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On the classification of equivariantly $\mathcal{O}_2$-stable amenable actions of locally compact groups

Matteo Pagliero

Dissertation presented in partial fulfillment
of the requirements for the degree of
Doctor of Science (PhD): Mathematics

Supervisor:
Prof. dr. Gábor Szabó

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Preface

Over the past few years, I have spent more time with operator algebras than I have with most of my family and friends. What lies on these pages is the proud product of a long and winding journey: rewarding, perplexing, and at times absurd, and though I shaped the words, I never walked the path alone.

First and foremost, I am deeply grateful to my supervisor, Gábor, for helping me navigate the thickets of abstract mathematics. Your patience, sharp insights, and steady presence kept this project from derailing more than once. You managed to make difficult things seem just slightly less impossible, which is perhaps the quintessential academic magic trick of all. I am also especially thankful to you for the trust you placed in me, not just in our joint research and the writing of our papers, but also in encouraging me to take on mentoring responsibilities with your Master's students. Your confidence in my abilities, often greater than my own, pushed me to grow in ways I had not anticipated. Those experiences were invaluable, and I learned as much from them as I did from any formal part of my training.

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Alla mia famiglia: Susanna, Maurilio, Sara, Davide & Giulia, Charlie, Patti, e a tutti i cugini, zie, zii, nonne e nonni. Non ci sono abbastanza parole per ringraziarvi davvero. Grazie per il vostro amore incondizionato, per la pazienza infinita, e per quella fiducia incrollabile in me che avete avuto anche quando io stesso vacillavo. So che quando cercavo di spiegarvi di cosa mi occupassi, le vostre espressioni oscillavano tra il perplesso e il diplomaticamente incoraggiante (e vi giuro che ho apprezzato ogni sorriso annuente). Anche se forse non era sempre chiarissimo cosa stessi facendo, è sempre stato chiarissimo che c'eravate e questo ha fatto tutta la differenza. Avete saputo sostenermi nei momenti bui, nei periodi in cui sembrava non finire mai, e avete festeggiato con me anche le più piccole conquiste, come se fossero traguardi epocali. Per questo, vorrei dedicare a voi il mio lavoro di dottorato. In un certo senso questo risultato è anche vostro.

Aan mijn vriendin, Marta. Dankjewel voor je eindeloze liefde, voor je rust

wanneer ik gestresseerd was, en voor je lach wanneer ik die het meest nodig had. Bedankt dat je begreep dat ik soms niet echt aan het "werken" was, maar gewoon naar een blad staarde in de hoop dat de wiskunde zich door goddelijke interventie zou openbaren. Bedankt om te luisteren naar mijn klaagzangen over groepen en symmetrieën en heel overtuigend te doen alsof het allemaal fascinerend is. En ten slotte, bedankt om me te introduceren in het rotsklimmen, een passie die mijn verticale ontsnapping uit de wiskunde is geworden. Tussen de ene en de andere muur heb je me geholpen om evenwicht, kracht en een mooie manier te vinden om mijn hoofd leeg te maken (gelukkig zonder al te vaak te vallen). Samen met Laura, Philippe, Federica, Céline & Axel, Sophie en Mira hebben jullie me allemaal zo hartelijk in jullie familie verwelkomd, en daar ben ik jullie enorm dankbaar voor. Bedankt voor jullie vriendelijkheid en generositeit om jullie huis voor me open te stellen, me uit te nodigen voor het weekend en me te betrekken bij jullie activiteiten. Tijd samen doorbrengen heeft me echt het gevoel gegeven deel uit te maken van de familie, en ik koester de momenten die we hebben gedeeld. Jullie warmte en gastvrijheid betekenen veel voor mij, en ik kijk uit naar nog veel meer prachtige tijden samen. Tot slot is één ding heel duidelijk geworden, Marta: deze PhD mag dan wel afgerond zijn, maar aan de andere kant, *we zijn nog maar begonnen*.

Abstract

This thesis investigates the classification and structure of group actions on C^*-algebras, with a focus on equivariant $\mathcal{O}_2$-stability, topological dynamics on primitive ideal spaces, and induced actions on corona algebras.

In the first part of the thesis, we establish classification results — up to cocycle conjugacy — for equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing, amenable actions of second-countable, locally compact groups on separable, nuclear C^*-algebras that are either stable or unital. The classification invariant in this result is the topological dynamical system induced on the primitive ideal space of the C^*-algebra. This result represents a novel contribution, offering the best analogue of the Gabe–Kirchberg $\mathcal{O}_2$-stable classification theorem.

In the second part, we establish a range-of-invariant result corresponding to the aforementioned classification theorem. Specifically, we show that every topological dynamical system on the primitive ideal space of a separable, nuclear C^*-algebra arises as the induced system of a C^*-action.

In the final part, we study dynamics on corona algebras, and use it to characterise absorption of certain strongly self-absorbing C^*-dynamical systems. As a key step towards this result, we prove that the corona algebra, when equipped with the induced algebraic action, possesses a certain dynamical lifting property, which is of independent interest.

Contents

Chapter 1

Introduction

This manuscript reports on the research carried out by the author during his PhD at KU Leuven. The main topic of the thesis is the classification of a certain class of group actions on C^*-algebras, and first appeared in [156], which was written in collaboration with Gábor Szabó. The range of the invariant used in the aforementioned classification theorem is also fully determined in this thesis, and was published in the author's article [155]. The last objective addressed in this manuscript concerns dynamical properties of corona algebras, and builds new connections with the theory of strongly self-absorbing C^*-dynamics. This part of the work was recently made available as a research article [134] co-authored with Xiuyuan Li and Gábor Szabó. This introduction serves as a brief, and certainly not exhaustive, overview of the state of the art in the classification theory of C^*-algebras and C^*-dynamics. Instead of starting the exposition in medias res, we begin with a short historical account. However, this does not mean that the discussion that follows proceeds in chronological order. We stress that parts of this introduction are contained in the introduction of the aforementioned articles.

C^*-algebras and their classification

The study of operator algebras emerged in the early 20^{th} century as part of a broader effort to unfold the interactions between three influential developments: *Hilbert space theory*, the *theory of group representations*, and the emerging mathematical descriptions of *quantum mechanics*. In particular, the seminal works of Schrödinger and Heisenberg about the mathematical formulation of quantum mechanics, called *wave mechanics* and *matrix mechanics* at the time, were pursued by von Neumann, who gave a firm mathematical footing to model quantum mechanics. The first operator algebras were initially studied under the name *rings of operators* in a series of articles by Murray and von Neumann [146–148, 151], which represents one of the most impactful mathematical pieces of work from the last century. Today, these are referred to as *von Neumann algebras*. Non-commutativity of so-called observables, such as position and momentum, is one of the key points of quantum theory, as for example it provides a straightforward explanation for Heisenberg's uncertainty principle. This lack of commutativity between two operators a and b is measured by the *commutator* $[a, b] = a \cdot b - b \cdot a$, and when this vanishes for every pair of operators, the ambient von Neumann algebra is *commutative*, and can be identified with $L^\infty(X, \mu)$ for some measure space (X, μ). For this reason, one may think of von Neumann algebras as *non-commutative measure spaces*.

In the 1940s, Gelfand and Naimark [77, 78] developed the theory of C*-*algebras* under the name *normed rings with involution*, as an abstract framework that does not necessarily assume that elements are operators on some Hilbert space. One of the foundational results of Gelfand and Naimark, which was later refined by Segal [177], is the fact that C*-algebras can always be faithfully represented as C*-subalgebras of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, which relates *abstract* objects to *concrete* ones. Just like von Neumann algebras can be thought of as non-commutative measure spaces, the theory of C*-algebras can be interpreted as *non-commutative topology*. In fact, for every commutative C*-algebra A there exists a locally compact Hausdorff space X such that A is naturally isomorphic to the C*-algebra $\mathcal{C}_0(X)$ of continuous functions on X that vanish at infinity.

A prime objective in the field of operator algebras is to understand their *fine structure*, and learn to distiguish them up to isomorphism by pursuing *classification* results. More precisely, the goal is to identify a collection of invariants — typically algebraic or topological in nature — that can distinguish C*-algebras up to isomorphism. A successful approach to the classification of von Neumann algebras was started early on by Murray and von Neumann, who identified separate classes, called *types*, of indecomposable objects, called *factors*. Arguably one of the greatest breakthroughs in this direction, Connes and Haagerup's classification of *injective* factors [38, 89] gave new impetus to the field, and served as a model approach in the neighbouring classification theory of C*-algebras. *Simple* C*-*algebras*, namely those that only admit trivial norm-closed two-sided ideals, are the right analogue of factors in von Neumann algebra theory. However, it is impossible to reduce the study of C*-algebras to simple ones in a straightforward manner. Nonetheless, the classification of simple C*-algebras that are moreover *separable* and *nuclear* — the appropriate analogue of injectivity — has emerged as an interesting and ambitious goal since the 1970s, when first results in this direction were obtained by Bratteli [20] and Elliott [50] following Glimm's classification of UHF algebras [81]. At the time, Elliott conjectured that *simple, separable, nuclear* C*-algebras could be classified by what is nowadays called the *Elliott invariant*, which consists of *K-theory* (a functorial invariant that generalises topological K-theory to the non-commutative setting), *tracial states* (positive linear functionals τ with the property that $\tau(a \cdot b) = \tau(b \cdot a)$ for all elements a, b of the C*-algebra), and the natural pairing between them; see [51]. This endeavour is commonly known as the *Elliott classification program*, and can be thought of as the conceptual successor of Connes and Haagerup's classification of injective factors [38, 89].

However, Elliott's conjecture in its original form was disproved when certain exotic C*-algebras that cannot be classified by the Elliott invariant were constructed [170, 201, 203]. In fact, these examples possess a fine structure that can only be captured by finer invariants, such as the *Cuntz semigroup*. The subsequent work on the classification program centered around finding a clear distinction between those simple, separable, nuclear C*-algebra that can be classified via the Elliott invariant, and those that are too wild to fit into the same scope. This led to the study of new regularity properties for C*-algebras that could grant access to classification. Great progress on this topic was made in the form of the *Toms–Winter conjecture* [49, 207, 209, 211], which states that three of these conditions are equivalent for simple, separable, nuclear and non-elementary C*-algebras. In a remarkable series of works [18, 30, 32, 139, 140, 175, 197], most of the conjecture has been confirmed, with only one missing implication in full generality. Recent work shows that the conjecture holds under some additional assumptions, such as when the C*-algebra has uniform property Gamma [31] or stable rank one and locally finite nuclear dimension [196]. One of the properties appearing in the Toms–Winter conjecture is tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$. A C*-algebra A is said to be *$\mathcal{Z}$-stable* (or $\mathcal{Z}$-absorbing) if it is isomorphic to $A \otimes \mathcal{Z}$. This is a key property that grants access to a refined version of the Elliott conjecture. More precisely, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras are classified by Elliott's invariant when they moreover satisfy the *Universal Coefficients Theorem* (UCT). Whether all separable, nuclear C*-algebras satisfy the UCT is one of the major open problems in the theory of C*-algebras. The aforementioned classification theorem splits into two parts. One deals with *purely infinite* C*-algebras and encompasses a significant body of work culminating in what is now known as the Kirchberg–Phillips theorem [114, 119, 160]. The second part handles *stably finite* C*-algebras, and follows from joint efforts of many people [29, 53, 54, 82–85, 176, 198]. The fact that simple, nuclear, $\mathcal{Z}$-stable C*-algebras are either purely infinite or stably finite follows from a result of Kirchberg (see [17] or [173]). A more extensive review of the known classification results for purely infinite C*-algebras will be given in the following paragraph. On the other hand, we will not give a presentation of the stably finite side of classification as it is less pertinent to this thesis, and refer the reader to the introduction of [29] for a thorough overview of the progress in this direction.

As mentioned above, one of the most impressive abstract classification results for C*-algebras is the *Kirchberg–Phillips theorem* [114, 119, 160], which states that simple, separable, nuclear and *purely infinite* C*-algebras, which are called *Kirchberg algebras* today, are classified via Kasparov's KK-theory [108]. Pure infiniteness for simple C*-algebras is a certain structural property that evolved from the work of Cuntz [41] on the C*-algebras $\mathcal{O}_n$ generated by n isometries with pairwise orthogonal range projections that sum to one, where $2 \leq n \leq \infty$. Nowadays, these are referred to as *Cuntz algebras*, and are the first concrete and well-studied examples of purely infinite, simple C*-algebras. Pure infiniteness can be viewed as a counterpart of being type III for von Neumann algebras, even though this analogy is not entirely accurate as there exist examples of simple C*-algebras that would not be considered analogs of type II nor type III factors [170]. As a key intermediate step towards their classification, Kirchberg and Phillips establish the famous *$\mathcal{O}_2$-embedding theorem.* This states that every separable, *exact* C*-algebra embeds into the Cuntz algebra $\mathcal{O}_2$, where a C*-algebra is said to be exact when it admits a nuclear embedding into some C*-algebra. Kirchberg and Phillips' classification result fits into the scope of the Elliott program because a classification up to KK-equivalence can be upgraded to a K-theoretical classification when C*-algebras additionally satisfy the UCT. As a byproduct of the $\mathcal{O}_2$-embedding theorem, a C*-algebra A is *tensorially absorbed* by $\mathcal{O}_2$, namely $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ exactly when it is simple, separable, nuclear and unital, which was a question predating the Kirchberg–Phillips theorem. A partial answer was already given by Rørdam in [173], where he also proved that there exists an isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ with a proof accredited to Elliott. On the other hand, as a consequence of the Kirchberg–Phillips theorem, A is a Kirchberg algebra precisely when it is simple, separable, nuclear and $A \cong A \otimes \mathcal{O}_\infty$. A C*-algebra possessing the final property mentioned above will be referred to as $\mathcal{O}_\infty$-stable. It became clear that $\mathcal{O}_2$ and $\mathcal{O}_\infty$ play a special role in the classification of purely infinite C*-algebras, and that their tensorial absorption should grant access to stronger results.

Shortly after these astonishing achievements, Kirchberg [116] outlined a far-reaching generalisation of the Kirchberg–Phillips theorem that does not ask for *simplicity*, and extends to all separable, nuclear, *stable/unital* C*-algebras that tensorially absorb $\mathcal{O}_\infty$. This is achieved via an ideal-related version of Kasparov's KK-theory. As a word of caution, we clarify that when we refer to "stable/unital" C*-algebras in a statement such as the one above, we mean that the classification separates into two distinct cases — one for stable C*-algebras and a second one for unital C*-algebras — rather than considering a single class of C*-algebras that are either stable or unital. The assumption that

C*-algebras are either unital or stable was implicitly already present in the Kirchberg–Phillips theorem because every simple, separable, purely infinite C*-algebra is either unital or stable by *Zhang's dichotomy* [212]. Moreover, one should view the $\mathcal{O}_\infty$-stability assumption as — mutatis mutandis — a pure infiniteness condition. In fact, the extensive work of Kirchberg and Rørdam on extending the notion of pure infiniteness beyond the simple case [121–123] led to the identification of a key condition, called strong pure infiniteness. For separable, nuclear C*-algebras, this condition turns out to be equivalent to $\mathcal{O}_\infty$-stability, as shown by combining results from [117] and [200]. The first full proof of Kirchberg's vast classification result had to wait until very recently, when Gabe presented the theorem with a series of technical improvements [66]. Gabe also published a new proof of a special case of this result, which he calls *Kirchberg's $\mathcal{O}_2$-stable classification* [64]. In fact, as an intermediate step of his classification theorem for non-simple C*-algebras, Kirchberg outlined an ideal-related version of the $\mathcal{O}_2$-embedding theorem that, in particular, gives direct access to the following classification theorem: Two separable, nuclear, stable/unital C*-algebras that tensorially absorb $\mathcal{O}_2$ are isomorphic precisely when they carry the same ideal structure. Within the class of separable, nuclear, strongly purely infinite C*-algebras, the $\mathcal{O}_2$-stable ones are exactly those whose closed, two-sided ideals are KK-contractible (meaning KK-equivalent to the zero C*-algebra) by [65] (and the same is true for hereditary subalgebras, quotients, inductive limits and extensions of such algebras [117, 200]).

In Gabe's proof of Kirchberg's $\mathcal{O}_2$-stable classification theorem, the *lattice of ideals* $\mathcal{I}(A)$ of a separable C*-algebra A is endowed with an *abstract Cuntz semigroup* structure in the sense of [40]. Moreover, $\mathcal{I}(-)$ can be viewed as a covariant functor that sends a $*$-homomorphism $\varphi : A \to B$ between C*-algebras to a morphism of abstract Cuntz semigroups $\mathcal{I}(\varphi) : \mathcal{I}(A) \to \mathcal{I}(B)$. As it turns out, for a purely infinite, separable C*-algebra A, $\mathcal{I}(A)$ is canonically isomorphic to the *Cuntz semigroup* $\mathrm{Cu}(A)$. As mentioned before, the Cuntz semigroup is a fine invariant that was used by Toms [201] to construct a large family of counterexamples to the original Elliott conjecture for simple, separable, nuclear C*-algebras. First introduced in the seminal work of Cuntz [43], the Cuntz semigroup inspired the definition and study of the abstract Cuntz semigroup category [40] (see also [5, 6]), to which it belongs. An alternative classification invariant for a separable, nuclear, stable/unital and $\mathcal{O}_2$-stable C*-algebra A is the topological space $\mathrm{Prim}(A)$ given by primitive ideals of A equipped with the Jacobson topology. Although it lacks a rich categorical structure, $\mathrm{Prim}(A)$ captures the isomorphisms class of $\mathcal{I}(A)$. For this reason, $\mathrm{Prim}(A)$ appears in the main theorem of [64], while proofs are carried out using the finer auxiliary invariant $\mathcal{I}(A)$.

Classification of C^*-dynamical systems

Group actions have played a central role in the field of operator algebras since their inception as an aid to describe natural physical phenomena occurring in quantum mechanics. If quantum systems are described by operator algebras, time evolution and symmetries are expressed in terms of group actions on them. Moreover, viewing von Neumann algebras and C^*-algebras as non-commutative analogues of measure spaces and topological spaces, respectively, group actions on them can be thought of as non-commutative ergodic theory and non-commutative topological dynamics. Group actions turned out to be fundamental in the purely mathematical context for understanding structural properties and the classification of operator algebras. A prime example is Connes and Haagerup's aforementioned classification of injective factors [38, 89], which relies on the classification of certain cyclic group actions. Consequently, Connes's result opened up a line of research that, through many subsequent works, culminated in the classification of actions of amenable groups on injective factors up to *cocycle conjugacy* [37, 39, 105, 109, 110, 137, 153, 180]. This represented a sufficiently significant milestone to spark a lot of interest in the classification of group actions on other classes of operator algebras, including C^*-algebras. Let us introduce cocycle conjugacy, a weaker notion than equivariant isomorphism (also referred to as conjugacy), which will serve as the primary equivalence relation for identifying C^*-dynamical systems throughout this work.

Definition. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C^*-algebras. A *cocycle conjugacy* from (A, α) to (B, β) is a pair $(\varphi, \mathbb{u})$ consisting of a $*$-isomorphism $\varphi : A \to B$ and a strictly continuous map $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ such that $\varphi \circ \alpha_g = \mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi$ and $\mathbb{u}_{gh} = \mathbb{u}_g \beta_g(\mathbb{u}_h)$ for all $g, h \in G$.

Informally speaking, two cocycle conjugate C^*-dynamical systems are equivalent up to appropriate unitary perturbations, and hence need not be identical in a strict sense. To illustrate why classification up to conjugacy is often too rigid a requirement, we present an example drawn from [100]. Recall that an action of a countable discrete group G on $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, determines a projective unitary representation of G on $\mathcal{H}$ because $\mathcal{B}(\mathcal{H})$ has trivial center and all automorphisms of $\mathcal{B}(\mathcal{H})$ are inner. Hence, a classification of G-actions on $\mathcal{B}(\mathcal{H})$ up to genuine conjugacy would require one to find all projective unitary representations of G on $\mathcal{H}$ up to unitary equivalence. Even in the case where $G = \mathbb{Z}^2$ and $\mathcal{H} = \ell^2(\mathbb{N})$, this is not feasible. On the other hand, for any countable discrete group G one has a bijection between cocycle conjugacy classes of G-actions on $\mathcal{B}(\ell^2(\mathbb{N}))$ and the second cohomology group $H^2(G, \mathbb{T})$, which is a computable object. In particular, cocycle conjugacy emerges as the more natural and tractable notion of isomorphism in classification theory, especially when working with general (not necessarily finite or compact) groups.

The historical overview of progress with respect to the classification of C*-dynamics can be found in Izumi's survey article [100], which describes the state of the art at the time of its publication in 2011. For an overview of more recent works, the reader can consult the introduction of [192] and the references therein. Although we do not give a complete historical account of the classification of C*-dynamical systems, we would like to mention the historical importance of the techniques developed by Evans and Kishimoto for classifying single automorphisms [57], which were followed by work of Nakamura [149] on single automorphisms of simple purely infinite C*-algebras. Partially in parallel to the study of single automorphisms, the Rokhlin property for finite group actions, which one may think of as some kind of approximate periodicity, was introduced by Izumi in [98,99], where he also gave a general classification of these up to conjugacy that is easily applicable to concrete situations. Subsequently, Izumi and Matui [103,104] established a groundbreaking classification of poly-$\mathbb{Z}$ group actions on Kirchberg algebras, generalising their previous result for $\mathbb{Z}^2$-actions [102], but with little hope that a direct adaptation of their methodology could work in general. In fact, in 2011 Izumi writes that "there is little hope to classify general outer actions of finite groups on Kirchberg algebras"; see [100, p. 1541]. However, Gabe and Szabó recently reached a major breakthrough, establishing a far-reaching generalisation of Izumi–Matui's result [67]. Indeed, their classification applies to (second-countable) locally compact group actions on Kirchberg algebras, and can be considered as the dynamical version of the Kirchberg–Phillips theorem [112, 160], in which the KK-theoretical invariant is replaced by equivariant Kasparov theory KK^G [108]. In particular, their work implies that outer G-actions on Kirchberg algebras can be classified not only when G is finite, but also for all countable, discrete, amenable groups, thus vastly exceeding Izumi's expectations expressed in [100].

Group actions on $\mathcal{O}_2$

There are several important examples of strongly self-absorbing C*-algebras [200] that play a key role in the classification of nuclear C*-algebras. Some examples, such as the Jiang–Su algebra $\mathcal{Z}$ and the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$, were already mentioned earlier. These objects are equally important in the dynamical setting, where strongly self-absorbing actions [185, 187, 188] generalise them in an appropriate sense. Since $\mathcal{O}_2$ will play a major role in this thesis, let us give a brief overview of the prominence of $\mathcal{O}_2$ in the theory of C*-dynamical systems. First of all, note that despite its classification invariant being trivial, $\mathcal{O}_2$ admits group actions in abundance, for instance in the sense that the relations of cocycle conjugacy of automorphisms of $\mathcal{O}_2$, and isomorphisms of simple crossed products of the form $\mathcal{O}_2 \rtimes \mathbb{Z}_2$ are not Borel [73]. In fact, Izumi [98] provided a hands-on construction of uncountably many mutually non-cocycle conjugate outer actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$. To show that these actions are not cocycle conjugate, Izumi computes the K-theory groups of their crossed product explicitly, showing that they are all different and therefore the actions in question cannot be cocycle conjugate. In point of fact, it is known that the behavior of actions of groups with torsion cannot generally be predicted by considering just the classification invariant of the underlying C*-algebra, as evidenced by the aforementioned K-theoretical obstructions for actions on $\mathcal{O}_2$. In contrast, for many torsion-free groups G, there is a unique outer G-action on $\mathcal{O}_2$ up to cocycle conjugacy. This is true for $G = \mathbb{Z}^k$ by work of Matui [138], although it should be noted that the case $k = 1$ follows from work of Nakamura [149]. More generally, it has been shown in [67] that there exists a unique amenable outer G-action on every strongly self-absorbing Kirchberg algebra (such as $\mathcal{O}_2$) when G is assumed to be countable, discrete, exact, torsion-free with the Haagerup property [33, 87].

In the following paragraph, we recall some of the seminal results involving $\mathcal{O}_2$ and explain their dynamical generalisations with special emphasis on the relevance of equivariantly $\mathcal{O}_2$-stable actions. Recall that the $\mathcal{O}_2$-absorption theorem [119, 169] states that a C*-algebra A is separable, nuclear, unital and simple precisely when $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. Its dynamical counterpart for actions of amenable groups was established by Szabó in [186]. This states that if G is a countable, discrete, amenable group, and $\delta : G \curvearrowright \mathcal{O}_2$ an outer action that is cocycle conjugate to $\delta \otimes \mathrm{id}_{\mathcal{O}_2}$, then any action $\alpha : G \curvearrowright A$ on a separable, nuclear, simple, unital C*-algebra is tensorially absorbed — up to cocycle conjugacy — by δ. (We note that this result had been known for finite groups earlier [98].)

Definition. A group action $\alpha : G \curvearrowright A$ on a C*-algebra is *equivariantly $\mathcal{O}_2$-stable* if it is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{O}_2} : G \curvearrowright A \otimes \mathcal{O}_2$.

As a direct consquence of what was said before, for any countable, discrete, amenable group G, $\mathcal{O}_2$ admits precisely one outer and equivariantly $\mathcal{O}_2$-stable G-action. This serves as evidence that equivariant $\mathcal{O}_2$-stability for actions is the right dynamical counterpart of $\mathcal{O}_2$-stability. One may notice that assuming equivariant $\mathcal{O}_2$-stability rules out the type of examples like Izumi's uncountably many non-cocycle conjugate $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ mentioned above. The dynamical $\mathcal{O}_2$-absorption theorem was then extended by Suzuki [182] to *amenable* actions of countable, exact groups. It should be noted that Suzuki's result is stated for actions with the *quasicentral approximation property* (abbreviated QAP). Roughly, an action $\alpha : G \curvearrowright A$ has the QAP if $L^2(G, A)$, viewed as an equivariant A-bimodule, admits a net of approximately fixed elements with certain desirable properties; see Definition 2.5.24 for the precise statement. Since the QAP and amenability have been shown to be equivalent for locally compact group actions as a result of work of Bearden–Crann [11, 12], Buss–Echterhoff–Willet

[26, 27], Suzuki [181] and Ozawa–Suzuki [154], one can use these two properties interchangeably. This turns out to be a useful new input for classifying amenable actions of locally compact groups, instead of restricting the attention to the more special case of actions of amenable groups.

Classification of equivariantly $\mathcal{O}_2$-stable actions

Thanks to the developments outlined in the previous sections, the classification of group actions in the simple, purely infinite setting can now be regarded as completed. Significantly less attention has been paid to the dynamics of group actions on non-simple C*-algebras as many of the tools and techniques developed for simple C*-algebras do not extend easily to the non-simple setting. The main thrust of Chapter 3, which also represents the primary goal of this thesis, is to bridge this gap by exploring the classification theory of group actions on non-simple, $\mathcal{O}_2$-stable C*-algebras. Let us give a more precise idea of our framework. We assume G to be a second-countable, locally compact group acting on a separable C*-algebra A by $\alpha : G \curvearrowright A$. An element $g \in G$ acts by order isomorphisms on the ideal lattice $\mathcal{I}(A)$ of A by associating

$$\alpha_g^\sharp(I) = \{\alpha_g(x) \mid x \in I\}$$

to the ideal I. It follows that the restriction of $\alpha^\sharp$ to $\mathrm{Prim}(A)$ is a continuous G-action with respect to the Jacobson topology (see [164] or Lemma 2.3.38), thus turning $(\mathrm{Prim}(A), \alpha^\sharp)$ into a topological dynamical system. Here, we use the term "topological dynamical system" in a broad sense, as the topological space $\mathrm{Prim}(A)$ can be highly non-Hausdorff. Since inner automorphisms of A induce the identity map on $\mathrm{Prim}(A)$, a cocycle conjugacy between C*-dynamical systems induces a conjugacy between the corresponding primitive ideals. This observation implies that the induced action on the primitive ideal space does not distinguish between cocycle conjugate C*-dynamical systems. Hence, the main problem under consideration may be stated as follows.

Question A. Let A and B be separable, nuclear, stable/unital, $\mathcal{O}_2$-stable C*-algebras. What are sufficient conditions on the actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ such that a conjugacy $(\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ lifts to a cocycle conjugacy $(A, \alpha) \to (B, \beta)$?

It is well-known that, in order to classify a sufficiently broad class of actions, one must impose some non-triviality conditions on the actions at hand, which is usually expressed as an outnerness-type property. The right outerness condition in the context of this manuscript is imported from [67], and is therein called *isometric shift-absorption*. Simply put, an action of G on A is isometrically shift-absorbing when A can be locally approximated by $L^2(G, A)$ in a suitable sense (see Definition 2.7.1 and Proposition 2.7.8), which forces at least $A \cong A \otimes \mathcal{O}_\infty$ for the underlying C*-algebra. In [102], it was shown that an action of a countable discrete group on a unital Kirchberg algebra is isometrically shift-absorbing if and only if it is outer. This result was later extended to the general (not necessarily unital) case in [67]. Although outerness is a strictly weaker assumption in greater generality, this particular case suggests that isometric shift-absorption is a reasonable condition to ask. In the same article it is observed that isometric shift-absorption is equivalent to the Rokhlin property for $\mathbb{R}^k$-actions on $\mathcal{O}_\infty$-stable C*-algebras. However, the Rokhlin property does not coincide with (because it is strictly stronger than) isometric shift-absorption for compact group actions as one might be tempted to guess from this. Since isometric shift-absorption and amenability can both be expressed as properties of the equivariant A-bimodule $L^2(G, A)$, they blend well together. In fact, when an action is both amenable and isometrically shift-absorbing, one gains access to a useful averaging argument developed in [67], which was a key piece of methodology to proving the dynamical Kirchberg–Phillips theorem and also plays a similar role in the present work. Let us briefly recall the appropriate notion of morphism between group actions that is suitable to develop a classification up to cocycle conjugacy.

Definition. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be group actions on C*-algebras. A *proper cocycle morphism* is a pair $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$, where $\varphi : A \to B$ is a $*$-homomorphism and $\mathbb{u} : G \to \mathcal{U}(\mathbf{1} + B)$ is a norm-continuous β-cocycle satisfying $\mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi = \varphi \circ \alpha_g$ for all $g \in G$.

As mentioned before, amenability and isometric shift-absorption are not sufficient conditions on α and β to ensure that the statement in Question A holds. This is because of K-theoretical obstructions that arise even in the case where $G = \mathbb{Z}_2$ and $A = B = \mathcal{O}_2$, as illustrated by Izumi's examples [98] discussed earlier. The central insight underpinning the main result of Chapter 3, which is the main theorem of [156], is that the additional assumption of equivariant $\mathcal{O}_2$-stability for α and β eliminates the aforementioned obstruction, thereby allowing for a complete resolution of Question A. The resulting theorem can be considered as a dynamical generalisation of Gabe and Kirchberg's $\mathcal{O}_2$-stable classification that uses the topological dynamical system given by $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$ as a classification invariant for $\alpha : G \curvearrowright A$. The following is a shortened rendition of the aforementioned theorem, and the reader is referred to Theorem 3.5.5 for the full version.

Theorem B. *Let G be a second-countable, locally compact group, and A, B two separable, nuclear, stable* C**-algebras. Then, two amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ are cocycle conjugate if and only if $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$ and $\beta^\sharp : G \curvearrowright \mathrm{Prim}(B)$ are conjugate.*

Our methodology behind the proof of this result blends together various techniques from [67] and [64]. As one of the extra complications compared to what happens in [67], we need to control in a suitable sense the ideal structure of the involved C*-algebras, which comes with additional technical challenges in light of the dynamical structure that we need to keep track of. The dynamical nature of the invariant makes it impossible to directly apply results from [64], hence we need to reprove a number of intermediate results suited for our context, which can be considered the most challenging component of the work related to this theorem. Before proving Theorem B in full generality, we begin by examining a few special cases where ad hoc arguments based on existing techniques from the literature suffice to prove the result. These special cases include actions of $G = \mathbb{Z}$, $G = \mathbb{R}$ and compact groups. However, we emphasise that even in seemingly simple extensions — such as $G = \mathbb{Z}^2$ — these methods are insufficient to establish the theorem.

The overall strategy employed to prove the Theorem B in full generality is based on an abstract classification approach that represents the foundation of several classification results in C*-algebra theory. Essentially, it relies on uniqueness and existence theorems for maps, which, in our case, yield a classification of proper cocycle morphisms up to proper approximate unitary equivalence. By combining this classification of maps with Elliott's intertwining argument, one arrives at the classification result for C*-dynamical systems. The uniqueness theorem underpinning our classification theory is proved in Section 3.3, and is reported below (see also Theorem 3.3.15). To prove this result, we adapt techniques from [68] and [67] to the non-simple setting, while also taking advantage of equivariant $\mathcal{O}_2$-stability, which allows for certain simplifications in the process. Moreover, note that in this theorem we assume β to be *strongly stable*, which means that it is conjugate to $\beta \otimes \mathrm{id}_{\mathcal{K}}$, where $\mathcal{K}$ denotes the C*-algebra of compact operators on a separable, infinite-dimensional Hilbert space. However, as observed in [67], this assumption can ultimately be removed in the classification theorem.

Theorem C. *Let $\alpha : G \curvearrowright A$ be an action on a separable, exact* C**-algebra, and $\beta : G \curvearrowright B$ a strongly stable, amenable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a separable* C**-algebra. If $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ are two proper cocycle morphisms with φ and ψ nuclear, then $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$ if and only if $(\varphi, \mathbb{u})$ is properly approximately unitarily equivalent to $(\psi, \mathbb{v})$, i.e., there exists a sequence of unitaries $v_n \in \mathcal{U}(\mathbf{1} + B)$ such that*

$$\psi(a) = \lim_{n\to\infty} v_n \varphi(a) v_n^*, \qquad \lim_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g - v_n \mathbb{u}_g \beta_g(v_n)^*\| = 0$$

for all $a \in A$ and compact sets $K \subseteq G$.

We also establish a unital version of the theorem above (see Corollary 3.3.16). However, this requires that the acting group G is *exact* because only exact groups can act amenably on unital C*-algebras by a result of Ozawa and Suzuki [154]. We remark that Theorem C differs from its published counterpart [156, Theorem 3.10] in that, in this version, uniqueness is established up to *proper approximate unitary equivalence*, whereas in the article it is proved up to *strong asymptotic unitary equivalence*, which is a priori a stronger notion. However, this distinction does not affect the final outcome in Theorem B.

Historically, the $\mathcal{O}_2$-embedding theorem [119] was proved as an intermediate step to the Kirchberg–Phillips theorem. Likewise, as an intermediate step to the dynamical Kirchberg–Phillips theorem [67], the appropriate dynamical counterpart of the $\mathcal{O}_2$-embedding theorem was established in [67, Theorem G]. This states that an amenable action on a separable, exact C*-algebra admits a proper cocycle embedding into any equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a Kirchberg algebra. Our main existence theorem for proper cocycle morphisms is proved in Section 3.4, along with a number of technical results. This existence result can be considered as a dynamical, ideal-related version of the $\mathcal{O}_2$-embedding theorem (see also Theorem 3.4.17).

Theorem D. *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C*-*algebra, and $\beta : G \curvearrowright B$ an isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable action on a separable* C*-*algebra. Then, for every equivariant* **Cu**-*morphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$, there exists a nuclear proper cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ such that $\mathcal{I}(\varphi) = \Phi$.*

Finally, combining the aforentioned theorems, we deduce a bijective correspondence between equivariant morphisms of abstract Cuntz semigroups of ideals and nuclear proper cocycle morphisms up to proper approximate unitary equivalence. From this we will deduce our main result, Theorem 3.5.5, the more detailed version of Theorem B. We can furthermore show that a unital version of this theorem holds when G is exact. In the case when the acting group G is compact, results in [67] pertaining to asymptotic coboundaries allow us to obtain Corollary 3.5.7, in which we achieve classification up to genuine conjugacy.

The range of the invariant for equivariantly $\mathcal{O}_2$-stable actions

Having obtained the aforementioned classification theorem for equivariantly $\mathcal{O}_2$-stable actions on C*-algebras, it is natural to investigate the range of the invariant. This leads to the following question.

Question E. Do continuous group actions on the primitive ideal space of a C^*-algebra lift to an action on a C^*-algebra with the same primitive ideal space?

To study our range-of-invariant problem, it is sufficient to answer Question E when the underlying C^*-algebra is separable and nuclear. Moreover, since the Cuntz algebra $\mathcal{O}_2$ and the C^*-algebra of compact operators $\mathcal{K}$ are simple, for any C^*-algebra B, the primitive ideal space of $B \otimes \mathcal{O}_2 \otimes \mathcal{K}$ is homeomorphic to $\mathrm{Prim}(B)$. Therefore, we may (and will) restrict our focus to the case where B is separable, nuclear, and satisfies $B \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$, without loss of generality for the problem under consideration. The following theorem, which represents the main result of Chapter 4, answers the refined version of Question E in the positive, and appeared as Theorem A in the author's article [155].

Theorem F. *Let G be a second-countable, locally compact group, and B a separable, nuclear C^*-algebra such that $B \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$. For every continuous action $\gamma : G \curvearrowright \mathrm{Prim}(B)$, there exists an action $\beta : G \curvearrowright B$ such that $\beta^\sharp = \gamma$.*

A similar result has recently been made available in the form of a preprint by Phillips [120], who proved, jointly with Kirchberg, a partial version of Theorem F. In particular, in [120, Theorem 5.9] they show the conclusion of Theorem F under the additional assumption that the primitive ideal space is homeomorphic to the (non-Hausdorff) compactification in the sense of [120, Definition 1.5] of a second-countable, non-compact, locally compact space, which, in particular, forces A to be prime (see [120, Corollary 1.7]). This overlaps with the result presented here (and in [155]), and it would not be hard to adapt their proof to obtain Theorem F in full generality, although this is not explicitly stated by the authors. In fact, the approach presented here is conceptually inspired by Kirchberg and Phillips' core idea, which is outlined in an extended abstract within an Oberwolfach report from 2008 [118]. However, the author was not aware of their subsequent work when he carried out the research pertatining to Chapter 4 (and [155]). Moreover, it should be noted that the approach presented here is self-contained and, contrarily to Kirchberg–Phillips', does not rely on the reconstruction theorem of Harnisch and Kirchberg [91], which has only very recently been made available in the form of a preprint. Conversely, the present work does not provide information about what topological spaces arise as primitive ideal spaces of separable nuclear C^*-algebras, which is the main novelty of Harnisch and Kirchberg's work.

Theorem F, combined with the existence of amenable actions on $\mathcal{O}_2 \otimes \mathcal{K}$ that can be deduced from [154, 183] (see Remark 2.7.9), yields an upgrade of Theorem B to a complete characterisation of cocycle conjugacy classes of equivariantly $\mathcal{O}_2$-stable, amenable, isometrically shift-absorbing actions on separable, nuclear, stable C^*-algebras.

Theorem G. *Let G be a second-countable, locally compact group, and A a separable, nuclear, stable, $\mathcal{O}_2$-stable* C**-algebra. Then associating $\alpha^\sharp$ to each action $\alpha : G \curvearrowright A$ gives the following bijection:*

$$\frac{\left\{\begin{matrix}\text{amenable, isometrically shift-absorbing,}\\ \text{equivariantly } \mathcal{O}_2\text{-stable actions } G \curvearrowright A\end{matrix}\right\}}{\text{cocycle conjugacy}} \longrightarrow \frac{\left\{\begin{matrix}\text{continuous actions}\\ G \curvearrowright \mathrm{Prim}(A)\end{matrix}\right\}}{\text{conjugacy}}$$

If G is moreover compact, "cocycle conjugacy" above can be replaced with "conjugacy".

Corona algebras and strongly self-absorbing actions

The work contained in Chapter 5 appeared in [134], which was co-authored with Xiuyuan Li and Gábor Szabó. We wish to emphasize that part of the material in this chapter originates from Xiuyuan Li's Master's thesis [133], completed as part of the Master of Mathematics program at KU Leuven. More specifically, the author collaborated closely with Gábor Szabó in supervising this thesis, which establishes a special case of Theorem H presented here. The remaining work required to extend the result to the full generality of Theorem H was carried out jointly by Gábor Szabó and the author after the completion of Xiuyuan Li's Master's project.

Here, we shift focus slightly to explore certain structural properties of corona algebras associated to C*-algebras with various absorption properties, which generalise to the dynamical setting certain results established in recent work of Farah [59], and Farah–Szabó [62]. Recall that the corona algebra of a C*-algebra is defined to be the quotient $\mathcal{Q}(A) = \mathcal{M}(A)/A$, where $\mathcal{M}(A)$ denotes the multiplier algebra of A. Most notably, the corona algebra generalizes the Stone–Čech boundary for topological spaces to the noncommutative setting. As mentioned before, strongly self-absorbing C*-algebras and the algebras tensorially absorbing them [200] play a crucial role in the structure and classification theory of C*-algebras. In a similar vein, given a second-countable locally compact group G, the notion of a strongly self-absorbing G-action was introduced and studied for a similar purpose [185,187,188]. Even in special cases, such as considering the trivial G-action on a strongly self-absorbing C*-algebra $\mathcal{D}$, it can be instructive to study the notion of equivariant $\mathcal{D}$-stability for group actions on C*-algebras. A clear illustration of this principle is provided by the classification result in Chapter 3, where, as previously discussed, equivariant $\mathcal{O}_2$-stability plays a central role. Similar absorption properties have also been exploited in various other dynamical classification results as an intermediate technical step [67,139,140,182,186,189].

As observed by Farah [59], and later refined together with Szabó [62], the absorption of a strongly self-absorbing C*-algebra by a stable C*-algebra can be detected at the level of its corona algebra. Given how most of the theory of strongly self-absorbing C*-algebras has some dynamical analogue, this raises the question whether a similar behaviour can be observed for group actions induced on corona algebras. In Chapter 5, we present such a dynamical analogue of these recent C*-algebraic results. To formulate these, we begin with a dynamical generalisation of a C*-algebraic property that is of interest in this context (see Definition 5.2.1 for the precise definition.)

Definition. Let $\alpha : G \curvearrowright A$ be an algebraic action on a unital C*-algebra, and $\gamma : G \curvearrowright D$ a continuous action on a separable unital C*-algebra. We say that (A, α) is *γ-saturated* if for every separable C*-subalgebra $C \subseteq A$, there exists an equivariant unital embedding $(D, \gamma) \to (A, \alpha)$ with range in $A \cap C'$.

The main results we obtain in this context can be summarized as follows (see Theorem 5.2.4 and Remark 5.2.7). While these results are established under more general assumptions, we shall present them here in the setting of separable C*-algebras.

Theorem H. *Let $\alpha : G \curvearrowright A$ be an action on a separable non-unital C*-algebra, $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing, unitarily regular action, and equip $\mathcal{Q}(A)$ with the algebraic action induced by α. If α is γ-absorbing, then $(\mathcal{Q}(A), \alpha)$ is γ-saturated. Furthermore, if α tensorially absorbs the trivial action on the compacts $\mathcal{K}$, then the converse holds as well.*

The "furthermore" part in the above theorem can be obtained rather easily from the results and techniques that were introduced recently in [62]. The rest of the statement, however, is proved via a careful dynamical extension of the ideas behind the proof of [62, Theorem A]. At least for actions of compact groups, this generalization is mostly technical in nature and can be achieved via the similar conceptual steps and extra bookkeeping. This has been accomplished in Xiuyuan Li's Master thesis work [133], which the article [134] extends on. However, some new conceptual steps are needed in the proof of the statement for general C*-dynamics, in particular to cover arbitrary actions of groups G that are non-compact. This stems from the added difficulty that one may only identify G-actions modulo cocycle conjugacy for such groups in various contexts, which cannot be improved to genuine conjugacy in general, as explained before.

In order to overcome this technical difficulty, we prove and make use of the so-called *dynamical folding property* for G-actions induced on corona algebras; see Definition 5.1.2 for the details. In somewhat oversimplified terms, a (not necessarily continuous) action $\alpha : G \curvearrowright A$ on a C*-algebra has the dynamical folding property if a given equivariant embedding $(D_0, \delta) \to (A, \alpha)$ of a separable system can be extended to an embedding of a surrounding system $(D, \delta) \supseteq (D_0, \delta)$ if and only if D can be embedded into the C^*-algebra of continuous paths with values in A in a way that preserves the embedding of (D_0, δ). Without the dynamics (i.e., with $G = \{1\}$), this property has been observed for corona algebras by Manuilov–Thomsen [136] and Phillips–Weaver [161]; the term "folding" is directly inspired by a comment about this property made in the latter article. This proved instrumental in the former's proof that the E-theory of any C^*-algebra pair (A, B) can be obtained as the KK-group of the associated pair $(\mathcal{C}_0(\mathbb{R}) \otimes A, \mathcal{Q}(B \otimes K))$. We prove the dynamical folding property for corona algebras in the following generality.

Theorem I. *Let $\alpha : G \curvearrowright A$ be an action on a σ-unital* C^**-algebra. Then $(\mathcal{Q}(A), \alpha)$ has the dynamical folding property.*

While itself of independent interest, Theorem I also plays a crucial role in the proof of Theorem H. As previously noted, this input is not required when G is compact. Apart from the application to our main result, the dynamical folding property has a few other immediate consequences. In the context of Theorem H, for instance, we show that all equivariant unital embeddings $(\mathcal{D}, \gamma) \to (\mathcal{Q}(A), \alpha)$ are mutually G-unitarily equivalent, i.e., conjugates of each other via unitaries in $\mathcal{Q}(A)$ that are fixed by α.

Chapter 2

Preliminaries

The concepts introduced in this chapter are, at least in part, well-known to anyone working with operator algebras. However, the concepts become increasingly specialised as we introduce more recent developments, such as the cocycle category for C^*-dynamics or the characterization of amenability via the quasicentral approximation property. We do not provide a comprehensive overview of the basic theory of C^*-algebras and instead refer the reader to classic textbooks such as [106, 145, 174] for an introduction to operator algebras. We refer the reader to more advanced textbooks such as [16, 24, 157] for some well-established results that we do not prove here.

Let us mention here that we adopt the convention that $0 \in \mathbb{N}$.

2.1 Basics of C^*-algebras and selected topics

We begin by recalling a few basic definitions before jumping straight to some foundational facts and results in the theory of C^*-algebras that will be needed later. The reader is assumed to be familiar with Banach space theory and other fundamental concepts in functional analysis.

Recall that a C^*-*algebra* is an associative algebra A over $\mathbb{C}$ equipped with a norm, denoted by $\|\cdot\|$ or $\|\cdot\|_A$, and a unary operation

$$* : A \to A, \quad a \mapsto a^*,$$

such that

- the normed space $(A, \|\cdot\|)$ is complete, and hence Banach;
- for all elements $a, b \in A$, one has that
$$\|ab\| \leq \|a\| \cdot \|b\|;$$
- the operation "$*$" is an *involution*, that is, for all $a, b \in A$ and all $\lambda \in \mathbb{C}$,
$$(\lambda a)^* = \bar{\lambda} a^*, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*;$$
- every element $a \in A$ satisfies the C**-identity*, namely,
$$\|a^* a\| = \|a\|^2.$$

Let us introduce here a piece of notation that will be used throughout the thesis. Namely, if two elements a, b of a C*-algebra A are ε-close in norm, i.e., $\|b - a\| \leq \varepsilon$, we simply write $a =_\varepsilon b$.

A map $\varphi : A \to B$ between two C*-algebras is said to be

(i) a $*$-*homomorphism* if it is a homomorphism of complex algebras such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$;

(ii) a $*$-*isomorphism*, which we usually call *isomorphism* for brevity, if it is a bijective $*$-homomorphism, and two C*-algebra are called *isomorphic* if there exists an isomorphism between them;

(iii) an *embedding* of A into B if it is an injective $*$-homomorphism.

An important fact about $*$-homomorphisms is that they are automatically *contractive*, i.e., $\|\varphi(a)\| \leq \|a\|$ for every $a \in A$, and embeddings are automatically *isometric*, namely $\|\varphi(a)\| = \|a\|$ for every $a \in A$. As a consequence, a C*-algebra has a unique norm $\|\cdot\|$, in the sense that whenever it is equipped with a second norm, say $\|\cdot\|'$, such that A satisfies all axioms of a C*-algebra with $\|\cdot\|'$ apart from completeness, then $\|\cdot\| = \|\cdot\|'$.

Example 2.1.1. Here we present a first list of examples.

(i) For a locally compact space X, consider the space of continuous complex-valued functions from $X \to \mathbb{C}$ that vanish at infinity, which is denoted by $\mathcal{C}_0(X)$. When equipped with pointwise operations, the uniform norm $\|\cdot\|_\infty$, and involution induced by complex-conjugation: $f^*(x) = \overline{f(x)}$ for $f \in \mathcal{C}_0(X)$ and $x \in X$, $\mathcal{C}_0(X)$ is a C*-algebra. Note that the multiplication operation is commutative. In fact, by Gelfand–Naimark's theorem, every commutative C*-algebra A is isomorphic to $\mathcal{C}_0(X)$ for some locally compact space X, or to $\mathcal{C}(K)$ for a compact space K if A is unital.

(ii) Consider now a generalisation of the previous example. For a locally compact space X and a C*-algebra A, denote by $\mathcal{C}_0(X, A)$ the space of continuous function from X to A that eventually vanish. This is a C*-algebra with operations induced pointwise by the operations of A and the norm is given by $\|f\| = \sup_{x\in X} \|f(x)\|$ for all $f \in \mathcal{C}_0(X, A)$. An important special case of this construction is the C*-algebra $\mathcal{C}_0(\mathbb{N}, A)$, which consists of sequences $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $a_n \to 0$ as $n \to \infty$. We will the latter by $c_0(\mathbb{N}, A)$. Moreover, one may consider a filter $\mathcal{U}$ over $\mathbb{N}$, i.e., a non-empty collection of subsets of $\mathbb{N}$ that is closed under finite intersections, upward directed, and $\emptyset \notin \mathcal{U}$. Then, one can form the algebra of sequences $(a_n)_{n\in\mathbb{N}} \subseteq A$ such that $\lim_{n\to\mathcal{U}} \|a_n\| = 0$. This algebra equipped with the pointwise operations coming from A is a C*-algebra, and is denoted by $c_{\mathcal{U}}(\mathbb{N}, A)$.

(iii) The algebra of bounded sequences $\ell^\infty(\mathbb{N})$ with the supremum norm and pointwise operations is a C*-algebra as well. As in the previous example, one may define the C*-algebra $\ell^\infty(\mathbb{N}, A)$ of bounded sequences with values in a C*-algebra A with operations induced by A.

(iv) The bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with operations given by sum and composition of operators is a C*-algebra when equipped with the operator norm, and involution given by the adjoint operation. More generally, any norm-closed and self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is a C*-algebra, and C*-algebras of this form are called *concrete*. By the Gelfand–Naimark (and Segal) theorem, every C*-algebra is isomorphic to a concrete one.

(v) The finite dimensional case of the previous example will occur quite often. More precisely, when $\mathcal{H} = \mathbb{C}^n$, $\mathcal{B}(\mathcal{H})$ is the space of $n \times n$ matrices with complex entries, which we denote by M_n. Note that the identity operator on $\mathcal{H}$, which we denote by $\mathbf{1}_{\mathcal{H}}$ is a multiplicative unit of $\mathcal{B}(\mathcal{H})$. We will denote the unit of M_n by $\mathbf{1}_n$.

(vi) In this thesis, one of the most popular concrete C*-algebras will be $\mathcal{K}(\mathcal{H})$, i.e., the compact operators on the Hilbert space $\mathcal{H}$. This is a C*-algebra because it is clearly norm-closed (being the closure of finite rank operators), self-adjoint, and a subalgebra (in fact, an ideal) of $\mathcal{B}(\mathcal{H})$. Note, moreover, that $\mathcal{K}(\mathcal{H})$ does not contain $\mathbf{1}_{\mathcal{H}}$ if $\mathcal{H}$ is infinite-dimensional.

Warning 2.1.2. We adopt the convention that a *locally compact space* must also satisfy the Hausdorff separation axiom. If it does not, we refer to it as a *locally quasi-compact space*. The same convention applies to compact spaces.

We will now omit much of the basic theory and move directly to key results that form the foundation of this manuscript. In particular, we assume the reader is familiar with positive (and normal) elements in a C*-algebra, as well as the continuous functional calculus. Throughout, we denote the set of positive elements in a C*-algebra A by A_+. Recall that every element of A can be expressed as a linear combination of at most four positive elements, and hence the span of A_+ is dense in A.

2.1.1 Ideals and hereditary C*-subalgebras

Note that a closed two-sided ideal $I \subseteq A$ is self-adjoint, which thus implies that it is a C*-subalgebra of A. One can then form the quotient A/I, which is a C*-algebra when equipped with the norm given by

$$\|a + I\| = \inf_{b \in I} \|a + b\|.$$

Given a subset $S \subseteq A$ of a C*-algebra, we will make extensive use of the notation

$$\overline{ASA} = \overline{\operatorname{span}}\{asb \mid s \in S,\ a, b \in A\},$$

which is the smallest closed two-sided ideal of A generated by S. If $S = \{a\}$, we simply denote $\overline{ASA}$ by $\overline{AaA}$. There is a special class of ideals, called *essential*, which will be relevant later. First, for any closed two-sided ideal $I \subseteq A$, consider the *two-sided annihilator* of I,

$$A \cap I^\perp = \{a \in A \mid ab = ba = 0, \text{ for all } b \in I\},$$

which is easily seen to be a closed two-sided ideal of A. A closed two-sided ideal $I \subseteq A$ is called essential if $A \cap I^\perp = \{0\}$. We note here, and use without further mention, that a closed two-sided ideal $I \subseteq A$ is essential if and only if the left or right annihilator is zero, i.e.,

$$\{a \in A \mid ab = 0, \text{ for all } b \in I\} = \{0\}, \quad \text{or}$$

$$\{a \in A \mid ba = 0, \text{ for all } b \in I\} = \{0\}.$$

This is moreover equivalent to saying that any non-zero ideal of A has non-zero intersection with I. A C*-algebra A is said to be *simple* if it only contains the closed, two-sided ideals $\{0\}, A \subseteq A$. The ideal structure of a C*-algebra can be described in many ways, which we will discuss in greater depth in Section 2.3.

One says that a C*-subalgebra $B \subseteq A$ is *hereditary* if whenever $a \in A$ and $b \in B$ satisfy $0 \le a \le b$, one has that $a \in B$. The smallest hereditary C*-subalgebra containing a positive element $a \in A_+$ is given by

$$\overline{aAa} = \overline{\operatorname{span}}\{aba \mid b \in A\},$$

and when A is *separable*, every hereditary C*-subalgebra of A has this form. More generally, any separable hereditary C*-subalgebra B of a (possibly non-separable) C*-algebra A is of the form $\overline{bAb}$ for some positive element $b \in B$ (see, e.g., [145, Theorem 3.2.5]). Moreover, there is a one-to-one correspondence between hereditary C*-subalgebras of A and closed left ideals of A given as follows (see [145, Theorem 3.2.1]),

$$B \subseteq A \text{ hereditary C}^*\text{-subalgebra } \mapsto \{a \in A \mid a^*a \in B\},$$

$$L \subseteq A \text{ closed left ideal} \mapsto L \cap L^*.$$

One can show that every closed two-sided ideal is an hereditary C*-algebra; for a proof, see [145, Corollary 3.2.3].

Example 2.1.3. We limit ourselves to a few examples for now, with more to follow later on.

