Quasi-Hopf algebras of dimension 6

Matteo Misurati

University of Ferrara

ItaCa 2024, Padova, December 20th

Based on a joint work with Daniel Bulacu[1]

[1] D. Bulacu, M.M., *Quasi-Hopf algebras of dimension* 6, arXiv:2410.03476 4 Oct 2024.

Summary

- We proved that all 6-dimensional quasi-Hopf algebras are semisimple, by classifying 3-dimensional bialgebras in the Yetter-Drinfeld module category.
- We then obtained an explicit list of 6-dimensional quasi-Hopf algebras, thanks to a classification result of fusion categories[2].
- We start by recalling some basic facts and results on quasi-Hopf algebras (qHas) and fusion categories.
- Unless otherwise specified, we denote by k the base field of characteristic 0 and algebraically closed.

[2] P. Etingof, S. Gelaki, V. Ostrik, *Classification of fusion categories of dimension pq*, Int Math Res Notices **57** (2004), 3041–3056.

Ingredients: Quasi-bialgebra

A quasi-bialgebra B is an associative unital algebra endowed with extra structure by a coproduct (an algebra morphism) $\Delta : B \to B \otimes B$, a counit $\epsilon : B \to k$ and a reassociator $\Phi \in \mathcal{U}(B \otimes B \otimes B)$ satisfying

$$(Id_B \otimes \Delta)(\Delta(b)) = \Phi(\Delta \otimes Id_B)(\Delta(b))\Phi^{-1},$$
$$(Id_B \otimes \epsilon)(\Delta(b)) = b \quad \text{and} \quad (\epsilon \otimes Id_B)(\Delta(b)) = b.$$

for any $b \in B$. The reassociator Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(Id_B \otimes \Delta \otimes Id_B)(\Phi)(\Phi \otimes 1)$$
$$= (Id_B \otimes Id_B \otimes \Delta)(\Phi)(\Delta \otimes Id_B \otimes Id_B)(\Phi),$$
$$(Id_B \otimes \epsilon \otimes Id_B)(\Phi) = 1 \otimes 1.$$

$$\Phi =: \sum_{i \in I} X_i^1 \otimes X_i^2 \otimes X_i^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3,$$

$$\Phi^{-1} =: \sum_{i \in I} x_i^1 \otimes x_i^2 \otimes x_i^3 = y^1 \otimes y^2 \otimes y^3, \quad \Delta(b) = \sum_{j \in J} (b_1)_j \otimes (b_2)_j.$$

Ingredients: Quasi-Hopf algebra

- A quasi-bialgebra H is a quasi-Hopf algebra if there exists an algebra antimorphism $S : H \to H$ and distinguished elements $\alpha, \beta \in H$ such that, for all $h \in H$, the following conditions hold:

$$S(h_1)\alpha h_2 = \epsilon(h)\alpha, \qquad h_1\beta S(h_2) = \epsilon(h)\beta,$$
$$X^1\beta S(X^2)\alpha X^3 = 1, \quad S(x^1)\alpha x^2\beta S(x^3) = 1.$$

- If $\Phi = 1 \otimes 1 \otimes 1$, H is a (usual) Hopf algebra, otherwise H is a genuine quasi-Hopf algebra.
- If it exists, an antipode (S, α, β) of a quasi-bialgebra H is unique up to conjugation by an element in H. Being "Hopf" is more a property than an additional structure.
- $H' \subseteq H$ is a sub-quasi-Hopf algebra of H if it is a quasi-Hopf algebra with the same reassociator and distinguished elements of H, and has structure maps induced by those of H.

Quasi-Hopf algebras: categorical interpretation

Let A be a unital associative algebra and let ${}_A\mathcal{M}$ be its category of left modules. By reconstruction theory, we have the following bijection of structures:

A		${}_A\mathcal{M}$
bialgebra	$\longleftrightarrow$	strict monoidal
Hopf algebra	$\longleftrightarrow$	left rigid strict monoidal
Hopf algebra with S bijective	$\longleftrightarrow$	rigid strict monoidal

$$\rightsquigarrow$$

A		${}_A\mathcal{M}$
quasi-bialgebra	$\longleftrightarrow$	monoidal
quasi-Hopf algebra	$\longleftrightarrow$	left rigid monoidal
quasi-Hopf algebra with S bijective	$\longleftrightarrow$	rigid monoidal

Ingredients: Fusion categories

Definition

A tensor category

- is rigid monoidal with semisimple unit object
- is abelian
- all objects have finite length

An abelian category $\mathcal{C}$ is said to be finite if

- has finitely many simple objects (up to isomorphism)
- objects have finite length
- all simple objects admit a projective cover

Definition

A fusion category is a semisimple finite tensor category.

