## An introduction to Nichols algebras and bosonization

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## Summary

- We will introduce all the ingredients necessary to the construction of Nichols algebras in Yetter-Drinfeld modules category.
- We define Nichols algebras and give some simple examples.
- We will show how regular and braided Hopf algebras are related through Radford-Majid bosonization.
- If time permits, we will conclude by showing the role of Nichols algebras in the classification of pointed Hopf algebras.

Unless otherwise stated, we denote by H a Hopf algebra with invertible antipode over an algebraically closed field k of characteristic 0. For this talk  $\mathbb{N} \ni 0$ .

## Braided monoidal category

A monoidal category  $\mathcal C$  is called braided when endowed with a natural isomorphism  $c_{X,Y}:X\otimes Y\to Y\otimes X$  such that the hexagon axioms hold:

- The hexagon axioms imply the Yang-Baxter equation for c.
- c plays in C the role of the transposition  $\tau$  in  $Vec_k$ .
- The braiding is, in general, not unique:  $(C, c) \leadsto (C, c^{-1})$ .

# (left) Center construction

Starting from any monoidal category C, we can construct a braided monoidal category  $\mathcal{L}_L(C)$ , where:

• Objects:  $(Z, \gamma)$ , where  $Z \in \mathcal{C}$  and  $\gamma_X : X \otimes Z \to Z \otimes X$  nat. iso. s.t.

• Morphisms between  $(Z, \gamma)$  and  $(Z', \gamma')$ : maps  $f: Z \to Z'$  in  $\mathcal C$  s.t.

$$(f \otimes id)\gamma_X = {\gamma'}_X(id \otimes f), \text{ for all } X \in \mathcal{C}.$$

- Tensor product:  $(Z \otimes Z', \tilde{\gamma})$ , where  $\tilde{\gamma}$  is defined by  $\gamma$ ,  $\gamma'$  and a.
- Braiding:  $c_{(Z,\gamma),(Z',\gamma')} = \gamma'_Z$ .

Example: 
$$\mathcal{L}_L(H\mathcal{M}) \cong {}_H^H \mathcal{YD}$$
.

# (left-left) Yetter-Drinfeld modules

Objects in  ${}_{H}^{H}\mathcal{YD}$  are vector spaces V endowed with

- a H-action  $\cdot : H \otimes V \rightarrow V : h \otimes v \mapsto h \cdot v$
- a H-coaction  $\lambda: V \to H \otimes V: v \mapsto v_{[-1]} \otimes v_{[0]}$

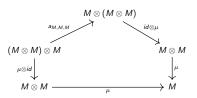
such that  $h_1v_{[-1]}\otimes (h_2\cdot v_{[0]})=(h_1\cdot v)_{[-1]}h_2\otimes (h_1\cdot v)_{[0]}$ , while morphism are H-linear and H-colinear maps.

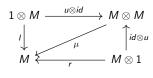
### ${}_{H}^{H}\mathcal{YD}$ is braided monoidal:

- ullet  $V\otimes W$  is the tensor product between vector spaces
- *H*-action on  $V \otimes W$ :  $h \cdot (v \otimes w) = h_1 \cdot v \otimes h_2 \cdot w$
- *H*-coaction on  $V \otimes W$ :  $\lambda(v \otimes w) = v_{[-1]}w_{[-1]} \otimes v_{[0]} \otimes w_{[0]}$
- Braiding:  $c_{V,W}(v \otimes w) = v_{[-1]} \cdot w \otimes v_{[0]}$
- Inverse of the braiding:  $c_{W,V}^{-1}(v\otimes w)=w_{[0]}\otimes S^{-1}(w_{[-1]})\cdot v$

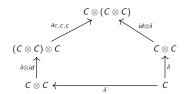
### Monoids and comonoids

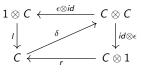
A monoid in a monoidal category  $\mathcal C$  is a triple  $(M,\mu,u)$ , where  $\mu:M\otimes M\to M$  and  $u:1\to M$  are morphisms in  $\mathcal C$  s.t.





A comonoid in a monoidal category  $\mathcal C$  is a triple  $(\mathcal C,\delta,\epsilon)$ , where  $\delta:\mathcal C\to\mathcal C\otimes\mathcal C$  and  $\epsilon:\mathcal C\to 1$  are morphisms in  $\mathcal C$  s.t.