(i) Let $A = \mathcal{C}_0(X)$ for some locally compact space X. Consider the set $I_x = \{f \in A \mid f(x) = 0\} \subseteq A$. It is straightforward that I_x is a two-sided closed ideal of A. In fact, I_x is the kernel of the point-evaluation map at x, which we denote by $\mathrm{ev}_x : A \to \mathbb{C}$. Since $I_x \cong \mathcal{C}_0(X \setminus \{x\})$, it follows that $A/I_x \cong \mathbb{C}$.

(ii) As briefly mentioned before, the compact operators $\mathcal{K}(\mathcal{H})$ on an infinite-dimensional, separable Hilbert space $\mathcal{H}$ form a closed two-sided ideal of $\mathcal{B}(\mathcal{H})$. The C*-algebra defined as the quotient $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is referred to as the *Calkin algebra*, and was first introduced by Calkin in [28].

(iii) The C*-algebra $c_0(\mathbb{N}, A)$ is easily seen to be a closed, two-sided ideal of $\ell^\infty(\mathbb{N}, A)$. The quotient C*-algebra $A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ is an important example, and is referred to as the *sequence algebra* of A. Analogously, for any filter $\mathcal{U}$ over $\mathbb{N}$, also $c_{\mathcal{U}}(\mathbb{N}, A)$ is a closed, two-sided ideal of $\ell^\infty(\mathbb{N}, A)$, and one may form the *ultrapower* of A as $A_{\mathcal{U}} = \ell^\infty(\mathbb{N}, A)/c_{\mathcal{U}}(\mathbb{N}, A)$.

Warning 2.1.4. From now on, we adopt the convention that a norm-closed and two-sided ideal I of A will simply be referred to as an *ideal*. In the rare cases where an ideal is not two-sided or not closed, this will be explicitly mentioned.

2.1.2 Approximate units

One says that a C*-algebra A is *unital* if it contains a mutliplicative unit $\mathbf{1}_A \in A$. (Note that $\mathbf{1}_A$ is automatically a norm-one, self-adjoint element.)

When a C*-algebra is not unital, one can resort to *approximate units*. Indeed, it is well-known that every C*-algebra admits an approximate unit (see, e.g., [111, Theorem I.4.8]), that is, an increasing[1] net $(e_i)_i \subseteq A$ of positive elements that are contractive, i.e., $\|e_i\| \leq 1$, and such that

$$\lim_i e_i a = \lim_i a e_i = a \quad \text{for all } a \in A.$$

Recall that a C*-algebra is said to be *σ-unital* if it contains a *strictly positive element*, i.e., a positive element $a \in A$ such that aAa is dense in A. It is well-known that a C*-algebra is *σ-unital* if and only if it contains a *sequential* approximate unit, which is an approximate unit as above where the index set can be chosen to be numerable. Moreover, it is a non-trivial result that if $I \subseteq A$ is an ideal of a C*-algebra, then I admits an approximate unit $(e_i)_i \subseteq I$ that additionally satisfies

$$\lim_i \|e_i a - a e_i\| = 0 \quad \text{for all } a \in A.$$

A proof can be found in [111, Theorem I.9.16] or [7, Theorem 1]. It is common to refer to approximate units that satisfy this proerty as *quasicentral*. It should be mentioned that an approximate unit is not necessarily quasicentral, but a quasicentral approximate unit can be extracted from its convex hull.

2.1.3 Unitisations

Starting from a (possibly non-unital) C*-algebra A, there are many different ways of embedding it as an ideal into a bigger unital C*-algebra. The two canonical choices are the *proper unitisation*, which is the smallest unitisation when A is non-unital, and the *multiplier algebra*, which represents the largest unitisation in a certain sense. In order to form the proper unitisation, one considers $A^\dagger = A \times \mathbb{C}$ with addition defined on the components, while mutliplication and involution are given by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu), \quad (a, \lambda)^* = (a^*, \bar{\lambda})$$

for all $(a, \lambda), (b, \mu) \in A \times \mathbb{C}$. The norm on $A^\dagger$ is defined by

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\| \mid b \in A,\ \|b\| \leq 1\}.$$

The unit of $A^\dagger$ is then $\mathbf{1}_{A^\dagger} = (0, 1)$, and each element may be written as $a + \lambda \mathbf{1}_{A^\dagger}$, which is arguably more pleasing to the eye than (a, λ). One may clearly see that A sits as an ideal inside of $A^\dagger$ with quotient $A^\dagger / A \cong \mathbb{C}$. Moreover, one

[1] Here, by increasing we mean that $e_{i_0} \leq e_{i_1}$ whenever $i_0 \leq i_1$.

may notice that, if A was already unital with unit $\mathbf{1}_A$ to start with, then the map
$$A \oplus \mathbb{C} \to A^\dagger, \quad (a, \lambda) \mapsto a - \lambda \mathbf{1}_A + \lambda \mathbf{1}_{A^\dagger}$$
is a $*$-isomorphism.

The multiplier algebra of A, denoted by $\mathcal{M}(A)$, on the other hand, enjoys the following universal property: For every C*-algebra D that contains A as an ideal, there exists a unique $*$-homomorphism $\varphi : D \to \mathcal{M}(A)$ extending the identity map on A, with kernel $D \cap A^\perp$. In particular, $\mathcal{M}(A)$ is the largest unital C*-algebra containing A as an essential ideal. There are various ways to show that such a C*-algebra exists. The first method dates back to Busby [25], who constructed $\mathcal{M}(A)$ as the space of *double centralisers* associated to A. A double centraliser is a pair of linear maps (L, R) on A satisfying $aL(b) = R(a)b$ for all $a, b \in A$. In particular, the latter condition implies that $L(ab) = L(a)b$ and $R(ab) = aR(b)$ (one says that L is a *left centraliser* and R a *right centraliser*), and mreover that L and R are bounded with norm $\|L\| = \|R\|$. The set of double centralisers becomes a C*-algebra if equipped with componentwise addition, norm given by $\|(L, R)\| = \|L\| = \|R\|$, and with product and involution given by
$$(L_1, R_1)(L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1), \quad (L, R)^* = (R^*, L^*),$$
where $T^* : A \to A$ is the linear map obtained from T by setting $T^*(a) = (T(a^*))^*$ for all $a \in A$. Now, one can define $\mathcal{M}(A)$ as the C*-algebra of double centralisers of A, in which A embeds via the map
$$a \mapsto (L_a, R_a), \quad L_a(b) = ab, \; R_a(b) = ba$$
for all $a, b \in A$. To prove that $\mathcal{M}(A)$ has the aforementioned universal property, note that for any C*-algebra D containing A as an ideal, one can define (L_d, R_d) for $d \in D$ in the same way. We usually equip $\mathcal{M}(A)$ with the *strict topology*, i.e., the locally convex topology generated by the family of seminorms $\{\|\cdot\|_{\ell_a}, \|\cdot\|_{r_a} \mid a \in A\}$, where
$$\|x\|_{\ell_a} = \|ax\|, \quad \|x\|_{r_a} = \|xa\|$$
for all $x \in \mathcal{M}(A)$. In other words, a net $(x_i)_i$ in $\mathcal{M}(A)$ converges to $x \in \mathcal{M}(A)$ in the strict topology if and only if
$$\|x_i a - xa\| \to 0 \text{ and } \|ax_i - ax\| \to 0$$
for all $a \in A$. Clearly, if A is already unital, then $\mathcal{M}(A) = A$.

It is an important consequence of Cohen's factorisation theorem [36, 92] (see also [93, Theorem 4.1] or [16, Corollary II.5.3.8]) that if $B \subseteq \mathcal{M}(A)$ is a C*-subalgebra of the multiplier algebra of A, then $\overline{BA} = BA$.

One can always extend a morphism $\varphi : A \to B$ to the proper unitisation $\varphi^\dagger : A^\dagger \to B^\dagger$ by setting $\varphi^\dagger(a + \lambda \mathbf{1}_{A^\dagger}) = \varphi(a) + \lambda \mathbf{1}_{B^\dagger}$; see [24, Proposition 2.2.1] for a proof. However, when it comes to multiplier algebras, this is not always possible.

Definition 2.1.5. Let A and B be C*-algebras. A $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ is said to be *non-degenerate* if $\overline{\varphi(A)B} = B$.

Proposition 2.1.6 (see [132, Propositions 2.1 and 2.5])**.** *Let A and B be a* C*-*algebras, and $\varphi : A \to \mathcal{M}(B)$ a $*$-homomorphism. Then the following are equivalent:*

(i) *φ is non-degenerate;*

(ii) *φ extends uniquely to a unital $*$-homomorphism $\Phi : \mathcal{M}(A) \to \mathcal{M}(B)$ that is strictly continuous on the unit ball;*

(iii) *for any approximate unit $(e_i)_i$, the net $(\varphi(e_i))_i$ converges to $\mathbf{1}$ in the strict topology.*

Moreover, if φ is injective, then the extension Φ in (ii) *is injective as well.*

Definition 2.1.7. For a unital C*-algebra A, denote by $\mathcal{U}(A)$ the collection of all the unitaries in A. Then $\mathcal{U}(A)$ is a group with the product operation of A, and is referred to as the *unitary group* of A.

If B is a (not necessarily unital) C*-algebra, then we often work with either one of the following unitary groups:

$$\mathcal{U}(\mathbf{1} + B) := \mathcal{U}(B^\dagger) \cap (\mathbf{1} + B), \quad \mathcal{U}(\mathcal{M}(B)).$$

Assume that B is a unital C*-algebra. It follows from what was said before that $\mathcal{U}(\mathcal{M}(B)) = \mathcal{U}(B)$. On the other hand, $\mathcal{U}(\mathbf{1} + B) \neq \mathcal{U}(B)$. However, the preimage of $\mathcal{U}(\mathbf{1}+B)$ under the canonical $*$-isomorphism $B \oplus \mathbb{C} \to B^\dagger$ described before is $\mathcal{U}(B) \oplus \mathbf{1}$, and thus $\mathcal{U}(\mathbf{1} + B) \cong \mathcal{U}(B)$.

Example 2.1.8. The following examples will come up quite regularly.

(i) Consider a commutative C*-algebra, hence $A = \mathcal{C}_0(X)$ for some locally compact space X. Then, the proper unitisation of A is given by $\mathcal{C}(X^\dagger)$, where $X^\dagger$ is the *one-point compactification* (or *Alexandroff compactification*) of X. On the other hand, the multiplier algebra of A is given by $\mathcal{C}(\beta X)$, where βX is the Stone–Čech compactification of X. We note that there exists a natural isomorphism $\mathcal{C}(\beta X) \cong \mathcal{C}_b(X)$, where the latter denotes the space of bounded continuous functions on X.

(ii) The previous example can be viewed as a special case of the multiplier algebra of $\mathcal{C}_0(X, B)$ for a locally compact space X and a C*-algebra B. In fact, it shown in [1, Corollary 3.4] that $\mathcal{M}(\mathcal{C}_0(X, B)) = \mathcal{C}_b^s(X, \mathcal{M}(B))$, where the latter is the C*-algebra of bounded strictly continuous functions on X with values in $\mathcal{M}(B)$.

(iii) Consider the compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. We already know that $\mathcal{K}(\mathcal{H})$ sits as an ideal inside $\mathcal{B}(\mathcal{H})$. It is moreover the case that the multiplier algebra of $\mathcal{K}(\mathcal{H})$ is precisely $\mathcal{B}(\mathcal{H})$.

2.1.4 Corona algebras

One of the basic examples of ideal in a C*-algebra, as noted above, is the C*-algebra of compact operators $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. Consider an infinite dimensional, separable Hilbert space $\mathcal{H}$, and recall that the *Calkin algebra* is defined as the quotient C*-algebra $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Generalising this example, one may consider any (non-unital) C*-algebra A, and form the quotient C*-algebra

$$\mathcal{Q}(A) = \mathcal{M}(A)/A,$$

which is called *corona algebra* of A. In the commutative setting, for a locally compact (non-compact) space X, the corona algebra of $A = \mathcal{C}_0(X)$ is given by

$$\mathcal{M}(\mathcal{C}_0(X))/\mathcal{C}_0(X) \cong \mathcal{C}(\beta X)/\mathcal{C}_0(X) \cong \mathcal{C}(\beta X \setminus X),$$

where the space $\beta X \setminus X$ is sometimes called the *corona set* (or *Stone–Čech boundary*) of X. The commutative case reveals that the corona can be viewed as what remains after removing a set from its compactification. We will discuss corona algebras in more detail in Chapter 5.

2.1.5 Inductive limits

An *inductive system* of C*-algebras $\{A_n, \varphi_n\}_{n\in\mathbb{N}}$ is a sequence of C*-algebras A_n and *connecting* $*$-homomorphisms $\varphi_n : A_n \to A_{n+1}$. It is a basic fact of C*-algebra theory that every inductive system of C*-algebras admits an *inductive limit*. This is a C*-algebra A together with a sequence of $*$-homomorphisms

$$\varphi_{n,\infty} : A_n \to A$$

such that $\varphi_{n,\infty} = \varphi_{n+1,\infty} \circ \varphi_n$ for all $n \in \mathbb{N}$, that satisfy the following universal property. If B is another C*-algebra and there are $*$-homomorphisms

$$\psi_{n,\infty} : A_n \to B$$

such that $\psi_{n,\infty} = \psi_{n+1,\infty} \circ \varphi_n$ for all $n \in \mathbb{N}$, then there exists a unique $*$-homomorphism $\psi : A \to B$ such that $\psi_{n,\infty} = \psi \circ \varphi_{n,\infty}$ for all $n \in \mathbb{N}$. It is common to use the notation $\varinjlim\{A_n, \varphi_n\}$ for the inductive limit of $\{A_n, \varphi_n\}_{n\in\mathbb{N}}$, or simply $\varinjlim A_n$ if the connecting maps are clear from context.

Example 2.1.9. A *uniformly hyperfinite* (*UHF*) algebra is a C*-algebra isomorphic to the inductive limit of an inductive system $\{M_{n_k}, \varphi_k\}_{k\in\mathbb{N}}$ consisting of matrix algebras and unital connecting maps. To each UHF-algebra, one associates a *supernatural number* in the following way. First, enumerate the prime numbers in ascending order as $\{p_1, p_2, \dots\}$. For each $k \in \mathbb{N}$, one may write n_k as the product of prime numbers (with multiplicity), i.e., $n_k = \prod_{i=1}^{\infty} p_i^{r_{k,i}}$, where al but finitely many $r_{k,i}$ are zero. If one thinks of $M := \varinjlim M_{n_k}$ as an infite matrix, then $r_i = \sup_{k\in\mathbb{N}}\{r_{k,i}\}$ represents how many times the prime number p_i divides the size of M, which may well be infinite, and measures the growth of the sequence of M_{n_k}'s. The supernatural number associated to the inductive system $\{M_{n_k}, \varphi_n\}_{n\in\mathbb{N}}$ is given by the formal product

$$\mathfrak{n} = \prod_{i=1}^{\infty} p_i^{r_i}.$$

Glimm proved in [81] that two UHF-algebras are isomorphic if and only if they have the same supernatural number. An important subclass of UHF-algebras is that of *infinite type*, i.e., those with supernatural number of the form $\prod_{i=1}^{\infty} p_i^{r_i}$, where r_i are either 0 or ∞. For instance, the UHF-algebra associated to the supernatural number $\mathfrak{n} = 2^\infty$ is — up to isomorphism — determined by the inductive sequence $\{M_{2^n}, \varphi_n\}_{n\in\mathbb{N}}$, where

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{for all } a \in M_{2^n}.$$

The UHF-algebra M_{2^∞} is referred to as the *CAR-algebra* (for *canonical anticommutation relations*). This terminology arises from the fact that, for any infinite-dimensional, separable Hilbert space $\mathcal{H}$, unital C*-algebra A and conjugate linear map $s : \mathcal{H} \to A$ satisfying the canonical anticommutation relations, i.e.,

$$s(\xi)s(\eta) + s(\eta)s(\xi) = 0 \quad \text{and} \quad s(\xi)^*s(\eta) + s(\eta)s(\xi)^* = \langle \xi, \eta\rangle \cdot \mathbf{1}_A$$

for all $\xi, \eta \in \mathcal{H}$, then M_{2^∞} is isomorphic to the C*-subalgebra of A generated by the range of s.

Example 2.1.10. Generalising the previous example, one says that a C*-algebra is *approximately finite-dimensional* (*AF*) if it is isomorphic to the inductive limit of finite-dimensional C*-algebras. Recall that a finite-dimensional C*-algebra A is isomorphic to a finite direct sum of matrix algebras by a Wedderburn-type theorem, namely, there exists some $m \in \mathbb{N}$ such that $A \cong \bigoplus_{k=1}^{m} M_{n_k}$, where $n_k \in \mathbb{N}$ for all $k = 1, \dots, m$.

Example 2.1.11. Adding another level of generality, one says that a C*-algebra A is *approximately homogeneous* (AH) if it is isomorphic to the inductive limit of an inductive system of the form $\{\mathcal{C}_0(X_n, M_{n_k}), \varphi_n\}_{n\in\mathbb{N}}$, where X_n is a locally compact space for every $n \in \mathbb{N}$. The definition as stated here is due to Rørdam [171], but AH-algebras were already defined (in a slightly different way) and studied by Blackadar in [15].

We want to single out a specific AH-algebra from [171], whose construction was inspired by Mortensen's work [144]. Consider a dense sequence $(t_n)_n$ in $[0,1)$, and C*-algebras of the form $A_n = \mathcal{C}_0([0,1), M_{2^n})$ with connecting maps

$$A_n \xrightarrow{\varphi_n} A_{n+1}, \quad (\varphi_n(f))(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(\max\{t_n, t\}) \end{pmatrix}.$$

The resulting AH-algebra is denoted by $\mathcal{A}_{[0,1]}$, and enjoys many interesting properties that are relevant to the classification program of C*-algebras, such as $\mathcal{O}_2$-stability. Moreover, we know from [171, Proposition 2.1] (cf. [144, Theorem 1.2.1]) that the ideals of $\mathcal{A}_{[0,1]}$ correspond to points in $[0,1]$. This example will appear mutliple times in this manuscript.

2.1.6 The GNS construction

A linear map $\varphi : A \to B$ between C*-algebras is said to be *positive* if $\varphi(A_+) \subseteq B_+$. In this case, φ is also bounded and $*$-preserving. A positive linear functional $\varphi : A \to \mathbb{C}$ is said to be a *state* if is has norm 1. If A is unital, this is the case exactly when $\varphi(\mathbf{1}_A) = 1$. More generally, this is the case when $\lim_i \varphi(e_i) = 1$ for every (or some) approximate unit $(e_i)_i$ of A. A state is said to be *faithful* if $\varphi(a^*a) = 0$ implies that $a = 0$. One denotes the collection of all states on A by $\mathcal{S}(A)$, which is usually equipped with the weak*-topology. Then, $\mathcal{S}(A)$ is weak*-compact if A is unital.

A $*$-*representation*, or just *representation*, of a C*-algebra A on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. The following terminology is widely used in the representation theory of C*-algebras. A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is said to be

- *faithful*, whenever π is injective;
- *cyclic*, whenever there exists a unit vector $\xi \in \mathcal{H}$, called *cyclic vector*, such that $\pi(A)\xi$ is dense in $\mathcal{H}$;
- *non-degenerate*, whenever $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

To every state $\varphi : A \to \mathbb{C}$, one may associate a representation via the GNS (Gelfand–Naimark–Segal) construction. Let us briefly outline this construction. First, consider
$$N_\varphi = \{a \in A \mid \varphi(a^*a) = 0\},$$
which is a closed left ideal of A. Note that $\varphi(a^*b) = 0$ if either $a \in N_\varphi$ or $b \in N_\varphi$ by the Cauchy–Schwarz inequality. Subsequently, form a vector space
$$\mathcal{H}_\varphi^0 = A/N_\varphi$$
and note that one can define an inner product on $\mathcal{H}_\varphi^0$ as
$$\langle a + N_\varphi, b + N_\varphi \rangle_\varphi = \varphi(b^*a), \quad \text{for all } a, b \in A.$$
The Hilbert space completion of $\mathcal{H}_\varphi^0$ with respect to the inner product $\langle \cdot, \cdot \rangle_\varphi$, which is denoted by $\mathcal{H}_\varphi$,[2] will be the Hilbert space of the GNS representation. Note that each element $a \in A$ naturally acts as a bounded operator on $\mathcal{H}_\varphi^0$ by
$$b + N_\varphi \mapsto ab + N_\varphi, \quad \text{for all } b \in A.$$
Let us call by $\pi_\varphi(a)$ the extension of this operator to a bounded operator on $\mathcal{H}_\varphi$. This yields a representation
$$\pi_\varphi : A \to \mathcal{B}(\mathcal{H}_\varphi), \quad a \mapsto \pi_\varphi(a),$$
which is called the *GNS representation* of A associated to φ. One may also notice that this representation is cyclic with respect to the unit vector
$$\xi_\varphi = \lim_i (e_i + N_\varphi)$$
for any approximate unit $(e_i)_i$ of A. Almost by definition, one can see that the original state φ can be recovered from π_φ as the *vector state* associated to it, namely,
$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle_\varphi \quad \text{for all } a \in A.$$
One can easily see form this construction that if φ is a state such that $\varphi(a) \neq 0$, then $\pi_\varphi(a) \neq 0$. Combined with the fact that there is an abundance of states on a C*-algebra, one can conclude that the *universal GNS representation* of A, which is given by
$$\pi_{\mathrm{u}} : A \to \mathcal{B}\Big(\bigoplus_{\varphi \in \mathcal{S}(A)} \mathcal{H}_\varphi \Big), \quad \pi_{\mathrm{u}}(a) = \bigoplus_{\varphi \in \mathcal{S}(A)} \pi_\varphi(a)$$
is faithful.

[2] Another common notation for $\mathcal{H}_\varphi$ is $L^2(A, \varphi)$.

2.1.7 Universal C^*-algebras

Given a non-empty set of variables $\mathcal{G}$, denote by $\mathcal{G}^*$ the set $\{x^* \mid x \in \mathcal{G}\}$, which we consider as a disjoint set from $\mathcal{G}$. One can define a *noncommutative* $*$-*polynomial* in $\mathcal{G}$ as a formal expression given by a finite combination of products and sums with complex coefficients of variables in $\mathcal{G} \cup \mathcal{G}^*$. A set of *relations* $\mathcal{R}$ on $\mathcal{G}$ consists of a set of formal statements of the form

$$\|p(\mathcal{G})\| \leq \eta,$$

where p is a noncommutative $*$-polynomial in $\mathcal{G}$, and $\eta \in [0, \infty)$.

Given $\mathcal{G}$ and $\mathcal{R}$ as above, a representation of $(\mathcal{G} \mid \mathcal{R})$ is a map $\pi : \mathcal{G} \to A$ for some C^*-algebra A such that the relations $\mathcal{R}$ are satisfied by $\pi(\mathcal{G})$. One says that $(\mathcal{G} \mid \mathcal{R})$ is *bounded* if for every $x \in \mathcal{G}$ one has that

$$\sup\{\|\pi(x)\| \mid \pi : \mathcal{G} \to A \text{ is a representation of } (\mathcal{G} \mid \mathcal{R})\} < \infty.$$

Moreover, one says that a representation π of $(\mathcal{G} \mid \mathcal{R})$ into A is *universal* if, whenever $\rho : \mathcal{G} \to B$ is another representation in a C^*-algebra B, there exists a unique $*$-homomorphism $\varphi : A \to B$ such that $\varphi \circ \pi = \rho$. If this is the case, $A = \mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$ is called the *universal* C^*-*algebra* generated by $(\mathcal{G} \mid \mathcal{R})$, which is unique up to isomorphism. The existence of $\mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$ is provided precisely when $(\mathcal{G} \mid \mathcal{R})$ is bounded; see [157, Theorem 2.9.3].

Example 2.1.12. There are a few examples of universal C^*-algebras that will be absolutely central in this exposition.

(i) Set $\mathcal{G} = \{u, \mathbf{1}\}$, and consider the relations $\mathcal{R} = \{u^*u = \mathbf{1}, uu^* = \mathbf{1}\}$. Then $(\mathcal{G} \mid \mathcal{R})$ is clearly bounded, and one has that $\mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$ is the universal unital C^*-algebra generated by a unitary element, i.e., $\mathcal{C}(\mathbb{T})$.

(ii) Take now $\mathcal{G} = \{e_{i,j} \mid 1 \leq i, j \leq n\}$ for some $n > 0$, and consider the set $\mathcal{R}$ containing the following relations,

$$e_{i,i} = e_{i,i}^* = e_{i,i}^2$$

$$e_{i,j} e_{k,\ell} = \delta_{j,k} e_{i,\ell}$$

The generators $\{e_{i,j} \mid 1 \leq i, j \leq n\}$ are usually called *matrix units*. Then, it is not hard to see that $\mathrm{C}^*(\mathcal{G} \mid \mathcal{R}) \cong M_n$ by sending each matrix unit $e_{i,j}$ to the matrix with 1 in the entry (i, j) and 0 elsewhere.

(iii) The countably-infinite analogue of the previous example, where one chooses an infinite set of *matrix units*, produces a copy of the compact operators on some separable, infinite-dimensional Hilbert space. We denote this universal C^*-algebra by $\mathcal{K}$.

(iv) By making use of some continuous functional calculus, one can see that the universal C*-algebra generated by a positive contraction can be identified with $\mathcal{C}_0(0,1]$, up to isomorphism, and the canonical generator is $\mathrm{id}_{(0,1]}$.

2.1.8 Nuclearity

Given two C*-algebras A and B, the algebraic tensor product $A \odot B$ can be equipped with the involution operation given by

$$(a \otimes b)^* = a^* \otimes b^*, \quad \text{for } a \in A,\, b \in B,$$

which turns it into a $*$-algebra. In general, there is no unique way of completing $A \odot B$ to a C*-algebra. We will only consider the following two norm-completions. Since they are both well-known constructions, we omit a lot of details that can be found, for instance, in Brown and Ozawa's book [24].

Definition 2.1.13. Let A and B be C*-algebras. Then the *maximal norm* (or *universal norm*) on $A \odot B$ is given by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| \mid \pi : A \odot B \to \mathcal{B}(\mathcal{H}) \text{ is a representation}\}$$

for all $x \in A \odot B$. The completion of $A \odot B$ with respect to this norm is called the *maximal tensor product*, and is denoted by $A \otimes_{\max} B$.

The advantage of working with the maximal norm is that it satisfies the following universal property, which follows from the definition, as well as from the universal property of $A \odot B$.

Proposition 2.1.14. *Let A, B and C be* C*-*algebras, and consider any $*$-homomorphism $\pi : A \odot B \to C$. Then there exists a unique $*$-homomorphism $A \otimes_{\max} B \to C$ extending π. In particular, for any pair of $*$-homomorphisms $\pi_A : A \to C$ and $\pi_B : B \to C$ with commuting ranges, there exists a unique $*$-homomorphism $A \otimes_{\max} B \to C$ given on elementary tensors by $a \otimes b \mapsto \pi_A(a)\pi_B(b)$ for all $a \in A$, $b \in B$.*

Recall that for any pair of representations $\pi_A : A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$, there exists a unique representation

$$\pi_A \otimes \pi_B : A \odot B \to \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

such that $(\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a)\pi_B(b)$ for all $a \in A$ and $b \in B$. Moreover, one can show that $\pi_A \otimes \pi_B$ is faithful whenever π_A and π_B are both faithful representations.

Definition 2.1.15. Let A and B be C*-algebras, and $\pi_A : A \to \mathcal{B}(\mathcal{H}_A)$, $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$ faithful representations. Then the *spatial norm* (or *minimal norm*) on $A \odot B$ is given by

$$\left\| \sum_{i=1}^{n} a_i \otimes b_i \right\| = \left\| (\pi_A \otimes \pi_B) \left(\sum_{i=1}^{n} a_i \otimes b_i \right) \right\| \leq \sum_{i=1}^{n} \|a_i\| \|b_i\|.$$

where $a_i \otimes b_i \in A \odot B$ are elementary tensors for all $i = 1, \ldots, n$. The completion of $A \odot B$ with respect to this norm is referred to as the *spatial tensor product*, denoted by $A \otimes B$.

At first glance, the spatial norm may appear to depend on the choice of representations. However, as a consequence of Takesaki's theorem, it actually induces the smallest C*-norm on $A \odot B$, which in turn ensures that it is independent of the choice of π_A and π_B. Note that the universal property of the maximal tensor product yields a canonical quotient map

$$\pi : A \otimes_{\max} B \to A \otimes B$$

for any pair of C*-algebras A, B.

We introduce nuclearity as a condition on tensor products.

Definition 2.1.16. A C*-algebra A is said to be *nuclear* (or *amenable*) if there exists a unique C*-norm on $A \odot B$ for every C*-algebra B.

Example 2.1.17. Consider a C*-algebra A, and denote by $M_n(A)$ the space of $n \times n$ matrices with entries in A, referred to as its *matrix amplification.* In this example, we show that $M_n(A)$ is a C*-algebra, that $M_n(A) \cong M_n \odot A$, and hence that M_n is nuclear. The adjoint of a matrix $(a_{i,j})_{i,j=1}^n$ in $M_n(A)$ is defined as $\left((a_{i,j})_{i,j=1}^n\right)^* = (a_{j,i}^*)_{i,j=1}^n$, and operations in $M_n(A)$ are induced by those in M_n and A. Hence, $M_n(A)$ is a $*$-algebra in a canonical way. In fact, $M_n(A)$ is a C*-algebra. To see this, remember that for any Hilbert space $\mathcal{H}$, one may identify $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^{\oplus n})$. Consider any faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, and define

$$\pi^{(n)} : M_n(A) \hookrightarrow M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^{\oplus n}), \quad (a_{i,j})_{i,j=1}^n \mapsto (\pi(a_{i,j}))_{i,j=1}^n.$$

One can show that $\pi^{(n)}$ is a faithful representation of $M_n(A)$ on $\mathcal{H}^{\oplus n}$, and hence induces a C*-norm on $M_n(A)$ by setting

$$\|(a_{i,j})_{i,j=1}^n\| = \|\pi^{(n)} \left((a_{i,j})_{i,j=1}^n\right)\|, \quad \text{for all } (a_{i,j})_{i,j=1}^n \in M_n(A).$$

It turns out that $M_n(A)$ endowed with this norm is closed. Indeed, if $\left((a_{i,j}^{(k)})_{i,j=1}^n\right)_k \subseteq M_n(A)$ is a sequence of matrices such that $\pi^{(n)}((a_{i,j}^{(k)})_{i,j=1}^n)$ converges to an operator in $\mathcal{B}(\mathcal{H}^{\oplus n})$, then $\pi(a_{i,j}^{(k)}) \xrightarrow{k\to\infty} T_{i,j} \in \mathcal{B}(\mathcal{H})$, and $T_{i,j} = \pi(a_{i,j})$ for some $a_{i,j} \in A$ because A is closed. Then, the map given by

$$M_n(A) \to M_n \odot A, \quad (a_{i,j})_{i,j=1}^n \mapsto \sum_{i,j=1}^n e_{i,j} \otimes a_{i,j},$$

where $e_{i,j}$ is a set of matrix units that generates M_n, is a $*$-isomorphism, and $M_n \odot A$ is already a C*-algebra with the norm coming from $M_n(A)$. In particular, it follows that M_n is nuclear.

Theorem 2.1.18. *Nuclearity enjoys the following permanence properties.*

(i) *The proper unitisation $A^\dagger$ of a nuclear* C**-algebra A is nuclear;*

(ii) *If A and B are stably isomorphic, i.e., $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, and A is nuclear, then B is nuclear as well;*

(iii) *If A and B are nuclear* C**-algebras, then $A \otimes B$ is nuclear as well;*

(iv) *If $\{A_n, \varphi_n\}_n$ is an inductive system of nuclear* C**-algebras, then $\varinjlim A_i$ is nuclear as well;*

(v) *Hereditary* C**-subalgebras of a nuclear* C**-algebra are nuclear;*

(vi) *Quotients of nuclear* C**-algebras are nuclear.*

2.1.9 Completely positive maps

When $*$-homomorphisms are too restrictive, one needs to work with completely positive maps, which, despite lacking multiplicativity, still encode significant information. In fact, one could argue that they are the appropriate morphisms when dealing with *operator systems*, which are unital, norm-closed, self-adjoint linear subspaces of unital C*-algebras. However, we will restrict our focus to the C*-algebra setting and, therefore, define completely positive maps only for C*-algebras.

Definition 2.1.19. Let A and B be C*-algebras. For any linear map $\varphi : A \to B$ and $n \geq 1$, let $\varphi^{(n)} : M_n(A) \to M_n(B)$ denote its *matrix amplification*, i.e.,

$$\varphi^{(n)} : M_n(A) \to M_n(B), \quad (a_{i,j})_{i,j=1}^n \mapsto \left(\varphi(a_{i,j})\right)_{i,j=1}^n .$$

A *completely positive* map (abbreaviated *c.p.* map) is a linear map $\varphi : A \to B$ such that its matrix amplification $\varphi^{(n)}$ is positive for all $n \geq 1$. If φ is also contractive, we refer to it as a *completely positive contractive* map (abbreaviated *c.p.c.* map). When A and B are unital, $\varphi : A \to B$ is said to be a *unital completely positive* map (abbreaviated *u.c.p.* map) if it is unital and c.p.

Using a famous theorem of Stinespring, one can show that c.p. maps are well-behaved with respect to tensor products.

Proposition 2.1.20. *Let A, B, C, and D be* C*-*algebras, and* $\varphi : A \to C$, $\psi : B \to D$ *c.p. maps. Then, there exists a unique c.p. map* $\varphi \otimes \psi : A \otimes B \to C \otimes D$ *such that* $(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \otimes \psi(b)$ *for all* $a \in A$ *and* $b \in B$, *and* $\|\varphi \otimes \psi\| = \|\varphi\|\|\psi\|$. *Moreover, the same holds if one replaces the spatial tensor product with the maximal tensor product.*

The following result of Arveson is one of the foundational results about c.p.c. maps; see [24, Theorem 1.6.1] for a proof.

Theorem 2.1.21 (Arveson's extension theorem)**.** *Let A be a unital* C*-*algebra, $\mathcal{H}$ a Hilbert space, and $E \subseteq A$ a closed, self-adjoint linear subspace containing the unit of A. Then, every c.p.c. map $\varphi : E \to \mathcal{B}(\mathcal{H})$ extends to a c.p.c. map $\tilde{\varphi} : A \to \mathcal{B}(\mathcal{H})$.*

Definition 2.1.22. A c.p. map between C*-algebras $\varphi : A \to B$ is said to be *factorable* if there exists $n \in \mathbb{N}$ and two c.p. maps $\psi : A \to M_n$, $\eta : M_n \to B$ such that $\varphi = \eta \circ \psi$.

Moreover, one says that a c.p. map $\varphi : A \to B$ is *nuclear* if there exists a net of factorable c.p. maps φ_i that converges to φ pointwise in norm, namely

$$\lim_i \|\varphi(a) - \varphi_i(a)\| = 0, \quad \text{for all } a \in A.$$

A C*-algebra A is said to have the *completely positive approximation property* (abbreviated *CPAP*) if id_A is nuclear.

The collection of all factorable c.p. maps is a *convex cone*, that is, for all $t \in \mathbb{R}_+$ and for all pairs φ, ψ of factorable maps, one has that $\varphi + t\psi$ is factorable as well; see [24, Lemma 2.3.6] for a proof. It follows that the collection of nuclear maps is a convex cone that is closed in the point-norm topology.

The following profound theorem by Choi, Effros, and Kirchberg characterizes nuclear C*-algebras and is highly valuable in applications.

Theorem 2.1.23 (see [35, Theorem 3.1] and [113, Corollary 1])**.** *Let A be a* C*-*algebra. Then, A is nuclear if and only if it has the CPAP.*

The following (non-trivial) lemma shows that nuclearity of a c.p. map $\varphi : A \to B$ can be phrased using tensor products. We omit its proof, which can be found in [24, Lemma 3.6.10 and Corollary 3.8.8].

Lemma 2.1.24. *Let A and B be unital* C**-algebras and* $\varphi : A \to B$ *a u.c.p. map. Then,* φ *is nuclear if and only if, for every* C**-algebra* D*, there exists a c.p. map* $\psi : D \otimes A \to D \otimes_{\max} B$ *such that the following diagram commutes,*

$$\begin{tikzcd} D \otimes_{\max} A \arrow[rr, "\mathrm{id}_D \otimes_{\max} \varphi"] \arrow[dr, "\pi"'] & & D \otimes_{\max} B \\ & D \otimes A \arrow[ur, "\psi"'] & \end{tikzcd}$$

where π *is the canonical quotient map. In particular, when such a* ψ *exists, one says that* $\mathrm{id}_D \otimes_{\max} \varphi$ *factors through* $D \otimes A$.

We record here an important fact about nuclear c.p. maps that will be used in Chapter 3.

Lemma 2.1.25 (see [64, Lemma 3.9])**.** *Let* A, B *and* C *be* C**-algebras, and assume that* C *is nuclear. Let* $\varphi : A \to B$ *and* $\psi : C \to B$ *be c.p. maps with commuting ranges. Then, if* φ *is nuclear, the product map* $\varphi \times \psi : A \otimes C \to B$ *is a nuclear c.p. map.*

Proof. Let us first assume that D is a unital C*-algebra, and consider the map

$$\mathrm{id}_D \otimes_{\max} (\varphi \times \psi) : D \otimes_{\max} (A \otimes C) \to D \otimes_{\max} B.$$

Since φ is nuclear, by Lemma 2.1.24, $\mathrm{id}_D \otimes_{\max} \varphi : D \otimes_{\max} A \to D \otimes_{\max} B$ factors through $D \otimes A$ via some map $\Phi : D \otimes A \to D \otimes_{\max} B$. By assumption φ and ψ have commuting ranges. Therefore, the range of $\mathbf{1}_D \otimes_{\max} \psi : C \to D \otimes_{\max} B$ commutes with the range of $\mathrm{id}_D \otimes_{\max} \varphi$, and hence also with the range of Φ. We may then form the product map

$$\Psi = \Phi \times (\mathbf{1}_D \otimes_{\max} \psi) : (D \otimes A) \otimes C \to D \otimes_{\max} B$$

By composing with the canonical isomorphism $(D \otimes A) \otimes C \cong D \otimes (A \otimes C)$, we have that Ψ fits into the following commutative diagram

$$\begin{tikzcd} D \otimes_{\max} (A \otimes C) \arrow[rr, "\mathrm{id}_D \otimes_{\max} (\varphi \times \psi)"] \arrow[dr, "\pi"'] & & D \otimes_{\max} B \\ & D \otimes (A \otimes C) \arrow[ur, "\Psi"'] & \end{tikzcd}$$

and thus $\mathrm{id}_D \otimes_{\max} (\varphi \times \psi)$ factors through $D \otimes (A \otimes C)$. It follows from Lemma 2.1.24 that $\varphi \times \psi$ is nuclear.

If D is not unital, then one may consider its proper unitisation $D^\dagger$. From the previous part of the proof, $\mathrm{id}_{D^\dagger} \otimes_{\max} (\varphi \times \psi)$ factors through $D^\dagger \otimes (A \otimes C)$, and since the restriction of π to $D \otimes_{\max} (A \otimes C)$ has range in $D \otimes (A \otimes C)$, this concludes the proof. □

Let us introduce another important regularity property of C*-algebras.

Definition 2.1.26. A C*-algebra A is said to be *exact* if, for any short exact sequence of C*-algebras

$$0 \longrightarrow I \xrightarrow{\iota} E \xrightarrow{\pi} Q \longrightarrow 0$$

the sequence

$$0 \longrightarrow A \otimes I \xrightarrow{\mathrm{id}_A \otimes \iota} A \otimes E \xrightarrow{\mathrm{id}_A \otimes \pi} A \otimes Q \longrightarrow 0$$

is a short exact sequence as well.[3]

We note right away that, in Definition 2.1.26, replacing $A \otimes -$ with the maximal tensor product $A \otimes_{\max} -$ automatically satisfies the condition; see [24, Proposition 3.7.1]. Therefore, if A is nuclear, it is also exact.

Definition 2.1.27. A C*-algebra A is said to be *nuclearly embeddable* if there exists a nuclear embedding $\iota : A \to B$ into some C*-algebra B.

The following important theorem is due to Kirchberg; see [115] or [163].

Theorem 2.1.28 (see [24, Theorem 3.9.1])**.** *A* C* *-algebra A is exact if and only if it is nuclearly embeddable.*

Recall that a c.p.c. map $\varphi : A \to B$ is said to be of *order zero* if, for any $a, b \in A_+$, one has that $ab = 0$ implies $\varphi(a)\varphi(b) = 0$. This means that φ preserves othogonality in A, bringing it one step closer to being a $*$-homomorphism. Order zero maps provide a highly useful characterisation of certain $*$-homomorphisms.

Theorem 2.1.29 (see [210, Corollary 4.1])**.** *Let A and B be* C* *-algebras, and $\varphi : A \to B$ a c.p.c. order zero map. The map defined as $\rho_\varphi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a)$ induces a $*$-homomorphism $\rho_\varphi : \mathcal{C}_0(0,1] \otimes A \to B$. Conversely, any $*$-homomorphism $\rho : \mathcal{C}_0(0,1] \otimes A \to B$ induces a c.p.c. order zero map $\varphi_\rho : A \to B$ via $\varphi_\rho(a) = \rho(\mathrm{id}_{(0,1]} \otimes a)$ for all $a \in A$.*

[3] In other words, the functor $A \otimes -$ is exact.

As a consequence of this result, one has a one-to-one correspondence between c.p.c. order zero maps $A \to B$ and $*$-homomorphisms $\mathcal{C}_0(0,1] \otimes A \to B$. We will often refer to $CA = \mathcal{C}_0(0,1] \otimes A$ as the *cone* over A.

Remark 2.1.30. By [63, Theorem 2.9] the one-to-one correspondence between c.p.c. order zero maps $A \to B$ and $*$-homomorphisms $CA \to B$ from Theorem 2.1.29 restricts to a one-to-one correspondence as follows,

$$\left\{ \begin{array}{c} \text{nuclear c.p.c. order zero maps} \\ A \to B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{nuclear } *\text{-homomorphisms} \\ CA \to B \end{array} \right\}.$$

The following lemma is well-known among experts. Since we struggled to pin down a direct reference, we give a short proof for the reader's convenience.

Lemma 2.1.31. *Let X be a locally compact space, and A, B C^*-algebras. A c.p. map $\varphi : A \to \mathcal{C}_0(X, B)$ is nuclear if and only if $\mathrm{ev}_x \circ \varphi : A \to B$ is nuclear for all $x \in X$.*

Proof. Only the "if" part is non-trivial. It is well-known (see [24, Proposition 2.4.2] and its proof) that the nuclearity of $\mathcal{C}_0(X)$ is witnessed by two nets of c.p.c. maps $\kappa_\lambda : \mathcal{C}_0(X) \to \mathbb{C}^{n_\lambda}$ and $\psi_\lambda : \mathbb{C}^{n_\lambda} \to \mathcal{C}_0(X)$ with $\psi_\lambda \circ \kappa_\lambda \to \mathrm{id}_{\mathcal{C}_0(X)}$ in point-norm, where for each λ the map κ_λ is of the form $\kappa_\lambda = \mathrm{ev}_{x_1} \oplus \mathrm{ev}_{x_2} \oplus \cdots \oplus \mathrm{ev}_{x_{n_\lambda}}$, for finitely many points $x_1, \ldots, x_{n_\lambda} \in X$. By tensoring this approximate factorization with B, we obtain that $(\psi_\lambda \otimes \mathrm{id}_B) \circ (\kappa_\lambda \otimes \mathrm{id}_B) \to \mathrm{id}_{\mathcal{C}_0(X,B)}$ in point-norm.

Assuming that $\mathrm{ev}_x \circ \varphi : A \to B$ is nuclear for all $x \in X$, it thus follows that $(\kappa_\lambda \otimes \mathrm{id}_B) \circ \varphi$ is nuclear for every λ, since it is a finite direct sum of compositions with evaluation maps. We can thus conclude that $\varphi = \lim_\lambda (\psi_\lambda \otimes \mathrm{id}_B) \circ (\kappa_\lambda \otimes \mathrm{id}_B) \circ \varphi$ is the point-norm limit of nuclear c.p. maps, so it is itself nuclear. $\square$

The following result, due to Choi and Effros (see [34, Theorem 3.10]), is one of the most powerful tools when working with quotients and nuclearity.

Theorem 2.1.32 (The Choi–Effros lifting theorem)**.** *Let A and B be C^*-algebras with A separable, and $I \subseteq B$ an ideal. Then, every nuclear c.p.c. map $\varphi : A \to B/I$ admits a nuclear c.p.c. lift, that is, a nuclear c.p.c. map $\tilde{\varphi} : A \to B$ that fits into the following commutative diagram.*

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & B/I \end{tikzcd}$$

The following definition is introduced, as in [156], to improve readability.

Definition 2.1.33. Let A and B be C*-algebras. We say that a c.p.c. map $\varphi : A \to B_\infty$ is *nuclearly liftable* if it admits a lift $(\varphi_n)_{n\in\mathbb{N}} : A \to \ell^\infty(\mathbb{N}, B)$ consisting of nuclear c.p.c. maps.

Nuclearly liftable maps are nuclear under the assumption that the domain C*-algebra is separable and exact. This is a fact that follows from the Choi–Effros lifting theorem, Theorem 2.1.32, and an argument of Dadarlat from [45]. We give a proof for the reader's convenience.

Proposition 2.1.34. *Let A and B be* C*-*algebras with A separable and exact, and $\varphi : A \to B_\infty$ a c.p.c. map. Then φ is nuclearly liftable precisely when it is nuclear.*

Proof. The Choi–Effros theorem (see Theorem 2.1.32) implies that if $\varphi : A \to B_\infty = \ell^\infty(\mathbb{N}, B)/c_0(\mathbb{N}, B)$ is nuclear, then it lifts to a nuclear c.p.c. map $A \to \ell^\infty(\mathbb{N}, B)$. Hence, we only need to show that if φ is nuclearly liftable, then it is nuclear. Let $\Phi = (\varphi_n)_{n\in\mathbb{N}} : A \to \ell^\infty(\mathbb{N}, B)$ be a lift of φ consisting of nuclear c.p.c. maps. We follow an adaptation of the argument used by Dadarlat to prove [45, Proposition 3.3]. In order to check that φ is nuclear, let us fix $\varepsilon > 0$, and an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ with dense union. Since A is exact, we may find a nuclear embedding $\psi : A \hookrightarrow C$ for some C*-algebra C. By nuclearity of φ_n, we may find, for each $n \in \mathbb{N}$, an integer $k_n \in \mathbb{N}$ and a pair of c.p.c. maps $\pi_n : \psi(A) \to M_{k_n}$ and $\rho_n : M_{k_n} \to B$ such that

$$\max_{a\in\mathcal{F}_n} \|\varphi_n(a) - (\rho_n \circ \pi_n \circ \psi)(a)\| \leq \varepsilon.$$

By Arveson's extension theorem (see Theorem 2.1.21), each map π_n admits an extension to a c.p.c. map

$$\hat{\pi}_n : C \to M_{k(n)} \quad \text{such that} \quad \hat{\pi}_n \restriction_{\psi(A)} = \pi_n.$$

Now, the map $\eta : C \to \ell^\infty(\mathbb{N}, B)$ given by $\eta(c) = ((\rho_n \circ \hat{\pi}_n)(c))_n$ for $c \in C$, satisfies

$$\|\Phi(a) - (\eta \circ \psi)(a)\| = \sup_{n\in\mathbb{N}} \|\varphi_n(a) - (\rho_n \circ \pi_n \circ \psi)(a)\| \leq \varepsilon$$

for all $a \in \bigcup_n \mathcal{F}_n$. Hence, density of $\bigcup_n \mathcal{F}_n$ in A and the triangle inequality ensure that

$$\|\varphi(a) - (\eta \circ \psi)(a)\| \leq 2\varepsilon$$

for all $a \in A$. Note that $\eta \circ \psi$ is nuclear because ψ is nuclear. Since ε was arbitrary, we have that φ is point-norm approximated by nuclear maps, and thus φ itself is nuclear. $\square$

2.1.10 The universal enveloping von Neumann algebra

The purpose of this section is to introduce a canonical von Neumann algebra associated with a C*-algebra. We will cover only the essential concepts needed to define this object, and the reader is referred to [174] for a comprehensive exposition on von Neumann algebras. We also assume that the reader is familiar with the standard topologies on Banach spaces with respect to their duals.

Recall that any norm-closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is a (concrete) C*-algebra. Consider the following topologies on $\mathcal{B}(\mathcal{H})$,

- the *strong operator topology* (abbreviated *SOT*) is the topology generated by the family of seminorms $\mathcal{B}(\mathcal{H}) \ni T \mapsto \|T\xi\|$ for all $\xi \in \mathcal{H}$.
- the *weak operator topology* (abbreviated *WOT*) is the topology generated by the family of seminorms $\mathcal{B}(\mathcal{H}) \ni T \mapsto |\langle T\xi, \eta\rangle|$ for all $\xi, \eta \in \mathcal{H}$.

A *von Neumann algebra* M is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology. In particular, M is automatically SOT-closed and norm-closed, and always contains a unit, which we always assume to be the unit of $\mathcal{B}(\mathcal{H})$ (i.e., $M \subseteq \mathcal{B}(\mathcal{H})$ is *non-degenerate*). As a result, every von Neumann algebra is a concrete C*-algebra, while the converse is clearly not true.

In general, given a subset S of a C*-algebra A, denote the *commutant* of S in A by

$$A \cap S' = \{a \in A \mid as = sa, \text{ for all } s \in S\},$$

and its *double commutant* (or *bicommutant*) by

$$A \cap S'' = A \cap (A \cap S')'.$$

When the ambient C*-algebra, in this case A, is understood from context, we simply write S' and S'' in place of $A \cap S'$ and $A \cap S''$, respectively. The following result is foundational for the whole theory of von Neumann algebras.

Theorem 2.1.35 (von Neumann's bicommutant theorem)**.** *Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra containing the unit of $\mathcal{B}(\mathcal{H})$. Then the following are equivalent,*

(i) *$M'' = M$;*

(ii) *M is WOT-closed;*

(iii) *M is SOT-closed.*

This means that a $*$-subalgebra $\mathbf{1}_{\mathcal{H}} \in M \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if one of the equivalent conditions in the von Neumann bicommutant theorem is satisfied.

Since any increasing bounded net $(x_i)_i \subseteq M_+$ is SOT-convergent to $\sup_i x_i$ (see [16, Proposition I.3.2.5]), one is often interested in *normal* positive linear maps, i.e., those $\varphi : M \to N$ such that $\varphi(\sup_i x_i) = \sup_i \varphi(x_i)$ for all increasing bounded nets $(x_i)_i \subseteq M_+$. One can show that a positive linear map $\varphi : M \to N$ between von Neumann algebras is normal precisely when it is continuous on the unit ball of M with respect to the weak operator topologies of M and N. If φ is also $*$-homomorphic, this is equivalent to being continuous on the unit ball of M with respect to the strong operator topologies of M and N; see [16, Proposition III.2.2] and [47, Proposition 2.5.8].

Definition 2.1.36. Let A be a C*-algebra. Let $\pi_{\mathrm{u}} : A \to \mathcal{B}(\mathcal{H}_{\mathrm{u}})$ denote the universal GNS representation of A. The (universal) *enveloping von Neumann algebra* of A is defined as

$$\pi_{\mathrm{u}}(A)'' = \overline{\pi_{\mathrm{u}}(A)}^{SOT} = \overline{\pi_{\mathrm{u}}(A)}^{WOT}.$$

Remark 2.1.37. By the Sherman–Takeda theorem [179,194], the double dual A^{**} of a C*-algebra A is a von Neumann algebra, which contains A as an ultraweakly dense subset by Goldstine's theorem.[4] In fact, one has an isomorphism of von Neumann algebras $A^{**} \cong \pi_{\mathrm{u}}(A)''$, which can be obtained as follows. Since $\pi_{\mathrm{u}}(A)''$ contains A as an ultraweakly dense $*$-subalgebra, any normal linear functional on $\pi_{\mathrm{u}}(A)''$ restricts to a bounded linear functional on A, and hence produces a Banach space map $\varphi : (\pi_{\mathrm{u}}(A)'')_* \to A^*$. As a consequence of the Kaplansky density theorem, this map is an isometry; see [106, Theorem 5.3.5]. We argue that it is in fact an isomorphism of Banach spaces. To see this, note that any state on A induces a vector state on $\pi_{\mathrm{u}}(A)$, which extends to a normal linear functional on $\pi_{\mathrm{u}}(A)''$. Then, the adjoint map $\varphi^* : A^{**} \to \pi_{\mathrm{u}}(A)''$ will also be an isomorphism of Banach spaces (in fact $(\varphi^*)^{-1} = (\varphi^{-1})^*$). Thus, φ^* defines operations on A^{**} by transport of structure, and these agree with the original operations when restricted to A. Since A is an ultraweakly dense $*$-subalgebra of both $\pi_u(A)''$ and A^{**}, we have an isomorphism of von Neumann algebras $\pi_u(A)'' \cong A^{**}$.[5]

Definition 2.1.38. Let $A \subseteq B$ be an inclusion of C*-algebras. A *projection* $E : B \to A$ is a linear map satisfying $E(a) = a$ for all $a \in A$. Consider the following conditions for a projection $E : B \to A$,

[4] Recall that the ultraweak topology on A^{**} is the weak* topology coming from the predual A^*, that is, the coarsest topology such that all elements of A^* are continuous when viewed as functionals on A^{**}. The right multiplication on A^{**} turning it into a von Neumann algebra is called Arens product.

[5] By the same argument, one may notice that for any norm-closed subspace $S \subseteq A$, the ultraweak closure of S is naturally isomorphic to the double dual S^{**}.

(i) E is c.p.c. and it satisfies the following property,

$$E(a_1 b a_2) = a_1 E(b) a_2, \quad \text{for all } a_1, a_2 \in A,\, b \in B,$$

(ii) E is c.p.c.,

(iii) E is contractive.

A famous theorem of Tomiyama [199] shows that for any projection E, the conditions above are all equivalent. A projection E satisfying one of the equivalent conditions above is referred to as a *conditional expectation.*

A central concept in the classification theory of C*-algebras is undoubtedly that of nuclear (or amenable) C*-algebras. The regularity condition underlying nuclearity, known as the completely positive approximation property, can be traced back to a condition for von Neumann algebras, making it, in a sense, a precursor to nuclearity.

Definition 2.1.39. A von Neumann algebra M is said to be

(i) *injective* if for every faithful representation $M \subseteq \mathcal{B}(\mathcal{H})$, there exists a conditional expectation $\mathcal{B}(\mathcal{H}) \to M$;[6]

(ii) *semidiscrete* if the identity map id_M is the weak*-limit of completely positive factorable normal maps.

The equivalence of these regularity properties for von Neumann algebras, along with others, is the outcome of a series of influential results by Connes [38] (and Haagerup [88]), Choi–Effros [35], Effros–Lance [48], Wassermann [205], and Lance [131].

Theorem 2.1.40. *Let M be a von Neumann algebra. Then M is semidiscrete if and only if it is injective. Moreover, a* C*-*algebra A is nuclear if and only if its universal enveloping von Neumann algebra A^{**} is injective (or, equivalently, semidiscrete).*

2.1.11 K-theory and traces

The presence (or absence) of traces has been a fundamental method for distinguishing between operator algebras, with its origins in the type

[6] The definition of injectivity given here is not the original one, but equivalent (see for instance [16, Proposition IV.2.1.4]).

classification of Murray and von Neumann. Naturally, traces also serve as a key invariant for differentiating C*-algebras.

Recall that a *bounded trace* on a C*-algebra A is a bounded positive linear map $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. When τ is also a state, it is called a *tracial state*. There is an extremely complex and rich theory around traces. However, we will not discuss this here as it is not relevant to the results that will be presented in this thesis.

