Quasi-Hopf algebras of dimension 6

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Based on joint works with Daniel Bulacu[1]

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[1] • *Quasi-Hopf algebras of dimension* 6, J. Pure Appl. Algebra **229** (2025).

• *Biproduct quasi-Hopf algebras of rank 2*, preprint coming soon!

Summary

- We proved that all 6-dimensional quasi-Hopf algebras are semisimple, by classifying 2 and 3 dimensional braided bialgebras in ${}_H^H\mathcal{YD}$.
- By using a classification result on fusion categories[2], it is possible to recover a complete list of 6 dimensional quasi-Hopf algebras, up to twist equivalence.
- In conclusion, we will see how the classification of 2-dimensional braided Hopf algebras can be also useful to produce examples of quasi-Hopf algebras.

[2] P. Etingof, S. Gelaki, V. Ostrik, *Classification of fusion categories of dimension pq*, Int Math Res Notices **57** (2004), 3041–3056.

Ingredients: Quasi-bialgebra

A quasi-bialgebra B is an associative unital algebra endowed with extra structure by a coproduct (an algebra morphism) $\Delta : B \to B \otimes B$, a counit $\epsilon : B \to k$ and a reassociator $\Phi \in \mathcal{U}(B \otimes B \otimes B)$ satisfying

$$(Id_B \otimes \Delta)(\Delta(b)) = \Phi(\Delta \otimes Id_B)(\Delta(b))\Phi^{-1},$$
$$(Id_B \otimes \epsilon)(\Delta(b)) = b \quad \text{and} \quad (\epsilon \otimes Id_B)(\Delta(b)) = b.$$

for any $b \in B$. The reassociator Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(Id_B \otimes \Delta \otimes Id_B)(\Phi)(\Phi \otimes 1)$$
$$= (Id_B \otimes Id_B \otimes \Delta)(\Phi)(\Delta \otimes Id_B \otimes Id_B)(\Phi),$$
$$(Id_B \otimes \epsilon \otimes Id_B)(\Phi) = 1 \otimes 1.$$

$$\Phi =: \sum_{i \in I} X_i^1 \otimes X_i^2 \otimes X_i^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3,$$

$$\Phi^{-1} =: \sum_{i \in I} x_i^1 \otimes x_i^2 \otimes x_i^3 = y^1 \otimes y^2 \otimes y^3, \quad \Delta(b) = \sum_{j \in J} (b_1)_j \otimes (b_2)_j.$$

Ingredients: Quasi-Hopf algebra

- A quasi-bialgebra H is a quasi-Hopf algebra if there exists an algebra antimorphism $S : H \to H$ and distinguished elements $\alpha, \beta \in H$ such that, for all $h \in H$, the following conditions hold:

$$S(h_1)\alpha h_2 = \epsilon(h)\alpha, \qquad h_1\beta S(h_2) = \epsilon(h)\beta,$$
$$X^1\beta S(X^2)\alpha X^3 = 1, \quad S(x^1)\alpha x^2\beta S(x^3) = 1.$$

- If it exists, an antipode (S, α, β) of a quasi-bialgebra H is unique up to conjugation by an element in H.
- If $\Phi = 1 \otimes 1 \otimes 1$, H is a Hopf algebra, otherwise H is a genuine quasi-Hopf algebra.

Quasi-Hopf algebras: categorical interpretation

Let A be a unital associative algebra and let ${}_A\mathcal{M}$ be its category of left modules. By reconstruction theory, we have the following bijection of structures:

A		${}_A\mathcal{M}$
bialgebra	$\longleftrightarrow$	strict monoidal
Hopf algebra	$\longleftrightarrow$	left rigid strict monoidal
Hopf algebra with S bijective	$\longleftrightarrow$	rigid strict monoidal

$$\rightsquigarrow$$

A		${}_A\mathcal{M}$
quasi-bialgebra	$\longleftrightarrow$	monoidal
quasi-Hopf algebra	$\longleftrightarrow$	left rigid monoidal
quasi-Hopf algebra with S bijective	$\longleftrightarrow$	rigid monoidal

Ingredients: Fusion categories

Definition

A tensor category

- is abelian
- is rigid monoidal with semisimple unit object
- all objects have finite length

Definition

A fusion category is a semisimple finite tensor category.

