Free biproduct quasi-Hopf algebras of rank 2

Matteo Misurati

University of Ferrara

GABY, June 17th to 21st, 2024

Based on a joint work in progress with Daniel Bulacu

Brief summary

- We obtain the structure of biproducts of rank 2 as free modules over a quasi-Hopf algebra H.
- This is achieved by considering 2-dimensional Hopf algebras in the category of left Yetter-Drinfeld modules over H (For the classical case, see Radford[1]).
- Examples of such biproducts are found, where H is taken as a quasi-Hopf algebra obtained from cyclic groups or the Klein four-group.
- In conclusion, we will see how this characterization can be applied to the classification of low dimensional quasi-Hopf algebras.

[1] Biproducts and Kashina's Examples, Commun. Algebra 44, (2015).

Motivation

- Hopf algebras with a projection were introduced by Radford[2]. Up to isomorphism, a Hopf algebra A with a projection over H is a biproduct $B \times H$, where B is a braided Hopf algebra in ${}_H^H\mathcal{YD}$.
- It was proven by Bulacu[3] that this holds even when H is a quasi-Hopf algebra.
- Knowing the structure of quasi-Hopf algebra with a projection can be a useful tool in the classification of quasi-Hopf algebras of low dimension.

[2] The structure of Hopf algebras with a projection. J. Algebra 92 (1985).

[3] A structure theorem for quasi-Hopf bimodule coalgebras, Theory Appl. Categ. 32 (2017).

Preliminaries: Quasi-bialgebra

A quasi-bialgebra is a datum $(H, \Delta, \epsilon, \Phi)$, where H is an algebra, Φ is an invertible element in $H \otimes H \otimes H$, called reassociator, and $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ are algebra morphism satisfying:

$$(Id_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes Id_H)(\Delta(h))\Phi^{-1},$$
$$(Id_H \otimes \epsilon)(\Delta(h)) = h \quad \text{and} \quad (\epsilon \otimes Id_H)(\Delta(h)) = h.$$

The reassociator Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(Id_H \otimes \Delta \otimes Id_H)(\Phi)(\Phi \otimes 1)$$
$$= (Id_H \otimes Id_H \otimes \Delta)(\Phi)(\Delta \otimes Id_H \otimes Id_H)(\Phi),$$
$$\text{and}$$
$$(Id_H \otimes \epsilon \otimes Id_H)(\Phi) = 1 \otimes 1.$$

The last two equalities imply $(\epsilon \otimes Id_H \otimes Id_H)(\Phi) = 1 \otimes 1$ and $(Id_H \otimes Id_H \otimes \epsilon)(\Phi) = 1 \otimes 1$.

- A quasi-bialgebra H is a quasi-Hopf algebra if there exists an algebra antimorphism $S : H \to H$ and distinguished elements $\alpha, \beta \in H$ such that, for all $h \in H$, the following conditions hold:

$$S(h_1)\alpha h_2 = \epsilon(h)\alpha,$$
$$h_1\beta S(h_2) = \epsilon(h)\beta,$$

and

$$X^1\beta S(X^2)\alpha X^3 = 1,$$
$$S(x^1)\alpha x^2\beta S(x^3) = 1.$$

- Let H be a quasi-Hopf algebra with reassociator Φ and antipode (S, α, β). $H' \subseteq H$ is a sub-quasi-Hopf algebra of H if it is a quasi-Hopf algebra with reassociator Φ, distinguished elements α, β and structure maps induced by those of H.

Preliminaries: Notation and conventions

- The following Sweedler-like notation is used for quasi-Hopf algebras and their reassociator:

$$(\Delta \otimes Id_H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2,$$
$$(Id_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)}.$$
$$\Phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3 = \dots$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = y^1 \otimes y^2 \otimes y^3 = z^1 \otimes z^2 \otimes z^3 = \dots$$

- We consider all (quasi-)Hopf algebras to have invertible antipode, and all fields to have characteristic different from 2.
- We call a quasi-Hopf algebra genuine if its reassociator is not $1 \otimes 1 \otimes 1$ (i.e., it is not a Hopf algebra).
- By Yetter-Drinfeld modules over H, we mean the left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$.

