An introduction to Nichols algebras and bosonization

Matteo Misurati

University of Ferrara

NCG&T Prague, Pre-Seminar, April 1st 2025

Summary

- We will introduce all the ingredients necessary to the construction of Nichols algebras in Yetter-Drinfeld modules category.
- We define Nichols algebras and give some simple examples.
- We will show how regular and braided Hopf algebras are related through Radford-Majid bosonization.
- If time permits, we will conclude by showing the role of Nichols algebras in the classification of pointed Hopf algebras.

Unless otherwise stated, we denote by H a Hopf algebra with invertible antipode over an algebraically closed field k of characteristic 0.
For this talk $\mathbb{N} \ni 0$.

Braided monoidal category

A monoidal category $\mathcal{C}$ is called braided when endowed with a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that the hexagon axioms hold:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "c_{X,Y} \otimes id"'] & X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y \otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "id \otimes c_{X,Z}"'] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "id \otimes c_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "c_{X \otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Y,Z}"'] & (X \otimes Z) \otimes Y \arrow[r, "c_{X,Z} \otimes id"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

- The hexagon axioms imply the Yang-Baxter equation for c.
- c plays in $\mathcal{C}$ the role of the transposition τ in Vec_k.
- The braiding is, in general, not unique: $(\mathcal{C}, c) \rightsquigarrow (\mathcal{C}, c^{-1})$.

(left) Center construction

Starting from any monoidal category $\mathcal{C}$, we can construct a braided monoidal category $\mathcal{L}_L(\mathcal{C})$, where:

- Objects: (Z, γ), where $Z \in \mathcal{C}$ and $\gamma_X : X \otimes Z \to Z \otimes X$ nat. iso. s.t.

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "id \otimes \gamma_Y"'] & (X \otimes Y) \otimes Z \arrow[r, "\gamma_{X \otimes Y}"] & Z \otimes (X \otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Y,Z}"'] & (X \otimes Z) \otimes Y \arrow[r, "\gamma_X \otimes id"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

- Morphisms between (Z, γ) and (Z', γ'): maps $f : Z \to Z'$ in $\mathcal{C}$ s.t.

$$(f \otimes id)\gamma_X = \gamma'_X(id \otimes f), \quad \text{for all } X \in \mathcal{C}.$$

- Tensor product: $(Z \otimes Z', \tilde{\gamma})$, where $\tilde{\gamma}$ is defined by γ, γ' and a.
- Braiding: $c_{(Z,\gamma),(Z',\gamma')} = \gamma'_Z$.

Example: $\mathcal{L}_L({}_H\mathcal{M}) \cong {}^H_H\mathcal{YD}$.

(left-left) Yetter-Drinfeld modules

Objects in ${}^H_H\mathcal{YD}$ are vector spaces V endowed with

- a H-action $\cdot : H \otimes V \to V : h \otimes v \mapsto h \cdot v$
- a H-coaction $\lambda : V \to H \otimes V : v \mapsto v_{[-1]} \otimes v_{[0]}$

such that $h_1 v_{[-1]} \otimes (h_2 \cdot v_{[0]}) = (h_1 \cdot v)_{[-1]} h_2 \otimes (h_1 \cdot v)_{[0]}$, while morphism are H-linear and H-colinear maps.

${}^H_H\mathcal{YD}$ is braided monoidal:

- $V \otimes W$ is the tensor product between vector spaces
- H-action on $V \otimes W$: $h \cdot (v \otimes w) = h_1 \cdot v \otimes h_2 \cdot w$
- H-coaction on $V \otimes W$: $\lambda(v \otimes w) = v_{[-1]} w_{[-1]} \otimes v_{[0]} \otimes w_{[0]}$
- Braiding: $c_{V,W}(v \otimes w) = v_{[-1]} \cdot w \otimes v_{[0]}$
- Inverse of the braiding: $c^{-1}_{W,V}(v \otimes w) = w_{[0]} \otimes S^{-1}(w_{[-1]}) \cdot v$

Monoids and comonoids

A monoid in a monoidal category $\mathcal{C}$ is a triple (M, μ, u), where $\mu : M \otimes M \to M$ and $u : 1 \to M$ are morphisms in $\mathcal{C}$ s.t.