An abelian category $\mathcal{C}$ is said to be finite if

- has finitely many simple objects (up to isomorphism)
- objects have finite length
- all simple objects admit a projective cover

Example: qHa and fusion categories

G finite group, $\omega \in Z^3(G, k^\times)$ normalized 3-cocycle. $H = k^G_\omega$ denotes the quasi-Hopf algebra k^G with the following structure maps, where $P_g \in k^G$ denotes the element dual to $g \in G$,

$$\Delta(P_g) = \sum_{x \in G} P_x \otimes P_{x^{-1}g}, \quad \epsilon(P_g) = P_g(e_G), \quad S(P_g) = P_{g^{-1}},$$

$$\Phi_\omega = \sum_{x,y,z \in G} \omega(x, y, z)\ P_x \otimes P_y \otimes P_z,$$

${}_H\mathcal{M}$ is a fusion category, with evaluation modules $\{V_g\}_{g \in G}$ as simple objects and associativity constraints given by

$$a_{V_g, V_h, V_l} = \omega(g, h, l) Id$$

If G is an abelian group, we can identify k^G_ω with $k_{\Phi_\omega}[G]$.

Fusion categories and the classification of qHas

Theorem (P. Etingof, V. Ostrik, '04)

Let $\mathcal{C}$ be a finite tensor category. $\mathcal{C} \cong Rep(H)$ as a tensor category for a finite dimensional quasi-Hopf algebra H if and only if every object X of $\mathcal{C}$ has an integer Frobenius-Perron dimension.

Moreover, $\mathcal{C}$ has FP dimension n if and only if H has linear dimension n.

The following results describe classes of semisimple quasi-Hopf algebras.

Corollary (P. Etingof, D. Nikshych, V. Ostrik, '05)

Let $\mathcal{C}$ be a fusion category with FP dimension p, where p is prime. Then $\mathcal{C}$ is equivalent to the category of representations of C_p, with associativity determined by a cohomology class in $H^3(C_p, k^\times)$.

Other classification results for quasi-Hopf algebras

Theorem (P. Etingof, S. Gelaki, V. Ostrik, '04)

Let $p < q$ be primes. Any fusion category of FP dimension pq, with integer dimension of simple objects is equivalent to one of the following:

- *A category with 1-dimensional simple objects.*
- *The category of representations of $C_p \ltimes C_q$.*
- *If $p = 2$, a category of bimodules in graded vector spaces with associativity controlled by $\omega \in H^3(C_2 \ltimes C_q, k^\times)$. This is a category of left modules over a genuine quasi-Hopf algebra.*

- P. Etingof and S. Gelaki classified[3] all quasi-Hopf algebras with two 1-dimensional simple modules.
- As a byproduct, they obtained the classification of quasi-Hopf algebras in dimension 4 (they are all semisimple or twist equivalent to H_4).

[3] P. Etingof, S. Gelaki, *Finite dimensional quasi-Hopf algebras with radical of codimension 2*, Math. Res. Lett. **11** (2004), 685–696.

Ingredients: Yetter-Drinfeld modules

Let H be a quasi-Hopf algebra. The (left) Yetter-Drinfeld module category ${}_H^H\mathcal{YD}$ over H is a subcategory of ${}_H\mathcal{M}$ such that each module in ${}_H^H\mathcal{YD}$ is equipped with a "coassociative" left coaction $\lambda_M(m) = m_{(-1)} \otimes m_{(0)}$ for which the Yetter-Drinfeld compatibility holds:

$$h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}$$

The category ${}_H^H\mathcal{YD}$ is monoidal in such a way that the forgetful functor ${}_H^H\mathcal{YD} \to {}_H\mathcal{M}$ is strong monoidal. Moreover, it is braided with braiding

$$m \otimes n \mapsto m_{(-1)} \cdot n \otimes m_{(0)}.$$

In a braided monoidal category $\mathcal{C}$, we can define bialgebras (and Hopf algebras) by taking objects in the category $\mathcal{C}$ and structure maps from morphisms in $\mathcal{C}$ such that they satisfy axioms analogous to those of usual bialgebras. They are called braided bialgebras and braided Hopf algebras.