Braided bialgebras in ${}_H^H\mathcal{YD}$

Let now H be a quasi-Hopf algebra over a field k. We recall that ${}_H^H\mathcal{YD}$ is braided monoidal in such a way that the forgetful functor ${}_H^H\mathcal{YD} \to {}_H\mathcal{M}$ is strong monoidal.

A bialgebra B in ${}_H^H\mathcal{YD}$ is defined as a bialgebra in a braided category; i.e.

1. B is an algebra in ${}_H^H\mathcal{YD}$, meaning it is equipped with a unit 1_B and a multiplication m_B such that both are H-(co)linear and m_B is associative in ${}_H\mathcal{M}$.
2. B is a coalgebra in ${}_H^H\mathcal{YD}$, and as such it is endowed with a counit map $\epsilon_B : B \to k$ and a coproduct Δ_B, both H-(co)linear and such that Δ_B is coassociative in ${}_H\mathcal{M}$.
3. The coproduct and the counit are required to be algebra morphisms, where the algebra structure on $B \otimes B$ is given by $m_{B\otimes B}$, defined naturally through the associativity constraints, m_B and the braiding of ${}_H^H\mathcal{YD}$.

Braided Hopf algebras in ${}_H^H\mathcal{YD}$

- Since the monoidal category ${}_H^H\mathcal{YD}$ is not strict, the product and coproduct of B are associative and coassociative only up to the associativity constraints in ${}_H^H\mathcal{YD}$, which coincide with those of ${}_H\mathcal{M}$, given on B by

$$a_{B,B,B} : (b \otimes b') \otimes b'' \mapsto X^1 \cdot b \otimes \left[(X^2 \cdot b') \otimes (X^3 \cdot b'') \right],$$

  where $X^1 \otimes X^2 \otimes X^3$ is the reassociator of H.
- In particular, B is an algebra and a coalgebra in ${}_H\mathcal{M}$.
- Moreover, B is a Hopf algebra in ${}_H^H\mathcal{YD}$ if it has an antipode S which is also a morphism in ${}_H^H\mathcal{YD}$.

2-dimensional braided Hopf algebras

The previous conditions result in the following structure for a 2-dimensional Hopf algebra B in ${}_H^H\mathcal{YD}$. B is isomorphic to one of the following:

1. The group Hopf algebra $k[C_2]$, seen as a YD-module via the trivial H-(co)action: $h \cdot g = \epsilon_H(h)g$, $g \mapsto 1_H \otimes g$, $h \in H$. We call this structure the trivial one.
2. The Hopf algebra $B_{\sigma,\nu}$ generated by $\{1, \theta\}$, with unit 1 and θ a primitive element s.t. $\theta^2 = 0$, with H-module and comodule structures given, respectively, by $\sigma \in \mathrm{Alg}_k(H, k)$ and $\nu \in H$ as follows:

$$h \cdot n = \sigma(h)n, \quad n \mapsto \nu \otimes n.$$

(σ, ν) has to satisfy the following relations:

$$\sigma(\nu) = -1, \quad \sigma(h_2)h_1\nu = \sigma(h_1)\nu h_2,$$
$$\Delta_H(\nu) = \sigma(y^1x^3X^2)(x^1 \otimes x^2)(X^1\nu \otimes \nu X^3)(y^2 \otimes y^3).$$

The braided antipode S is determined by $S(n) = -n$.

