



A Compact Course on Linear Algebra

**Laboratorio di Intelligenza Artificiale e
Grafica Interattiva
Anno Accademico 2019/20**

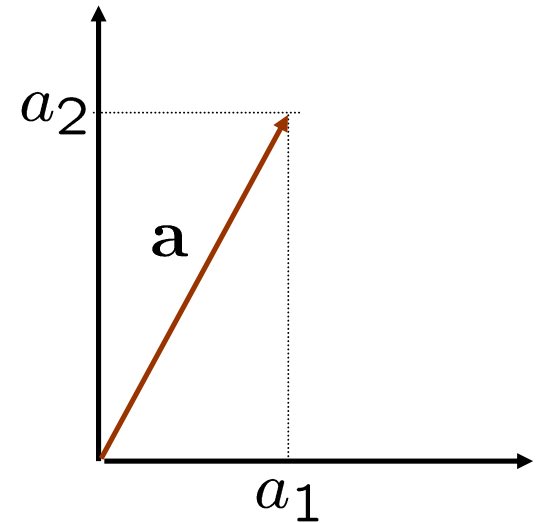
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Part of the material of this course is taken from the Robotics 2 lectures given by G.Grisetti, W.Burgard, C.Stachniss, K.Arras, D. Tipaldi and M.Bennewitz

Vectors

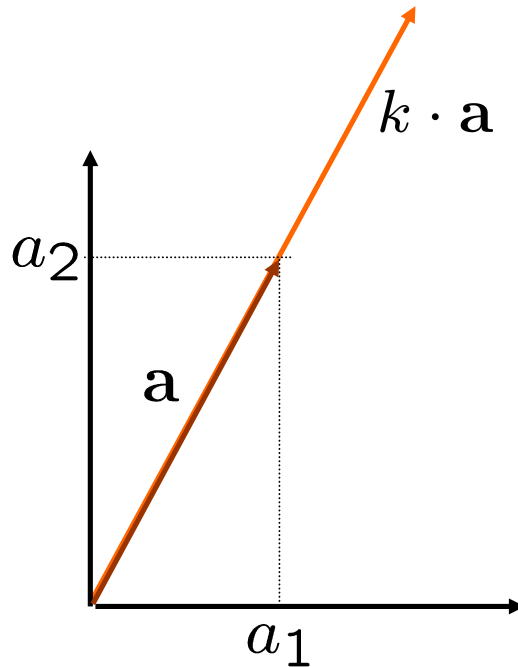
- Arrays of numbers
- They represent a point in a n dimensional space

$$(a_1) \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Vectors: Scalar Product

- Scalar-Vector Product $k \cdot \mathbf{a}$
- Changes the length of the vector, but **not** its direction

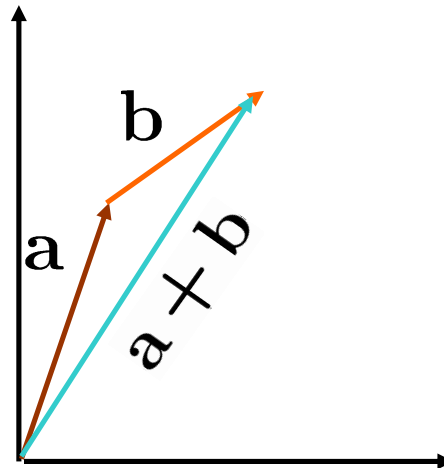


Vectors: Sum

- Sum of vectors (is commutative)

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Can be visualized as “chaining” the vectors.

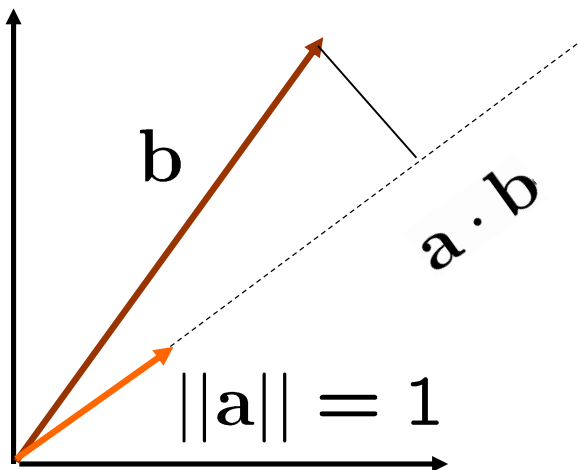


Vectors: Dot Product

- Inner product of vectors (is a scalar)

