Exponentials & logarithms

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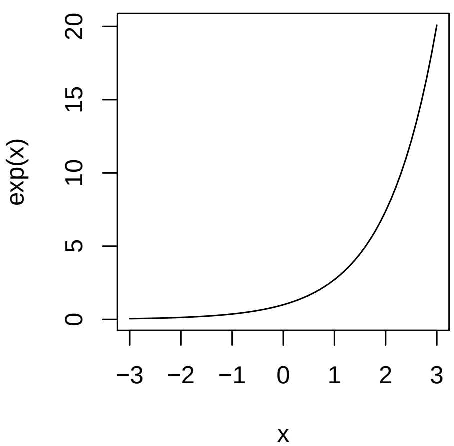
Linear equationssuch as $y = mx + c$ are powerful tools for describing many relationships (and as such they are the foundation of methods such as linear regression and ANOVA); however they they are not sufficient for capturing the full range of patterns we may observe in the world. Some variables, especially those that take only positive values (such as response times and income), often have relationships that don't fit well onto a straight line. In other situations, we may have phenomena where measures increase or decay at rates that are proportional to their current values, which linear models cannot adequately represent.

An example of this is wealth: as wealth tends to generate wealth, the wealth of richer people tends to increase faster than that of poorer people. This suggests that the relationship between wealth and anything that increases is unlikely to be well described by a linear equation. Another example is the number of people infected during an epidemic: this grows slowly at the beginning, when the number of infected people is small, and accelerates as the number increases. A similar *exponential growth* is often observed in the context of the spread of information through social networks: in these cases the rate at which information spreads is proportional to the number of people who are already aware of it. The *logarithms* and *exponential* functions are useful for helping to deal with these situations. These two functions are related to one another and are illustrated in Figure 1.



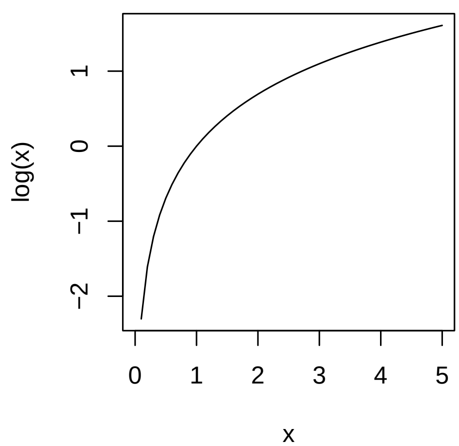


Figure 1: (*left*) An example of an exponential relationship. As x increases the values of y =exp(x) increase faster than faster. (*right*) A logarithmic relationship. As x increases, so does y =log(x), but at a slower and slower rate.

Exponential functions

Think about the relationship between addition, $+$, and multiplication, $\times$: we can define multiplication in terms of repeated additions. For example, 2×3 is the same as adding 2 to itself 3 times ($2 + 2 + 2$). With

exponential functions, we take the same idea of repeating a process multiple times and apply it to multiplication. You will have come across simple versions of this in school. The *square* a number is equal to that number multiplied by itself ($x^2 = x \times x$). You may have also come across the concept of *cubes*, defined as $x^3 = x \times x \times x$. The general form of this operation is known as an *exponential*, and we write it as b^x:

- x is called the **exponent**.
- b is known as the **base**.

As exponents indicate repeated multiplication, the numbers can increase rapidly. For instance 2^3 means we're multiplying the base, 2, by itself three times, yielding 8. If we increase the exponent by one, from 3 to 4, the result doubles (since we're multiplying by 2 once more). $2^4 = 16$, $2^5 = 32$ and $2^6 = 64$. This is why we say that the expression 2^x grows exponentially with x. Conversely, if we use a value between 0 and 1 such as $\frac{1}{2}$ as the base, the function will exhibit exponential decay. In this case, the value halves with each unit increase in x. Importantly, exponential functions are defined for any value of the exponent x, not only for whole numbers like 1, 2, 3, and so on. While it is less intuitive to interpret something like $2^{3.5}$ as repeated multiplication, for our purposes it is enough to know that the answer falls somewhere between 2^3 and 2^4. (As exponential growth gets faster and faster, $2^3.5 = 11.32$ is closer to $2^3 = 8$ than 2^4.)

You will often see exponential functions written as $y = \exp(x) = e^x$: this notation corresponds to a function where the base e is assumed to be *Euler's number*, a mathematical constant approximately equal to 2.718[1]. Much like π, this is one of these strange numbers that crop up frequently in mathematics and science. For us, it is enough to remember that e is a number around 2.7.

[1] Euler's number is denoted as e and is essential in various fields of mathematics, including statistics. Technically, this is defined as the limit of $\left(1 + \frac{1}{n}\right)^n$ as n approaches infinity, an expression that arise for example in the computation of compound interest.

Example: exponential growth in social media

Consider a scenario where a new recipe is shared on social media, such as a failproof method for making perfect poached eggs. If we assume that the number of shares the post receives follows an exponential growth pattern, doubling each day, we can calculate the expected number of shares at time t (expressed in days) as $N \times 2^t$, where N is the initial number of shares. If the recipe was initially shared by 3 accounts ($N = 3$) the expected number of shares is 48 after 4 days, 96 after 5 days, and so on. Moreover, this formula allows us to compute expected shares for fractional periods, such as hours. For instance, after 4.5 days (or 4 days and 12 hours), the expected number of shares would be approximately 68 (obtained calculating the value of the expression for $N = 3$ and $t = 4.5$).

Quick rules for exponentials

If you end up working with exponential relationships, there are some mathematical rules that are worth knowing about. There is no need to memorise these at this stage, but they are often taken for granted in more advanced statistical texts.

