A. Church
1903 - 1995

λ-calcolo ('33 e seg.)

$$M, N ::= x \mid \lambda x.M \mid MN$$

$$x \in \{x_0, x_1, x_2, \dots\} \ni x, y, z, \dots$$

notaz. $MNL \equiv (MN)L$ $\qquad \lambda x y. M \equiv \lambda x.(\lambda y.M)$

$\lambda x. \dots x \dots x \dots$ funzione anonima con param. formale x

MN applicazione funzionale di M ad N

Sostituzione (1)

$$(\ldots x \ldots x \ldots)[N/x] \equiv (\ldots N \ldots N \ldots)$$

$$\beta: \quad (\lambda x.M)N = M[N/x]$$

Problema: $\lambda x.y$ funzione costante con val. y

$$(\lambda x.y)[w/y] \equiv \lambda x.w \quad \checkmark \quad (\text{fun. cost.} = w)$$

$$(\lambda x.y)[x/y] \equiv \lambda x.x \quad \times \quad (\text{identità})$$

Senza restrizioni λx dipende dalla scelta di x

Variabili libere e vincolate

$FV(M)$ insieme delle var. libere in M

$$FV(x) = \{x\}$$
$$FV(MN) = FV(M) \cup FV(N)$$
$$FV(\lambda x. M) = FV(M) - \{x\}$$

$BV(M)$ insieme delle var. vincolate in M

$$BV(M) = Var(M) - FV(M)$$

Sostituzione (2)

$$x[N/x] \equiv N$$

$$y[N/x] \equiv y \qquad \text{se} \quad x \not\equiv y$$

$$(ML)[N/x] \equiv (M[N/x])(L[N/x])$$

$$(\lambda y.M)[N/x] \equiv \lambda y.M[N/x] \qquad x \not\equiv y,\ y \notin FV(N)$$

sempre definita a meno di ridenom. delle var. vincolate:

$$\lambda x.M =_\alpha \lambda y.M[y/x] \qquad y \notin FV(M) \cup BV(M)$$

Riduzione

$$\beta_{red}: \quad (\lambda x.M)\,N \longrightarrow M[N/x]$$

$$\frac{M \longrightarrow N}{C[M] \longrightarrow C[N]}$$

dove $C[\;] \equiv \ldots[\;]\ldots$ e $C[M] \equiv \ldots M \ldots$

nota: le var. libere in M possono essere vincolate in $C[M]$

$$\xi \quad \frac{M \longrightarrow N}{\lambda x.M \longrightarrow \lambda x.N}$$

Teorema di Church-Rosser

$$\forall M, N, L.\ M \xrightarrow{*} N \wedge M \xrightarrow{*} L \Longrightarrow$$
$$\exists P.\ N \xrightarrow{*} P \wedge L \xrightarrow{*} P$$

J. B. Rosser

1907 - 1994

$$\begin{tikzcd} & M \arrow[dl, "*"'] \arrow[dr, "*"] & \\ N \arrow[dr, dashed, "*"'] & & L \arrow[dl, dashed, "*"] \\ & P & \end{tikzcd}$$

Corollario Se $=_\beta$ è la chiusura simm. di $\xrightarrow{*}$

allora $M =_\beta N \Longrightarrow \exists L.\ M \xrightarrow{*} L \wedge N \xrightarrow{*} L$

Booleani

$$\underline{true} \equiv \lambda x\, y.\, x \qquad (\text{comb. } K)$$

$$\underline{false} \equiv \lambda x\, y.\, y \qquad (\text{comb. } O)$$

$$\text{if-then-else} \equiv \lambda x\, y\, z.\, x\, y\, z \qquad (\text{estensionalmente equivalente a } I)$$

$$\text{if-then-else } \underline{true}\ M\, N =_\beta \underline{true}\ M\, N =_\beta M$$

$$\text{if-then-else } \underline{false}\ M\, N =_\beta \underline{false}\ M\, N =_\beta N$$

Esercizio: def. neg t.c. neg $\underline{true}$ = false, neg $\underline{false}$ = $\underline{true}$

Numerali di Church

$\underline{n} \equiv \lambda x y . \underbrace{x (\ldots (x}_{n} y) \ldots)$

es. $\underline{0} \equiv \lambda x y . y \qquad \underline{1} \equiv \lambda x y . x (x y)$

