Free biproduct quasi-Hopf algebras of rank 2

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The content of the work

- We obtain the structure of biproduct of rank 2 as free modules over a quasi-Hopf algebra H.
- This is achieved by considering 2-dimensional Hopf algebras in the category of left Yetter-Drinfeld modules over H (For the classical case, see Radford[1]).
- These structures are exemplified in the case when $H = k\left[C_n\right]$, seen as a quasi-Hopf through the use of its non-trivial 3-cocycles.
  Concrete examples are given for $n = 4$.

[1] Biproducts and Kashina's Examples, Commun. Algebra 44, (2015).

Projections, biproducts and Yetter-Drinfeld modules

- Hopf algebras with a projection where introduced by Radford[2]. Up to an isomorphism, B is a biproduct $A \times H$, where A is an Hopf algebra in the Yetter-Drinfeld module category ${}_H^H\mathcal{YD}$.
- It was proven by Bulacu[3] that this holds even when H is a quasi-Hopf algebra.

[2] The structure of Hopf algebras with a projection. J. Algebra 92 (1985).

[3] A structure theorem for quasi-Hopf bimodule coalgebras, Theory Appl. Categ. 32 (2017).

Preliminaries: Quasi-bialgebra

A quasi-bialgebra is a quadruple $(H, \Delta, \epsilon, \Phi)$, where H is an algebra and Φ is an invertible element in $H \otimes H \otimes H$, called reassociator, and $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ are algebra morphism satisfying

$$(Id_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes Id_H)(\Delta(h))\Phi^{-1},$$
$$(Id_H \otimes \epsilon)(\Delta(h)) = h \text{ and}$$
$$(\epsilon \otimes Id_H)(\Delta(h)) = h.$$

The reassociator Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(Id_H \otimes \Delta \otimes Id_H)(\Phi)(\Phi \otimes 1)$$
$$= (Id_H \otimes Id_H \otimes \Delta)(\Phi)(\Delta \otimes Id_H \otimes Id_H)(\Phi),$$
$$(Id_H \otimes \epsilon \otimes Id_H)(\Phi) = 1 \otimes 1.$$

The last two equalities imply $(\epsilon \otimes Id_H \otimes Id_H)(\Phi) = 1 \otimes 1$ and $(Id_H \otimes Id_H \otimes \epsilon)(\Phi) = 1 \otimes 1$.

Preliminaries: Quasi-Hopf algebra

- A quasi-bialgebra H is a quasi-Hopf algebra if there exists an algebra antimorphism $S : H \to H$ and distinguished elements $\alpha, \beta \in H$ such that, for all $h \in H$, the following hold:

$$S(h_1)\alpha h_2 = \epsilon(h)\alpha,$$
$$h_1\beta S(h_2) = \epsilon(h)\beta,$$

$$X^1\beta S(X^2)\alpha X^3 = 1,$$
$$S(x^1)\alpha x^2\beta S(x^3) = 1.$$

- Let H be a quasi-Hopf algebra with reassociator Φ and antipode (S, α, β). $H' \subseteq H$ is a sub-quasi-Hopf algebra of H if it is a quasi-Hopf algebra with reassociator Φ, distinguished elements α, β and structure maps induced by those of H.

Preliminaries: Notation and conventions

- The following Sweedler-like notation is used for quasi-Hopf algebras and their reassociator:

$$(\Delta \otimes Id_H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2,$$
$$(Id_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)}.$$

$$\Phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3 = \dots$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = y^1 \otimes y^2 \otimes y^3 = z^1 \otimes z^2 \otimes z^3 = \dots$$

- We consider all (quasi-)Hopf algebras to have invertible antipode.
- We take all fields with characteristic different from 2.
- By Yetter-Drinfeld modules over H, we mean left Yetter-Drinfeld modules ${}_H^H\mathcal{YD}$.

Quasi-Hopf case: Braided Hopf algebras in ${}_H^H\mathcal{YD}$

Let now H be a quasi-Hopf algebra over a field k.
A bialgebra B in ${}_H^H\mathcal{YD}$ follows the definition of bialgebras in braided categories. Therefore:

1. B is subject to the compatibility relations of left H-module algebra and coalgebra.
2. Since the monoidal category ${}_H^H\mathcal{YD}$ is not strict, the product and coproduct of B are associative and coassociative only up to the associativity constraint in ${}_H^H\mathcal{YD}$ on B, given by

$$a_{B,B,B} : (b \otimes b') \otimes b'' \mapsto X^1 \cdot b \otimes \left[(X^2 \cdot b') \otimes (X^3 \cdot b'')\right],$$

where $X^1 \otimes X^2 \otimes X^3$ is the reassociator of H.