Topological K-theory associates specific abelian groups, known as K-groups, to topological spaces. Operator K-theory is defined for operator algebras in a similar fashion, generalising its topological counterpart to non-commutative objects. In contrast to traces, K-theory does not offer additional information in the von Neumann setting. However, it serves as one of the key invariants for C*-algebras. While operator K-theory will not play a central role — at least not directly — in this manuscript, we provide a brief outline of its definition and highlight the few results that will be used later. A great introductory resource on this topic is the book by Larsen, Laustsen, and Rørdam [172].

Recall that two projections p and q in a C*-algebra A are said to be *Murray–von Neumann* equivalent, denoted $p \sim_{\mathrm{MvN}} q$, if there exists an element $v \in A$ such that $v^*v = p$ and $vv^* = q$. Note that, in particular, $p \sim_{\mathrm{MvN}} q$ whenever $q = upu^*$ for some $u \in \mathcal{U}(\mathbf{1} + A)$ (see, e.g., [172, Proposition 2.2.7]). The latter condition forms an equivalence relation, which we denote by "$\sim_{\mathrm{pu}}$", and is referred to as *proper unitary equivalence*.

The starting point for defining the abelian group $K_0(A)$ associated to a C*-algebra A is the family of projections given by the disjoint union

$$\mathcal{P}_\infty(A) = \bigsqcup_{n=1}^{\infty} \mathcal{P}(M_n(A)),$$

where $\mathcal{P}(M_n(A))$ denotes the collection of all projections in $M_n(A)$. If $p, q \in \mathcal{P}_\infty(A)$, where $p \in M_n(A)$ and $q \in M_m(A)$, one writes

$$p \sim_0 q$$

if there exists an $m \times n$ matrix with entries in A, $v \in M_{m,n}(A)$, such that

$$p = v^*v \quad \text{and} \quad q = vv^*.$$

This defines an equivalence relation on $\mathcal{P}_\infty(A)$, and if p and q happen to be projections in the same matrix amplification $M_n(A)$, $\sim_0$ coincides with Murray–von Neumann equivalence. Moreover, one defines the *sum* of projections $p \in M_n(A)$ and $q \in M_m(A)$ as projection given by the block diagonal matrix

$p \oplus q \in M_{n+m}(A)$. Now, the set

$$\mathcal{P}_\infty(A)/\sim_0$$

forms a semigroup with the operation given by

$$[p]_{\sim_0} + [q]_{\sim_0} = [p \oplus q]_{\sim_0}$$

for all $p, q \in \mathcal{P}_\infty(A)$. Observe that when p and q are orthogonal projections in $M_n(A)$, then $p+q$ is a projection and $p+q \sim_0 p \oplus q$, which in particular entails that (see [172, Propositions 2.3.2+3.1.7])

$$[p+q]_0 = [p \oplus q]_{\sim_0} = [p]_{\sim_0} + [q]_{\sim_0}.$$

Definition 2.1.41. Let A be a unital C*-algebra. The K_0-group of A, denoted by $K_0(A)$, is the abelian group obtained from the semigroup $\mathcal{P}_\infty(A)/\sim_0$ via the Grothendieck construction.

The definition of $K_0(A)$ as given above is deliberately vague, and we refer the reader to [172, Section 3.1.1] for details about the Grothendieck construction and $K_0(A)$. What is important for our discussion is that there exists an abelian group $K_0(A)$ that is equipped (by construction) with a map

$$[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A), \quad p \mapsto [p]_0$$

that factors through $\mathcal{P}_\infty(A)/\sim_0$. Moreover, using this map, one may write $K_0(A)$ in the following way (see [172, Proposition 3.1.7])

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\}.$$

We also mention without proof that $K_0(-)$ is a functor from the category of C*-algebras with $*$-homomorphisms to the category of abelian groups with homomorphisms. In particular, for every $*$-homomorphism $\varphi : A \to B$, there exists a group homomorphism $K_0(\varphi) : K_0(A) \to K_0(B)$ such that

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0, \quad \text{for all } p \in \mathcal{P}_\infty(A)$$

We do not discuss functoriality further, and refer the reader to [172, Section 3.2] for more information. We also recall here that, for a non-unital C*-algebra A, $K_0(A)$ is defined as the kernel of the homomorphism $K_0(\pi)$ induced by the canonical quotient map $A^\dagger \to \mathbb{C}$.

In general, two elements x and y of a topological space X are said to be *homotopic* if there exists a continuous map $h : [0,1] \to X$ such that $h(0) = x$ and $h(1) = y$. Homotopy induces an equivalence relation on X, which is denoted by "$\sim_{\mathrm{h}}$". We remark here the relation between the equivalence relations presented so far on the family of projections in a C*-algebra. Let $p, q \in \mathcal{P}(A)$ for a C*-algebra A. Then one has that

$$\|p - q\| < 1 \quad \Rightarrow \quad p \sim_{\mathrm{h}} q \quad \Rightarrow \quad p \sim_{\mathrm{pu}} q \quad \Rightarrow \quad p \sim_{\mathrm{MvN}} q.$$

A proof of these facts can be found in [172]; more precisely, the first implication is [172, Proposition 2.2.4], the second and third are [172, Proposition 2.2.7].

Recall that two $*$-homomorphisms $\varphi, \psi : A \to B$ between C*-algebras are said to be *homotopic*, denoted by $\varphi \sim_{\mathrm{h}} \psi$, if there exists a map $\theta : [0,1] \times A \to B$ such that $[t \mapsto \theta_t(a)]$ is continuous for all $a \in A$, each $\theta_t : A \to B$ is a $*$-homomorphism, $\theta_0 = \varphi$, and $\theta_1 = \psi$. This is an equivalence relation on the family of $*$-homomorphisms $A \to B$. An important property of K-theory is that two homotopic $*$-homomorphisms $\varphi, \psi : A \to B$ will induce the same map between their K_0-groups, i.e.,

$$\varphi \sim_{\mathrm{h}} \psi \quad \Rightarrow \quad K_0(\varphi) = K_0(\psi).$$

See [172, Propositions 3.2.6+4.1.4] for a proof.

Remark 2.1.42. Let A be a unital C*-algebra. It is very useful to know that, if p and q are projection in $\mathcal{P}_\infty(A)$, one has that $[p]_0 = [q]_0$ if and only if there exists a projection $s \in \mathcal{P}_\infty(A)$ such that $p \oplus s \sim_0 q \oplus s$; for a proof, see [172, Proposition 3.1.7(v)]. Since A is unital, one can always find some $n \in \mathbb{N}$ such that $s = \mathbf{1}_n \in M_n(A)$.

Definition 2.1.43. Let A be a unital C*-algebra. A projection $p \in A$ is said to be *properly infinite* if there exist mutually orthogonal projections $p_1, p_2 \in A$ such that both are Murray von Neumann equivalent to p, and $p_1, p_2 \leq p$.

Moreover, A is said to be a *properly infinite* C*-*algebra* if $\mathbf{1}_A$ is properly infinite.

It is not hard to show that if $p \in \mathcal{P}_\infty(A)$ is a properly infinite and *full* projection, then for every projection $q \in \mathcal{P}_\infty(A)$, there exists $q' \in \mathcal{P}_\infty(A)$ such that $q \oplus q' \sim_0 p$; see, for instance, [172, Exercise 4.9(i)].

The following observation is basically contained in Cuntz' groundbreaking article [44], which in turns refers back to his previous work [42]. (We also give [172, Exercise 4.9(ii)+(iii)] as a textbook reference.)

Lemma 2.1.44. *Let A be a unital* C*-*algebra, and p and q be projections in $\mathcal{P}_\infty(A)$. Assume that $\mathcal{P}_\infty(A)$ contains a properly infinite, full projection. Then, for every properly infinite, full projection $r \in \mathcal{P}_\infty(A)$, one has that*

$$[p]_0 = [q]_0 \Leftrightarrow p \oplus r \sim_0 q \oplus r.$$

In fact, if p and q are properly infinite, full projections to begin with, one has that

$$[p]_0 = [q]_0 \Leftrightarrow p \sim_0 q.$$

Proof. Clearly, both statements only require a proof for the "$\Rightarrow$" implication. Assume that p and q are projections in $\mathcal{P}_\infty(A)$, and $r \in \mathcal{P}_\infty(A)$ is properly infinite and full. First of all, $[p]_0 = [q]_0$ means that there exists $s \in \mathcal{P}_\infty(A)$ such that $p \oplus s \sim_0 q \oplus s$. Note that, by the aforementioned observation about properly infinite full projections, we know that there exists some $s' \in \mathcal{P}_\infty(A)$ such that $s \oplus s' \sim_0 r$. Hence, we may conclude that

$$p \oplus r \sim_0 p \oplus s \oplus s' \sim_0 q \oplus s \oplus s' \sim_0 q \oplus r.$$

Assume now that p and q are properly infinite and full. Again using the observation above, we may find some $s \in \mathcal{P}_\infty(A)$ such that $q \sim_0 p \oplus s$. Since one always has that $p \oplus 0 \sim_0 p$, by using the first part of the proof we may infer that

$$q \sim_0 p \oplus s \sim_0 p \oplus 0 \sim_0 p.$$

$\square$

Moreover, the K_0-group takes a simpler form when A contains a properly infinite projection thanks to a characterisation of Cuntz; see [44, Theorem 1.4] (or [173, Proposition 4.1.4]) and Lemma 2.1.44).

Theorem 2.1.45. *Let A be a unital* C**-algebra contaning a properly infinite, full projection. Then,*

$$\begin{aligned} K_0(A) &= \{[p]_0 \mid p \textit{ is a properly infinite, full projection in } A\} \\ &= \{p \in A \mid p \textit{ is a properly infinite, full projection}\}/\sim_{\mathrm{MvN}} . \end{aligned}$$

The second part of operator K-theory is described by another abelian group, denoted by $K_1(A)$. Unlike $K_0(A)$, which is constructed from the set of projections, $K_1(A)$ is defined using the group of unitaries in the C*-algebra.

Recall that for a unital C*-algebra A, one denotes by $\mathcal{U}(A)$ the set of unitaries in A. Similarly as before, we define

$$\mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}(M_n(A)),$$

and denote by $u \oplus v$ the block-diagonal matrix obtained from $u, v \in \mathcal{U}_\infty(A)$. Recall that $\mathbf{1}_n$ denotes the unit of the matrix algebra M_n, where $n \in \mathbb{N} \setminus \{0\}$. We adopt the convention that every unitary $u \in \mathcal{U}_\infty(A)$ is identified with $u \oplus \mathbf{1}_0$.

Now, one defines a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows. If u is a unitary in $M_n(A)$ and v a unitary in $M_m(A)$, then

$$u \sim_1 v \quad \text{if} \quad u \oplus \mathbf{1}_{k-n} \sim_{\mathrm{h}} v \oplus \mathbf{1}_{k-m}$$

for some $k \in \mathbb{N}$ such that $k \geq \max\{n, m\}$. (Note that the homotopy relation is well-defined on $\mathcal{U}(A)$ for any unital C*-algebra A.) The relation $\sim_1$ is an equivalence relation with several desirable properties, which we do not describe here, and instead refer the reader to [172, Chapter 8]. The equivalence class of a unitary $u \in \mathcal{U}_\infty(A)$ up to $\sim_1$ is denoted by $[u]_1$. Note that $[\cdot]_1$ enjoys the following interesting properties:

$$[\mathbf{1}_n]_1 = 0, \quad [u^*]_1 = -[u]_1, \quad [uv]_1 = [vu]_1 = [u]_1 + [v]_1$$

for every $n \in \mathbb{N}$ and every $u, v \in \mathcal{U}(M_n(A))$.

Definition 2.1.46. Let A be a C*-algebra. The K_1-group of A is defined as the quotient

$$K_1(A) = \mathcal{U}_\infty(A^\dagger)/\sim_1,$$

which is an abelian group with respect to the binary operation given by $[u]_1 + [v]_1 = [u \oplus v]_1$ for all $u, v \in \mathcal{U}_\infty(A^\dagger)$.

When A is unital, there is a canonical isomorphism between $K_1(A) = \mathcal{U}_\infty(A^\dagger)/\sim_1$ and $\mathcal{U}_\infty(A)/\sim_1$ (see [172, Proposition 8.1.6]). Again, we only mention en passant that $K_1(-)$ is a functor from the category of C*-algebras with $*$-homomorphisms to the category of abelian groups with group homomorphisms. The group homomorphism induced by a $*$-homomorphism $\varphi : A \to B$ is denoted by $K_1(\varphi) : K_1(A) \to K_1(B)$, and satisfies

$$K_1(\varphi)([u]_1) = [\varphi^\dagger(u)]_1, \quad \text{for all } u \in \mathcal{U}_\infty(A).$$

We refer the reader to [172, Section 8.2] for mote details on the functorial description of $K_1(-)$. Just like for the functor $K_0(-)$, we have that

$$\varphi \sim_{\mathrm{h}} \psi \quad \Rightarrow \quad K_1(\varphi) = \mathcal{K}_1(\psi),$$

and the reader is referred to [172, Proposition 8.2.2(v)] for a proof.

Recall that, for a unital C*-algebra A, the normal subgroup of $\mathcal{U}(A)$ consisting of unitaries that are homotopic to the unit of A is referred to as the *connected component of the unit*, and is denoted by $\mathcal{U}_0(A)$. By definition, one has that $[\mathbf{1}_A]_1 = 0$, and it is readily verified that $[u]_1 = 0$ for all $u \in \mathcal{U}_0(A)$. Consequently, one has a group homomorphism $\kappa : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ that ensures commutativity of the following diagram.

$$
\begin{tikzcd}
\mathcal{U}(A) \arrow[d] \arrow[dr, "{[\cdot]_1}"] & \\
\mathcal{U}(A)/\mathcal{U}_0(A) \arrow[r, "\kappa"'] & K_1(A)
\end{tikzcd}
$$

In general the homomorphism κ needs not be injective nor surjective. However, if A is properly infinite, κ is surjective; see [44, Theorem 1.9] or [173, Proposition 4.1.6].

2.1.12 Cuntz algebras

We define here certain universal C*-algebras associated with isometries, which are referred to as *Cuntz algebras*. They were first introduced by Cuntz in his seminal article [41], and have since then sparked inspiration for numerous other constructions. Cuntz proved in [41] that these C*-algebras are purely infinite and simple (see Definition 2.1.55), thereby introducing the first class of concrete examples with this property.

For $2 \leq n < \infty$, consider a set of generators $\mathcal{G} = \{s_1, \dots, s_n\} \cup \{\mathbf{1}\}$ consisting of n different elements. As set of relations $\mathcal{R}$ choose

$$s_i^* s_i = \mathbf{1}, \sum_{i=1}^{n} s_i s_i^* = \mathbf{1}$$

where the first condition asserts that $\mathcal{G}$ consists of isometries, while the second states that the sum of all the *range projections* is maximal. Clearly, for any representation π of $(\mathcal{G} \mid \mathcal{R})$, it follows that $\|\pi(s_i)\| \leq 1$ for all $i = 1, \dots, n$, and thus $(\mathcal{G} \mid \mathcal{R})$ is bounded. The universal C*-algebras $\mathcal{O}_n = \mathrm{C}^*(\mathcal{G} \mid \mathcal{R})$ for $n \geq 2$, referred to as *Cuntz algebras*, are therefore well-defined and unique up to isomorphism.

Similarly, consider an infinite sequence of isometries $(s_i)_{i\in\mathbb{N}}$, and set $\mathcal{G} = (s_i)_{i\in\mathbb{N}} \cup \{\mathbf{1}\}$ with the countable set of relations $\mathcal{R}$ given by

$$s_i^* s_j = \delta_{i,j} \cdot \mathbf{1} \quad \text{for all } i, j \in \mathbb{N},$$

where $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. The Cuntz algebra associated to $(\mathcal{G} \mid \mathcal{R})$ is denoted by $\mathcal{O}_\infty$.

The Cuntz algebra $\mathcal{O}_2$ is a cornerstone object in the theory of C*-algebras. In particular, as an intermediate step to obtain their classification theorem, Kirchberg and Phillips established the following deep result [119], which is known as the $\mathcal{O}_2$*-embedding theorem*.

Theorem 2.1.47 (The $\mathcal{O}_2$-embedding theorem)**.** *Let A be a separable, exact* C*-*algebra. Then there exists an embedding $A \hookrightarrow \mathcal{O}_2$. If A is unital, the embedding can be chosen to be unital.*

We do not give a proof of the $\mathcal{O}_2$-embedding theorem. In fact, we give *two* proofs of it. Firstly, it will follow as a special case of Theorem 3.1.18, which is an ideal-related version of it due to Gabe [64] and Kirchberg [116] (see Remark 3.1.19 to know why this recovers the $\mathcal{O}_2$-embedding theorem). Secondly, we will prove a dynamical generalisation of both the ideal-related $\mathcal{O}_2$-embedding theorem, Theorem 3.4.17, which is one of the main results we establish in Chapter 3 (and [156]). The $\mathcal{O}_2$-embedding theorem has many applications in the theory of C*-algebras, and primarily in classification theory. As one of the first deep (and highly non-trivial) results proved with the help of the $\mathcal{O}_2$-embedding theorem, one has a beautiful characterisation referred to as the $\mathcal{O}_2$*-absorption theorem*.

Theorem 2.1.48 (The $\mathcal{O}_2$-absorption theorem)**.** *A* C*-*algebra A is simple, separable, nuclear and unital precisely when there exists an isomorphism $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.*

As a special case of this, one has that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, which had already been established in [169] before Kirchberg and Phillips proved their theorem.

2.1.13 The K-theory groups of $\mathcal{O}_2$

In this section, the K-groups of the Cuntz algebra $\mathcal{O}_2$ are computed. The next result is due to Cuntz [44].

Theorem 2.1.49. *One has that $K_0(\mathcal{O}_2) = 0$ and $K_1(\mathcal{O}_2) = 0$.*

Proof. Define the map $\varphi : \mathcal{O}_2 \to \mathcal{O}_2$ as $\varphi(x) = s_1 x s_1^* + s_2 x s_2^*$. One can show that φ is a $*$-homomorphism by using the properties of s_1 and s_2. Now, if u denotes the unitary given by $\varphi(s_1)s_1^* + \varphi(s_2)s_2^*$, then $\varphi(s_i) = us_i$ for $i = 1, 2$. Hence, by the universal property of $\mathcal{O}_2$, φ is the unique $*$-homomorphism satisfying this property. (In fact, every unital endomorphism of $\mathcal{O}_2$ arises this way.) Consider now the group homomorphism

$$K_0(\varphi) : K_0(\mathcal{O}_2) \to K_0(\mathcal{O}_2)$$

induced by φ. Observe that, for any isometry $s \in \mathcal{O}_2$ and any projection $p \in \mathcal{O}_2$, one has that $sps^* \sim_{\mathrm{MvN}} p$ via the element $v := sp$. In particular, for any projection $p \in \mathcal{O}_2$ one has that

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0$$

$$
\begin{aligned}
&= [s_1 p s_1^* + s_2 p s_2^*]_0 \\
&= [s_1 p s_1^*]_0 + [s_2 p s_2^*]_0 \\
&= 2[p]_0.
\end{aligned}
$$

Now, notice that

$$
\begin{aligned}
u &= \varphi(s_1)s_1^* + \varphi(s_2)s_2^* \\
&= s_1^2(s_1^*)^2 + s_2 s_1 s_2^* s_1^* + s_1 s_2 s_1^* s_2^* + s_2^2(s_2^*)^2 \\
&= u^*.
\end{aligned}
$$

In particular, since u is self-adjoint, it must satisfy $u \sim_{\mathrm{h}} \mathbf{1}_{\mathcal{O}_2}$. (One way of seeing this is from the fact that $u = u^*$ cannot have full spectrum $\mathbb{T}$.) In particular, this homotopy shows that $\varphi \sim_{\mathrm{h}} \mathrm{id}_{\mathcal{O}_2}$, and hence that $K_0(\varphi) = \mathrm{id}_{K_0(\mathcal{O}_2)}$. Combined with the conclusion from before, one has that $[p]_0 = 2[p]_0$, and consequently that $[p]_0 = 0$ for every projection $p \in \mathcal{O}_2$. Therefore, $K_0(\mathcal{O}_2) = 0$.

Let us now turn to $K_1(\mathcal{O}_2)$. By the same argument as above, the behaviour of the functor $K_1(-)$ with respect to homotopy equivalence implies that $K_1(\varphi) = K_1(\mathrm{id}_{\mathcal{O}_2})$. Next, we show that for every unitary $u \in \mathcal{O}_2$ and every isometry $s \in \mathcal{O}_2$, one has $[sus^* + \mathbf{1}_{\mathcal{O}_2} - ss^*]_1 = [u]_1$. The unitary given by

$$
w = \begin{pmatrix} s & \mathbf{1}_{\mathcal{O}_2} - ss^* \\ 0 & s^* \end{pmatrix} \in M_2(\mathcal{O}_2)
$$

satisfies the condition

$$
w(u \oplus \mathbf{1}_{\mathcal{O}_2})w^* = (sus^* + \mathbf{1}_{\mathcal{O}_2} - ss^*) \oplus \mathbf{1}_{\mathcal{O}_2}.
$$

Using the properties of the relation $\sim_1$, it follows that $[sus^* + \mathbf{1}_{\mathcal{O}_2} - ss^*]_1 = [u]_1$. Notice also that, for any element $a \in \mathcal{O}_2$,

$$
s_1 a s_1^* + s_2 a s_2^* = (s_1 a s_1^* + \mathbf{1}_{\mathcal{O}_2} - s_1 s_1^*) \cdot (s_2 a s_2^* + \mathbf{1}_{\mathcal{O}_2} - s_2 s_2^*).
$$

One may then argue using the properties of $[\cdot]_1$ that, for any unitary $u \in \mathcal{U}(\mathcal{O}_2)$,

$$
\begin{aligned}
[\varphi(u)]_1 &= [s_1 u s_1^* + s_2 u s_2^*]_1 \\
&= [(s_1 a s_1^* + \mathbf{1}_{\mathcal{O}_2} - s_1 s_1^*) \cdot (s_2 a s_2^* + \mathbf{1}_{\mathcal{O}_2} - s_2 s_2^*)]_1 \\
&= [(s_1 a s_1^* + \mathbf{1}_{\mathcal{O}_2} - s_1 s_1^*)]_1 + [(s_2 a s_2^* + \mathbf{1}_{\mathcal{O}_2} - s_2 s_2^*)]_1 \\
&= 2[u]_1,
\end{aligned}
$$

thus implying that $K_1(\varphi)([u]_1) = 2[u]_1$. In conclusion, the two descriptions of $K_1(\varphi)$ imply that $K_1(\mathcal{O}_2) = 0$. $\square$

In [44], Cuntz also computes the K-theory of the Cuntz algebras with $n \geq 3$, obtaining

$$K_0(\mathcal{O}_n) = \{0, [\mathbf{1}]_0, 2[\mathbf{1}]_0, \ldots, (n-2)[\mathbf{1}]_0\} \cong \mathbb{Z}/(n-1)\mathbb{Z},$$

$$K_1(\mathcal{O}_n) = 0,$$

and

$$K_0(\mathcal{O}_\infty) \cong \mathbb{Z},$$

$$K_1(\mathcal{O}_n) = 0.$$

Remark 2.1.50. Recall that there exists a group homomorphism

$$\kappa : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A).$$

We have already mentioned earlier that κ is surjective when A is properly infinite. It is moreover an isomorphism when A is purely infinite and simple; see [44, Theorem 1.9] or [173, Proposition 4.1.6]. In particular, together with the fact that $K_1(\mathcal{O}_n) = 0$, one may conclude that $\mathcal{U}(\mathcal{O}_n) = \mathcal{U}_0(\mathcal{O}_n)$, i.e., the unitary group of the Cuntz algebras is connected to the unit.

2.1.14 Comparison of positive elements and the Cuntz semigroup

The theory of comparison of positive elements in a C*-algebra was started by Cuntz with the purpose of constructing dimension functions and traces on C*-algebras [43], and later developed further by Blackadar [14], and Rørdam [167]. Probably the most striking application of this theory is the construction of a new invariant for C*-algebras, known as the *Cuntz semigroup*, which played a crucial role in Toms' counterexamples in classification theory [201]. One may notice that comparison of positive elements generalises the comparison of projections that was introduced in the previous sections.

Definition 2.1.51 (Cuntz)**.** Let A be a C*-algebra, and a, b positive elements in A. Then one says that a is *Cuntz subequivalent* to b, denoted by $a \precsim b$, if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_n^* b x_n \to a$ for $n \to \infty$. If $a \in M_n(A)_+$ and $b \in M_m(A)_+$, one writes $a \precsim b$ if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $M_{m,n}(A)$ such that $x_n^* b x_n \to a$ for $n \to \infty$. Moreover, a and b are said to be *Cuntz equivalent*, written $a \sim_{\mathrm{Cu}} b$, if $a \precsim b$ and $b \precsim a$.

Observe that, for any pair of positive elements $a, b \in A_+$, the following implications hold,

$$a \leq b \implies a \in \overline{bAb} \implies a \precsim b \implies a \in \overline{AbA},$$

where the implication in the middle, the only non-trivial one, is proved in [121, Proposition 2.7(i)]. In the following lemma, we rephrase some important technical results of Rørdam (see [167, Section 2]), and Kirchberg–Rørdam (see [121, Section 2] and [122, Section 2]), that relate certain approximate properties and Cuntz subequivalence. Let us first fix an important piece of notation.

Notation 2.1.52. For every $\varepsilon > 0$, define $h_\varepsilon, f_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ to be the continuous functions

$$h_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq \varepsilon \\ t-\varepsilon & \text{if } t > \varepsilon \end{cases} \quad \text{and} \quad f_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq \varepsilon \\ \frac{t-\varepsilon}{\varepsilon} & \text{if } \varepsilon < t \leq 2\varepsilon \\ 1 & \text{if } t \geq 2\varepsilon \end{cases}$$

If a is a positive element of a C*-algebra A, one writes $h_\varepsilon(a)$ and $f_\varepsilon(a)$ for the elements obtained via functional calculus. We will, moreover, use the additional notation $h_\varepsilon(a) = (a-\varepsilon)_+$ and refer to it as the *epsilon cutdown* of a.

Lemma 2.1.53. *Let a and b be positive elements of a* C**-algebra A.*

(i) *If $\varepsilon > 0$ and $\|a-b\| < \varepsilon$, then there exists a contraction $c \in A$ such that $(a-\varepsilon)_+ = c^*bc$. In particular, $(a-\varepsilon)_+ \precsim b$ if $a =_\varepsilon b$.*

(ii) *If $a \precsim b$, then for every $\varepsilon > 0$ there exist $d_0, d_1 \in A$ and $n \in \mathbb{N}$ such that $(a-\varepsilon)_+ = d_0^* b d_0 = d_1^*(b-2^{-n})_+ d_1$.*

(iii) *If $a \precsim b$, then for every $\varepsilon > 0$ there exists $\delta > 0$ and $r \in A$ such that $f_\varepsilon(a) = r^* f_\delta(b) r$.*

Proof. Statement (i) is [122, Lemma 2.2] (see also [167, Proposition 2.2] and [121, Lemma 2.5(ii)]), and (iii) is [167, Lemma 2.4]. The only condition that does not appear precisely as written here in the literature is (ii). Hence, we argue how to obtain it. By [121, Proposition 2.6], for every $\varepsilon > 0$ there exists $\delta > 0$ and $d \in A$ such that $(a-\varepsilon)_+ = d^*(b-\delta)_+ d$. Then one may argue with [122, Lemma 2.4(i)] to conclude that there exists $d_0 \in A$ such that $(a-\varepsilon)_+ = d_0^* b d_0$. Choose $n \in \mathbb{N}$ such that $2^{-n} < \delta$, and set $\delta' := \delta - 2^{-n}$. It follows from (i) that there exists a contraction $c \in A$ such that $(b-\delta)_+ = ((b-2^{-n})_+ - \delta')_+ = c^*(b-2^{-n})_+ c$. Therefore, we have that $(a-\varepsilon)_+ = d_1^*(b-2^{-n})_+ d_1$, where $d_1 := cd$. □

Similarly as before, if $a \in M_n(A)_+$ and $b \in M_m(A)_+$, we denote by $a \oplus b$ the diagonal matrix in $M_{n+m}(A)$. By viewing each matrix amplification $M_n(A)$ inside $A \otimes \mathcal{K}$, one realises that Cuntz equivalence of positive elements in $(A \otimes \mathcal{K})_+$ incorporates Cuntz equivalence of elements in $\bigcup_{n=1}^\infty M_n(A)_+$. For $a \in (A \otimes \mathcal{K})_+$, let $[a]_{\mathrm{Cu}}$ denote the class of elements in $(A \otimes \mathcal{K})_+$ that are Cuntz equivalent to a.

The following definition should be regarded as a modern version of [43].

Definition 2.1.54. The *Cuntz semigroup* of a C*-algebra A is the abelian monoid given by[7]

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})/\sim_{\mathrm{Cu}},$$

equipped with the operation $[a]_{\mathrm{Cu}} + [b]_{\mathrm{Cu}} = [a \oplus b]_{\mathrm{Cu}}$ for all $a, b \in (A \otimes \mathcal{K})_+$,[8] with zero element $0 = [0]_{\mathrm{Cu}}$.

The idea that $\mathrm{Cu}(A)$ could serve as part of the classification invariant for certain C*-algebras emerged soon after Toms' groundbreaking contribution [201], as evidenced by [40] and [6]. One of the most significant developments in this direction remains Robert's classification theorem of certain inductive limit C*-algebras [166]. We will return to $\mathrm{Cu}(A)$ in Section 2.3, where we will see that, in some cases relevant to this thesis, the Cuntz semigroup is completely determined by the ideal structure of the algebra. More importantly, we will discover that the abstract category to which $\mathrm{Cu}(A)$ belongs also provides the natural setting for the ideal-related invariant that plays a central role in this work.

2.1.15 Pure infiniteness and absorbing $\mathcal{O}_2$ or $\mathcal{O}_\infty$

The concept of purely infinite C*-algebras was first introduced by Cuntz in his seminal work on the Cuntz algebras [41]. However, his original definition applies exclusively to the case of simple C*-algebras.

Definition 2.1.55 (see [173, Definition 4.1.2])**.** A simple C*-algebra A is said to be *purely infinite* if $A \ncong \mathbb{C}$ and for every pair of non-zero elements a and b in A, there are elements x and y in A such that $b = xay$.

Remark 2.1.56. If A is a separable and simple purely infinite C*-algebra, then it must be either unital, or it is *stable*, i.e., $A \cong A \otimes \mathcal{K}$. This result was proved by Zhang in [212], and is known as *Zhang's dichotomy*.

Numerous other conditions that are equivalent to pure infiniteness for simple C*-algebras were found later by work of Cuntz, Zhang, H. Lin, and Kirchberg. In

[7] If one chooses instead to identify $\bigcup_{n=1}^{\infty} M_n(A)_+$ up to Cuntz equivalence, then one arrives at another object, sometimes called the *classical Cuntz semigroup*, denoted by $W(A)$, which has its own well-studied theory.

[8] In order to define this, one should fix an isomorphism $M_2 \otimes \mathcal{K} \cong \mathcal{K}$ and use it to identify $\mathrm{Cu}(A) \cong \mathrm{Cu}(M_2(A))$. However, it is not hard to prove that this identification is independent of the choice of the isomorphism.

the non-simple setting, the definition of purely infinite C*-algebra, which reduces to the one above for simple C*-algebras, is due to Kirchberg and Rørdam [121].

Definition 2.1.57 (see [121, Definition 3.2])**.** Let A be a C*-algebra. A positive element $a \in A_+$ is said to be

- *infinite* if there exists a non-zero positive element $b \in A_+$ such that $a \oplus b \precsim a$;
- *finite* if it is not infinite;
- *properly infinite* if a is non-zero and $a \oplus a \precsim a$.[9]

Definition 2.1.58 (see [121, Definition 4.1+Theorem 4.16])**.** A C*-algebra A is said to be *purely infinite* if A admits no non-zero abelian quotients and for every pair of positive elements $a, b \in A_+$ such that $a \in \overline{AbA}$, one has that $a \precsim b$. Equivalently, A is purely infinite if every non-zero positive element in A is properly infinite.

Proposition 2.1.59. *Purely infinite* C**-algebras enjoy the following permanence properties.*

(i) *Non-zero hereditary* C**-subalgebras of purely infinite* C**-algebras are purely infinite; see [121, Proposition 4.17].*

(ii) *Quotients of purely infinite* C**-algebras are purely infinite; see [121, Proposition 4.3].*

(iii) *Extensions of purely infinite* C**-algebras by purely infinite* C**-algebras are again purely infinite; see [121, Proposition 4.19].*

(iv) *The inductive limit of an inductive system consisting of purely infinite* C**-algebras is purely infinite; see [121, Proposition 4.18].*

(v) *If A is purely infinite, then $\ell^\infty(\mathbb{N}, A)$ and $A_{\mathcal{U}}$ are purely infinite as well, where $\mathcal{U}$ is any filter on $\mathbb{N}$; see [121, Proposition 4.20]. In particular, A_∞ is purely infinite.*

(vi) *If A is purely infinite, and B is a* C**-algebra such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then B is purely infinite as well; see [121, Proposition 4.23].*

In [122], Kirchberg and Rørdam introduce more notions of pure infiniteness for non-simple C*-algebras. These are called *weak pure infiniteness* and *strong pure infiniteness*. As the names suggest, these are *weaker* and *stronger* properties than pure infiniteness, respectively. We will be more interested in the latter.

[9] If a is a projection in A, this definition coincides with the definition of properly infinite projection used in previous sections by [121, Lemma 3.1].

Definition 2.1.60 (see [122, Definition 5.1])**.** A C*-algebra A is said to be *strongly purely infinite* if for every $\varepsilon > 0$ and every matrix

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)_+$$

there exist $d_1, d_2 \in A$ such that

$$\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| \leq \varepsilon.$$

Proposition 2.1.61. *Strongly purely infinite* C*-*algebras enjoy the following permanence properties.*

(i) *Non-zero hereditary* C*-*subalgebras of strongly purely infinite* C*-*algebras are strongly purely infinite; see [122, Proposition 5.11].*

(ii) *Non-zero quotients of strongly purely infinite* C*-*algebras are strongly purely infinite; see [122, Proposition 5.11].*

(iii) *Extensions of strongly purely infinite* C*-*algebras by strongly purely infinite* C*-*algebras are again strongly purely infinite; see [121, Proposition 4.19].*

(iv) *The inductive limit of an inductive system consisting of strongly purely infinite* C*-*algebras is strongly purely infinite; see [122, Proposition 5.11].*

(v) *A* C*-*algebra A is strongly purely infinite if and only if $\ell^\infty(\mathbb{N}, A)$ is strongly purely infinite, if and only if $A_\mathcal{U}$ is strongly purely infinite (for some or any filter $\mathcal{U}$ on* $\mathbb{N}$*); see [122, Proposition 5.12]. In particular, A_∞ is strongly purely infinite precisely when A is a strongly purely infinite* C*-*algebra.*

(vi) *If A is a strongly purely infinite* C*-*algebra, and B is a* C*-*algebra such that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, then B is strongly purely infinite as well; see [122, Proposition 5.11].*

Let us consider two special classes of strongly purely infinite C*-algebras.

Definition 2.1.62. A C*-algebra A is said to be $\mathcal{O}_\infty$*-stable* if $A \cong A \otimes \mathcal{O}_\infty$. Analogously, A is said to be $\mathcal{O}_2$*-stable* if $A \cong A \otimes \mathcal{O}_2$; cf. Section 2.6.

One has that the classes of $\mathcal{O}_\infty$-stable and $\mathcal{O}_2$-stable C*-algebras are closed under taking hereditary C*-subalgebras (and in particular, ideals), quotients, and inductive limits by results from [117, 122, 200]. Moreover, by [121, Proposition 4.5], every $\mathcal{O}_\infty$-stable C*-algebra is automatically purely infinite. In fact, the following more general result holds true.

Theorem 2.1.63 (see [122, Theorem 8.6])**.** *Every $\mathcal{O}_\infty$-stable* C*-*algebra A is strongly purely infinite.*

Surprisingly enough, the converse of Theorem 2.1.63 also holds under the additional assumptions that A is separable and nuclear. The result is due to Kirchberg and Rørdam, [122, Theorem 8.6], when A is unital or stable, and to obtain the general version one can use [117, Proposition 4.4(4,5)] or [200, Corollary 3.2]. We will not illustrate the proof of this theorem, and refer the reader to the aforementioned references.

Theorem 2.1.64. *Let A be a separable, nuclear* C*-*algebra. Then A is strongly purely infinite if and only if it is $\mathcal{O}_\infty$-stable.*

Example 2.1.65. It is shown in [171, Theorem 3.2 and Proposition 4.2] that the AH-algebra from Example 2.1.11, $\mathcal{A}_{[0,1]}$, is $\mathcal{O}_\infty$-stable. To show this, Rørdam uses heavy structure theory of purely infinite C*-algebras. In fact, $\mathcal{A}_{[0,1]}$ is even $\mathcal{O}_2$-stable, and this is shown by Kirchberg and Rørdam in [123, Proposition 6.1]. We point out that the proof of this fact given there relies on Kirchberg's classification of $\mathcal{O}_\infty$-stable C*-algebras [66, 116].

We conclude this section with a discussion about the unitary group of $\mathcal{O}_2$-stable C*-algebras. More precisely, we will present an elementary proof that, for any unitary u in a unital C*-algebra A, $u \otimes \mathbf{1}_{\mathcal{O}_2} \in A \otimes \mathcal{O}_2$ is homotopic to the unit $\mathbf{1}_A \otimes \mathbf{1}_{\mathcal{O}_2}$.

The following result appeared, in one form or another, in many works, such as Haagerup–Rørdam [90], Phillips [159], Nakamura [149], and Szabó [191].

Lemma 2.1.66. *Let A be any unital* C*-*algebra. For every unitary $u \in \mathcal{U}(A)$, one has that $u \otimes \mathbf{1}_{\mathcal{O}_2} \in \mathcal{U}_0(A \otimes \mathcal{O}_2)$.*

Proof. We follow parts of the proof of [191, Lemma 5.1]. Let $s_1, s_2 \in \mathcal{O}_2$ be two isometries satisfying $s_1 s_1^* + s_2 s_2^* = \mathbf{1}_{\mathcal{O}_2}$. Define two unital $*$-homomorphisms $\theta_1, \theta_2 : M_3 \to \mathcal{O}_2$ given on matrix units by

$$\theta_1(e_{1,1}) = s_1 s_1^*,\ \theta_1(e_{2,1}) = s_2 s_1 s_1^*,\ \theta_1(e_{3,1}) = s_2^2 s_1^*$$

and

$$\theta_2(e_{1,1}) = s_2 s_2^*,\ \theta_2(e_{2,1}) = s_1 s_2 s_2^*,\ \theta_2(e_{3,1}) = s_1^2 s_2^*.$$

The unitary given by the permutation matrix

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3,$$

which has eigenvalues $\{1, \exp(2\pi i/3), \exp(-2\pi i/3)\}$ can then be written as $w = \exp(ih)$ for some self-adjoint element $h \in M_3$. To simplify notation, define the unitary

$$v = \begin{pmatrix} u^* & 0 & 0 \\ 0 & \mathbf{1}_A & 0 \\ 0 & 0 & u \end{pmatrix} \in M_3(A).$$

Now, let $u : [0,1] \to \mathcal{U}(A \otimes \mathcal{O}_2)$ be the continuous path of unitaries given by

$$u_t = (\mathrm{id}_A \otimes \theta_1)\left(\exp(ith)v\exp(-ith)v^*\right) \cdot (\mathrm{id}_A \otimes \theta_2)\left(\exp(ith)v\exp(-ith)v^*\right).$$

We have that $u_0 = \mathbf{1}$, and also that

$$\begin{aligned} u_1 &= (\mathrm{id}_A \otimes \theta_1)(wvw^*v^*) \cdot (\mathrm{id}_A \otimes \theta_2)(wvw^*v^*) \\ &= (\mathrm{id}_A \otimes \theta_1)(u^2 \otimes e_{1,1} + u^* \otimes (e_{2,2} + e_{3,3})) \\ &\quad \cdot (\mathrm{id}_A \otimes \theta_2)(u^2 \otimes e_{1,1} + u^* \otimes (e_{2,2} + e_{3,3})) \\ &= (u^2 \otimes s_1 s_1^* + u^* \otimes s_2 s_2^*) \cdot (u^2 \otimes s_2 s_2^* + u^* \otimes s_1 s_1^*) \\ &= u \otimes \mathbf{1}. \end{aligned}$$

□

2.2 Group actions on C*-algebras

When approaching abstract classification results, it is essential to work within the appropriate category, beginning with a careful choice of the most suitable notion of morphisms. Furthermore, one must consider the correct notion of unitary equivalence between maps, an issue that becomes particularly subtle in the non-unital setting, where unitaries must be taken in a larger ambient C*-algebra. The goal of this section is to outline the categorical framework for C*-dynamical systems from [192] that will be used in the following chapter to obtain abstract classification results for group actions on C*-algebras.

Warning 2.2.1. From now on, G will always denote a *second-countable locally, compact group*, unless speficied otherwise. Moreover, recall that by *locally compact* we really mean locally compact and *Hausdorff*.

Just like C*-algebras can be thought of as non-commutative topological spaces via the Gelfand–Naimark theorem, C*-dynamical systems provide a non-commutative generalisation of topological dynamics. Let us first clarify what we mean by topological dynamics.

Definition 2.2.2. Let X be a topological space. An *action* of G on X, denoted by $\alpha : G \curvearrowright X$, is a group homomorphism from $\alpha : G \to \mathrm{Homeo}(X)$, which we write as

$$G \times X \ni (g, x) \mapsto \alpha_g(x) \in X.$$

One says that the action α is *continuous* if it is jointly continuous when viewed as a map $\alpha : G \times X \to X$, and that it is *algebraic* if one wants to stress that it is not necessarily continuous. The pair (X, α) is also referred to as a *dynamical system*, or G-dynamical system if one wants to highlight the acting group.

Remark 2.2.3. Assume that X is a locally compact space. Note that, in particular, this applies also to the case when $X = A$ is a C*-algebra. Then, an action $\alpha : G \curvearrowright X$ is continuous if and only if

$$G \to X, \quad g \mapsto \alpha_g(x)$$

is a continuous map for every $x \in X$; see [55].

Example 2.2.4. One of the most basic examples of continuous actions on a topological space is the left-translation action of G on itself. We will denote this action by $\mathtt{lt} : G \curvearrowright G$, namely,

$$\mathtt{lt}_g(h) = g^{-1}h \quad \text{for all } g, h \in G.$$

It is clear that the action induced by $\mathtt{lt}$ on the (commutative) C*-algebra $\mathcal{C}_0(G)$ satisfies the continuity condition. With slight abuse of notation, we denote this action by $\mathtt{lt} : G \curvearrowright \mathcal{C}_0(G)$ as well, i.e.,

$$(\mathtt{lt}_g(f))(h) = f(g^{-1}h), \quad \text{for all } f \in \mathcal{C}_0(G),\ g, h \in G.$$

Similarly, one defines the right-translation action $\mathtt{rt} : G \curvearrowright \mathcal{C}_0(G)$ by

$$\mathtt{rt}_g(f)(h) = f(hg) \quad \text{for all } g, h \in G.$$

Despite being so simple to describe, one can get a lot mileage out of these two examples.

Warning 2.2.5. It is an established convention that actions on C*-algebras are assumed to be continuous. Throughout this thesis, we will normally follow this tradition. Hence, an action $\alpha : G \curvearrowright A$ on a C*-algebra is a group homomorphism $\alpha : A \to \mathrm{Aut}(A)$ such that the function $[g \mapsto \alpha_g(a)]$ is continuous for every $a \in A$.

Whenever an action $\alpha : G \curvearrowright A$ may fail the continuity condition, we say that it is an *algebraic* action.

Notation 2.2.6. If $\alpha : G \curvearrowright A$ is an action on a C*-algebra, the pair (A, α) is referred to as a (G-)C*-*dynamical system* or G-C*-*algebra*.

Given two actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$, one can form their induced *tensor product action* $\alpha \otimes \beta : G \curvearrowright A \otimes B$ by setting $(\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ for all $g \in G$. The same holds if one replaces $A \otimes B$ with the maximal tensor product.

Warning 2.2.7. For a C*-algebra A, we denote by id_A both the identity automorphism of A, and the trivial G-action $\mathrm{id}_A : G \curvearrowright A$ given by $(\mathrm{id}_A)_g = \mathrm{id}_A$ for all $g \in G$. This does not usually give rise to confusion; but if that is the case, we will introduce ad hoc notation to remove ambiguity.

Notation 2.2.8. Let $\alpha : G \curvearrowright A$ be an algebraic action on a C*-algebra. The *fixed-point algebra* of A is denoted by A^α, namely

$$A^\alpha := \{a \in A \mid \alpha_g(a) = a \text{ for all } a \in A\}.$$

Note that, whenever $\alpha : G \curvearrowright A$ is an action on a C*-algebra, one can induce an algebraic action on the multiplier algebra by setting

$$\alpha_g(x)a = \alpha_g(x\alpha_{g^{-1}}(a)), \quad a\alpha_g(x) = \alpha_g(\alpha_{g^{-1}}(a)x),$$

for all $g \in G$, $x \in \mathcal{M}(A)$ and $a \in A$. Moreover, this action induces an algebraic action on the corona algebra $\mathcal{Q}(A) = \mathcal{M}(A)/A$ in a canonical way.

Warning 2.2.9. For an action $\alpha : G \curvearrowright A$ on a C*-algebra, we routinely denote the induced actions on $\mathcal{M}(A)$ and $\mathcal{Q}(A)$ again by α.

Example 2.2.10. Not every algebraic action on a C*-algebra is continuous, as we showcase below. Consider again the left-translation action of $\mathbb{R}$ on itself, $\mathtt{lt} : \mathbb{R} \curvearrowright \mathcal{C}_0(\mathbb{R})$. We know that the multiplier C*-algebra of $\mathcal{C}_0(\mathbb{R})$ is isomorphic to the set of bounded continuous functions $\mathcal{C}_b(\mathbb{R})$. We remark here that the induced algebraic action $\mathtt{lt} : \mathbb{R} \curvearrowright \mathcal{C}_b(\mathbb{R})$ is not continuous. In fact, this is just a consequence of the fact that not all continuous bounded functions on $\mathbb{R}$ are *uniformly* continuous (e.g., $f(t) = \sin(t^2)$). Of course, the same is true for more general groups as well.

In classical dynamics, the right notion of isomorphism between dynamical systems is the following.

Definition 2.2.11. Let $\alpha : G \curvearrowright X$ and $\beta : G \curvearrowright Y$ be continuous actions on topological spaces. A *conjugacy* $\varphi : (X, \alpha) \to (Y, \beta)$ is an homeomorphism $X \to Y$ that is additionally equivariant with respect to α and β, namely

$$\beta_g \circ \varphi = \varphi \circ \alpha_g, \quad \text{for all } g \in G.$$

The systems (X, α) and (Y, β) are said to be *conjugate* if there exists a conjugacy between them.

In the preceding section on fundamental concepts in C*-algebra theory, we noted that every C*-algebra possesses an increasing approximate unit, which for σ-unital C*-algebras can be chosen to be countable. We will need approximate units that are moreover approximately fixed by an action, which exist thanks to the following important result of Kasparov.

Lemma 2.2.12 (see [108, Lemma 1.4])**.** *Let $\beta : G \curvearrowright B$ be an action on a σ-unital* C**-algebra, and let $(e_n)_n$ be a countable increasing approximate unit of positive contractions $e_n \in B$. Then, for any separable* C**-subalgebra $D \subseteq \mathcal{M}(B)$, there exists a countable increasing approximate unit of positive contractions $b_n \in B$ that belong to the convex hull of $(e_n)_n$, and such that*

$$\lim_{n\to\infty} \|[b_n, d]\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \max_{g\in K} \|b_n - \beta_g(b_n)\| = 0$$

for all $d \in D$ and all compact subsets $K \subseteq G$.

Moreover, if $e_n = e_n e_{n+1}$ for all $n \geq 1$, then $(b_n)_n$ can be chosen so that it additionally satisfies $b_n = b_n b_{n+1}$ for all $n \geq 1$.

2.2.1 The (proper) cocycle category

Recall that an automorphism $\varphi \in \mathrm{Aut}(A)$ of a C*-algebra A is *inner* when it is of the form $\varphi = \mathrm{Ad}(u)$ for some unitary $u \in \mathcal{M}(A)$. An automorphism that is not inner is said to be *outer*. When working with certain C*-dynamical systems, demanding the existence of conjugacies is often too strong a requirement, and it is more reasonable to ask for a notion of conjugacy that "forgets" about inner automorphisms. Formalizing this intuition inevitably leads to the construction of an appropriate categorical framework for C*-dynamical systems.

Definition 2.2.13. Let $\beta : G \curvearrowright B$ be an action on a C*-algebra. A strictly continuous map $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ is a *β-cocycle* (or β-1-cocycle) if it satisfies $\mathbb{u}_{gh} = \mathbb{u}_g \beta_g(\mathbb{u}_h)$ for all $g, h \in G$.

The following notion of map will be used throughout the manuscript.

Definition 2.2.14 (cf. [192, Definition 1.10])**.** Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras. A *cocycle representation* from (A, α) to $(\mathcal{M}(B), \beta)$ is a pair $(\varphi, \mathbb{u})$ consisting of a $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ and a β-cocycle $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(B))$ satisfying the equivariance condition

$$\varphi \circ \alpha_g = \mathrm{Ad}(\mathbb{u}_g) \circ \beta_g \circ \varphi$$

for all $g \in G$. If $\varphi(A) \subseteq B$, then $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is said to be a *cocycle morphism*. Moreover, if $(\varphi, \mathbb{u})$ is a cocycle morphism with the additional requiremenet that $\mathbb{u}$ is a norm-continuous β-cocycle with values in $\mathcal{U}(\mathbf{1}+B)$, then $(\varphi,\mathbb{u})$ is said to be a *proper cocycle morphism*.

When the cocycle is trivial, namely $\mathbb{u}_g=\mathbf{1}$ for all $g\in G$, then φ is an equivariant $*$-homomorphism, and sometimes we denote $(\varphi,\mathbf{1})$ simply by φ if no confusion may arise. Moreover, when the cocycle $\mathbb{u}$ is not trivial, φ becomes an equivariant $*$-homomorphism between α and the perturbed action $\beta^{\mathbb{u}}: G \curvearrowright B$ given by

$$\beta^{\mathbb{u}}_g := \mathrm{Ad}(\mathbb{u}_g)\circ\beta_g \quad \text{for all } g\in G.$$

We record here a few important notational conventions that will be used throughout the manuscript.

- A cocycle representation $(\varphi,\mathbb{u}):(A,\alpha)\to(\mathcal{M}(B),\beta)$ is said to be *non-degenerate* if φ is non-degenerate.
- A (proper) cocycle morphism $(\varphi,\mathbb{u}):(A,\alpha)\to(B,\beta)$ with φ injective is referred to as a *(proper) cocycle embedding.*
- In the same spirit, $(\varphi,\mathbb{u})$ is said to be a *(proper) cocycle conjugacy* if φ is an isomorphism.
- We will write $\alpha\simeq_{\mathrm{cc}}\beta$, and say that (A,α) and (B,β) are *cocycle conjugate* if there exists a cocycle conjugacy between them.
- A cocycle morphism $(\varphi,\mathbb{u}):(A,\alpha)\to(B,\beta)$ with $\mathbb{u}=\mathbf{1}$ is simply denoted by φ and referred to as an *equivariant $*$-homomorphism.*

Example 2.2.15. Let $\alpha: G\curvearrowright A$ be an action on a C*-algebra. Every unitary $u\in\mathcal{U}(\mathcal{M}(A))$ induces an *inner cocycle conjugacy* given by

$$(\mathrm{Ad}(u), u\alpha_\bullet(u)^*):(A,\alpha)\to(A,\alpha).$$

If $u\in\mathcal{U}(\mathbf{1}+A)$, then this becomes a proper cocycle conjugacy.

Warning 2.2.16. Let $\alpha: G\curvearrowright A$ be an action on a C*-algebra. Unless confusion may arise, for any unitary $u\in\mathcal{U}(\mathcal{M}(A))$, we abuse notation and denote by $\mathrm{Ad}(u)$ the inner cocycle conjugacy given by $(\mathrm{Ad}(u), u\alpha_\bullet(u)^*)$. Note that when $u\in\mathcal{M}(A)^\alpha$, then $\mathrm{Ad}(u)=(\mathrm{Ad}(u),u\alpha_\bullet(u)^*)=(\mathrm{Ad}(u),\mathbf{1})$.

To complete the categorical picture, we need the right notion of composition.

Definition 2.2.17. Let A and B be C*-algebras. A $*$-homomorphism $\varphi: A\to\mathcal{M}(B)$ is said to be *extendible* if for any increasing approximate unit $(e_\lambda)_{\lambda\in\Lambda}\subseteq A$, the net $(\varphi(e_\lambda))_{\lambda\in\Lambda}$ converges strictly to a projection in $p\in\mathcal{M}(B)$.

In particular, this means that φ factors through $\mathcal{M}(pBp) \cong p\mathcal{M}(B)p$, and its corestriction $\varphi_p : A \to \mathcal{M}(pBp)$ is non-degenerate. Now, by Proposition 2.1.6 φ_p uniquely extends to a $*$-homorphism $\mathcal{M}(A) \to \mathcal{M}(pBp)$ that is strictly continuous on the unit ball. Therefore, one may define a strictly continuous homomorphism between unitary groups:

$$\varphi^+ : \mathcal{U}(\mathcal{M}(A)) \to \mathcal{U}(\mathcal{M}(B)), \quad \varphi^+(u) = \varphi_p(u) + (\mathbf{1}_{\mathcal{M}(B)} - p).$$

This map has the property that $\varphi^+(u)\varphi(a) = \varphi(ua)$ for all $u \in \mathcal{U}(\mathcal{M}(A))$ and $a \in A$.

On the other hand, if $u \in \mathcal{U}(\mathbf{1} + A)$, then one may consider the image of u under the unique extension of φ to the unitisation of A, i.e., $\varphi^\dagger : A^\dagger \to B^\dagger$. Then, the unitary multiplier in $\mathcal{M}(B)$ arising from $\varphi^\dagger(u) \in \mathcal{U}(\mathbf{1} + B)$ agrees with $\varphi^+(u)$. In this situation, we always prefer the notation $\varphi^\dagger(u)$ to $\varphi^+(u)$.

Finally, whenever φ is non-degenerate to begin with (which is stronger than being extendible), it extends to a unital $*$-homomorphism $\mathcal{M}(A) \to \mathcal{M}(B)$. In this case, by slight abuse of notation, we write $\varphi(u)$ instead of $\varphi^+(u)$.

Definition 2.2.18. Let $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright C$ be actions on C*-algebras. The composition of two extendible cocycle morphisms $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ and $(\psi, \mathbb{v}) : (B, \beta) \to (C, \gamma)$, is given by the pair

$$(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) = (\psi \circ \varphi, \psi^+(\mathbb{u})\mathbb{v}) : (A, \alpha) \to (C, \gamma).$$

The proof that this is indeed an extendible cocycle morphism, and that this binary operation is associative is contained in [192, Proposition 1.15(i+ii)].

In light of the discussion in Definition 2.2.17, the composition of two proper cocycle morphisms $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ and $(\psi, \mathbb{v}) : (B, \beta) \to (C, \gamma)$, is given by the pair

$$(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u}) = (\psi \circ \varphi, \psi^\dagger(\mathbb{u})\mathbb{v}) : (A, \alpha) \to (C, \gamma).$$

The fact that this is a proper cocycle morphism, and that this binary operation is associative is proved in [192, Proposition 1.15(iii)].

