathcal{L}_Y(NX) - N\mathcal{L}_Y(X)) \\ &= \mathcal{L}_{NY}(N)X - N(\mathcal{L}_Y(N)X). \end{aligned} \tag{5.3.61}$$

Thus when N is Nijenhuis, the thesis follows. □

Lemma 5.23. *Let N be Nijenhuis and let $\{X_1, \ldots, X_n, \ldots\}$ be a system of vector fields in involution and preserving N ($\mathcal{L}_{X_i} N = 0$ for any n). Then*

$$[N^k X_i, N^h X_j] = 0, \tag{5.3.62}$$

where exponents denote repeated application.

Proof. We get the claim by repeated applications of the Leibniz rule, and in the last step

using Equation 5.3.60:

$$\begin{aligned}
[N^k X_i, N^h X_j] &= \mathcal{L}_{N^k X_i}(N^h X_j) \\
&= \mathcal{L}_{N^k X_i}(N^h) X_j + N^h \mathcal{L}_{N^k X_i}(X_j) \\
&= \left(\sum_{n=0}^{h-1} N^n \mathcal{L}_{N^k X_i}(N) N^{h-n-1} X_j \right) - N^h \mathcal{L}_{X_j}(N^k X_i) \\
&= \left(\sum_{n=0}^{h-1} N^n N^k \underbrace{\mathcal{L}_{X_i}(N)}_{0} N^{h-n-1} X_j \right) - N^h \underbrace{\mathcal{L}_{X_j}(N^k)}_{0} X_i - N^{h+k} \underbrace{\mathcal{L}_{X_j} X_i}_{0} \\
&= 0.
\end{aligned} \tag{5.3.63}$$

□

A peculiar fact about the hierarchy of Poisson tensors we get out of a Poisson–Nijenhuis structure is that

$$[\Pi_i, \Pi_j] = 0, \quad \forall i, j; \tag{5.3.64}$$

so Nijenhuis tensor become a rich source of bihamiltonian structures. Then Lemma 5.23 tells us that when we find a system of commuting fields which preserve N, we get out a whole hierarchy of them. One can show that X_i is Hamiltonian with respect to each Π_j, $j \leq i$, and that this fact characterizes completely integrable systems.

5.3.4 The recursion

Theorem 5.24. *The functions*

$$h_{p+1} := \frac{1}{p+1} \operatorname{Tr} N^{p+1}, \tag{5.3.65}$$

for positive p, satisfy the recursion

$$dh_{p+1} = N^* dh_p. \tag{5.3.66}$$

Remark 5.24.1. Notice that, when $N = \Pi\omega$, $N^* = \omega\Pi$ by antisymmetry. This means the recursion is

$$\begin{aligned}
dh_{p+1} &= \omega \Pi dh_p \\
\omega^{-1}(dh_{p+1}) &= \Pi(dh_p) \\
\{-, h_{p+1}\}_{\omega^{-1}} &= \{-, h_p\}_\Pi
\end{aligned} \tag{5.3.67}$$

that is, the Hamiltonian recursion of Magri's Theorem! However, the result is completely general since we do not assume N to be in a particular form, indeed, we do not even assume the manifold to be Poisson or symplectic.

Proof. Recall contractions commute with Lie derivatives, then we compute

$$\begin{aligned}
\langle dh_{p+1}, X\rangle = X(h_{p+1}) &= \mathcal{L}_X h_{p+1} \\
&= \mathcal{L}_X \left(\operatorname{Tr} \frac{1}{p+1} N^{p+1} \right) \\
&= \frac{\cancel{p+1}}{\cancel{p+1}} \operatorname{Tr}(N^p \mathcal{L}_X N) \\
&= \operatorname{Tr}(N^{p-1} \mathcal{L}_{NX} N) \qquad \text{by using (5.3.60)} \\
&= \mathcal{L}_{NX} (\operatorname{Tr} \frac{1}{p} N^p) \\
&= \langle dh_p, NX\rangle \\
&= \langle N^* dh_p, X\rangle, \quad \forall X \in \mathfrak{X}(M).
\end{aligned} \tag{5.3.68}$$

□

Hence with this theorem we found the hierarchy of symmetries we were looking for in the symplectic (non-degenerate) case of a bihamiltonian structure. In fact, if (Π_0, Π_1) is a compatible pair of Poisson tensors where Π_0 is non-degenerate, $N = \Pi_0 \Pi_0^{-1}$ is Nijenhuis and the hierarchy is given by the spectrum of N (as per previous theorem).

Chapter 6

Deformation quantization

For this chapter, see [**cattaneo2004formality**], [**kontsevich2003deformation**] and [**gerstenhaber1964deformation**].

In classical mechanics the state of a physical system is represented by a point $p \in M$ on the phase space M, which is moreover equipped with a Poisson structure Π. Its evolution is described by the flow $\phi^{X_h} : M \times \mathbb{R} \to M$ of the vector field associated to the Hamiltonian $h \in \mathcal{C}^\infty(M)$:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = X_h|_{p(t)}. \tag{6.0.1}$$

An **observable** of the system can be thought as any other function $f \in \mathcal{C}^\infty(M)$: its value at $p \in M$ is the result of measuring the corresponding physical quantity when the system is in the state p. Clearly its evolution can be computed by simply probing f at $p(t)$, but we can also compute directly its evolution using the Poisson brackets:

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \{f, h\}. \tag{6.0.2}$$

Hence, for a classical system, **the algebra of the observables is** $\mathcal{C}^\infty(M)$.

In quantum mechanics, what happens is similar but the geometry is different. Traditionally the state of a quantum system is encoded by a point in an Hilbert space[1] $\mathcal{H}$. The Hamiltonian function gets translated to an Hamiltonian *operator* $H \in \mathcal{L}(\mathcal{H})$ (the latter being the space of continuous linear operators of $\mathcal{H}$ into itself), and time evolution is given in terms of the flow

$$\begin{aligned} \phi^H : \mathcal{H} \times \mathbb{R} &\longrightarrow \mathcal{H} \\ (v, t) &\longmapsto \phi_t^H v =: v(t) \end{aligned} \tag{6.0.3}$$

of the *linear* vector field (i.e., whose components are linear functions)

$$V|_v = -\frac{i}{\hbar} H(v). \tag{6.0.4}$$

[1] An **Hilbert space** is a complex, infinite-dimensional, vector space equipped with an Hermitian inner product making it into a complete metric space. In practice, states on the same line have the same physical meaning hence we actually deal with the projectivization $\mathbb{P}\mathcal{H}$ of the Hilbert space.)

This is summarized by the equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{i}{\hbar}H(v(t)), \quad \forall v \in H, \tag{6.0.5}$$

In this sense, the mathematical setup is remindful of the mechanics we've seen so far: given an observable, we produce dynamics, even tough now everything is linear.

Analogously to what happened to the Hamiltonian, observables become operators too. However, **in QM the value of an observable $\mathcal{O} \in \mathcal{L}(\mathcal{H})$ on a state $v \in \mathcal{H}$ is generally not uniquely defined**, as each measurement can give a different outcome, altough you can still speak of its **statistical expectation value** given by

$$\langle \mathcal{O} \rangle_v := \langle v \,|\, \mathcal{O}v \rangle, \tag{6.0.6}$$

where $\langle - \,|\, - \rangle$ is the hermitian product of $\mathcal{H}$ (here we use normalized $v \in \mathcal{H}$, i.e. $\langle v \,|\, v \rangle = 1$). Now, since $\langle \mathcal{O} \rangle_v$ should be a real number, **we require additionally that physical observables have to be represented by self-adjoint[2] operators**. Thus **the algebra of observables is now given by self-adjoint linear operators $\mathcal{L}^\dagger(\mathcal{H})$**.

