Background.We use notation similar to the Supplemental Material of [1]. In brief, to model behaviour in the dual-decision task, we assumed that the observer has only access to r_i, a corrupted version of the true stimulus value s_i, defined as $r_i = s_i + \eta_i$, where the subscript $i = 1, 2$ indicate the decision at hand (first or second) and η_i is additive noise. Here we relax the assumption that the noise is Gaussian, and we simply assume that the noise comes from a distribution $\eta_i \sim f(\mu, \Theta)$, which is symmetric and unimodal. The first parameter of the distribution is the mean (μ) and Θ is another, possibly vector valued, parameter that controls other features of the distribution such as variance and kurtosis.

The assumption of symmetry implies that $\exists\, x_0$ such that $f(x_0 - \delta) = f(x_0 + \delta)$, $\forall \delta \in \mathbb{R}$.

The cumulative distribution function (CDF) is $F(x) = \int_{-\infty}^{x} f(u)\, du$.

Since the distribution is continuous and symmetric, we also have that $F(x_0 + \delta) = 1 - F(x_0 - \delta)$, $\forall \delta \in \mathbb{R}$. Note also that since the distribution is also unimodal, the point x_0 must be the mean and the median of the distribution.

Proposition. Regardless of the precise form of the internal noise distribution f, as long as it is symmetric and the stimuli and internal noise parameters Θ are the same for the first and second decisions, the optimal strategy is to use the internal response to the first stimulus, r_1, to set the criterion for the second decision. That is, the observer will decide optimally in the second decision if they choose *'right'* whenever $r_2 \geq -|r_1|$.

Proof. By construction, the probabilities that the stimulus s_2 is sampled from the positive or negative semi-axis is not even, as it depends on the accuracy of first decision. Let us denote the prior probability that the second stimulus is positive as $P(s_2 \geq 0) = \alpha$; it follows that $P(s_2 < 0) = 1 - \alpha$. The optimal decision rule then can be expressed as the ratio of posterior probabilities:

$$\frac{p(r_2 \mid s_2 \geq 0)\alpha}{p(r_2 | s_2 < 0)(1-\alpha)} \geq 1. \tag{1}$$

The likelihoods of the observation given the hypothesis of a positive or negative stimulus can be obtained by integrating the probability density function over the corresponding semi-axis. This can be expressed in terms of the CDF:

$$\begin{aligned} p(r_2 \mid s_2 \geq 0) &= \int_0^{\infty} f(r_2 \mid s_2, \Theta)\, ds_2 \\ &= \int_0^{\infty} f(r_2 - s_2 \mid 0, \Theta)\, ds_2 \\ &= \int_{-\infty}^{r_2} f(u \mid 0, \Theta)\, du = F(r_2 \mid 0, \Theta). \end{aligned}$$

Note that we have performed the substitution $u = r_2 - s_2$. This yielded the differential $ds_2 = -du$ and changed the limits of integration as follow: when $s_2 = 0$, $u = r_2$; and when $s_2 \to \infty$, $u \to -\infty$. We also reversed the limits of integration to remove the negative sign from the differential.

Applying the same steps we can also obtain: $p(r_2 \mid s_2 < 0) = 1 - F(r_2 \mid 0, \Theta)$.

Plugging these in (1) yields:

$$\frac{F(r_2 \mid 0, \Theta)\alpha}{[1 - F(r_2 \mid 0, \Theta)]\,(1-\alpha)} \geq 1.$$

With few algebraic manipulations we obtain:

$$F(r_2 \mid 0, \Theta) \geq 1 - \alpha.$$

In the dual-decision setting, the prior α of the second decision depends on the confidence that the first decision is correct. The confidence in the first decision is a function of the internal response r_1 and can be again expressed in terms of the CDF:

$$\alpha = \begin{cases} F(r_1 \mid 0, \Theta), & \text{if } r_1 \geq 0 \text{ (i.e. participants responded } \textit{'right'}) \\ 1 - F(r_1 \mid 0, \Theta), & \text{if } r_1 < 0 \text{ (i.e. participants responded } \textit{'left'}) \end{cases}$$

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Since f is symmetrical and continuous, we have that $1 - F(r_1 \mid 0, \Theta) = F(-r_1 \mid 0, \Theta)$.

The optimal rule for second decisions after a *'right'* first decision ($r_1 \geq 0$) is then:

$$F(r_2 \mid 0, \Theta) \geq F(-r_1 \mid 0, \Theta)$$

Applying F^{-1}, the inverse of the cumulative function, to both sides of the inequality yields:

$$r_2 \geq -r_1$$

as the decision rule for trials with $r_1 \geq 0$.

Conversely, in trials with $r_1 < 0$ we obtain $F(r_2 \mid 0, \Theta) \geq F(r_1 \mid 0, \Theta)$, leading to: $r_2 \geq r_1$.

Putting together both cases where $r_1 < 0$ and $r_1 \geq 0$, we obtain that the observer should decide *'right'* in the second decision whenever $r_2 \geq -|r_2|$. This concludes the proof.

References

[1] M. Lisi, G. Mongillo, G. Milne, T. Dekker, and A. Gorea. Discrete confidence levels revealed by sequential decisions. 5(2):273–280.