We are now ready to introduce the ambient categories for C*-dynamical systems that will play a central role in the following chapters.

Definition 2.2.19. The *cocycle category* associated with G, denoted by C^*_G as in [192], consists of C*-dynamical systems as objects and *extendible* cocycle morphisms as maps. Moreover, the *proper cocycle category* associated with G, denoted by $\mathrm{C}^*_{G,\mathrm{p}}$, has C*-dynamical systems as objects and proper cocycle morphisms as maps.

A few remarks and observations regarding the distinctive features of these categories are in order.

Remark 2.2.20. It might seem natural to include extendibility in the definition of cocycle morphisms, as is done in the original formulation of the cocycle category in [192, Section 1]. However, we choose not to adopt this approach — following instead the convention used in [192, Section 6] — and take care to ensure that the composition of two cocycle morphisms is well defined. In fact, the only instances in this dissertation where we need to compose cocycle morphisms that are not proper involve morphisms that are already non-degenerate. More precisely, we only compose cocycle conjugacies with proper cocycle conjugacies, which clearly yield cocycle conjugacies.

Remark 2.2.21. In the (proper) cocycle category, a (proper) cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is invertible if and only if φ is an isomorphism of C*-algebras, and hence $(\varphi, \mathbb{u})$ is a (proper) cocycle conjugacy. The inverse (proper) cocycle morphism is then given by

$$(\varphi, \mathbb{u})^{-1} = (\varphi^{-1}, \varphi^{-1}(\mathbb{u}_\bullet)^*) : (B, \beta) \to (A, \alpha).$$

Remark 2.2.22. In [192, Definition 2.5], Szabó defines metric structures on the sets of homomorphism in C^*_G and $\mathrm{C}^*_{G,\mathrm{p}}$ between two C*-dynamical systems (A, α) and (B, β). When $(A, \alpha), (B, \beta)$ are objects in C^*_G, he considers the topology on the collection of all extendible cocycle morphisms from (A, α) to (B, β) generated by the family of pseudometrics

$$d_{\mathcal{F}^A, \mathcal{F}^B, K}((\psi, \mathbb{v}), (\varphi, \mathbb{u})) = \max_{a \in \mathcal{F}^A} \|\psi(a) - \varphi(a)\| + \max_{g \in K} \max_{b \in \mathcal{F}^B} \|b(\mathbb{v}_g - \mathbb{u}_g)\|$$

for all compact sets $\mathcal{F}^A \subseteq A$, $\mathcal{F}^B \subseteq B$, $1_G \in K \subseteq G$, and extendible cocycle morphisms $(\psi, \mathbb{v}), (\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$. For two objects $(A, \alpha), (B, \beta)$ in $\mathrm{C}^*_{G,\mathrm{p}}$, he considers the topology on the collection of all proper cocycle morphisms between from (A, α) to (B, β) generated by the family of pseudometrics

$$d_{\mathcal{F}^A, K}((\psi, \mathbb{v}), (\varphi, \mathbb{u})) = \max_{a \in \mathcal{F}^A} \|\psi(a) - \varphi(a)\| + \max_{g \in K} \|\mathbb{v}_g - \mathbb{u}_g\|$$

for all compact sets $\mathcal{F}^A \subseteq A$, $1_G \in K \subseteq G$, and proper cocycle morphisms $(\psi, \mathbb{v}), (\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$. It follows from [192, Lemma 2.6] that the sets of arrows in C^*_G and $\mathrm{C}^*_{G,\mathrm{p}}$ are Polish spaces when one restricts to the full subcategories of separable objects. Moreover, by [192, Lemma 2.7] the composition operation of Definition 2.2.18 is jointly continuous with respect to the topologies defined above.

Next, we recall various notions of unitary equivalence between maps, which play a crucial role in the classification of C*-algebras and C*-dynamical systems. We present only those equivalence relations that are relevant to the work in this thesis; for a more comprehensive treatment, the reader is referred to [192, Section 2].

Definition 2.2.23 (cf. [192, Definitions 2.8+4.1] and [67, Definition 1.15])**.** Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras, and consider two cocycle morphisms

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta).$$

Then $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are said to be

(i) *(properly) unitarily equivalent*, denoted by $(\varphi, \mathbb{u}) \sim_{\mathrm{u}} (\psi, \mathbb{v})$ (respectively, $(\varphi, \mathbb{u}) \sim_{\mathrm{pu}} (\psi, \mathbb{v})$), if there exists a unitary $u \in \mathcal{U}(\mathcal{M}(B))$ (respectively, $u \in \mathcal{U}(\mathbf{1} + B)$) such that

$$\psi(a) = u\varphi(a)u^*, \text{ and } \mathbb{v}_g = u\mathbb{u}_g\beta_g(u)^*$$

for all $a \in A$ and for all $g \in G$.

(ii) *properly approximately unitarily equivalent*, denoted by $(\varphi, \mathbb{u}) \approx_{\mathrm{pu}} (\psi, \mathbb{v})$, if there exists a net of unitaries $u_\lambda \in \mathcal{U}(\mathbf{1} + B)$ that satisfies

$$\psi(a) = \lim_{\lambda \to \infty} u_\lambda \varphi(a) u_\lambda^*$$

for all $a \in A$, and

$$\lim_{\lambda \to \infty} \max_{g \in K} \|\mathbb{v}_g - u_\lambda \mathbb{u}_g \beta_g(u_\lambda)^*\| = 0$$

for all compact subsets $K \subseteq G$.

When C*-algebras are separable, one can replace nets in (ii) with sequences. Moreover, we remark that the same equivalence relations are automatically defined also for $*$-homomorphisms, as this corresponds to the case where $G = \{1\}$.

Remark 2.2.24. By [192, Proposition 2.11], proper approximate unitary equivalence passes to compositions in the following sense: If $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent cocycle morphisms from (A, α) to (B, β), and $(\varphi', \mathbb{u}')$ and $(\psi', \mathbb{v}')$ are properly approximately unitarily equivalent cocycle morphisms from (B, β) to (C, γ), then the compositions $(\varphi', \mathbb{u}') \circ (\varphi, \mathbb{u})$ and $(\psi', \mathbb{v}') \circ (\psi, \mathbb{v})$ are properly approximately unitarily equivalent as well.

In the next chapter, we will need to upgrade unitaries from the multiplier algebra to the proper unitisation in order to obtain stronger uniqueness theorems for maps. Here, we carry out the groundwork for these types of results.

Definition 2.2.25. Let $\beta : G \curvearrowright B$ be an action on a C^*-algebra. One denotes by $\mathcal{M}^\beta(B)$ the C^*-subalgebra of the multiplier algebra of B consisting of elements $x \in \mathcal{M}(B)$ with the property that $\{x - \beta_g(x) \mid g \in G\} \subseteq B$. We warn the reader not to confuse $\mathcal{M}^\beta(B)$ with the fixed point algebra $\mathcal{M}(B)^\beta$, which is clearly contained in it.

For sake of completeness, we also write out the following basic fact about Banach algebras.

Lemma 2.2.26. *Let A be a unital Banach algebra. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that the following statements are true for any pair of elements $a, b \in A$:*

- *if $\|a\| \leq 1$, $\|b\| \leq 2\pi$ and $\|ab\| \leq \delta$, then $\|(\exp(b) - \mathbf{1})a\| \leq \varepsilon$ and $\|a(\exp(b) - \mathbf{1})\| \leq \varepsilon$,*
- *if $\|a\|, \|b\| \leq \pi$ and $\|[a, b]\| \leq \delta$, then $\|\exp(a)\exp(b) - \exp(a + b)\| \leq \varepsilon$,*
- *if $\|a\| \leq 2\pi$ and $\|b\| \leq \delta$, then $\|\exp(a + b) - \exp(a)\| \leq \varepsilon$.*

Lemma 2.2.27. *Let $\beta : G \curvearrowright B$ be an action on a σ-unital C^*-algebra, $h \in \mathcal{M}^\beta(B)$ a self-adjoint element with $\|h\| \leq 2\pi$, and define $u := \exp(ih)$. Then, for every $\varepsilon > 0$, and compact subsets $\mathcal{F} \subseteq B$, $K \subseteq G$, there exists a unitary $v \in \mathcal{U}(\mathbf{1} + B)$ such that*

$$\max_{b \in \mathcal{F}} \big(\|(v^*u - \mathbf{1})b\|, \|b(v^*u - \mathbf{1})\| \big) \leq \varepsilon, \text{ and } \max_{g \in K} \|v^*u - \beta_g(v^*u)\| \leq \varepsilon.$$

Proof. Fix $\mathcal{F} \subseteq B$ and $K \subseteq G$ as in the statement and note that, by a rescaling argument, we may assume that every element of $\mathcal{F}$ is a contraction. By Lemma 2.2.12 we may find an increasing approximate unit $b_n \in B$ such that $\lim_{n\to\infty} \max_{g \in K} \|b_n - \beta_g(b_n)\| = 0$ and $\lim_{n\to\infty} \|[h, b_n]\| = 0$. In particular, observe that for every compact subset $K \subseteq G$,

$$\lim_{n\to\infty} \max_{g \in K} \|h - b_n h b_n - \beta_g(h - b_n h b_n)\| = \lim_{n\to\infty} \max_{g \in K} \|(\mathbf{1} - b_n^2)(h - \beta_g(h))\| = 0,$$

where the limit is zero because $h - \beta_g(h) \in B$ for all $g \in G$ by assumption. Fix $\varepsilon > 0$ as in the statement, and choose $\delta > 0$ that satisfies the three conditions in Lemma 2.2.26 with respect to $\varepsilon/3$. We may find some $n_0 \in \mathbb{N}$ such that, using the notation $b_0 := b_{n_0}$,

$$\max_{b \in \mathcal{F}} \|(h - b_0 h b_0)b\| \leq \delta, \tag{2.1}$$

$$\|[a, hah]\| \leq \delta, \tag{2.2}$$

$$\max_{g\in K}\|h-b_0hb_0-\beta_g(h-b_0hb_0)\|\leq\delta, \tag{2.3}$$

Now we set $v := \exp(ib_0hb_0)$, and proceed to check that v satisfies the statement. First, using Lemma 2.2.26, note that 2.1 and 2.2 give the following,

$$\max_{b\in\mathcal{F}}\|(v^*u-\mathbf{1})b\| = \max_{b\in\mathcal{F}}\|(\exp(-ib_0hb_0)\exp(ih)-\mathbf{1})b\|$$

$$\leq \frac{\varepsilon}{3}+\max_{b\in\mathcal{F}}\|(\exp(i(h-b_0hb_0))-\mathbf{1})b\|\leq\frac{2\varepsilon}{3},$$

and the analogous calculation for $\max_{b\in\mathcal{F}}\|b(v^*u-\mathbf{1})\|$. Second, with Lemma 2.2.26 again, condition 2.1 followed by 2.3 gives that

$$\max_{g\in K}\|\beta_g(v^*u)-v^*u\|$$

$$= \max_{g\in K}\|\beta_g(\exp(-ib_0hb_0)\exp(ih))-\exp(-ib_0hb_0)^*\exp(ih)\|$$

$$\leq \frac{2\varepsilon}{3}+\max_{g\in K}\|\beta_g(\exp(i(h-b_0hb_0)))-\exp(i(h-b_0hb_0))\|$$

$$= \frac{2\varepsilon}{3}+\max_{g\in K}\|\exp(i\beta_g(h-b_0hb_0))-\exp(i(h-b_0hb_0))\|\leq\varepsilon.$$

□

Remark 2.2.28. Let A be a unital C*-algebra A. It is a basic fact that a unitary $u\in A$ belongs to $\mathcal{U}_0(A)$ if and only if it can be written as

$$u=\exp(ih_0)\cdots\exp(ih_n)$$

for some $n\in\mathbb{N}$ and self-adjoint elements $h_0,\dots,h_n\in A$ with $\max_{0\leq k\leq n}\|h_k\|\leq 2\pi$; see for instance [172, Proposition 2.1.6(iii)].

Lemma 2.2.29. *Let $\beta: G\curvearrowright B$ be an action on a separable* C**-algebra, and $u\in\mathcal{U}_0(\mathcal{M}^\beta(B))$. Then, the inner cocycle conjugacy given by $(\mathrm{Ad}(u), u\beta_\bullet(u)^*)$ is properly approximately unitarily equivalent to* id_B.

Proof. By Remark 2.2.28, u may be written as

$$u=u_0\cdots u_k,$$

where $u_j=\exp(ih_j)$ for self-adjoint elements h_j in $\mathcal{M}^\beta(B)$ with $\|h_j\|\leq 2\pi$ for all $0\leq j\leq k$. In particular, it suffices to show that for any unitary of the form $w=\exp(ih)$ for a self-adjoint element $h\in\mathcal{M}^\beta(B)$ with $\|h\|\leq 2\pi$, the inner cocycle conjugacy given by $(\mathrm{Ad}(w), w\beta_\bullet(w)^*)$ is properly approximately unitarily equivalent to id_B. This is because a composition of cocycle conjugacies that are properly approximately unitarily equivalent to id_B will still have the same property.

Now, fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq B$ with dense union, and an increasing sequence of compact subsets $K_n \subseteq G$ such that $\bigcup_{n\in\mathbb{N}} K_n^\circ = G$. Moreover, as in the previous paragraph, set $w = \exp(ih)$ for a self-adjoint element $h \in \mathcal{M}^\beta(B)$ with $\|h\| \leq 2\pi$. It follows from Lemma 2.2.27 that we may find a sequence of unitaries $v_n \in \mathcal{U}(\mathbf{1} + B)$ such that

$$\lim_{n\to\infty} \max_{b\in\mathcal{F}_n} (\|(v_n^* w - \mathbf{1})b\| + \|b(v_n^* w - \mathbf{1})\|) = 0,$$

$$\lim_{n\to\infty} \max_{g\in K_n} \|v_n^* w - \beta_g(v_n^* w)\| = 0.$$

In particular, the first limit implies that

$$\lim_{n\to\infty} \|v_n b v_n^* - wbw^*\| = \lim_{n\to\infty} \|b(v_n^* w) - (v_n^* w)b\| = 0$$

for all $b \in B$, while the second limit ensures that for every compact set $K \subseteq G$

$$\lim_{n\to\infty} \max_{g\in K} \|v_n\beta_g(v_n)^* - w\beta_g(w)^*\| \leq \lim_{n\to\infty} \max_{g\in K_n} \|\beta_g(v_n^* w) - v_n^* w\| = 0$$

because there exists some $n \in \mathbb{N}$ such that $K \subseteq K_n$ by out choice of the sequence $(K_n)_n$. In other words, $(\mathrm{Ad}(w), w\beta_\bullet(w)^*)$ is properly approximately unitarily equivalent to id_B. Hence, the composition

$$(\mathrm{Ad}(u), u\beta_\bullet(u)^*) = (\mathrm{Ad}(u_{k_0}), u_{k_0}\beta_\bullet(u_{k_0})^*) \circ \cdots \circ (\mathrm{Ad}(u_{k_n}), u_{k_n}\beta_\bullet(u_{k_n})^*)$$

is properly approximately unitarily equivalent to the identity as well. □

2.2.2 Elliott's two-sided intertwining

It was observed in [192, Theorem 3.1] that Elliott's high-level abstract framework [52, pp. 35–36] can be applied in the proper cocycle category to obtain a two-sided intertwining argument for proper cocycle morphisms. We first give an overview of the proof as carried out in the aforementioned reference, using Elliott's theorem. Subsequently, at the end of this section, we give a different and relatively straightforward proof of the same result, in the style of [173, Section 2.3].

Theorem 2.2.30 (see [192, Theorem 3.1])**.** *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable* C**-algebras. Let*

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta), \quad (\psi, \mathbb{v}) : (B, \beta) \to (A, \alpha)$$

be proper cocycle morphisms such that $(\psi, \mathbb{v}) \circ (\varphi, \mathbb{u})$ *and* $(\varphi, \mathbb{u}) \circ (\psi, \mathbb{v})$ *are properly approximately unitarily equivalent to* id_A *and* id_B*, respectively. Then,* $(\varphi, \mathbb{u})$ *and* $(\psi, \mathbb{v})$ *are properly approximately unitarily equivalent to mutually inverse proper cocycle conjugacies.*

Proof. The category we choose is the full subcategory of $\mathrm{C}^*_{G,\mathrm{p}}$ consisting of separable C*-dynamical systems with proper cocycle morphisms. Here, the inner cocycle automorphisms of an object (B, β) are of the form $(\mathrm{Ad}(u), u\beta_\bullet(u)^*)$ for some unitary $u \in \mathcal{U}(\mathbf{1} + B)$, and they form a normal subgroup of the cocycle automorphism group of (A, α) by [192, Proposition 1.24]. We equip the collection of all homomorphisms between two objects with proper approximate unitary equivalence, which is compatible with the notion of inner cocycle automorphism above. Now, if $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ are actions on separable C*-algebras, we put a metric on the set of proper cocycle morphisms $(A, \alpha) \to (B, \beta)$ as follows. Fix a sequence of contractions $a_n \in A$ that is dense in the unit ball of A, and an increasing sequence of compact subsets $K_n \subseteq G$ whose union is G. Then, the distance between two proper cocycle morphisms $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ is given by

$$\sum_{n \in \mathbb{N}} 2^{-n} \left(\|\psi(a_n) - \varphi(a_n)\| + \max_{g \in K_n} \|\mathbb{v}_g - \mathbb{u}_g\| \right),$$

and it defines a metric that induces the topology from Remark 2.2.22 on the same set. We also know that composition with an inner cocycle morphism from the left is isometric, and that composition of proper cocycle morphisms is jointly continuous (see Remark 2.2.22). Finally, this shows that [52, Theorem 2, pp. 35–36] applies, and gives the desired result. $\square$

We are now ready to give a more direct proof of Theorem 2.2.30. The type of argument presented here is a very special case of the cocycle intertwining machinery [192, Theorem 3.6], which is a refined analogue of [173, Corollary 2.3.3].

Notation 2.2.31. Throughout the manuscript, for any compact subset $K \subset G$, we will denote by K° its interior.

Alternative proof of Theorem 2.2.30. Fix increasing sequences of compact sets $\mathcal{G}_n^A \subseteq A$, $\mathcal{G}_n^B \subseteq B$ with dense union in A and B, respectively, and a sequence of compact sets $K_n \subseteq G$ such that $\bigcup_n K_n = G$ and $K_n \subseteq K_{n+1}^\circ$ for all $n \in \mathbb{N}$. Start by setting $(\kappa_0, \mathbb{x}^{(0)}) := (\psi, \mathbb{v})$, and $\mathcal{F}_0^B := \mathcal{G}_0^B$. By assumption, $(\varphi, \mathbb{u}) \circ (\kappa_0, \mathbb{x}^{(0)})$ is properly approximately unitarily equivalent to id_B, hence we may find a unitary $u_1 \in \mathcal{U}(\mathbf{1}+B)$ such that

$$\max_{b\in\mathcal{F}_0^B} \|\mathrm{Ad}(u_1)\circ\varphi\circ\kappa_0(b) - b\| \le 1,$$

$$\max_{g\in K_0} \|u_1\varphi^\dagger(\mathbb{x}_g^{(0)})\mathbb{u}_g\beta_g(u_1)^* - \mathbf{1}\| \le 1.$$

We then set $(\theta_0, \mathbb{y}^{(0)}) := \mathrm{Ad}(u_1)\circ(\varphi,\mathbb{u})$, and $\mathcal{F}_0^A := \mathcal{G}_0^A \cup \kappa_0(\mathcal{F}_0^B)$. Now, since proper approximate unitary equivalence passes to compositions (see Remark 2.2.24), we know that $(\psi,\mathbb{v})\circ(\theta_0,\mathbb{y}^{(0)})$ is properly approximately unitarily equivalent to id_A. Hence, we may find a unitary $v_1 \in \mathcal{U}(\mathbf{1}+A)$ that satisfies the following approximate conditions,

$$\max_{a\in\mathcal{F}_0^A} \|\mathrm{Ad}(v_1)\circ\psi\circ\theta_0(a) - a\| \le 1,$$

$$\max_{g\in K_0} \|v_1\psi^\dagger(\mathbb{y}^{(0)})\mathbb{v}_g\alpha_g(v_1)^* - \mathbf{1}\| \le 1,$$

and we define a proper cocycle morphism $(\kappa_1, \mathbb{x}^{(1)}) := \mathrm{Ad}(v_1)\circ(\psi,\mathbb{v})$, and a finite set

$$\mathcal{F}_1^B := \mathcal{G}_1^B \cup \mathcal{F}_0^B \cup \theta_0(\mathcal{F}_0^A).$$

We then proceed recursively. Suppose we have found unitaries $u_1,\dots,u_n \in \mathcal{U}(\mathbf{1}+B)$, $v_1,\dots,v_n \in \mathcal{U}(\mathbf{1}+A)$, increasing sequences of finite sets $\mathcal{F}_0^A \subseteq \cdots \subseteq \mathcal{F}_{n-1}^A$ and $\mathcal{F}_0^B \subseteq \cdots \subseteq \mathcal{F}_n^B$ with the property that $\kappa_{i-1}(\mathcal{F}_{i-1}^B)\cup\mathcal{G}_i^A\cup\mathcal{F}_{i-1}^A \subseteq \mathcal{F}_i^A$ and $\theta_{i-1}(\mathcal{F}_{i-1}^A)\cup\mathcal{G}_i^B\cup\mathcal{F}_{i-1}^B \subseteq \mathcal{F}_i^B$, and proper cocycle morphisms $(\kappa_i,\mathbb{x}^{(i)}) = \mathrm{Ad}(v_i)\circ(\psi,\mathbb{v})$ and $(\theta_i,\mathbb{y}^{(i)}) = \mathrm{Ad}(u_{i+1})\circ(\varphi,\mathbb{u})$ such that

$$\max_{b\in\mathcal{F}_i^B} \|\theta_i\circ\kappa_i(b) - b\| \le 2^{-i},$$

$$\max_{g\in K_i} \|\varphi^\dagger(\mathbb{x}_g^{(i)})\mathbb{y}_g^{(i)*} - \mathbf{1}\| \le 2^{-i},$$

and

$$\max_{a\in\mathcal{F}_i^A} \|\kappa_{i+1}\circ\theta_i(a) - a\| \le 2^{-i},$$

$$\max_{g\in K_i} \|\psi^\dagger(\mathbb{y}^{(i)})\mathbb{x}_g^{(i+1)*} - \mathbf{1}\| \le 2^{-i},$$

for all $i \le n-1$. A visual representation of this is given by the following (not necessarily commutative) diagram

$$\begin{tikzcd}
& \cdots \arrow[r] & (A,\alpha) \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "(\theta_i,\mathbb{y}^{(i)})"] & & (A,\alpha) \arrow[r] & \cdots \\
\cdots \arrow[r] & (B,\beta) \arrow[ur, "(\kappa_i,\mathbb{x}^{(i)})"] \arrow[rr, "\mathrm{id}_B"'] & & (B,\beta) \arrow[ur, "(\kappa_{i+1},\mathbb{x}^{(i+1)})"'] \arrow[r] & \cdots &
\end{tikzcd}$$

that continues up to the n-th step. The goal of the next part of the proof is to prove that the diagram is approximately commutative in the sense that at the step $(n+1)$ commutation is ensured with upper bound 2^{-n} and uniformly over the finite sets $\mathcal{F}_n^B, \mathcal{F}_n^A$. To show this, we use proper approximate unitary equivalence to find a unitary $u_{n+1} \in \mathcal{U}(\mathbf{1}+B)$ such that

$$\max_{b \in \mathcal{F}_n^B} \|\mathrm{Ad}(u_{n+1}) \circ \varphi \circ \kappa_n(b) - b\| \leq 2^{-n},$$

$$\max_{g \in K_n} \|u_{n+1}\varphi^\dagger(\mathbb{x}_g^{(n)})\mathbb{u}_g\beta_g(u_{n+1})^* - \mathbf{1}\| \leq 2^{-n}.$$

Then, set $(\theta_n, \mathbb{y}^{(n)}) := \mathrm{Ad}(u_{n+1}) \circ (\varphi, \mathbb{u})$, and $\mathcal{F}_n^A := \mathcal{F}_{n-1}^A \cup \mathcal{G}_n^A \cup \kappa_n(\mathcal{F}_n^B)$. Similarly, we find a unitary $v_{n+1} \in \mathcal{U}(\mathbf{1}+A)$ such that

$$\max_{a \in \mathcal{F}_n^A} \|\mathrm{Ad}(v_{n+1}) \circ \psi \circ \theta_n(a) - a\| \leq 2^{-n},$$

$$\max_{g \in K_n} \|v_{n+1}\psi^\dagger(\mathbb{y}_g^{(n)})\mathbb{v}_g\alpha_g(v_{n+1})^* - \mathbf{1}\| \leq 2^{-n},$$

and set $(\kappa_{n+1}, \mathbb{x}^{(n+1)}) := \mathrm{Ad}(v_{n+1}) \circ (\psi, \mathbb{v})$. Moreover, we choose $\mathcal{F}_{n+1}^B = \mathcal{F}_n^B \cup \mathcal{G}_{n+1}^B \cup \kappa_{n+1}(\mathcal{G}_n^A)$. Now, via this procedure we find two sequences of unitaries $u_n \in \mathcal{U}(\mathbf{1}+B)$ and $v_n \in \mathcal{U}(\mathbf{1}+A)$, respectively, that yield proper cocycle morphisms $(\kappa_n, \mathbb{x}^{(n)}) : (B, \beta) \to (A, \alpha)$ and $(\theta_i, \mathbb{y}^{(i)}) : (A, \alpha) \to (B, \beta)$ that are properly approximately unitarily equivalent to $(\psi, \mathbb{v})$ and $(\varphi, \mathbb{u})$, respectively. Moreover these maps satisfy the approximate conditions above, which we now use to prove that there exist mutually inverse proper cocycle conjugacies

$$(\theta, \mathbb{y}) : (A, \alpha) \to (B, \beta), \text{ and } (\kappa, \mathbb{x}) : (B, \beta) \to (A, \alpha)$$

with the property that $(\theta, \mathbb{y}) \approx_{\mathrm{pu}} (\varphi, \mathbb{u})$ and $(\kappa, \mathbb{x}) \approx_{\mathrm{pu}} (\psi, \mathbb{v})$. We claim that these are given by

$$\theta(a) = \lim_{k \to \infty} \theta_k(a), \text{ and } \mathbb{y}_g = \lim_{k \to \infty} \mathbb{y}_g^{(k)}, \qquad a \in A, g \in G$$

$$\kappa(b) = \lim_{k \to \infty} \kappa_k(b), \text{ and } \mathbb{x}_g = \lim_{k \to \infty} \mathbb{x}_g^{(k)}. \qquad b \in B, g \in G,$$

and start by proving that these limits exist. Let $b \in \bigcup_n \mathcal{G}_n^B$, and use density find to find some $m \in \mathbb{N}$ such that $a \in \kappa_n^{-1}(\mathcal{F}_n^A)$ for all $n \geq m$. Then, for all $k \geq m$

$$\|\theta_k(b) - \theta_{k+1}(a)\| = \|\theta_k \circ \kappa_k(a) - \theta_{k+1} \circ \kappa_{k+1}(a)\| \leq 3 \cdot 2^{-k-1},$$

which means that $(\theta_k(a))_k$ forms a Cauchy sequence and therefore admits a limit in B. A density argument show that θ is defined on the entire domain A, the existence of κ is proved in an analogous manner. It is a direct consequence that θ and κ are mutually inverse isomorphisms. When it comes to the cocycles, we note that for any compact subset $K \subseteq G$, there exists some $n \in \mathbb{N}$ such that $K \subseteq K_n$. Hence, we may observe that by the conditions above that for all $k \geq n$

$$\begin{aligned}
\max_{g \in K} \|\mathbb{y}_g^{(k)} - \mathbb{y}_g^{(k+1)}\| &\leq \max_{g \in K_k} \|\mathbb{y}_g^{(k)} - \mathbb{y}_g^{(k+1)}\| \\
&\leq \max_{g \in K_k} \left(\|\mathbb{y}_g^{(k)} - \mathbf{1}\| + \|\mathbf{1} - \mathbb{y}_g^{(k+1)}\| \right) \\
&= \max_{g \in K_k} \left(\|\varphi^\dagger(\mathbb{x}_g^{(k)})\mathbb{y}_g^{(k)*} - \mathbf{1}\| + \|\varphi^\dagger(\mathbb{x}_g^{(k+1)})\mathbb{y}_g^{(k+1)*} - \mathbf{1}\| \right) \\
&\leq 3 \cdot 2^{-k-1}.
\end{aligned}$$

Hence, $(\mathbb{y}_g^{(k)})_k$ is Cauchy uniformly over $K \subseteq G$, and since K was arbitrary, it converges to an element in $\mathcal{U}(\mathbf{1} + B)$ for all $g \in G$. Moreover, it follows from continuity and the cocycle condition of $\mathbb{y}^{(n)}$ that $\mathbb{y} : G \to \mathcal{U}(\mathbf{1} + B)$ is a β-cocycle and satisfies the equivariance condition with respect to θ. The analogous statement clearly holds for $\mathbb{x}$ as well. Finally, we have that $(\theta, \mathbb{y}) \approx_{\mathrm{pu}} (\varphi, \mathbb{u})$ and $(\kappa, \mathbb{x}) \approx_{\mathrm{pu}} (\psi, \mathbb{v})$ by construction, and the proof is concluded. □

2.2.3 Sequence algebras, path algebras, and induced dynamics

Sequence algebras play a foundational role in the theory of C*-algebras. In fact, whenever a property is formulated via approximate conditions, it can typically be more conveniently expressed — and often more clearly understood — through the language of sequence algebras.

Definition 2.2.32. Let $\alpha : G \curvearrowright A$ be an algebraic action on a C*-algebra. Then we denote by A_α the subset of A containing all elements a on which α is continuous, i.e.,

$$A_\alpha = \{a \in A \mid [g \mapsto \alpha_g(a)] \text{ is continuous}\}.$$

This is a C*-subalgebra of A, which we refer to as the *(α-)continuous part of* A.

Definition 2.2.33. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. We have already encountered the *sequence algebra* of A,

$$A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

Note that A embeds into A_∞ as constant sequences. We denote this embedding by ι_∞ when A is understood from the context.

The *α-continuous sequence algebra* of A is given by

$$A_{\infty,\alpha} := (A_\infty)_{\alpha_\infty} = \big\{x \in A_\infty \mid [g \mapsto \alpha_{\infty,g}(x)] \text{ is continuous}\big\},$$

where $\alpha_\infty : G \to \mathrm{Aut}(A_\infty)$ is the algebraic action obtained by applying α componentwise. By construction, α_∞ is a continuous action on $A_{\infty,\alpha}$. Moreover, as a consequence of [23, Theorem 2], $A_{\infty,\alpha}$ coincides with the quotient C*-algebra

$$\ell^\infty_\alpha(\mathbb{N}, A)/c_0(\mathbb{N}, A),$$

where $\ell^\infty_\alpha(\mathbb{N}, A) = \big\{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A) \mid [g \mapsto (\alpha_g(x_n))_{n\in\mathbb{N}}] \text{ is continuous}\big\}$.

Let B be an α_∞-invariant C*-subalgebra of $A_{\infty,\alpha}$. Then, α_∞ restricts to an action on the *α-continuous relative central sequence algebra*,

$$A_{\infty,\alpha} \cap B' = \{x \in A_{\infty,\alpha} \mid xb = bx \text{ for all } b \in B\},$$

as well as on the annihilator of B in $A_{\infty,\alpha}$,

$$A_{\infty,\alpha} \cap B^\perp = \{x \in A_{\infty,\alpha} \mid xb = bx = 0 \text{ for all } b \in B\}.$$

The induced action on $F(B, A_{\infty,\alpha}) = (A_{\infty,\alpha} \cap B')/(A_{\infty,\alpha} \cap B^\perp)$ is denoted by $\tilde\alpha_\infty$. If $A = B$, then one writes $F_{\infty,\alpha}(A) = F(A, A_{\infty,\alpha})$.

Remark 2.2.34 (see [117, Proposition 1.9(3)]). In the context of Definition 2.2.33, note that for any action $\alpha : G \curvearrowright A$, and σ-unital α_∞-invariant C*-subalgebra $B \subseteq A_{\infty,\alpha}$, the C*-algebra $F(B, A_{\infty,\alpha})$ is unital. Let us show why this is the case. With the help of Lemma 2.2.12, find an approximate unit $(e_n)_n \subseteq B$ consisting of positive contractions such that $\max_{g\in K} \|\alpha_{\infty,g}(e_n) - e_n\| \to 0$ for all compact subsets $K \subseteq G$, and lift each element to an α-continuous sequence of positive contractions $(e_n^{(k)})_k \in \ell^\infty_\alpha(\mathbb{N}, A)$. Since B is σ-unital, there exists a strictly positive contraction $h \in B$ such that hBh is dense in B. Let us lift h to a sequence of positive contractions $(h_k)_k$. Then, one has that

$$\lim_{n\to\infty} \limsup_{k\to\infty} \|e_n^{(k)} h_k - h_k\| = 0, \quad \lim_{n\to\infty} \limsup_{k\to\infty} \|h_k e_n^{(k)} - h_k\| = 0$$

and also that for all compact sets $K \subseteq G$,

$$\lim_{n\to\infty} \limsup_{k\to\infty} \max_{g\in K_i} \|\alpha_g(e_n^{(k)}) - e_n^{(k)}\| = 0.$$

Note that the last condition is possible because each e_n lifts to an α-continuous sequence as remarked in Definition 2.2.33. Choose an increasing sequence of compact subsets $K_i \subseteq G$ whose union is G. It follows that for any $i \in \mathbb{N}$, one may find some $n_i \in \mathbb{N}$ and some $k_i \in \mathbb{N}$ such that, for all $k \geq k_i$

$$\sup_{k\geq k_i} \|e_{n_i}^{(k)} h_k - h_k\| \leq 2^{-i}, \quad \sup_{k\geq k_i} \|h_k e_{n_i}^{(k)} - h_k\| \leq 2^{-i}$$

and

$$\sup_{k \geq k_i} \max_{g \in K} \|\alpha_g(e_{n_i}^{(k)}) - e_n^{(k)}\| \leq 2^{-i}.$$

Upon replacing k_i with a larger natural number, one may also assume that $k_{i+1} > k_i$ for all $i \in \mathbb{N}$. Let e be the element of A_∞ represented by the sequence $(e_{n_i}^{(k_i)})_{i\in\mathbb{N}} \subseteq A$. We have that, by construction, $eh = h = he$ and $\alpha_{\infty,g}(e) = e$ for all $g \in G$. Therefore, e acts like as a unit on $\overline{hBh} = B$, and in particular belongs to $A_{\infty,\alpha} \cap B'$. It follows that the image of e inside $F(B, A_{\infty,\alpha})$ under the canonical quotient map is the unit of $F(B, A_{\infty,\alpha})$.

Remark 2.2.35. Let B be a σ-unital C*-algebra equipped with an action $\beta : G \curvearrowright B$. It is a non-trivial fact following from [184, Proposition 1.5] (cf. [117, Proposition 1.9(4)+(5)]) that there exists a canonical isomorphism

$$\nu : F_\infty(B) \to \mathcal{M}(\overline{BB_\infty B}) \cap B'$$

that is automatically equivariant with respect the respective algebraic actions induced by β.

The path algebra associated to a C*-algebra is somewhat reminiscent of the sequence algebra, and one should really think of it as a continuous version of the latter. This object will play a central role throughout Chapter 5.

Definition 2.2.36. Let A be a C*-algebra. Denote by $\mathcal{C}_b([0,\infty), A)$ the C*-algebra of bounded continuous functions from $[0,\infty)$ to A and by $\mathcal{C}_0([0,\infty), A)$ the ideal of continuous functions vanishing at infinity. We denote by

$$A_\mathfrak{c} = \mathcal{C}_b([0,\infty), A)/\mathcal{C}_0([0,\infty), A)$$

the *path algebra* of A.[10] Note that A can be naturally identified with the C*-subalgebra of constant paths in $A_\mathfrak{c}$. When we need to make this embedding explicit, we use the notation $\iota_\mathfrak{c}$. We moreover call by $\pi_\mathfrak{c}$ the quotient map $\mathcal{C}_b([0,\infty), A) \to A_\mathfrak{c}$.

Remark 2.2.37. Let $\sigma \in \mathrm{Aut}(A)$ be an automorphism of a C*-algebra. Then, by sending each $f \in \mathcal{C}_b([0,\infty), A)$ to the function $\sigma \circ f \in \mathcal{C}_b([0,\infty), A)$, one obtains an automorphism σ_b of $\mathcal{C}_b([0,\infty), A)$. This induces an automorphism $\sigma_\mathfrak{c}$ of $A_\mathfrak{c}$ because $\sigma \circ f$ vanishes at infinity when f does. The resulting assignments $\mathrm{Aut}(A) \to \mathrm{Aut}\big(\mathcal{C}_b([0,\infty), A)\big)$ and $\mathrm{Aut}(A) \to \mathrm{Aut}(A_\mathfrak{c})$ are clearly both multiplicative. Moreover, if $C \subseteq A_\mathfrak{c}$ is a $\sigma_\mathfrak{c}$-invariant C*-subalgebra, then $\sigma_\mathfrak{c}$ restricts to an automorphism of $A_\mathfrak{c} \cap C'$ as well.

Assume now that A is unital, and that $\sigma = \mathrm{Ad}(u)$ for a unitary $u \in \mathcal{U}(A)$, then $\sigma_\mathfrak{c}$ is the identity map on $A_\mathfrak{c} \cap A'$. Indeed, if $f \in \mathcal{C}_b([0,\infty), A)$ is the

[10] Note that the path algebra does not have a commonly agreed upon notation in the literature.

representative of an element $x \in A_{\mathfrak{c}} \cap A'$, then $[f(t), a] \xrightarrow{t\to\infty} 0$ for all $a \in A$. As a result, $\|uf(t)u^* - f(t)\| \xrightarrow{t\to\infty} 0$ and therefore $\sigma_{\mathfrak{c}}(x) = x$.

Remark 2.2.38. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. In light of the above, α induces algebraic G-actions α_b on $\mathcal{C}_b([0,\infty), A)$ and $\alpha_{\mathfrak{c}}$ on $A_{\mathfrak{c}}$. In general these actions may fail to be continuous. However, they restrict to continuous actions on their respective continuous parts, which we denote by

$$\mathcal{C}_{b,\alpha}([0,\infty), A) := (\mathcal{C}_b([0,\infty), A))_{\alpha_b} \quad \text{and} \quad A_{\mathfrak{c},\alpha} := (A_{\mathfrak{c}})_{\alpha_{\mathfrak{c}}}.$$

Note that as a C*-subalgebra of $A_{\mathfrak{c}}$, $A_{\mathfrak{c},\alpha}$ agrees with $\mathcal{C}_{b,\alpha}([0,\infty), A)/\mathcal{C}_0([0,\infty), A)$ thanks to [23, Theorem 2].

2.2.4 Lifting approximate proper cocycle morphisms

The goal of this section is to turn an *approximate* proper cocycle morphism, that is, a proper cocycle morphism with range in some dynamical sequence algebra $(B_{\infty,\beta}, \beta_\infty)$ into a proper cocycle morphism with image in (B, β) that can be obtained as a unitary perturbation of the former. We present a critical theorem that allows one to achieve this goal, which will play a pivotal role in existence-type results of Chapter 3. The theorem we are going to present in this section is a dynamical generalisation of a result of Phillips [160, Proposition 1.3.7] (see also [64, Theorem 4.3]), and first appeared in Szabó work [192, Theorem 4.10].

First, we recall that, given a C*-algebra B, every sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \eta(n) = \infty$ induces a $*$-homomorphisms on the sequence algebra of B by

$$B_\infty \to B_\infty, \quad [(x_n)_{n\in\mathbb{N}}] \mapsto [(x_{\eta(n)})_{n\in\mathbb{N}}],$$

where we denote by $[(x_n)_n]$ the class in B_∞ of a sequence $(x_n)_n \in \ell^\infty(\mathbb{N}, B)$. Then notice that, for any action $\beta : G \curvearrowright B$, the $*$-endomorphism η^* is equivariant with respect to β_∞, and therefore restricts to an equivariant $*$-endomorphism on $B_{\infty,\beta}$.

The following result is a special case of [192, Theorem 4.10]. In fact, the statement in full generality holds for any proper cocycle morphism. However, since we are only going to apply it to equivariant $*$-homomorphisms, we only give a proof of this sub-case.

Theorem 2.2.39. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on* C**-algebras with A separable, and $\varphi : (A, \alpha) \to (B_{\infty,\beta}, \beta_\infty)$ an equivariant $*$-homomorphism. Then, the following are equivalent:*

(i) φ *is properly unitarily equivalent to* $\iota_\infty \circ (\psi, \mathbb{v})$ *for a proper cocycle morphism* $(\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$.[11]

(ii) For every sequence $\eta : \mathbb{N} \to \mathbb{N}$ *such that* $\lim_{n\to\infty} \eta(n) = \infty$, φ *is properly unitarily equivalent to* $\eta^* \circ \varphi$.

(iii) Suppose that $\varphi_n : A \to B$ *is a sequence of* $*$*-linear maps that form a lift of* φ. *Then, for every* $\varepsilon > 0$, *finite subset* $\mathcal{F} \subseteq A$, *compact set* $K \subseteq G$ *and* $m \in \mathbb{N}$, *there exists some* $k \geq m$ *such that for every* $n \geq k$, *there exists a unitary* $v \in \mathcal{U}(\mathbf{1} + B)$ *such that*

$$\max_{a\in\mathcal{F}} \|v^*\varphi_n(a)v - \varphi_k(a)\| \leq \varepsilon \quad \textit{and} \quad \max_{g\in K} \|v^*\beta_g(v) - \mathbf{1}\| \leq \varepsilon.$$

Proof. Throughout the proof, $\mathcal{F}_n \subseteq A$ denotes an increasing sequence of finite sets with dense union, and $K_n \subseteq G$ an increasing sequence of compact sets such that $\bigcup_n K_n^\circ = G$.

The implication (i)⇒(ii) is automatically true because if φ is properly unitarily equivalent to $\iota_\infty \circ (\psi, \mathbb{v})$, then for any sequence $\eta : \mathbb{N} \to \mathbb{N}$ with $\eta(n) \to \infty$ as $n \to \infty$ one has that

$$\eta^* \circ \varphi \sim_{\mathrm{pu}} \eta^* \circ \iota_\infty \circ (\psi, \mathbb{v}) = \iota_\infty \circ (\psi, \mathbb{v}) \sim_{\mathrm{pu}} \varphi.$$

It is also not hard to see that (iii)⇒(ii) as we argue below. Assume that (iii) holds, and consider any sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\eta(n) \to \infty$ as $n \to \infty$. Then, let $(\varphi_n)_n$ be a lift of $*$-linear maps of φ, and note that $(\varphi_{\eta(n)})_n$ is a lift of $\eta^* \circ \varphi$. Then, by assumption, we have that for each $n \in \mathbb{N}$ we may find some $k_n \geq \eta(n)$ such that for every $\ell \geq k_n$ there exists a unitary $v_\ell \in \mathcal{U}(\mathbf{1} + B)$ with

$$\max_{a\in\mathcal{F}_n} \|v_\ell^*\varphi_\ell(a)v_\ell - \varphi_{k_n}(a)\| \leq 2^{-n} \quad \text{and} \quad \max_{g\in K_n} \|v_\ell^*\beta_g(v_\ell) - \mathbf{1}\| \leq 2^{-n}.$$

In particular, the unitary $v \in \mathcal{U}(\mathbf{1} + B_\infty)$ induced by the sequence $(v_{k_n})_{n\in\mathbb{N}}$ witnesses proper approximate unitary equivalence between φ and $\eta^* \circ \varphi$.

We proceed to show the implication (ii)⇒(iii). Assume, towards a contradiction, that (iii) does not hold, i.e., that there exists some $\varepsilon > 0$, finite subset $\mathcal{F} \subseteq A$, compact set $K \subseteq G$, and $m \in \mathbb{N}$, such that for every $k \geq m$, there exists some $n_k \geq k$ such that no unitary $v \in \mathcal{U}(\mathbf{1} + B)$ satisfies the condition in the statement where n is replaced by n_k. Let now $\eta : \mathbb{N} \to \mathbb{N}$ denote the sequence given by

$$\eta(k) := \begin{cases} 1 & \text{if } k < m \\ n_k & \text{otherwise} \end{cases}$$

[11]Here ι_∞ denotes the equivariant diagonal embedding $B \hookrightarrow B_{\infty,\beta}$.

which satisfies $\eta(k) \to \infty$ as $k \to \infty$ because $n_k \geq k$. Since (ii) holds, we know that $\eta^* \circ \varphi \sim_{\mathrm{pu}} \varphi$, and hence there exists a unitary $v \in \mathcal{U}(\mathbf{1} + B_{\infty,\beta})$, which is not necessarily β_∞-invariant, such that $\eta^* \circ \varphi = \mathrm{Ad}(v) \circ \varphi$. Let us lift v to a sequence of unitaries $v_k \in \mathcal{U}(\mathbf{1} + B)$, and note that

$$0 = \|\eta^* \circ \varphi(a) - v\varphi(a)v^*\| = \limsup_{k\to\infty} \|\varphi_{n_k}(a) - v_k\varphi_k(a)v_k^*\|,$$

and

$$0 = \max_{g\in K} \|\mathbf{1} - v\beta_{\infty,g}(v)^*\| = \limsup_{k\to\infty} \max_{g\in K} \|\mathbf{1} - v_k\beta_g(v_k)^*\|,$$

a contradiction.

Now, we wish to prove that (ii)+(iii)$\Rightarrow$(i), which would conclude the proof. Start by applying (iii) inductively to find a sequence $k_n \in \mathbb{N}$ starting at $k_0 = 1$ such that for every $n \geq 1$, the number k_n satisfies the conclusion of (iii) with k_n in place of k with respect to $\varepsilon = 2^{-n}$, $\mathcal{F} = \mathcal{F}_n$, $K = K_n$, and $m = k_{n-1} + 1$. This implies that there exists a sequence of unitaries $v_n \in \mathcal{U}(\mathbf{1} + B)$ such that

$$\max_{a\in\mathcal{F}_n} \|v_n^*\varphi_{k_n}(a)v_n - \varphi_{k_{n-1}}(a)\| \leq 2^{-n}$$

and that

$$\max_{g\in K_n} \|v_n^*\beta_g(v) - \mathbf{1}\| \leq 2^{-n}.$$

Now, consider the unitary V in $\mathcal{U}(\mathbf{1} + B_\infty)$ induced by the sequence given by $V_n := v_n v_{n-1} \cdot \ldots \cdot v_1 \in \mathcal{U}(\mathbf{1} + B)$ for all $n \geq 1$. It follows that for every $a \in \bigcup_n \mathcal{F}_n$, the sequence $n \mapsto V_n^*\varphi_{k_n}(a)V_n$ is Cauchy, and therefore admits a limit in B. Let $\eta : \mathbb{N} \to \mathbb{N}$ be the sequence defined by $\eta(n) = k_n$, which satisfies $\eta(n) \to \infty$ as $n \to \infty$ because we chose k_n to be strictly increasing. The conclusion from before implies that $\mathrm{Ad}(V^*) \circ \eta^* \circ \varphi$ maps a dense subset of A to B, and hence its whole image is contained in B as well. It follows that the assigment $\psi(a) = \lim_{n\to\infty} V_n^*\varphi_{k_n}(a)V_n$ for all $a \in A$ defines a $*$-homomorphism $\psi : A \to B$. Now, the map $\mathbb{v} : G \to \mathcal{U}(\mathbf{1} + B)$ given by

$$\mathbb{v}_g = \lim_{n\to\infty} V_n^*\beta_g(V_n) \quad \text{for all } g \in G$$

is norm-continuous because β is continuous on each V_n. One therefore obtains

$$\beta_{\infty,g}(V) = V\mathbb{v}_g \quad \text{for all } g \in G.$$

Since $\mathbb{v}$ is continuous in norm, this identity implies that $V \in \mathcal{U}(\mathbf{1} + B_{\infty,\beta})$. To sum up, we have shown that the unitary V witnesses proper unitary equivalence between φ and $\iota_\infty \circ (\psi, \mathbb{v})$, where $(\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ is the proper cocycle morphism constructed above. $\square$

2.3 The ideal structure of C^*-algebras and their dynamics

The goal of this section is to give a brief presentation of the ideal structure associated with a C*-algebra. We will describe two key objects associated with ideals: the lattice of closed two-sided ideals and the topological space of primitive ideals. While the latter is a desirable object because it carries a natural topological structure, and recovers the spectrum of a commutative C*-algebra, the former will play a central role throughout the manuscript thanks to its functorial properties. The topological and order-theoretic notions needed to present certain peculiarities of these objects will be mainly recalled from textbooks such as [79] and [135].

2.3.1 The ideal lattice

Notation 2.3.1. Let A be a C*-algebra. The collection of all ideals[12] of A is denoted by $\mathcal{I}(A)$. Note that the partial order induced by inclusion of ideals turns $\mathcal{I}(A)$ into a partially ordered set. One may moreover define the join (or supremum) of two ideals $I, J \in \mathcal{I}(A)$ to be $I + J \in \mathcal{I}(A)$ and their meet (or infimum) to be $I \cap J = I \cdot J$. Consequently, $\mathcal{I}(A)$ is an order-theoretic lattice, called the *ideal lattice* of A. Furthermore, $\mathcal{I}(A)$ is complete, i.e., every family of ideals has an infimum and a supremum in $\mathcal{I}(A)$.

In the sequel, we will often need to consider ideals of (spatial) tensor products, and ideals of multiplier algebras. We recall here some basic facts about these two cases. First, notice that if A and B are C*-algebras with B simple, then $\mathcal{I}(A \otimes B) \cong \mathcal{I}(A)$ canonically.

Lemma 2.3.2 (see [24, Corollary 9.4.6])**.** *Let A and B be* C*-*algebras, and assume that at least one of them is exact. Then, every ideal $I \subseteq (A \otimes B)$ can be written as*

$$I = \overline{\operatorname{span}}\{a \otimes b \mid a \in A,\ b \in B, \text{ and } a \otimes b \in I\}.$$

Definition 2.3.3 (see [91, Definition 2.9])**.** Let A be a C*-algebra, and $J \in \mathcal{I}(A)$. Define $\mathcal{M}(A, J)$ to be the subalgebra of $\mathcal{M}(A)$ consisting of multipliers $x \in \mathcal{M}(A)$ that satisfy $xA \subseteq J$.

Remark 2.3.4. Let us observe a few basic facts that appeared already in [120, Lemma 5.4] and in the comment after [91, Definition 2.9]. In the setting of

[12] Recall that by *ideal* we mean *closed two-sided ideal*.

Definition 2.3.3, let $\pi : \mathcal{M}(A) \to \mathcal{M}(A/J)$ be the unique unital extension of the canonical quotient map $A \to A/J$ that is strictly continuous on the unit ball, which exists by Proposition 2.1.6. Note that π is surjective when A is σ-unital as a consequence of the non-commutative Tietze extension theorem (see, e.g., [132, Proposition 6.8]), which goes back to [1, Theorem 4.2] (where it appeared in the separable case). One may observe that $\mathcal{M}(A, J) = \ker(\pi)$, and therefore it is a strictly closed, two-sided *ideal* of the multiplier algebra $\mathcal{M}(A)$. This also shows that one can equivalently define $\mathcal{M}(A, J)$ as $\{x \in \mathcal{M}(A) \mid Ax \subseteq J\}$. Furthermore, one has the following straightfoward inclusions

$$J \subseteq \mathcal{M}(A,J) \cap A = \overline{\mathcal{M}(A,J) \cap A} \subseteq (\overline{\mathcal{M}(A,J) \cap A})^2 \subseteq \overline{\mathcal{M}(A,J) \cdot A} \subseteq J$$

and thus

$$J = \mathcal{M}(A,J) \cap A = \overline{\mathcal{M}(A,J) \cdot A}.$$

Finally, we also observe that J is strictly dense in $\mathcal{M}(A, J)$. Indeed, if $x \in \mathcal{M}(A, J)$, we will prove that x is contained in the strict closure of J. Fix a quasicentral approximate unit $(e_i)_i$ of A, and note that $xe_i \in J$ by definition. Now, one has that, for any $a \in A$

$$\lim_i \|(xe_i - x)a\| = \lim_i \|x(e_i a) - xa\| = 0$$

and

$$\lim_i \|a(xe_i - x)\| = \lim_i \|(ax)e_i - ax\| = \lim_i \|(e_i a)x - ax\| = 0,$$

which implies that $xe_i \to x$ in the strict topology.

If $A \subseteq B$ is an inclusion of C*-algebras, and I is an ideal of B, then it is clear that $A \cap I$ is an ideal of A. It is natural, in this setting, to measure how well A is able to discern ideals of B.

Definition 2.3.5. Let B be a C*-algebra, $A \subseteq B$ a C*-subalgebra. One says that A *detects ideals in* B, or that the inclusion $A \subseteq B$ has the *ideal intersection property*,[13] if for every non-zero ideal $I \subseteq B$ one has that $A \cap I \neq \{0\}$.

Assume now that $\mathcal{L} \subseteq \mathcal{I}(B)$ is a non-empty subset of the ideal lattice of B. Then we say that A *detects ideals in* $\mathcal{L}$ if every non-zero ideal in $\mathcal{L}$ has non-zero intersection with A.[14]

[13] In the literature, one may also find that an inclusion $A \subseteq B$ with the ideal intersection property is called essential. We avoid this terminology here because use *essential* for another property of ideals.

[14] Note that if $\mathcal{L}$ only contains the zero ideal, this is a vacuous condition.

Note that the second part of the definition above generalises the first one: A detects ideals in B precisely when it detects ideals in $\mathcal{L} = \mathcal{I}(B)$. We argue that detection of ideals is ultimately a representation theoretic property. Indeed, it is well known that a C*-subalgebra A detects ideals in B if and only if every $*$-representation of B with faithful restriction to A was already faithful on B. In the following lemma, we generalise this fact.

Lemma 2.3.6. *Let B be a* C^**-algebra, $A \subseteq B$ a* C^**-subalgebra, and $\mathcal{L} \subseteq \mathcal{I}(B)$ be a non-empty subset. A detects ideals in $\mathcal{L}$ if and only if every $*$-representation of B which is faithful on A and whose kernel is in $\mathcal{L}$ is in fact faithful on B.*

Proof. When $\mathcal{L}$ only contains the trivial ideal, the equivalence is a tautology. Hence, assume that $\mathcal{L}$ contains at least one non-zero ideal.

Suppose that A detects ideals in $\mathcal{L}$, and pick a $*$-representation ρ of B whose restriction to A is faithful and $\ker(\rho) \in \mathcal{L}$. Since $\ker(\rho) \cap A = \{0\}$, it must be that $\ker(\rho) = \{0\}$.

For the converse, pick a non-zero ideal $I \in \mathcal{L}$, and assume that every $*$-representation of B that is faithful on A and whose kernel is in $\mathcal{L}$ is faithful on B. Choose a faithful representation of B/I, and denote by ρ its composition with the quotient map $B \to B/I$. We have that ρ is not faithful and $\ker(\rho) = I \in \mathcal{L}$. Therefore, ρ cannot be faithful on A, which implies that $I \cap A = \ker(\rho) \cap A \neq \{0\}$. $\square$

Let us now recall some useful properties of the ideal lattice of a C*-algebra from [64, Section 2].

Definition 2.3.7. Let I and J be ideals of a C*-algebra A. One says that I is *compactly contained* in J, and write $I \Subset J$, if for any family of ideals $\{I_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{I}(A)$ such that $J \subseteq \overline{\sum_\lambda I_\lambda}$ there exist finitely many $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $I \subseteq \sum_{i=1}^n I_{\lambda_n}$.