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \sum_i a_i \cdot b_i$$

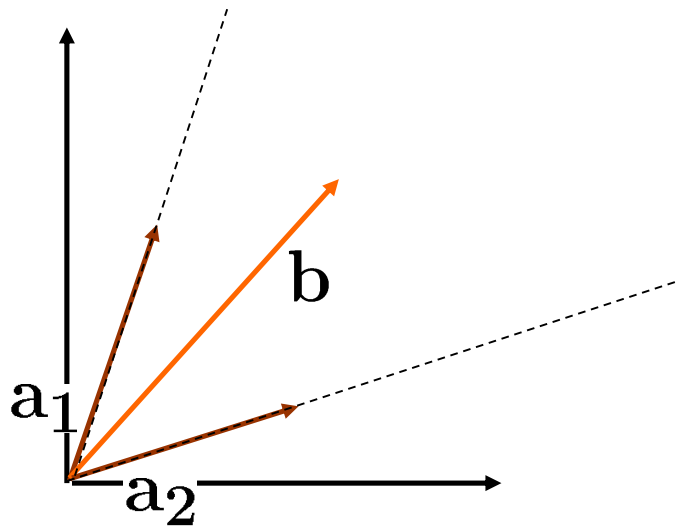
- If one of the two vectors has $\|\mathbf{a}\| = 1$ the inner product $\mathbf{a} \cdot \mathbf{b}$ returns the length of the projection of \mathbf{b} along the direction of \mathbf{a}



- If $\mathbf{a} \cdot \mathbf{b} = 0$ the two vectors are **orthogonal**

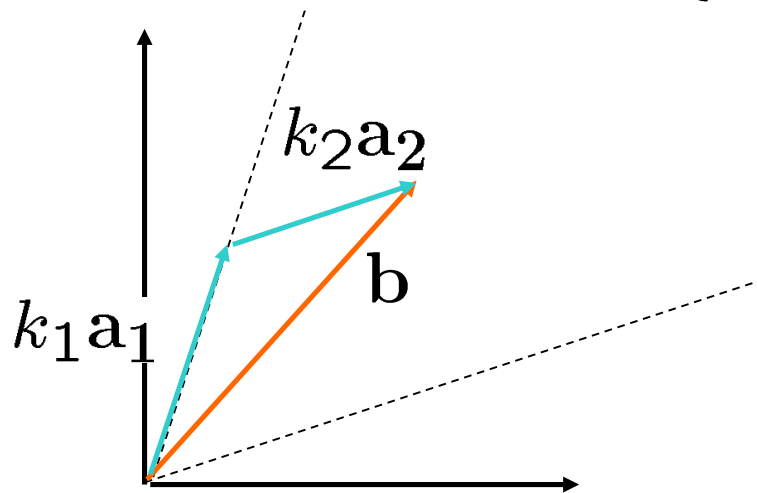
Vectors: Linear (In)Dependence

- A vector b is **linearly dependent** from $\{a_1, a_2, \dots, a_n\}$ if $b = \sum_i k_i \cdot a_i$
- In other words if b can be obtained by summing up the a_i properly scaled.
- If do not exist $\{k_i\}$ such that $b = \sum_i k_i \cdot a_i$ then b is independent from $\{a_i\}$



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Matrices

- A matrix is written as a table of values
- Can be used in many ways:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

- Note: a d-dimensional vector is equivalent to a $d \times 1$ matrix

Matrices as Collections of Vectors

- Column vectors

$$\mathbf{A} = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{matrix}} \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{a}_{*1} & \mathbf{a}_{*2} & \cdots & \mathbf{a}_{*m} \end{matrix}$

Matrices as Collections of Vectors

- Row Vectors

$$\mathbf{A} = \left(\begin{array}{c} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{array} \right) \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \left(\begin{array}{c} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{*n}^T \end{array} \right)$$