$$\begin{tikzcd}
& M \otimes (M \otimes M) \arrow[dr, "id\otimes\mu"] & \\
(M \otimes M) \otimes M \arrow[ur, "a_{M,M,M}"] \arrow[d, "\mu\otimes id"'] & & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[rr, "\mu"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes M \arrow[r, "u\otimes id"] \arrow[d, "l"'] & M \otimes M \arrow[dl, "\mu"] \\
M & M \otimes 1 \arrow[l, "r"] \arrow[u, "id\otimes u"']
\end{tikzcd}$$

A comonoid in a monoidal category $\mathcal{C}$ is a triple (C, δ, ϵ), where $\delta : C \to C \otimes C$ and $\epsilon : C \to 1$ are morphisms in $\mathcal{C}$ s.t.

$$\begin{tikzcd}
& C \otimes (C \otimes C) & \\
(C \otimes C) \otimes C \arrow[ur, "a_{C,C,C}"] & & C \otimes C \arrow[ul, "id\otimes\delta"'] \\
C \otimes C \arrow[u, "\delta\otimes id"] & & C \arrow[ll, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes C \arrow[d, "l"'] & C \otimes C \arrow[l, "\epsilon\otimes id"'] \arrow[d, "id\otimes\epsilon"] \\
C \arrow[ur, "\delta"] & C \otimes 1 \arrow[l, "r"]
\end{tikzcd}$$

Example: A monoid in Vec_k^G is a G-graded algebra.

Tensor product of monoids

Given two monoids (M, μ_M, u_M) and (N, μ_N, u_N) in a (strict) braided monoidal category $\mathcal{C}$, we can give a monoid structure to the tensor product $M \otimes N$:

$$M \underline{\otimes} N = (M \otimes N, \mu_{M\otimes N}, u_{M\otimes N})$$

where $\mu_{M\otimes N}$ and $u_{M\otimes N}$ are morphisms defined by the compositions

$$\begin{tikzcd}
M \otimes N \otimes M \otimes N \arrow[rr, "\mu_{M\otimes N}"] \arrow[dr, "id_M \otimes c_{N,M} \otimes id_N"'] & & M \otimes N \\
& M \otimes M \otimes N \otimes N \arrow[ur, "\mu_M \otimes \mu_N"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes N \arrow[rr, "u_{M\otimes N}"] \arrow[dr, "u_M \otimes u_N"'] & & 1. \\
& 1 \otimes 1 \arrow[ur, "l"'] &
\end{tikzcd}$$

We call monoids and comonoids in ${}^H_H\mathcal{YD}$, respectively, algebras and coalgebras in ${}^H_H\mathcal{YD}$.

Braided Hopf algebras

A bialgebra in ${}^H_H\mathcal{YD}$ is a collection $(B, \mu, u, \Delta, \epsilon)$ such that

- (B, μ, u) is an algebra (monoid) in ${}^H_H\mathcal{YD}$,
- (B, Δ, ϵ) is a coalgebra (comonoid) in ${}^H_H\mathcal{YD}$,
- $\Delta : B \to B \underline{\otimes} B$ is a morphism of algebras.

A Hopf algebra in ${}^H_H\mathcal{YD}$ is a bialgebra B in ${}^H_H\mathcal{YD}$ such that $id_B \in Hom_{\mathcal{C}}(B, B)$ has an inverse $\underline{S}$ w.r.t. the convolution product $*$, meaning a morphism $\underline{S} : B \to B$ such that $\underline{S} * id_B = id_B * \underline{S} = u \circ \epsilon$,

$$\text{i.e. } \underline{S}(b_1)b_2 = b_1\underline{S}(b_2) = \epsilon(b)u(1_k) \text{ for all } b \in B.$$

Hopf algebras in braided monoidal categories are usually called braided Hopf algebras.

We will see noan example of a braided Hopf algebra in ${}^H_H\mathcal{YD}$ that will be central in defining Nichols algebras.

Tensor algebra

Let $V \in {}^H_H\mathcal{YD}$. Set $T^0(V) = k$ and $T^{n+1}(V) = V \otimes T^n(V)$.
The tensor algebra over V is $T(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)$, and is a graded algebra in ${}^H_H\mathcal{YD}$ with multiplication obtained from the (trivial) associativity constraints on the graded components of $T(V) \otimes T(V)$

$$\begin{aligned} \mu_{m,n} : T^m(V) \otimes T^n(V) &\longrightarrow T^{m+n}(V) \\ (v_1 \otimes \ldots \otimes v_m) \otimes (v_{m+1} \otimes \ldots \otimes v_{m+n}) &\longmapsto v_1 \otimes \ldots \otimes v_{m+n} \end{aligned}$$

and unit $1_k \in T^0(V) \subseteq T(V)$

Let $i : V = T^1(V) \hookrightarrow T(V)$ be the inclusion. Every linear map $\varphi : V \to A$, where A is an algebra, factorizes through a morphism of algebras $\Phi : T(V) \to A$ s.t. $\Phi \circ i = \varphi$. $(\Phi(v_1 \otimes \ldots \otimes v_n) = \varphi(v_1) \ldots \varphi(v_n))$