Quasi-Hopf case: Braided Hopf algebras in ${}_H^H\mathcal{YD}$

3. B is also subject to relations analogous to those holding for comodule algebras and coalgebras, where the reassociator of H plays an heavy role.
4. The coproduct and the counit are required to be algebra morphisms, where the product on $B \otimes B$ is given by $m_{B\otimes B}$ defined through the associativity constraint.
5. If it exists, the antipode S of B has to be a morphism in the category.

Braided Hopf algebras of dimension 2

These relations result in the following structure for a 2-dimensional bialgebra B in ${}^H_H\mathcal{YD}$. For each B, one of the following holds:

1. B has trivial H-(co)action. It is generated by $S = \{1, s\}$ as an algebra, with s grouplike. Moreover:
   - $B \cong k[S]$, if $s^2 = s$. (not an Hopf algebra)
   - $B \cong k[C_2]$, if $s^2 = 1$. (Hopf algebra)
2. $B \cong B_{\alpha,y}$, the Hopf algebra generated by $\{1, n\}$ with unit 1 and n primitive element s.t. $n^2 = 0$, with H-module and comodule structures determined, respectively, by $\alpha \in \mathrm{Alg}_k(H, k)$ and $y \in H$ as follows:

$$h \cdot n = \alpha(h)n, \quad \rho(n) = y \otimes n$$

satisfying compatibility relations

$$\alpha(y) = -1, \quad \alpha(h_2)h_1 y = \alpha(h_1)yh_2, \quad \epsilon_H(y) = 1,$$
$$\Delta_H(y) = \alpha(y^1 x^3 X^2)(x^1 \otimes x^2)(X^1 y \otimes yX^3)(y^2 \otimes y^3).$$

Braided Hopf algebras of dimension 2

Explicitly, the structure of non-trivial 2-dimensional braided Hopf algebras $B_{\alpha,y}$ in ${}^H_H\mathcal{YD}$ is as follows.

- As a k-algebra, it is generated by the unit 1 and the nilpotent element n of degree 2.
- Its coalgebra structure is determined by

$$\begin{aligned} \Delta(n) &= 1 \otimes n + n \otimes 1, & \Delta(1) &= 1 \otimes 1, \\ \epsilon(n) &= 0, & \epsilon(1) &= 1_k. \end{aligned}$$

- The antipode is determined by $S(n) = -n$ and $S(1) = 1$.
- The Yetter-Drinfeld H-module structure is determined by

$$\begin{aligned} h \cdot n &= \alpha(h)n, & \rho(n) &= y \otimes n, \\ h \cdot 1 &= \epsilon(h)1, & \rho(1) &= 1 \otimes 1. \end{aligned}$$

Note that $(B_{\alpha,y}, \Delta, \epsilon, S)$, as defined above, is also an Hopf algebra in the usual sense.

Remark on braided Hopf algebras of dimension 2

- The difference between the classical and quasi-Hopf case results in the condition on $B_{\alpha,y}$:

$$\Delta_H(y) = \alpha(y^1x^3X^2)(x^1 \otimes x^2)(X^1y \otimes yX^3)(y^2 \otimes y^3).$$

- When H is a commutative quasi-Hopf algebra with reassociator such that $x^1 \otimes x^2 \otimes x^3 = x^1 \otimes x^3 \otimes x^2$, the condition becomes much simpler:

$$\Delta_H(y) = \alpha(y^1)(y^2 \otimes y^3)(y \otimes y).$$

- The difference between cases is substantial; even in the first non-trivial example, given by $H = k\,[C_2]$ seen as a quasi-Hopf algebra with reassociator $\Phi = 1_H \otimes 1_H \otimes 1_H - 2p_- \otimes p_- \otimes p_-$, where $p_- = \frac{1}{2}(1_H - g)$, there are no nontrivial 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$, while in the classical case, i.e. in ${}_{k[C_2]}^{k[C_2]}\mathcal{YD}$, there is one, namely $B_{\tilde{\alpha},g}$ where $\tilde{\alpha} \in Alg_k(H,k)$ is determined by $\tilde{\alpha}(g) = -1_k$.

Biproduct quasi-Hopf algebras

Quasi-Hopf algebras with a projection on H are obtained, through the biproduct construction, from Hopf algebras B in ${}_H^H\mathcal{YD}$. The structure [4] of the biproduct $B \times H$ is as follows:

$$B \times H = B \otimes H \text{ (as a vector space)}$$

$$(b \times h)(b' \times h') = (x^1 \cdot b)(x^2 h_1 \cdot b') \times x^3 h_2 h'$$

$$\begin{aligned}\Delta(b \times h) =& y^1 X^1 \cdot b_1 \times y^2 Y^1 (x^1 X^2 \cdot b_2)_{-1} x^2 X_1^3 h_1 \\ & \otimes y_1^3 Y^2 \cdot (x^1 X^2 \cdot b_2)_0 \times y_2^3 Y^3 x^3 X_2^3 h_2\end{aligned}$$

$$\epsilon(h \times b) = \epsilon_B(h)\epsilon_H(h)$$

$$\Phi_{B \times H} = 1 \times X^1 \otimes 1 \times X^2 \otimes 1 \times X^3$$

$$S(b \times h) = (1 \times S_H(X^1 x_1^1 b_{-1} h)\alpha_H)(X^2 x_2^1 \cdot S_B(b_0) \times X^3 x^2 \beta_H S_H(x^3))$$

with distinguished elements: $\alpha_{B \times H} = 1 \times \alpha_H, \quad \beta_{B \times H} = 1 \times \beta_H.$