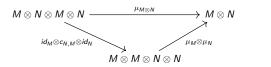
Example: A monoid in  $Vec_k^G$  is a G-graded algebra.

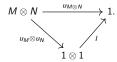
### Tensor product of monoids

Given two monoids  $(M, \mu_M, u_M)$  and  $(N, \mu_N, u_N)$  in a (strict) braided monoidal category C, we can give a monoid structure to the tensor product  $M \otimes N$ :

$$M \underline{\otimes} N = (M \otimes N, \mu_{M \otimes N}, u_{M \otimes N})$$

where  $\mu_{M\otimes N}$  and  $u_{M\otimes N}$  are morphisms defined by the compositions





We call monoids and comonoids in  ${}^H_H\mathcal{YD}$ , respectively, algebras and coalgebras in  ${}^H_H\mathcal{YD}$ .

# Braided Hopf algebras

A bialgebra in  ${}^H_H\mathcal{YD}$  is a collection  $(B, \mu, u, \Delta, \epsilon)$  such that

- $(B, \mu, u)$  is an algebra (monoid) in  ${}^H_H \mathcal{YD}$ ,
- $(B, \Delta, \epsilon)$  is a coalgebra (comonoid) in  ${}^H_H \mathcal{YD}$ ,
- $\Delta: B \to B \underline{\otimes} B$  is a morphism of algebras.

A Hopf algebra in  ${}^H_H\mathcal{YD}$  is a bialgebra B in  ${}^H_H\mathcal{YD}$  such that  $id_B \in Hom_{\mathcal{C}}(B,B)$  has an inverse  $\underline{S}$  w.r.t. the convolution product \*, meaning a morphism  $\underline{S}: B \to B$  such that  $\underline{S}*id_B = id_B*\underline{S} = u \circ \epsilon$ ,

i.e. 
$$\underline{S}(b_1)b_2 = b_1\underline{S}(b_2) = \epsilon(b)u(1_k)$$
 for all  $b \in B$ .

Hopf algebras in braided monoidal categories are usually called braided Hopf algebras.

We will see noan example of a braided Hopf algebra in  ${}^H_H\mathcal{YD}$  that will be central in defining Nichols algebras.

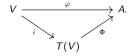
### Tensor algebra

Let  $V \in {}_{H}^{H}\mathcal{YD}$ . Set  $T^{0}(V) = k$  and  $T^{n+1}(V) = V \otimes T^{n}(V)$ . The tensor algebra over V is  $T(V) = \bigoplus_{n \in \mathbb{N}} T^{n}(V)$ , and is a graded algebra in  ${}_{H}^{H}\mathcal{YD}$  with multiplication obtained from the (trivial) associativity constraints on the graded components of  $T(V) \otimes T(V)$ 

$$\mu_{m,n}: T^m(V) \otimes T^n(V) \longrightarrow T^{m+n}(V)$$
$$(v_1 \otimes \ldots \otimes v_m) \otimes (v_{m+1} \otimes \ldots \otimes v_{m+n}) \longmapsto v_1 \otimes \ldots \otimes v_{m+n}$$

and unit  $1_k \in T^0(V) \subseteq T(V)$ 

Let  $i: V = T^1(V) \hookrightarrow T(V)$  be the inclusion. Every linear map  $\varphi: V \to A$ , where A is an algebra, factorizes through a morphism of algebras  $\Phi: T(V) \to A$  s.t.  $\Phi \circ i = \varphi$ .  $(\Phi(v_1 \otimes \ldots \otimes v_n) = \varphi(v_1) \ldots \varphi(v_n))$ 



## Tensor coalgebra 1

By the universal property, the linear map  $\delta: V \to T(V) \underline{\otimes} T(V)$ , given by  $\delta(v) = v \otimes 1_k + 1_k \otimes v$ , extends to  $\Delta: T(V) \to T(V) \underline{\otimes} T(V)$ , which induces a (graded) Hopf algebra structure on T(V) in  ${}^H_H \mathcal{YD}$ , with S(v) = -v.