Frobenius-Perron dimension

The Grothendieck ring $Gr(\mathcal{C})$ of a tensor category is the ring with $\mathbb{Z}$-basis given by the isomorphism classes of simple objects, with multiplication

$$X \cdot Y = \sum_{S \text{ simple}} [X \otimes Y : S]\, S,$$

where $[X \otimes Y : S]$ denotes the number of occurrences of S in $X \otimes Y$.

Let $\mathcal{C}$ be a finite tensor category with simple objects $X_1, \ldots, X_n$. For any $X \in Obj(\mathcal{C})$, there exits a square matrix with non-negative entries M_X such that, in $Gr(\mathcal{C})$,

$$X \cdot X_i = \sum_{j=1}^{n} (M_X)_{ij} X_j$$

The FP dimension of X is the largest non-negative real eigenvalue $\lambda(X)$ of M_X. The FP dimension of $\mathcal{C}$ is $\sum_i \lambda(X_i)\lambda(P_i)$, where P_i is the projective cover of X_i.

Example: fusion categories

G finite group, $\omega \in Z^3(G, k^\times)$ normalized 3-cocycle. $H = Fun_\omega(G)$ denotes the quasi-Hopf algebra which is k^G as an algebra and has structure maps determined as follows, where $P_g \in k^G$ denotes the element dual to $g \in G$.

$$\Delta(P_g) = \sum_{x\in G} P_x \otimes P_{x^{-1}g}, \quad \epsilon(P_g) = P_g(e_G), \quad S(P_g) = P_{g^{-1}},$$

$$\Phi_\omega = \sum_{x,y,z\in G} \omega(x,y,z)\ P_x \otimes P_y \otimes P_z,$$

${}_H\mathcal{M}$ is a fusion category with the evaluation modules $\{V_g\}_{g\in G}$ as simple objects. Since $V_g \otimes V_h = V_{gh}$, $Gr({}_H\mathcal{M}^{fd}) = \mathbb{Z}[G]$.
As a monoidal category, its associativity isomorphism Ψ is determined by

$$\Psi_{V_g, V_h, V_l} = \omega(g,h,l) Id$$

If G is an abelian group, we can identify $Fun_\omega(G)$ with $k_{\Phi_\omega}[G]$, the Hopf algebra $k[G]$ seen as "quasi" with reassociator induced by ω.

Fusion categories and the classification of qHas

Theorem (P. Etingof, V. Ostrik, '04)

Let $\mathcal{C}$ be a finite tensor category. $\mathcal{C} \cong Rep(H)$ as a tensor category for a finite dimensional quasi-Hopf algebra H if and only if every object X of $\mathcal{C}$ has an integer Frobenius-Perron dimension.

Moreover, $\mathcal{C}$ has FP dimension n if and only if H has linear dimension n.

The following result describe classes of semisimple quasi-Hopf algebras.

Corollary (P. Etingof, D. Nikshych, V. Ostrik, '05)

Let $\mathcal{C}$ be a fusion category with FP dimension p, where p is prime. Then $\mathcal{C}$ is equivalent to the category of representations of C_p, with associativity determined by a cohomology class in $H^3(C_p, k^\times)$.

Other classification results for quasi-Hopf algebras

Theorem (P. Etingof, S. Gelaki, V. Ostrik, '04)

Let $p < q$ be primes. Any fusion category of FP dimension pq, with integer dimension of simple objects is equivalent to one of the following:

- *A category with 1-dimensional simple objects.*
- *The category of representations of $C_p \ltimes C_q$.*
- *If $p = 2$, a category of bimodules in graded vector spaces with associativity controlled by $\omega \in H^3(C_2 \ltimes C_q, k^\times)$. This is a category of left modules over a genuine quasi-Hopf algebra.*

- P. Etingof and S. Gelaki classified[3] all quasi-Hopf algebras with two 1-dimensional simple modules.
- As a byproduct, they obtained the classification of quasi-Hopf algebras in dimension 4 (they are all semisimple or twist equivalent to H_4).