Remark 2.3.8. Observe that two ideals I, J of a C*-algebra A satisfy $I \Subset J$ if and only if for every upward directed family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ with $J \subseteq \overline{\bigcup_{\lambda\in\Lambda} I_\lambda}$, there exists $\lambda \in \Lambda$ such that $I \subseteq I_\lambda$. Let us show the nontrivial implication. Assume that $J \subseteq \overline{\sum_\lambda I_\lambda}$ for some (not necessarily directed) family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$. The upward directed family of ideals given by $\{\sum_{\gamma\in\Gamma} I_\gamma \mid \Gamma \subseteq \Lambda \text{ finite}\}$ has the same supremum as $\{I_\lambda\}_{\lambda\in\Lambda}$. Hence, there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $I \subseteq \sum_{i=1}^n I_{\lambda_i}$, i.e., we have verified that $I \Subset J$.

Remark 2.3.9 (see [64, Lemma 2.2+Proposition 6.5]). Let A be a C*-algebra. In this remark we verify the following properties:

(i) For any pair of ideals $I, J \in \mathcal{I}(A)$, $I \Subset J$ if and only if there exists a positive element $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$. In particular, for any positive element $a \in A$, one has that $\overline{A(a-\varepsilon)_+A} \Subset \overline{AaA}$.

(ii) If A is purely infinite[15] and $I \in \mathcal{I}(A)$, then

$$\overline{A_\infty I A_\infty} = \overline{\bigcup_{J \Subset I} J_\infty}.$$

For the "if" part of (i), suppose there exist $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$, and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of A such that $J \subseteq \overline{\sum_\lambda I_\lambda}$. Then, there exist $\lambda_1, \dots, \lambda_n \in \Lambda$ and a positive element $b \in \sum_{i=1}^n I_{\lambda_i}$ such that $\|a-b\| \leq \frac{\varepsilon}{2}$. One has that $a - \frac{\varepsilon}{2}\mathbf{1}_{A^\dagger} \leq b$, and therefore

$$(a-\varepsilon)_+ \leq f_{\frac{\varepsilon}{2}}(a)(a - \frac{\varepsilon}{2}\mathbf{1}_{A^\dagger})f_{\frac{\varepsilon}{2}}(a) \leq f_{\frac{\varepsilon}{2}}(a) b f_{\frac{\varepsilon}{2}}(a).$$

Hence, $I \subseteq \overline{A(a-\varepsilon)_+A} \subseteq \overline{AbA} \subseteq \sum_{i=1}^n I_{\lambda_i}$. Note that this also implies the "in particular" part of the statement. For the "only if" part of (i), suppose that $I \Subset J$. We show that the family of ideals given by $\{\overline{A(a-\varepsilon)_+A}\}_{a\in J_+,\varepsilon>0}$ is upwards directed. For $a_1, a_2 \in J_+$ and $\varepsilon_1, \varepsilon_2 > 0$, one has that

$$\overline{A(a_i-\varepsilon_i)_+A} \Subset \overline{AaA} = \overline{\bigcup_{\varepsilon>0} \overline{A(a-\varepsilon)_+A}},$$

where $a = a_1 + a_2$ and $i = 1, 2$. Hence, we conclude by picking some $\varepsilon_i' > 0$ for $i = 1, 2$ such that

$$\overline{A(a_i-\varepsilon_i)_+A} \subseteq \overline{A(a-\varepsilon_i')_+A} \subseteq \overline{A(a-\varepsilon)_+A}$$

where $\varepsilon := \min(\varepsilon_1', \varepsilon_2')$. Finally, since $J = \overline{\sum_{a\in J_+,\varepsilon>0} \overline{A(a-\varepsilon)_+A}}$, compact containment provides some $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$. To prove $\overline{A_\infty I A_\infty} \subseteq \overline{\bigcup_{J\Subset I} J_\infty}$ in (ii), it is sufficient to show that, for every positive $a \in I_+$ and $\varepsilon > 0$, one has that $(a-\varepsilon)_+ \in \overline{\bigcup_{J\Subset I} J_\infty}$. By (i), we know that $\overline{A(a-\varepsilon)_+A} \Subset \overline{AaA} \subseteq I$, and thus

$$(a-\varepsilon)_+ \in (\overline{A(a-\varepsilon)_+A})_\infty \subseteq \overline{\bigcup_{J\Subset I} J_\infty}.$$

For the opposite inclusion, it is sufficient to show that $J_\infty \subseteq \overline{A_\infty I A_\infty}$ for all $J \Subset I$. Pick such a J, and note that by (i) there exists $a \in I_+$ and $\varepsilon > 0$ such

[15] In [64], Gabe proves this fact more generally for *weakly purely infinite* C*-algebras (see [17, Definition 1.2]). However, we will not need this level of generality in this thesis.

that $J \subseteq \overline{A(a-\varepsilon_+)A}$. Fix now a positive contraction $x \in J_\infty$, and lift it to a sequence of positive contractions $(x_n)_n \subseteq J$. Since A is assumed to be purely infinite and $x_n \in \overline{A(a-\varepsilon)_+A}$, then $x_n \precsim (a-\varepsilon)_+$ for all $n \in \mathbb{N}$. It follows that we may find a sequence $d_n \in A$ such that $x_n =_{2^{-n}} d_n^*(a-\varepsilon)_+ d_n$. Define a new sequence $y_n := (a-\varepsilon)_+^{1/2} d_n$ for $n \in \mathbb{N}$, and note that these elements are bounded. Therefore, the image of $(y_n)_n \in \ell^\infty(\mathbb{N}, A)$ under the canonical quotient yields an element $y \in A_\infty$. Observe that, by definition of $(a-\varepsilon)_+$, if one considers a continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ that is constantly 1 on $[\varepsilon, \infty)$, then $g(a)(a-\varepsilon)_+ = (a-\varepsilon)_+$. Hence,

$$x = y^* y = y^* g(a) y \in \overline{A_\infty a A_\infty} \subseteq \overline{A_\infty I A_\infty}.$$

Since x was arbitrarily chosen, one has that $J_\infty \subseteq \overline{A_\infty I A_\infty}$.

Corollary 2.3.10 (see [64, Corollary 2.3])**.** *Let A be a* C*-*algebra, and $I \in \mathcal{I}(A)$. Then I has a full element if and only if there exists an $\Subset$-increasing sequence of ideals $(I_n)_{n\in\mathbb{N}} \subseteq \mathcal{I}(A)$ such that*

$$I = \overline{\bigcup_{n\in\mathbb{N}} I_n}.$$

Proof. Assume $a \in I$ is a full element. Then the sequence of ideals given by $I_n := \overline{A(a^*a - 2^{-n})_+ A}$ is the desired sequence because by Remark 2.3.9(i) $I_n \Subset I_{n+1}$ for all $n \in \mathbb{N}$, and clearly $I = \overline{\bigcup_n I_n}$. For the converse, assume that there exists a $\Subset$-increasing sequence of ideals such that $I = \overline{\bigcup_n I_n}$. Using Remark 2.3.9(i) once more, we may find positive contractions $a_n \in A$ such that $I_n \subseteq \overline{A a_n A} \subseteq I_{n+1}$. Then, the element given by $\sum_{n\in\mathbb{N}} 2^{-n} a_n$ is full in I. □

We define here the category of abstract Cuntz semigroups because it will be used as ambient category for the ideal lattice of separable C*-algebras.

Definition 2.3.11. Let S be a positively ordered abelian monoid, i.e., an abelian monoid $(S, +, 0)$ with partial order $\leq$ such that $0 \leq x$ for all $x \in S$, and $x + y \leq x' + y'$ whenever $x, x', y, y' \in S$ and $x \leq x'$ and $y \leq y'$. Let $\ll$ denote the way-below relation on S, which is defined as follows: For any $x, y \in S$, one writes $x \ll y$ if for every increasing sequence $(z_n)_{n\in\mathbb{N}}$ in S with supremum z such that $y \leq z$, there exists $n \in \mathbb{N}$ such that $x \leq z_n$. One says that S is an *abstract Cuntz semigroup* if the following conditions are satisfied:

(i) Every increasing sequence in S admits a supremum.

(ii) For every $x \in S$, there exists an increasing sequence $(x_n)_n$ in S with $x_n \ll x_{n+1}$ for all $n \in \mathbb{N}$ such that $x = \sup_n x_n$.

(iii) If $x \ll x'$ and $y \ll y'$ for x, x', y and y' in S, then $x + y \ll x' + y'$.

(iv) For every pair of increasing sequences $(x_n)_n$ and $(y_n)_n$ in S, $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$.

We denote by **Cu** the category of abstract Cuntz semigroups with morphisms given by ordered monoid homomorphisms that preserve countable increasing suprema and way-below relation.

The Cuntz semigroup of a C*-algebra is the motivating example for the category of abstract Cuntz semigroups defined above.

The object we name below will allow us to view (a subset of) the ideal lattice as an object in **Cu**.

Notation 2.3.12. Let A be a C*-algebra. We denote by $\mathcal{I}_\sigma(A) \subseteq \mathcal{I}(A)$ the subset of ideals I of A that contain a full element in I.

Remark 2.3.13. Let A be a C*-algebra, $I \in \mathcal{I}(A)$, and $J \in \mathcal{I}_\sigma(A)$. We observe that $I \Subset J$ if and only if for every increasing sequence of ideals $(I_n)_{n\in\mathbb{N}}$ of A such that $J \subseteq \overline{\bigcup_{n\in\mathbb{N}} I_n}$, there exists $n \in \mathbb{N}$ such that $I \subseteq I_n$. Let us show the nontrivial implication. Assume that $J \subseteq \overline{\bigcup_\lambda I_\lambda}$ for an upward directed family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of A, and write $J = \overline{AaA}$ for some $a \in A_+$. By Remark 2.3.9(i) we have that $\overline{A(a-2^{-n})_+A} \Subset J$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \Lambda$ such that $\overline{A(a-2^{-n})_+A} \subseteq I_{\lambda_n}$. This implies that

$$J = \overline{\bigcup_{n\in\mathbb{N}} \overline{A(a-2^{-n})_+A}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} I_{\lambda_n}},$$

and by assumption there exists $n \in \mathbb{N}$ such that $I \subseteq I_{\lambda_n}$. In light of Remark 2.3.8, we have shown the equivalence.

Proposition 2.3.14 (cf. [64, Proposition 2.5]). *Let A be a* C*-*algebra. Then $\mathcal{I}_\sigma(A)$ is an abstract Cuntz semigroup. In particular, $\mathcal{I}(A)$ is an abstract Cuntz semigroup if and only if $\mathcal{I}(A) = \mathcal{I}_\sigma(A)$.*

Proof. We have that $(\mathcal{I}_\sigma(A), \subseteq, +, 0)$ is a positively ordered submonoid of $(\mathcal{I}(A), \subseteq, +, 0)$, and the way-below relation is equivalent to compact containment by Remark 2.3.13. In the following paragraph, we show that $\mathcal{I}_\sigma(A)$ is an abstract Cuntz semigroup by showing that all the attributes from Definition 2.3.11 hold true. Let us first show that it is closed under passing to suprema of increasing sequences. Fix an increasing sequence of ideals $(I_n)_{n\in\mathbb{N}}$, where $I_n = \overline{Aa_nA}$ and $a_n \in A$ are positive contractions of A for all $n \in \mathbb{N}$. Then, it follows that $\sum_{n\in\mathbb{N}} 2^{-n} a_n$ is a full element of $\overline{\bigcup_{n\in\mathbb{N}} I_n}$, which is therefore contained in

$\mathcal{I}_\sigma(A)$. Now, let $I = \overline{AaA} \in \mathcal{I}_\sigma(A)$ for some positive element $a \in A$. Then, the sequence in $\mathcal{I}_\sigma(A)$ given by $I_n = \overline{A(a-2^{-n})_+A}$ for $n \in \mathbb{N}$ is such that $\overline{A(a-2^{-n})_+A} \Subset \overline{A(a-2^{-(n+1)})_+A}$ for all $n \in \mathbb{N}$. Since $\overline{\bigcup_{n\in\mathbb{N}} I_n} = I$, we have shown the second point in the definition of abstract Cuntz semigroups. Suppose now that I, I' are compactly contained in J, J', respectively, where $I, I', J, J' \in \mathcal{I}_\sigma(A)$. For any increasing sequence $(K_n)_{n\in\mathbb{N}}$ in $\mathcal{I}_\sigma(A)$ such that $(I'+J') \subseteq \overline{\bigcup_{n\in\mathbb{N}} K_n}$, there exist $n_I, n_J \in \mathbb{N}$ for which $I \subseteq K_{n_I}$, and $J \subseteq K_{n_J}$. This implies that $(I+J) \subseteq K_{\max(n_I,n_J)}$, and thus $(I+J) \Subset (I'+J')$. Assume now that $(I_n)_n$ and $(J_n)_n$ are increasing sequences in $\mathcal{I}_\sigma(A)$. One has that $\overline{\bigcup_n I_n}, \overline{\bigcup_n J_n} \subseteq \overline{\bigcup_n (I_n+J_n)}$. Moreover, $\overline{\bigcup_n (I_n+J_n)} \subseteq (\overline{\bigcup_n I_n} + \overline{\bigcup_n J_n})$ because the former is the smallest ideal containing $I_n + J_n$ for all $n \in \mathbb{N}$. Since from Corollary 2.3.10 an ideal $I \in \mathcal{I}(A)$ has a full element if and only if it is the supremum of a $\Subset$-increasing sequence of ideals, it follows that $\mathcal{I}(A)$ is an abstract Cuntz semigroup if and only if $\mathcal{I}(A) = \mathcal{I}_\sigma(A)$. $\square$

Definition 2.3.15. Let S be an abstract Cuntz semigroup. An *ideal* I of S is a submonoid $I \subseteq S$ closed under passing to suprema of increasing sequences, and such that $a \leq b$ and $b \in I$ imply that $a \in I$. The set of ideals of S is denoted by $\mathcal{I}(S)$, and it is a complete lattice.

Remark 2.3.16. Contrarily to $\mathcal{I}(A)$, the Cuntz semigroup of A is always an object in the category **Cu** by [40, Theorem 1]. Therefore, they can be substantially different in general. However, the ideal lattice of $\mathrm{Cu}(A)$ is always order-isomorphic to $\mathcal{I}(A)$ via the map that associates $\mathrm{Cu}(I) \in \mathcal{I}(\mathrm{Cu}(A))$ to the ideal $I \in \mathcal{I}(A)$.

We note here that, as observed by Gabe in [64], if B is another C*-algebra, any order-theoretic isomorphism $\mathcal{I}_\sigma(A) \to \mathcal{I}_\sigma(B)$ is automatically an isomorphism of abstract Cuntz semigroups.

As a direct consequence of Definition 2.1.58, if A is a purely infinite C*-algebra, two elements of A_+ are Cuntz equivalent precisely when they generate the same ideal in A. Therefore, the map $\mathrm{Cu}(A) \to \mathcal{I}(A)$, realised by associating to the class of an element in $(A \otimes \mathcal{K})_+$ the ideal it generates as an element of $\mathcal{I}(A \otimes \mathcal{K}) \cong \mathcal{I}(A)$, is an order-theoretic isomorphism onto its image. In particular, we claim that $\mathrm{Cu}(A) \cong \mathcal{I}_\sigma(A)$ under this map. In fact, an equivalence class $[a]_{\mathrm{Cu}} \in \mathrm{Cu}(A)$, where $a \in (A \otimes \mathcal{K})_+$, is mapped to the ideal in $\mathcal{I}_\sigma(A \otimes \mathcal{K}) \cong \mathcal{I}_\sigma(A)$ that contains a as a full element. Furthermore, for any $I \in \mathcal{I}_\sigma(A \otimes \mathcal{K}) \cong \mathcal{I}_\sigma(A)$ with full element a, we have by pure infiniteness that the only class in $\mathrm{Cu}(A)$ mapping to I is $[a]_{\mathrm{Cu}}$.

On the other hand, under the assumption that $\mathcal{I}_\sigma(A) = \mathcal{I}(A)$, such as when A is separable, $\mathcal{I}(A)$ is an abstract Cuntz semigroup. If A is additionally purely infinite, one has that $\mathcal{I}(A)$ is order isomorphic to $\mathrm{Cu}(A)$, which implies that they are isomorphic as abstract Cuntz semigroups.

Definition 2.3.17 (see [64, Definition 2.6]). Let A, B be C*-algebras. A monoid order homomorphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ is said to be a **Cu**-*morphism* if it preserves increasing suprema and compact containment.

Notation 2.3.18. Let A, B be C*-algebras, and $\varphi : A \to B$ a c.p. map (usually a $*$-homomorphism). We denote by $\mathcal{I}(\varphi) : \mathcal{I}(A) \to \mathcal{I}(B)$ the order preserving map given by

$$\mathcal{I}(\varphi)(I) = \overline{B\varphi(I)B} \in \mathcal{I}(B)$$

for all $I \in \mathcal{I}(A)$. Moreover, the notation $\mathcal{I}_\sigma(\varphi)$ refers to the restriction of $\mathcal{I}(\varphi)$ to $\mathcal{I}_\sigma(A)$.

Consider now a $*$-homomorphism $\psi : A \to \mathcal{M}(B)$. We denote by $\mathcal{I}_B(\psi)$ the map $\mathcal{I}(A) \to \mathcal{I}(B)$ induced by ψ by taking

$$\mathcal{I}_B(\psi)(I) = \overline{B\psi(I)B}$$

for all $I \in \mathcal{I}(A)$.

We point out the elementary fact that for any $*$-homomorphism $\varphi : A \to B$, one has that $\mathcal{I}(\varphi)(I) = \overline{B\varphi(I_+)B}$.

Lemma 2.3.19 (see [64, Proposition 2.15]). *Let A, B and C be* C*-*algebras, and $\varphi : A \to B$ and $\psi : B \to C$ $*$-homomorphisms. Then*

$$\mathcal{I}(\psi \circ \varphi) = \mathcal{I}(\psi) \circ \mathcal{I}(\varphi).$$

Proof. Pick $I \in \mathcal{I}(A)$. The inclusion

$$\mathcal{I}(\psi) \circ \mathcal{I}(\varphi)(I) \subseteq \mathcal{I}(\psi \circ \varphi)(I)$$

follows from the fact that $\psi(B\varphi(I)B) \subseteq \overline{C\psi(\varphi(I))C}$ and continuity of ψ. For the opposite inclusion, note that $\varphi(I) \subseteq \overline{B\varphi(I)B}$. In fact, for any element $x \in I$, one can use an approximate unit $(e_i)_i$ of B to see that $e_i\varphi(x)e_i \to \varphi(x)$ and $e_i\varphi(x)e_i \in B\varphi(I)B$. Using this fact, we have that $\overline{C\psi(\varphi(I))C} \subseteq \overline{C\psi(\overline{B\varphi(I)B})C}$, which concludes the proof. ☐

For the following proposition, recall that a non-degenerate $*$-homomorphism $\psi : B \to \mathcal{M}(C)$ extends uniquely to a unital $*$-homomorphism $\mathcal{M}(B) \to \mathcal{M}(C)$ that is strictly continuous on the unit ball by Proposition 2.1.6. Slightly abusing notation, we denote this extension by ψ as well.

Lemma 2.3.20. *Let A, B and C be* C*-*algebras, and* $\varphi : A \to \mathcal{M}(B)$, $\psi : B \to \mathcal{M}(C)$ *two* $*$*-homomorphisms. Assume that* ψ *is non-degenerate. Then*

$$\mathcal{I}_C(\psi \circ \varphi) = \mathcal{I}_C(\psi) \circ \mathcal{I}_B(\varphi).$$

Proof. Let $I \in \mathcal{I}(A)$. For the inclusion

$$\overline{C\psi(\overline{B\varphi(I)B})C} \subseteq \overline{C\psi(\varphi(I))C}$$

note that, for every $b_1, b_2 \in B$, $x \in I$ and $c_1, c_2 \in C$, one has that

$$c_1\psi(b_1\varphi(x)b_2)c_2 = c_1\psi(b_1)\psi(\varphi(x))\psi(b_2)c_2 \in \overline{C\psi(\varphi(I))C}.$$

Hence, $C\psi(\overline{B\varphi(I)B})C \subseteq \overline{C\psi(\varphi(I))C}$ and the aforementioned inclusion follows. For the opposite inclusion, fix an approximate unit $(e_i)_i$ of B, and note that by Proposition 2.1.6, $\psi(e_i) \to \mathbf{1}_{\mathcal{M}(C)}$ in the strict topology. Then, for any $c_1, c_2 \in C$, and $a \in I$, we have that

$$c_1\psi(\varphi(a))c_2 = \lim_i c_1\psi(e_i)\psi(\varphi(a))\psi(e_i)c_2 = \lim_i c_1\psi(e_i\varphi(a)e_i)c_2.$$

Note that, in particular, every element $c_1\psi(e_i\varphi(a)e_i)c_2$ belongs to $C\psi(B\varphi(I)B)C$. Therefore, we have showed that $c_1\psi(\varphi(a))c_2$ is in $\overline{C\psi(\overline{B\varphi(I)B})C}$, which concludes the proof. □

Lemma 2.3.21 (cf. [64, Lemma 2.12(iii)])**.** *Let A and B be* C*-*algebras. For any* $*$*-homomorphism* $\varphi : A \to B$, *the map* $\mathcal{I}(\varphi)$ *is a* **Cu**-*morphism. In particular,* $\mathcal{I}_\sigma(\varphi)$ *is a* **Cu**-*morphism with range in* $\mathcal{I}_\sigma(B)$.

Proof. It is clear that $\mathcal{I}(\varphi)$ preserves the order, and sends the zero element to zero. It follows that $\mathcal{I}(\varphi)(I + J) = \overline{B\varphi(I \cup J)B} \subseteq \mathcal{I}(\varphi)(I) + \mathcal{I}(\varphi)(J)$, Moreover, since $\mathcal{I}(\varphi)$ is order-preserving, one has that $\mathcal{I}(\varphi)(I), \mathcal{I}(\varphi)(J) \subseteq \mathcal{I}(\varphi)(I + J)$. Hence, $\mathcal{I}(\varphi)$ is an ordered monoid morphism. Consider now an increasing net of ideals $(I_\lambda)_\lambda \subseteq \mathcal{I}(A)$, and let $I = \overline{\bigcup_\lambda I_\lambda}$. It is easily seen that $\overline{\bigcup_\lambda \mathcal{I}(\varphi)(I_\lambda)} \subseteq \mathcal{I}(\varphi)(I)$. For the other inclusion, let $x \in I$ and pick a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \bigcup_\lambda I_\lambda$ satisfying $\|x_n - x\| \to 0$ as $n \to \infty$. Then, we have a sequence $\varphi(x_n) \in \bigcup_\lambda \mathcal{I}(\varphi)(I_\lambda)$ such that $\|\varphi(x_n) - \varphi(x)\| \to 0$ as $n \to \infty$, and hence $\varphi(x) \in \overline{\bigcup_\lambda \mathcal{I}(\varphi)(I_\lambda)}$. In other words, $\mathcal{I}(\varphi)$ preserves increasing suprema. Finally, we need to show that it preserves compact containment. Pick two ideals $I, J \in \mathcal{I}(A)$ such that $I \Subset J$. By Remark 2.3.9(i), there exist $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a - \varepsilon)_+A}$. Then,

$$\begin{aligned}\mathcal{I}(\varphi)(I) &\subseteq \mathcal{I}(\varphi)(\overline{A(a-\varepsilon)_+A}) \\ &= \overline{B(\varphi(a)-\varepsilon)_+B}\end{aligned}$$

$$\Subset \overline{B\varphi(a)B}$$
$$= \mathcal{I}(\varphi)(\overline{AaA})$$
$$\subseteq \mathcal{I}(\varphi)(J),$$

and thus $\mathcal{I}(\varphi)$ is a **Cu**-morphism. It follows from Corollary 2.3.10, that the image of an ideal with a full element under a **Cu**-morphism contains a full element, and thus $\mathcal{I}_\sigma(\varphi)$ is a **Cu**-morphism with range in $\mathcal{I}_\sigma(B)$. □

Remark 2.3.22 (see [64, Remark 2.8]). Note that, for any pair of C*-algebras A, B, a map $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ is a **Cu**-morphism if and only if it preserves suprema of arbitrary families of ideals, and compact containment. In fact, if Φ preserves suprema, then it must be an ordered monoid homomorphism as a consequence of the following basic observations. First, we have that Φ preserves zero element and sums because $\sup\big\{\{0\}\big\} = \{0\}$, and $\sup\{I, J\} = I + J$ for all $I, J \in \mathcal{I}(A)$. Moreover, the order is preserved because, when $I \subseteq J$, one has that $\Phi(I) \subseteq \sup\{\Phi(I), \Phi(J)\} = \Phi(\sup\{I, J\}) = \Phi(J)$.

Notation 2.3.23. Let $\Phi, \Psi : \mathcal{I}(A) \to \mathcal{I}(B)$ be **Cu**-morphisms. One writes $\Phi \leq \Psi$ whenever $\Phi(I) \subseteq \Psi(I)$ for all $I \in \mathcal{I}(A)$.

Observe that by Lemma 2.3.19 and Lemma 2.3.21 $\mathcal{I}_\sigma(-)$ is functorial on the category of C*-algebras with $*$-homomorphisms as arrows to the category **Cu**. If one restricts to the category of separable C*-algebras, then the $\mathcal{I}(-)$ is functorial.

2.3.2 The primitive ideal space

Definition 2.3.24 (see [157, Definition 3.13.7])**.** Let A be a C*-algebra and $I \subseteq A$ a closed two-sided ideal. I is said to be *prime* if whenever $J, K \subseteq A$ are ideals such that $J \cap K \subseteq I$, then $J \subseteq I$ or $K \subseteq I$. I is said to be *primitive* if there exists a non-zero irreducible representation π of A such that $\ker(\pi) = I$.

Definition 2.3.25 (see [157, Definition 4.1.2])**.** We will denote by $\mathrm{Prim}(A)$ the *primitive ideal space* (or *primitive spectrum*) of A, i.e., the set containing all primitive ideals of A. Given a subset $U \subseteq \mathrm{Prim}(A)$, the closure of U is given by

$$\overline{U} = \left\{\mathfrak{p} \in \mathrm{Prim}(A) \mid \mathfrak{p} \supseteq \bigcap_{\mathfrak{q}\in U} \mathfrak{q}\right\}.$$

The closure operation defined above satisfies Kuratowski's closure axioms, and thus defines a topology on $\mathrm{Prim}(A)$, which is usually called Jacobson (or hull-kernel) topology. Throughout the present article, we endow $\mathrm{Prim}(A)$ with the Jacobson topology.

Note that one can define in an analogous fashion the *prime ideal space*, which consists of all prime ideals of A. This ideal-related invariant will seldom be mentioned here, because when A is a separable C*-algebra, it coincides with $\mathrm{Prim}(A)$; see [157, Proposition 4.3.6] for a proof.

Example 2.3.26. Let X be a locally compact space. It is a classical result that every irreducible representation of a commutative C*-algebra is one-dimensional, and more precisely given by point-evaluation maps. Therefore, one can identify each $\mathfrak{p} \in \mathrm{Prim}(\mathcal{C}_0(X))$ with $\ker(\mathrm{ev}_x) = \{f \in \mathcal{C}_0(X) \mid f(x) = 0\}$ for some point $x \in X$. One may then check that when $\mathrm{Prim}(\mathcal{C}_0(X))$ is endowed with the Jacobson topology, the map

$$X \to \mathrm{Prim}(\mathcal{C}_0(X)), \quad x \mapsto \ker(\mathrm{ev}_x),$$

is a homeomorphism.

Definition 2.3.27. Let X be a topological space. An open subset $P \subseteq X$ is said to be *prime* if for every pair of open subsets $V, W \subseteq X$ such that $V \cap W \subseteq P$, it holds that $V \subseteq P$ or $W \subseteq P$. A prime set P is *proper* when $P \neq X$.

Definition 2.3.28 (see [135, Definition 2, p. 477])**.** A topological space X is said to be *sober*[16] if for each proper prime open subset P of X there exists a unique point $x \in X$ such that $P = X \setminus \overline{\{x\}}$.

Recall that a topological space X is said to be

- T_0 (or *Kolmogorov*) if for any pair of points $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$ implies that $x = y$;
- T_1 (or *Fréchet*) if for any $x \in X$, one has $\overline{\{x\}} = \{x\}$.

Note that any Hausdorff space is sober, and any sober space is T_0 (see [135, Theorem 3, p. 477]). Hence, we can draw the following diagram of implication:

$$\begin{tikzcd}
& \text{Hausdorff } (T_2) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
T_1 \arrow[dr, Rightarrow] & & \text{sober} \arrow[dl, Rightarrow] \\
& T_0 &
\end{tikzcd}$$

[16] The names *point-complete space* and *spectral space* are also used to refer to sober spaces in the literature.

However, a sober space need not be T_1, and vice versa. An example of a sober space that is not T_1 is the two-point space $\{a,b\}$ with open sets given by

$$\{\emptyset, \{a\}, \{a,b\}\},$$

which is sometimes called *Sierpiński space*. It is not T_1 because $\overline{\{a\}} = \{a,b\}$. Any infinite set X endowed with the cofinite topology is an example of a T_1 space that is not sober because the empty set is a proper open prime subset of X that cannot be written as $X \setminus \overline{\{x\}} = X \setminus \{x\}$ for any point $x \in X$.

Note that, for a topological space X, the collection $\mathcal{O}(X)$ of all open sets in X carries a natural lattice structure induced by inclusion, with infimum and supremum given by intersection and union, respectively. The supremum of arbitrary families of open sets is always well-defined, while the infimum of an infinite family of open sets is defined to be the interior of the intersection. One can check that, in general, $\mathcal{O}(X)$ is closed under arbitrary unions and finite intersections, and they enjoy the obvious distributivity law. Hence, the lattice $\mathcal{O}(X)$ is not automatically a completely distributive lattice, but rather can be viewed as an object in a number of different categories, such as that of *frames* (or dually, *locales*), or *complete Heyting algebras* (all of which are different categories). We will not discuss these special classes of lattices, as we will focus on spaces X such that $\mathcal{O}(X)$ is a complete lattice. However, we do want to stress that when X is a sober space, X is determined up to homeomorphism by the lattice $\mathcal{O}(X)$; see [135, Proposition 2, p. 479].

We now turn to the space of primitive ideals of a C*-algebra, and recall its topological properties.

Remark 2.3.29. Let A be a C*-algebra. Recall from [157, Theorem 4.1.3] that the lattice of open subsets of $\mathrm{Prim}(A)$ is order-isomorphic to $\mathcal{I}(A)$ via the following map,

$$\mathcal{O}(\mathrm{Prim}(A)) \to \mathcal{I}(A), \quad U \mapsto \bigcap_{\mathfrak{p} \in \mathrm{Prim}(A) \setminus U} \mathfrak{p},$$

whose inverse is

$$\mathcal{I}(A) \to \mathcal{O}(\mathrm{Prim}(A)), \quad I \mapsto \{\mathfrak{p} \mid \mathfrak{p} \not\supseteq I\}.$$

We also remark that, if B is another C*-algebra, and $f : \mathrm{Prim}(A) \to \mathrm{Prim}(B)$ a homeomorphism, then there exists a unique order-isomorphism between $\mathcal{I}(A)$ and $\mathcal{I}(B)$ that sends $\mathfrak{p} \in \mathrm{Prim}(A)$ to $f(\mathfrak{p}) \in \mathrm{Prim}(B)$.

The prime ideal space of *any* C*-algebra is sober and locally quasi-compact (see [157, Theorem 4.4.13]). On the other hand, $\mathrm{Prim}(A)$ is only a *locally quasi-compact* space in general[17]; see [157, Proposition 4.4.4] for a proof. However,

[17]We follow the convention that a locally compact space is Hausdorff, while a locally quasi-compact space need not be Hausdorff.

if A is *separable*, $\operatorname{Prim}(A)$ coincides with the set of prime ideals of A,[18] and it is therefore also sober. There is also a converse to this fact: $\operatorname{Prim}(A)$ is sober precisely when it coincides with the prime ideal space; see [157, Proposition 4.4.14]. Under the separability assumption, $\operatorname{Prim}(A)$ is also second-countable by [157, Theorem 4.4.13].

To sum up, for a separable C*-algebra A, the primitive ideal space $\operatorname{Prim}(A)$ is

- locally quasi-compact;
- sober;
- second-countable.

It is not known whether all locally quasi-compact, second-countable, sober spaces can be realised as primitive ideal spaces of separable C*-algebras. However, in [91] Harnisch and Kirchberg obtain a characterisation of primitive spectra of separable *nuclear* C*-algebras.

Theorem 2.3.30 (cf. [91, Corollary 1.5]). *Let X be a topological space satisfying the T_0 separation axiom. Then, X arises as the primitive ideal space of a separable, nuclear* C**-algebra precisely when*

(i) *X is locally quasi-compact, second-countable, sober, and*

(ii) *there exists a locally compact Polish space*[19] *P, and an order-embedding*

$$\Phi : \mathcal{O}(X) \hookrightarrow \mathcal{O}(P) \cong \mathcal{I}(\mathcal{C}_0(P))$$

that preserves arbitrary suprema and infima, and such that $\Phi(\emptyset) = \emptyset$, and $\Phi^{-1}(P) = \{X\}$.

If X additionally satisfies the T_1 separation axiom, then (ii) *is equivalent to the existence of a continuous open surjection $P \to X$.*

The following example is based on [118] (see also [120]).

Example 2.3.31. Let X be a second-countable, locally compact space, and recall that we have adopted the convention that locally compact spaces are Hausdorff. Let

$$\mathcal{N}(X) = \{(X \setminus K) \cup \{\infty\} \mid K \subseteq X \text{ compact subset}\}$$

[18] Every primitive ideal is prime, but the converse is not true in general.
[19] Recall that a *Polish space* is a separable, completely metrisable topological space.

and consider the topological space $\tilde{X}$ given as follows (see [120, Definition 1.5]). Let $\tilde{X} = X \cup \{\infty\}$ as a set, and endow $\tilde{X}$ with the topology determined by the open sets

$$\mathcal{O}(\tilde{X}) = \{\emptyset, \tilde{X}\} \cup \mathcal{N}(X).$$

The fact that these open sets form a *quasi-compact* and *sober* topology is proved in [120, Lemmas 1.6+1.8]. Hence, one may view $\tilde{X}$ as a *non-Hausdorff* (yet sober) *compactification* of X. In [120], Kirchberg and Phillips prove that there exists a separable, nuclear C*-algebra A with $\mathrm{Prim}(A) \cong \tilde{X}$. Moreover, as observed in [120, Corollary 1.7], such a C*-algebra A must be *prime*, i.e., the product of any two non-zero ideals is non-zero, and hence has trivial center.

We proceed with a short list of examples of C*-algebras with a computable primitive ideal space.

Example 2.3.32. Consider $\mathcal{B}(\mathcal{H})$ for an infinite-dimensional, separable Hilbert space $\mathcal{H}$. Then $\mathrm{Prim}(\mathcal{B}(\mathcal{H}))$ contains only two elements: $\{0\}$ and $\mathcal{K}$. When endowed with the Jacobson topology, $\mathrm{Prim}(\mathcal{B}(\mathcal{H}))$ is homeomorphic to the Sierpiński space, and therefore it fails to be Hausdorff (or even T_1).

Example 2.3.33. Consider the C*-subalgebra of $\mathcal{C}([0,1], M_2)$ defined as

$$D = \left\{ f \in \mathcal{C}([0,1], M_2) \mid f(0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ a, b \in \mathbb{C} \right\}$$

Since every irreducible representation of D necessarily factors through a point-evaluation map on $\mathcal{C}([0,1], M_2)$, one would be tempted to say that D has primitive spectrum heomeomorphic to $[0,1]$, just like $\mathcal{C}[0,1]$. However, there are two irreducible representations factoring through ev_0 that have different kernels. In fact, if $\pi_1, \pi_2 : M_2 \to \mathbb{C}$ denote the projections onto the upper-left and lower-right corners, respectively, the representations $\rho_1 = \pi_1 \circ \mathrm{ev}_0$ and $\rho_2 = \pi_2 \circ \mathrm{ev}_0$ give rise to different primitive ideals, denoted by 0_1 and 0_2, respectively. In particular, one can show that $\mathrm{Prim}(D) = \{0_1, 0_2\} \cup (0,1]$, which can be thought of as the *unit interval with two origins*. On the part of the primitive ideal space given by $(0,1]$, the Jacobson topology agrees with the usual topology. However, sets of the form $\{0_1\} \cup (0,\varepsilon)$ or $\{0_2\} \cup (0,\varepsilon)$ for $\varepsilon > 0$ are Jacobson-open "around zero" in $\mathrm{Prim}(D)$. Hence, any sequence converging to zero in $(0,1]$ converges to both 0_1 and 0_2 in $\mathrm{Prim}(D)$. In particular, $\mathrm{Prim}(D)$ is not Hausdorff, but it is T_1 because the singletons corresponding to the "two origins" are both closed. Observe that this also suggests that even spaces that are both sober and T_1 need not be Hausdorff.

Example 2.3.34. Recall that the AH-algebra $\mathcal{A}_{[0,1]}$ from Example 2.1.11 is defined as the inductive limit C*-algebra of $A_n = \mathcal{C}_0([0,1), M_{2^n})$ with connecting maps $\varphi_n(f)(t) = f(t) \oplus f(\max\{t, t_n\})$ for $f \in A_n$, where $(t_n)_n \subseteq [0,1)$ is a dense sequence. It was shown it [171, Proposition 2.1] (cf. [144, Theorem 1.2.1]) that $\mathcal{I}(\mathcal{A}_{[0,1]})$ is order-isomorphic to $[0,1]$ via the map

$$[0,1] \ni t \mapsto I_t = \overline{\bigcup \varphi_{n,\infty}(I_t^{(n)})} \in \mathcal{I}(\mathcal{A}_{[0,1]}),$$

where

$$I_t^{(n)} = \{f \in A_n \mid f(s) = 0 \text{ for all } s \geq t\} \cong \mathcal{C}_0([0,t), M_{2^n}).$$

In particular, the primitive spectrum of $\mathcal{A}_{[0,1]}$ is $[0,1)$ with Jacobson-open subsets given by[20]

$$\{[0,t) \mid t \in (0,1]\} \cup \emptyset$$

and hence $\mathrm{Prim}(\mathcal{A}_{[0,1]})$ is not Hausdorff.

Remark 2.3.35. In [17, Propositions 2.16(iii)+2.17(2)], Blanchard and Kirchberg prove that if A and B are separable C*-algebras, and at least one of them is exact, then the map given by

$$\mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \mathrm{Prim}(A \otimes B), \quad (\mathfrak{p}, \mathfrak{q}) \mapsto (\mathfrak{p} \otimes B) + (A \otimes \mathfrak{q})$$

is a homeomorphism. Without the separability assumption, one still has the same conclusion for the prime ideal spaces.

A particularly relevant example where the previous remark applies is the tensor product $\mathcal{C}_0(X) \otimes B \cong \mathcal{C}_0(X, B)$ for a second-countable, locally compact space X and a separable C*-algebra B. In this case, one may conclude that $\mathrm{Prim}(\mathcal{C}_0(X, B)) \cong X \times \mathrm{Prim}(B)$.

2.3.3 Dynamics on ideal structures

Definition 2.3.36. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. Denote by $\alpha^\sharp$ the set-theoretic action induced by α on the ideal lattice of A, i.e.,

$$\alpha^\sharp : G \curvearrowright \mathcal{I}(A), \quad \alpha^\sharp_g(I) = \{\alpha_g(a) \mid a \in I\}$$

for every $g \in G$ and $I \in \mathcal{I}(A)$.

With slight abuse of notation, we also denote by $\alpha^\sharp$ the restriction of $\alpha^\sharp$ to the primitive ideal space of A, which may equivalently be described by

$$\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A), \quad \alpha^\sharp_g(\ker(\pi)) = \ker(\pi \circ \alpha_{g^{-1}})$$

for all $g \in G$ and non-zero irreducible representations π of A.

[20] The topology on $[0,1)$ described here is sometimes referred to as the *order topology*.

Remark 2.3.37. Let X be a locally compact space equipped with a continuous action $\alpha : G \curvearrowright X$. Then one may induce an action on $\mathcal{C}_0(X)$ as follows,

$$\alpha^\flat : G \curvearrowright \mathcal{C}_0(X), \quad \alpha^\flat_g(f)(x) = f(\alpha_{g^{-1}}(x)).$$

Viceversa, if $\beta : G \curvearrowright \mathcal{C}_0(X)$ is an action, then we have defined an induced action

$$\beta^\sharp : G \curvearrowright X \cong \mathrm{Prim}(\mathcal{C}_0(X)), \quad \beta^\sharp_g(\ker(\mathrm{ev}_x)) = \mathrm{ev}_x \circ \beta_g^{-1},$$

where X is identified with $\mathrm{Prim}(\mathcal{C}_0(X))$ via the homeomorphism that sends a point $x \in X$ to $\ker(\mathrm{ev}_x)$. One may then show that there are natural identifications

$$(\alpha^\flat)^\sharp = \alpha, \quad (\beta^\sharp)^\flat = \beta.$$

Moreover, note that $\beta^\sharp : G \curvearrowright X$ is continuous whenever β is a continuous action.

This example naturally leads to the following question related to a C*-action $\alpha : G \curvearrowright A$. Is the induced algebraic action $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$ always *continuous*? This is a non-trivial fact, a proof of which can be found in [164, Lemma 7.1], where $\mathrm{Prim}(A)$ is realised as a quotient of the *pure states space* of A. Here is a different, more direct, proof.

Lemma 2.3.38. *Let $\alpha : G \curvearrowright A$ be an action on a* C**-algebra. Then $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$ is continuous with respect to the Jacobson topology.*

Proof. We want to prove that, for any given open set $U \subseteq \mathrm{Prim}(A)$, the set given by

$$\{(g, \mathfrak{p}) \in G \times \mathrm{Prim}(A) \mid \alpha^\sharp_g(\mathfrak{p}) \in U\}$$

is open in $G \times \mathrm{Prim}(A)$ endowed with the product topology. Equivalently, one can show that the set

$$W = \{(g, \mathfrak{p}) \in G \times \mathrm{Prim}(A) \mid \alpha^\sharp_g(\mathfrak{p}) \in \mathrm{Prim}(A) \setminus U\}$$

is closed. Let us verify that this is true.

First of all, note that from Remark 2.3.29 there exists a unique ideal $I \in \mathcal{I}(A)$ corresponding to U, and W can be written as

$$W = \{(g, \mathfrak{p}) \in G \times \mathrm{Prim}(A) \mid \alpha^\sharp_g(\mathfrak{p}) \supseteq I\}.$$

Consider a net $\{(g_\lambda, \mathfrak{p}_\lambda)\}_{\lambda \in \Lambda}$ in W converging to an element $(g, \mathfrak{p}) \in G \times \mathrm{Prim}(A)$. In particular, it follows that $g_\lambda \to g$, and $\mathfrak{p}_\lambda \to \mathfrak{p}$. Let $e \in I$ be a positive element. By continuity of α, we have that for any $\varepsilon > 0$ there exists an open neighbourhood H around 1_G such that $\alpha_h(e) =_\varepsilon e$ for all $h \in H$. It follows from

Lemma 2.1.53(i) that $(e-\varepsilon)_+$ is Cuntz subequivalent to $\alpha_h(e)$ for all $h \in H$. Hence, we have that $(e-\varepsilon)_+ \in \bigcap_{h\in H} \alpha_h(I)$. By passing to a subnet, we may assume that $gg_\lambda^{-1} \in H$ for all $\lambda \in \Lambda$.

Now, since $\mathfrak{p}_\lambda \to \mathfrak{p}$, it is certainly true that $\mathfrak{p}$ belongs to the closure of $\{\mathfrak{p}_\lambda \mid \lambda \in \Lambda\}$, which means that $\bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda \subseteq \mathfrak{p}$. Note moreover that $(g_\lambda, \mathfrak{p}_\lambda) \in W$ means that $\alpha^\sharp_{g_\lambda}(\mathfrak{p}_\lambda) \supseteq I$, or equivalently, $\mathfrak{p}_\lambda \supseteq (\alpha^\sharp_{g_\lambda})^{-1}(I)$ for all $\lambda \in \Lambda$. Hence, we obtain that

$$\alpha^\sharp_g(\mathfrak{p}) \supseteq \bigcap_{\lambda\in\Lambda} \alpha^\sharp_g(\mathfrak{p}_\lambda) \supseteq \bigcap_{\lambda\in\Lambda} \alpha^\sharp_{gg_\lambda^{-1}}(I) \ni (e-\varepsilon)_+.$$

Since ε was arbitrarily chosen at the beginning, we get that $e \in \alpha^\sharp_g(\mathfrak{p})$. This shows that $I_+ \subseteq \alpha^\sharp_g(\mathfrak{p})$, and therefore we get that $I \subseteq \alpha^\sharp_g(\mathfrak{p})$, and $(g,\mathfrak{p}) \in W$. $\square$

Remark 2.3.39. Observe that for every cocycle morphism $(\varphi, \mathbb{u}) : (A,\alpha) \to (B,\beta)$, the **Cu**-morphism $\mathcal{I}(\varphi) : \mathcal{I}(A) \to \mathcal{I}(B)$ is equivariant with respect to $\alpha^\sharp$ and $\beta^\sharp$. This follows from the fact that unitarily equivalent $*$-homomorphisms induce the same **Cu**-morphism. Clearly, the same comment about being equivariant and stable under unitary equivalence also applies to $\mathcal{I}_B(\varphi)$ for a $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$. As one may expect, $\mathcal{I}_\sigma(-)$ (respectively, $\mathcal{I}(-)$) becomes a functor from the category of (separable) G-C*-algebras and proper cocycle morphisms as arrows to the category of abstract Cuntz semigroups endowed with order-theoretic G-actions and arrows given by G-equivariant **Cu**-morphisms.

Remark 2.3.40. We record here an observation that will be used in Theorem 3.5.5. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras. Let $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ be a conjugacy. As a consequence of Remark 2.3.29, f induces a unique order isomorphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ such that $\Phi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$. Moreover, Φ is $\alpha^\sharp$-to-$\beta^\sharp$ equivariant because

$$\begin{aligned}
\beta^\sharp_g \circ \Phi(I) &= \beta^\sharp_g \circ \Phi\Big(\bigcap\{\mathfrak{p} \in \mathrm{Prim}(A) \mid \mathfrak{p} \supseteq I\}\Big) \\
&= \bigcap\{\beta^\sharp_g \circ f(\mathfrak{p}) \in \mathrm{Prim}(A) \mid \mathfrak{p} \supseteq I\} \\
&= \bigcap\{f \circ \alpha^\sharp_g(\mathfrak{p}) \in \mathrm{Prim}(A) \mid \mathfrak{p} \supseteq I\} = \Phi \circ \alpha^\sharp_g(I)
\end{aligned}$$

for all $g \in G$ and $I \in \mathcal{I}(A)$.

2.4 Crossed products and aperiodicity

This is an expository section about crossed product arising from C*-dynamical systems. At the end of the section, we will provide certain results about the ideal structure of reduced crossed product C*-algebras by discrete groups that will be needed in Chapter 4.

Warning 2.4.1. Sometimes we will consider a *countable* group endowed with the *discrete topology.* Instead of writing this assumption every time, we consistently denote such a group by Γ, instead of G.

At various stages of this thesis, we will need to integrate compactly supported continuous functions on the group G. For this reason, we briefly recall a few fundamental facts about vector-valued integration over G with respect to a Haar measure, which we fix for the rest of the chapter, and consistently denote by μ. We refer the reader to, e.g., [164, Appendix C.2], for a complete overview of the topic and for proofs of the results stated here.

Let A be a C*-algebra, μ a Haar measure on G, and consider the collection $\mathcal{C}_c(G, A)$ of all compactly supported continuous functions from G to A. Then, for any $f \in \mathcal{C}_c(G, A)$, the map $[g \mapsto \|f(g)\|]$ belongs to $\mathcal{C}_c(G)$, which implies that

$$\|f\|_1 := \int_G \|f(g)\| \, d\mu(g)$$

is finite, and $\|f\|_1 \leq \|f\|_\infty \mu(\mathrm{supp} f)$.

The following statement contains all the properties that will be needed to carry out computations involving integrals. For a proof, we refer the reader to [164, Lemma C.3].

Lemma 2.4.2. *Let A be a* C**-algebra. There is a unique linear map $\int_G (\cdot) d\mu : \mathcal{C}_c(G, A) \to A$ satisfying the following properties.*

(i) *For any representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and any $f \in \mathcal{C}_c(G, A)$,*

$$\left\langle \pi\Big(\int_G f d\mu \Big) \xi, \eta \right\rangle = \int_G \langle \pi(f(g))\xi, \eta \rangle \, d\mu(g) \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

(ii) *For any $f \in \mathcal{C}_c(G, A)$,*

$$\left\| \int_G f(g) \, d\mu(g) \right\| \leq \|f\|_1, \tag{2.4}$$

$$\left(\int_G f(g) \, d\mu(g) \right)^* = \int_G f(g)^* \, d\mu(g). \tag{2.5}$$

(iii) *For any $a \in \mathcal{M}(A)$ and any $f \in \mathcal{C}_c(G, A)$,*

$$\left(\int_G f(g)\,d\mu(g)\right)a = \int_G f(g)a\,d\mu(g), \tag{2.6}$$

$$a\left(\int_G f(g)\,d\mu(g)\right) = \int_G af(g)\,d\mu(g). \tag{2.7}$$

(iv) *For any* C*-*algebra B, bounded, linear map $\varphi : A \to B$, an any $f \in \mathcal{C}_c(G, A)$,*

$$\varphi\left(\int_G f(g)\,d\mu(g)\right) = \int_G \varphi(f(g))\,d\mu(g). \tag{2.8}$$

Now, denote by $\mathcal{C}_c^s(G, \mathcal{M}(A))$ the set of compactly supported fuctions from G to the multipler algebra of A that are continuous with respect to the strict topology. In particular, for a function $f \in \mathcal{C}_c^s(G, \mathcal{M}(A))$, the set $\{\|f(g)a\| \mid g \in G\}$ is bounded for each $a \in A$. Then by the Banach–Steinhaus theorem, the set $\{\|f(g)\| \mid g \in G\}$ is bounded as well. Hence, all functions in $\mathcal{C}_c^s(G, \mathcal{M}(A))$ are automatically norm-bounded.

Lemma 2.4.3 (see [164, Lemma C.11])**.** *Let A be a* C*-*algebra. There is a unique linear map $\int_G(\cdot)d\mu : \mathcal{C}_c^s(G, \mathcal{M}(A)) \to \mathcal{M}(A)$ such that for any non-degenerate representation $\pi : A \to \mathcal{B}(\mathcal{H})$, whose extension to $\mathcal{M}(A)$ is also denoted by π, one has that*

$$\left\langle \pi\left(\int_G f d\mu\right)\xi, \eta\right\rangle = \int_G \langle \pi(f(g))\xi, \eta\rangle\,d\mu(g) \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

Moreover, one has that

$$\left\|\int_G f(g)\,d\mu(g)\right\| \leq \|f\|_\infty \mu(\operatorname{supp} f),$$

and that the formulas (2.5), (2.6), (2.7) *hold true. If B is a* C*-*algebra and $\varphi : A \to B$ a non-degenerate $*$-homomorphism, then* (2.8) *holds as well.*

Let us now continue with the definition of crossed products.

Remark 2.4.4. Let A be a C*-algebra equipped with an action $\alpha : G \curvearrowright A$. Then $\mathcal{C}_c(G, A)$ is a $*$-algebra when endowed with multiplication[21] and involution given by

$$f_1 * f_2(g) = \int_G f_1(h)\alpha_h(f_2(h^{-1}g))\,dh, \quad f^*(g) = \Delta(g^{-1})\alpha_g(f(g^{-1}))^*,$$

[21] Here, the operation giving rise to multiplication in $\mathcal{C}_c(G, A)$ is commonly referred to as *convolution product.*

for all $f, f_1, f_2 \in \mathcal{C}_c(G, A)$ and $g \in G$, where integration is taken with respect to a left Haar measure μ on G and $\Delta : G \to \mathbb{R}_+$ is the modular function on G, i.e., a continuous homomorphism such that for all $g \in G$ and measurable subset $A \subseteq G$, $\mu(Ag) = \Delta(g^{-1})\mu(A)$.

Note that when we consider a discrete group Γ, $\mathcal{C}_c(\Gamma, A)$ is the $*$-algebra of finitely supported functions on Γ with values in A, and write its elements as formal sums $\sum_{g\in\Gamma} a_g g$, where $a_g = 0$ for every but finitely-many $g \in \Gamma$. Product and involution in $\mathcal{C}_c(\Gamma, A)$ then become

$$\left(\sum_{g\in\Gamma} a_g g\right) \cdot \left(\sum_{g\in\Gamma} b_g g\right) = \sum_{g,h\in\Gamma} a_g \alpha_g(b_h) gh$$

and

$$\left(\sum_{g\in\Gamma} a_g g\right)^* = \sum_{g\in\Gamma} \alpha_{g^{-1}}(a_g^*) g^{-1}.$$

Definition 2.4.5. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and D a C*-algebra. A cocycle representation $(\pi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(D), \mathrm{id}_D)$ is said to be a *covariant representation.*[22] Note that this means that $\pi : A \to \mathcal{M}(D)$ is a $*$-homomorphism and $\mathbb{u} : G \to \mathcal{U}(\mathcal{M}(D))$ a strictly continuous *group homomorphism* such that the following condition holds,

$$\pi(\alpha_g(a)) = \mathbb{u}_g \pi(a) \mathbb{u}_g^* \quad \text{for all } g \in G, a \in A.$$

Every covariant representation gives rise to a $*$-representation $\pi \rtimes \mathbb{u} : \mathcal{C}_c(G, A) \to \mathcal{M}(D)$ in the following way,[23]

$$(\pi \rtimes \mathbb{u})(f) = \int_G \pi(f(g))\mathbb{u}_g \, dg, \qquad \text{for all } f \in \mathcal{C}_c(G, A).$$

Notation 2.4.6. Recall that, for any C*-algebra A, the multiplier algebra of $\mathcal{C}_0(G, A) \cong \mathcal{C}_0(G) \otimes A$ is naturally identified with the C*-algebra of bounded and stricly-continuous functions $G \to \mathcal{M}(A)$, which is denoted by $\mathcal{C}_b^s(G, \mathcal{M}(A))$. Next, let $\alpha : G \curvearrowright A$ be an action. We denote by $\bar{\alpha} : G \curvearrowright \mathcal{C}_b^s(G, \mathcal{M}(A))$ the strictly continuous action given by

$$\bar{\alpha}_g(f)(h) = \alpha_g(f(g^{-1}h)), \quad \text{for all } f \in \mathcal{C}_b^s(G, \mathcal{M}(A)),\ g, h \in G.$$

Note that this is the action induced on the multplier algebra by $\mathtt{lt} \otimes \alpha : G \curvearrowright \mathcal{C}_0(G) \otimes A$.