T(V) can be endowed with another graded coalgebra structure in  ${}^H_H\mathcal{YD}$ , denoted by  $T^c(V)$ , through the map  $\Delta^c: T(V) \to T(V) \underline{\otimes} T(V)$  defined by

 $\Delta^{c}(v_{1}\otimes \ldots \otimes v_{n}) := \sum_{j=0}^{n} (v_{1}\otimes \ldots \otimes v_{j}) \otimes (v_{j+1}\otimes \ldots \otimes v_{n}).$ 

While  $\Delta^c(v) = v \otimes 1_k + 1_k \otimes v$ , we have  $\Delta^c \neq \Delta$  and  $\Delta^c$  is not multiplicative w.r.t. the standard product in T(V), indeed:

$$\Delta^{c}(v \otimes w) = (v \otimes w) \otimes 1_{k} + v \otimes w + 1_{k} \otimes (v \otimes w) \neq \Delta^{c}(v)\Delta^{c}(w)$$
$$= (v \otimes w) \otimes 1_{k} + v \otimes w + 1_{k} \otimes (v \otimes w) + v_{-1} \cdot w \otimes v_{0}$$

### Tensor coalgebra 2

However,  $T^c(V)$  is a braided Hopf algebra when considering the multiplication  $\mu^c: T(V) \underline{\otimes} T(V) \to T(V)$ , defined as

$$(v_1 \otimes \ldots \otimes v_i) \cdot (v_{i+1} \otimes \ldots \otimes v_n) = \sum_{\omega \in \mathbb{S}_{i,n-i}} c_{\omega}(v_1 \otimes \ldots \otimes v_n)$$

where  $\mathbb{S}_{i,n-i}$  denotes (i,n-i) shuffles,  $\omega \in \mathbb{S}_n$  such that  $\omega(1) < \omega(2) < \ldots < \omega(i)$  and  $\omega(i+1) < \omega(i+2) < \ldots < \omega(n)$ 

$$\omega = \left(\begin{array}{ccc|c} 1 & 2 & \dots & i & i+1 & \dots & n-1 & n \\ \omega(1) & \omega(2) & \dots & \omega(i) & \omega(i+1) & \dots & \omega(n-1) & \omega(n) \end{array}\right).$$

To define the morphism  $c_{\omega}: T^n(V) \to T^n(V)$ , we need a small digression.

## Braid group

Let n > 0. The braid group  $\mathbb{B}_n$  is the group generated by  $\sigma_i$ , where  $i \in \{1, \dots, n\}$ , with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i - j| \ge 2$$
  
 $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad \text{for } |i - j| = 1$ 

The symmetric group  $\mathbb{S}_n$  is generated by the simple transpositions  $\tau_i = (i \ i + 1)$ , with relations

$$au_i^2 = e, \quad \text{for all } 1 \le i < n$$
 $au_i au_j = au_j au_i, \quad \text{for } |i - j| \ge 2$ 
 $au_i au_j au_i = au_j au_i au_j, \quad \text{for } |i - j| = 1$ 

The length function  $I: \mathbb{S}_n \to \mathbb{N}$  measures the minimum decomposition of a permutation as a product of simple transpositions.

#### Matsumoto section

By this presentation of  $S_n$ , there is a surjective morphism of groups

$$\pi: \mathbb{B}_n \longrightarrow \mathbb{S}_n: \sigma_i \longmapsto \tau_i.$$

At the level of sets,  $\pi$  admits a section  $M: \mathbb{S}_n \to \mathbb{B}_n$  such that  $\tau_i \mapsto \sigma_i$ , called the Matsumoto section, which is not a group homomorphism. However, for any  $\omega, \omega' \in \mathbb{S}_n$ :

$$M(\omega \circ \omega') = M(\omega)M(\omega'), \text{ if } I(\omega \circ \omega') = I(\omega) + I(\omega')$$

*Example:* Consider  $\omega = (1\ 3\ 4\ 2) \in \mathbb{S}_4$ . A minimal length decomposition of  $\omega$  is  $(1\ 3\ 4\ 2) = (2\ 3)(1\ 2)(3\ 4) = \tau_2\tau_1\tau_3$ . Then

$$M((1\ 3\ 4\ 2)) = M(\tau_2\tau_1\tau_3) = M(\tau_2)M(\tau_1)M(\tau_3) = \sigma_2\sigma_1\sigma_3,$$

since 
$$I(\tau_2\tau_1\tau_3) = 3 = I(\tau_2) + I(\tau_1) + I(\tau_3)$$
.