Successore $\quad succ\ \underline{n} =_\beta \underline{n+1} \equiv \lambda x y . \underbrace{x (x \ldots (x}_{n+1} y) \ldots)$

ma $\quad \underline{n}\, x y =_\beta \underbrace{x (\ldots (x}_{n} y) \ldots)$

quindi def. il successore con

$$suc \equiv \lambda z x y . x (z x y)$$

Numerali di Church

$$\underline{n} \equiv \lambda x y . \underbrace{x (\dots (}_{n} x y) \dots)$$

Somma

$$add\ \underline{n}\ \underline{m} = \underline{n+m}$$

oss. $\quad \underline{n}\, f\, x = \underbrace{f \dots (}_{n} f\ x) \dots$

quindi $\quad \underline{n}\ suc\ \underline{m} = \underbrace{suc\ (\dots (suc}_{n}\ \underline{m}) \dots)$

da cui $\quad add \equiv \lambda u v . u\ suc\ v$

Esercizio: trovare mult t.c. $mult\ \underline{n}\ \underline{m} = \underline{n \cdot m}$

Test per zero

$$\text{is-zero}\ \underline{0} = \underline{\text{true}}$$

$$\text{is-zero}\ \underline{n+1} = \underline{\text{false}}$$

$$\underline{0}\, x\, y = y$$

$$\underline{1}\, x\, y = x\, y$$

$$\underline{2}\, x\, y = x\,(x\, y)$$

$$\dots$$

$$y \equiv \underline{\text{true}}$$

$$x \equiv K\, \underline{\text{false}} = \lambda z.\, \underline{\text{false}}$$

$$\text{is-zero} = \lambda n.\, n\,(\lambda z.\, \underline{\text{false}})\, \underline{\text{true}}$$

Ricorsione

$$\begin{cases} fact\ \underline{0} = \underline{1} \\ fact\ \underline{n+1} = mult\ (\underline{n+1})\ (fact\ \underline{n}) \end{cases}$$

supponendo di aver def. il predecessore

$$pred\ \underline{0} = \underline{0} \qquad pred\ \underline{n+1} = \underline{n}$$

$$fact\ \underline{n} = if\text{-}then\text{-}else\ (is\text{-}zero\ \underline{n})\ \underline{1}\ (mult\ \underline{n}\ (fact\ (pred\ \underline{n})))$$

Idea: $\quad F\ f\ n = if\text{-}then\text{-}else\ \cdots\ (f\ (pred\ n))$

allora $\quad fact = fix\ F$ dove $\quad fix\ F = F\ (fix\ F)$

Teorema del punto fisso

$$\forall F \, \exists X \, . \, FX = X$$

H. B. Curry
1900 – 1982

Dim. leggiamo l'eq. alla rovescia:

$$X = FX$$

poniamo $\quad X \equiv WW \quad$ ottenendo

$$WW = F(WW)$$

quindi $\quad W \equiv \lambda w. \, F(ww) \qquad \square$

Operatore a punto fisso (Y)

$$fix \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

allora

$$\begin{aligned} fix\,F &= (\lambda x.F(xx))(\lambda x.F(xx)) \\ &= F((\lambda x.F(xx))(\lambda x.F(xx))) \\ &= F(fix\,F) \end{aligned}$$

in letteratura fix si scrive Y_{curry} o semplicemente Y.

Conseguenze:

$$fix\,(\lambda y.y)$$

$$= (\lambda x.(\lambda y.y)(xx))\,(\lambda x.(\lambda y.y)(xx))$$

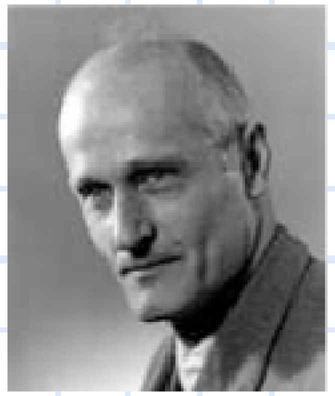
$$= (\lambda x.xx)(\lambda x.xx) \equiv \Omega$$

ma

$$\Omega \equiv (\lambda x.xx)(\underline{\lambda x.xx}) \longrightarrow (\lambda x.xx)(\lambda x.xx)$$

Dunque il λ-calcolo non tipato puro (senza cost.) non è SN.

Teorema di Kleene



S. C. Kleene
1909 - 1994

Per ogni f calcolabile parziale esiste $F \in \Lambda^0$ t.c.

$$f(n_1, \dots, n_k) \simeq m \iff F\,\underline{n}_1 \cdots \underline{n}_k =_\beta \underline{m}$$

dove $f(\vec{n}) \simeq m$ significa che $f(\vec{m})$ è def.
e uguale ad m