[22] It is somewhat more common to find *concrete* covariant representations in the literature (e.g., [206]). Namely, the definition would be different in the sense that, instead of considering any C*-algebra D, one would take $D = \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

[23] The representation of $\mathcal{C}_c(G, A)$ induced by a covariant representation is usually called *integrated form.*

Moreover, if B is another C*-algebra, equipped with an action $\beta : G \curvearrowright B$, and $\psi : A \to B$ is a $*$-homomorphism, we denote by $\hat{\psi} : A \to \mathcal{C}_b^s(G, \mathcal{M}(B))$ the $*$-homomorphism defined by

$$\hat{\psi}(a)(g) = \beta_g \circ \psi \circ \alpha_{g^{-1}}(a), \quad \text{for all } a \in A,\, g \in G.$$

The abuse of notation is reasonable as the actions appearing in the definition of $\hat{\psi}$ will be clear from context. By construction, we have that $\hat{\psi}$ is α-to-$\bar{\beta}$-equivariant. Indeed,

$$\begin{aligned} \bar{\beta}_g(\hat{\psi}(a))(h) &= \beta_g(\hat{\psi}(a)(g^{-1}h)) \\ &= \beta_g \circ \beta_{g^{-1}h} \circ \psi(\alpha_{(g^{-1}h)^{-1}}(a)) \\ &= \hat{\psi}(\alpha_g(a))(h) \end{aligned}$$

for all $a \in A$ and $g, h \in G$.

Given an action $\alpha : G \curvearrowright A$, one has that id_A (viewed as an automorphism) induces a $*$-homomorphism

$$\widehat{\mathrm{id}_A} : A \to \mathcal{C}_b(G, A) \subseteq \mathcal{C}_b^s(G, \mathcal{M}(A)) \cong \mathcal{M}(\mathcal{C}_0(G) \otimes A), \quad \widehat{\mathrm{id}_A}(a)(g) = \alpha_{g^{-1}}(a)$$

that is equivariant with respect to α and id_A (viewed as a G-action). In order to avoid confusion, in this special case we will use the less ambiguous notation $\mathrm{id}_\alpha = \widehat{\mathrm{id}_A}$.

Example 2.4.7. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and D any C*-algebra. For any $*$-homomorphism $\pi : A \to \mathcal{M}(D)$, one can produce a covariant representation of (A, α) in the following way. Let $M : \mathcal{C}_0(G) \to \mathcal{B}(L^2(G))$ denote the representation as multiplication operators, and consider the $*$-homomorphism $\pi^\alpha : A \to \mathcal{M}(\mathcal{K}(L^2(G)) \otimes D)$ given by the compositions

$$A \xrightarrow{\mathrm{id}_\alpha} \mathcal{C}_b(G, A) \subseteq \mathcal{C}_b^s(G, \mathcal{M}(A)) \xrightarrow{M \otimes \pi} \mathcal{M}(\mathcal{K}(L^2(G)) \otimes D).$$

Then, if $\lambda : G \to \mathcal{U}(L^2(G))$ denotes the left-regular representation, the homomorphism

$$\lambda \otimes \mathbf{1}_{\mathcal{M}(D)} : G \to \mathcal{U}(\mathcal{M}(\mathcal{K}(L^2(G)) \otimes D)), \quad (\lambda \otimes \mathbf{1}_{\mathcal{M}(D)})_g = \lambda_g \otimes \mathbf{1}_{\mathcal{M}(D)}.$$

is strictly continuous, and the pair $(\pi^\alpha, \lambda \otimes \mathbf{1}_{\mathcal{M}(D)})$ is covariant representation of (A, α), and it is called the *regular representation* associated to π. We also remark here that if π is faithful, the resulting regular covariant representation is faithful as well.

Note that, for a representation $\pi : A \to \mathcal{B}(\mathcal{H}) = \mathcal{M}(\mathcal{K}(\mathcal{H}))$ on some Hilbert space $\mathcal{H}$, one has that $\pi^\alpha : A \to \mathcal{B}(L^2(G, \mathcal{H}))$ is given by

$$\pi^\alpha(a)(\xi)(g) = \pi(\alpha_{g^{-1}}(a))(\xi), \quad \text{for all } \xi \in L^2(G, \mathcal{H}),\ a \in A,\ g \in G.$$

Moreover, $\lambda \otimes \mathbf{1}_\mathcal{H}$ is just the diagonal repeat of λ, which is explicitly given by

$$(\lambda \otimes \mathbf{1}_\mathcal{H})_g(\xi)(h) = \xi(g^{-1}h), \quad \text{for all } \xi \in L^2(G, \mathcal{H}),\ g, h \in G.$$

Definition 2.4.8. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra.

- The *full* or *universal crossed product* associated to α, denoted by $A \rtimes_{\alpha,\mathrm{f}} G$ is the completion of $\mathcal{C}_c(G, A)$ with respect to the norm given by

$$\|f\|_\mathrm{f} = \sup\{\|(\pi \rtimes \mathbb{u})(f)\| \mid (\pi, \mathbb{u}) \text{ is a covariant representation}\}$$

  for all $f \in \mathcal{C}_c(G, A)$.

- The *reduced crossed product* of A by G, denoted by $A \rtimes_{\alpha,\mathrm{r}} G$, is given by

$$\overline{(\mathrm{id}_A^\alpha \rtimes (\lambda \otimes \mathbf{1}_{\mathcal{M}(A)}))(\mathcal{C}_c(G, A))} \subseteq \mathcal{M}(\mathcal{K}(L^2(G)) \otimes A).$$

Clearly, both completions give rise to C*-algebras, and they also admit a *concrete* definition. In particular, $A \rtimes_{\alpha,\mathrm{r}} G$ can be defined as the completion of the image of $\mathcal{C}_c(G, A)$ under the norm

$$\|f\|_\mathrm{r} = \|(\pi^\alpha \rtimes (\lambda \otimes \mathbf{1}_\mathcal{H}))(f)\|, \quad f \in \mathcal{C}_c(G, A)$$

for any faithful representation π of A.[24]

It follows from the definition of $A \rtimes_{\alpha,\mathrm{f}} G$ that it enjoys the following universal property (see [206, Section 2.6]). For any covariant representation $(\pi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(D), \mathrm{id}_D)$, the associated integrated form $\pi \rtimes \mathbb{u}$ extends to a $*$-homomorphism

$$\pi \rtimes \mathbb{u} : A \rtimes_{\alpha,\mathrm{f}} G \to \mathcal{M}(D).$$

The converse also holds, in the sense that any non-degenerate covariant $*$-homomorphism

$$\varphi : A \rtimes_{\alpha,\mathrm{f}} G \to \mathcal{M}(D)$$

is the integrated form $\pi \rtimes \mathbb{u}$ associated to the non-degenerate covariant representation $(\pi, \mathbb{u})$ given as follows. First, construct the *canonical covariant*

[24] One can show that this construction of $A \rtimes_{\alpha,\mathrm{r}} G$ is independent of the choice of the representation π; see, e.g., [206].

representation (ι_A, ι_G) of (A, α) on $\mathcal{M}(A \rtimes_{\alpha,\mathrm{f}} G)$ as follows. The $*$-homomorphism $\iota_A : A \to \mathcal{M}(A \rtimes_{\alpha,\mathrm{f}} G)$ is given by

$$(\iota_A(a) \cdot f)(g) = af(g), \quad (f \cdot \iota_A(a))(g) = f(g)\alpha_g(a)$$

for all $f \in \mathcal{C}_c(G, A)$, $g \in G$, and $a \in A$. The map $\iota_G : G \to \mathcal{U}(\mathcal{M}(A \rtimes_{\alpha,\mathrm{f}} G))$ is given by

$$(\iota_G(g) \cdot f)(h) = \alpha_g(f(g^{-1}h)), \quad (f \cdot \iota_G(g))(h) = \Delta(g^{-1})f(hg^{-1})$$

for all $f \in \mathcal{C}_c(G, A)$, and $g, h \in G$. Note that these maps extend to left and right multiplications of ι_A and ι_G with elements in $A \rtimes_{\alpha,\mathrm{f}} G$ by density of $\mathcal{C}_c(G, A)$. Finally, one can see that $\varphi = \pi \rtimes \mathbb{u}$, where

$$\pi = \varphi \circ \iota_A, \quad \text{and} \quad \mathbb{u} = \varphi \circ \iota_G,$$

which are well-defined thanks to non-degeneracy of φ.

In particular, the integrated form associated to the canonical regular covariant representation used in the definition of $A \rtimes_{\alpha,\mathrm{r}} G$ extends to a $*$-homomorphism

$$q : A \rtimes_{\alpha,\mathrm{f}} G \to A \rtimes_{\alpha,\mathrm{r}} G \subseteq \mathcal{M}(A \rtimes_{\alpha,\mathrm{r}} G)$$

that is clearly surjective, and therefore $\frac{A \rtimes_{\alpha,\mathrm{f}} G}{\ker(q)} \cong A \rtimes_{\alpha,\mathrm{r}} G$; see [206, Lemma 7.8].

Remark 2.4.9. Observe that, since

$$(\mathrm{id}_A^\alpha \rtimes (\lambda \otimes \mathbf{1}_{\mathcal{M}(A)})) \circ \iota_A = \mathrm{id}_A^\alpha$$

and the right-hand side is injective, it follows that ι_A is injective as well and defines a canonical embedding

$$A \hookrightarrow \mathcal{M}(A \rtimes_{\alpha,\mathrm{f}} G).$$

Remark 2.4.10. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on C*-algebras, consider any non-degenerate cocycle representation $(\varphi, \mathbb{u}) : (A, \alpha) \to (\mathcal{M}(B), \beta)$. One has an induced non-degenerate $*$-homomorphism

$$(\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G : A \rtimes_{\alpha,\mathrm{f}} G \to B \rtimes_{\beta,\mathrm{f}} G$$

given by

$$\big((\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G\big) \circ \iota_A = \iota_B \circ \varphi, \quad \big((\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G\big) \circ \iota_G^{(A)} = \iota_B(\mathbb{u}_\bullet)\iota_G^{(B)},$$

where $(\iota_A, \iota_G^{(A)})$ and $(\iota_B, \iota_G^{(B)})$ denote the canonical convariant representations of (A, α) and (B, β), respectively. By the universal property of $A \rtimes_{\alpha,\mathrm{f}} G$, $(\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G$

is the unique non-degenerate $*$-homomorphism induced by the non-degenerate covariant representation $(\iota_B \circ \varphi, \iota_B(\mathbb{u}_\bullet)\iota_G^{(B)})$ of (A, α) on $(\mathcal{M}(B \rtimes_{\beta,\mathrm{f}} G), \mathrm{id})$. Clearly, if $(\varphi, \mathbb{u})$ has range in (B, β), then $(\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G$ has image in $B \rtimes_{\beta,\mathrm{f}} G$.

In an analogous manner, one may define a $*$-homomorphism

$$(\varphi, \mathbb{u}) \rtimes_{\mathrm{r}} G : A \rtimes_{\alpha,\mathrm{r}} G \to \mathcal{M}(B \rtimes_{\beta,\mathrm{r}} G)$$

which has range in $B \rtimes_{\beta,\mathrm{r}} G$ whenever $\varphi(A) \subseteq B$. Moreover, $(\varphi, \mathbb{u}) \rtimes_{\mathrm{r}} G$ is injective precisely when φ is, while this is not always true for $(\varphi, \mathbb{u}) \rtimes_{\mathrm{f}} G$.

The full and reduced crossed product constructions give rise to two functors from the category C^*_G where one restricts to non-degenerate maps, and the category of C*-algebras (see [192, Section 1.4]). When one considers the special case of G-equivariant maps, the same observations hold true with the appropriate simplifications. In particular, in that case non-degeneracy is not needed anymore, and the crossed product constructions are functorial with respect to G-equivariant morphisms; see [206, Corollary 2.48] for a proof.

We now turn our attention to discrete groups.[25]

Remark 2.4.11. In the special case when $A = \mathbb{C}$, the constructions in Definition 2.4.8 yield the full and reduced *group* C*-*algebras* associated to Γ, respectively. These are denoted by $A \rtimes_{\alpha,\mathrm{f}} \Gamma = \mathrm{C}^*(\Gamma)$ and $A \rtimes_{\alpha,\mathrm{r}} \Gamma = \mathrm{C}^*_{\mathrm{r}}(\Gamma)$.

Remark 2.4.12. Let $\alpha : \Gamma \curvearrowright A$ be an action on a C*-algebra. Note that A is naturally embedded in $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ by simply sending each $a \in A$ to $\mathrm{id}_A^\alpha(a)$, which sits inside $\mathcal{C}_c(\Gamma, A)$ as the image of $a \cdot 1_\Gamma$. Analogously, one has that A sits inside $A \rtimes_{\alpha,\mathrm{f}} \Gamma$ via the canonical inclusion ι_A. We will always identify A with its isomorphic copy inside $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ and $A \rtimes_{\alpha,\mathrm{f}} \Gamma$.

If A is unital, one also has a copy of Γ inside $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ and $A \rtimes_{\alpha,\mathrm{f}} \Gamma$. In the reduced crossed product, this is witnessed by $\lambda \otimes \mathbf{1}_{\mathcal{M}(A)}$, while for the full crossed product one takes ι_Γ. In particular, this entails that the full crossed product is realised as the universal C*-algebra given by

$$\mathrm{C}^*(A \cup \iota_\Gamma(\Gamma) \mid \iota_\Gamma(g)a = \alpha_g(a)\iota_\Gamma(g),\ g \in \Gamma, a \in A).$$

Remark 2.4.13. Let $\alpha : \Gamma \curvearrowright A$ be a discrete group action on a C*-algebra. The map $E : \mathcal{C}_c(G, A) \to A$ given by $E(\sum_{g\in\Gamma} a_g g) = a_{1_\Gamma}$ extends to a *faithful* conditional expectation from $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ onto A. For a proof of this fact, see [24, Proposition 4.1.9]. We will refer to E as the *canonical faithful conditional expectation* from $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ onto A.

[25] Recall that, to remark the discreteness assumption, we denote by Γ every group with discrete topology.

In the following remark, we record an application of a theorem of Green about certain crossed product of commutative C*-algebras [86]. This particular example will be relevant in Chapter 3, where it is used to carry out existence-type results for maps. First, let us recall that an action $\sigma : G \curvearrowright X$ on a locally compact space X is said to be

- *free* if, for all $x \in X$, $\sigma_g(x) = x$ implies that $g = 1_G$;
- *proper* if the map $G \times X \to X \times X$, $(g, x) \mapsto (\sigma_g(x), x)$ is proper, i.e., the preimage of every compact set is compact.

Remark 2.4.14. By a result of Green, [86, Corollary 3.15], if an action $\sigma : G \curvearrowright X$ is free and proper, then there exists an isomorphism

$$\mathcal{C}_0(X) \rtimes_{\sigma,\mathrm{f}} G \cong \mathcal{C}(X/G) \otimes \mathcal{K}(L^2(G)),$$

provided that G is an infinite, second-countable, locally compact group, X is a second-countable, locally compact space, and the orbit space X/G with the quotient topology has finite covering dimension.

Let us choose, as a demostrative example, the action of $\mathbb{Z}$ on $\mathbb{R}$ induced by the shift homeomorphism $\sigma(t) = t - 1$ for all $t \in \mathbb{R}$. The associated automorphism of $\mathcal{C}_0(\mathbb{R})$ is given by $\sigma(f)(t) = f(t+1)$. It is readily verified that σ induces a free and proper $\mathbb{Z}$-action, and thus

$$\mathcal{C}_0(\mathbb{R}) \rtimes_{\sigma,\mathrm{f}} \mathbb{Z} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}(\ell^2(\mathbb{Z})).$$

2.4.1 Aperiodic actions of discrete groups

Definition 2.4.15. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. A (closed and two-sided) ideal $I \subseteq A$ is said to be α*-invariant* if $\alpha_g(I) \subseteq I$ for every $g \in G$. The collection of all α-invariant ideals of A is denoted by $\mathcal{I}(A)^\alpha$, and forms a complete sublattice of $\mathcal{I}(A)$.

Any α-invariant ideal $I \subseteq A$ endowed with the induced action can be used to form a $*$-algebraic ideal $\mathcal{C}_c(G, I) \subseteq \mathcal{C}_c(G, A)$, whose completion yields an ideal

$$I \rtimes_{\alpha,\mathrm{r}} G \subseteq A \rtimes_{\alpha,\mathrm{r}} G.$$

This assignment immediately gives us an order embedding

$$\mathcal{I}(A)^\sigma \hookrightarrow \mathcal{I}(A \rtimes_{\alpha,\mathrm{r}} G), \quad I \mapsto I \rtimes_{\alpha,\mathrm{r}} G.$$

Assume now that $G = \Gamma$ is discrete. Note that if $I \subseteq A \rtimes_{\alpha,\mathrm{r}} \Gamma$ is an ideal, then $I \cap A \subseteq A$ is an α-invariant ideal of A because α becomes inner in the crossed product, and $uIu^* = I$ for any unitary $u \in \mathcal{M}(A \rtimes_{\alpha,\mathrm{r}} \Gamma)$ (remember that I is an ideal). Hence, we have a well-defined order map in the other direction given by

$$\mathcal{I}(A \rtimes_{\alpha,\mathrm{r}} \Gamma) \to \mathcal{I}(A)^\sigma, \quad I \mapsto I \cap A.$$

Here, the reader should be warned that this map is not necessarily injective (see the example below).

Example 2.4.16. Let α be the trivial action of $\Gamma = \mathbb{Z}$ on the complex numbers $A = \mathbb{C}$. Then we have that $\mathbb{C} \rtimes_{\alpha,\mathrm{r}} \mathbb{Z} = \mathrm{C}^*_\mathrm{r}(\mathbb{Z}) \cong \mathcal{C}(\mathbb{T})$. Hence, for every proper ideal $I \subseteq \mathcal{C}(\mathbb{T})$, we have that $I \cap \mathbb{C} = \{0\}$, thus implying that the assignment $I \mapsto I \cap \mathbb{C}$ is not injective.

In the rest of this section we present a sufficient condition ensuring that the order map $\mathcal{I}(A \rtimes_{\alpha,\mathrm{r}} \Gamma) \to \mathcal{I}(A)^\sigma$ presented above takes non-zero ideals to non-zero ideals.

Definition 2.4.17 (cf. [130, Definitions 2.8+2.15])**.** Let A be a C*-algebra. One says that an automorphism $\sigma \in \mathrm{Aut}(A)$ satisfies *Kishimoto's condition* if for every $\varepsilon > 0$, $x \in A$, and non-zero hereditary C*-subalgebra H of A, there exists a positive, norm-one element $a \in H$ such that $\|ax\sigma(a)\| < \varepsilon$.

An action $\alpha : \Gamma \curvearrowright A$ of a discrete group Γ is said to be *aperiodic* if α_g satisfies Kishimoto's condition for all $g \in \Gamma \setminus \{1_\Gamma\}$.

Remark 2.4.18. Let us first clarify the terminology in Definition 2.4.17. Kwaśniewski and Meyer introduced "Kishimoto's condition" inspired by Kishimoto's original work on outer automorphisms [125, Lemma 1.1], thus the name. One should also note that variations of the same property appeared elsewhere, for instance in [80, 101, 129].

The relation between aperiodicity and detection of ideals was established in [80, Theorem 3.12] (see also [130, Theorem 2.18]). In that context, they work with partial actions, a more general setting than what we need here. Hence, for the reader's convenience, we will replicate their proof in the discrete group action case.

Theorem 2.4.19. *Let $\alpha : \Gamma \curvearrowright A$ be an aperiodic action on a* C*-*algebra. Then A detects ideals*[26] *in the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} \Gamma$.*

Behind the scenes, the real centerpiece making this result possible is the canonical faithful conditional expectation from $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ onto A. This will become clear with in the next result, which is a special case of Giordano and Sierakowski's [80, Lemma 3.11], which in turn is a non-commutative version of Exel, Laca and Quigg's [58, Proposition 2.4].

[26] Recall detection of ideals from Definition 2.3.5.

Lemma 2.4.20. *Let $\alpha : \Gamma \curvearrowright A$ be an aperiodic action on a C^*-algebra. Then, for every $\varepsilon > 0$ and for every positive element $a \in (A \rtimes_{\alpha,\mathrm{r}} \Gamma)_+$, there exists a positive contraction $x \in A_+$ such that*

$$\|xE(a)x - xax\| \leq \varepsilon, \ \textit{ and } \ \|E(a)\| - \|xE(a)x\| \leq \varepsilon.$$

Proof. We first show a few reduction arguments. Assume that the statement holds for all $a \in \mathcal{C}_c(\Gamma, A)_+$. Then, if $b \in (A \rtimes_{\alpha,\mathrm{r}} \Gamma)_+$ and $\varepsilon > 0$ are fixed, by density we may find $a \in \mathcal{C}_c(\Gamma, A)_+$ such that $a =_\varepsilon b$, and a contraction $x \in A_+$ such that

$$\|xE(a)x - xax\| \leq \frac{\varepsilon}{3}, \ \text{ and } \ \|E(a)\| - \|xE(a)x\| \leq \frac{\varepsilon}{3}.$$

It follows that

$$\|xE(b)x - xbx\| \leq \|xE(b)x - xE(a)x\| + \|xE(a)x - xax\| + \|xbx - xax\| \leq \varepsilon,$$

and

$$\|E(b)\| \leq \|E(b-a)\| + \frac{\varepsilon}{3} + \|xE(a)x\| \leq \varepsilon + \|xE(b)x\|.$$

Furthermore, the statement holds if and only if it holds when $\|E(a)\| = 1$. (Note that when $a = 0$ the statement always holds.) We show this fact here. Pick any $b \neq 0$ in $(A \rtimes_{\alpha,\mathrm{r}} \Gamma)_+$, then $\|E(b)\| \neq 0$ because E is faithful, and we may set $a := b/\|E(b)\|$. If for any ε there exists a contraction $x \in A_+$ such that

$$\|xE(a)x - xax\| \leq \frac{\varepsilon}{\|E(b)\|}, \ \text{ and } \ \|E(a)\| - \|xE(a)x\| \leq \frac{\varepsilon}{\|E(b)\|},$$

then

$$\|xE(b)x - xbx\| \leq \varepsilon, \ \text{ and } \ \|E(b)\| - \|xE(b)x\| \leq \varepsilon.$$

Thanks to these two reduction arguments, it is sufficient to prove that for every $\varepsilon > 0$ and $a \in \mathcal{C}_c(\Gamma, A)_+$ with $\|E(a)\| = 1$, there exists a positive contraction $x \in A_+$ such that

$$\|xE(a)x - xax\| \leq \varepsilon, \ \text{ and } \ \|E(a)\| - \|xE(a)x\| \leq \varepsilon.$$

Hence, fix $0 < \varepsilon < 1$ and $a \in \mathcal{C}_c(\Gamma, A)_+$ with $\|E(a)\| = 1$. If we write a as a finite sum

$$a = a_{1_\Gamma} + \sum_{g \in F} a_g u_g, \quad F \subseteq \Gamma \setminus \{1_\Gamma\} \text{ finite, } a_g \in A,$$

Let $n = |F| < \infty$, and enumerate the elements in F, i.e., $F = \{g_1, \ldots, g_n\}$. Then we want to prove that there exists a positive contraction $x \in A_+$ such that

$$\|x \sum_{k=1}^{n} a_{g_k} u_{g_k} x\| \leq \varepsilon, \ \text{ and } \ 1 - \|x a_{1_\Gamma} x\| \leq \varepsilon.$$

(Note that a_{1_Γ} is positive because $E(a) = a_{1_\Gamma} \in A_+$, and by assumption $\|a_{1_\Gamma}\| = 1$.) Observe that, for the first inequality, it is sufficient to show that

$$\|xa_{g_k}u_{g_k}x\| \leq \frac{\varepsilon}{n}$$

and then use triangle inequality to get the statement. We start by choosing two continuous increasing functions defined as follows,

$$f : [0,1] \to [0,1], \text{ such that } f \restriction_{[0,1-\varepsilon]} = 0 \text{ and } f \restriction_{[1-\frac{\varepsilon}{2},1]} = 1;$$

$$h : [0,1] \to [0,1], \text{ such that } h \restriction_{[0,1-\frac{\varepsilon}{2}]} = 0 \text{ and } h \restriction_{[1-\frac{\varepsilon}{4},1]} = 1.$$

(Note that $fh = hf = h$ and $\|h(a_{1_\Gamma})\| = 1$.) Define $x_0 = f(a_{1_\Gamma})$ via continuous functional calculus and consider the subset of A on which x_0 acts as a unit, namely,

$$A_1 = \{x \in A \mid xx_0 = x_0x = x\}.$$

Since $h(a_{1_\Gamma}) \in A_1$, we know that A_1 is non-zero. Furthermore, the subset $L = \{x \in A \mid x^*x \in A_1\}$ is a closed left ideal of A. Clearly $A_1 \subseteq L \cap L^*$, and a simple calculation shows that, in fact, $A_1 = L \cap L^*$. Hence, A_1 is a non-zero hereditary C*-subalgebra of A. By aperiodicity, there exists a positive element $y_1 \in (A_1)_+$ with $\|y_1\| = 1$ such that $\|y_1a_{g_1}\alpha_{g_1}(y_1)\| < \varepsilon$, and therefore that

$$\|y_1a_{g_1}u_{g_1}y_1u_{g_1}^*\| = \|y_1a_{g_1}u_{g_1}y_1\| = \|y_1a_{g_1}\alpha_{g_1}(y_1)\| \leq \varepsilon.$$

Fix now any continuous increasing function with the following properties,

$$\ell : [0,1] \to [0,1], \text{ such that } \ell \restriction_{[0,1-\varepsilon]} = \mathrm{id} \text{ and } \ell \restriction_{[1-\frac{\varepsilon}{2},1]} = 1,$$

and define $x_1 = \ell(y_1)$ via continuous functional calculus. Clearly, $x_1 \in (A_1)_+$ with $\|x_1\| = 1$, and $\|x_1 - \ell(y_1)\| \leq \varepsilon$. Hence,

$$\|x_1a_{g_1}x_1\| \leq (2\|a_{g_1}\| + 1)\varepsilon.$$

Define, as above, the subset

$$A_2 = \{x \in A_1 \mid xx_1 = x_1x = x\},$$

which is a non-zero hereditary C*-subalgebra of A because $\ell h = h\ell = h$ and hence it contains $h(y_1) \neq 0$. Repeating the same procedure as above, we may find, for $k \leq n$, $x_k \in (A_k)_+$ with $\|x_k\| = 1$ and

$$\|x_ka_{g_k}u_{g_k}x_k\| \leq (2\|a_{g_k}\| + 1)\varepsilon.$$

Finally, let $x = x_n \in (A_n)_+$. Then, $xx_k = x_kx = x$ and

$$\|xa_{g_k}u_{g_k}x\| \leq (2\|a_{g_k}\| + 1)\varepsilon$$

for all $0 \leq k \leq n$. Now, since we have that $f \cdot \mathrm{id} \cdot f = f^2 \cdot \mathrm{id} \geq (1-\varepsilon)f^2$ as elements of $\mathcal{C}([0,1])_+$, then

$$xa_{1_\Gamma}x = xx_0a_{1_\Gamma}x_0x \geq (1-\varepsilon)xx_0^2x = (1-\varepsilon)x^2$$

and hence

$$1 - \|xa_{1_\Gamma}x\| \leq \varepsilon.$$

□

Proof of Theorem 2.4.19. The strategy of the proof is the same as the one used by Exel, Laca and Quigg in the commutative setting. Consider a non-zero ideal $I \subseteq A \rtimes_{\alpha,\mathrm{r}} \Gamma$ and assume that, towards a contradiction, $I \cap A = \{0\}$. Let us denote the quotient map by

$$\pi : A \rtimes_{\alpha,\mathrm{r}} \Gamma \to \frac{A \rtimes_{\alpha,\mathrm{r}} \Gamma}{I}.$$

Pick a positive element $a \in I_+$. For a fixed $\varepsilon > 0$, we may apply Lemma 2.4.20 and find a positive contraction $x \in A_+$ such that

$$\|xE(a)x - xax\| \leq \varepsilon, \text{ and } \|E(a)\| - \|xE(a)x\| \leq \varepsilon.$$

Since $a \in I_+$, one has that $\pi(a) = 0$, which together with the first inequality above implies that

$$\|\pi(xE(a)x)\| = \|\pi(xE(a)x - xax)\| \leq \varepsilon.$$

Moreover, π is isometric on A because $I \cap A = \{0\}$ by assumption. This means, in particular, that

$$\|xE(a)x\| = \|\pi(xE(a)x)\| \leq \varepsilon.$$

However, by the second inequality coming from Lemma 2.4.20, we have that

$$\|E(a)\| < \varepsilon + \|xE(a)x\| \leq 2\varepsilon,$$

which implies that $E(a) = 0$ because $\varepsilon > 0$ was arbitrarily chosen. Since E is faithful, this means that $a = 0$, thus $I = \{0\}$, a contradiction. □

Remark 2.4.21. When $\Gamma = \mathbb{Z}$ or $\Gamma = \mathbb{Z}/p\mathbb{Z}$, where p is a square-free number,[27] the converse to Theorem 2.4.19 is also true, and one can find a proof of this fact in [130, Theorem 9.12] when A contains an essential ideal that is either separable or type I, or in [76, Corollary 9.6] for the general case.[28]

[27] The integer p is said to be square-free if no non-trivial perfect square divides p.

[28] Note that in [76, Corollary 9.6], detection of ideals is called ideal intersection property, and aperiodicity is equivalent to proper outerness.

2.5 A brief history of amenability

There are many equivalent notions of amenability for groups. We only recall a few of them here.

Definition 2.5.1. A locally compact group G with Haar measure μ is amenable if one of the following equivalent properties holds.

(i) There exists a state m on $L^\infty(G)$ that is left-invariant (or, alternatively, right-invariant) with respect to the shift action,[29]

(ii) Every continuous G-action by affine automorphisms on a compact convex set has a fixed point.

(iii) For any compact subset $K \subseteq G$, and $\varepsilon > 0$, there exists a subset $F \subseteq G$ with finite positive Haar measure such that

$$\sup_{g \in K} \frac{\mu(gF\Delta F)}{\mu(F)} \leq \varepsilon,$$

where $gF\Delta F$ denotes the symmetric difference of $gF = \{gh \mid h \in F\}$ and F, i.e., $(gF \cup F) \setminus (gF \cap F)$. This is referred to as the *Følner condition.*

The way Zimmer first introduced amenability for actions, sometimes called *Zimmer amenability*, extends Definition 2.5.1(ii) for groups. Let us give another (equivalent) definition, which is sometimes referred to as *Reiter's condition* (or *Day's condition*).[30] We remark that, here, $\mathrm{Prob}(G) \subseteq \mathcal{C}_0(G)^*$ denotes the set of probability measures on G, and is equipped with the left translation action of G given by $\mathtt{lt}_g(m)(f) = m(\mathtt{lt}_{g^{-1}}(f))$; cf. Example 2.2.4.

Definition 2.5.2. A continuous action $\alpha : G \curvearrowright X$ on a locally compact space X is *amenable* if there exists a net of continuous maps $m_\lambda : X \to \mathrm{Prob}(G)$ such that

$$\|m_\lambda(\alpha_g(x)) - (\mathtt{lt}_g(m_\lambda))(x)\|_1 \to 0 \tag{2.9}$$

uniformly on compact subsets of $G \times X$.

Remark 2.5.3. Note that in the limit condition 2.9 of Definition 2.5.2 we consider the $L^1(G, \mu)$-norm $\|\cdot\|_1$. One might wonder whether it is necessary to restrict $\mathrm{Prob}(G)$ to the subset of measures that are absolutely continuous with respect to the Haar measure. However, this is not required, since even

[29] A state satisfying these conditions is called an *invariant mean*.

[30] Amenability also generalises to topological groupoids as per the work of Renault; see [165, Definition 3.6].

when $m_i(x) \in \mathrm{Prob}(G)$ are not absolutely continuous with respect to the Haar measure μ on G, one can replace $m_i(x)$ by the convolution measure

$$m_i'(x) := m_i(x) * \eta$$

for any $\eta \in \mathrm{Prob}(G)$ that *is* absolutely continuous with respect to μ. It is straightforward to verify that this procedure preserves all the properties of the net m_i, and the resulting probability measures $m_i'(x)$ are absolutely continuous with respect to the Haar measure μ.

To see how this generalises group amenability, observe that the trivial G-space $X = \{x\}$ is amenable if and only if G is an amenable group (this is [215, Corollary 1.6]). Zimmer's work [213, 215] also contains the first results connecting amenability with injectivity of the associated crossed product of a commutative von Neumann algebra.

An example (probably the first one) of an amenable action of a non-amenable group on a von Neumann algebra was given by Zimmer in [214]. Moreover, Zimmer shows that the von Neumann algebra crossed product arising from his example is injective, thus stimulating the question whether injectivity can be ensured for actions of non-amenable groups. Shortly after, Anantharaman-Delaroche defined amenability of G-actions on von Neumann algebras as a generalisation of (i) of Definition 2.5.1 for groups. Recall that an action $\alpha : G \curvearrowright M$ on a von Neumann algebra assumed to be point-ultraweakly continuous, that is, the map $[g \mapsto \alpha_g(x)]$ from G to M is ultraweakly continuous for all $x \in M$.[31]

Definition 2.5.4 (see [2, Définition 3.4])**.** Denote the left-translation action of G on $L^\infty(G)$ by $\mathtt{lt}$, i.e., $(\mathtt{lt}_g(f))(h) = f(g^{-1}h)$ for all $g, h \in G$ and $f \in L^\infty(G)$.

An action $\alpha : G \curvearrowright M$ on a von Neumann algebra is *amenable* if there exists a conditional expectation E from $L^\infty(G)\bar{\otimes}M$ onto M (identified with the subalgebra $1\bar{\otimes}M$) such that $E \circ (\mathtt{lt} \otimes \alpha)_g = \alpha_g \circ E$ for all $g \in G$.

For a general injective von Neumann algebra M, an amenable action $\alpha : \Gamma \curvearrowright M$ of a discrete group still yields an injective crossed product; cf. [38, Proposition 6.8].

Remark 2.5.5. Consider the same setup as in Definition 2.5.4 If one chooses $M = \mathbb{C}$, then a G-action on M is amenable if and only if there exists a left-invariant mean on $L^\infty(G)$, i.e., G is amenable. On the other hand, if $G = \Gamma$ is discrete and $M \cong L^\infty(X, \mu)$ is an abelian von Neumann algebra, then the

[31] This continuity condition is equivalent to the a priori stronger condition that for all $\varphi \in M_*$ (the predual of M) the map $[g \mapsto \varphi \circ \alpha_g]$ from G to M_* is continuous in norm (see [195, Proposition X.1.2]).

induced action of Γ on the measure space (X, μ) is amenable in the sense of Zimmer; see [2, Remarques 3.5(a)+(b)]).

Finally, note that $\mathtt{lt} : G \curvearrowright L^\infty(G)$ is clearly amenable.

The following result is due to Anantharaman-Delaroche.[32]

Theorem 2.5.6 (cf. [2, Proposition 3.12 + Corollaire 4.2])**.** *Let $\alpha : G \curvearrowright M$ be an action on a von Neumann algebra. If α is amenable and M is injective, then $M \rtimes_\alpha G$ is injective. Moreover, when $G = \Gamma$ is discrete, the converse is also true.*

Nevertheless, the moreover part of the theorem above is not true for general topological groups as the next example shows.

Example 2.5.7. There exist non-amenable locally compact (non-discrete) groups, such as $SL_2(\mathbb{C})$, that act non-amenably on $M = \mathbb{C}$, despite the crossed product $M \rtimes G = L(G)$ being injective by [48, Theorem 6.4]; see [2, Remarque 4.4 (b)].

After this quick dip in the realm of von Neumann actions, we turn our attention back to the world of C*-dynamics. The following definition is due to Anantharaman-Delaroche.

Definition 2.5.8 (see [3, Définition 4.1])**.** Let $\alpha : \Gamma \curvearrowright A$ be an action on a C*-algebra. Then α is said to be *amenable* if $\alpha^{**} : \Gamma \curvearrowright A^{**}$ is amenable in the sense of Definition 2.5.4.

Remark 2.5.9. Anantharaman-Delaroche's notion of amenability for C*-actions given in Definition 2.5.8 coincides with Zimmer amenability in the commutative setting from Definition 2.5.2 by [3, Théorème 4.9+Remarque 4.10]. However, the assumption that Γ is discrete in Definition 2.5.8 requires an immediate word of caution. In fact, one would be tempted to define amenability in the same fashion for general actions of locally compact groups. However, this is not possible because, if G is not a discrete group, $\alpha^{**} : G \curvearrowright A^{**}$ is not necessarily continuous. A good case in point is the algebraic action induced by the left-translation action $\mathtt{lt} : G \curvearrowright \mathcal{C}_0(G)$ on the double dual, namely, $\mathtt{lt}^{**} : G \curvearrowright \mathcal{C}_0(G)^{**}$. This is because $\mathcal{C}_0(G)^{**}$ contains discontinuous non-zero elements with respect to the left-translation action $\mathtt{lt}^{**}$, such as indicators functions over singletons (and more generally all Borel indicator functions).

Remark 2.5.10. Clearly, a discrete group action on $A = \mathbb{C}$ is amenable if and only if the acting group is amenable.

[32] We remark that one can also give a short proof of the theorem using functoriality of the crossed product with respect to completely positive G-equivariant maps.

In order to define amenability for general actions on C*-algebras, it is convenient to build a new kind of enveloping von Neumann algebra of A that depends on the action $\alpha : G \curvearrowright A$.

Definition 2.5.11. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and denote by $\pi : A \to \mathcal{B}(\mathcal{H})$ the universal GNS representation of A. With slight abuse of notation, identify the full crossed product C*-algebra with its faithfully represented copy under the universal GNS representation, i.e., $A \rtimes_{\alpha,f} G \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{f}})$. One can then consider the image of A under the inclusion coming from the canonical covariant representation (ι_A, ι_G) of (A, α) as a faithful representation of A on $\mathcal{H}_{\mathrm{f}}$, namely

$$\iota_A : A \hookrightarrow \mathcal{M}(A \rtimes_{\alpha,f} G) \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{f}}).$$

The von Neumann algebra generated by the isomorphic image of A in $\mathcal{B}(\mathcal{H}_{\mathrm{f}})$ is called *G-enveloping von Neumann algebra* of (A, α), and denoted by

$$A''_{\alpha} = \iota_A(A)'' \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{f}}).$$

Remark 2.5.12. Although the G-enveloping von Neumann algebra A''_{α} of a C*-dynamical system (A, α) in Definition 2.5.11 emerged in recent work of Buss–Echterhoff–Willett [26, 27], it coincides with a certain enveloping von Neumann algebra that can be found in Ikunishi's work [97]; cf. [97, Theorem 1]. However, Ikunishi's approach is evidently different. In fact, in his construction, instead of defining A''_{α}, Ikunishi defines its predual as the set of G-continuous linear functionals in A^*.

The crucial feature of the G-enveloping von Neumann algebra A''_{α} of (A, α) is that α naturally extends to a (point-ultraweakly continuous) action $\alpha'' : G \curvearrowright A''_{\alpha}$ by

$$\alpha''_g(a) = \mathrm{Ad}(\iota_G(g))(a), \quad \text{for all } g \in G,\ a \in A''_{\alpha},$$

where $\iota_G : G \to \mathcal{U}(\mathcal{H}_{\mathrm{f}})$ is the canonical map arising from the universal crossed product construction.

Remark 2.5.13. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. One should note that when $\alpha^{**} : G \curvearrowright A^{**}$ is point-ultraweakly continuous, then it follows from [27, Proposition 2.2] that there exists an α^{**}-to-α'' equivariant isomorphism $A^{**} \cong A''_{\alpha}$. In particular, this is the case when $G = \Gamma$ is discrete.

Moreover, by the Stone–von Neumann theorem [178, Section 10.4], which can be proved with Green's theorem (see Remark 2.4.14), one has that $\mathcal{C}_0(G) \rtimes_{\mathtt{lt},\mathrm{f}} G \cong \mathcal{K}(L^2(G))$, and hence $\mathcal{C}_0(G)''_{\mathtt{lt}} \cong L^\infty(G)$. In particular, $\mathtt{lt} : G \curvearrowright \mathcal{C}_0(G)$ is always amenable.

We are now ready to define amenability for general C*-dynamics in the sense of Buss–Echterhoff–Willett, which in light of Remark 2.5.13, generalises Definition 2.5.8

Definition 2.5.14 (see [27, Definition 2.1])**.** We say that an action $\alpha : G \curvearrowright A$ on a C*-algebra A is amenable if the corresponding action on the G-enveloping von Neumann algebra $\alpha'' : G \curvearrowright A''_\alpha$ is amenable in the sense of Definition 2.5.4.

Note that, when G is a dicrete group, $A''_\alpha = A^{**}$ and the definition becomes the same as Anantharaman-Delaroche's given above.

The aforementioned results that an injective von Neumann algebra equipped with an amenable action yields an injective crossed product von Neumann algebra has a clear C*-algebraic analogue.

Theorem 2.5.15. *Let $\alpha : G \curvearrowright A$ be a continuous action on a* C*-*algebra. If α is amenable and A is nuclear, then $A \rtimes_{\alpha,r} G$ is nuclear.*

Remark 2.5.16. If $G = \Gamma$ is discrete, then for any amenable action $\alpha : \Gamma \curvearrowright A$ on a C*-algebra, one has that $A \rtimes_{\alpha,\mathrm{f}} \Gamma \cong A \rtimes_{\alpha,\mathrm{r}} \Gamma$; see [3, Proposition 4.8]. Moreover, in the discrete case, the converse to Theorem 2.5.15 holds as well. In fact, in this case, given a nucelar C*-algebra A the following are equivalent (see [3, Théorème 4.5]):

(i) α is amenable,

(ii) $A \rtimes_{\alpha,\mathrm{f}} \Gamma$ is nuclear,

(iii) $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ is nuclear,

(iv) $A^{**} \rtimes_{\alpha^{**}} \Gamma$ is injective.

We record here a useful application of amenability in the context of nuclear maps on crossed products.

Lemma 2.5.17 (cf. [64, Lemma 6.10])**.** *Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete amenable group on a* C*-*algebra, B a* C*-*algebra, and $\theta : A \rtimes_\alpha \Gamma \to B$ a $*$-homomorphism. Then, θ is nuclear precisely when $\theta \restriction_A$ is nuclear.*

Proof. Using the definition of amenability for groups via Følner sets, one may find a sequence of finite subsets $F_n \subseteq G$ such that

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0, \quad \text{for all } g \in \Gamma.$$

By [24, Lemma 4.2.3], one may find c.p.c. maps $\varphi_n : A \rtimes_\alpha \Gamma \to M_{|F_n|}(A)$ and $\psi_n : M_{|F_n|}(A) \to \mathcal{C}_c(\Gamma, A)$ such that

$$\psi_n \circ \varphi_n(a\lambda_g) = \frac{|gF_n \cap F_n|}{|F_n|} a\lambda_g.$$

In particular, $\psi_n \circ \varphi_n \to \mathrm{id}_{A \rtimes_\alpha \Gamma}$ in the point-norm topology. For this proof, we are only concerned with the maps ψ_n, which are explicitly given as follows. For $a \in A$ and a set of matrix units $e_{g,h} \in M_{|F_n|}$,

$$\psi_n(a \otimes e_{g,h}) = \frac{1}{|F_n|}\alpha_g(a)\lambda_{gh^{-1}} = \frac{1}{|F_n|}\lambda_g a \lambda_h^*.$$

In other words, if one considers the row vector $(\lambda_g)_{g\in F_n}$, the map ψ_n acts on a matrix $(a_{g,h})_{g,h\in F_n} \in M_{|F_n|}(A)$ as

$$(a_{g,h})_{g,h\in F_n} \overset{\psi_n}{\longmapsto} \frac{1}{|F_n|}(\lambda_g)_g(a_{g,h})_{g,h}(\lambda_g)_g^*.$$

Fix now an approximate unit $(e_i)_i \subseteq A$ of A. We define new maps $\psi_{n,i} : M_{|F_n|}(A) \to \mathcal{C}_c(\Gamma, A)$ by

$$(a_{g,h})_{g,h\in F_n} \overset{\psi_{n,i}}{\longmapsto} \frac{1}{|F_n|}(\lambda_g)_g(e_i a_{g,h} e_i)_{g,h}(\lambda_g)_g^* = \frac{1}{|F_n|}(\lambda_g e_i)_g(a_{g,h})_{g,h}(e_i\lambda_g)_g^*.$$

Clearly, also the net of maps$\psi_{n,i} \circ \varphi_n$ point-norm converges to $\mathrm{id}_{A\rtimes_\alpha\Gamma}$. Since, by assumption, $\theta\restriction_A$ is nuclear, its matrix amplification to a map $M_{|F_n|}(A) \to M_{|F_n|}(B)$, which we denote by $\theta^{(n)}$ with slightly abusing of notation, is also nuclear. Then, we have that

$$\begin{aligned}\theta^{(n)} \circ \psi_{n,i}\left((a_{g,h})_{g,h\in F_n}\right) &= \frac{1}{|F_n|}(\theta(\lambda_g e_i))_g\theta((a_{g,h})_{g,h})(\theta(e_i\lambda_g))_g^* \\ &= \frac{1}{|F_n|}\mathrm{Ad}\left((\theta(\lambda_g e_i))_g\right) \circ \theta^{(n)}\left((a_{g,h})_{g,h\in F_n}\right),\end{aligned}$$

is nuclear as well. Hence, one may conclude that $\theta \circ \psi_{n,i} \circ \varphi_n$ is nuclear and point-norm convergent to θ, which is therefore nuclear. $\square$

2.5.1 The quasicentral approximation property

We now continue the history of amenable C*-actions by giving a very short outline of its latest characterisation, namely the so-called quasicentral approximation property. It is thanks to this equivalent notion of amenability that the classification in the following chapter can be achieved in the general setting of second-countable, locally compact group actions. Before proceeding, we introduce some background material, most of which can be found in Lance's textbook [132].

Definition 2.5.18. Let A be a C*-algebra, and $\mathcal{E}$ a linear space equipped with a right A-module structure that is compatible with scalar multiplication (i.e.,

$\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $\lambda \in \mathbb{C}$, $x \in \mathcal{E}$, and $a \in A$), and with an A-valued inner product, namely a map

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to A$$

such that

- $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$ for all $\lambda, \mu \in \mathbb{C}$, and $x, y, z \in \mathcal{E}$;
- $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in \mathcal{E}$ and $a \in A$;
- $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in \mathcal{E}$;
- $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{E}$;
- $\langle x, x \rangle = 0$ if and only if $x = 0$.

The inner product induces a norm on $\mathcal{E}$ given by

$$\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2},$$

which satisfies $\|\langle x, y \rangle\| \leq \|x\|\|y\|$ and $\|xa\| \leq \|x\|\|a\|$ for all $x, y \in \mathcal{E}$, and $a \in A$ (see [132, Proposition 1.1]). If $\mathcal{E}$ is moreover complete with respect to this norm, then it is said to be a *right Hilbert A-module.*

The collection of adjointable (bounded A-linear) maps from $\mathcal{E}$ to itself, denoted by $\mathcal{B}_A(\mathcal{E})$, is a C*-algebra (see [132, Chapter 1]). If A is a right Hilbert module, and there is a $*$-homomorphism $\varphi : A \to \mathcal{B}_A(\mathcal{E})$, which defines a left action of A on $\mathcal{E}$, one says that $\mathcal{E}$ is a Hilbert *A-bimodule.*[33]

The Hilbert modules encountered in this thesis are very basic. Indeed, they will either be Hilbert spaces, which are mutatis mutandis Hilbert $\mathbb{C}$-modules, or the trivial A-module, namely a C*-algebra A with inner product given by $\langle a, b \rangle = a^* b$ for all $a, b \in A$, or modules arising as a combination of these.

Definition 2.5.19. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and $(\mathcal{E}, \varphi)$ a Hilbert A-bimodule. Let $\bar{\alpha} : G \curvearrowright \mathcal{E}$ be a continuous action by linear isometries satisfying

- $\langle \bar{\alpha}_g(x), \bar{\alpha}(y) \rangle = \alpha_g(\langle x, y \rangle)$ for all $g \in G$ and $x, y \in \mathcal{E}$;
- $\bar{\alpha}_g(xa) = \bar{\alpha}_g(x)\alpha_g(a)$ for all $g \in G$, $x \in \mathcal{E}$ and $a \in A$;

[33] As a word of caution, note that the terminology Hilbert bimodules can have a different meaning in the literature. What here is called Hilbert bimodule can sometimes be referred to as a C*-correspondence.

- $\bar{\alpha}_g(\varphi(a)x) = \varphi(\alpha_g(a))\bar{\alpha}_g(x)$ for all $g \in G$, $x \in \mathcal{E}$ and $a \in A$.

Then, $(\mathcal{E}, \varphi)$ is referred to as a *Hilbert* (A, α)*-bimodule.*

Remark 2.5.20. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and $\mathcal{H}$ a Hilbert space. We consider the external tensor product $\mathcal{H} \otimes A$ (see [132, Chapter 4]), where A is viewed as a G-Hilbert bimodule with its identity (A, α)-bimodule structure. Let $\sigma : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation that is continuous with respect to the strong operator topology. It follows that $\sigma \otimes \alpha : G \curvearrowright \mathcal{H} \otimes A$ is an action by linear isometries turning $\mathcal{H} \otimes A$ into a Hilbert (A, α)-bimodule as well.

Notation 2.5.21. Let μ be a left Haar measure on G. We denote the Hilbert spaces $L^2(G, \mu)$ and $\ell^2(\mathbb{N})\hat{\otimes}L^2(G, \mu)$ by $\mathcal{H}_G$ and $\mathcal{H}_G^\infty$, respectively. Moreover, the infinite repeat of the left-regular representation $\lambda : G \to \mathcal{U}(\mathcal{H}_G)$ is denoted by λ^∞, i.e., $\lambda^\infty = \mathbf{1}_{\ell^2(\mathbb{N})} \otimes \lambda : G \to \mathcal{U}(\mathcal{H}_G^\infty)$.

Remark 2.5.22. Consider the left-regular representation $\lambda : G \to \mathcal{U}(\mathcal{H}_G)$. By Fell's absorption principle (see [24, Theorem 2.5.5]), for any other continuous unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$, $\lambda \otimes \rho$ is unitarily equivalent to $\lambda \otimes \mathbf{1}_{\mathcal{H}}$. In particular, this implies that the infinite repeat of the left-regular representation $\lambda^\infty : G \to \mathcal{U}(\mathcal{H}_G^\infty)$ is unitarily equivalent to $\lambda \otimes \lambda$. As a direct consequence, one may conclude that the action induced by λ^∞ on the compact operators via conjugation, i.e., $\mathrm{Ad}(\lambda^\infty) : G \curvearrowright \mathcal{K}(\mathcal{H}_G^\infty)$ is conjugate to $\mathrm{Ad}(\lambda^\infty) \otimes \mathrm{Ad}(\lambda^\infty)$. In other works, the action $\mathrm{Ad}(\lambda^\infty)$ always absorbs itself tensorially.

Definition 2.5.23. Consider the construction described in Remark with $\mathcal{H} = \mathcal{H}_G$ and $\mathcal{H} = \mathcal{H}_G^\infty$, and define the following (A, α)-bimodules

$$L^2(G, A) := \mathcal{H}_G \otimes A, \quad L^2_\infty(G, A) := \mathcal{H}_G^\infty \otimes A,$$

with actions given by $\lambda \otimes \alpha$ and $\lambda^\infty \otimes \alpha$, respectively. One has that $\mathcal{C}_c(G, A)$ is densely contained in $L^2(G, A)$ as a pre-Hilbert module with inner product given by

$$\langle \xi, \eta \rangle = \int_G \xi(g)^* \eta(g)\, d\mu(g),$$

for all $\xi, \eta \in \mathcal{C}_c(G, A)$, and norm given by $\| \cdot \|_2 = \| \langle \cdot, \cdot \rangle \|^{1/2}$. Moreover, if we denote $\bar{\alpha} = \lambda \otimes \alpha$, we have the following,

$$\bar{\alpha}_g(a\xi) = \alpha_g(a)\bar{\alpha}_g(\xi), \quad \bar{\alpha}_g(\xi a) = \bar{\alpha}_g(\xi)\alpha_g(a), \quad \langle \bar{\alpha}_g(\xi), \bar{\alpha}_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle)$$

for all $g \in G$, $a \in A$ and $\xi \in L^2(G, A)$. The restriction of $\bar{\alpha}$ to $\mathcal{C}_c(G, A)$ can be explicitly described by $\bar{\alpha}_g(\xi)(h) = \alpha_g(\xi(g^{-1}h))$ for all $g, h \in G$, and $\xi \in \mathcal{C}_c(G, A)$.

The quasicentral approximation property already appeared, in one form or another, in various works. A partial list includes Suzuki, Buss–Echterhoff–Willett, Bearden–Crann and finally Ozawa–Suzuki's works [11, 12, 27, 154, 181].

Definition 2.5.24. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra. One says that α has the *quasicentral approximation property* (shortened QAP) if there exists a net of contractions $(\zeta_i)_i$ in $L^2(G, A)$ satisfying the following,

$$\langle \zeta_i, \zeta_i \rangle \to \mathbf{1} \text{ strictly}, \qquad \|a\zeta_i - \zeta_i a\|_2 \to 0, \qquad \max_{g \in K} \|(\zeta_i - \bar{\alpha}_g(\zeta_i))a\|_2 \to 0$$

for all $a \in A$ and compact sets $K \subseteq G$.

Remark 2.5.25. Note that in the setting of Definition 2.5.24, one could replace the net $(\zeta_i)_i$ in $L^2(G, A)$ with one in $\mathcal{C}_c(G, A)$ by a density argument.

Theorem 2.5.26 (see [154, Theorems 3.2])**.** *Let $\alpha : G \curvearrowright A$ be an action on a* C**-algebra. Then the following are equivalent,*

(i) *α has the QAP,*

(ii) *α is amenable,*

(iii) *there exists an equivariant conditional expectation of (algebraic) von Neumann dynamical systems*[34]

$$(L^\infty(G)\bar{\otimes}Z(A^{**}), \mathtt{lt} \otimes \alpha^{**}) \to (Z(A^{**}), \alpha^{**}).$$

Remark 2.5.27. Amenability has the following permanence properties:

(i) A G-inductive limit of C*-algebras equipped with amenable actions is equipped with an amenable action.

(ii) The restriction of an amenable action to a G-invariant hereditary C*-subalgebra is amenable.

(iii) The extension of an amenable G-C*-algebra by an amenable G-C*-algebra is amenable.

(iv) A quotient of amenable G-C*-algebras is amenable.

(v) The image of an amenable G-C*-algebra under a G-conditional expectation is again an amenable G-C*-algebra.

[34] Recall that $Z(A^{**})$ denotes the center of the von Neumann algebra A^{**}, i.e., $A^{**} \cap (A^{**})'$, and that $\mathtt{lt}$ is the left-shift action of G on $L^\infty(G)$.

2.5.2 Exact groups

In this section we define a well-studied class of groups, called *exact*, which was introduced by Kirchberg and S. Wassermann. They consider a C*-dynamical system (A, α) equipped with an α-invariant ideal $I \subseteq A$. Functoriality of the reduced crossed product induces two maps

$$I \rtimes_{\alpha,\mathrm{r}} G \xrightarrow{\iota} A \rtimes_{\alpha,\mathrm{r}} G \xrightarrow{\pi} (A/I) \rtimes_{\alpha,\mathrm{r}} G,$$

where we denote α also the actions induced on I and A/I. One has that ι is injective, π surjective, and $\mathrm{im}(\iota) \subseteq \ker(\pi)$. However, the latter need not be an equality. This motivated Kirchberg–Wassermann to give the following definition.

Definition 2.5.28 (see [124, Introduction])**.** A locally compact group G is said to be *exact* if the reduced crossed product functor is *short-exact*, i.e., any equivariant short exact sequence of G-C*-algebras

$$0 \longrightarrow (A, \alpha) \longrightarrow (B, \beta) \longrightarrow (C, \gamma) \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow A \rtimes_{\alpha,\mathrm{r}} G \longrightarrow B \rtimes_{\beta,\mathrm{r}} G \longrightarrow C \rtimes_{\gamma,\mathrm{r}} G \longrightarrow 0.$$

Note that the full crossed product functor is always short-exact thanks to the universal property. Amenable groups are therefore always exact, but the converse is not true. For instance, the free group on two generators $\mathbb{F}_2$ is exact, but is not amenable.