Matrices Operations

- Sum (commutative, associative)
- Product (not commutative)
- Inversion (square, full rank)
- Transposition
- Multiplication by a scalar
- Multiplication by a vector

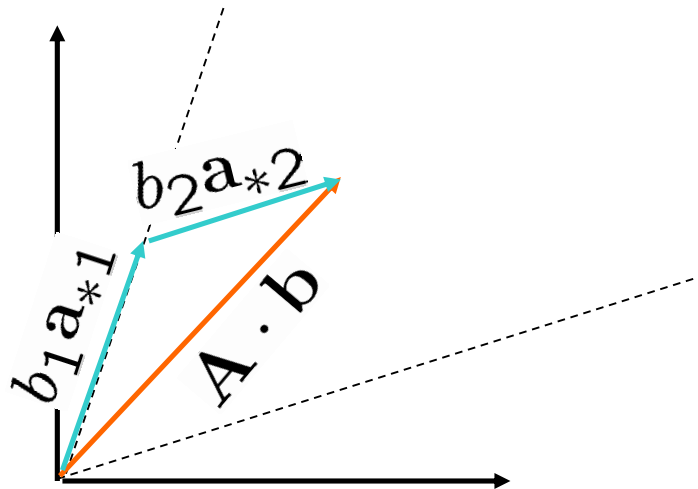
Matrix Vector Product

- The i component of $\mathbf{A} \cdot \mathbf{b}$ is the dot product $\mathbf{a}_{i*}^T \cdot \mathbf{b}$
- The vector $\mathbf{A} \cdot \mathbf{b}$ is linearly dependent from $\{\mathbf{a}_{*i}\}$ with coefficients $\{b_i\}$

$$\mathbf{A} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{n*}^T \end{pmatrix} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \cdot \mathbf{b} \\ \mathbf{a}_{2*}^T \cdot \mathbf{b} \\ \vdots \\ \mathbf{a}_{n*}^T \cdot \mathbf{b} \end{pmatrix} = \sum_k \mathbf{a}_{*k} \cdot b_k$$

Matrix Vector Product

- If the column vectors represent a reference system, the product $A \cdot b$ computes the global transformation of the vector b according to $\{a_{*i}\}$



Matrix Vector Product

- Each $a_{i,j}$ can be seen as a linear mixing coefficient that quantifies the contribution to $(A \cdot b)_j$
- Example: Jacobian of a multi-dimensional function

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix} \quad \mathbf{J}_f = \begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_m} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_m} \\ \vdots & \ddots & & \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_m} \end{pmatrix}$$

Matrix Matrix Product

- Can be defined through
 - the dot product of row and column vectors
 - the linear combination of the columns of **A** scaled by the coefficients of the columns of **B**

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\ &= \begin{pmatrix} \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*m} \\ \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*m} \\ \vdots & & & \\ \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*m} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} \cdot \mathbf{b}_{*1} & \mathbf{A} \cdot \mathbf{b}_{*2} & \cdots & \mathbf{A} \cdot \mathbf{b}_{*m} \end{pmatrix} \end{aligned}$$

Matrix Matrix Product

- If we consider the second interpretation we see that the columns of **C** are the projections of the columns of **B** through **A**
- All the interpretations made for the matrix vector product hold.

$$\begin{aligned} \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \\ &= \left(\mathbf{A} \cdot \mathbf{b}_{*1} \quad \mathbf{A} \cdot \mathbf{b}_{*2} \quad \dots \mathbf{A} \cdot \mathbf{b}_{*m} \right) \\ \mathbf{c}_{*i} &= \mathbf{A} \cdot \mathbf{b}_{*i} \end{aligned}$$

Linear Systems

$$\mathbf{Ax} = \mathbf{b}$$

- Interpretations:
 - Find the coordinates \mathbf{x} in the reference system of \mathbf{A} such that \mathbf{b} is the result of the transformation of \mathbf{Ax} .
 - Many efficient solvers
 - Conjugate gradients
 - Sparse Cholesky Decomposition (if SPD)
 - ...
 - One can obtain a reduced system $(\mathbf{A}' \mathbf{b}')$ by considering the matrix $(\mathbf{A} \mathbf{b})$ and suppressing all the rows which are linearly dependent
 - Let $\mathbf{A}'\mathbf{x}=\mathbf{b}'$ the reduced system with $\mathbf{A}':m'\times n$ and $\mathbf{b}':m'\times 1$ and rank $\mathbf{A}' = \min(m',n)$
 - The system might be either over-constrained ($m'>n$) or under-constrained ($m'<n$)