[3] P. Etingof, S. Gelaki, *Finite dimensional quasi-Hopf algebras with radical of codimension 2*, Math. Res. Lett. **11** (2004), 685–696.

Ingredients: Yetter-Drinfeld modules

Let H be a quasi-Hopf algebra. The (left) Yetter-Drinfeld module category ${}_H^H\mathcal{YD}$ over H is a subcategory of ${}_H\mathcal{M}$ such that each module in ${}_H^H\mathcal{YD}$ is equipped with a "coassociative" left coaction $\lambda_M(m) = m_{(-1)} \otimes m_{(0)}$ for which the Yetter-Drinfeld compatibility holds:

$$h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}$$

The category ${}_H^H\mathcal{YD}$ is monoidal in such a way that the forgetful functor ${}_H^H\mathcal{YD} \to {}_H\mathcal{M}$ is strong monoidal. Moreover, it is braided with braiding

$$m \otimes n \mapsto m_{(-1)} \cdot n \otimes m_{(0)}.$$

In a braided monoidal category $\mathcal{C}$, we can define bialgebras (and Hopf algebras) by taking objects in the category $\mathcal{C}$ and structure maps from morphisms in $\mathcal{C}$ such that they satisfy axioms analogous to those of usual bialgebras. They are called braided bialgebras and braided Hopf algebras.

Left Yetter-Drinfeld modules, projections and "four corners"

Let H be a quasi-Hopf algebra. We donote by ${}_H^H\mathcal{M}_H^H$ the category of H-bicomodules in ${}_H\mathcal{M}_H$.
The (braided) monoidal equivalence between ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{YD}$ was proven by P. Schauenburg for Hopf algebras in 1994, ad by D. Bulacu in 2009 for quasi-Hopf algebras[4].
In the same work, he proved the isomorphism between quasi-bialgebras with a projection $A \underset{i}{\overset{\pi}{\rightleftarrows}} H$ and bialgebras in ${}_H^H\mathcal{M}_H^H$.

$$\begin{tikzcd}
\mathrm{Bialg}({}_H^H\mathcal{YD}) \arrow[r] \arrow[dr, "\text{biproduct construction}"'] & \mathrm{Bialg}({}_H^H\mathcal{M}_H^H) \arrow[d] \\
& H\text{-qBialProj}
\end{tikzcd}$$

[4] D. Bulacu, *Quasi-quantum groups obtained from tensor braided Hopf algebras*, Math. Res. Lett. **11** (2004), 685–696.

Biproduct quasi-Hopf algebras

A quasi-Hopf algebra A with a projection $A \underset{i}{\overset{\pi}{\rightleftarrows}} H$ is obtained, through the biproduct construction, from a Hopf algebra B in ${}_H^H\mathcal{YD}$. The structure of the biproduct $B \times H$ is as follows[5]:

$$B \times H = B \otimes H \text{ (as a vector space)},$$
$$(b \times h)(b' \times h') = (x^1 \cdot b)(x^2 h_1 \cdot b') \times x^3 h_2 h',$$
$$\begin{aligned}\Delta(b \times h) =& y^1 X^1 \cdot b_1 \times y^2 Y^1 (x^1 X^2 \cdot b_2)_{-1} x^2 X_1^3 h_1 \\ & \otimes y_1^3 Y^2 \cdot (x^1 X^2 \cdot b_2)_0 \times y_2^3 Y^3 x^3 X_2^3 h_2\end{aligned},$$
$$\epsilon(h \times b) = \epsilon_B(h)\epsilon_H(h), \quad \Phi_{B\times H} = 1 \times X^1 \otimes 1 \times X^2 \otimes 1 \times X^3,$$
$$S(b \times h) = (1 \times S_H(X^1 x_1^1 b_{-1} h)\alpha_H)(X^2 x_2^1 \cdot S_B(b_0) \times X^3 x^2 \beta_H S_H(x^3)),$$
$$\text{with distinguished elements: } \alpha_{B\times H} = 1 \times \alpha_H, \quad \beta_{B\times H} = 1 \times \beta_H.$$

[5] D. Bulacu, E. Nauwelaerts, *Radford's biproduct for quasi-Hopf algebras and bosonization*, J. Pure Appl. Algebra **174** (2002), 1–42.