As in the classical case, the evolution of a system can be computed with two different strategies. In the first (the **Schrödinger picture**) we imagine states evolve in time while observables (as operators) remain fixed. In the second (the **Heisenberg picture**), observables evolve and states remain fixed. To deal with both cases, we start from Schrödinger's equation to define a **time evolution operator**:

$$U(t) = \exp(-itH/\hbar). \tag{6.0.7}$$

This operator enjoys various properties: it is unitary ($U^\dagger U = I$), it is "additive" ($U(t_1 + t_2) = U(t_2)U(t_1)$) and its is the identity at $t = 0$. In terms of U, Equation (6.0.5) can be expressed as

$$v(t) = U(t)v(0) = \exp(-itH/\hbar)\, v(0). \tag{6.0.8}$$

Thus U directly evolves states. On the other hand, if $\mathcal{O}$ is an observable, its evolution is defined to be

$$\mathcal{O}(t) := U^\dagger(t)\, \mathcal{O}\, U(t). \tag{6.0.9}$$

Hence

$$\begin{aligned} \langle \mathcal{O}(t) \rangle_v &= \langle \mathcal{O} \rangle_{v(t)} \\ &= \langle v(t) \,|\, \mathcal{O}v(t) \rangle \\ &= \langle \exp(-itH/\hbar)\, v \,|\, \exp(-itH/\hbar)\, v \rangle \\ &= \langle v \,|\, \underbrace{\exp(itH/\hbar)\ \mathcal{O}\ \exp(-itH/\hbar)}_{\mathcal{O}(t)} v \rangle. \end{aligned} \tag{6.0.10}$$

[2]The precise definition of self-adjointness is subtly technical: a self-adjoint operator is a symmetric one ($\langle Tv \,|\, u \rangle = \langle v \,|\, Tu \rangle$) such that the domains of T and $T^\dagger$ coincide.

The action on $\mathcal{O}$ by U looks like the adjoint action (Ad) of the group $\mathcal{L}(\mathcal{H})$ on the observables. We know what is the vector field which generates the action, so that we can write

$$\frac{\mathrm{d}\mathcal{O}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[\mathcal{O}(t), H]. \tag{6.0.11}$$

This equation is suspectingly similar to the classical evolution equation (6.0.2) (indeed, the brackets $[-,-]$ can be interpreted as linear Poisson brackets, but since we are in the infinite-dimensional context it is difficult to work with them).

Is this similarity just an accident, or can we actually find classical mechanics inside quantum mechanics (as a sort of "limit case")? The answer is known to be *yes* (essentially, taking the limit $\hbar \to 0$). The reverse procedure is called **quantization**, and recovers a quantum description from a classical one. It can be made in various ways, through analysis and geometry, but here we'll explore a purely algebraic way to deal with this problem, called, indeed, **deformation quantization**.

The idea of deformation quantization is to do without the explicit *representation* (in the technical sense) of the "algebra of observables" as self-adjoint operators on an Hilbert space and just describe evolution of elements of an abstract non-commutative associative algebra through Equation (6.0.11) (which doesn't involve the Hilbert structure). Hence: start with an Hilbert space description of the system, build the algebra of observables and represent it on another non-commutative (but still associative) algebra.

In fact, the non-commutativity relations among the observables (i.e. Heisenberg uncertainity) can be seen as to arise from a non-commutativity of the product of the observables:

$$[f,g] = fg - gf = 0 \iff \text{the product is commutative} \tag{6.0.12}$$

We observe, moreover, is that this non-commutativity is first-order in $\hbar$: it means that it vanishes when $\hbar$ does, and that its principal part is already 0. This suggest a way to deal with quantization by deformation of the product of the algebra of observables itself, by "expansion in power series of $\hbar$".

6.1 Deformation theory for commutative rings

Throughout this section, let R be a commutative ring[3]. See [**gerstenhaber1964deformation**] for a rigorous algebraic treatment of deformation.

[3] If you are not familiar with rings, look up the (simple) definition. But you can go on and just think R to be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Moreover, later on, R will always be a field (usually $\mathbb{R}$ or $\mathbb{C}$).

Definition 6.1. A **formal power series** is a sequence of elements $\{a_n\}_{n\in\mathbb{N}} \subseteq R$ which we write as $\sum_{n=0}^{\infty} a_n \hbar^n$. The set of such formal expressions is denoted by $R[[\hbar]]$, and has the structure of an R-algebra: for any $a, b \in R[[\hbar]], r \in R$:

$$\begin{aligned} a + b &:= \sum_{n=0}^{\infty} (a_n + b_n)\, \hbar^n, \\ ab &:= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \hbar^n, \\ r \cdot a &:= \sum_{n=0}^{\infty} (r \cdot a_n)\, \hbar^n. \end{aligned} \tag{6.1.1}$$

Notice that R is naturally embedded in its algebra of formal power series, with the product being the same as in R.

The construction of the ring of formal power series passes functorially to its category of modules, simply by extension of scalars:

$$A[[\hbar]] := A \otimes_R R[[\hbar]]. \tag{6.1.2}$$

Hence also R-linear maps between modules can be promoted automatically to $R[[\hbar]]$-linear maps. We are interested, in particular, in bilinear forms:

Definition 6.2. A $R[[\hbar]]$-linear map $\mu_\hbar : A[[\hbar]] \to B[[\hbar]]$ is said to be **defined over** R if it is the $R[[\hbar]]$-bilinear extension of an R-bilinear map $\mu : A \to B$.

The map $\mu_\hbar$ is defined from μ by imposing the following:

$$\mu_\hbar \left(\sum_{n=0}^{\infty} a_n \hbar^n \right) := \sum_{n=0}^{\infty} \mu(a_n)\, \hbar^n, \quad \forall \sum_{n=0}^{\infty} a_n \hbar^n \in A[[\hbar]]. \tag{6.1.3}$$

Definition 6.3. Let A be an associative R-algebra with unit over a commutative ring. A **formal deformation** of A is an associative algebra[4] $(A[[\hbar]], \star)$ over $R[[\hbar]]$, having the following form

$$a \star b = ab + \sum_{n=1}^{\infty} \mu_n(a, b)\, \hbar^n, \quad \forall a, b \in A[[\hbar]] \tag{6.1.4}$$

where ab is the canonical power series product (6.1.1) and the coefficients μ_n are bilinear forms defined over R.

Remark 6.3.1. Intuitively, a formal deformation is the same product of A "up to higher-order terms in $\hbar$". All the fuss about $R[[\hbar]]$-bilinearity is there to mean that, in practice, the product (in particular, the coefficients μ_n) is defined on elements of A and then extended to the whole power series algebra.

[4] We are not assuming unitariety.

Proposition 6.4. *Given a formal deformation $(A[[\hbar]], \star)$ of an associative unital R-algebra A then*

$$\sum_{k=0}^{n} \mu_{n-k}(\mu_k(a,b),c) - \mu_{n-k}(a,\mu_k(b,c)) = 0, \quad \forall n \in \mathbb{N}, a,b,c \in A. \tag{6.1.5}$$

Proof. The stated identity is derived from the associativity of $\star$, written "degree-wise". Notice the role of $R[[\hbar]]$-bilinearity of the coefficients in the penultimate step:

$$\begin{aligned}
(a \star b) \star c &= a \star (b \star c) \\
\left(\sum_{n=0}^{\infty} \mu_n(a,b)\,\hbar^n\right) \star c &= a \star \left(\sum_{n=0}^{\infty} \mu_n(b,c)\,\hbar^n\right) \\
\sum_{n=0}^{\infty} \mu_n\left(\sum_{k=0}^{\infty} \mu_k(a,b)\,\hbar^k, c\right)\hbar^n &= \sum_{n=0}^{\infty} \mu_n\left(a, \sum_{k=0}^{\infty} \mu_k(b,c)\,\hbar^k\right)\hbar^n \\
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mu_n(\mu_k(a,b),c)\,\hbar^{k+n} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mu_n(a,\mu_k(b,c))\,\hbar^{k+n} \\
\sum_{k=0}^{n} \mu_{n-k}(\mu_k(a,b),c) &= \sum_{k=0}^{n} \mu_{n-k}(a,\mu_k(b,c)), \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{6.1.6}$$

□

6.1.1 Star-products

Let M be a finite-dimensional smooth manifold, $\dim M = n$. From now on, $\mathcal{C}^\infty(M)$ will denote the algebra of *complex-valued* smooth functions.