We donote by ${}^H_H\mathcal{M}^H_H$ the category of H-bicomodules in ${}_H\mathcal{M}_H$.
The braided monoidal equivalence between ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$ was proven by P. Schauenburg for Hopf algebras in 1994, and also holds for quasi-Hopf[4].
Moreover, quasi-bialgebras with a projection $A \underset{i}{\overset{\pi}{\rightleftarrows}} H$ are isomorphic to bialgebras in ${}^H_H\mathcal{M}^H_H$.

$$\begin{tikzcd}
\mathrm{Bialg}({}^H_H\mathcal{YD}) \arrow[r] \arrow[dr, "\text{biproduct construction}"'] & \mathrm{Bialg}({}^H_H\mathcal{M}^H_H) \arrow[d] \\
 & H\text{-qBialProj}
\end{tikzcd}$$

[4] D. Bulacu, *Quasi-quantum groups obtained from tensor braided Hopf algebras*, Math. Res. Lett. **11** (2004), 685–696.

A quasi-Hopf algebra A with a projection $A \underset{i}{\overset{\pi}{\rightleftarrows}} H$ is obtained, through the biproduct construction, from a Hopf algebra B in ${}_H^H\mathcal{YD}$. The structure of the biproduct $B \times H$ is as follows[5]:

$$B \times H = B \otimes H \text{ (as a vector space)},$$
$$(b \times h)(b' \times h') = (x^1 \cdot b)(x^2 h_1 \cdot b') \times x^3 h_2 h',$$
$$\begin{aligned}\Delta(b \times h) =& y^1 X^1 \cdot b_1 \times y^2 Y^1 (x^1 X^2 \cdot b_2)_{-1} x^2 X_1^3 h_1 \\ & \otimes y_1^3 Y^2 \cdot (x^1 X^2 \cdot b_2)_0 \times y_2^3 Y^3 x^3 X_2^3 h_2\end{aligned},$$
$$\epsilon(h \times b) = \epsilon_B(h)\epsilon_H(h), \quad \Phi_{B\times H} = 1 \times X^1 \otimes 1 \times X^2 \otimes 1 \times X^3,$$
$$S(b \times h) = (1 \times S_H(X^1 x_1^1 b_{-1} h)\alpha_H)(X^2 x_2^1 \cdot S_B(b_0) \times X^3 x^2 \beta_H S_H(x^3)),$$
$$\text{with distinguished elements: } \alpha_{B\times H} = 1 \times \alpha_H, \quad \beta_{B\times H} = 1 \times \beta_H.$$

[5] D. Bulacu, E. Nauwelaerts, *Radford's biproduct for quasi-Hopf algebras and bosonization*, J. Pure Appl. Algebra **174** (2002), 1–42.

2-dimensional braided Hopf algebras in ${}^H_H\mathcal{YD}$

Theorem (D. Bulacu, MM)

Let H be a quasi-Hopf algebra with bijective antipode. If B is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ of dimension 2 then B is isomorphic to a braided Hopf algebra of type $B_{\sigma,v}$ or to the group Hopf algebra $k[C_2]$ considered in ${}^H_H\mathcal{YD}$ via the trivial module structure.

In particular, the couple (σ, v), composed by $\sigma \in Alg(H,k)$ and $v \in H$, which determines each bialgebra $B_{\sigma,v}$ has to satisfy the following properties:

$$\Delta(v) = \sigma(x^3X^2y^1)x^1X^1vy^2 \otimes x^2vX^3y^3,$$
$$hu = \sigma(h_1)uh_2,\ \sigma(h_2)h_1v = \sigma(h_1)vh_2,\ \forall\ h \in H,$$

which determine a 1-dimensional YD module structure on a vector space ku by $h \cdot u = \sigma(h)u$ and $\lambda(u) = v \otimes u$, and $\sigma(v) = -1$

2-dimensional braided Hopf algebras in ${}^H_H\mathcal{YD}$

The structure of the braided Hopf algebras $B_{\sigma,v}$ in ${}^H_H\mathcal{YD}$ is as follows:

- As a unital associative algebra, it is generated by the unit 1 and the nilpotent element n of degree 2.
- Its coalgebra structure is determined by

$$\begin{aligned} \Delta(n) &= 1 \otimes n + n \otimes 1, & \qquad \Delta(1) &= 1 \otimes 1, \\ \epsilon(n) &= 0, & \epsilon(1) &= 1_k. \end{aligned}$$

- The antipode is determined by $S(n) = -n$ and $S(1) = 1$.
- The Yetter-Drinfeld H-module structure is determined by

$$\begin{aligned} h \cdot n &= \sigma(h)n, & \qquad \rho(n) &= v \otimes n, \\ h \cdot 1 &= \epsilon(h)1, & \rho(1) &= 1 \otimes 1. \end{aligned}$$

Note that, by what was said in the previous slide, kn has the 1-dimensional Yetter-Drinfeld module structure associated to the couple (σ, v).