2-dimensional braided Hopf algebras in ${}_H^H\mathcal{YD}$

Theorem (D. Bulacu, MM)

Let H be a quasi-Hopf algebra with bijective antipode. If B is a braided Hopf algebra in ${}_H^H\mathcal{YD}$ of dimension 2 then B is isomorphic to a braided Hopf algebra of type $B_{\sigma,v}$ or to the group Hopf algebra $k[C_2]$ considered in ${}_H^H\mathcal{YD}$ via the trivial module structure.

In particular, the couple (σ, v), composed by $\sigma \in Alg(H, k)$ and $v \in H$, which determines each bialgebra $B_{\sigma,v}$ has to satisfy the following properties:

$$\Delta(v) = \sigma(x^3X^2y^1)x^1X^1vy^2 \otimes x^2vX^3y^3,$$
$$hu = \sigma(h_1)uh_2,\ \sigma(h_2)h_1v = \sigma(h_1)vh_2,\ \forall\ h \in H,$$

which determine a 1-dimensional YD module structure on a vector space ku by $h \cdot u = \sigma(h)u$ and $\lambda(u) = v \otimes u$, and $\sigma(v) = -1$

2-dimensional braided Hopf algebras in ${}^H_H\mathcal{YD}$

The structure of the braided Hopf algebras $B_{\sigma,v}$ in ${}^H_H\mathcal{YD}$ is as follows:

- As a unital associative algebra, it is generated by the unit 1 and the nilpotent element n of degree 2.
- Its coalgebra structure is determined by

$$\begin{aligned} \Delta(n) &= 1 \otimes n + n \otimes 1, & \Delta(1) &= 1 \otimes 1, \\ \epsilon(n) &= 0, & \epsilon(1) &= 1_k. \end{aligned}$$

- The antipode is determined by $S(n) = -n$ and $S(1) = 1$.
- The Yetter-Drinfeld H-module structure is determined by

$$\begin{aligned} h \cdot n &= \sigma(h)n, & \rho(n) &= v \otimes n, \\ h \cdot 1 &= \epsilon(h)1, & \rho(1) &= 1 \otimes 1. \end{aligned}$$

Note that, by what was said in the previous slide, kn has the 1-dimensional Yetter-Drinfeld module structure associated to the couple (σ, v).

Biproduct quasi-Hopf algebras of rank 2

By the biproduct construction, we obtain that all quasi-Hopf algebras with a projection of rank 2 as a free H-module are isomorphic to one of the following:

- $H_g := k[C_2] \otimes H$, generated as an algebra by H and the grouplike element g, with relations

$$g^2 = 1 \quad \text{and} \quad gh = hg$$

  for all $h \in H$, such that it has H as a sub-quasi-Hopf algebra.
- $H(\theta)_{\sigma,v}$, the algebra generated by H and θ with relations

$$\theta^2 = 0, \quad h\theta = \sigma(h_1)\theta h_2,$$

  for all $h \in H$. Its quasi-Hopf algebra structure is determined by H being its sub-quasi-Hopf algebra and

$$\Delta(\theta) = \sigma(X^2x^1)vX^1x^2 \otimes \theta X^3x^3 + \sigma(x^1)\theta x^2 \otimes x^3,$$
$$S(\theta) = -\sigma(X^2x_2^1)S(X^1x_1^1v)\sigma\theta X^3x^2\beta S(x^3).$$

Classification of quasi-Hopf algebras of dimension 6

- The main motivation for the previous result was gaining a tool to construct examples of non-commutative, non-cocommutative genuine quasi-Hopf algebras of arbitrary (even) dimension.
- We were able to apply the classication result on biproduct quasi-Hopf algebras of rank 2 to the classification of general quasi-Hopf algebras of dimension 6.
- Indeed, to classify such quasi-Hopf algebras, we started by characterizing all 6-dimensional biproducts.