Definition 6.5. A **multi-differential operator of rank** k on M is a k-multilinear map $D : \mathcal{C}^\infty(M)^k \to \mathcal{C}^\infty(M)$ such that there exists a $N \in \mathbb{N}$ for which in each chart $(U, x^1, \ldots, x^n)$ of M

$$D(f_1,\ldots,f_k)|_U = \sum_{\substack{I_1,\ldots,I_k \in \mathbb{N}^n, \\ |I_i| \le N}} D^{I_1\ldots I_k} \frac{\partial f_1|_U}{\partial x^{I_1}} \cdots \frac{\partial f_k|_U}{\partial x^{I_k}} \tag{6.1.7}$$

where $|I| := \sum_{j\le n} I_j$, $D^{I_1\ldots I_k} \in \mathcal{C}^\infty(U)$, and

$$\frac{\partial}{\partial x^I} := \frac{\partial^{I_1}}{\partial (x^i)^{I_1}} \cdots \frac{\partial^{I_n}}{\partial (x^n)^{I_n}}. \tag{6.1.8}$$

The smallest N for which this condition holds for D its called its **order** (basically it measures the highest-order derivation D depends on).

Definition 6.6. A **$\star$-product on** M is a formal deformation of $\mathcal{C}^\infty(M)$ given by

$$f \star g = fg + \sum_{n=1}^{\infty} B_n(f,g)\hbar^n, \quad f,g \in \mathcal{C}^\infty(M)[[\hbar]] \tag{6.1.9}$$

where B_n are bidifferential operators on M and such that

$$1 \star f = f \star 1 = f, \quad \forall f \in \mathcal{C}^\infty(M)[[\hbar]]. \tag{6.1.10}$$

A $\star$-product on M with B_n of order at most n is called **natural**.

Remark 6.6.1. Compared to the general case, the requirement of on the coefficients of the product is there to impose local dependence of the product on the factors, as phyiscs should depend only locally on observales. The additional condition $1 \star f = f$ implies the deformation does not even involve the values of the factors, but just their derivatives. Indeed, it could be restated as $B_n(1, f) = B_n(f, 1) = 0$ for $n > 0$, hence **each B_n in a $\star$-product has no zeroth order term for $n > 0$.**

There are three most prominent examples of $\star$-products, related to what are called **ordering operators**, which are, intuitively, formal symbolic operations which reorder products of operators in a suitable way.

Example 6.1. On $\mathbb{R}^2$ with coordinates (q, p), we define the **standard product**:

$$f \star_S g = f \, \exp\left(\frac{\hbar}{i} \overleftarrow{\partial}_p \overrightarrow{\partial}_q\right) g = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r}. \tag{6.1.11}$$

The arrow on a differential operators means it acts on the left or the right argument, respectively.

It is immediate to see that $f \star_S g = fg + \mathcal{O}(\hbar)$. Associativity is easier to show using the following form for the product:

$$(f \star_S g)(q, p) = \int dq' \, dp' \, \delta(q' - q) \, \delta(p' - p) \, \exp\left(\frac{\hbar}{i} \partial_{p'} \partial_{q'}\right) f(q', p) \, g(q, p'). \tag{6.1.12}$$

Let us try and understand where this $\star$-product comes from and how it is related to the so-called *standard ordering of operators* in quantum mechanics.

On the subalgebra of polynomials $\mathbb{C}[p, q] \subseteq \mathcal{C}^\infty(\mathbb{R}^2)$, consider the map defined as the $\mathbb{C}$-linear extension of

$$\begin{aligned} \rho_S : \mathbb{C}[p, q] &\longrightarrow \mathrm{Diff}_{pol}(\mathbb{R}) \\ q^m p^n &\longmapsto q^m \left(\frac{\hbar}{i} \frac{\partial}{\partial q}\right)^n, \end{aligned} \tag{6.1.13}$$

where $\mathrm{Diff}_{pol}(\mathbb{R})$ is the space of differential operators acting on $\mathcal{C}^\infty(\mathbb{R})$ with polynomial coefficients in the variable q.

Notice ρ_S is not a morphism of $\mathbb{C}$-algebras, only $\mathbb{C}$-linear, because its domain is commutative while its codomain is not. Therefore, it fails to be a representation.

It follows that this map ρ_S explicitly makes an ordering choice, since operators don't commute: calling $Q = \rho_S(q) = q\cdot$ (the operator that multiplies by q) and $P = \rho_S(p) = \dfrac{\hbar}{i} \dfrac{\partial}{\partial q}$,

ρ_S brings all the Qs to the left and all the Ps to the right. For instance:

$$\rho_S(q^2p) = \rho_S(pq^2) = Q^2P \neq PQ^2. \tag{6.1.14}$$

Indeed, the whole point of quantization is to pass from a commutative regime (here, the algebra of polynomials) to a non-commutative one (its deformation).

Remarkably, the standard ordering of differential operators and the standard product are tightly bound together, as it holds

$$f \star_S g = \rho_S^{-1}(\rho_S(f)\rho_S(g)), \quad \forall f, g \in \mathbb{C}[p, q]. \tag{6.1.15}$$

By a density argument, we could actually extend this definition to the whole algebra of functions. Let us also point out that associativity of the star-product in this form is trivial to prove.

Let us verify this relation in the simpler cases:

1. $q \star_S q = qq$. On the other hand, given $\psi \in \mathcal{C}^\infty(\mathbb{R})$,

$$\begin{aligned}(\rho_S(q) \circ \rho_S(q))\psi = \rho_S(q)(q\psi) = (qq)\psi \\ \Rightarrow \rho_S(q) \circ \rho_S(q) = qq \\ \Rightarrow \rho_S^{-1}(\rho_S(q) \circ \rho_S(q)) = qq = q \star_S q,\end{aligned} \tag{6.1.16}$$

   as we wanted.

2. $p \star_S q = pq + \frac{\hbar}{i}$. On the other hand

$$\begin{aligned}(\rho_S(p) \circ \rho_S(q))\psi = \frac{\hbar}{i}\frac{\partial}{\partial(q\psi)} = \left(\frac{\hbar}{i} + \frac{\hbar}{i}q\frac{\partial}{\partial q}\right)\psi \\ \Rightarrow \rho_S^{-1}(\rho_S(p) \circ \rho_S(q)) = \frac{\hbar}{i} + qp,\end{aligned} \tag{6.1.17}$$

   as required.

3. $q \star_S p = qp$. Untouched, since the q's are already all on the left. Let us point out that, then,

$$[q, p]_{\star_S} = i\hbar, \tag{6.1.18}$$

   which is exactly the commutator we wish to reproduce in Quantum Mechanics between the position operator Q and the momentum operator P.