[4] D.Bulacu, E.Nauwelaerts, Radford's biproduct for quasi-Hopf algebras and bosonization, J. Pure Appl. Algebra 174, 2002

Biproduct quasi-Hopf algebras

- By identifying H with the image of $i : h \mapsto 1 \times h$, we get, from the structure maps seen before, that H is a sub-quasi-Hopf algebra of $B \times H$.
- If B is finite dimensional with $\dim(B) = n$, $B \times H$ is a right free H-module of rank n.
- The projection on H is determined by $b \times h \mapsto \epsilon_B(b)h$.

Biproduct quasi-Hopf algebras of rank 2: Trivial case

From the characterization of 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$, we obtain the structure of biproduct quasi-Hopf algebras of rank 2. We have two cases.

Case 1: Let $B = k[C_2]$, with trivial Yetter-Drinfeld H-module structures.

- We identify H with $i(H)$.
- We get that $k[C_2] \times H$ is the algebra $k[C_2] \otimes H$.
- As a quasi-Hopf algebra, its structure maps are determined in the usual way, with reassociator and distinguished elements given by

$$1 \otimes \Phi_H, \quad 1 \otimes \alpha_H, \quad 1 \otimes \beta_H.$$

Biproduct quasi-Hopf algebras of rank 2: Non-trivial case

Case 2: Let now $B = B_{\alpha,y}$, for an admissible pair $\alpha \in \mathrm{Alg}_k(H,k)$, $y \in H$.

- Let $x = n \times 1$ and identify $h = 1 \times h$.
- We obtain that $B_{\alpha,y} \times H$ is generated, as an algebra, by x and H, with relations

$$x^2 = 0, \quad hx = \alpha(h_1)xh_2.$$

- As a quasi-Hopf algebra, $B_{\alpha,y} \times H$ has H as a sub-quasi-Hopf algebra, and its structure maps are determined by their value on x. Such values are:

$$\Delta(x) = \alpha(x^1Y^2)\ Y^1\, y\, x^2 \otimes x\, Y^3x^3 + \alpha(y^1)\ x\, y^2 \otimes y^3,$$

which implies $\epsilon(x) = 0$,

$$S(x) = -\,\alpha(X^2x^1_2)\ S_H(y)S_H(X^1x^1_1)\alpha_H\, x\, X^3x_2\beta_H S_H(x^3).$$

Cyclic group algebra, classical case

Fix $n \geq 2$ and let k be a field with primitive n^2-th roots of the unit, let g be a generator of C_n.
To exemplify the previous structures, we analyze the case $H = k[C_n]$, seen, at first, as an Hopf algebra.

- All 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$ are isomorphic to one of the following:
  - $k[C_2]$, with trivial H-(co)module structures,
  - B_{α_ω, g^z}, where ω is an n-th root of the unit, $1 \leq z \leq n-1$ s.t. $\omega^z = -1_k$ and $\alpha_\omega \in \mathrm{Alg}_k(H, k)$ is determined by $\alpha_\omega(g) = \omega$.
- An example of such ω and z is given, for the case $k = \mathbb{C}$ and $n = 4$, by the pairs $(i, 2)$, $(-1, 1)$ and $(-1, 3)$.

Cyclic group algebra, classical case

- All biproduct Hopf algebras of rank 2 with a projection over $k[C_n]$ are isomorphic to one of the following
  - $k[C_2 \times C_n] \cong k[C_2] \otimes k[C_n]$,
  - $B_{\alpha_\omega, g^z} \times H$, generated as an algebra by x and g with relations

$$x^2 = 0, \quad gx = \omega xg, \quad g^n = 1$$

having H as a sub-Hopf algebra, with structure maps determined by

$$\Delta(x) = g^z \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -g^{-z}x.$$

Cyclic group algebra, primitive root reassociator

Given a primitive n-th root of the unit $\mathfrak{q}$, $k[C_n]$ can be seen as a quasi-Hopf algebra[5] H with reassociator

$$\Phi = \sum_{i,j,l=0}^{n-1} \mathfrak{q}^{i\lfloor \frac{j+l}{n} \rfloor} 1_i \otimes 1_j \otimes 1_l$$

and distinguished elements $\alpha_H = g$ and $\beta_H = 1$.
Here 1_j are the orthogonal idempotents of $k[C_n]$, defined by

$$1_j = \frac{1}{n} \sum_{t=0}^{n-1} \mathfrak{q}^{(n-t)j} g^j .$$

We proved that there are no non-trivial 2-dimensional Hopf algebras in ${}_H^H\mathcal{YD}$.