Over-constrained Systems

- An **over-constrained** does not admit an exact solution however if *rank* $\mathbf{A}' = \text{cols}(\mathbf{A})$ one may find a minimum norm solution by closed form pseudo inversion

$$\mathbf{x} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{A}'\mathbf{x} - \mathbf{b}'\| = (\mathbf{A}'^T \mathbf{A}')^{-1} \mathbf{A}'^T \mathbf{b}'$$

Linear Systems

- The system is **under-constrained** if the number of linearly independent columns (or rows) of \mathbf{A}' is smaller than the dimension of \mathbf{b}'
- An under-constrained system admits infinite solutions. The degree of infinity is $cols(\mathbf{A}') - rows(\mathbf{A}')$

Matrix Inversion

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{I}$$

- If \mathbf{A} is a square matrix of full rank, then there is a unique matrix $\mathbf{B} = \mathbf{A}^{-1}$ such that the above equation holds.
- The i^{th} row of \mathbf{A} is and the j^{th} column of \mathbf{A}^{-1} are:
 - orthogonal, if $i \neq j$
 - their scalar product is 1, otherwise
- The j^{th} column of \mathbf{A}^{-1} can be found by solving the following system:

$$\mathbf{A} \cdot \mathbf{a}^{-1}_{*j} = \mathbf{i}_{*j} \text{ ————— This is the } j^{th} \text{ column of the identity matrix}$$

Trace

- Only defined for **square matrices**
- **Sum** of the elements on the main diagonal, that is

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^n a_{ii}$$

- It is a linear operator with the following properties
 - Additivity: $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
 - Homogeneity: $\text{tr}(c \cdot A) = c \cdot \text{tr}(A)$
 - Pairwise commutative: $\text{tr}(AB) = \text{tr}(BA)$, $\text{tr}(ABC) \neq \text{tr}(ACB)$
- Trace is similarity invariant $\text{tr}(P^{-1}AP) = \text{tr}((AP^{-1})P) = \text{tr}(A)$
- Trace is transpose invariant $\text{tr}(A) = \text{tr}(A^T)$
- Given two vectors **a** and **b**, $\text{tr}(\mathbf{a}^T \mathbf{b}) = \text{tr}(\mathbf{a} \mathbf{b}^T)$

Rank

- **Maximum** number of linearly independent rows (columns)
- Dimension of the **image** of the transformation $f(\mathbf{x}) = A\mathbf{x}$
- When A is $m \times n$ we have
 - $\text{rank}(A) \geq 0$ and the equality holds iff A is the null matrix
 - $\text{rank}(A) \leq \min(m, n)$
 - $f(\mathbf{x})$ is **injective** iff $\text{rank}(A) = n$
 - $f(\mathbf{x})$ is **surjective** iff $\text{rank}(A) = m$
 - if $m = n$, $f(\mathbf{x})$ is **bijective** and A is **invertible** iff $\text{rank}(A) = n$
- Computation of the rank is done by
 - Gaussian elimination on the matrix
 - Counting the number of non-zero rows

Determinant

- Only defined for **square matrices**
- Remember? $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$ if and only if $\det(\mathbf{A}) \neq 0$
- For 2×2 matrices:

Let $\mathbf{A} = [a_{ij}]$ and $|\mathbf{A}| = \det(\mathbf{A})$, then

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- For 3×3 matrices the Sarrus rule holds:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Determinant

- For **general** $n \times n$ matrices?