Remark on braided Hopf algebras of dimension 2

- The following condition on (σ, ν) differentiates the Hopf case from the genuine quasi-Hopf case:

$$\Delta_H(\nu) = \sigma(y^1 x^3 X^2)(x^1 \otimes x^2)(X^1\nu \otimes \nu X^3)(y^2 \otimes y^3).$$

- The difference between cases is substantial; even in the first non-trivial example, given by $H = k\left[C_2\right]$ seen as a quasi-Hopf algebra with reassociator $\Phi = 1_H \otimes 1_H \otimes 1_H - 2p_- \otimes p_- \otimes p_-$, where $p_- = \frac{1}{2}(1_H - g)$, there are no nontrivial 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$, while in the classical case, i.e. in ${}_{k[C_2]}^{k[C_2]}\mathcal{YD}$, there is one, namely $B_{\bar{\sigma},g}$ where $\bar{\sigma} \in Alg_k(H,k)$ is determined by $\bar{\sigma}(g) = -1_k$.
- When H is a commutative quasi-Hopf algebra with reassociator such that $x^1 \otimes x^2 \otimes x^3 = x^1 \otimes x^3 \otimes x^2$ (such will be the case in our examples), the condition becomes much simpler:

$$\Delta_H(\nu) = \sigma(y^1)(y^2 \otimes y^3)(\nu \otimes \nu).$$

Biproduct quasi-Hopf algebras

A quasi-Hopf algebra A with a projection $A \underset{\pi}{\overset{i}{\rightleftarrows}} H$ is obtained, through the biproduct construction, from a braided Hopf algebra B in ${}_H^H\mathcal{YD}$. The structure of the biproduct $B \times H$ is as follows: [4]

$$\begin{aligned} B \times H = B \otimes H \text{ (as a vector space)}, \\ (b \times h)(b' \times h') = (x^1 \cdot b)(x^2 h_1 \cdot b') \times x^3 h_2 h', \\ \Delta(b \times h) = & y^1 X^1 \cdot b_1 \times y^2 Y^1 (x^1 X^2 \cdot b_2)_{-1} x^2 X_1^3 h_1 \\ & \otimes y_1^3 Y^2 \cdot (x^1 X^2 \cdot b_2)_0 \times y_2^3 Y^3 x^3 X_2^3 h_2, \\ \epsilon(h \times b) = \epsilon_B(h)\epsilon_H(h), \quad \Phi_{B\times H} = 1 \times X^1 \otimes 1 \times X^2 \otimes 1 \times X^3, \\ S(b \times h) = (1 \times S_H(X^1 x_1^1 b_{-1} h)\alpha_H)(X^2 x_2^1 \cdot S_B(b_0) \times X^3 x^2 \beta_H S_H(x^3)), \end{aligned}$$

with distinguished elements: $\alpha_{B\times H} = 1 \times \alpha_H, \quad \beta_{B\times H} = 1 \times \beta_H.$

[4] D.Bulacu, E.Nauwelaerts, Radford's biproduct for quasi-Hopf algebras and bosonization, J. Pure Appl. Algebra 174, 2002

Biproduct quasi-Hopf algebras

- By identifying H with the image of $i : h \mapsto 1 \times h$, we get, from the structure maps seen before, that H is a sub-quasi-Hopf algebra of $B \times H$.
- If B is finite dimensional with $\dim(B) = n$, $B \times H$ is a right free H-module of rank n. We will call any such structure a biproduct of rank n, for short.
- The projection on H is determined by $b \times h \mapsto \epsilon_B(b)h$.

Biproduct quasi-Hopf algebras of rank 2

Through the biproduct construction, we obtain the next quasi-Hopf algebras, respectively associated to $k[C_2]$ and $B_{\sigma,\nu}$.

- $H_g := k[C_2] \otimes H$, generated as a k-algebra by H and the grouplike element g, with relations

$$g^2 = 1 \quad \text{and} \quad gh = hg,$$

  such that it has H as a sub-quasi-Hopf algebra.