Remark 2.5.29. It should be mentioned that, before Kirchberg–Wassermann, Ng defined a group G to be exact when its reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ is exact; see [152]. One may show that any exact group is also exact in the sense of Ng by equipping a short exact sequence of C*-algebras with the trivial G-action. The converse also holds if $G = \Gamma$ is discrete (see [24, Theorem 5.1.10]), while it is not known whether it is true for general locally compact groups.

The following amenability condition was introduced by Anantharaman-Delaroche:

Definition 2.5.30 (see [3, Definition 3.1])**.** A locally compact group G is said to be *amenable at infinity* if it admits an amenable action on a compact space.

A good way to describe the following result is by the sententious statement: Groups, like people, are judged by their actions.

Theorem 2.5.31 (see [154, Proposition 2.4] and [21, Theorem 5.8]). *A locally compact group G is exact if and only if it is amenable at infinity.*

Moreover, Ozawa and Suzuki establish the following obstruction (cf. [4]), which implies that Theorem 2.5.31 has a C^*-algebraic analogue.

Corollary 2.5.32 (see [154, Corollary 3.6]). *If G admits an amenable action on a unital C^*-algebra, then G is exact.*

Moreover, Ozawa–Suzuki show the following further characterisation of amenable action of exact groups.

Theorem 2.5.33 (see [154, Theorem 4.4]). *Assume G is exact,*[35] *and let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra. Then α is amenable if and only if, for every α_∞-invariant separable C^*-subalgebra $B \subseteq A_{\infty,\alpha}$,the action $\tilde{\alpha}_\infty : G \curvearrowright F(B, A_{\infty,\alpha})$ is amenable.*

2.6 Strongly self-absorbing C^*-dynamics

Strongly self-absorbing C^*-algebras were formally introduced, and systematically studied, by Toms and Winter in [200].

Definition 2.6.1 (see [200, Definition 1.3]). Let $\mathcal{D}$ be a separable unital C^*-algebra. The C^*-algebra $\mathcal{D}$ is said to be *strongly self-absorbing* if $\mathcal{D} \ncong \mathbb{C}$ and there exists an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent[36] to the first factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}, \qquad\qquad d \mapsto d \otimes \mathbf{1}_{\mathcal{D}}.$$

Moreover, one says that a C^*-algebra A is *$\mathcal{D}$-stable* (or *$\mathcal{D}$-absorbing*) if there exists an isomorphism $A \cong A \otimes \mathcal{D}$.

We turn directly to strongly self-absorbing C^*-dynamical systems, which generalise the situation above to the level of group actions. We remark here that the theory of strongly self-absorbing C^*-dynamics that will be presented below was first developed in the non-dynamical setting [200]. We will recall several results (without proof) from [185, 187, 188] that can be considered dynamical versions of foundational results from [200], which we do not report here.

[35] Here, the assumption that G is exact in the previous theorem is necessary to have an equivalence. Indeed, $\tilde{\mathtt{lt}}_\infty : G \curvearrowright F_{\infty,\mathtt{lt}}(\mathcal{C}_0(G))$ is not amenable when G is not exact; see [154, Remark 4.7].

[36] Approximate unitary equivalence for $*$-homomorphisms is defined analogously as for cocycle morphisms, Definition 2.2.23, where one has $G = \{\mathbf{1}\}$.

Definition 2.6.2 (see [187, Definition 3.1] and [192, Definition 5.3])**.** Let $\delta : G \curvearrowright \mathcal{D}$ be an action on a separable unital C*-algebra with $\mathcal{D} \not\cong \mathbb{C}$. The action δ is said to be *strongly self-absorbing* if there exists a cocycle conjugacy

$$(\varphi, \mathbb{u}) : (\mathcal{D}, \delta) \to (\mathcal{D} \otimes \mathcal{D}, \delta \otimes \delta)$$

that is properly approximately unitarily equivalent to the equivariant first factor embedding

$$\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : (\mathcal{D}, \delta) \to (\mathcal{D} \otimes \mathcal{D}, \delta \otimes \delta).$$

An action $\alpha : G \curvearrowright A$ on a C*-algebra is said to be *δ-stable*, or *δ-absorbing*, if α is cocycle conjugate to $\alpha \otimes \delta$.

Remark 2.6.3. We want to clarify here that the Definition of strongly self-absorbing action used in [185, 187, 188] is not the same defined here, which was introduced in this precise form in [192]. In the aforementioned former works, in fact, strong self-absorption referred to a more restrictive definition, which we do not consider here. This was studied side by side with the weaker notion of *semi-strong self-absorption.* However, the characterisation of semi-strong self-absorption given in [187, Theorem 4.6] compared with the characterisation of strong self-absorption as defined here established in [192, Proposition 5.5] shows that they are equivalent. (The aforementioned characterisation is recalled below as Theorem 2.6.7.)

Definition 2.6.4. Let D be a unital, separable C*-algebra equipped with an action $\delta : G \curvearrowright D$. The action δ is said to have *approximately G-inner flip* if there exists a sequence of unitaries $u_n \in \mathcal{U}(D \otimes D)$ such that

$$\|u_n(d_1 \otimes d_2)u_n^* - d_2 \otimes d_1\| \xrightarrow{n\to\infty} 0$$

for all $d_1, d_2 \in D$, and

$$\max_{g \in K} \|(\delta \otimes \delta)_g(u_n) - u_n\| \xrightarrow{n\to\infty} 0$$

for all compact subsets $K \subseteq G$.

The action δ is said to have *approximately G-inner half-flip* if there exists a sequence of unitaries $u_n \in \mathcal{U}(D \otimes D)$ such that

$$\|u_n(d \otimes \mathbf{1}_D)u_n^* - \mathbf{1}_d \otimes d\| \xrightarrow{n\to\infty} 0$$

for all $d \in D$, and

$$\max_{g \in K} \|(\delta \otimes \delta)_g(u_n) - u_n\| \xrightarrow{n\to\infty} 0$$

for all compact subsets $K \subseteq G$.

Analogously, the C*-algebra D is said to have *approximately inner (half-)flip* if it has *approximately G-inner (half-)flip* with $G = \{1\}$.

The following is a standard definition, and can be found in [13] along with a generalised version.

Definition 2.6.5. Let A be a unital C*-algebra. In this definition we only consider spatial tensor products. The *infinite tensor product* of A, denoted by $A^{\otimes\infty}$ or $\bigotimes_{\mathbb{N}} A$, is the C*-algebra arising as the inductive limit of the inductive system $\{A_n, \varphi_n\}_{n\in\mathbb{N}}$ given by

$$A_n := A^{\otimes n}, \quad \varphi_n := \mathrm{id}_{A_n} \otimes \mathbf{1}_A$$

for all $n \in \mathbb{N}$, where $A^{\otimes n}$ is the n-fold tensor product of A.

Proposition 2.6.6 (see [188, Proposition 3.3])**.** *Let D be a separable, unital* C**-algebra equipped with an action $\delta : G \curvearrowright D$ that has approximately G-inner half-flip. Then the action $\delta^{\otimes\infty} : G \curvearrowright D^{\otimes\infty}$ is strongly self-absorbing.*

Starting from the seminal work of Kirchberg [117], sequence algebras have been occupying a pivotal role in the theory of strongly self-absorbing C*-algebras [200]. The unfolding of the analogous theory for C*-dynamics [185,187,188] has brought to light that certain sequence algebras with additional dynamical structure are equally important for characterising strongly self-absorbing actions on C*-algebras. Here, we will recall the basic dictionary relevant for this topic, and a small collection of results on the theory of strongly self-absorbing C*-dynamics.

Theorem 2.6.7 (see [187, Theorem 4.6])**.** *Let $\mathcal{D}$ be a separable, unital* C**-algebra equipped with an action $\delta : G \curvearrowright \mathcal{D}$. The following are equivalent:*

(i) *δ is strongly self-absorbing;*

(ii) *δ has approximately G-inner half-flip and there exists a unital, equivariant $*$-homomorphism $(\mathcal{D}, \delta) \to (\mathcal{D}_{\infty,\delta} \cap \mathcal{D}', \delta_\infty)$;*

(iii) *δ has approximately G-inner half-flip and is cocycle conjugate to $(\mathcal{D}^{\otimes\infty}, \delta^{\otimes\infty})$.*

One can conclude the following as a consequence of [188, Theorem 4.7 and Lemma 2.12].

Theorem 2.6.8. *Let $\alpha : G \curvearrowright A$ be an action on a separable* C**-algebra, and $\delta : G \curvearrowright \mathcal{D}$ a strongly self-absorbing action on a separable unital* C**-algebra. Then the following are equivalent,*

(i) *α is δ-stable,*

(ii) *there exists an equivariant unital* $*$*-homomorphism*

$$(\mathcal{D}, \delta) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty),$$

(iii) *for every separable* α_∞*-invariant* C^**-subalgebra* $C \subseteq A_{\infty,\alpha}$*, there exists an equivariant unital* $*$*-homomorphisms*

$$(\mathcal{D}, \delta) \to (F(C, A_{\infty,\alpha}), \tilde{\alpha}_\infty).$$

Remark 2.6.9. Let A be a C^*-algebra, and $B \subseteq A_\infty$ a C^*-subalgebra. Then, by viewing B as a C^*-subalgebra of $\mathcal{M}(A)_\infty$, it is clear that

$$A_\infty \cap B' = A_\infty \cap (\mathcal{M}(A)_\infty \cap B^\perp),$$

and that

$$A_\infty \cap B^\perp = A_\infty \cap (\mathcal{M}(A)_\infty \cap B^\perp).$$

This means that we have a natural embedding $F(B, A_\infty) \hookrightarrow F(B, \mathcal{M}(A)_\infty)$. The same map becomes an isomorphism if B is σ-unital as explained in [187, Remark 1.11].

Theorem 2.6.10. *Let* $\alpha : G \curvearrowright A$ *be an action on a separable* C^**-algebra, and* $\gamma : G \curvearrowright \mathcal{D}$ *a strongly self-absorbing action on a separable unital* C^**-algebra. The following are equivalent:*

(i) α *is* γ*-absorbing,*

(ii) *there exists a sequence of unital* $*$*-homomorphisms* $\varphi_n : \mathcal{D} \to \mathcal{M}(A)$ *such that*

- $\|[\varphi_n(d), a]\| \xrightarrow{n\to\infty} 0$ *for all* $a \in A$ *and* $d \in \mathcal{D}$*, and*
- $\alpha_g(\varphi_n(d)) - \varphi_n(\gamma_g(d)) \xrightarrow{n\to\infty} 0$ *uniformly on compact subsets of* G *in the strict topology for all* $d \in \mathcal{D}$*.*

(iii) *there exists an equivariant unital* $*$*-homomorphism*[37]

$$\varphi : (\mathcal{D}, \gamma) \to (F(A, \mathcal{M}(A)_\infty), \tilde{\alpha}_\infty).$$

Having recalled some of the fundamental theorems regarding strongly self-absorbing (G-)C^*-algebras, we give brief overview of known examples, starting with strongly self-absorbing C^*-algebras.

[37]Here, note that $\tilde{\alpha}_\infty$ only induces an algebraic action on $F(A, \mathcal{M}(A)_\infty)$. However, the range of φ will inevitably lie in the continuous part of $F(A, \mathcal{M}(A)_\infty)$.

Example 2.6.11. All UHF algebras of infinite type (see 2.1.9) are strongly self-absorbing. This is because the flip automorphism on a matrix algebra of any size is inner, and hence that UHF algebras have approximately inner flip. Being of infinite type is the necessary condition that allows a UHF algebra to be isomorphic to its infinite tensor product. Hence, by Theorem 2.6.7, they are strongly self-absorbing.

Example 2.6.12. The Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$ are strongly self-absorbing C*-algebras. However, it is a lot harder to show this if compared with the previous example. Let us give an idea of how one can prove this rather non-trivial fact.

In the case of $\mathcal{O}_2$, Elliott's theorem that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ [169] mentioned earlier yields that $\mathcal{O}_2^{\otimes\infty} \cong \mathcal{O}_2$ (see [173, Corollary 5.2.4]). Then, a theorem of Rørdam [168, Theorem 3.6] (see also [167, Theorem 5.1.1]) shows that any pair of unital $*$-homomorphisms from $\mathcal{O}_2$ into any unital, simple, purely infinite C*-algebra are approximately unitarily equivalent. Hence, $\mathcal{O}_2$ has approximately inner half-flip, and Theorem 2.6.7 allows one to conclude that $\mathcal{O}_2$ is strongly self-absorbing.

The proof for $\mathcal{O}_\infty$ can be outlined as follows. Again, one has that any two unital $*$-homomorphisms from $\mathcal{O}_\infty$ into some unital, simple, purely infinite C*-algebra are approximately unitarily equivalent, a result that is proved in [158, Theorem 3.3]. Hence, $\mathcal{O}_\infty$ has approximately inner half-flip. Then, by [119, Proposition 3.4], one has that for any unital Kirchberg algebra A, for any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$ the C*-algebra $A_\mathcal{U} \cap A'$ is unital, simple and purely infinite. Hence, the unit of $A_\mathcal{U} \cap A'$ is properly infinite, which entails that there exists a unital embedding $\mathcal{O}_\infty \hookrightarrow A_\mathcal{U} \cap A'$ (see [173, Proposition 1.1.2]). Hence, choosing $A = \mathcal{O}_\infty$ shows that $\mathcal{O}_\infty$ fits into the scope of Theorem 2.6.7, and is therefore strongly self-absorbing.

Remark 2.6.13. Another known example of strongly self-absorbing C*-algebra is the Jiang–Su algebra $\mathcal{Z}$. Its definition is more complex than that of the other examples, and as it will not feature in this thesis, we omit it. As a consequence of the groundbreaking work of Tikuisis, White and Winter (see [198, Corollary 6.7]), we know that all the strongly self-absorbing C*-algebras satisfying the Universal Coefficients Theorem are, up to isomorphism, $\mathcal{Z}$, the UHF algebras of infinite type, $\mathcal{O}_2$, $\mathcal{O}_\infty$, and tensor products of $\mathcal{O}_\infty$ with UHF-algebras of infinite type. It should be mentioned that the last three were already known to be the only purely infinite ones by [200, Corollary 5.2].

The first trivial, although important, example of strongly self-absorbing action is the following.

Example 2.6.14. Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. The trivial action $\mathrm{id}_\mathcal{D} : G \curvearrowright \mathcal{D}$ is strongly self-absorbing. This is because by strong self-absorption of $\mathcal{D}$, there exists an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ that is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$, and both maps are automatically equivariant with respect to $\mathrm{id}_{\mathcal{D}}$.

The following definition is a special case of Definition 2.6.2, and is the quintessential assumption of this work.

Definition 2.6.15. Let $\alpha : G \curvearrowright A$ be an action on a C*-algebra, and $\mathcal{D}$ a separable, unital strongly self-absorbing C*-algebra. The action α is said to be *equivariantly $\mathcal{D}$-stable* if it is $\mathrm{id}_{\mathcal{D}}$-stable.

Remark 2.6.16. It will be silently assumed at various places throughout the exposition that Definition 2.6.15, in the absence of dynamics — namely when $G = \{1\}$ — coincides with $\mathcal{D}$-stability (see Definition 2.6.1). Let us unpack this statement here, once and for all. Suppose that a C*-algebra A is equipped with the only action α of $G = \{1\}$, i.e., $\alpha = \mathrm{id}_A$. Then, α is equivariantly $\mathcal{D}$-stable precisely when there exists a cocycle conjugacy $(\varphi, \mathbb{u}) : (A, \alpha) \to (A \otimes \mathcal{D}, \alpha \otimes \mathrm{id}_{\mathcal{D}})$. Note that since $G = \{1\}$, the cocycle must be trivial: $\mathbb{u} = \mathbf{1}$. Hence the cocycle conjugacy above is nothing but an isomorphism $\varphi : A \to A \otimes \mathcal{D}$. This means that the action of $G = \{1\}$ is equivariantly $\mathcal{D}$-stable exactly when A is $\mathcal{D}$-stable. As a result, we will be able to apply key results such as Theorem 2.6.8 to $\mathcal{D}$-stable C*-algebras in absence of dynamics, with the appropriate adjustments in the statement.

There are many more examples of strongly self-absorbing actions. We give an abridged overview of these, starting with a proposition from [188] that gives access to a wide class of examples.

Proposition 2.6.17 (see [188, Proposition 6.3])**.** *Let $\mathcal{D}$ be a separable, unital C*-algebra with approximately inner flip, and $u : G \to \mathcal{U}(\mathcal{D})$ a continuous unitary representation of G. Then the action*

$$\bigotimes_{\mathbb{N}} \mathrm{Ad}(u) : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{D}$$

is strongly self-absorbing.

Proof. Let $v_n \in \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ be a sequence of unitaries that implements the approximate unitary equivalence between the flip automorphism on $\mathcal{D} \otimes \mathcal{D}$ and $\mathrm{id}_{\mathcal{D} \otimes \mathcal{D}}$. For any compact subset $K \subseteq G$, the set

$$\{u_g \otimes u_g \mid g \in K\} \subseteq \mathcal{D} \otimes \mathcal{D}$$

is compact as well. In particular, for every $\varepsilon > 0$, we may find finitely many group elements $\{g_i\}_{i=1}^k \subseteq K$ such that for every $g \in K$, $u_g \otimes u_g =_\varepsilon u_{g_i} \otimes u_{g_i}$ for some i. By assumption, we may find $n \in \mathbb{N}$ such that

$$\max_{1 \le i \le k} \|v_n (u_{g_i} \otimes u_{g_i}) v_n^* - u_{g_i} \otimes u_{g_i}\| \le \varepsilon.$$

It follows that

$$\begin{aligned}\max_{g\in K}\|\mathrm{Ad}(u_g\otimes u_g)(v_n)-v_n\| &= \max_{g\in K}\|v_n(u_g\otimes u_g)v_n^*-u_g\otimes u_g\|\\ &\le 2\varepsilon+\max_{1\le i\le k}\|v_n(u_{g_i}\otimes u_{g_i})v_n^*-u_{g_i}\otimes u_{g_i}\|\\ &\le 3\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we have that

$$\max_{g\in K}\|\mathrm{Ad}(u_g\otimes u_g)(v_n)-v_n\| \xrightarrow{n\to\infty} 0$$

and thus $\mathrm{Ad}(u):G\curvearrowright\mathcal{D}$ has approximately G-inner flip, and in particular also approximately G-inner half-flip. Finally, by Proposition 2.6.6, the action $\bigotimes_{\mathbb{N}}\mathrm{Ad}(u)$ is strongly self-absorbing. □

Example 2.6.18 (see [187, Example 5.3]). Assume Γ is finite. Let $n\in\mathbb{N}$, $\sigma:\Gamma\curvearrowright\{1,\dots,n\}$ an action, and $\mathfrak{n}$ any supernatural number of infinite type. Then σ induces an action $\beta^\sigma:\Gamma\curvearrowright M_{\mathfrak{n}}^{\otimes n}$ given by shifting tensor factors as follows,

$$\beta^\sigma_g(a_1\otimes\cdots\otimes a_n)=a_{\sigma_g(1)}\otimes\cdots\otimes a_{\sigma_g(n)}$$

for all $g\in\Gamma$ and $a_1\otimes\cdots\otimes a_n\in M_{\mathfrak{n}}^{\otimes n}$. The action β^σ is strongly self-absorbing.

The following notions are at the core of many classification results in C*-dynamics. However, they will never be assumed in results contained in this thesis, which is why we do not recall prominently any of the classification results involving them.

Definition 2.6.19. Let A be a separable C*-algebra.

- An automorphism $\alpha\in\mathrm{Aut}(A)$, or its induced $\mathbb{Z}$-action, is said to have the *Rokhlin property* if there is a partition of $\mathbf{1}\in F_\infty(A)$ into projections $e_0,\dots,e_{n-1},f_0,\dots,f_n\in F_\infty(A)$ such that $\tilde{\alpha}_\infty(e_j)=e_{j+1}$ for $j<n-1$ and $\tilde{\alpha}_\infty(f_j)=f_{j+1}$ for all $j<n$. This traces back to [127] (see also [95, Definition 4.1]).

- A compact group action $\alpha:G\curvearrowright A$ is said to have the *Rokhlin property* if there exists a unital equivariant embedding

  $$(\mathcal{C}(G),\mathtt{lt})\hookrightarrow(F_{\infty,\alpha}(A),\tilde{\alpha}_\infty).$$

  When A is unital, this is [95, Definition 3.2]; for the non-unital case, see [69, Definition 2.2].

- A *flow*, i.e., an action of the real numbers $\alpha : \mathbb{R} \curvearrowright A$ is said to be a *Rokhlin flow* if for every $p > 0$, there exists a unitary $u \in F_{\infty,\alpha}(A)$ such that $\tilde{\alpha}_\infty(u) = \exp(ipt)u$ for all $t \in \mathbb{R}$. This originated from [126] (see also [95, Definition 5.1]) in the unital setting; see [94] for its non-unital counterpart.

The Rokhlin property in the compact group setting grants access to strong self-absorption as explained below.

Notation 2.6.20. Let $\alpha : G \curvearrowright A$ be an action of a compact group on a C^*-algebra. We denote by $\alpha_{\mathrm{co}} : A \to \mathcal{C}(G, A) \cong \mathcal{C}(G) \otimes A$ the $*$-homomorphism induced the action α, namely

$$\alpha_{\mathrm{co}}(a)(g) = \alpha_g(a) \quad \text{for all } a \in A,\, g \in G.$$

The following result follows from [9, Theorem 5.10] (see also [98, Theorem 3.5], [150, Theorem 3.5] and [75, Theorem 3.4] for finite group actions).

Theorem 2.6.21. *Assume G is compact, and let $\alpha, \beta : G \curvearrowright A$ be actions with the Rokhlin property on a separable* C^**-algebra. Then $\alpha_{\mathrm{co}} \approx_{\mathrm{u}} \beta_{\mathrm{co}}$ if and only if there exists a conjugacy $\varphi : (A, \alpha) \to (B, \beta)$ that is approximately inner as an automorphism of A.*

Lemma 2.6.22 (see [187, Example 5.1])**.** *Assume G is compact. Then, every Rokhlin action on a strongly self-absorbing* C^**-algebra is strongly self-absorbing.*

Proof. Assume $\alpha : G \curvearrowright \mathcal{D}$ is a compact group action on a strongly self-absorbing C^*-algebra. First notice that, by [200, Corollary 1.12], two $*$-homomorphisms $\mathcal{D} \to A \otimes \mathcal{D}$ are approximately unitarily equivalent for any separable, unital C^*-algebra A. Consider the action β on $\mathcal{D}$ induced by $\alpha \otimes \alpha : G \curvearrowright \mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}$, which is therefore conjugate to $\alpha \otimes \alpha$ by construction. It follows that the $*$-homomorphisms $\alpha_{\mathrm{co}}, \beta_{\mathrm{co}} : \mathcal{D} \to \mathcal{C}(G) \otimes \mathcal{D}$ are approximately unitarily equivalent. Now, since α (and hence β) have the Rokhlin property, the fact that $\alpha_{\mathrm{co}} \approx_{\mathrm{u}} \beta_{\mathrm{co}}$ implies that $\alpha \simeq_{\mathrm{cc}} \beta$ by Theorem 2.6.21. Hence, there exists a conjugacy $\varphi : (\mathcal{D}, \alpha) \to (\mathcal{D} \otimes \mathcal{D}, \alpha \otimes \alpha)$, which is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ when viewed as $*$-homomorphisms by strong self-absorption of $\mathcal{D}$. Then, again using that α has the Rokhlin property, we may conclude with [9, Corollary 5.9] that φ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ as equivariant maps, and hence α is strongly self-absorbing. $\square$

In the specific case of $\mathcal{O}_2$, we present some more examples.

Example 2.6.23. If G is finite and one defines the shift-action $\beta^\sigma : G \curvearrowright \mathcal{O}_2^{\otimes n}$ induced by a G-action σ on n-points analogously as in Example 2.6.18, then β^σ

has the Rokhlin property by [98, Remark 5.4]. By the previous lemma, β^σ is also strongly self-absorbing.

Example 2.6.24 (see [72, Examples 2.7 and 2.9]). Assume G is compact, and consider the inductive system given by $A_n := \mathcal{C}(G) \otimes M_{2^n}$ with action $\alpha^{(n)} := \mathtt{lt} \otimes \mathrm{id}_{M_{2^n}} : G \curvearrowright A_n$, and connecting maps defined as follows. Fix a countable dense subset $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ such that every tail is dense, i.e., $\{g_n\}_{n\geq m}$ is dense in G for all $m \in \mathbb{N}$. Then, the (unital, injective) connecting maps $\varphi_n : A_n \to A_{n+1}$ are given by

$$\varphi_n(f) = \begin{pmatrix} f & 0 \\ 0 & \mathtt{rt}_{g_n}(f) \end{pmatrix} \quad \text{for all } f \in A_n.$$

The resulting inductive limit C*-algebra $A = \varinjlim(A_n, \varphi_n)$ is clearly an AH-algebra, and it is moreover simple by [46, Proposition 2.1]. Moreover, the connecting maps φ_n are equivariant, and thus induce an action α on A. Now, by the $\mathcal{O}_2$-absorption theorem one may find an isomorphism $\psi : A \otimes \mathcal{O}_2 \to \mathcal{O}_2$, and use it to define an action on $\mathcal{O}_2$ by

$$\beta : G \curvearrowright \mathcal{O}_2, \quad \beta_g = \varphi \circ (\alpha_g \otimes \mathrm{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$$

for all $g \in G$. Since the actions $\alpha^{(n)}$ clearly have the Rokhlin property, one may infer by [71, Theorem 3.8] and [70, Proposition 5.1] (see also [72, Theorem 2.5]) that α has the Rokhlin property. From the same result, one may conclude that β has the Rokhlin property as well.

Example 2.6.25. Fix an irrational number $\lambda > 0$, and let $\gamma : \mathbb{R} \curvearrowright \mathcal{O}_2$ be the action defined on the generators by $\gamma_t(s_1) = \exp(it)s_1$ and $\gamma(s_2) = \exp(-i\lambda t)s_2$. Then γ is strongly self-absorbing as shown in [187, Example 5.2]. Actually, in [187] Szabó observes that this is a special case of a family of strongly self-absorbing actions, i.e., so-called *quasi-free Rokhlin flows* on $\mathcal{O}_2$; cf. [126, 128].

Example 2.6.26. Assume Γ is a countable, discrete, exact group. Then, by the $\mathcal{O}_2$-embedding theorem (see [119]) there exists an embedding $\mathrm{C}^*_{\mathrm{r}}(G) \hookrightarrow \mathcal{O}_2$. As a result, this induces a unitary representation $u : G \to \mathcal{U}(\mathcal{O}_2)$. Hence, the action on $\mathcal{O}_2$ given by

$$\delta : G \curvearrowright \bigotimes_{\mathbb{N}} \mathcal{O}_2 \cong \mathcal{O}_2, \quad \delta_g = \bigotimes_{\mathbb{N}} \mathrm{Ad}(u_g)$$

is strongly self-absorbing by Proposition 2.6.17. Observe that δ is also pointwise outer because the representation u is faithful and, when G is finite, it satisfies the Rokhlin property by [98, 4.2].

Finally, we recall the following characterisation of equivariant $\mathcal{O}_2$-stability, which will be important in the sequel.

Theorem 2.6.27 (see [192, Theorem 5.6])**.** *Let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra. Then, α is equivariantly $\mathcal{O}_2$-stable if and only if the equivariant first factor embedding*

$$\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{O}_2} : (A, \alpha) \to (A \otimes \mathcal{O}_2, \alpha \otimes \mathrm{id}_{\mathcal{O}_2})$$

is properly approximately unitarily equivalent to a proper cocycle conjugacy.

2.7 Isometric shift-absorption

Isometric shift-absorption was recently introduced by Gabe and Szabó [67] to provide the right outerness-type condition for actions on purely infinite C*-algebras.

Definition 2.7.1 (see [67, Definition 3.7])**.** Let $\alpha : G \curvearrowright A$ be an action on a separable C*-algebra. We say that α is *isometrically shift-absorbing* if there exists an equivariant linear map

$$\mathfrak{s} : (\mathcal{H}_G, \lambda) \to (F_{\infty,\alpha}(A), \tilde{\alpha}_\infty)$$

that satisfies $\mathfrak{s}(\xi)^*\mathfrak{s}(\eta) = \langle \xi, \eta \rangle \cdot \mathbf{1}$ for all $\xi, \eta \in \mathcal{H}_G$.

It was observed by Gabe and Szabó [67] that equivariant $\mathcal{O}_\infty$-stability and isometric shift-absorption are related notions.

Proposition 2.7.2 (see [67, Proposition 3.9])**.** *Assume $G \neq \{1\}$, and let $\beta : G \curvearrowright B$ be an action on a separable C^*-algebra. If β is amenable and isometrically shift-absorbing, then it is equivariantly $\mathcal{O}_\infty$-stable.*

The following result clarifies the relation between isometric shift-absorption and outerness. We would like to point out that the same result for unital Kirchberg algebras was already proved in [102, Lemma 6.2].

Proposition 2.7.3 (see [67, Proposition 3.15])**.** *Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete group on a Kirchberg algebra.*[38] *Then α is isometrically shift-absorbing if and only if it is pointwise outer.*

In general, however, isometric shift-absorption is stronger than (pointwise) outerness. Consider the following example.

[38] Recall that a C*-algebra is said to be a Kirchberg algebra if it is separable, nuclear, simple and purely infinite.

Example 2.7.4. Consider the *gauge action* on $\mathcal{O}_2$, namely the circle action $\delta : \mathbb{T} \curvearrowright \mathcal{O}_2$ defined by $\delta_z(s_i) = zs_i$ for $z \in \mathbb{T}$, where s_i for $i = 1, 2$ are Cuntz isometries generating $\mathcal{O}_2$. It is a well-known fact that the fixed point algebra $\mathcal{O}_2^\delta$ is the UHF-algebra with supernatural number $\mathfrak{n} = 2^\infty$, i.e., the *CAR algebra* M_{2^∞}. The action δ is pointwise outer, but it fails to be isometrically shift-absorbing. This is because isometric shift-absorption of δ, together with amenability of $\mathbb{T}$ and Proposition 2.7.2, would imply that $\delta \simeq_{\mathrm{cc}} \delta \otimes \mathrm{id}_{\mathcal{O}_\infty}$, and therefore that $\mathcal{O}_2^\delta \cong M_{2^\infty}$ is $\mathcal{O}_\infty$-stable (as it embeds in the crossed product of $\mathcal{O}_2$ by an equivariantly $\mathcal{O}_\infty$-stable action δ as a corner). The latter is clearly a contradition because, for instance, the CAR algebra has a (unique) trace.

Remark 2.7.5. It was shown in [67, Corollary 6.15] that when $G = \mathbb{R}$, an action on a separable $\mathcal{O}_\infty$-stable C*-algebra is isometrically shift-absorbing if and only if it is a Rokhlin flow in the sense of Definition 2.6.19.

There is a deeper connection between $\mathcal{O}_\infty$ and isometric shift-absorption, as hinted at in the previous remark. We will give an overview of their relation in the remaining part of this section.

Remark 2.7.6 (see [67, Remark 3.1]). Recall [56] that for any separable, infinite-dimensional Hilbert space $\mathcal{H}$, the Cuntz algebra $\mathcal{O}_\infty$ is isomorphic to $\mathcal{O}_\mathcal{H}$, the universal unital C*-algebra generated by the range of a linear map $\mathfrak{s} : \mathcal{H} \to \mathcal{O}_\mathcal{H}$ subject to the relation $\mathfrak{s}(\xi)^*\mathfrak{s}(\eta) = \langle \xi, \eta \rangle \cdot \mathbf{1}$ for all $\xi, \eta \in \mathcal{H}$. As a consequence, every unitary U on $\mathcal{H}$ gives rise to a unique automorphism on $\mathcal{O}_\mathcal{H}$ such that $\mathfrak{s}(\xi)$ is sent to $\mathfrak{s}(U\xi)$ for all $\xi \in \mathcal{H}$. The resulting assignment $\mathcal{U}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{O}_\mathcal{H})$ is a group homomorphism that is continuous with respect to the strong operator topology on $\mathcal{U}(\mathcal{H})$ and the point-norm topology on $\mathrm{Aut}(\mathcal{O}_\mathcal{H})$. Any group action on $\mathcal{O}_\infty$ that is conjugate to one factoring through this homomorphism is said to be *quasi-free.*

Definition 2.7.7 (see [67, Definition 3.4]). Define $\gamma : G \curvearrowright \mathcal{O}_\infty \cong \mathcal{O}_{\mathcal{H}_G^\infty}$ to be the quasi-free action determined by $\gamma_g \circ \mathfrak{s} = \mathfrak{s} \circ \lambda_g^\infty$ for all $g \in G$.

The following proposition tells us that isometric shift-absorption is equivalent to approximately absorbing a certain dynamical system on the Cuntz algebra $\mathcal{O}_\infty$, and will be a key ingredient in Section 3.4.

Proposition 2.7.8 (see [67, Proposition 3.8]). *Assume that $G \neq \{1\}$.*[39] *Let $\beta : G \curvearrowright B$ be an action on a separable* C**-algebra, and $\gamma : G \curvearrowright \mathcal{O}_\infty$ the quasi-free action from Definition 2.7.7. The following are equivalent.*

(i) *The action β is isometrically shift-absorbing.*

[39] As Gabe and Szabó notice in [67], this assumption makes the statement cleaner. In fact, for the trivial group one would have to adjust the statement due to the fact that in that case isometric shift-absorption is a vacuous condition.

(ii) *There exists an equivariant linear map*

$$\mathfrak{s} : (\mathcal{H}_G^\infty, \lambda^\infty) \to (F_{\infty,\beta}(B), \tilde{\beta}_\infty)$$

such that $\mathfrak{s}(\xi)^*\mathfrak{s}(\eta) = \langle \xi, \eta \rangle \cdot \mathbf{1}$ *for all* ξ *and* η *in* $\mathcal{H}_G^\infty$.

(iii) *There exists a unital equivariant $*$-homomorphism*

$$(\mathcal{O}_\infty, \gamma) \to (F_{\infty,\beta}(B), \tilde{\beta}_\infty).$$

(iv) *There exists an equivariant linear B-bimodule map*

$$\theta : (L^2_\infty(G,B), \bar{\beta}) \to (B_{\infty,\beta}, \beta_\infty)$$

such that $\theta(\xi)^*\theta(\eta) = \langle \xi, \eta \rangle_B$ *for all* $\xi, \eta \in L^2_\infty(G,B)$.

(v) *There exists an equivariant linear B-bimodule map*

$$\theta : (L^2(G,B), \bar{\beta}) \to (B_{\infty,\beta}, \beta_\infty)$$

such that $\theta(\xi)^*\theta(\eta) = \langle \xi, \eta \rangle_B$ *for all* $\xi, \eta \in L^2(G,B)$.

Proof. Since in this thesis we will only need to use

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)}$$

we only show this chain of implications and refer the reader to [67, Proposition 3.8] for a proof of the last implication (v)⇒(i).

Let us start by showing that (i)⇒(ii). Assume that $\mathfrak{s} : (L^2(G), \lambda) \to (F_{\infty,\beta}(B), \tilde{\beta}_\infty)$ is as in the definition of isometric shift-absorption. If $1 < |G| < \infty$, then we may fix any element $g \in G \setminus \{1_G\}$, and set

$$s_1 = \frac{1}{|G|^{1/2}} \sum_{h \in G} \mathfrak{s}(\delta_h)\mathfrak{s}(\delta_h) \in F_{\infty,\beta}(B)^{\tilde{\beta}_\infty},$$

$$s_2 = \frac{1}{|G|^{1/2}} \sum_{h \in G} \mathfrak{s}(\delta_h)\mathfrak{s}(\delta_{hg}) \in F_{\infty,\beta}(B)^{\tilde{\beta}_\infty}.$$

Observing that

$$\mathfrak{s}(\delta_h)^*\mathfrak{s}(\delta_g) = \langle \delta_h, \delta_g \rangle = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise} \end{cases}$$

one may see that s_1 and s_2 are orthogonal isometries. Hence, $F_{\infty,\beta}(B)^{\tilde{\beta}_\infty}$ is properly infinite, which is equivalent to the existence of an equivariant unital embedding $(\mathcal{O}_\infty, \mathrm{id}_{\mathcal{O}_\infty}) \hookrightarrow (F_{\infty,\beta}(B), \tilde{\beta}_\infty)$, and hence an equivariant linear map

$$\mathfrak{t} : (\ell^2(\mathbb{N}), \mathbf{1}) \to (F_{\infty,\beta}(B), \tilde{\beta}_\infty)$$

with the property that $\mathfrak{t}(\xi)^*\mathfrak{t}(\eta) = \langle \xi, \eta \rangle \cdot \mathbf{1}$ for all vectors ξ, η in $\ell^2(\mathbb{N})$. Consider the product map

$$\mathfrak{t} \times \mathfrak{s} : \ell^2(\mathbb{N}) \odot \mathcal{H}_G \to F_{\infty,\beta}(B)$$

defined on elementary tensors by $\mathfrak{t} \times \mathfrak{s}(\xi \otimes \eta) = \mathfrak{t}(\xi)\mathfrak{s}(\eta)$ for $\xi \in \ell^2(\mathbb{N})$ and $\eta \in \mathcal{H}_G$. This map also satisfies the following,

$$(\mathfrak{t} \times \mathfrak{s})(\xi)^*(\mathfrak{t} \times \mathfrak{s})(\eta) = \langle \xi, \eta \rangle \cdot \mathbf{1}, \qquad \text{for all } \xi, \eta \in \ell^2(\mathbb{N}) \odot \mathcal{H}_G,$$

$$\tilde{\beta}_{\infty,g} \circ (\mathfrak{t} \times \mathfrak{s}) = (\mathfrak{t} \times \mathfrak{s}) \circ \lambda^\infty \qquad \text{for all } g \in G$$

Hence, $\mathfrak{t} \times \mathfrak{s}$ extends to a λ^∞-to-$\tilde{\beta}_\infty$ equivariant linear map $\ell^2(\mathbb{N}) \otimes \mathcal{H}_G = \mathcal{H}_G^\infty \to F_{\infty,\beta}(B)$ satisfying the inner product condition, i.e., the statement in (ii). Let us now assume that G is infinite. This time we define, in the same way as above, an equivariant linear map $\mathfrak{s} \times \mathfrak{s}$ from $(\mathcal{H}_G \otimes \mathcal{H}_G, \lambda \times \lambda)$ to $(F_{\infty,\beta}(B), \tilde{\beta}_\infty)$ with the inner product condition. It then follows from Fell's absorption principle that $(\mathcal{H}_G \otimes \mathcal{H}_G, \lambda \times \lambda) \cong (\mathcal{H}_G^\infty, \lambda^\infty)$ as G is infinite dimensional. Therefore, we have again that (ii) is satisfied, which finishes the proof also for the infinite case.

Now, by looking at the definition of $\gamma : G \curvearrowright \mathcal{O}_\infty$, one notes right away that (ii) and (iii) are equivalent.

We proceed with (ii)$\Rightarrow$(iv). As discussed in Remark 2.2.35, there exists a natural isomorphism

$$\nu : F_\infty(B) \to \mathcal{M}(\overline{BB_\infty B}) \cap B'.$$

In particular, one may consider the linear map given by

$$\theta : \mathcal{H}_G^\infty(G) \odot B \to B_\infty, \quad \xi \otimes b \mapsto \nu(\mathfrak{s}(\xi)) \cdot b$$

for all $\xi \in \mathcal{H}_G^\infty$ and $b \in B$. By Remark 2.2.35, the restriction of ν to $F_{\infty,\beta}(B)$ is equivariant with respect to $\tilde{\beta}_\infty$ and β_∞, which implies that $\nu \circ \mathfrak{s}$ is λ^∞-to-β_∞ equivariant. Therefore, θ is $\bar{\beta}$-to-β_∞ equivariant, and the its range is contained in $B_{\infty,\beta}$. Since the image of ν commutes with B, we see that θ must be a B-bimodule map:

$$b'\theta(\xi \otimes b) = b'\nu(\mathfrak{s}(\xi))b = \nu(\mathfrak{s}(\xi))b'b = \theta(\xi \otimes b'b)$$

and similarly $\theta(\xi \otimes b)b' = \theta(\xi \otimes bb')$ for all $\xi \in \mathcal{H}_G^\infty$ and $b, b' \in B$. Let us show that θ satisfies the desired inner product formula. Let $n, m \in \mathbb{N}$ and

$\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m \in \mathcal{H}_G^\infty$, $b_1, \dots, b_n, c_1, \dots, c_m \in B$. Then, we verify that the formula holds because

$$\begin{aligned}\theta\left(\sum_{i=1}^n \xi_i \otimes b_i\right)^* \theta\left(\sum_{j=1}^m \eta_j \otimes c_j\right) &= \sum_{i=1}^n \sum_{j=1}^m b_i^* \nu(\mathfrak{s}(\xi_i)^* \mathfrak{s}(\eta_j)) c_j \\ &= \sum_{i=1}^n \sum_{j=1}^m b_i^* c_j \langle \xi_i, \eta_j \rangle \\ &= \langle \sum_{i=1}^n \xi_i \otimes b_i, \sum_{j=1}^m \eta_j \otimes c_j \rangle_B.\end{aligned}$$

Finally, the formula above θ is isometric with respect to the norm whose completion is $L^2_\infty(G, B)$, and hence it yields the desired equivariant linear B-bimodule map, and shows that (iv) holds.

To conclude, note that (iv)$\Rightarrow$(v) because $L^2(G, B)$ equivariantly embeds into $L^2_\infty(G, B)$ as a direct summand. □

We conclude this section with the following observation that amenable, isometric shift-absorbing, equivariantly $\mathcal{O}_2$-stable actions on $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes \mathcal{K}$ exist.

Remark 2.7.9. We argue that every second-countable locally compact group G admits an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action on $\mathcal{O}_2 \otimes \mathcal{K}$. The same holds for $\mathcal{O}_2$ as well if G is exact. We give a short proof of these facts in the following paragraph.

First, recall that by [154, Theorem 6.1], for any amenable action $\alpha : G \curvearrowright A$ on a separable, nuclear C*-algebra, there exists an amenable, outer action $\beta : G \curvearrowright B$ on a purely infinite, simple, separable, nuclear C*-algebra containing (A, α) as a subsystem. In particular, since any locally compact, second-countable group G acts amenably on $A = \mathcal{C}_0(G)$, one may find an amenable action $\beta : G \curvearrowright B$ as above. Moreover, we may pick any isometrically shift-absorbing action $\delta : G \curvearrowright \mathcal{O}_\infty$ such as $\delta = \gamma^{\otimes\infty}$ from Definition 2.7.7. Since amenability and isometric shift-absorption are preserved under tensoring with an action, then $\beta \otimes \delta \otimes \mathrm{id}_{\mathcal{O}_2 \otimes \mathcal{K}}$ is an amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable action on $B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong \mathcal{O}_2 \otimes \mathcal{K}$, where the last isomorphism follows from Kirchberg's $\mathcal{O}_2$-absorption theorem [112, 119] and Zhang's dichotomy [212]. If the acting group is moreover exact, one can always find an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action on $\mathcal{O}_2$. One way of seeing this is by picking an amenable G-action on $\mathcal{O}_\infty$, which exists by [183, Theorem B], and then tensoring this action with $\delta \otimes \mathrm{id}_{\mathcal{O}_2}$, where $\delta : G \curvearrowright \mathcal{O}_\infty$ is as above.

2.8 Strong stability

As mentioned before, the C*-algebra of compact operators $\mathcal{K}$ plays a central role in the overarching theory of C*-algebras, and is one of the basic objects in classification. Recall that a C*-algebra A is said to be *stable* if there exists an isomorphism $A \otimes \mathcal{K} \cong A$.

Lemma 2.8.1. *A* C*-*algebra B is stable if and only if there exists a sequence of isometries $t_i \in \mathcal{M}(B)$ such that*

$$\sum_{i \in \mathbb{N}} t_i t_i^* = \mathbf{1} \quad \textit{strictly.}$$

Proof. Suppose B is stable, and fix an isomorphism $\varphi : \mathcal{M}(B) \to \mathcal{M}(B \otimes \mathcal{K})$. Note that if we can find a sequence of isometries $s_i \in \mathcal{M}(\mathcal{K})$ with $\sum_{i \in \mathbb{N}} s_i s_i^* = \mathbf{1}$ in the strict topology, then the sequence given by $t_i = \varphi^{-1}(\mathbf{1}_{\mathcal{M}(B)} \otimes s_i) \in \mathcal{M}(B)$ clearly satisfies the condition in the statement. As a result, it suffices to find such a sequence $(s_i)_{i \in \mathbb{N}} \in \mathcal{M}(\mathcal{K})$. Without loss of generality, we may concretely represent $\mathcal{K}$ as $\mathcal{K}(\ell^2(\mathbb{N}))$ and find the desired sequence $(s_i)_{i \in \mathbb{N}}$ inside $\mathcal{B}(\ell^2(\mathbb{N}))$. Fix an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$, and define operators $s_i \in \mathcal{B}(\ell^2(\mathbb{N}))$ as follows,

$$s_i(e_n) := e_{2^i(2n+1)} \quad \text{for all } i, n \in \mathbb{N}.$$

It is readily checked that these operators are isometries. Note that they have pairwise disjoint ranges, and are therefore their range projections $(s_i s_i^*)_{i \in \mathbb{N}}$ are mutually orthogonal. Finally, we argue that $\sum_{i \in \mathbb{N}} s_i s_i^* = \mathbf{1}$ in the strict topology of $\mathcal{B}(\ell^2(\mathbb{N})) = \mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N})))$ because the limit holds in the strong* topology, and the sequence $(\sum_{i=1}^k s_i s_i^*)_k$ is uniformly norm-bounded (in fact, $\sum_{i=1}^k s_i s_i^* \leq \mathbf{1}$ for all $k \in \mathbb{N}$). For the converse, assume that there exists a sequence of isometries $t_i \in \mathcal{M}(B)$ such that $\sum_{i \in \mathbb{N}} t_i t_i^* = \mathbf{1}$ strictly. Fix a set of matrix units $\{e_{i,j} \mid i, j \in \mathbb{N}\}$ that generate $\mathcal{K}$. Then, define a map $\Lambda : B \otimes \mathcal{K} \to B$ given on elementary tensors by

$$\Lambda(b \otimes e_{i,j}) = t_i b t_j^*.$$

The map Λ is clearly $*$-preserving, and additive. Multiplicativity follows from the fact that

$$\begin{aligned} \Lambda((b \otimes e_{i,j})(c \otimes e_{k,\ell})) &= t_i (bc) t_j^* \\ &= t_i b t_j^* t_k b t_j^* \\ &= \Lambda(b \otimes e_{i,j}) \Lambda(c \otimes e_{k,\ell}) \end{aligned}$$

when $j = k$, and

$$\Lambda((b \otimes e_{i,j})(c \otimes e_{k,\ell})) = 0 = \Lambda(b \otimes e_{i,j})\Lambda(c \otimes e_{k,\ell})$$

when $j \neq k$. To check injectivity of Λ, first note that $\ker(\Lambda)$ is an ideal of $B \otimes \mathcal{K}$, and is therefore generated by the linear span of elementary tensors in $\ker(\Lambda) \cap B \odot \mathcal{K}$. In fact, $\ker(\Lambda)$ must be of the form $I \otimes \mathcal{K}$ for some ideal $I \in \mathcal{I}(B)$ because $\mathcal{K}$ is simple. As a result, it suffices to consider an element of the form $x = \sum_{i,j \leq n} b_{i,j} \otimes e_{i,j} \in \ker(\Lambda) \cap B \odot \mathcal{K}$, and show that $\Lambda(x) = 0$ implies $x = 0$. Assuming $\Lambda(x) = \sum_{i,j \leq n} t_i b_{i,j} t_j^* = 0$, we see that

$$b_{i,j} = t_i^* \left(\sum_{i,j \leq n} t_i b_{i,j} t_j^* \right) t_j = 0 \quad \text{for all } i, j \leq n.$$

Therefore, one may conclude that Λ is injective. One can then see that the map Λ must be also surjective because for every $b \in B$, one can consider

$$x_n = \sum_{i \leq n} t_i^* b t_i \otimes e_{i,i} \in B \otimes \mathcal{K}, \text{ for } n \geq 1,$$

and it is clear that using the property of $(t_i)_i$ in the strict topology that

$$\lim_{n \to \infty} \|\Lambda(x_n) - b\| = 0.$$

Hence Λ is the isomorphism we are looking for. □

Example 2.8.2. Once again, recall the C*-algebra $\mathcal{A}_{[0,1]}$ defined in Example 2.1.11. This is stable by [173, Proposition 5.1 or Proposition 5.2].

Stability, and some of its relations with dynamics, were investigated profusely in [202]. We record here (without a proof) some results from that article that will be used later.

Theorem 2.8.3 (see [202, Theorem 2.1]). *Let B be a σ-unital* C*-*algebra. Then B is stable if and only if, for every $\varepsilon > 0$ and every norm-one, positive element $h \in B$, there exists a contraction $b \in B$ such that $h =_\varepsilon b^* b$ and $\|hbb^*\| \leq \varepsilon$.*

Proposition 2.8.4 (see [202, Proposition 4.4, Corollaries 4.1 and 4.5]). *The following statements hold true.*

(i) *If A is a σ-unital* C*-*algebra, and $B \subseteq A$ a non-degenerate inclusion of a stable* C*-*subalgebra, then A is stable*

(ii) *If A is the inductive limit of a sequence of stable, σ-unital* C*-*algebras, then A is stable.*

(iii) *If A is a σ-unital, stable* C^**-algebra, and $\alpha \in \mathrm{Aut}(A)$ an automorphism, then $A \rtimes_\alpha \mathbb{Z}$ is stable.*

In the next chapter, to carry out some of the intermediate results we will need to assume that actions satisfy the following dynamical stability property with respect to the compact operators $\mathcal{K}$.

Definition 2.8.5. Let $\beta : G \curvearrowright B$ be an action on a C^*-algebra. One says that β is *strongly stable* if it is conjugate to $\beta \otimes \mathrm{id}_{\mathcal{K}} : G \curvearrowright B \otimes \mathcal{K}$.

By virtually the same proof as Lemma 2.8.1, one may conclude the following.

Lemma 2.8.6. *Let $\beta : G \curvearrowright B$ be an action. Then β is strongly stable if and only if there exists a sequence of isometries $t_i \in \mathcal{M}(B)^\beta$ such that*

$$\sum_{i \in \mathbb{N}} t_i t_i^* = \mathbf{1} \quad \textit{strictly.}$$

The following key observation from [67] essentially allows one to access strong stability for actions on stable C^*-algebras at the cost of applying a cocycle conjugacy.

Proposition 2.8.7 (see [67, Proposition 1.4])**.** *Let $\beta : G \curvearrowright B$ be an action on a stable* C^**-algebra. Then β is cocycle conjugate to $\beta \otimes \mathrm{id}_{\mathcal{K}} : G \curvearrowright B \otimes \mathcal{K}$. In particular, every action on a stable* C^**-algebra is cocycle conjugate to a strongly stable action.*

Proof. Since B is stable, we may fix a sequence of isometries $t_n \in \mathcal{M}(B)$ such that

$$\sum_{n \in \mathbb{N}} t_n t_n^* = \mathbf{1}, \text{ strictly.}$$

Fix a system of matrix units $\{e_{i,j} \mid i, j \geq 1\}$ generating $\mathcal{K}$. Then consider the isomorphism $\Lambda : B \otimes \mathcal{K} \to B$ defined on elementary tensors by

$$\Lambda(b \otimes e_{i,j}) = t_i b t_j^*, \quad \text{for } b \in B,\, i, j \in \mathbb{N}.$$

Define the strictly continuous map $\mathbb{U} : G \to \mathcal{U}(\mathcal{M}(B))$ given by

$$\mathbb{U}_g = \sum_{i \in \mathbb{N}} t_i \beta_g(t_i)^*, \quad \text{for } g \in G.$$

In order to show that the pair $(\Lambda, \mathbb{U})$ is a cocycle conjugacy, we only need to prove that $\mathbb{U}$ is cocycle, and that the equivariance condition is satisfied. Let us compute, for $g, h \in G$,

$$\mathbb{U}_g \beta_g(\mathbb{U}_h) \mathbb{U}_{gh}^* = \sum_{i,j \in \mathbb{N}} t_i \beta_g(t_i)^* \beta_g(t_j) \beta_{gh}(t_j)^* \mathbb{U}_{gh}$$

$$
\begin{aligned}
&= \sum_{i\in\mathbb{N}} t_i\beta_{gh}(t_i)^*\mathbb{U}_{gh} \\
&= \sum_{i\in\mathbb{N}} t_i t_i^* \\
&= \mathbf{1}.
\end{aligned}
$$

The equivariance condition follows from the fact that, for all $g \in G$, $b \in B$ and $i, j \in \mathbb{N}$,

$$
\mathrm{Ad}(\mathbb{U}_g)\circ\beta_g\circ\Lambda(b\otimes e_{i,j}) = \mathbb{U}_g\beta_g(t_i b t_j^*)\mathbb{U}_g = t_i\beta_g(b)t_j^* = \Lambda\circ(\beta_g\otimes\mathrm{id}_{\mathcal{K}})(b\otimes e_{i,j}).
$$

□

The following are key properties of strongly stable actions that will be used on various occasions in the next chapters.