Let \mathbf{A}_{ij} be the submatrix obtained from \mathbf{A} by deleting the i -th row and the j -th column

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & 3 & 4 & -1 \\ -5 & 8 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{bmatrix} \rightarrow \mathbf{A}_{23} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Rewrite determinant for 3×3 matrices:

$$\begin{aligned} \det(\mathbf{A}_{3 \times 3}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11} \\ &= a_{11} \cdot \det(\mathbf{A}_{11}) - a_{12} \cdot \det(\mathbf{A}_{12}) + a_{13} \cdot \det(\mathbf{A}_{13}) \end{aligned}$$

Determinant

- For **general** $n \times n$ matrices?

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}\det(\mathbf{A}_{11}) - a_{12}\det(\mathbf{A}_{12}) + \dots + (-1)^{1+n}a_{1n}\det(\mathbf{A}_{1n}) \\ &= \sum_{j=1}^n (-1)^{1+j}a_{1j}\det(\mathbf{A}_{1j}) \end{aligned}$$

Let $\mathbf{C}_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$ be the (i,j) -cofactor, then

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}\mathbf{C}_{11} + a_{12}\mathbf{C}_{12} + \dots + a_{1n}\mathbf{C}_{1n} \\ &= \sum_{j=1}^n a_{1j}\mathbf{C}_{1j} \end{aligned}$$

This is called the **cofactor expansion** across the first row

Determinant

- **Problem:** Take a 25 x 25 matrix (which is considered small). The cofactor expansion method requires $n!$ multiplications. For $n = 25$, this is 1.5×10^{25} multiplications for which a today supercomputer would take **500,000 years**.
- There are **much faster methods**, namely using **Gauss elimination** to bring the matrix into **triangular form**.

Then:

$$\mathbf{A} = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{bmatrix} \quad \det(\mathbf{A}) = \prod_{i=1}^n d_i$$

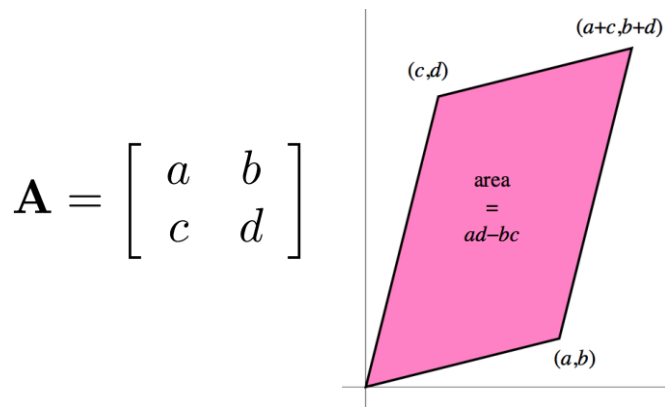
Because for **triangular matrices** the determinant is the product of diagonal elements

Determinant: Properties

- **Row operations** (\mathbf{A} still a $n \times n$ square matrix)
 - If \mathbf{B} results from \mathbf{A} by interchanging two rows, then $\det(\mathbf{B}) = -\det(\mathbf{A})$
 - If \mathbf{B} results from \mathbf{A} by multiplying one row with a number c , then $\det(\mathbf{B}) = c \cdot \det(\mathbf{A})$
 - If \mathbf{B} results from \mathbf{A} by adding a multiple of one row to another row, then $\det(\mathbf{B}) = \det(\mathbf{A})$
- **Transpose:** $\det(\mathbf{A}^T) = \det(\mathbf{A})$
- **Multiplication:** $\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$
- Does **not** apply to addition! $\det(\mathbf{A} + \mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$

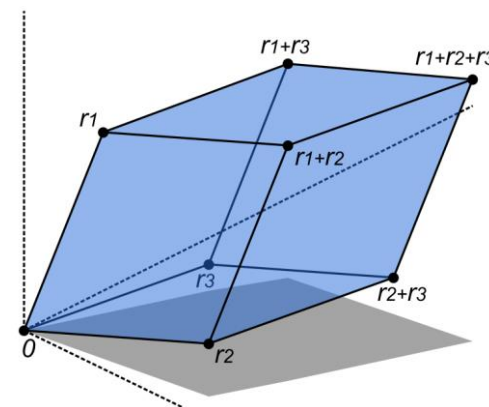
Determinant: Applications

- Find **the inverse** \mathbf{A}^{-1} using Cramer's rule $\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}$ with $\text{adj}(\mathbf{A})$ being the adjugate of \mathbf{A}
- Compute **Eigenvalues**
Solve the characteristic polynomial $\det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$
- Area and Volume:** $\text{area} = |\det(\mathbf{A})|$