- $H(\theta)_{\sigma,\nu}$, the k-algebra generated by H and θ with relations

$$\theta^2 = 0, \quad h\theta = \sigma(h_1)\theta h_2,$$

  for all $h \in H$. Its quasi-Hopf algebra structure is determined by H being its sub-quasi-Hopf algebra and

$$\Delta(\theta) = \sigma(X^2x^1)\nu X^1x^2 \otimes \theta X^3x^3 + \sigma(x^1)\theta x^2 \otimes x^3,$$
$$S(\theta) = -\sigma(X^2x^1_2)S(X^1x^1_1\nu)\alpha\theta X^3x^2\beta S(x^3).$$

Biproduct quasi-Hopf algebras of rank 2: Remarks

From the classification of 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$, we obtain the following characterization for quasi-Hopf algebras A with a projection on H, of rank 2 as free H-modules. There are 3 cases.

1. If A is semisimple, so is H. Moreover, $A \cong H_g$ (as quasi-Hopf algebras).
2. If A is not semisimple and H is semisimple, then $A \cong H(\theta)_{\sigma,\nu}$.
3. If both A and H are not semisimple, then $A \cong H_g$ or $A \cong H(\theta)_{\sigma,\nu}$

A nice property of biproducts quasi-Hopf algebras $A = B \times H$ of rank 2 is that A is basic if and only if H is basic. (A is basic if all of its left simple modules are 1 dimensional)

Example: cyclic group quasi-Hopf algebra

Fix $n \geq 2$ and $1 \leq t \leq n$. Let k be a field with n^2-th primitive roots of the unit. We fix a primitive n-th root of the unit $\mathfrak{q}$.
We denote by $k_{\Phi_t}[C_n]$ the quasi-Hopf algebra with the same k-algebra structure and structure maps as $k[C_n]$, with reassociator

$$\Phi_t = \sum_{i,j,l=0}^{n-1} \mathfrak{q}^{ti\lfloor \frac{j+l}{n} \rfloor} 1_i \otimes 1_j \otimes 1_l, \quad \text{where } 1 \leq t \leq n,$$

$$1_j = \frac{1}{n} \sum_{a=0}^{n-1} \mathfrak{q}^{(n-a)j} g^j,$$

and distinguished elements α, β such that $\alpha\beta = g^{-t}$.

Example: cyclic group quasi-Hopf algebra

We proved that all biproduct quasi-Hopf algebras of rank 2 over $H = k_{\Phi_t}[C_n]$ are either

1. the trivial one $k[C_2] \otimes k_{\Phi_t}[C_n]$, or
2. $H(\theta)_{\sigma_z, g^d}$, where (σ_z, g^d) is determined by two integers $0 \le z, d \le n-1$, such that $\sigma_z(g) = \mathfrak{q}^z$,

$$m \mid z \quad \text{and} \quad \mathfrak{q}^{zd} = -1$$

where $m = \frac{n}{gcd(n,t)}$.

We notice that, if n is odd or $gcd(n,t) = 1$, we have only the first, semisimple, type of biproduct.

Example: Klein four-group

Denote by g, h the generators of $C_2 \times C_2$ and suppose k is a field with primitive 4th roots of the unit and fix $a, b, c \in \{0, 1\}$.
The quasi-Hopf algebra structures[5] on the Klein group algebra are $k_{\Phi_{(a,b,c)}}[C_2 \times C_2]$, where

$$\Phi_{(a,b,c)} = \sum_{i,j,p,q,r,s=0}^{1} (-1)^{ai\lfloor \frac{r+p}{2} \rfloor + bj\lfloor \frac{r+p}{2} \rfloor + cj\lfloor \frac{q+s}{2} \rfloor} 1_{ij} \otimes 1_{pq} \otimes 1_{rs},$$

$$1_{ij} = \frac{1}{4}(1 + (-1)^i g)(1 + (-1)^j h)$$

$$\text{and} \quad \alpha\beta = g^a h^c \sum_{ij} (-1)^{bij} 1_{ij}.$$

[5] HL.Huang, G.Liu, Y.Ye, The Braided Monoidal Structures on a Class of Linear Gr-Categories, Algebr Represent Theor 17, (2014)