Example 6.2. On $\mathbb{R}^2$, equipped with a Poisson tensor $\Pi^{ij}\partial_{x^i} \otimes \partial_{x^j}$, we can define the **Moyal product**:

$$\begin{aligned}f \star_M g &= f \exp\left(\frac{\hbar}{i}\Pi^{ij}\overleftarrow{\partial}_{x^i}\overrightarrow{\partial}_{x^j}\right) g \\ &= \sum_{r=0}^{\infty} \frac{(i\hbar/2)^r}{r!} \sum_{a=0}^{r} \binom{r}{a} (-1)^{r-a} \frac{\partial^r f}{\partial q^a \partial p^{r-a}} \frac{\partial^r g}{\partial q^{r-a} \partial p^a}.\end{aligned} \tag{6.1.19}$$

Again, the product comes from an ordering (the **Weyl ordering**) which can be defined on polynomials as the $\mathbb{C}$-linear extension of

$$\begin{aligned} \rho_M : \mathbb{C}[p,q] &\longrightarrow \mathrm{Diff}_{\mathrm{pol}}(\mathbb{R}) \\ q^m p^n &\longmapsto \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} A_{\sigma(1)} \circ \ldots \circ A_{\sigma(m+n)} \end{aligned} \tag{6.1.20}$$

where

$$A_k = \begin{cases} q\cdot & \text{if } 1 \leq k \leq m \\ \frac{\hbar}{i}\frac{\partial}{\partial q} & \text{if } m+1 \leq k \leq m+n \end{cases} \tag{6.1.21}$$

Basically, the Weyl ordering resolves noncommutativity by "symmetrizing" the product (which turns ρ_M into a proper map of algebras). For instance, the function $q^2 p \in \mathbb{C}[p,q]$ is mapped to

$$\rho_M(q^2 p) = \underbrace{\frac{1}{3}}_{=2/6} (Q^2 P + QPQ + PQ^2) \tag{6.1.22}$$

where we set $Q = q\cdot$ and $P = \frac{\hbar}{i}\partial_q$. It can be shown to have the property

$$\overline{f \star_M g} = \overline{g} \star_M \overline{f}. \tag{6.1.23}$$

Example 6.3. On $\mathbb{C}$, we have the Wick product:

$$f \star_W g = f \exp(2\hbar \overleftarrow{\partial}_z \overrightarrow{\partial}_{\overline{z}}) g = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r f}{\partial z^r} \frac{\partial^r g}{\partial \overline{z}^r}. \tag{6.1.24}$$

It comes from the **Wick ordering**:

$$\begin{aligned} \rho_W : \mathbb{C}[z,\overline{z}] &\longrightarrow \mathrm{Diff}(\mathbb{C}) \\ 1 &\longmapsto 1 \\ z &\longmapsto A := 2\hbar \frac{\partial}{\partial \overline{z}} \qquad \text{the "annihilator"} \\ \overline{z} &\longmapsto A^\dagger := \overline{z}\,\cdot \qquad \text{the "creator"} \\ \overline{z}^m z^n &\longmapsto (A^\dagger)^m A^n. \end{aligned} \tag{6.1.25}$$

It has the following property:

$$f \star_W g = \overline{g} \star_W \overline{f}. \tag{6.1.26}$$

6.1.2 Relation to Poisson manifolds

Proposition 6.7. *If A is any commutative associative R-algebra and $a \star b = \sum_{n=0}^{\infty} \mu_n(a,b)\,\hbar^n$ is a formal deformation, then the bracket*

$$\{a,b\}_\star := \mu_1(a,b) - \mu_1(b,a), \quad \forall f,g \in A \tag{6.1.27}$$

defines the structure of a Poisson algebra on A.

Proof. Linearity and skew-symmetry are evident. The Jacobi identity follows from the second-order associativity identity:

$$\mu_2(ab,c)+\mu_1(\mu_1(a,b),c)+\mu_2(a,b)c=\mu_2(a,bc)+\mu_1(a,\mu_1(b,c))+a\mu_2(b,c) \tag{6.1.28}$$

Its total antisymmetrization with respect to a,b,c gives the sought identity. Similarly, Leibniz rules comes from the first-order associativity identity:

$$\mu_1(a,b)c-\mu_1(a,bc)+\mu_1(ab,c)-a\mu_1(b,c)=0 \tag{6.1.29}$$

By adding the same expression with a exchanged with c:

$$-a\{b,c\}_\star+\{ab,c\}_\star-\{a,bc\}_\star+\{a,b\}_\star c=0. \tag{6.1.30}$$

Now add to this the same equation the one with b exchanged with c and substract the one with a exchanged with b. □

Corollary 6.7.1. *Given a $\star$-product $f\star g=\sum_{n=0}^{\infty}B_n(f,g)\hbar^n$ on a smooth manifold M, the bracket $\{f,g\}=B_1(f,g)-B_1(g,f)$ gives M the structure of a Poisson manifold.*

Remark 6.7.1. We have

$$[f,g]_\star=f\star g-g\star f=\hbar\{f|_{\hbar=0},\,g|_{\hbar=0}\}+O(\hbar^2) \tag{6.1.31}$$

so that

$$\frac{1}{\hbar}[f,g]_\star\underset{\hbar\to 0}{\longrightarrow}\{f,g\} \tag{6.1.32}$$

which means **quantum mechanical dynamics converges to classical dynamics in the classical limit** $\hbar\to 0$.

6.1.3 Equivalence of $\star$-products

The last round of results linked deformations of an algebra to its admissible Poisson structures. Yet this link is still quite weak, as we don't know (1) if different deformations give different structures and (2) if every Poisson structure derives from a suitable deformation.

We'll explore these two problems in the context of $\star$-products (i.e. "differential" deformations of the smooth functions algebra on a manifold), starting with a notion of equivalence between them.

Definition 6.8. A **formal linear operator** $D:A[[\hbar]]\to B[[\hbar]]$, where both A and B are algebras on the commutative ring R, is given by a sequence $\{D_n\}_{n\in\mathbb{N}}$ of $R[[\hbar]]$-linear operators defined over R:

$$D=\sum_{n=0}^{\infty}D_n\,\hbar^n. \tag{6.1.33}$$

Definition 6.9. Two star products $\star$ and $\star'$ on a smooth manifold M are said to be **equivalent** if there exists a formal linear operator $D : \mathcal{C}^\infty(M)[[\hbar]] \to \mathcal{C}^\infty(M)[[\hbar]]$, of the form

$$D = I + \sum_{n=1}^{\infty} D_n \hbar^n \tag{6.1.34}$$

where I is the identity, such that the following commutes:

$$\begin{array}{ccccc} \mathcal{C}^\infty(M)[[\hbar]] & \times & \mathcal{C}^\infty(M)[[\hbar]] & \xrightarrow{\star'} & \mathcal{C}^\infty(M)[[\hbar]] \\ \downarrow D & & \downarrow D & & \downarrow D \\ \mathcal{C}^\infty(M)[[\hbar]] & \times & \mathcal{C}^\infty(M)[[\hbar]] & \xrightarrow{\star} & \mathcal{C}^\infty(M)[[\hbar]]. \end{array} \tag{6.1.35}$$

Remark 6.9.1. Notice an equivalence operator D is invertible as a formal power series (because it starts with $D_0 = I$) which means commutativity of the diagram entails:

$$f \star' g = D^{-1}(D(f) \star D(g)). \tag{6.1.36}$$

However, D is not really a morphism of algebras since it is defined as a power series, so it slightly more general than that.

Remark 6.9.2. If B_n and B'_n are the bidifferential operators for the $\star$-product $\star$ and $\star'$ respectively, then it follows from the fact that B_n and B'_n have no zeroth order terms (Remark 6.6.1) that **the D_n are differential operators with no zeroth order terms (for $n > 0$)**. This is proved, for example, in [**gutt1999equivalence**].