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "i"'] & & A. \\ & T(V) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Tensor coalgebra 1

By the universal property, the linear map $\delta : V \to T(V)\underline{\otimes}T(V)$, given by $\delta(v) = v \otimes 1_k + 1_k \otimes v$, extends to $\Delta : T(V) \to T(V)\underline{\otimes}T(V)$, which induces a (graded) Hopf algebra structure on $T(V)$ in ${}^H_H\mathcal{YD}$, with $S(v) = -v$.

$T(V)$ can be endowed with another graded coalgebra structure in ${}^H_H\mathcal{YD}$, denoted by $T^c(V)$, through the map $\Delta^c : T(V) \to T(V)\underline{\otimes}T(V)$ defined by

$$\Delta^c(v_1 \otimes \ldots \otimes v_n) := \sum_{j=0}^{n}(v_1 \otimes \ldots \otimes v_j) \otimes (v_{j+1} \otimes \ldots \otimes v_n).$$

While $\Delta^c(v) = v \otimes 1_k + 1_k \otimes v$, we have $\Delta^c \neq \Delta$ and Δ^c is not multiplicative w.r.t. the standard product in $T(V)$, indeed:

$$\begin{aligned}\Delta^c(v \otimes w) =&(v \otimes w) \otimes 1_k + v \otimes w + 1_k \otimes (v \otimes w) \neq \Delta^c(v)\Delta^c(w)\\ =&(v \otimes w) \otimes 1_k + v \otimes w + 1_k \otimes (v \otimes w) + v_{-1} \cdot w \otimes v_0\end{aligned}$$

Tensor coalgebra 2

However, $T^c(V)$ is a braided Hopf algebra when considering the multiplication $\mu^c : T(V)\underline{\otimes}T(V) \to T(V)$, defined as

$$(v_1 \otimes \ldots \otimes v_i) \cdot (v_{i+1} \otimes \ldots \otimes v_n) = \sum_{\omega \in \mathbb{S}_{i,n-i}} c_\omega(v_1 \otimes \ldots \otimes v_n)$$

where $\mathbb{S}_{i,n-i}$ denotes $(i, n-i)$ shuffles, $\omega \in \mathbb{S}_n$ such that $\omega(1) < \omega(2) < \ldots < \omega(i)$ and $\omega(i+1) < \omega(i+2) < \ldots < \omega(n)$

$$\omega = \left(\begin{array}{cccc|cccc} 1 & 2 & \ldots & i & i+1 & \ldots & n-1 & n \\ \omega(1) & \omega(2) & \ldots & \omega(i) & \omega(i+1) & \ldots & \omega(n-1) & \omega(n) \end{array} \right).$$

To define the morphism $c_\omega : T^n(V) \to T^n(V)$, we need a small digression.

Braid group

Let $n > 0$. The braid group $\mathbb{B}_n$ is the group generated by σ_i, where $i \in \{1, \ldots, n\}$, with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } \mid i - j \mid \geq 2$$
$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \text{for } \mid i - j \mid = 1$$

The symmetric group $\mathbb{S}_n$ is generated by the simple transpositions $\tau_i = (i \ i+1)$, with relations

$$\tau_i^2 = e, \quad \text{for all } 1 \leq i < n$$
$$\tau_i \tau_j = \tau_j \tau_i, \quad \text{for } \mid i - j \mid \geq 2$$
$$\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j, \quad \text{for } \mid i - j \mid = 1$$

The length function $l : \mathbb{S}_n \to \mathbb{N}$ measures the minimum decomposition of a permutation as a product of simple transpositions.