### Braid group representation

Let  $V \in {}_{H}^{H}\mathcal{YD}$ . We can come back to define  $c_{\omega} : T^{n}(V) \to T^{n}(V)$ . The following assignment defines a representation  $\rho_{n} : \mathbb{B}_{n} \to GL(T^{n}(V))$ 

$$\sigma_{i} \longmapsto c_{i} = id_{\mathcal{T}^{j-1}(V)} \otimes c_{V,V} \otimes id_{\mathcal{T}^{n-j-1}(V)}$$

$$c_{i}(v_{1} \otimes \ldots \otimes v_{n}) = \ldots v_{j-1} \otimes v_{i[-1]} \cdot v_{i+1} \otimes v_{i[0]} \otimes v_{j+2} \ldots$$

Through the Matsumoto section, we can now define  $c_{\omega} := \rho_n(M(\omega))$  for all  $\omega \in \mathbb{S}_n$ . In particular, if  $\omega = \tau_{i_1} \dots \tau_{i_m}$  is a minimal decomposition,  $c_{\omega} = \rho_n(M(\omega)) = \rho_n(M(\tau_{i_1} \dots \tau_{i_m})) = \rho_n(M(\tau_{i_1})) \dots \rho_n(M(\tau_{i_m})) = \rho_n(\sigma_{i_1}) \dots \rho_n(\sigma_{i_m}) = c_{i_1} \dots c_{i_m}$ .

Example: Consider  $\omega = (1 \ 3 \ 4 \ 2) = \tau_2 \tau_1 \tau_3 \in \mathbb{S}_4$ . Then  $c_\omega = \rho_n(M(\omega)) = \rho_n(M(\tau_2 \tau_1 \tau_3)) = \rho_n(\sigma_2)\rho_n(\sigma_1)\rho_n(\sigma_3) = c_2 c_1 c_3$ .

## Nichols algebras 1

Let  $V \in {}^H_H \mathcal{YD}$ . Consider T(V) and  $T^c(V)$  in  ${}^H_H \mathcal{YD}$  and the inclusion  $V \hookrightarrow T^c(V)$ ; by the universal property of T(V), there exist an algebra map  $\Omega: T(V) \to T^c(V)$  s.t.  $\Omega(v) = v$  for all  $v \in V$ .

The map has additionally the following properties:

- $\Omega = \sum_{n} \Omega_n$  is a graded map, with  $\Omega_n = \Omega_{|T^n(V)}$ ,
- $\Omega$  is a coalgebra morphism between T(V) and  $T^c(V)$ ,
- $\Omega$  is a morphism in  ${}_{H}^{H}\mathcal{YD}$ .

# Nichols algebras 2

#### **Definition**

The Nichols algebra  $\mathcal{B}(V)$  is the image of the map  $\Omega$  in  $T^c(V)$ .

If 
$$I(V) := \ker \Omega$$
 and  $I^n(V) := \ker \Omega_n$ , then  $I(V) = \bigoplus_{n \geq 2} I^n(V)$ .

Then 
$$\mathcal{B}(V) = \bigoplus_{n \geq 0} \mathcal{B}^n(V) \simeq \frac{T(V)}{I(V)}$$
, where  $\mathcal{B}^n(V) \simeq \frac{T^n(V)}{I^n(V)}$ 

By induction, one can characterize the graded components of  $\boldsymbol{\Omega}$  as follows.

### Proposition

For all 
$$n \ge 2$$

$$\Omega_n = \sum_{\omega \in \mathbb{S}_n} c_\omega : \mathcal{T}^n(V) o \mathcal{T}^n(V)$$

#### Example:

- As we have seen before,  $\Omega_2 = id + c_1$ .
- $\Omega_3 = id + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1$ .

## Alternative characterizations of Nichols algebras

### Proposition

I(V) is the maximal element in the set

$${J\subseteq\bigoplus_{n\geq 2}T^n(V)\mid}$$

J is a graded Hopf ideal and a submodule of T(V) in  ${}_H^H \mathcal{YD}$   $\}$ .

### Proposition

A graded Hopf algebra B in  ${}^H_H\mathcal{YD}$  is isomorphic to  $\mathcal{B}(V)$  if and only if:

- B is generated as an algebra by V.
- B is coradically graded, meaning the coradical filtration and the filtration induced by the grading coincide.