Example 6.4. The Moyal product is equivalent to the standard product through the linear operator

$$N := \exp\left(\frac{\hbar}{2i}\frac{\partial^2}{\partial q \partial p}\right). \tag{6.1.37}$$

This means, moreover, than the two ordering ρ_S and ρ_M are related

$$\rho_M(f) = \rho_S(N(f)), \quad \forall f \in \mathcal{C}^\infty(M). \tag{6.1.38}$$

Example 6.5. The operator

$$M := \exp\left(\frac{\hbar}{4}(\partial_q^2 + \partial_p^2)\right) \tag{6.1.39}$$

witnesses the equivalence of Moyal's and Wick's products:

$$M(f \star_M g) = M(f) \star_W M(g). \tag{6.1.40}$$

Then **all three examples we made are equivalent**. However, in infinite dimension the standard product converges and the Wick product diverges. Hence changing to an equivalent $\star$-product can renormalize the theory.

Lemma 6.10. *Let $\star$ and $\star'$ be $\star$-products on a smooth manifold M, with coefficients B_n and B'_n respectively. Then*

1. *if $\star$ and $\star'$ are equivalent the Poisson brackets associated to them are the same,*
2. *there is an equivalent $\star$-product $\star'$ with $B'_1(f,g)$ skew-symmetric, so that*

$$\{f,g\}_{\star'} = 2B'_1(f,g). \tag{6.1.41}$$

Proof.

1. Let D be an equivalence operator between $\star$ and $\star'$. Computing explicitly the first term of $D(f\star g)$ gives

$$\begin{aligned} D(f\star g) &= f\star g + \sum_{n=1}^{\infty} D_n(f\star g)\,\hbar^n \\ &= fg + (B_1(f,g) + D_1(fg))\,\hbar + O(\hbar^2), \end{aligned} \tag{6.1.42}$$

   and likewise

$$D(f)\star' D(g) = fg + (B'_1(f,g) + D_1(f)g + fD_1(g))\,\hbar + O(\hbar^2). \tag{6.1.43}$$

   Then comparing the two, we get:

$$B_1(f,g) - B'_1(f,g) = D_1(fg) + D_1(f)g + fD_1(g) \tag{6.1.44}$$

   Subtracting from this equation the same one with exchanged arguments, on the left we get $\{f,g\}_\star - \{f,g\}_{\star'}$, while the right hand side vanishes as it is symmetric under exchange of arguments, proving the claim.

2. Our strategy will be to use (6.1.36) as a *definition* for $\star'$, showing a suitable equivalence operator D exists. Equation (6.1.44) gives an expression for D_1:

$$D_1(fg) = B'_1(f,g) - B_1(f,g) + D_1(f)g + fD_1(g). \tag{6.1.45}$$

   By using commutativity of $\mathcal{C}^\infty(M)$ as $\mathbb{C}$-algebra, we can sum this equation to itself but with f exchanged with g, to get

$$\begin{aligned} 2D_1(fg) &= D_1(fg) + D_1(gf) \\ &= \underbrace{B'_1(f,g) + B'_1(g,f)}_{=B'^{+}_1(f,g)} - \underbrace{(B_1(f,g) + B_1(g,f))}_{=B^{+}_1(f,g)} + 2D_1(f)g + 2fD_1(g), \end{aligned} \tag{6.1.46}$$

   where $\cdot^+$ denotes the symmetric part of its argument. Assuming B'_1 to be skew-symmetric, we get

$$D_1(fg) = D_1(f)g + fD_1(g) - \frac{1}{2}B^+_1(f,g). \tag{6.1.47}$$

Let us choose D_1 to vanish on linear functions of some local coordinates $(x^1, \dots, x^n)$, i.e. $D(x^i) = 0$ for $i = 1, \dots, n$. Then on quadratic monomials

$$D_1(x^i x^j) = \cancel{D_1(x^i)}x^j + x^i \cancel{D_1(x^j)} - \frac{1}{2}B_1^+(x^i, x^j) = -B_1^+(x^i, x^j), \tag{6.1.48}$$

where B_1^+ denotes the symmetric part of B_1. We can go on on cubics, but here we have to check our definition of D_1 respects associativity:

$$\begin{aligned} D_1((x^i x^j)x^k) &= D_1(x^i x^j)x^k + x^i x^j D_1(x^k) - B_1^+(x^i x^j, x^k) \\ &= -B_1^+(x^i, x^k)x^k - B_1^+(x^i x^j, x^k) \\ &= -B_1^+(x^i, x^j x^k) - x^i B_1^+(x^k, x^k) \qquad \text{by first-order assoc.} \\ &= x^i D_1(x^j x^k) + D_1(x^i)x^j x^k - B_1^+(x^i, x^j x^k) \\ &= D_1(x^i(x^j x^k)). \end{aligned} \tag{6.1.49}$$

By induction, this guarantees the well-definedness of D_1 on any product of coordinates. Then D_1 exists and it is uniquely defined locally on polynomials. To get a full definition for D_1, we appeal to density of polynomials to extend to the whole $\mathcal{C}^\infty(U)$, and then we can use the machinery of partitions of unity to glue all the local definitions to a global one.

□

This result established that equivalent $\star$-products induce the same Poisson brackets (answering problem (1) abovementioned), but we can't say nothing about the converse (problem (2)). Infact the second point of the lemma isn't strong enough to allow us to appeal to transitivity and say

$$\star \underset{(2)}{\simeq} \star^- = \star'^- \underset{(2)}{\simeq} \star'. \tag{6.1.50}$$

In fact the middle equality above isn't given at all. Indeed, the lemma constrains only the first coefficients of the "antisymmetrization" of each $\star$-product, and we don't know nothing about the rest of them. Hence we are left with a classification issue for deformations of a Poisson algebra:

Problem. *Given a Poisson manifold $(M, \{-,-\})$, classify (up to equivalence) the $\star$-products on M such that*

$$f \star g = fg + \frac{1}{2}\{f, g\}\hbar + \sum_{n=2}^{\infty} B_n(f, g)\,\hbar^n. \tag{6.1.51}$$

6.2 Formal deformations of Poisson structures

Actual deformations of Poisson structures Π on a smooth manifold M are classified at infinitesimal level by $H^2_\Pi(M)$ as bivectors $\Delta \in \mathfrak{X}^2(M)$ such that $[\Pi, \Delta] = 0$ up to bivectors of the form $\mathcal{L}_X\Pi$ for some $X \in \mathfrak{X}^1(M)$. Basically, this corresponds to study the tangent space at Π of the space of Poisson structures on M, so this can tell us some things about admissible deformations but doesn't tell the full story.

The full story is told by *finite deformations*, which are then harder to study because we do not have a great control or insight into the global structure of the manifold of Poisson structures. We would like to study them up to diffeomorphism, i.e. we consider Π and Π' equivalent if there is a diffeomorphism $\phi : M \to M$ which sends one into the other:

$$\Pi \sim \Pi' \iff \Pi' = \phi_*\Pi. \tag{6.2.1}$$

If we let $X \in \mathfrak{X}^1(M)$ be the derivative of ϕ, then we can express the same condition as

$$\Pi' = \exp(t\mathcal{L}_X)\Pi. \tag{6.2.2}$$

This suggest that we can see such a deformation as a Taylor series, so we can try a formal approach to such objects:

Definition 6.11. A **formal Poisson structure** is a Poisson $\mathbb{R}[[\hbar]]$-bilinear product on the associative commutative unital algebra $\mathcal{C}^\infty(M)[[\hbar]]$ of the form:

$$\{f, g\}_\hbar = \sum_{n=0}^{\infty} \Pi_n(df, dg)\,\hbar^n \tag{6.2.3}$$

where $\Pi_i \in \mathfrak{X}^2(M)$.