Matsumoto section

By this presentation of $\mathbb{S}_n$, there is a surjective morphism of groups

$$\pi : \mathbb{B}_n \longrightarrow \mathbb{S}_n : \sigma_i \longmapsto \tau_i.$$

At the level of sets, π admits a section $M : \mathbb{S}_n \to \mathbb{B}_n$ such that $\tau_i \mapsto \sigma_i$, called the Matsumoto section, which is not a group homomorphism. However, for any $\omega, \omega' \in \mathbb{S}_n$:

$$M(\omega \circ \omega') = M(\omega)M(\omega'), \quad \text{if } l(\omega \circ \omega') = l(\omega) + l(\omega')$$

Example: Consider $\omega = (1\ 3\ 4\ 2) \in \mathbb{S}_4$. A minimal length decomposition of ω is $(1\ 3\ 4\ 2) = (2\ 3)(1\ 2)(3\ 4) = \tau_2\tau_1\tau_3$. Then

$$M((1\ 3\ 4\ 2)) = M(\tau_2\tau_1\tau_3) = M(\tau_2)M(\tau_1)M(\tau_3) = \sigma_2\sigma_1\sigma_3,$$

since $l(\tau_2\tau_1\tau_3) = 3 = l(\tau_2) + l(\tau_1) + l(\tau_3)$.

Braid group representation

Let $V \in {}^H_H\mathcal{YD}$. We can come back to define $c_\omega : T^n(V) \to T^n(V)$.
The following assignment defines a representation $\rho_n : \mathbb{B}_n \to GL(T^n(V))$

$$\sigma_i \longmapsto c_i = id_{T^{j-1}(V)} \otimes c_{V,V} \otimes id_{T^{n-j-1}(V)}$$

$$c_i(v_1 \otimes \ldots \otimes v_n) = \ldots v_{j-1} \otimes v_{i[-1]} \cdot v_{i+1} \otimes v_{i[0]} \otimes v_{j+2} \cdots \ .$$

Through the Matsumoto section, we can now define $c_\omega := \rho_n(M(\omega))$ for all $\omega \in \mathbb{S}_n$. In particular, if $\omega = \tau_{i_1} \ldots \tau_{i_m}$ is a minimal decomposition, $c_\omega = \rho_n(M(\omega)) = \rho_n(M(\tau_{i_1} \ldots \tau_{i_m})) = \rho_n(M(\tau_{i_1})) \ldots \rho_n(M(\tau_{i_m})) = \rho_n(\sigma_{i_1}) \ldots \rho_n(\sigma_{i_m}) = c_{i_1} \ldots c_{i_m}$.

Example: Consider $\omega = (1\ 3\ 4\ 2) = \tau_2\tau_1\tau_3 \in \mathbb{S}_4$. Then $c_\omega = \rho_n(M(\omega))$ $= \rho_n(M(\tau_2\tau_1\tau_3)) = \rho_n(\sigma_2)\rho_n(\sigma_1)\rho_n(\sigma_3) = c_2c_1c_3$.

Nichols algebras 1

Let $V \in {}^H_H\mathcal{YD}$. Consider $T(V)$ and $T^c(V)$ in ${}^H_H\mathcal{YD}$ and the inclusion $V \hookrightarrow T^c(V)$; by the universal property of $T(V)$, there exist an algebra map $\Omega : T(V) \to T^c(V)$ s.t. $\Omega(v) = v$ for all $v \in V$.

$$\textit{Example: } \Omega(v \otimes w) = \Omega(v) \cdot \Omega(w) = v \cdot w = \sum_{\omega \in \mathbb{S}_{1,1}} c_\omega(v \otimes w)$$

$$= \sum_{\omega \in \{id,\ (1\ 2)\}} c_\omega(v \otimes w) = v \otimes w + v_{[-1]} \cdot w \otimes v_{[0]}$$

The map has additionally the following properties:

- $\Omega = \sum_n \Omega_n$ is a graded map, with $\Omega_n = \Omega_{|T^n(V)}$,
- Ω is a coalgebra morphism between $T(V)$ and $T^c(V)$,
- Ω is a morphism in ${}^H_H\mathcal{YD}$.

Nichols algebras 2

Definition

The Nichols algebra $\mathcal{B}(V)$ is the image of the map Ω in $T^c(V)$.

If $I(V) := ker\,\Omega$ and $I^n(V) := ker\,\Omega_n$, then $I(V) = \bigoplus_{n\geq 2} I^n(V)$.
Then $\mathcal{B}(V) = \bigoplus_{n\geq 0} \mathcal{B}^n(V) \simeq \frac{T(V)}{I(V)}$, where $\mathcal{B}^n(V) \simeq \frac{T^n(V)}{I^n(V)}$

By induction, one can characterize the graded components of Ω as follows.

Proposition

For all $n \geq 2$

$$\Omega_n = \sum_{\omega\in\mathbb{S}_n} c_\omega : T^n(V) \to T^n(V)$$

Example:

- As we have seen before, $\Omega_2 = id + c_1$.
- $\Omega_3 = id + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1$.

