Background. We use notation similar to the Supplemental Material of [1]. In brief, to model behaviour in the dual-decision task, we assumed that the observer has only access to r_i, a corrupted version of the true stimulus value s_i, defined as $r_i = s_i + \eta_i$, where the subscript $i = 1, 2$ indicate the decision at hand (first or second) and η_i is additive noise. Here we relax the assumption that the noise is Gaussian, and we simply assume that the noise comes from a distribution $\eta_i \sim f(\mu, \Theta)$, which is symmetric and unimodal. The first parameter of the distribution is the mean (μ) and Θ is another, possibly vector valued, parameter that controls other features of the distribution such as variance and kurtosis.

The assumption of symmetry implies that $\exists x_0$ such that $f(x_0 - \delta) = f(x_0 + \delta)$, $\forall \delta \in \mathbb{R}$.

The cumulative distribution function (CDF) is $F(x) = \int_{-\infty}^{x} f(u)\, du$.

Since the distribution is continuous and symmetric, we also have that $F(x_0 + \delta) = 1 - F(x_0 - \delta)$, $\forall \delta \in \mathbb{R}$. Note also that since the distribution is also unimodal, the point x_0 must be the mean and the median of the distribution.

> **Proposition.** Regardless of the precise form of the internal noise distribution f, as long as it is symmetric and the stimuli and internal noise parameters Θ are the same for the first and second decisions, the optimal strategy is to use the internal response to the first stimulus, r_1, to set the criterion for the second decision. That is, the observer will decide optimally in the second decision if they choose *'right'* whenever $r_2 \geq -|r_2|$.

Proof. By construction, the probabilities that the stimulus s_2 is sampled from the positive or negative semi-axis is not even, as it depends on the accuracy of first decision. Let us denote the prior probability that the second stimulus is positive as $P(s_2 \geq 0) = \alpha$; it follows that $P(s_2 < 0) = 1 - \alpha$. The optimal decision rule then can be expressed as the ratio of posterior probabilities:

$$\frac{p(r_2 \mid s_2 \geq 0)\alpha}{p(r_2 \mid s_2 < 0)(1 - \alpha)} \geq 1. \tag{1}$$

The likelihoods of the observation given the hypothesis of a positive or negative stimulus can be obtained by integrating the probability density function over the corresponding semi-axis. This can be expressed in terms of the CDF:

$$p(r_2 \mid s_2 \geq 0) = \int_0^{\infty} f(r_2 \mid s_2, \Theta)\, ds_2$$
$$= \int_0^{\infty} f(r_2 - s_2 \mid 0, \Theta)\, ds_2$$
$$= \int_{-\infty}^{r_2} f(u \mid 0, \Theta)\, du = F(r_2 \mid 0, \Theta).$$

Note that we have performed the substitution $u = r_2 - s_2$. This yielded the differential $ds_2 = -du$ and changed the limits of integration as follow: when $s_2 = 0$, $u = r_2$; and when $s_2 \to \infty$, $u \to -\infty$. We also reversed the limits of integration to remove the negative sign from the differential.

Applying the same steps we can also obtain: $p(r_2 \mid s_2 < 0) = 1 - F(r_2 \mid 0, \Theta)$.

Plugging these in (1) yields:

$$\frac{F(r_2 \mid 0, \Theta)\alpha}{[1 - F(r_2 \mid 0, \Theta)](1 - \alpha)} \geq 1.$$

With few algebraic manipulations we obtain:

$$F(r_2 \mid 0, \Theta) \geq 1 - \alpha.$$

In the dual-decision setting, the prior α of the second decision depends on the confidence that the first decision is correct. The confidence in the first decision is a function of the internal response r_1 and can be again expressed in terms of the CDF:

$$\alpha = \begin{cases} F(r_1 \mid 0, \Theta), & \text{if } r_1 \geq 0 \text{ (i.e. participants responded *'right'*)} \\ 1 - F(r_1 \mid 0, \Theta), & \text{if } r_1 < 0 \text{ (i.e. participants responded *'left'*)} \end{cases}$$

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Since f is symmetrical and continuous, we have that $1 - F(r_1 \mid 0, \Theta) = F(-r_1 \mid 0, \Theta)$.

The optimal rule for second decisions after a *'right'* first decision ($r_1 \geq 0$) is then:

$$F(r_2 \mid 0, \Theta) \geq F(-r_1 \mid 0, \Theta)$$

Applying F^{-1}, the inverse of the cumulative function, to both sides of the inequality yields:

$$r_2 \geq -r_1$$

as the decision rule for trials with $r_1 \geq 0$.

Conversely, in trials with $r_1 < 0$ we obtain $F(r_2 \mid 0, \Theta) \geq F(r_1 \mid 0, \Theta)$, leading to: $r_2 \geq r_1$.

Putting together both cases where $r_1 < 0$ and $r_1 \geq 0$, we obtain that the observer should decide *'right'* in the second decision whenever $r_2 \geq -|r_2|$. This concludes the proof.

References

[1] M. Lisi, G. Mongillo, G. Milne, T. Dekker, and A. Gorea. Discrete confidence levels revealed by sequential decisions. 5(2):273–280.