Proofs for both can be found in section 2 of <sup>1</sup> or section 1.6 of <sup>2</sup>.

<sup>&</sup>lt;sup>1</sup>N. Andruskiewitsch, H.-J. Schneider, *Pointed Hopf algebras*, (2002).

<sup>&</sup>lt;sup>2</sup>I. Heckenberger, H.-J. Schneider, "Hopf algebras and root systems", 2020.

## Examples 1

Consider a vector space  $V \in {}^H_H \mathcal{YD}$  with trivial H-action and coaction, meaning  $h \cdot v = \epsilon_H(h)v$  and  $\lambda(v) = 1_H \otimes v$ . Then  $c_{V,V} = \tau$  and

$$\Omega_n(v_1 \otimes \ldots \otimes v_n) = \sum_{\omega \in \mathbb{S}_n} v_{\omega^{-1}(1)} \otimes \ldots \otimes v_{\omega^{-1}(n)}.$$

In particular,  $\Omega_2(v_1 \otimes v_2) = v_1 \otimes v_2 + v_2 \otimes v_1$ , therefore  $\ker \Omega_2 = \{v \otimes w - w \otimes v \mid v, w \in V\}$ . However, it can be proven that, in this case,  $\ker \Omega_n = \langle \ker \Omega_2 \rangle \cap T^n(V)$ , and so

$$I(V) = \ker \Omega = < v \otimes w - w \otimes v \mid v, w \in V > .$$

Therefore

$$\mathcal{B}(V) \simeq \frac{T(V)}{\langle v \otimes w - w \otimes v \mid v, w \in V \rangle} = \mathit{Sym}(V).$$

## Examples 2

Let  $\bar{V} \in {}_{k[C_2]}^{k[C_2]} \mathcal{YD}$  s.t.  $g \cdot \bar{v} = -\bar{v}$  and  $\lambda(\bar{v}) = g \otimes \bar{v}$ , therefore  $c_{V,V} = -\tau$ . Then

$$\Omega_n(\bar{v}_1 \otimes \ldots \otimes \bar{v}_n) = \sum_{\omega \in \mathbb{S}_n} (-1)^{l(\omega)} \bar{v}_{\omega^{-1}(1)} \otimes \ldots \otimes \bar{v}_{\omega^{-1}(n)}.$$

In particular  $\Omega_2(v \otimes w) = v \otimes w - w \otimes v$ , so  $\ker \Omega_2 = \{v \otimes w + w \otimes v\}$ . As before,  $\ker \Omega = \langle \ker \Omega_2 \rangle = \langle v \otimes w + w \otimes v \mid v, w \in \overline{V} \rangle$ , therfore

$$\mathcal{B}(\bar{V}) \simeq \frac{T(\bar{V})}{< v \otimes w + w \otimes v \mid v, w \in \bar{V} >} = \Lambda(\bar{V}).$$

Let now  $V_0, V_1 \in {}_{k[C_2]}^{k[C_2]}\mathcal{YD}$ , such that  $g \cdot v_i = (-1)^i v_i$  and  $\lambda(v_i) = g^i \otimes v_i$ . Then  $V = V_0 + V_1 \in {}_{k[C_2]}^{k[C_2]}\mathcal{YD}$  is a super vector space, with braiding  $c_{V,V}(v_i \otimes v_j) = (-1)^{ij}(v_j \otimes v_i)$ . In this case  $\mathcal{B}(V) \simeq \mathit{Sym}(V_0) \otimes \Lambda(V_1)$ .

## Bosonization/Radford's biproduct

We denote by H-BialProj the category of bialgebras with a projection:

- Objects: triples  $(A, i, \pi)$ , where A is a bialgebra and  $i : H \hookrightarrow A$  and  $\pi : A \to H$  are bialgebra morphisms such that  $id_A = i \circ \pi$ .
- Morphisms:  $f:(A,i,\pi)\to (A',i',\pi')$ , bialgebra maps  $f:A\to A'$  such that  $f\circ i=i'$  and  $\pi'\circ f=\pi$ .