Alternative characterizations of Nichols algebras

Proposition

$I(V)$ is the maximal element in the set

$$\{J \subseteq \bigoplus_{n\geq 2} T^n(V) \mid \begin{array}{c} J \text{ is a graded Hopf ideal and} \\ \text{a submodule of } T(V) \text{ in } {}^H_H\mathcal{YD} \end{array}\}.$$

Proposition

A graded Hopf algebra B in ${}^H_H\mathcal{YD}$ is isomorphic to $\mathcal{B}(V)$ if and only if:

- *B is generated as an algebra by V.*
- *B is coradically graded, meaning the coradical filtration and the filtration induced by the grading coincide.*

Proofs for both can be found in section 2 of [1] or section 1.6 of [2].

[1] N. Andruskiewitsch, H.-J. Schneider, *Pointed Hopf algebras*, (2002).
[2] I. Heckenberger, H.-J. Schneider, "Hopf algebras and root systems", 2020.

Examples 1

Consider a vector space $V \in {}_H^H\mathcal{YD}$ with trivial H-action and coaction, meaning $h \cdot v = \epsilon_H(h)v$ and $\lambda(v) = 1_H \otimes v$. Then $c_{V,V} = \tau$ and

$$\Omega_n(v_1 \otimes \ldots \otimes v_n) = \sum_{\omega \in \mathbb{S}_n} v_{\omega^{-1}(1)} \otimes \ldots \otimes v_{\omega^{-1}(n)}.$$

In particular, $\Omega_2(v_1 \otimes v_2) = v_1 \otimes v_2 + v_2 \otimes v_1$, therefore $ker\,\Omega_2 = \{v \otimes w - w \otimes v \mid \ v, w \in V\}$. However, it can be proven that, in this case, $ker\,\Omega_n =< ker\,\Omega_2 > \cap T^n(V)$, and so

$$I(V) = ker\,\Omega =< v \otimes w - w \otimes v \mid \ v, w \in V > .$$

Therefore

$$\mathcal{B}(V) \simeq \frac{T(V)}{< v \otimes w - w \otimes v \mid \ v, w \in V >} = Sym(V).$$

Examples 2

Let $\bar{V} \in {}^{k[C_2]}_{k[C_2]}\mathcal{YD}$ s.t. $g \cdot \bar{v} = -\bar{v}$ and $\lambda(\bar{v}) = g \otimes \bar{v}$, therefore $c_{V,V} = -\tau$. Then

$$\Omega_n(\bar{v}_1 \otimes \ldots \otimes \bar{v}_n) = \sum_{\omega \in \mathbb{S}_n} (-1)^{l(\omega)} \, \bar{v}_{\omega^{-1}(1)} \otimes \ldots \otimes \bar{v}_{\omega^{-1}(n)}.$$

In particular $\Omega_2(v \otimes w) = v \otimes w - w \otimes v$, so $ker\,\Omega_2 = \{v \otimes w + w \otimes v\}$. As before, $ker\,\Omega =< ker\,\Omega_2 >=< v \otimes w + w \otimes v \mid v, w \in \bar{V} >$, therfore

$$\mathcal{B}(\bar{V}) \simeq \frac{T(\bar{V})}{< v \otimes w + w \otimes v \mid v, w \in \bar{V} >} = \Lambda(\bar{V}).$$

Let now $V_0, V_1 \in {}^{k[C_2]}_{k[C_2]}\mathcal{YD}$, such that $g \cdot v_i = (-1)^i v_i$ and $\lambda(v_i) = g^i \otimes v_i$. Then $V = V_0 + V_1 \in {}^{k[C_2]}_{k[C_2]}\mathcal{YD}$ is a super vector space, with braiding $c_{V,V}(v_i \otimes v_j) = (-1)^{ij}(v_j \otimes v_i)$. In this case $\mathcal{B}(V) \simeq Sym(V_0) \otimes \Lambda(V_1)$.

Bosonization/Radford's biproduct

We denote by H-BialProj the category of bialgebras with a projection:

- Objects: triples (A, i, π), where A is a bialgebra and $i : H \hookrightarrow A$ and $\pi : A \to H$ are bialgebra morphisms such that $id_A = i \circ \pi$.
- Morphisms: $f : (A, i, \pi) \to (A', i', \pi')$, bialgebra maps $f : A \to A'$ such that $f \circ i = i'$ and $\pi' \circ f = \pi$.