Example: Klein four-group

Depending on the value of (a, b, c), we can have up to three types of non-trivial rank 2 biproduct structures on $k_{\Phi_{(a,b,c)}}[C_2 \times C_2]$:

1. If $a = 0$, denoting $B_b = \sum_{ijpq} (-1)^{bjp}\, 1_{ij} \otimes 1_{pq}$ and as $\bar{m}$ the multiplication on the biproduct:

$$\nu \in \{g, gh\}, \quad \begin{aligned} g\theta =& -\theta g, & \Delta(\theta) =& (\nu \otimes \theta)B_b + \theta \otimes 1, \\ h\theta =& \theta h, & S(\theta) =& -\nu\alpha\theta\alpha^{-1}\bar{m}(B_b). \end{aligned}$$

2. If $c = 0$, denoting $E_b = \sum_{ijpq}(-1)^{b\lfloor \frac{i+p}{2} \rfloor}\, 1_{ij} \otimes 1_{pq}$:

$$\nu \in \{h, gh, \nu_2^+, \nu_2^-\}, \quad \begin{aligned} g\theta =& \theta g, & \Delta(\theta) =& (\nu \otimes \theta + \theta \otimes 1)\, E_b, \\ h\theta =& -\theta h, & S(\theta) =& -\nu\alpha\theta\alpha^{-1}. \end{aligned}$$

   where

$$\nu_2^{\pm} = \frac{1 \pm \mathfrak{q}^b}{2}\, h + \frac{1 \mp \mathfrak{q}^b}{2}\, gh.$$

Example: Klein four-group

The type and amount of structures for non-trivial biproducts of rank 2 over $k_{\Phi_{(a,b,c)}}[C_2 \times C_2]$ depends on the value of (a, b, c). Namely, we can have up to three types of non-trivial biproduct structures:

3. If $a + b + c$ is even, defining

$$C_{(a,b,c)} = \sum_{ij,pq} (-1)^{aip+bjp+cjq} \; 1_{ij} \otimes 1_{pq} \quad \text{and}$$

$$D_{(a,b,c)} = \sum_{ij,pq} (-1)^{(a+b)\lfloor \frac{i+p}{2} \rfloor + c \lfloor \frac{j+q}{2} \rfloor} \; 1_{ij} \otimes 1_{pq},$$

we have: $\nu \in \{g, h, \nu_4^+, \nu_4^-\}$,

$$g\theta = -\,\theta g, \quad \Delta(\theta) = \big((\nu \otimes \theta) C_{(a,b,c)} + \theta \otimes 1\big)\, D_{(a,b,c)},$$
$$h\theta = -\,\theta h, \quad S(\theta) = -\,\nu\; \alpha\theta\beta\; \bar{m}(C_{(a,b,c)}).$$

$$\text{where} \quad \nu_4^{\pm} = \pm\frac{\mathfrak{q}}{2}1 + \frac{1}{2}g + \frac{1}{2}h \mp \frac{\mathfrak{q}}{2}gh.$$

Application: Classification of 6 dimensional qHas 1

- We are applying the characterization of free biproduct quasi-Hopf algebras of rank 2 to the classification of all 6 dimensional quasi-Hopf algebras.
- From the example over cyclic groups, it follows that all biproducts of rank 2 in dimension 6 are semisimple.
- We characterize biproduct quasi-Hopf algebras of rank 3 over a quasi-Hopf algebra of dimension 2. All are semisimple.
- Then, we found that all 6 dimensional quasi-Hopf algebras with a projection are semisimple.

Application: Classification of 6 dimensional qHas 2

- In order to classify 6 dimensional quasi-Hopf algebras H, we consider the graded algebra $gr(H)$ associated to the radical filtration, which is a quasi-Hopf algebra when H is basic.
- We obtain that if H is not semisimple, $gr(H)$ is a non-semisimple quasi-Hopf algebras with a projection in dimension 6, which contradicts the characterization of biproducts in dimension 6.
- Therefore, all 6 dimensional quasi-Hopf algebras are semisimple.

Thank you for your attention!