Biproduct quasi-Hopf algebras of rank 2

By the biproduct construction, we obtain that all quasi-Hopf algebras with a projection of rank 2 as a free H-module are isomorphic to one of the following:

- $H_g := k[C_2] \otimes H$, generated as an algebra by H and the grouplike element g, with relations

$$g^2 = 1 \quad \text{and} \quad gh = hg$$

  for all $h \in H$, such that it has H as a sub-quasi-Hopf algebra.
- $H(\theta)_{\sigma,v}$, the algebra generated by H and θ with relations

$$\theta^2 = 0, \quad h\theta = \sigma(h_1)\theta h_2,$$

  for all $h \in H$. Its quasi-Hopf algebra structure is determined by H being its sub-quasi-Hopf algebra and

$$\Delta(\theta) = \sigma(X^2x^1)vX^1x^2 \otimes \theta X^3x^3 + \sigma(x^1)\theta x^2 \otimes x^3,$$
$$S(\theta) = -\sigma(X^2x_2^1)S(X^1x_1^1v)\sigma\theta X^3x^2\beta S(x^3).$$

Classification of quasi-Hopf algebras of dimension 6

- The main motivation for the previous result was gaining a tool to construct examples of non-commutative, non-cocommutative genuine quasi-Hopf algebras of arbitrary (even) dimension.

- We will see some such examples later, on finite abelian groups.

- The classication result on biproduct quasi-Hopf algebras of rank 2 can however be applied to the classification of general quasi-Hopf algebras of dimension 6.

- Indeed, to classify such quasi-Hopf algebras, we started by characterizing all 6-dimensional biproducts.

Biproduct quasi-Hopf algebras in dimension 6

6-dimensional biproducts are of 2 types:

- The product of a 2-dimensional braided Hopf in ${}^H_H\mathcal{YD}$ and a 3-dimensional quasi-Hopf algebra H (that is, $H = k_\Phi[C_3]$)

  Through the previous classification, we found that all of these biproducts are of the trivial type, and therefore semisimple.

- The product between a 3-dimensional braided Hopf in ${}^H_H\mathcal{YD}$ and a 2-dimensional quasi-Hopf algebra H. There are two cases:
  - $H = k[C_2]$, then the biproduct are all Hopf algebras of dimension 6. By the classification of regular Hopf algebras, we know that they are all semisimple. ($\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$, $\langle g \rangle = C_2$)
  - $H = H(2)$, that is the genuine quasi-Hopf algebra with maps and algebra structure of $k[C_2]$, but seen as a quasi-Hopf algebra with reassociator $1 \otimes 1 \otimes 1 - 2p_- \otimes p_- \otimes p_-$, where $p_\pm := \frac{1}{2}(1 \pm g)$, with g generator of C_2.

To prove that all biproducts in dimension 6 are semisimple, we worked on the last case, by classifying 3-dimensional bialgebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$.

3-dimensional modules in ${}^{H(2)}_{H(2)}\mathcal{YD}$

Let $\mathfrak{q}$ be a primitive 3rd root of the unit. By the isomorphism of categories ${}^{H(2)}_{H(2)}\mathcal{YD} \cong {}_{H(2)}\mathcal{YD}^{H(2)} \cong {}_{D(H(2))}\mathcal{M}$ and $D(H(2)) \cong k[C_4]$ as algebras[6], we were able to prove the following.

Proposition

${}^{H(2)}_{H(2)}\mathcal{YD}$ is a semisimple monoidal category, with 4 simple objects, M_i, $0 \leq i \leq 3$. Each M_i is one dimensional and if m_i is a generator of M_i, M_i is a module in ${}^{H}_{H}\mathcal{YD}$ with structure given by

$$g \cdot m_i = (-1)^i m_i \quad \textit{and} \quad \lambda(m_i) = (p_+ + \mathfrak{q}^i p_-) \otimes m_i.$$

Lemma

Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. B is not isomorphic as a Yetter-Drinfeld module to $M_0 \oplus M_{2i+1} \oplus M_{2j+1}$, $i, j \in \{0, 1\}$.