$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(r_i is i -th row)



Orthonormal matrix

- A matrix Q is **orthonormal** iff its column (row) vectors represent an **orthonormal** basis

$$q_{*i}^T \cdot q_{*j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \forall i, j$$

- As linear transformation, it is **norm** preserving, and acts as an isometry in Euclidean space (rotation, reflection)
- Some properties:
 - The transpose is the inverse $QQ^T = Q^T Q = I$
 - Determinant has unity norm (§ 1)

$$1 = \det(I) = \det(Q^T Q) = \det(Q)\det(Q^T) = \det(Q)^2$$

Rotation matrix

- A Rotation matrix is an orthonormal matrix with $\det = +1$

- 2D Rotations $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

- 3D Rotations along the main axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- IMPORTANT: Rotations are **not commutative**

$$R_x\left(\frac{\pi}{4}\right) \cdot R_y\left(\frac{\pi}{4}\right) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \quad R_x\left(\frac{\pi}{4}\right) \cdot R_y\left(\frac{\pi}{4}\right) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y\left(\frac{\pi}{4}\right) \cdot R_x\left(\frac{\pi}{4}\right) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \quad R_y\left(\frac{\pi}{4}\right) \cdot R_x\left(\frac{\pi}{4}\right) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Matrices to represent Affine Transformations

- A general and easy way to describe a 3D transformation is via matrices

$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} \mathbf{t} \\ 1 \end{pmatrix}$$

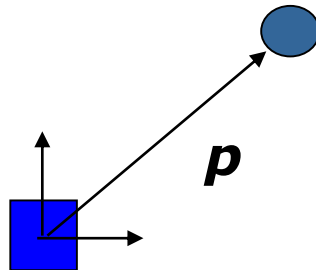
Translation Vector

Rotation Matrix

- Homogeneous behavior in 2D and 3D
- Takes naturally into account the non-commutativity of the transformations

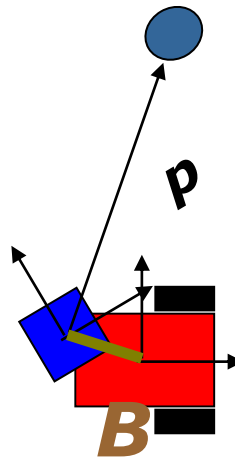
Combining Transformations

- A simple interpretation: chaining of transformations (represented as homogeneous matrices)
 - Matrix **A** represents the pose of a **robot** in the space
 - Matrix **B** represents the position of a sensor on the robot
 - The **sensor** perceives an **object** at a given location **p** , in its own frame [the sensor has no clue on where it is in the world]
 - Where is the object in the global frame?



Combining Transformations

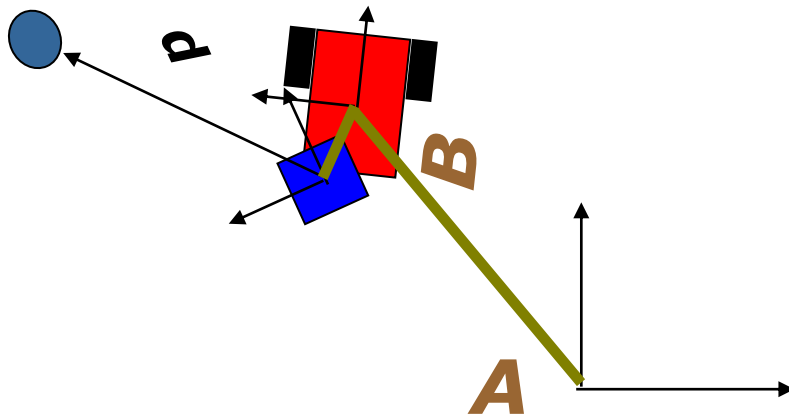
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Bp gives me the pose of the object wrt the robot

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Bp gives me the pose of the object wrt the robot