To a bialgebra with a projection  $(A, i, \pi)$  one can associate a braided bialgebra structure on  $B = A^{co(H)}$ , where the right coaction on A is given by  $\rho(a) = a_1 \otimes \pi(a_2)$ , and the structure maps are induced by those of A and by i and  $\pi$ .

$$\begin{array}{c} \mathsf{Bialg}(^H_{\mathcal{H}}\mathcal{YD}) & \xrightarrow{\mathsf{biproduct}} & H\text{-BialProj} \\ & \mathsf{construction} \\ & (\mathsf{bosonization}) \end{array}$$

## Biproduct Hopf algebras

Indeed, we can associate to each Hopf algebra B in  ${}^H_H\mathcal{YD}$  a Hopf algebra  $A=B\times H$  with a projection on H.

The structure of the biproduct  $B \times H$  is as follows:

$$\begin{split} B\times H &= B\otimes H \text{ (as a vector space)},\\ (b\times h)(b'\times h') &= b\left(h_1\cdot b'\right)\times h_2h',\\ \Delta(b\times h) &= b_1\times (b_2)_{[-1]}h_1\otimes (b_2)_{[0]}\times h_2\quad,\\ \epsilon(h\times b) &= \epsilon_B(h)\epsilon_H(h),\quad 1_{B\times H} = 1_B\times 1_H,\\ S(b\times h) &= (1\times S_H(b_{-1}h))(S_B(b_0)\times 1_H). \end{split}$$

Moreover, the injection and projection on  $B \times H$  are as follows:

$$i: H \hookrightarrow B \times H : h \mapsto 1 \times h$$
  
 $\pi: B \times H \rightarrow H : b \times h \mapsto \epsilon_B(b)h$ 

Via this structure, we can associate a Hopf algebra  $\mathcal{B}(V) \times H$  to each Nichols algebra  $\mathcal{B}(V)$  in  ${}^H_H \mathcal{YD}$ .

## Examples of biproducts

One can think of the biproduct construction as a generalization of the semidirect product for groups. Consider, for instance,  $\mathbb{S}_3 \simeq C_3 \rtimes C_2$ ; we have  $k[\mathbb{S}_3] \simeq k[C_3] \times k[C_2]$ , where  $k[C_3]$  is seen as a Hopf algebra in  $k[C_2] \mathcal{YD}$  with trivial coaction and  $g \cdot x = x^2$ , where  $< x >= C^3$ .

The simplest non-commutative non-cocommutative Hopf algebra is a biproduct; indeed

$$H_4 = \bar{B} \times k[C_2],$$

where  $\bar{B}=k<1,\ n>$  is a 2-dimensional braided Hopf algebra in  ${}^{k[C_2]}_{k[C_2]}\mathcal{YD}$ , generated by the unit and a nilpotent primitive element n with  $\lambda(n)=g\otimes n$  and  $g\cdot n=-g$ .

## Nichols and pointed Hopf algebras

We conclude with a brief explenation of the role of Nichols algebras in the classification of pointed Hopf algebras.

Let A be a pointed Hopf algebra, meaning all simple A-comodules are 1-dimensional, with coradical filtration  $\{A_i\}_{i\in\mathbb{N}}$ . The graded Hopf algebra associated to the coradical filtration, fixing  $A_{-1}=\{0\}$ , is

$$gr(A) := \bigoplus_{n \in \mathbb{N}} A_n/A_{n-1}.$$

gr(A) is a Hopf algebra with a projection on  $A_0 = k[G(A)]$ . Therefore,

$$gr(A) \simeq \mathcal{R} \times k[G(A)],$$

where  $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}^n$  is a coradically graded Hopf algebra in  ${}^{A_0}_{A_0}\mathcal{YD}$ . By setting  $V = \mathcal{R}^1$ , the algebra generated by V is isomorphic to the Nichols algebra  $\mathcal{B}(V)$  and a subalgebra of  $\mathcal{R}$ .

# References and further reading

#### Nichols algebras:

- N. Andruskiewitsch, An introduction to Nichols algebras, Quantization, Geometry and Noncommutative Structures in Mathematics and Physics. Springer, 135–195 (2017)
- I. Heckenberger, H.-J. Schneider, "Hopf algebras and root systems", *American Mathematical Soc.* **247**, 2020.
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#### Biproduct/Bosonization:

- D.E. Radford, "Hopf algebras", Series on Knots and Everything, University of Illinois at Chicago 2011.
- D.E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **92**, 322–347, (1985).