Biproduct quasi-Hopf algebras in dimension 6

6-dimensional biproducts are of 2 types:

- The product of a 2-dimensional braided Hopf in ${}^H_H\mathcal{YD}$ and a 3-dimensional quasi-Hopf algebra H (that is, $H = k_\Phi[C_3]$)

  Through the previous classification, we found that all of these biproducts are of the trivial type, and therefore semisimple.

- The product between a 3-dimensional braided Hopf in ${}^H_H\mathcal{YD}$ and a 2-dimensional quasi-Hopf algebra H. There are two cases:
  - $H = k[C_2]$, then the biproduct are all Hopf algebras of dimension 6. By the classification of regular Hopf algebras, we know that they are all semisimple. ($\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$, $\langle g \rangle = C_2$)
  - $H = H(2)$, that is the genuine quasi-Hopf algebra with maps and algebra structure of $k[C_2]$, but seen as a quasi-Hopf algebra with reassociator $1 \otimes 1 \otimes 1 - 2p_- \otimes p_- \otimes p_-$, where $p_\pm := \frac{1}{2}(1 \pm g)$, with g generator of C_2.

To prove that all biproducts in dimension 6 are semisimple, we worked on the last case, by classifying 3-dimensional bialgebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$.

3-dimensional modules in ${}^{H(2)}_{H(2)}\mathcal{YD}$

Let $\mathfrak{q}$ be a primitive 3rd root of the unit. By the isomorphism of categories ${}^{H(2)}_{H(2)}\mathcal{YD} \cong {}_{H(2)}\mathcal{YD}^{H(2)} \cong {}_{D(H(2))}\mathcal{M}$ and $D(H(2)) \cong k[C_4]$ as algebras[6], we were able to prove the following.

Proposition

${}^{H(2)}_{H(2)}\mathcal{YD}$ is a semisimple monoidal category, with 4 simple objects, M_i, $0 \leq i \leq 3$. Each M_i is one dimensional and if m_i is a generator of M_i, M_i is a module in ${}^{H}_{H}\mathcal{YD}$ with structure given by

$$g \cdot m_i = (-1)^i m_i \quad \textit{and} \quad \lambda(m_i) = (p_+ + \mathfrak{q}^i p_-) \otimes m_i.$$

Lemma

Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. B is not isomorphic as a Yetter-Drinfeld module to $M_0 \oplus M_{2i+1} \oplus M_{2j+1}$, $i, j \in \{0, 1\}$.

[6] D. Bulacu, S. Caenepeel, B. Torrecillas, *Involutory Quasi-Hopf Algebras*, Algebr. Represent. Theory **12** (2019), 257–285.

3-dimensional bialgebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$

Lemma

Let B be a 3-dimensional Hopf algebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$ such that $B \cong M_0 \oplus M_{2i} \oplus M_{2j}$ as a Yetter-Drinfeld module. Then B is isomorphic as a braided Hopf algebra to one of the following:

- *B_{C_6}, the group Hopf algebra $k[C_3]$ viewed as an object of ${}^{H}_{H}\mathcal{YD}$ via trivial H-action and H-coaction.*
- *B_*, the group Hopf algebra $k[C_3]$ regarded as an object of ${}^{H}_{H}\mathcal{YD}$ via the trivial H-action and H-coaction determined by*

$$\lambda(x^i) = p_+ \otimes x^i + p_- \otimes x^{(2i)'},$$

  where x is the generator of C_3 and $(2i)'$ is the remainder of the division of $2i$ by 3, $i \in \{0, 1, 2\}$.

We have that $B_{C_6} \times H(2) \cong k_\Phi[C_6]$ and $B_* \times H(2) \cong Fun_\omega(S_3)$ as quasi-Hopf algebras, both semisimple.