ABp gives me the pose of the object wrt the world

Symmetric matrix

- A matrix A is **symmetric** if $A = A^T$, e.g. $\begin{bmatrix} 1 & 4 & -2 \\ 4 & -1 & 3 \\ -2 & 3 & 5 \end{bmatrix}$
- A matrix A is **skew-symmetric** if $A = -A^T$, e.g. $\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$
- **Every** symmetric matrix:
 - is **diagonalizable** $D = QAQ^T$, where D is a diagonal matrix of **eigenvalues** and Q is an orthogonal matrix whose columns are the **eigenvectors** of A
 - define a **quadratic form** $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \sum_{i,j=1}^n a_{ij} x_i x_j$

Positive definite matrix

- The analogous of positive number
- Definition $M > 0$ iff $z^T M z > 0 \forall z \neq 0$

- Examples

- $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0$

- $M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1 z_2 < 0, z_1 = -z_2$

Positive definite matrix

- Properties
 - **Invertible**, with positive definite inverse
 - All real **eigenvalues** > 0
 - **Trace** is > 0
- **Cholesky** decomposition $A = LL^T$
- **Partial ordering**: $M > N$ iff $M - N > 0$
- If $M > N > 0$, we have $N^{-1} > M^{-1} > 0$
- If $M, N > 0$, then
 - $M + N > 0$

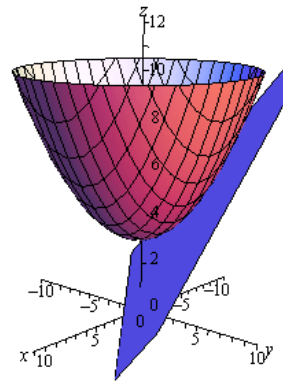
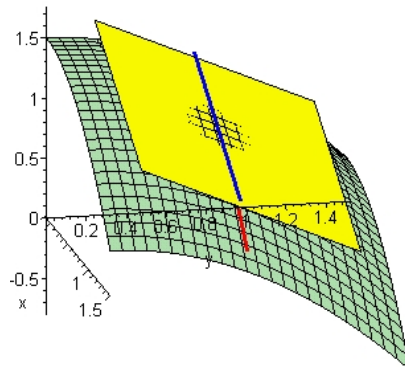
Jacobian Matrix

- It's a **non-square matrix** $n \times m$ in general
- Suppose you have a vector-valued function $f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$
- Then, the **Jacobian matrix** is defined as

$$\mathbf{F}_{\mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It's the orientation of the **tangent plane** to the vector-valued function at a given point



- **Generalizes the gradient** of a scalar valued function
- Heavily used for **first-order error propagation**

$$\mathbf{C}_{out} = \mathbf{F} \cdot \mathbf{C}_{in} \cdot \mathbf{F}^T$$

→ See later in the course

Quadratic Forms

- Many important functions can be locally approximated with a quadratic form.

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c \\ &= \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b} \mathbf{x} + c \end{aligned}$$

- Often one is interested in finding the minimum (or maximum) of a quadratic form.

$$\hat{\mathbf{x}} = \operatorname{argmin}_{\mathbf{x}} f(\mathbf{x})$$

Quadratic Forms

- How can we use the matrix properties to quickly compute a solution to this minimization problem?

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$$

- At the minimum we have $f'(\hat{\mathbf{x}}) = 0$
- By using the definition of matrix product we can compute f'

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b} \mathbf{x} + c \\ f'(\mathbf{x}) &= \mathbf{A}^T \mathbf{x} + \mathbf{A} \mathbf{x} + \mathbf{b} \end{aligned}$$

Quadratic Forms

- The minimum of $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b} \mathbf{x} + c$ is where its derivative is set to 0

$$0 = \mathbf{A}^T \mathbf{x} + \mathbf{A} \mathbf{x} + \mathbf{b}$$

- Thus we can solve the system

$$(\mathbf{A}^T + \mathbf{A})^T \mathbf{x} = -\mathbf{b}$$

- If the matrix is symmetric, the system becomes

$$2\mathbf{A} \mathbf{x} = -\mathbf{b}$$

- Solving that, leads to the minimum

Further Reading

- A “quick and dirty” guide to matrices is The Matrix Cookbook, see:
<http://matrixcookbook.com>