To a bialgebra with a projection (A, i, π) one can associate a braided bialgebra structure on $B = A^{co(H)}$, where the right coaction on A is given by $\rho(a) = a_1 \otimes \pi(a_2)$, and the structure maps are induced by those of A and by i and π.

$$\mathrm{Bialg}({}^H_H\mathcal{YD}) \underset{\substack{\text{biproduct}\\ \text{construction}\\ \text{(bosonization)}}}{\overset{\longleftarrow}{\longrightarrow}} H\text{-BialProj}$$

Biproduct Hopf algebras

Indeed, we can associate to each Hopf algebra B in ${}^H_H\mathcal{YD}$ a Hopf algebra $A = B \times H$ with a projection on H.
The structure of the biproduct $B \times H$ is as follows:

$$
\begin{gathered}
B \times H = B \otimes H \text{ (as a vector space)}, \\
(b \times h)(b' \times h') = b\,(h_1 \cdot b') \times h_2 h', \\
\Delta(b \times h) = b_1 \times (b_2)_{[-1]} h_1 \otimes (b_2)_{[0]} \times h_2 \quad , \\
\epsilon(h \times b) = \epsilon_B(h)\epsilon_H(h), \quad 1_{B\times H} = 1_B \times 1_H, \\
S(b \times h) = (1 \times S_H(b_{-1}h))(S_B(b_0) \times 1_H).
\end{gathered}
$$

Moreover, the injection and projection on $B \times H$ are as follows:

$$
\begin{gathered}
i : H \hookrightarrow B \times H : h \mapsto 1 \times h \\
\pi : B \times H \to H : b \times h \mapsto \epsilon_B(b)h
\end{gathered}
$$

Via this structure, we can associate a Hopf algebra $\mathcal{B}(V) \times H$ to each Nichols algebra $\mathcal{B}(V)$ in ${}^H_H\mathcal{YD}$.

Examples of biproducts

One can think of the biproduct construction as a generalization of the semidirect product for groups. Consider, for instance, $\mathbb{S}_3 \simeq C_3 \rtimes C_2$; we have $k[\mathbb{S}_3] \simeq k[C_3] \times k[C_2]$, where $k[C_3]$ is seen as a Hopf algebra in ${}^{k[C_2]}_{k[C_2]}\mathcal{YD}$ with trivial coaction and $g \cdot x = x^2$, where $< x >= C^3$.

The simplest non-commutative non-cocommutative Hopf algebra is a biproduct; indeed

$$H_4 = \bar{B} \times k[C_2],$$

where $\bar{B} = k < 1, n >$ is a 2-dimensional braided Hopf algebra in ${}^{k[C_2]}_{k[C_2]}\mathcal{YD}$, generated by the unit and a nilpotent primitive element n with $\lambda(n) = g \otimes n$ and $g \cdot n = -g$.

Nichols and pointed Hopf algebras

We conclude with a brief explenation of the role of Nichols algebras in the classification of pointed Hopf algebras.

Let A be a pointed Hopf algebra, meaning all simple A-comodules are 1-dimensional, with coradical filtration $\{A_i\}_{i\in\mathbb{N}}$. The graded Hopf algebra associated to the coradical filtration, fixing $A_{-1} = \{0\}$, is

$$gr(A) := \bigoplus_{n\in\mathbb{N}} A_n/A_{n-1}.$$

$gr(A)$ is a Hopf algebra with a projection on $A_0 = k[G(A)]$. Therefore,

$$gr(A) \simeq \mathcal{R} \times k[G(A)],$$

where $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}^n$ is a coradically graded Hopf algebra in ${}^{A_0}_{A_0}\mathcal{YD}$.
By setting $V = \mathcal{R}^1$, the algebra generated by V is isomorphic to the Nichols algebra $\mathcal{B}(V)$ and a subalgebra of $\mathcal{R}$.

References and further reading

Nichols algebras:

- N. Andruskiewitsch, *An introduction to Nichols algebras*, Quantization, Geometry and Noncommutative Structures in Mathematics and Physics. Springer, 135–195 (2017)
- I. Heckenberger, H.-J. Schneider, "Hopf algebras and root systems", *American Mathematical Soc.* **247**, 2020.
- N. Andruskiewitsch, H.-J. Schneider, *Pointed Hopf algebras*, New Directions in Hopf Algebras **43**, (2002).

Biproduct/Bosonization:

- D.E. Radford, "Hopf algebras", *Series on Knots and Everything*, University of Illinois at Chicago 2011.
- D.E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **92**, 322–347, (1985).