Remark 6.11.1. The condition for $\{-,-\}$ to be a Poisson bracket imposes conditions on the Π_is: they have to be anti-symmetric and $\Pi_\hbar = \sum_{i=0}^\infty \Pi_i\,\hbar^i$ (the **formal Poisson tensor**) has to satisfy the Jacobi identity, while Leibniz rule is satisfied automatically by the requirement of the Π_i to be bivectors, hence differential operators on each argument. If we previously fixed a Poisson tensor Π, then any formal Poisson structure with $\Pi_0 = \Pi$ its called a **formal deformation of** Π.

To give an updated definition of equivalent Poisson structures, we "formalize" the group of diffeomorphisms M too, by first formalizing vector fields:

Definition 6.12. A **formal vector field** is a formal sum $X = \sum_{k=0}^\infty X_k\hbar^k \in \mathfrak{X}^1(M)[[\hbar]]$ where $X_k \in \mathfrak{X}^1(M)$ for every $k \in \mathbb{N}$.

The algebra $\mathfrak{X}^1(M)[[\hbar]]$ is a Lie algebra when equipped with the $\hbar$-linear extension of the usual Lie brackets of fields. Then we can recover the group of **formal diffeomorphisms**

on M as its Lie group. It is defined to be the set of symbols $\exp(\hbar X)$ where $X \in \mathfrak{X}^1(M)[[\hbar]]$, with group structure given by a formal extension of the Baker–Campbell–Hausdorff formula:

$$\exp(\hbar X) \cdot \exp(\hbar Y) := \exp\left(\hbar\, \mathrm{BCH}(X,Y)\right) = \exp(\hbar(X+Y) + \frac{\hbar^2}{2}[X,Y] + O(\hbar^3)). \tag{6.2.4}$$

The exact expression of BCH is not relevant here, let us only observe that all terms of the formula are given by iterated Lie brackets. This group acts on formal Poisson tensors by **formal pushforward**:

$$\exp(\hbar X)_* \Pi_\hbar := \exp(\hbar\, \mathcal{L}_X)\Pi_\hbar. \tag{6.2.5}$$

Definition 6.13. Two formal Poisson structures Π and Π' are equivalent if there exists a formal vector field $X \in \mathfrak{X}^1(M)[[\hbar]]$ such that

$$\Pi' = \exp(\hbar X)_\star \Pi. \tag{6.2.6}$$

6.2.1 Kontsevich's Theorem

The seminal result for the theory, solving our classification problem, is the following:

Theorem 6.14 ([**kontsevich2003deformation**])**.** *For any manifold M there is an explicit bijection between the set of equivalence classes of formal Poisson structures on M and the set of equivalence classes of $\star$-products on M, such that if $\Pi_\hbar$ corresponds to $\star$, then*

$$\{f,g\}_{\Pi_\hbar} = B_1(f,g) - B_1(g,f). \tag{6.2.7}$$

In particular if M comes equipped with a Poisson structure Π, the $\star$-product associated with the null deformation ($\Pi_\hbar = \Pi$) is called ***canonical****.*

Kontsevich's Theorem is a very beautiful and deep result. The proof is not straightforward and requires some advanced techniques so we'll skip over it, but the formula for the bijection —however convoluted— can be given nonetheless. The rest of the chapter will be devoted to this goal.

We will construct the operators B_n in $\star$ in local coordinates[5] $(x^1, \ldots, x^d)$ as a sum of contributions involving $\Pi_\hbar^{ij}$ over a certain set (finite at every order) of special graphs (related to Feynman diagrams, although in an unusual way).

[5] To get an intrisic formulation, we'd have to work a little more. Kontsevich himself proved its theorem in $\mathbb{R}^n$, hence in local coordinates, hinted at an intrisic formulation, but left the problem to be solved by others. Cattaneo provides such a formulation.

6.2.1.1 Admissible graphs

Definition 6.15. A **quiver** Γ is the datum of

1. a set V_Γ whose elements are called **vertices** of the quiver,
2. a set E_Γ whose elements are called **arrows** or **edges** of the quiver,
3. two structure maps $s, t : E_\Gamma \to V_\Gamma$ (called the **source** and **target**).

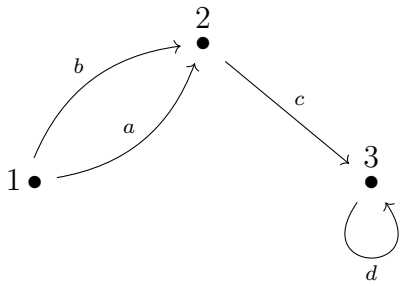
Remark 6.15.1. Basically, quivers are graphs decorated with directions for the edges (oriented graphs).

Example 6.6. The quiver defined by

$$\begin{aligned} V_\Gamma =& \{1, 2, 3\} \\ E_\Gamma =& \{a, b, c, d\} \end{aligned}$$

$$\begin{aligned} s : a \mapsto 1, \quad & t : a \mapsto 2 \\ b \mapsto 1, \quad & b \mapsto 2 \\ c \mapsto 2, \quad & c \mapsto 3 \\ d \mapsto 3, \quad & d \mapsto 3 \end{aligned} \tag{6.2.8}$$

can be represented graphically as



(6.2.9)

Clearly, d is a loop and a and b make a double arrow.

Definition 6.16. Given a quiver Γ, a **(small) loop** in Γ is an arrow $a \in E_\Gamma$ with $s(a) = t(a)$. A **double arrow** is a pair $a, b \in E_\Gamma$ with $s(a) = s(b)$, $t(a) = t(b)$.

Definition 6.17. An **admissible graph** (or **admissible quiver**) **of order** n is a quiver Γ such that

1. $V_\Gamma = \{1, \ldots, n\} \cup \{L, R\}$,
2. $E_\Gamma = \{a_1, b_1, \ldots, a_n, b_n\}$,
3. $s(a_i) = s(b_i) = i$ for any $i \in V_\Gamma$,

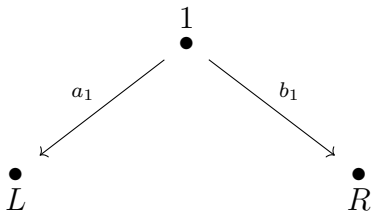
4. Γ has no loops and no double arrows.

The set of admissible graphs of order n is denoted G_n.

Example 6.7. G_0 is made by only one element,

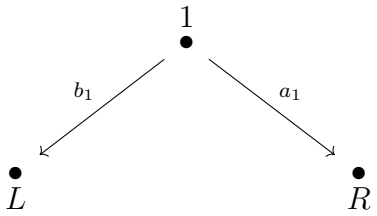
(6.2.10)

G_1 is made by two elements:



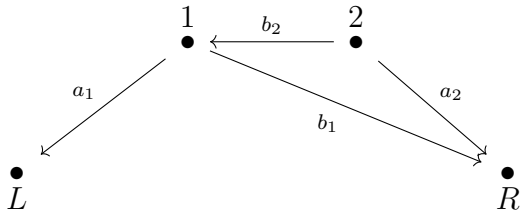
(6.2.11)

and



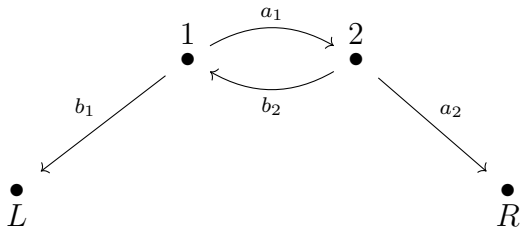
(6.2.12)