- an *exponent of 1* gives us the base: $b^1 = b$.
- an *exponent of 0* is defined to be 1: $b^0 = 1$.
- *negative exponents* lead to numbers between 0 and 1.
- the output of an exponential is always positive.
- multiplying two exponentials leads to addition: $b^{x_1} \times b^{x_2} = b^{x_1+x_2}$.

Logarithms

Just as subtraction is the inverse of addition, and division is the inverse of multiplication, **logarithms** are the inverse operation of exponentiation. As is commonly the case, undoing something is harder than doing it in the first place[2]. This means we are stuck with a somewhat complicated definition: the logarithm of a x is the exponent to which another fixed number, the base, must be raised to produce x. Technically, if $y = b^x$, then x is the logarithm of y with base b, and is written as $x = \log_b(y)$. When the base is Euler's number e, the logarithm is simply written as $x = \log(x)$ or $x = \ln(x)$ and is referred to as the *natural logarithm*.

[2] For example, breaking your favorite coffee mug is depressingly easy. Putting the pieces back together again is much harder. The same is true for many mathematical operations. Division is harder than multiplication, and logarithms are harder than exponents. This asymmetry in forward and backward computations also enables encryption. In cryptography, *hash* functions rely on this principle — they allow for quick 'forward' computations (creating a hash, a string of characters that can be thought of as an encrypted version of its input, often a password), but make it nearly impossible to reverse the process (reconstructing the original input), ensuring security in applications like password protection and digital signatures.

For our purposes, all you really need to know about logarithms is that they undo an exponential function. This leads to a curve in which each additional increase in x leads to a smaller and smaller increases in y (see Figure 1). Think of a logarithm as a transformation that compresses the scale of large numbers while expanding the scale of small numbers. For example, the logarithm with base 10 transforms each value into its order of magnitude, where each unit increase corresponds to multiplying the value by 10. Taking the logarithm of a variable of interest can be helpful helpful in several ways:

- **Handling wide-ranging data**: When we deal with data that spans a large range, such as income ranging from a few thousand to millions, logarithmic transformations help by compressing the scale, making the data more manageable and easier to analyze.

- **Expanding the scale of small values**: Logarithmic transformations are unique in the way they expand the range of small values between 0 and 1 into negative numbers. This characteristic is useful when dealing with variables that are bound to be positive. For instance, in a dataset where most values are clustered close to zero but a few are much larger, a logarithmic transformation can provide a clearer view of the data's spread.

- **Linearising relationships**: Some non-linear relationships can be (approximately) linearized with a logarithmic transformation, making linear modeling techniques applicable. By transforming one or both variables in a relationship, we can sometimes reveal a linear relationship that was somewhat hidden in the original scale.

Example

Let us consider the recipe sharing example again. We can use R to create a dataframe that tracks how the number of shares increases over time:

```
days <- seq(0, 5, by = 0.1)
initial_shares <- 3

d <- data.frame(days = days,
                shares = initial_shares * 2^days)
```

If we plot this data, we see the classic exponential curve, with the number of shares increasing by larger and larger amounts with each passing day. If instead we plot the logarithm of the number of shares, we obtain a linear relationship:

```
plot(d$days, log(d$shares),
     type = "l",
     xlab = "number of days", ylab = "log(shares)")
```

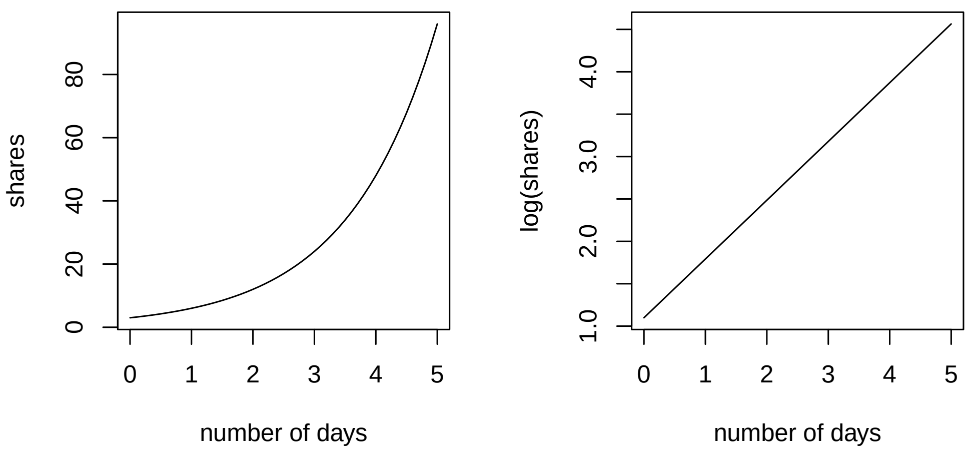


Figure 2: (*left*) The exponential relationship between `days` and `shares`. (*right*) If we replace shares with `log(shares)` we obtain a linear relationship.

Such manipulations of our variables are known as *log-transforms* and are used commonly in statistical analysis. Importantly, these transforms are reversible by applying the `exp()` function. This means that no information is lost.

Quick rules for logarithms

As logarithms are the inverse of exponential, we can reverse the rules from Section 2 to give us the following:

- the *logarithm of 1* gives us 0: $\log(1) = 0$.
- the *logarithm of* e is 1: $\log(e) = 1$ (assuming the natural logarithm in base e; more in general, we have that $\log_b(b) = 1$).
- *logarithms of numbers between 0 and 1* give negative values.
- the addition of two logarithms can be expressed as the logarithm of their product: $\log(x_1 \times x_2) = \log(x_1) + log(x_2)$.