[6] D. Bulacu, S. Caenepeel, B. Torrecillas, *Involutory Quasi-Hopf Algebras*, '19

Lemma

Let B be a 3-dimensional Hopf algebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$ such that $B \cong M_0 \oplus M_{2i} \oplus M_{2j}$ as a Yetter-Drinfeld module. Then B is isomorphic as a braided Hopf algebra to one of the following:

- *B_{C_6}, the group Hopf algebra $k[C_3]$ viewed as an object of ${}^{H}_{H}\mathcal{YD}$ via trivial H-action and H-coaction.*
- *B_*, the group Hopf algebra $k[C_3]$ regarded as an object of ${}^{H}_{H}\mathcal{YD}$ via the trivial H-action and H-coaction determined by*

$$\lambda(x^i) = p_+ \otimes x^i + p_- \otimes x^{(2i)'},$$

where x is the generator of C_3 and $(2i)'$ is the remainder of the division of $2i$ by 3, $i \in \{0, 1, 2\}$.

We have that $B_{C_6} \times H(2) \cong k_\Phi[C_6]$ and $B_* \times H(2) \cong k_\omega^{S_3}$ as quasi-Hopf algebras, both semisimple.

3-dimensional bialgebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$

Proposition

Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. If $B \cong M_0 \oplus M_{2i} \oplus M_{2j+1}$ then B is isomorphic to one of the following braided bialgebras:

- B^{10}_{0o}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ y^2 = 0,\ xy = y,\ yx = 0$
- B^{10}_{0o}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ y^2 = 0,\ xy = 0,\ yx = y$
- B^{10c}_{0o}, *generated by* $1, x, y$ *with relations* $x^2 = x,\ y^2 = 0,\ xy = 0,\ yx = 0$

for all 3*, the other structure maps are determined by:*

$$\Delta(x) = x \otimes x,\ \epsilon(x) = 1,\ \Delta(y) = x \otimes y + y \otimes x,\ \epsilon(y) = 0.$$

All 3 braided bialgebras are not braided Hopf algebras.
Therefore, all biproducts in dimension 6 are semisimple.

Quasi-Hopf algebras of dimension 6

Theorem (D. Bulacu, MM, '25)

Any 6-dimensional quasi-Hopf algebra is semisimple.

Sketch of the proof. We considered case by case the dimension of the Jacobson radical of a qHa H.

- Show that all non-semisimple 6-dimensional quasi-Hopf algebras H have to be basic (all simple H-modules are 1-dimensional).
- Prove that all non-semisimple basic 6-dimensional quasi-Hopf algebras have to be biproducts.
- Since all biproducts in dimension 6 are semisimple, there cannot be a non-semisimple 6-dimensional quasi-Hopf algebra.

Corollary

Let $\mathcal{C}$ be a finite tensor category of Frobenius-Perron dimension 6 such that every object of $\mathcal{C}$ has integer Frobenius-Perron dimension. Then $\mathcal{C}$ is a fusion category.

Quasi-Hopf algebras of dimension 6: complete list

Theorem (D. Bulacu, MM, '25)

Any 6 dimensional quasi-Hopf algebra is twist equivalent to either:

- $(k[C_6], \Phi_a)$, $k[C_6]$ *with reassociator*

$$\Phi_a = \sum_{i,j,l=0}^{5} \xi^{ai\lfloor \frac{j+l}{6} \rfloor} 1_i \otimes 1_j \otimes 1_l, \quad 0 \le a \le 5.$$

- (k^{S_3}, Φ_p), k^{S_3} *with reassociator*, $0 \le p \le 5$,

$$\Phi_p = \sum_{I,J,L=0}^{1} \sum_{i,j,l=0}^{2} \mathfrak{q}^{p(-1)^{J+L} i \lfloor \frac{(-1)^L j + l}{3} \rfloor} (-1)^{pIJL} P_{\tau^I \sigma^i} \otimes P_{\tau^J \sigma^j} \otimes P_{\tau^L \sigma^l}.$$