The set G_2 is made by 36 elements, two of them are



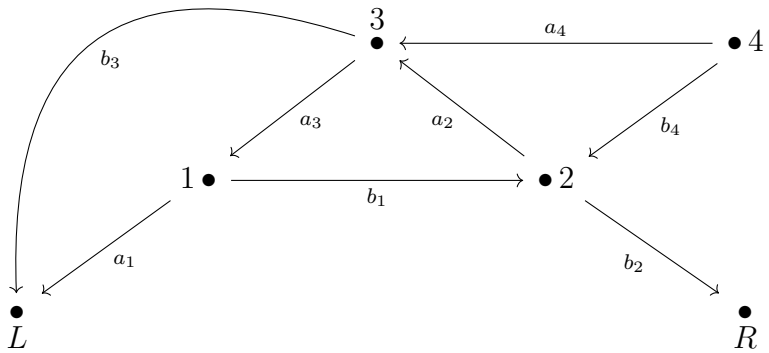
(6.2.13)

and



(6.2.14)

A G_4 admissible graph is



(6.2.15)

and there are other 159 999 graphs of this order. The number of distinct order n admissible graphs is in fact

$$|G_n| = (n(n+1))^n. \tag{6.2.16}$$

Definition 6.18. The bidifferential operator $B_{\Gamma,\Pi_\hbar}$ associated to an admissible graph $\Gamma \in G_n$ and the formal Poisson structure $\Pi_\hbar$ on a d-dimensional manifold M is expressed in local coordinates $(x^1, \dots, x^d)$ as

$$B_{\Gamma,\Pi_\hbar}(f,g) := \sum_{I:E_\Gamma \to \{1,\dots,d\}} \left(\prod_{i=1}^{n} \left(\prod_{a \in t^{-1}(i)} \partial_{I(a)} \right) \Pi_\hbar^{I(a_i)I(b_i)} \right) \left(\prod_{a \in t^{-1}(L)} \partial_{I(a)} \right) f \left(\prod_{a \in t^{-1}(R)} \partial_{I(a)} \right) g, \tag{6.2.17}$$

where the product for differential operators is intended as composition.

Example 6.8. Let's find the bidifferential operator associated to (6.2.15) on an arbitrary manifold M. Let's start by unpacking the first product of derivations, which is a product on the "numbered" vertices, hence has four factors: we thus put four Poisson factors in place.

$$\Pi \quad \Pi \quad \Pi \quad \Pi. \tag{6.2.18}$$

Then we put the two factor coming from L and R, thus given by the last two products. They make f and g factors respectively, so we put those in place as well:

$$\Pi \quad \Pi \quad \Pi \quad \Pi \quad f \quad g. \tag{6.2.19}$$

Then we can put indices on the Poisson tensors, which are completely determined by their positions. For conciseness, we denote $I(a_k)$ with i_k and $I(b_k)$ with j_k:

$$\Pi^{i_1 j_1} \quad \Pi^{i_2 j_2} \quad \Pi^{i_3 j_3} \quad \Pi^{i_4 j_4} \quad f \quad g. \tag{6.2.20}$$

Now we look at the derivative operators. We put them by successively look at each vertex and the arrows going **into** it:

$$(\partial_{i_3}\Pi^{i_1 j_1})(\partial_{j_1}\partial_{j_4}\Pi^{i_2 j_2})(\partial_{i_2}\partial_{i_4}\Pi^{i_3 j_3}\Pi^{i_4 j_4})(\partial_{i_1}\partial_{j_3} f)(\partial_{j_2} g). \tag{6.2.21}$$

The summation is then already implied by the summation convention on repeated indices. Clearly, it will have more and more terms as d grows.

The coefficients B_n of the $\star$-product associated to $\Pi_\hbar$ come from these operators, however they are also weighted in a particular way.

6.2.1.2 The weights

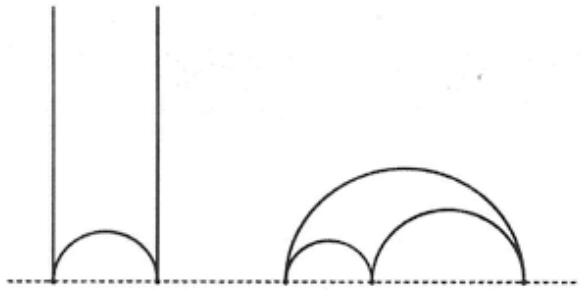
Definition 6.19. The **Poincarè half-plane** is the manifold supported on the upper part of the Gauss' plane

$$\mathbb{H} = \{p \in \mathbb{C} \,|\, \mathfrak{Im}\,(p) > 0\} \tag{6.2.22}$$

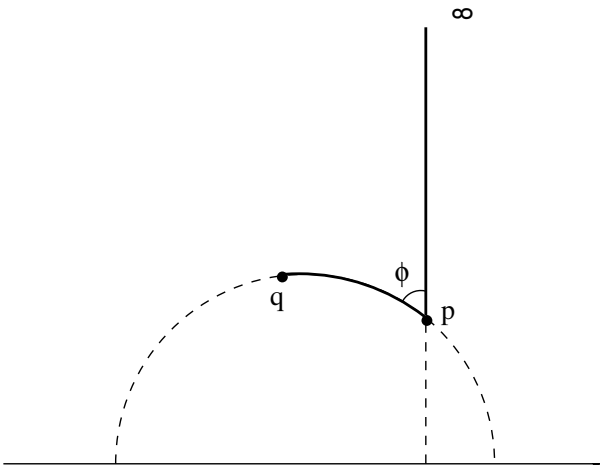
endowed with the **Lobachevsky metric**:

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{6.2.23}$$

The Poincarè half-plane is a model of hyperbolic geometry. The given metric has half-circles with center on real line $\mathfrak{Im}\,(p) = 0$ as geodesics, including those with infinite radius which degenerate to vertical lines.



We define a function $\phi : \mathbb{H}^2 \to \mathbb{R}/2\pi\mathbb{Z}$ which to each pair (p, q) of points in the Poincarè half-plane associates the angle between the vector $\vec{qp}$ and the vector $\vec{qp}$ (in this order) or, equivalently, to the angle between the vertical line at p and the geodesic between p and q (in this order):



Analytically, ϕ is given by

$$(p, q) \longmapsto \arg\left(\frac{q-p}{q-\overline{p}}\right) = \frac{1}{2i}\log\left(\frac{q-p}{q\overline{p}} \cdot \frac{\overline{q}-p}{\overline{q}-\overline{p}}\right). \tag{6.2.24}$$

This function is holomorphic over $\mathbb{H}^2$ and extends continuously to $\mathfrak{Im}(p) = 0$ or $\mathfrak{Im}(q) = 0$.

Now let $\Gamma \in G_n$ be an admissible graph. Let us denote with R_n the set of possible isometrical embeddings of Γ in $\mathbb{H}$ sending L to 0 and R to 1, i.e. the set of possible n-tuple of distinct points from which we realize geometrically the graph Γ using arcs of geodesics to draw the edges. Notice the set R_n is actually independent of Γ, as any n-tuple of distinct points can be used to draw isometrically any graph in G_n.