[5] S.Gelaki, Basic quasi-Hopf algebras of dimension n^3, J. Pure Appl. Algebra 198, (2005).

Cyclic group algebra, non-primitive root reassociator

If we fix a non-primitive n-th root of the unit q such that $q = \mathfrak{q}^{\frac{n}{m}}$ and $m \neq 1$, we have that $H = k\left[C_n\right]$ can be seen as a quasi-Hopf algebra with reassociator

$$\Phi_{\frac{n}{m}} = \sum_{i,j,l=0}^{n-1} q^{i\left\lfloor \frac{j+l}{n} \right\rfloor} 1_i \otimes 1_j \otimes 1_l$$

and distinguished elements $\alpha_H = g^{\frac{n}{m}}$ and $\beta_H = 1$.
All 2-dimensional Hopf algebras in ${}^H_H\mathcal{YD}$ are isomorphic to one of the following:

- The usual trivial object $k\left[C_2\right]$,
- The Hopf algebra B_{α_r, g^z}, where $1 \leq z \leq n-1$ is such that $\mathfrak{q}^{z^2 m} = -1_k$, r is the reminder of zm modulo n and α_r is determined by $\alpha_r(g) = \mathfrak{q}^r = \mathfrak{q}^{zm}$.

Cyclic group algebra, non-primitive root reassociator

- Then, fixing as H the quasi-Hopf algebra obtained by $k[C_n]$, the primitive root $\mathfrak{q}$ and an integer $m \mid n$, where $m \neq 1$,
- all biproduct over H of rank 2 are isomorphic to either $k[C_2 \times C_n]$ or $H_{\mathfrak{q},m,z}$.
- $H_{\mathfrak{q},m,z}$ is the quasi-Hopf algebra generated by x and g with relations

$$x^2 = 0, \quad gx = \mathfrak{q}^r xg, \quad g^n = 1,$$

such that H is a sub-quasi-Hopf algebra and the structure maps are determined by

$$\Delta(x) = \sum_{j,k=0}^{n-1} q^{j\lfloor \frac{r+k}{n} \rfloor} g^z 1_j \otimes x 1_k + x \otimes 1_H$$

$$S(x) = -\sum_{j=0}^{n-1} q^{-j\lfloor \frac{r+j}{n} \rfloor} 1_{(r+j)'} g^{-z} x$$

Cyclic group algebra, concrete example

- Consider $\mathbb{C}[C_4]$, seen as a quasi-Hopf algebra with the reassociator constructed using $\mathfrak{q} = i$ and $m = 2$.
- The non-trivial 2-dimensional Hopf algebras in ${}^{\mathbb{C}[C_4]}_{\mathbb{C}[C_4]}\mathcal{YD}$ are only, up to isomorphism, $B_{\alpha_2,g}$ and B_{α_2,g^3}.
- The biproducts of rank 2 over $\mathbb{C}[C_4]$, not isomorphic to a group algebra, are $H_{i,2,1}$ and $H_{i,2,3}$,
- The structure of $H_{i,2,1}$ summarized is

$$x^2 = 0, \quad gx = -xg, \quad g^4 = 1,$$

$$\Delta(x) = \sum_{j,k=0}^{3} (-1)^{j\lfloor \frac{2+k}{4} \rfloor}\, g1_j \otimes x1_k + x \otimes 1$$

$$S(x) = -\sum_{j=0}^{3} (-1)^{j\lfloor \frac{2+j}{4} \rfloor}\, 1_{(j+2)'} g^3 x$$

Conclusion: Possible ideas for further research

- The structure of $H_{\mathrm{q},m,z}$ can be studied further.
- So far, we know that $H_{\mathrm{q},m,z}$ is always not quasi-triangular, this is not the case for its classical counterpart.
- For the Klein group, the normalized 3-cocycles, and therefore the quasi-Hopf algebra structures over its group algebra, are well known[6].
- This could be used to produce new examples for the structures presented today.

[6] HL.Huang, G.Liu, Y.Ye, The Braided Monoidal Structures on a Class of Linear Gr-Categories, Algebr Represent Theor 17, (2014),

D.Bulacu, S.Caenepeel, B.Torrecillas, The braided monoidal structures on the category of vector spaces graded by the Klein group, Proceedings of the Edinburgh Mathematical Society 54 (2011).

Thank You!