- *The group Hopf algebra* $k[S_3]$.
- $(k[S_3], \Psi_a)$, $k[S_3]$ *with reassociator* Ψ_a, *obtained from a non-trivial 3-cocycle of* $C_3 = \langle \sigma \rangle$ *which commutes with* $\tau \otimes \tau \otimes \tau$, *where* $a \in \{1, 2\}$:

$$\Psi_a = \sum_{i,j,l=0}^{2} \mathfrak{q}^{ai\lfloor \frac{j+l}{3} \rfloor + ai(\lfloor \frac{(j+l)'}{2} \rfloor - \lfloor \frac{j}{2} \rfloor - \lfloor \frac{l}{2} \rfloor)} 1_i \otimes 1_j \otimes 1_l.$$

Appendix: quasi-Hopf algebras of even dimension

In conclusion, we see concrete examples of $B_{\sigma,v} \in {}^H_H\mathcal{YD}$, for an abelian group algebra H. Through the biproduct construction, this produces numerous classes of non-commutative, non-cocommutative qHas.

Let G be a finite group. Recall that k^G_ω is a quasi-Hopf algebra for any $\omega \in Z^3(G, k^\times)$. If G is abelian, $k^G_\omega \overset{ident.}{=} k_\omega[G]$.

Lemma (D. Bulacu, MM)

Let ω be a normalized 3-cocycle on G and $\mathfrak{g} \in G$. Define, for all $x, y \in G$,

$$\flat_{\mathfrak{g}}\omega : G \times G \to k^\times, \quad \flat_{\mathfrak{g}}\omega(x, y) = \frac{\omega(\mathfrak{g}, x, y)\omega(x, y, \mathfrak{g})}{\omega(x, \mathfrak{g}, y)}.$$

$\flat_{\mathfrak{g}}\omega$ is a normalized 2-cocycle on G if and only if, for all $x, y, z \in G$,

$$\omega(\mathfrak{g}x, y, z)\omega(x, \mathfrak{g}y, z)\omega(x, y, \mathfrak{g}z) = \omega(x\mathfrak{g}, y, z)\omega(x, y\mathfrak{g}, z)\omega(x, y, z\mathfrak{g}).$$

Consequently, $\flat_{\mathfrak{g}}\omega$ is a normalized 2-cocycle on G, if $\mathfrak{g} \in Z(G)$.

Appendix: quasi-Hopf algebras of even dimension

Recall that each $B_{\sigma,v}$ is determined by a couple $(\sigma, v) \in Alg(H) \otimes H$ s.t.

$$\left.\begin{array}{l} \Delta(v) = \sigma(x^3X^2y^1)x^1X^1vy^2 \otimes x^2vX^3y^3, \\ hu = \sigma(h_1)uh_2,\ \sigma(h_2)h_1v = \sigma(h_1)vh_2, \end{array}\right. \quad \forall\ h \in H, \quad \text{and} \quad \sigma(v) = -1$$

Proposition (D. Bulacu, MM)

$B_{\sigma,v}$ in ${}^{k_\omega^G}_{k_\omega^G}\mathcal{YD}$ are in a one to one correspondence with couples $(\mathfrak{g}, \rho) \in Z(G) \times (k^\times)^G$, such that $\rho(e) = 1$, $\rho(\mathfrak{g}) = -1$ and $\flat_{\mathfrak{g}}\omega = \partial\rho$, where $\partial\rho(x, y) := \rho(xy)^{-1}\rho(x)\rho(y)$.

Let $G = C_{m_1} \times \ldots \times C_{m_n}$ be an abelian group. Denote by ξ_{m_j} a primitive m_jth root of unit in k. Then $1_{j_1,\ldots,j_n}$, given by,

$$1_{j_1,\ldots,j_n} := \frac{1}{m_1 \ldots m_n} \prod_{l=1}^{n} \left(\sum_{t=0}^{m_l-1} \xi_{m_l}^{-tj_l} g_l^t \right)$$

defines a family of orthogonal idempotents of $k[G]$.