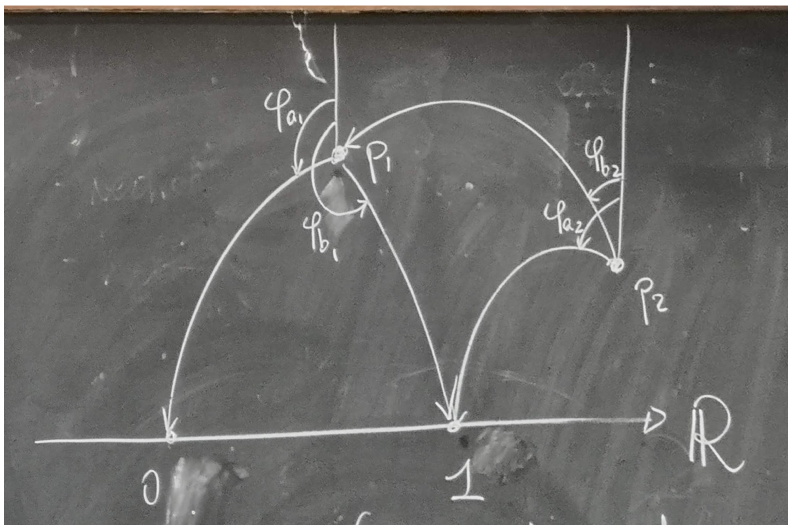


Figure 6.1: *A realization of the admissible graph* (6.2.13).

Given a realization $V_\Gamma \ni m \mapsto \overline{m} \in \mathbb{H}$, define ϕ_a to be $\phi(\overline{s(a)}, \overline{t(a)})$. Thus ϕ_a measure the abovementioned angle for the edge a. These are represented in Figure 6.1.

Definition 6.20. The weight w_Γ associated to $\Gamma \in G_n$ is defined as

$$w_\Gamma := \frac{1}{(2\pi)^{2n}} \int_{R_n} \bigwedge_{a \in E_\Gamma} d\phi_a. \tag{6.2.25}$$

Then the integral can be done, with a change of variables, in the ϕ_{a_i}, ϕ_{b_i} coordinates and it can be proven that

Lemma 6.21. *The integral defining w_Γ converges absolutely for every admissible graph Γ.*

6.2.1.3 The formula

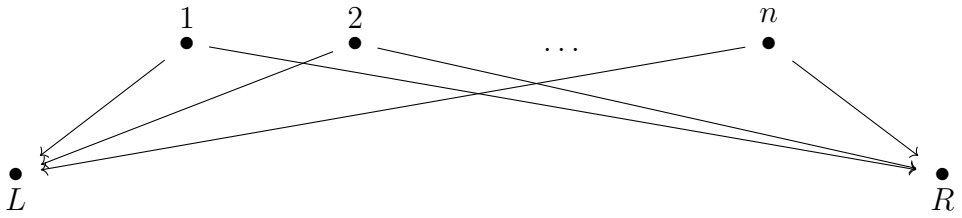
Finally, we can write down the sought formula.

Theorem 6.22. *To a formal Poisson structure $\Pi_\hbar$ it is associated the $\star$-product given by*

$$f \star_\hbar g := \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma, \Pi_\hbar}(f, g). \tag{6.2.26}$$

Moreover, this definition is stable (up to equivalence of $\star$-products) under change of coordinates.

Example 6.9. Let $\Pi_\hbar = \Pi = \Pi^{ij}\partial_i\partial_j$, with $\partial_k\Pi^{ij} = 0$ for any k. This means that only those $\Gamma \in G_N$ with $t(E_\Gamma) \subseteq \{L, R\}$ contribute to the associated star product, as this guarantees no derivatives of the Poisson tensors appear in their term. Such graphs look like this:



(6.2.27)

where we didn't label the arrows as for each vertex there are two choices, so for each n we get, in total, 2^n non-vanishing terms for the nth degree coefficient. The symmetry of exchanging a_k with b_k changes sign to $d\phi_{a_1} \wedge d\phi_{b_1} \wedge \ldots \wedge d\phi_{a_n} \wedge d\phi_{b_n}$, and so does w_Γ. On the other hand, the corresponding term also changes sign under this exchange, since a term $\Pi^{i_k j_k}$ changes to $\Pi^{j_k i_k} = -\Pi^{i_k j_k}$, by skew-symmetry. In the end, this gets us

$$B_n(f,g) = \frac{1}{n!}\sum_{\Gamma\in G_n} w_\Gamma B_{\Gamma,\Pi} = 2^n w_{\Gamma_n} B_{\Gamma_n,\Pi} \tag{6.2.28}$$

where Γ_n is the graph (6.2.27) labelled as to have all a_k edges going into L and all the b_k into R.

We know what $B_{\Gamma_n,\Pi}$ looks like:

$$B_{\Gamma_n,\Pi} = \Pi^{i_1 j_1} \cdot \ldots \cdot \Pi^{i_n j_n}\, \partial_{i_1} \ldots \partial_{i_n} f\, \partial_{j_1} \ldots \partial_{j_n} g. \tag{6.2.29}$$

To compute the weight, we solve the relevant integral. Notice each vertex can be positioned in the same points (i.e. the condition $\overline{1} \neq \overline{2}$ is virtual), and that all angle variables relative to different vertices are independent as there are no edges between them. With these considerations, we can already simplify the computation to the following:

$$w_\Gamma = \frac{1}{(2\pi)^{2n}}\int_{\mathcal{H}} d\phi_{a_1} \wedge d\phi_{b_1} \wedge \ldots \wedge d\phi_{a_n} \wedge d\phi_{b_b} = \frac{1}{(2\pi)^{2n}}\left(\int_{\mathcal{H}} d\phi_{a_1} \wedge d\phi_{b_1}\right)^n. \tag{6.2.30}$$

Then by fiddling around in the Poincarè upper half-plane, we see that (1) $\phi_a \leq \phi_b$ for any position and (2) all angles are achievable by a suitable configuration of the vertices. Therefore:

$$\frac{1}{(2\pi)^{2n}}\left(\int_{\mathcal{H}} d\phi_{a_1} \wedge d\phi_{b_1}\right)^n = \left(\int_0^{2\pi}\int_0^{\phi_b} d\phi_a d\phi_b\right)^n = \frac{1}{(2\pi)^{2n}}\left(\frac{2\pi \times 2\pi}{2}\right)^n = \frac{1}{2^n}. \tag{6.2.31}$$

So, in the end we get the Moyal product:

$$f \star g = \frac{1}{n!}\Pi^{i_1 j_1} \cdot \ldots \cdot \Pi^{i_n j_n}\, \partial_{i_1} \ldots \partial_{i_n} f\, \partial_{j_1} \ldots \partial_{j_n} g = f\, \exp\left(\Pi^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right)\, g = f \star_M g. \tag{6.2.32}$$

Remark 6.22.1. The Moyal product is the $\star$-product associated to $\Pi = \Pi^{ij}\partial_i \wedge \partial_j$, where Π^{ij} are constants. It's associated to the null deformation. Locally every constant rank Poisson tensor is expressible like that, in Darboux–Weinstein coordinates. This means that, locally, the first-order expansion of the associated product is:

$$f \star g = fg + \hbar\,\Pi^{ij}\partial_i f\,\partial_j g + O(\hbar^2). \tag{6.2.33}$$

6.3 Retrospect

Our goal was to produce $(\mathcal{C}^\infty(M), \star)$ so as to have observables f in there which obey the equation

$$\partial_t f = \frac{i}{\hbar}[f, h]_\star. \tag{6.3.1}$$

In the undeformed setting, the evolution of f was given by the time evolution operator (ref):

$$f(t) = U(t)^\dagger\, f(0)\, U(t) \tag{6.3.2}$$

In the deformed case we need to correct the operator, starting from the exponential:

$$\exp_\star(g) = 1 + g + \frac{g \star g}{2} + \frac{g \star g \star g}{6} + \ldots = \sum_{n=0}^{\infty} \frac{g^{\star n}}{n!}. \tag{6.3.3}$$

To get predictions out of the theory, we need necessarily some states on which evaluate observables. Where do we get them? It turns out we can interpret some functions of the algebra as states (the so-called Wigner functions).