3-dimensional bialgebras in ${}_{H(2)}^{H(2)}\mathcal{YD}$

Proposition

Let B be a 3-dimensional bialgebra in ${}_{H(2)}^{H(2)}\mathcal{YD}$. If $B \cong M_0 \oplus M_{2i} \oplus M_{2j+1}$ then B is isomorphic to one of the following braided bialgebras:

- B_{0o}^{10}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ \ y^2 = 0,\ \ xy = y,\ \ yx = 0$
- B_{0o}^{10}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ \ y^2 = 0,\ \ xy = 0,\ \ yx = y$
- B_{0o}^{10c}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ \ y^2 = 0,\ \ xy = 0,\ \ yx = 0$

for all 3, *the other structure maps are determined by:*

$$\Delta(x) = x \otimes x,\ \ \epsilon(x) = 1,\ \ \Delta(y) = x \otimes y + y \otimes x,\ \ \epsilon(y) = 0.$$

All 3 braided bialgebras are not braided Hopf algebras.
Therefore, all biproducts in dimension 6 are semisimple.

Quasi-Hopf algebras of dimension 6

Theorem (D. Bulacu, MM, '24)

Any 6-dimensional quasi-Hopf algebra is semisimple.

Sketch of the proof. We considered case by case the dimension of the Jacobson radical of a qHa H.

- We show that all non-semisimple 6-dimensional quasi-Hopf algebras H have to be basic (all simple H-modules are 1-dimensional).
- We prove that all non-semisimple basic 6-dimensional quasi-Hopf algebras have to be biproducts.
- Since all biproducts in dimension 6 are semisimple, there cannot be a non-semisimple 6-dimensional quasi-Hopf algebra.

Corollary

Let $\mathcal{C}$ be a finite tensor category of Frobenius-Perron dimension 6 such that every object of $\mathcal{C}$ has integer Frobenius-Perron dimension. Then $\mathcal{C}$ is a fusion category.

Quasi-Hopf algebras of dimension 6

Since semisimple quasi-Hopf algebras of dimension pq were characterized at the categorical level in [7] and 3-cocycles of S_3 were computed in [8], we are able to give a complete list of 6-dimensional quasi-Hopf algebras.

Theorem (D. Bulacu, MM, '24. Pt.1)

Any 6 dimensional quasi-Hopf algebra is twist equivalent to either:

- $(k[C_6], \Phi_a)$, *the group Hopf algebra* $k[C_6]$, *with reassociator*

$$\Phi_a = \sum_{i,j,l=0}^{5} \xi^{ai\lfloor \frac{j+l}{6} \rfloor} 1_i \otimes 1_j \otimes 1_l.$$

There are 6 quasi-Hopf algebras of this type, given by $0 \leq a \leq 5$. [...]

[7] P. Etingof, S. Gelaki, V. Ostrik, Classification of fusion categories of dimension pq, Int Math Res Notices 57 (2004), 3041–3056

[8] W. Propitius, Topological Interactions in Broken Gauge Theories, PhD Thesis, University of Amsterdam; arXiv:hep-th/9511195v1 27 Nov 1995.

Quasi-Hopf algebras of dimension 6

Theorem (D. Bulacu, MM, '24. Pt.2)

• (k^{S_3}, Φ_p), *the function algebra* $Fun(S_3)$ *endowed with the reassociator*

$$\Phi_p = \sum_{I,J,L=0}^{1} \sum_{i,j,l=0}^{2} \mathfrak{q}^{p(-1)^{J+L} i \lfloor \frac{(-1)^L j + l}{3} \rfloor} (-1)^{pIJL} \, P_{\tau^I \sigma^i} \otimes P_{\tau^J \sigma^j} \otimes P_{\tau^L \sigma^l}.$$

There are 6 *quasi-Hopf algebras of this type, given by* $0 \le p \le 5$.

• *The group Hopf algebra* $k[S_3]$.

• $(k[S_3], \Psi_a)$, *the group Hopf algebra* $k[S_3]$ *with reassociator* Ψ_a *obtained from a non-trivial* 3*-cocycle of* $C_3 = \langle \sigma \rangle$ *which commutes with* $\tau \otimes \tau \otimes \tau$*:*

$$\Psi_a = \sum_{i,j,l=0}^{2} \mathfrak{q}^{ai \lfloor \frac{j+l}{3} \rfloor + ai (\lfloor \frac{(j+l)'}{2} \rfloor - \lfloor \frac{j}{2} \rfloor - \lfloor \frac{l}{2} \rfloor)} \, 1_i \otimes 1_j \otimes 1_l,$$

There are 2 *quasi-Hopf algebras of this type, determined by* $a \in \{1, 2\}$.

Thank you for your attention!