Appendix: quasi-Hopf algebras of even dimension

Denote by $\mathcal{A}$ the set of sequences $\underline{a}$ with integer elements, having the form

$$(c_1, \ldots, c_l, \ldots, c_n, c_{12}, \ldots, c_{ij}, \ldots, c_{n-1,n}, c_{123}, \ldots, c_{rst}, \ldots, c_{n-2,n-1,n})$$

with $0 \leq c_l \leq m_l - 1$, $0 \leq c_{ij} \leq (m_i, m_j)$ and $0 \leq c_{rst} \leq (m_r, m_s, m_t)$, for all $1 \leq l \leq n$, $1 \leq i < j \leq n$ and $1 \leq r < s < t \leq n$. c_{ij} and c_{rst}

$$\omega_{\underline{a}}(g_1^{i_1} \ldots g_n^{i_n}, g_1^{j_1} \ldots g_n^{j_n}, g_1^{k_1} \ldots g_n^{k_n}) := \\ = \prod_{l=1}^{n} \xi_{m_l}^{c_l i_l \lfloor \frac{j_l + k_l}{m_l} \rfloor} \prod_{1 \leq s < t \leq n} \xi_{m_t}^{c_{st} i_t \lfloor \frac{j_s + k_s}{m_s} \rfloor} \prod_{1 \leq r < s < t \leq n} \xi_{(m_r, m_s, m_t)}^{-c_{rst} k_r j_s i_t}.$$

Owing to [7], $\{\omega_{\underline{a}}\}_{\underline{a} \in \mathcal{A}}$ is a set of representative 3-cocycles for G.

[7] HL. Huang, G. Liu, Y. Yang, Y. Ye, *Finite quasi-quntum groups of diagonal type*, '20

Appendix: quasi-Hopf algebras of even dimension

Proposition

Let $G = C_{m_1} \times \ldots \times C_{m_n}$ *be an abelian group and* $\omega_{\underline{a}}$ *a 3-cocycle with* $\underline{a} \in \mathcal{A}$. $B_{(\sigma,v)}$ *in* ${}^{k_{\omega_{\underline{a}}}[G]}_{k_{\omega_{\underline{a}}}[G]}\mathcal{YD}$ *are associated to each couple* (σ, v) *such that, for some fixed* $0 \leq f_j, \lambda_j \leq m_j - 1$,

$$\sigma(g_1^{j_1} \cdots g_n^{j_n}) = \prod_{l=1}^{n} \xi_{m_l}^{f_l j_l}, \ \forall \ g_1^{j_1} \cdots g_n^{j_n} \in G,$$

$$v = \sum_{a_1,\ldots,a_n} \prod_{j=1}^{n} \xi_{m_j}^{\lambda_j a_j} \xi_{m_j^2}^{c_j f_j a_j} \prod_{1 \leq j < t \leq n} \xi_{m_t m_j}^{c_{jt} f_t a_j} 1_{a_1,\cdots,a_n},$$

satisfying

$$v(\mathfrak{g}) = -1 \quad \text{and} \quad M | \sum_{u=1}^{r-1} M^c_{urs} f_u - \sum_{v=r+1}^{s-1} M^c_{rvs} f_v + \sum_{w=s+1}^{n} M^c_{rsw} f_w.$$

Appendix: quasi-Hopf algebras of even dimension

If G is the Klein group, the 2-dimensional non-trivial Hopf algebras in ${}^{k_{\Phi_a}[G]}_{k_{\Phi_a}[G]}\mathcal{YD}$ are of the form $B_{\sigma,v}$, with $a \in \mathcal{A}$, σ and v given by the table:

$a=(c_1,c_2,c_3)$	$\sigma_{f_1f_2}$	(λ_1,λ_2)	v	$\sigma_{f_1f_2}$	(λ_1,λ_2)	v
$(0,0,0)$	σ_{01}	$(0,1)$	g_2	σ_{10}	$(1,0)$	g_1
		$(1,1)$	g_1g_2		$(1,1)$	g_1g_2
	σ_{11}	$(0,1)$	g_2	σ_{11}	$(1,0)$	g_1
$(0,0,1)$	σ_{01}	$(0,1)$	$\mathfrak{g}^+ := \frac{1+i}{2}g_2 + \frac{1-i}{2}g_1g_2$	σ_{10}	$(1,0)$	g_1
		$(1,1)$	$\mathfrak{g}^- := \frac{1-i}{2}g_2 + \frac{1+i}{2}g_1g_2$		$(1,1)$	g_1g_2
$(0,1,1)$	σ_{11}	$(0,0)$	$\mathfrak{h}^+ := \frac{1}{2}(g_1+g_2) + \frac{i}{2}(1-g_1g_2)$	σ_{10}	$(1,0)$	g_1
		$(1,1)$	$\mathfrak{h}^- := \frac{1}{2}(g_1+g_2) - \frac{i}{2}(1-g_1g_2)$		$(1,1)$	g_1g_2
$(1,0,0)$	σ_{01}	$(0,1)$	g_2	σ_{01}	$(1,1)$	g_1g_2
$(1,0,1)$	σ_{01}	$(0,1)$	$\mathfrak{g}^+$	σ_{11}	$(0,0)$	g_1
		$(1,1)$	$\mathfrak{g}^-$		$(1,1)$	g_2
$(0,1,0)$	σ_{10}	$(1,0)$	g_1	σ_{10}	$(1,1)$	g_1g_2
$(1,1,0)$	σ_{11}	$(0,0)$	$\mathfrak{h}^+$	σ_{11}	$(1,1)$	$\mathfrak{h}^-$

Appendix: quasi-Hopf algebras of even dimension

Fix q a primitive mth root of unity. For $1 \leq c \leq m-1$, $k_{\Phi_c}[C_m]$ is $k[C_m]$, considered as a quasi-Hopf algebra via the reassociator

$$\Phi_c = \sum_{i,j,l=0}^{m-1} \mathfrak{q}^{ci\lfloor \frac{j+l}{m} \rfloor} 1_i \otimes 1_j \otimes 1_l.$$

As before, $1_j = \frac{1}{m}\sum_{s=1}^{m-1} q^{-js} g^s$, for all $0 \leq j \leq m-1$.

Proposition (D. Bulacu, MM)

If there exists a non-trivial 2-dimensional Hopf algebra within ${}^{k_{\Phi_c}[C_m]}_{k_{\Phi_c}[C_m]}\mathcal{YD}$ *then* (c, m) *is even.* $B_{\sigma,v}$ *in* ${}^{k_{\Phi_c}[C_m]}_{k_{\Phi_c}[C_m]}\mathcal{YD}$ *is determined by an algebra morphism* $\sigma : k[C_m] \to k$ *and* $v \in k[C_m]$*, where*

$$\sigma(g^j) = q^{fj},\ \ v = \sum_{j=1}^{m} \mathfrak{q}^{m\lambda j + cfj} 1_j, \quad s.t. \quad \frac{\lambda f}{m} + \frac{cf^2}{m^2} - \frac{1}{2} \in \mathbb{Z}.$$

for some $f, \lambda \in \{0, \cdots, m-1\}$.

Appendix: quasi-Hopf algebras of even dimension

If $m = 2$, the only non trivial 2-dimensional braided Hopf algebra is the one associated to H_4.

If $m = 4$, c has to be 2, then $\frac{\lambda f}{2} + \frac{f^2}{8} - \frac{1}{2} \in \mathbb{Z}$ is equivalent to $\frac{(2\lambda+f)f}{4}$, i.e. $(\lambda, f) \in \{(0,2),(2,2)\}$. Non trivial 2-dimensional braided Hopf algebras are isomorphic to either $B_{(\sigma,1)}$ and $B_{(\sigma,g)}$.
The 8 dimensional quasi-Hopf algebra associated to $B_{(\sigma,g^{\pm 1})}$ is generated as an algebra by g and x with relations $x^2 = 0$, $gx = -xg$ and $g^4 = 1$, with structure maps extended from $k_{\Phi_2}[C_4]$ by

$$\Delta(x) = \sum_{j,t=0}^{3} (-1)^{j\lfloor \frac{2+t}{n} \rfloor} g^{\pm 1} 1_j \otimes x 1_t + x \otimes 1,$$

$$S(x) = -\sum_{j=0}^{3} (-1)^{j\lfloor \frac{2+j}{n} \rfloor} g^{\mp 1} x 1_j.$$

$$\Phi_2 = \sum_{i,j,l=0}^{m-1} (-1)^{i\lfloor \frac{j+l}{m} \rfloor} 1_i \otimes 1_j \otimes 1_l.$$

Thank you for your attention!