Lemma 2.8.8 (see [67, Lemma 1.21])**.** *Let $\beta : G \curvearrowright B$ be a strongly stable action on a* C**-algebra. Then, the following properties hold true.*

(i) *There exist two sequences of isometries $r_1^{(n)}, r_2^{(n)} \in \mathcal{M}(B)^\beta$ such that*
   - $r_1^{(n)}r_1^{(n)*} + r_2^{(n)}r_2^{(n)*} = \mathbf{1}$ *for all $n \in \mathbb{N}$, and*
   - $r_1^{(n)} \to \mathbf{1}$ *in the strict topology.*

(ii) *For every σ-unital β_∞-invariant* C**-subalgebra $D \subseteq B_{\infty,\beta}$, there exists an equivariant $*$-homomorphism $\iota : (D\otimes\mathcal{K}, \beta_\infty\otimes\mathrm{id}_{\mathcal{K}}) \to (B_{\infty,\beta}, \beta_\infty)$ such that $\iota(d\otimes e_{1,1}) = d$ for all $d \in D$.* [40]

Proof. By strong stability of β we may fix a sequence of isometries $t_i \in \mathcal{M}(B)^\beta$ such that $\sum_{i\in\mathbb{N}} t_i t_i^* = \mathbf{1}$ in the strict topology. This will remain the same in the proof of both statements. Let us start by proving (i). Define, for each $n \in \mathbb{N}$, two new sequences by

$$
r_1^{(n)} = \sum_{i=0}^{n} t_i t_i^* + t_{n+1}\sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*, \quad \text{and} \quad r_2^{(n)} = \sum_{i\in\mathbb{N}} t_{i+n+2}t_i^*.
$$

These are isometries by simply carrying out the following computations for all $n \in \mathbb{N}$,

$$
r_1^{(n)*}r_1^{(n)} = \left(\sum_{i=0}^{n} t_i t_i^* + t_{n+1}\sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*\right)^* \left(\sum_{i=0}^{n} t_i t_i^* + t_{n+1}\sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*\right)
$$

[40] Here, $e_{1,1}$ denotes the rank-one projection belonging to a set of generators $e_{k,\ell}$ of $\mathcal{K}$.

$$= \sum_{i=0}^{n} t_i t_i^* + \sum_{i=0}^{n} t_i t_i^* t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*$$

$$+ \sum_{i\in\mathbb{N}} t_{i+n+1} t_i^* t_{n+1}^* \sum_{i=0}^{n} t_i t_i^* + \sum_{i\in\mathbb{N}} t_{i+n+1} t_i^* t_{n+1}^* t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*$$

$$= \sum_{i\in\mathbb{N}} t_i t_i^*$$

$$= \mathbf{1}.$$

Analogously,

$$r_2^{(n)*} r_2^{(n)} = \sum_{i\in\mathbb{N}} t_i t_{i+n+2}^* \sum_{i\in\mathbb{N}} t_{i+n+2} t_i^* = \mathbf{1}.$$

Note that, for each $n \in \mathbb{N}$ we have that

$$r_1^{(n)} r_1^{(n)*} = \sum_{i=0}^{n} t_i t_i^* \left(\sum_{i=0}^{n} t_i t_i^* \right)^* + \sum_{i=0}^{n} t_i t_i^* \left(t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \right)^*$$

$$+ t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \left(t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \right)^* + t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \left(\sum_{i=0}^{n} t_i t_i^* \right)^*$$

$$= \sum_{i=0}^{n} t_i t_i^* + \sum_{i=0}^{n} t_i t_i^* \sum_{i\in\mathbb{N}} t_{i+n+1} t_i^* t_{n+1}^*$$

$$+ t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \sum_{i\in\mathbb{N}} t_{i+n+1} t_i^* t_{n+1}^* + t_{n+1} \sum_{i\in\mathbb{N}} t_i t_{i+n+1}^* \sum_{i=0}^{n} t_i t_i^*$$

$$= \sum_{i=0}^{n} t_i t_i^* + t_n t_0^* t_{n+1}^* + t_{n+1} t_{n+1}^* + t_{n+1} t_0 t_n^*$$

$$= \sum_{i=0}^{n+1} t_i t_i^*.$$

Similarly, for all $n \in \mathbb{N}$, we have that

$$r_2^{(n)} r_2^{(n)*} = \sum_{i\in\mathbb{N}} t_{i+n+2} t_i^* \left(\sum_{i\in\mathbb{N}} t_{i+n+2} t_i^* \right)^*$$

$$= \sum_{i\in\mathbb{N}} t_{i+n+2}t_i^* \sum_{i\in\mathbb{N}} t_i t_{i+n+2}^*$$

$$= \sum_{i\geq n+2} t_i t_i^*.$$

Putting these calculations together gives that

$$r_1^{(n)}r_1^{(n)*} + r_2^{(n)}r_2^{(n)*} = \mathbf{1}.$$

Furthermore, we check that $r_1^{(n)} \to \mathbf{1}$ strictly. Fix $b \in B$ and $\varepsilon > 0$. Since the sequence $(\sum_{i=0}^n t_i t_i^*)_n$ strictly converges to $\mathbf{1}$, we may find $N \in \mathbb{N}$ such that $\sum_{i=0}^N t_i t_i^* b =_\varepsilon b$. Hence, we have that if $n \geq N+1$

$$\|(\mathbf{1} - r_1^{(n)})b\| =_{2\varepsilon} \|(\mathbf{1} - \sum_{i=0}^n t_i t_i^* + t_{n+1}\sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*)\sum_{i=0}^N t_i t_i^* b\|$$

$$= \|(\sum_{i\geq n+1} t_i t_i^* + t_{n+1}\sum_{i\in\mathbb{N}} t_i t_{i+n+1}^*)\sum_{i=0}^N t_i t_i^* b\|$$

$$= 0,$$

which shows the desired convergence.

Let us now show (ii). Define $r_1^{(n)}$ as above, and for all $k \geq 2$ set

$$r_k^{(n)} = t_{n+k}.$$

Since D is assumed to be σ-unital, and hence contains a strictly positive element, we may find an increasing sequence of numbers $(n_i)_{i\in\mathbb{N}}$ such that, if we put

$$s_k \in \mathcal{M}(B)_\infty^\beta \text{ induced by } (r_k^{(n_i)})_{i\in\mathbb{N}},$$

then $s_1 d = ds_1 = d$ for all $d \in D$. It is now possible to define a $*$-homomorphism

$$\iota : D\otimes\mathcal{K} \to B_{\infty,\beta}, \quad \iota(d\otimes e_{i,j}) = s_i d s_j^*,$$

where $\{e_{i,j} \mid i,j \geq 1\}$ is a system of matrix units generating $\mathcal{K}$. This is the map we were looking for because $\iota(d\otimes e_{1,1}) = s_1 d s_1^* = d$ for all $d \in D$. □

Finally, we record here a result from [67]. This will play a key role when we consider actions of *compact* groups.

Proposition 2.8.9 (see [67, Proposition 6.3]). *When G is a second-countable, compact group, the following is true. If $\beta : G \curvearrowright B$ is an isometrically shift-absorbing action on a separable, stable* C^**-algebra, then it is strongly stable.*

Proof. We may assume that $B \neq 0$. First, observe that the inclusion $B^\beta \subseteq B$ is non-degenerate. We see this via approximate units: Fix an increasing approximate unit $(b_n)_n \subseteq B$, and with Lemma 2.2.12 find an approximate unit $(e_n)_n \subseteq B$ such that

$$\max_{g\in G} \|\beta_g(e_n) - e_n\| \xrightarrow{n\to\infty} 0.$$

Set $b_n := \int_G \beta_g(e_n)\, d\mu(g) \in B^\beta$, where μ denotes the normalised Haar measure on G. Then, $(b_n)_n$ is an approximate unit of B because

$$\begin{aligned}
\lim_{n\to\infty} \|b_n b - b\| &= \lim_{n\to\infty} \left\| \int_G \beta_g(e_n) b\, d\mu(g) b \right\| \\
&= \lim_{n\to\infty} \left\| \int_G (\beta_g(e_n)b - b)\, d\mu(g) \right\| \\
&\leq \lim_{n\to\infty} \max_{g\in G} \|\beta_g(e_n)b - b\| \\
&= \lim_{n\to\infty} \|e_n b - b\| \\
&= 0
\end{aligned}$$

and similarly for $\lim_{n\to\infty} \|b - bb_n\| = 0$. As a result, the inclusion $B^\beta \subseteq B$ is non-degenerate, and by Proposition 2.1.6 there is a unique inclusion at the level of multiplier algebras $\mathcal{M}(B^\beta) \subseteq \mathcal{M}(B)^\beta$ that is strictly continuous on the unit ball. Hence, if B^β was stable, by Lemma 2.8.1 we would be able to find a sequence of isometries $t_i \in \mathcal{M}(B^\beta) \subseteq \mathcal{M}(B)^\beta$ such that $\sum_{i\in\mathbb{N}} t_i t_i^* = \mathbf{1}$ strictly, which by Lemma 2.8.6 would imply that β is strongly stable. Therefore, the goal is to show that B^β is stable. By Theorem 2.8.3, it suffices, for each $\varepsilon > 0$ and norm-one, positive element $h \in B^\beta$, that we find a contraction $b \in B^\beta$ such that $h =_\varepsilon b^*b$ and $\|hbb^*\| \leq \varepsilon$. Fix (ε, h) as above, and use stability of B to find a contraction $y \in B$ such that $\|h - y^*y\| + \|hyy^*\| \leq \varepsilon$. Since h is in the fixed-point algebra, we have that, for every $g \in G$

$$\|h - \beta_g(y^*y)\| + \|h\beta_g(yy^*)\| = \|h - y^*y\| + \|hyy^*\| \leq \varepsilon.$$

Then, consider $L^2(G, B)$, and define $\zeta \in \mathcal{C}(G, B) \subseteq L^2(G, B)$ given by $\zeta(g) = \beta_g(y)$ for all $g \in G$. It follows that ζ is a contraction since $\|y\| \leq 1$, and it is also fixed by the action $\bar{\beta}$ because, for every $g, s \in G$

$$\bar{\beta}_g(\zeta)(s) = \beta_g(\zeta(g^{-1}s)) = \beta_g(\beta_{g^{-1}s}(y)) = \beta_s(y) = \zeta(s).$$

Moreover, ζ the following properties,

$$\|h - \langle \zeta, \zeta \rangle\| = \left\| \int_G (h - \beta_g(y^* y))\, d\mu \right\| \leq \varepsilon,$$

and

$$\|h\zeta\|_2^2 = \left\| \int_G h\beta_g(yy^*)h\, d\mu \right\| \leq \int_G \|h\beta_g(yy^*)h\|\, d\mu \leq \varepsilon.$$

Then, by Proposition 2.7.8 there exists an equivariant linear B-bimodule map

$$\theta : (L^2(G, B), \bar{\beta}) \to (B_{\infty,\beta}, \beta_\infty)$$

such that $\theta(\xi)^*\theta(\eta) = \langle \xi, \eta \rangle_B$ for all $\xi, \eta \in L^2_\infty(G, B)$. Hence, we may observe that $z := \theta(\zeta) \in (B_{\infty,\beta})^{\beta_\infty} = (B^\beta)_\infty$ is a contraction and the computations above show that it satisfies

$$z^*z = \theta(\zeta)^*\theta(\zeta) = \langle \zeta, \zeta \rangle =_\varepsilon h$$

and

$$\|hz\|^2 = \|h\theta(\zeta)\|^2 = \|\theta(h\zeta)\|^2 = \|h\zeta\|_2^2 \leq \varepsilon.$$

Lift z to a sequence of contractions $(z_n)_n \in B^\beta$. After passing to a subsequence, if necessary, we may ensure that $(z_n)_n$ satisfies

$$\|z_n^* z_n - h\| \leq \varepsilon + 2^{-n} \quad \text{and} \quad \|hz_n\| \leq \varepsilon + 2^{-n}.$$

Therefore, there exists $n \in \mathbb{N}$ large enough such that $z_n^* z_n =_{2\varepsilon} h$ and $\|hz_n z_n^*\| \leq \|hz_n\| \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we may conclude that B^β is stable, and hence that β is strongly stable. $\square$

Chapter 3

Classification of equivariantly $\mathcal{O}_2$-stable actions

The aim of this chapter is to outline the results present in the published research article [156], where we prove a classification result for equivariantly $\mathcal{O}_2$-stable amenable actions on nuclear C^*-algebras. In particular, we stress that the main theorem of this chapter represents the full-fledged generalisation of Gabe–Kirchberg's classification of $\mathcal{O}_2$-stable C^*-algebras [64, 116]. It seems then reasonable, before diving into the core of our research, to explain the classification of the underlying C^*-algebras that we are going to generalise later.

3.1 The classification of $\mathcal{O}_2$-stable C^*-algebras

The purpose of this section is to outline Gabe's proof of Kirchberg's classification of separable, nuclear, stable (or unital) $\mathcal{O}_2$-stable C^*-algebras via their ideal structure. The goal is not to reproduce a detailed proof of the abovementioned result, but rather to convey its main conceptual steps in broad strokes to the reader.

Kirchberg's $\mathcal{O}_2$-stable classificaiton theorem, as it appeared in Gabe's work in the form of a strong classification result, reads as follows.

Theorem 3.1.1 (see [64, Theorem 6.13]). *Let A and B be separable, nuclear, $\mathcal{O}_2$-stable C^*-algebras that are either both stable or both unital. Then one has that the following equivalent statements hold.*

(i) *For every order isomorphism* $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$*, there exists an isomorphism* $\varphi : A \to B$ *such that* $\mathcal{I}(\varphi) = \Phi$*;*

(ii) *For every homeomorphism* $f : \mathrm{Prim}(A) \to \mathrm{Prim}(B)$*, there exists an isomorphism* $\varphi : A \to B$ *such that* $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Prim}(A)$*.*

The fact that one obtains a two-fold classification result, one via the ideal lattice and another via the primitive spectrum, is justified by the fact that they share the same isomorphisms classes, even though $\mathrm{Prim}(-)$ is not functorial; cf. Section 2.3. As a result, when it comes to proving Theorem 3.1.1, one could choose to either prove the statement with ideal lattices or the one with primitive ideal spaces. Since the classification approach adopted by Gabe in [64] relies on functoriality to lift maps between C*-algebras to maps between the invariants, the canonical choice is to prove part (i).

The strategy behind the proof of Theorem 3.1.1(i) can be (very roughly) divided in steps as follows,

(i) Classify $*$-homomorphisms $A \to B$, which is divided into two equally fundamental parts: existence and uniqueness,

   (a) The uniqueness part of classification provides sufficient conditions ensuring that, if $\varphi, \psi : A \to B$ are $*$-homomorphisms with $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$, then φ and ψ are (properly) approximately unitarily equivalent;[1]

   (b) The existence result allows one to lift maps on the invariants to $*$-homomorphisms. Namely, for every **Cu**-morphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$, one aims to find a $*$-homomorphism $\varphi : A \to B$ such that $\mathcal{I}(\varphi) = \Phi$.[2]

   To achieve classification, the uniqueness and existence results need to be strong enough to ensure that, when A and B are C*-algebras in the scope of the theorem above, they yield a one-to-one correspondence

$$\frac{\left\{A \xrightarrow{\varphi} B \text{ } *\text{-homomorphism}\right\}}{\approx_{\mathrm{pu}}} \xrightarrow{\mathcal{I}(-)} \left\{\mathcal{I}(A) \xrightarrow{\mathcal{I}(\varphi)} \mathcal{I}(B) \text{ } \mathbf{Cu}\text{-morphism}\right\}$$

(ii) In order to lift the classification of maps to a classification of C*-algebras, one employs Elliott's intertwining argument.[3] Let us briefly show how. If

[1]Here, note that all equivalence relations defined in Definition 2.2.23 are well-defined for $*$-homomorphisms by simply putting $G = \{1\}$.

[2]Recall from Section 2.3 that the right categorical framework for ideal lattices is the abstract Cuntz semigroup category Cu with **Cu**-morphisms.

[3]In this context, one does not need the full force of Theorem 2.2.30 because there are no group actions involved. Indeed, in this case it is sufficient to use [173, Corollary 2.3.4]

$\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ is an order isomorphism, by the existence result obtained in the previous step, one may find two $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ inducing Φ and Φ^{-1}, respectively. By the uniqueness result in the previous step φ and ψ satisfy

$$\psi \circ \varphi \approx_{\mathrm{pu}} \mathrm{id}_A \text{ and } \varphi \circ \psi \approx_{\mathrm{pu}} \mathrm{id}_B.$$

It is then thanks to Elliott's intertwining argument that there exists an isomorphism $A \to B$ that is approximately unitarily equivalent to φ, and its inverse to ψ, thus proving that Theorem 3.1.1 holds.

In light of the previous paragraph, the entire classification depends on the classification of $*$-homomorphisms in point (i) above. Let us explain how uniqueness and existence results for maps are obtained by Gabe in [64].[4]

First, we introduce a certain sub-equivalence relation that will return later and play a key role in the dynamical setting as well.

Definition 3.1.2. Let $\varphi, \psi : A \to B$ be $*$-homomorphisms. One says that φ *weakly approximately dominates*[5] ψ if, for any finite subset $\mathcal{F} \subseteq A$, and any $\varepsilon > 0$, there exist $n > 0$ and $\{c_i\}_{i=1}^n \subseteq B$ such that

$$\max_{a \in \mathcal{F}} \left\| \psi(a) - \sum_{i=1}^{n} c_i^* \varphi(a) c_i \right\| \leq \varepsilon.$$

Moreover, we say that φ *weakly approximately 1-dominates* ψ if one can always choose $n = 1$ in the condition above.

Remark 3.1.3. As stated in Definition 3.1.2, $\varphi : A \to B$ weakly approximately 1-dominates $\psi : A \to B$ if for every finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, there exists an element $c \in B$ such that

$$\max_{a \in \mathcal{F}} \|\psi(a) - c^* \varphi(a) c\| \leq \varepsilon.$$

We record here for later use that, if φ weakly approximately 1-dominates ψ, for each pair $(\mathcal{F}, \varepsilon)$ one can find c as above that is additionally contractive. To show this fact, let us fix a finite set $\mathcal{F} \subseteq A$ and $\varepsilon > 0$. Then, let $\eta > 0$ be any number satisfying $(1 + \max_{a \in \mathcal{F}} \|a\|)\eta \leq \varepsilon$, and find a positive contraction $b \in B$ such that $\max_{a \in \mathcal{F}} \|bab - a\| \leq \eta/2$. Weak approximate 1-domination applied

[4] In [64], Gabe obtains a more general classification of maps than the one exposed here, and the interested reader is referred to that article for more information.

[5] The original terminology does not contain the prefix "weakly". However, we use this ad hoc notation since we will introduce a different notion of approximate domination for $*$-homomorphisms.

to the pair $(\mathcal{F} \cup b\mathcal{F}b \cup \{b^2\}, \eta/2)$ ensures that there exists some element $c_1 \in B$ satisfying

$$\max_{a \in \mathcal{F} \cup b\mathcal{F}b \cup \{b^2\}} \|\psi(a) - c_1^* \varphi(a) c_1\| \leq \eta/2.$$

Setting $c_0 := \varphi(b)c_1$, we have that

$$\|c_0\|^2 = \|c_1^* \varphi(b^2) c_1\| \leq \|\psi(b^2) - c_1^* \varphi(b^2) c_1\| + \|\psi(b^2)\| \leq 1 + \eta,$$

and

$$\max_{a \in \mathcal{F}} \|\psi(a) - c_0^* \varphi(a) c_0\| \leq \eta/2 + \|\psi(bab) - c_1^* \varphi(bab) c_1\| \leq \eta.$$

Now, the element $c := (1+\eta)^{-1/2} c_0$ is a contraction, and it satisfies the desired condition because

$$\begin{aligned} \max_{a \in \mathcal{F}} \|\psi(a) - c^* \varphi(a) c\| &\leq \eta + \max_{a \in \mathcal{F}} \|c^* \varphi(a) c((1+\eta) - 1)\| \\ &\leq (1 + \max_{a \in \mathcal{F}} \|a\|)\eta \leq \varepsilon. \end{aligned}$$

Remark 3.1.4. Let A, B be C*-algebras, and $\varphi, \psi : A \to B$ two $*$-homomorphisms. Note that if φ weakly approximately dominates ψ, then $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$. Let us show why. Fix an ideal $I \in \mathcal{I}(A)$. For any element $a \in I$ and $\varepsilon > 0$, there exists $\{c_i\}_{i=1}^n \subseteq B$ such that

$$\psi(a) =_\varepsilon \sum_{i=1}^n c_i^* \varphi(a) c_i \in \overline{B\varphi(I)B}.$$

Since $a \in I$ and $\varepsilon > 0$ were arbitrary, it follows that $\mathcal{I}(\psi)(I) \subseteq \mathcal{I}(\varphi)(I)$.

The following result is Gabe's [64, Theorem 3.3], and is a slight modification of Kirchberg–Rørdam's Hahn–Banach separation theorem for completely positive maps; cf. [122, Lemma 7.18] (or [123, Proposition 4.2]). Apart from underpinning Gabe's uniqueness theorem, this statement will be used in our dynamical generalisation as well. Moreover, its proof is an incredible display of the mileage one can get out of the Hahn–Banach theorem, and hence we take the opportunity to outline its proof here.

Theorem 3.1.5 (see [64, Theorem 3.3])**.** *Let A and B be* C**-algebras with A exact, and $\varphi, \psi : A \to B$ nuclear $*$-homomorphisms. Then, φ weakly approximately dominates ψ if and only if $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$.*

Before diving into the proof of Theorem 3.1.5, we recall two lemmas from [122] without proving them. The first one is proved by Kirchberg and Rørdam using (a linear version of) Sakai's Radon–Nikodym theorem (see [107, Proposition 7.3.5]).

Lemma 3.1.6 (see [122, Lemma 7.17(i)]). *Let B be a* C*-*algebra, and* $f_1, \ldots, f_n \in B^*$.[6] *Then, there eists a cyclic representation $\pi : B \to \mathcal{B}(\mathcal{H})$ with cyclic vector $\xi \in \mathcal{H}$, and elements $c_1, \ldots, c_n \in \mathcal{B}(\mathcal{H}) \cap \pi(B)'$ such that*

$$f_i(b) = \langle \pi(b) c_i \xi, \xi \rangle, \quad \textit{for all } b \in B.$$

The second lemma we recall here is proved using the Hahn–Banach theorem.

Lemma 3.1.7 (see [122, Lemma 7.17(ii)]). *Let B be a* C*-*algebra, and $\mathcal{S}$ a weak*-closed convex subcone of B^* consisting of positive linear functionals, and assume that $\mathcal{S}$ is closed under pre-composing with* $\mathrm{Ad}(b)$ *for all $b \in B$. Then one has an ideal of B given by*

$$J_{\mathcal{S}} = \{ b \in B \mid \rho(b^* b) = 0 \textit{ for all } \rho \in \mathcal{S} \}.$$

Moreover, if ρ is a positive linear functional on B, and $\rho(b^*b) = 0$ for all $b \in J_{\mathcal{S}}$, then $\rho \in \mathcal{S}$.

Proof of Theorem 3.1.5. The "only if" part of the theorem holds almost by definition. Hence, it suffices to show the "if" statement. Assume that $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$, i.e., $\overline{B\psi(I)B} \subseteq \overline{B\varphi(I)B}$ for all $I \in \mathcal{I}(A)$. It follows that, for every $a \in A_+$, one has $\psi(a) \in \overline{B\varphi(\overline{AaA})B} \subseteq \overline{B\varphi(a)B}$. We will use this fact below. Now, set

$$\mathcal{C} = \{ \theta : A \to B \text{ c.p. map} \mid \varphi \text{ weakly approximately dominates } \theta \}.$$

Clearly, our objective is to show that $\psi \in \mathcal{C}$. Note that $\mathcal{C}$ is a convex cone, i.e., for any $\theta, \eta \in \mathcal{C}$ and $t \in [0, \infty)$, one has that $t\theta + \eta \in \mathcal{C}$. Moreover, by the very definition of nuclearity of φ via the completely positive approximation property, we know that any c.p. map that belongs to $\mathcal{C}$ must be nuclear as well. If endowed with the point-norm topology, $\mathcal{C}$ is clearly closed. A consequence of the Hahn–Banach theorem then tells us that $\mathcal{C}$ is also closed when endowed with the point-weak topology. Consequently, if suffices to prove that for any $\varepsilon > 0$, $a_1, \ldots, a_n \in A$, and $f_1, \ldots, f_n \in B^*$, there exists some $\theta \in \mathcal{C}$ such that

$$|f_i(\psi(a_i)) - f_i(\theta(a_i))| \leq \varepsilon \tag{3.1}$$

for all $i = 1, \ldots, n$. Using Lemma 3.1.6, there exists a cyclic representation $\pi : B \to \mathcal{B}(\mathcal{H})$ with cyclic vector $\xi \in \mathcal{H}$, and $c_1, \ldots, c_n \in \mathcal{B}(\mathcal{H}) \cap \pi(B)'$ such that

$$f_i(b) = \langle \pi(b) c_i \xi, \xi \rangle, \quad i = 1, \ldots, n.$$

[6] Here, by B^* we mean the dual of B.

Now, denote by C the C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $c_1, \dots, c_n$, and by ι the inclusion $C \subseteq \mathcal{B}(\mathcal{H})$. For every c.p. map $\eta : A \to B$, one may define a positive linear functional on $A \otimes_{\max} C$ via the compositions

$$A \otimes_{\max} C \xrightarrow{\eta \otimes \mathrm{id}_C} B \otimes_{\max} C \xrightarrow{\pi \times \iota} \mathcal{B}(\mathcal{H}) \xrightarrow{\omega_\xi} \mathbb{C},$$

where ω_ξ denotes the vector state associated to ξ by $\omega_\xi(T) = \langle T\xi, \xi\rangle$ for all $T \in \mathcal{B}(\mathcal{H})$. When the c.p. map η is nuclear, the map $\eta \otimes \mathrm{id}_C$ factors through $A \otimes C$ (see [24, Lemma 3.6.10]) and hence one has a positive linear functional induced by η, which we call by $\gamma_\eta : A \otimes C \to \mathbb{C}$. More generally, one has a mapping from nuclear c.p. maps $A \to B$ to positive linear functionals on $A \otimes C$

$$[\eta : A \to B] \overset{\gamma}{\mapsto} [\gamma_\eta : A \otimes C \to \mathbb{C}].$$

Since φ is assumed to be nuclear, and any $\theta \in \mathcal{C}$ is nuclear as well, they induce positive linear functionals $\gamma_\varphi, \gamma_\theta : A \otimes C \to \mathbb{C}$. Denote now by $\mathcal{S}$ the weak*-closure of $\{\gamma_\theta \mid \theta \in \mathcal{C}\} \subseteq (A \otimes C)^*$. The proof then reduces to showing that $\gamma_\psi \in \mathcal{S}$. This is because if one has that $|\gamma_\psi(a_i \otimes c_i) - \gamma_\theta(a_i \otimes c_i)| \leq \varepsilon$ for some $\theta \in \mathcal{C}$, then

$$\begin{aligned} f_i(\psi(a_i)) &= \langle \pi(\psi(a_i))c_i\xi, \xi\rangle = \gamma_\psi(a_i \otimes c_i) \\ &=_\varepsilon \gamma_\theta(a_i \otimes c_i) = \langle \pi(\theta(a_i))c_i\xi, \xi\rangle \\ &= f_i(\theta(a_i)) \end{aligned}$$

for all $i = 1, \dots, n$, which gives (3.1). Hence, let us prove that $\gamma_\psi \in \mathcal{S}$. It is readily checked that $\mathcal{S}$ forms a convex cone of positive linear functionals. Moreover, let us show that, if $\gamma \in \mathcal{S}$ and $d \in A \otimes C$, then $\gamma \circ \mathrm{Ad}(d^*) \in \mathcal{S}$. Since $\mathcal{S}$ is weak*-closed, it is sufficient to show that this is true for $\gamma = \gamma_\theta$, where $\theta = \sum_{i=1}^m \theta \circ \mathrm{Ad}(e_i^*)$ for some $e_1, \dots, e_m \in B$, and $d = \sum_{j=1}^k x_j \otimes y_j$ for $x_1, \dots, x_k \in A$ and $y_1, \dots, y_k \in C$. Fix a finite set of elementary tensors $\mathcal{F} \subseteq A \otimes C$, and $\varepsilon > 0$. Sice $\xi \in \mathcal{H}$ is cyclic for π, one can find elements $b_1, \dots, b_k \in B$ such that $\|\pi(b_j)\xi - y_j\xi\| \leq \frac{\varepsilon}{Mk^2}$ for all $j = 1, \dots, k$, where

$$M = \max_{1 \leq i,j \leq k} \{\|\pi(\theta(x_i a x_j))c y_j \xi\|, \|\pi(\theta(x_i a x_j))c\| \cdot \|y_i\xi\|\}.$$

Then, for all $a \otimes c \in \mathcal{F}$,

$$\begin{aligned} \gamma_\theta(d^*(a \otimes c)d) &= \sum_{i,j=1}^k \gamma_\theta((x_i^* a x_j) \otimes (y_i^* c y_j)) \\ &= \sum_{i,j=1}^k \langle \pi(\theta(x_i^* a x_j))c y_j \xi, y_i \xi\rangle \end{aligned}$$

$$
\begin{aligned}
&=_\varepsilon \sum_{i,j=1}^{k} \langle \pi(\theta(x_i^* a x_j)) c \pi(b_j)\xi, \pi(b_i)\xi\rangle \\
&= \left\langle \pi\left(\sum_{i,j=1}^{k} b_i^* \theta(x_i^* a x_j)) b_j\right) c\xi, \xi\right\rangle \\
&= \left\langle \pi\left(\sum_{\ell=1}^{m}\sum_{i,j=1}^{k} b_i^* e_\ell^* \varphi(x_i^* a x_j)) e_\ell b_j\right) c\xi, \xi\right\rangle \\
&= \left\langle \pi\left(\sum_{\ell=1}^{m}\sum_{i,j=1}^{k} b_i^* e_\ell^* \varphi(x_i)^* \varphi(a)\varphi(x_j)) e_\ell b_j\right) c\xi, \xi\right\rangle .
\end{aligned}
$$

If we set

$$f_\ell = \sum_{i}^{k} \varphi(x_j) e_\ell b_j, \quad \text{for all } \ell = 1, \dots, m,$$

and

$$\theta_0 = \sum_{\ell=1}^{m} \mathrm{Ad}(f_\ell^*) \circ \varphi \in \mathcal{C},$$

the calculation indicates that

$$\gamma_\theta(d^*(a \otimes c)d) =_\varepsilon \gamma_{\theta_0}(a \otimes c)$$

and thus that $\theta \circ \mathrm{Ad}(d^*) \in \mathcal{S}$ for any $\theta \in \mathcal{S}$ and $d \in A \otimes C$. As a result, $\mathcal{S}$ satisfies the assumptions of Lemma 3.1.7, and we only need to show that $\gamma_\psi(d^*d) = 0$ for all $d \in J_{\mathcal{S}}$, then $\gamma_\psi \in \mathcal{S}$, where

$$J_{\mathcal{S}} = \{d \in A \otimes C \mid \gamma(d^*d) = 0\}$$

is a closed two-sided ideal of $A \otimes C$. Since A is exact, by Lemma 2.3.2, every ideal of $A \otimes C$ is generated by elementary tensors in $A \odot C$. In particular,

$$J_{\mathcal{S}} = \overline{\mathrm{span}}\{a \otimes c \mid a \in A,\, c \in C,\, a \otimes c \in J_{\mathcal{S}}\},$$

and it suffices to show that $\gamma_\psi(d^*d) = 0$ when $d = a \otimes c \in J_{\mathcal{S}}$ for $a \in A$ and $c \in C$. Recall that by assumption $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$, which implies that $\psi(a^*a) \in \overline{B\varphi(a^*a)B}$ for all $a \in A$. Hence, for any $\delta > 0$ we may find $b_1, \dots, b_m \in B$ such that

$$\left\| \psi(a^*a) - \sum_{i=1}^{m} b_i^* \varphi(a^*a) b_i \right\| \leq \varepsilon.$$

Then the c.p. map given by $\theta = \sum_{i=1}^m \varphi \circ \mathrm{Ad}(b_i^*)$, which is in $\mathcal{C}$ by definition, satisfies $\|\psi(a^*a) - \theta(a^*a)\| \leq \varepsilon$. Since by definition of $J_{\mathcal{S}}$ we have that $\gamma_\theta(a^*a \otimes c^*c) = 0$, it follows that

$$\begin{aligned}|\gamma_\psi(a^*a \otimes c^*c)| &= |\gamma_\psi(a^*a \otimes c^*c) - \gamma_\theta(a^*a \otimes c^*c)| \\ &|\langle \pi(\psi(a^*a) - \theta(a^*a))c\xi, c\xi\rangle| \\ &\leq \varepsilon \cdot \|c\xi\|^2.\end{aligned}$$

As ε was arbitrary, we may conclude that $\gamma_\psi(a^*a \otimes c^*c) = 0$ for all a, c such that $a \otimes c \in J_{\mathcal{S}}$, which concludes the proof. ☐

Lemma 3.1.8. *Let A be a* C**-algebra, and B an $\mathcal{O}_\infty$-stable* C**-algebra. If $\varphi, \psi : A \to B_\infty$ are $*$-homomorphisms such that φ weakly approximately dominates ψ, then φ weakly approximately 1-dominates ψ.*

Proof. To prove that φ weakly approximately 1-dominates ψ, fix a finite set $\mathcal{F} \subseteq A$, and $\varepsilon > 0$. By assumption, there exist $c_i \in B_\infty$, for $1 \leq i \leq n$, such that

$$\left\| \sum_{i=1}^n c_i^* \varphi(a) c_i - \psi(a) \right\| \leq \varepsilon$$

for all $a \in \mathcal{F}$. Since B is $\mathcal{O}_\infty$-stable, by Theorem 2.6.8 there are elements $(s_k)_{k\in\mathbb{N}}$ in $B_\infty \cap \varphi(C^*(\mathcal{F}))'$ such that $s_i + B_\infty \cap \varphi(C^*(\mathcal{F}))^\perp$ form a sequence of isometries with orthogonal range projections. Note that

$$\sum_{i=1}^n c_i^* \varphi(a) c_i = \sum_{i=1}^n c_i^* s_i^* \varphi(a) s_i c_i = c^* \varphi(a) c,$$

where $c = \sum_{i=1}^n s_i c_i$. It follows that

$$\psi(a) =_\varepsilon \sum_{i=1}^n c_i^* \varphi(a) c_i = c^* \varphi(a) c$$

for all $a \in \mathcal{F}$, and thus c is the element that ensures weak approximate 1-domination. ☐

Now, weak approximate domination is not quite enough yet to obtain approximate unitary equivalence, and hence the uniqueness theorem. There is another intermediate equivalence relation that we need to consider.

Definition 3.1.9 (see [64, Definition 3.4 and Observation 3.7]). Let A, B be C*-algebras. Two $*$-homomoprhisms $\varphi, \psi : A \to B$ are said to be *approximately Murray–von Neumann equivalent* if there exists a contraction $v \in B_\infty$ such that

$$v^*\varphi(a)v = \psi(a), \text{ and } \varphi(a) = v\psi(a)v^*, \text{ for all } a \in A.$$

There are a few key facts about approximate Murray–von Neumann equivalence for maps that we recall from [64] without proofs. The reader should be warned that, while the first result is not too hard to show, the second lemma is proved via a two by two matrix trick à la Connes.

Lemma 3.1.10 (see [64, Lemma 3.8]). *Let A, B be* C*-*algebras, and* $\varphi, \psi : A \to B$ *two $*$-homomomomorphisms. Suppose that there exists a contraction $v \in B_\infty$ such that $v^*\varphi(a)v = \psi(a)$ for all $a \in A$. Then the following statements hold true.*

(i) $vv^* \in B_\infty \cap \varphi(A)'$,

(ii) $v^*v\psi(a) = \psi(a)$ *for all* $a \in A$,

(iii) $\varphi(a)v = v\psi(a)$ *for all* $a \in A$.

Proposition 3.1.11 (see [64, Proposition 3.10]). *Let A and B be* C*-*algebras with A separable, and $\varphi, \psi : A \to B$ two $*$-homomorphisms. Then the following are equivalent.*

(i) *φ and ψ are approximately Murray–von Neumann equivalent,*

(ii) *$\varphi \oplus 0, \psi \oplus 0 : A \to M_2(B)$ are properly approximately unitarily equivalent,*

(iii) *The projections $\mathbf{1} \oplus 0$ and $0 \oplus \mathbf{1}$ are Murray–von Neumann equivalent inside $F((\varphi \oplus \psi)(A), M_2(B_\infty))$.*

We are almost ready to promote approximate 1-domination to Murray–von Neumann equivalence. First, we need an intermediate result.

Theorem 3.1.12 (see [64, Theorem 3.22]). *Let A and B be* C*-*algebras with A separable and exact and B $\mathcal{O}_\infty$-stable. Let $\varphi, \psi : A \to B$ be nuclear $*$-homomorphisms such that $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$. Then, there exists a contraction $v \in B_\infty$ such that $v^*\varphi(a)v = \psi(a)$ for all $a \in A$.*

Proof. By combining Theorem 3.1.5 and Lemma 3.1.8, we know that φ weakly approximately 1-dominates ψ. Fix an increasing sequence of finite subsets

$\mathcal{F}_n \subseteq A$ whose union is dense in A. By weak approximate 1-domination we may find a sequence $c_n \in B$ such that

$$\max_{a\in\mathcal{F}_n} \|\psi(a) - c_n^*\varphi(a)c_n\| \leq 2^{-n}, \text{ for all } n \in \mathbb{N}.$$

By Remark 3.1.3 we may assume that each c_n is a contraction. Next, by a density argument, we have that $c_n^*\varphi(a)c_n \to \psi(a)$ for all $a \in A$. Fix now an approximate unit of positive contractions $a_n \in A$. After passing to a subsequence of $(c_n)_n$, we may assume that $c_n^*\varphi(a_n^2)c_n \to \psi(a_n^2)$. Set $v_n = \varphi(a_n)c_n$ for all $n \in \mathbb{N}$. Since $\sup_{n\in\mathbb{N}} \|v_n\| \leq 1$, v_n induces a contraction $v \in B_\infty$ such that $v^*\varphi(a)v = \psi(a)$ for all $a \in A$. □

We are ready to present the main uniqueness theorem.

Theorem 3.1.13 (see [64, Theorem 3.23])**.** *Let A be a separable exact* C*-*algebra, and B an $\mathcal{O}_2$-stable, separable* C*-*algebra. Let $\varphi, \psi : A \to B$ be nuclear $*$-homomorphisms. Then, $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$ if and only if φ and ψ are approximately Murray–von Neumann equivalent.*

Moreover, if either

(i) *B is stable, or*

(ii) *A, B, φ and ψ are all unital,*

then approximate Murray–von Neumann equivalence between φ and ψ in the statement above can be upgraded to proper approximate unitary equivalence.

Proof. By Remark 3.1.4, and recalling that approximate Murray–von Neumann equivalence is stronger than approximate domination, we only need to prove the "only if" statement. Assume then that $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$. By Theorem 3.1.12, there exists a contraction $v \in B_\infty$ such that $v^*\varphi(a)v = \psi(a)$ for all $a \in A$. Hence, if one sets $D = F((\varphi \oplus \psi)(A), M_2(B_\infty))$, then $v \otimes e_{1,2}$ induces an element $V \in D$ because of the relation $\varphi(a)v = v\psi(a)$ for all $a \in A$ that follows from Lemma 3.1.10. Using that v^*v acts as a unit on the image of ψ, which also follows from Lemma 3.1.10, we have that $V^*V = 0 \oplus \mathbf{1}$ and $VV^* \leq \mathbf{1} \oplus 0$, i.e., that $0 \oplus \mathbf{1}$ is Murray–von Neumann subequivalent to $\mathbf{1} \oplus 0$. By an analogous argument, one proves also that $\mathbf{1} \oplus 0$ is Murray–von Neumann subequivalent to $0 \oplus \mathbf{1}$. In particular, it follows that these two projections generate the same ideal inside D, and since they sum to $\mathbf{1}_D$, they are both full projections. Now, note that

$$(\mathbf{1} \oplus 0)D(\mathbf{1} \oplus 0) = F(\varphi(A), B_\infty) \oplus 0,$$

and that by $\mathcal{O}_2$-stability of B, and Theorem 2.6.8 applied in absence of group actions, $\mathcal{O}_2$ embeds unitally into $(\mathbf{1} \oplus 0)D(\mathbf{1} \oplus 0)$. Hence, we have that $\mathbf{1} \oplus 0$ is a properly infinite, full projection in D with $[1 \oplus 0]_0 = 0$ in $K_0(D)$. Clearly, one can conclude the same for $0 \oplus \mathbf{1}$. Hence, by Lemma 2.1.44, $\mathbf{1} \oplus 0$ is Murray–von Neumann equivalent to $0 \oplus \mathbf{1}$, and by Proposition 3.1.11, φ and ψ are approximately Murray–von Neuman equivalent.

For the "moreover" part, consider the case where A, B, φ and ψ are all unital. Then by approximate Murray–von Neumann equivalent there exists a contraction $v \in B_\infty$ such that

$$v^*\varphi(a)v = \psi(a), \quad \varphi(a) = v\psi(a)v^*$$

for all $a \in A$. In particular, taking $a = \mathbf{1}_A$, we discover that v was in fact a unitary to begin with, and shows proper approximate unitary equivalence.

Assume now that B is stable. We will first show that φ and ψ approximately unitarily equivalent (with unitaries in $\mathcal{M}(B)$), and at the very end lift this to a proper approximate unitary equivalence. Note that we may replace the codomain of φ and ψ with $B \otimes \mathcal{K}$ without loss of generality. Let $\mathcal{K}$ be concretely represented as $\mathcal{K}(\ell^2(\mathbb{N}))$, and fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $\ell^2(\mathbb{N})$. Define the isometries $s_1, s_2 \in \mathcal{B}(\ell^2(\mathbb{N}))$ given by

$$s_1(e_n) = e_{2n-1}, \quad s_2(e_n) = e_{2n}$$

for all $n \in \mathbb{N}$. We now proceed to show that $\mathrm{Ad}(s_1)$ is approximately unitarily equivalent to $\mathrm{id}_\mathcal{K}$. Let u_k be the unitary operator on $\ell^2(\mathbb{N})$ such that $u_k(e_n) = s_1(e_n) = e_{2n-1}$ and $u_k(e_{2n-1}) = e_n$ for all $n \leq k$. The C*-algebra of compact operators is generated by the operators $E_{\xi,\eta}$ given by $E_{\xi,\eta}(\zeta) = \langle \zeta, \eta\rangle \cdot \xi$, where $\xi, \eta, \zeta \in \ell^2(\mathbb{N})$. For this reason, we only need to check that $\mathrm{Ad}(s_1)$ is approximately unitarily equivalent to $\mathrm{id}_\mathcal{K}$ on these operators. This is easily seen to be true via the sequence of unitaries $(u_k)_k$, because

$$u_k s_1 E_{e_n,e_m} s_1^* u_k^* = E_{u_k s_1 e_n, u_k s_1 e_m} = E_{e_n,e_m} \text{ if } k \geq \max\{n, m\}.$$

If now one chooses $t_i = \mathbf{1}_{\mathcal{M}(B)} \otimes s_i$ for $i = 1, 2$, then $\mathrm{Ad}(t_1)$ is approximately unitarily equivalent to $\mathrm{id}_{B\otimes\mathcal{K}}$. It follows that $\mathrm{Ad}(t_1) \circ \varphi$ is approximately unitarily equivalent to φ and the same is true for ψ. Note that $t_1t_1^* + t_2t_2^* = \mathbf{1}$, and therefore the map $M_2(B \otimes \mathcal{K}) \to B \otimes \mathcal{K}$ defined on elementary tensors by

$$b \otimes e_{i,j} = t_i b t_j^*, \quad b \in B,\, i, j \in \mathbb{N},$$

is an isomorphism. Since, by assumption, φ and ψ are approximately Murray–von Neumann equivalent, it follows from Proposition 3.1.11 that $\varphi \oplus 0$ and $\psi \oplus 0$ are approximately unitarily equivalent as maps into $M_2(B \otimes \mathcal{K})$. The isomorphism above tells us precisely that $\mathrm{Ad}(t_1) \circ \varphi$ is approximately unitarily equivalent to $\mathrm{Ad}(t_1) \circ \psi$. By transitivity of approximate unitary equivalence, we conclude that φ and ψ are approximately unitarily equivalent via a sequence of unitaries $w_n \in \mathcal{U}(\mathcal{M}(B))$. Now, since B is $\mathcal{O}_2$-stable, we conclude with Theorem 2.6.27, applied to the case where $G = \{1\}$, that the first factor embedding

$$\mathrm{id} \otimes \mathbf{1}_{\mathcal{O}_2} : B \to B \otimes \mathcal{O}_2,$$

is properly approximately unitarily equivalent to an isomorphism $\theta : B \to B \otimes \mathcal{O}_2$. Hence, by transitivity it is sufficient to prove that the $*$-homomorphisms

$$\varphi_1 = (\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2}) \circ \varphi \quad \text{and} \quad \psi_1 = (\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2}) \circ \psi$$

are properly approximately unitarily equivalent. Define a new sequence of unitaries given by

$$u_n := w_n \otimes \mathbf{1}_{\mathcal{O}_2} \in \mathcal{U}(\mathcal{M}(B) \otimes \mathcal{O}_2) \subseteq \mathcal{U}(\mathcal{M}(B \otimes \mathcal{O}_2)).$$

We know by Lemma 2.1.66 that the unitaries u_n lie in the connected component of the unit, namely $\mathcal{U}_0(\mathcal{M}(B) \otimes \mathcal{O}_2)$, which is contained in $\mathcal{U}_0(\mathcal{M}(B \otimes \mathcal{O}_2))$. By Lemma 2.2.29, we thus have that $\mathrm{Ad}(u_n)$ is properly approximately inner for all $n \in \mathbb{N}$. Fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ with dense union. It follows from what was said before that, for each $n \in \mathbb{N}$, there exists a unitary $v_n \in \mathcal{U}(\mathbf{1} + B \otimes \mathcal{O}_2)$ such that

$$\max_{a \in \mathcal{F}_n} \| v_n \varphi_1(a) v_n^* - u_n \varphi_1(a) u_n^* \| \leq 2^{-n}.$$

Thus, for any $a \in A$ and $\varepsilon > 0$, one may find some $n \in \mathbb{N}$ and $a_n \in \mathcal{F}_n$ such that $a =_{\varepsilon/4} a_n$ and $2^{-n} \leq \varepsilon/4$, so that

$$v_n \varphi_1(a) v_n^* =_{\varepsilon/4} v_n \varphi_1(a_n) v_n^* =_{\varepsilon/4} u_n \varphi_1(a_n) u_n^* =_{\varepsilon/4} \psi_1(a_n) =_{\varepsilon/4} \psi_1(a),$$

which shows that φ_1 is properly approximately unitarily equivalent to ψ_1, and concludes the proof. □

We now pass to the existence part of the classification. In Gabe's original proof from [64], the existence relies on (a version of) Michael's selection theorem; see [141–143].[7] This is a very powerful result that allows one to lift certain maps. However, for this strategy of proof to work out, one needs the target C*-algebra to be nuclear. Later, Gabe gave a different proof of the existence theorem in [66], which does not rely on Michael's selection theorem, and where nuclearity of the target C*-algebra is not needed in the assumptions. The new approach is based on a deep result of Bosa–Gabe–Sims–White [19], which is built on Voiculescu's quasidiagonality theorem [204]. We record this result here without proof.

[7] Another application of Michael's selection theorem will appear also in Chapter 5, when we need to lift maps from a quotient.

Lemma 3.1.14 (see [19, Lemma 3.5])**.** *Let A be a separable C^*-algebra, and B an $\mathcal{O}_\infty$-stable C^*-algebra. Suppose that $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ is a* **Cu***-morphism. Then, there exists a $*$-homomorphism $\varphi : \mathcal{C}_0(0,1] \otimes A \to B_\infty$ such that the c.p.c. order zero map $\varphi \circ (\mathrm{id}_{(0,1]} \otimes -) : A \to B_\infty$ is nuclearly liftable, and*

$$\mathcal{I}(\varphi)(I \otimes J) = \mathcal{I}(\iota_\infty) \circ \Phi(J)$$

for all $J \in \mathcal{I}(A)$ and all non-zero $I \in \mathcal{I}(\mathcal{C}_0(0,1])$.

Remark 3.1.15. Note that, in Lemma 3.1.14, if one additionally asks that A is exact, then it follows from Proposition 2.1.34 that $\varphi \circ (\mathrm{id}_{(0,1]} \otimes -) : A \to B_\infty$ is nuclear. Then, by Remark 2.1.30, the corresponding $*$-homomorphism, which is $\varphi : CA \to B_\infty$ in Lemma 3.1.14, is also nuclear.

The following uniqueness-type result is another key ingredient in the existence theorem. Let us also note that this result can be viewed as a consequence of the Löwenheim–Skolem theorem (cf. Blackadar's method).

Lemma 3.1.16. *Let A and B be C^*-algebras with A separable, and $\varphi, \psi : A \to B$ two nuclear $*$-homomorphisms such that φ weakly approximately 1-dominates ψ. Then, there exists a separable C^*-subalgebra $D \subseteq B$ containing the range of φ and ψ such that, when corestricted to D, φ and ψ are nuclear, and φ weakly approximately 1-dominates ψ.*

Proof. Fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ whose union is dense in A. Since we assumed that φ weakly approximately 1-dominates ψ, we may find elements $c_n \in B$, which we can assume to be contraction as observed in Remark 3.1.3, such that

$$\max_{a \in \mathcal{F}_n} \|\psi(a) - c_n^* \varphi(a) c_n\| \leq 2^{-n}$$

for all $n \in \mathbb{N}$. By nuclearity of φ and ψ, we may also find two sequences of natural numbers $(k_\varphi^{(n)})_{n\in\mathbb{N}}$ and $(k_\psi^{(n)})_{n\in\mathbb{N}}$, and c.p.c. maps $\chi_\varphi^{(n)}$, $\theta_\varphi^{(n)}$, $\chi_\psi^{(n)}$ and $\theta_\psi^{(n)}$ such that for each $n \in \mathbb{N}$ the following diagrams

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\chi_\varphi^{(n)}"'] & & B \\ & M_{k_\varphi^{(n)}} \arrow[ur, "\theta_\varphi^{(n)}"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\chi_\psi^{(n)}"'] & & B \\ & M_{k_\psi^{(n)}} \arrow[ur, "\theta_\psi^{(n)}"'] & \end{tikzcd}$$

commute up to 2^{-n} uniformly over $\mathcal{F}_n$, i.e.,

$$\max_{a\in\mathcal{F}_n} \|\varphi(a) - \theta_\varphi^{(n)} \circ \chi_\varphi^{(n)}(a)\| \leq 2^{-n} \text{ and } \max_{a\in\mathcal{F}_n} \|\psi(a) - \theta_\psi^{(n)} \circ \chi_\psi^{(n)}(a)\| \leq 2^{-n}.$$

Then, let D be the separable C*-subalgebra of B generated by the set

$$\varphi(A) \cup \psi(A) \cup \{c_n\}_{n\in\mathbb{N}} \cup \left\{\theta_\varphi^{(n)}(M_{k_\varphi^{(n)}})\right\}_{n\in\mathbb{N}} \cup \left\{\theta_\psi^{(n)}(M_{k_\psi^{(n)}})\right\}_{n\in\mathbb{N}}.$$

Let us check that, when corestricted to D, φ weakly approximately 1-dominates ψ, and that they are nuclear with this codomain. To this end, fix a finite subset $\mathcal{F} \subseteq A$, and $\varepsilon > 0$. (We use the same $\mathcal{F}$ and ε for both approximate 1-domination and nuclearity.) We can find some $n \in \mathbb{N}$ such that every element of $\mathcal{F}$ is $\frac{\varepsilon}{3}$-close to an element of $\mathcal{F}_n$. Then, after possibly taking a larger n, we may also assume that $2^{-n} \leq \frac{\varepsilon}{3}$. Hence, by construction, D contains the contraction c_n, which satisfies

$$\max_{a\in\mathcal{F}} \|\psi(a) - c_n^*\varphi(a)c_n\| \leq 2\frac{\varepsilon}{3} + 2^{-n} \leq \varepsilon.$$

Let us also show that φ is nuclear when corestricted to D. (The proof for ψ is the same, word for word.) By construction, we have a diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\chi_\varphi^{(n)}"'] & & D \\ & M_{k_\varphi^{(n)}} \arrow[ur, "\theta_\varphi^{(n)}"'] & \end{tikzcd}$$

that commutes up to 2^{-n} uniformly over $\mathcal{F}_n$. It follows that

$$\max_{a\in\mathcal{F}} \|\varphi(a) - \theta_\varphi^{(n)} \circ \chi_\varphi^{(n)}(a)\| \leq 2\frac{\varepsilon}{3} + 2^{-n} \leq \varepsilon,$$

which shows nuclearity, and hence ends the proof. □

The following lemma follows from a reindexation argument. We omit the proof as we will give many similar ones in the sequel. We also mention here that proofs that require some kind of diagonal sequence argument (or reindexation) can also be viewed as a consequence of saturation (see [61]).

Lemma 3.1.17 (c.f. [64, Lemma 4.1]). *Let A and B be* C*-*algebras with A separable, and $\varphi, \psi : A \to B_\infty$ two $*$-homomorphisms. Then φ and ψ are Murray–von Neumann equivalent if and only if they are approximately Murray–von Neumann equivalent. The same is true if one replaces (approximate) Murray–von Neumann equivalence with (approximate) unitary equivalence.*

We are now ready to present the existence theorem.

Theorem 3.1.18 (see [66, Theorem 14.1] and [64, Theorem 6.1])**.** *Let A be a separable exact* C*-*algebra, and B an $\mathcal{O}_\infty$-stable, separable* C*-*algebra. Then, for every* **Cu**-*morphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$, there exists a nuclear $*$-homomorphism $\varphi : A \to B$ such that $\mathcal{I}(\varphi) = \Phi$.*

Proof. First of all, pick any non-zero embedding $\mathcal{O}_2 \otimes \mathcal{K} \hookrightarrow \mathcal{O}_\infty$, and notice that, on ideals, this induces an order isomorphism

$$\mathcal{I}(B \otimes \mathcal{O}_2 \otimes \mathcal{K}) \to \mathcal{I}(B \otimes \mathcal{O}_\infty), \quad I \otimes \mathcal{O}_2 \otimes \mathcal{K} \mapsto I \otimes \mathcal{O}_\infty.$$

Hence, we may assume that B is stable and $\mathcal{O}_2$-stable. By Lemma 3.1.14 and the discussion in Remark 3.1.15, we know that there exists a nuclear $*$-homomorphism

$$\rho : CA = \mathcal{C}_0(0,1] \otimes A \to B_\infty$$

with the property that

$$\mathcal{I}(\rho)(I \otimes J) = \mathcal{I}(\iota_\infty) \circ \Phi(J),$$

for all $J \in \mathcal{I}(A)$ and all non-zero ideals $I \in \mathcal{I}(\mathcal{C}_0(0,1])$. Now, call by ψ_1 the restriction of ρ to the suspension $SA = \mathcal{C}_0(0,1) \otimes A$ of A. Clearly, this is still nuclear, and satisfies

$$\mathcal{I}(\psi_1)(I \otimes J) = \mathcal{I}(\iota_\infty) \circ \Phi(J),$$

for all $J \in \mathcal{I}(A)$ and all non-zero ideals $I \in \mathcal{I}(\mathcal{C}_0(0,1))$, where $\mathcal{I}(\iota_\infty)$ is the map on ideals induced by the canonical inclusion as constant sequences $\iota_\infty : B \to B_\infty$. Consider now an automorphism $\alpha \in \mathrm{Aut}(\mathcal{C}_0(0,1))$ inducing an isomorphism $\mathcal{C}_0(0,1) \rtimes_\alpha \mathbb{Z} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}$.[8] Now, set $\beta = \alpha \otimes \mathrm{id}_A : G \curvearrowright SA$. We know that $\psi_1 \circ \beta$ is nuclear, and that $\mathcal{I}(\psi_1 \circ \beta) = \mathcal{I}(\psi_1)$ thanks to the ideal relation of ψ_1. By the uniqueness theorem, that is, Theorem 3.1.13, we have that ψ_1 and $\psi_1 \circ \beta$ are approximately Murray–von Neumann equivalent. By Lemma 3.1.17, ψ_1 and $\psi_1 \circ \beta$ are Murray–von Neumann equivalent, i.e., there exists a contraction $u \in B_\infty$ such that

$$u\psi_1(x)u^* = (\psi_1 \circ \beta)(x) \quad \text{and} \quad u^*(\psi_1 \circ \beta(x))u = \psi_1(x)$$

for all $x \in SA$. One may therefore infer with Lemma 3.1.10 that u satisfies the following conditions,

$$uu^*\psi_1(x) = \psi_1(x) \quad \text{and} \quad u^*u\psi_1(x) = u^*(\psi_1 \circ \beta(x))u = \psi_1(x)$$

[8] A possible choice for such an automorphism α on $\mathcal{C}_0(0,1) \cong \mathcal{C}_0(\mathbb{R})$ is given by $\alpha(f)(t) = f(t+1)$ for $f \in \mathcal{C}_0(\mathbb{R})$ and $t \in \mathbb{R}$.

for all $x \in SA$. In other words, uu^* and u^*u act as a unit on the hereditary C*-subalgebra of B_∞ generated by $\psi_1(SA)$, which is given by

$$D = \overline{\psi_1(SA)B_\infty\psi_1(SA)}.$$

Then, again using the conclusions of Lemma 3.1.10, we have that

$$u\psi_1(x) = (\psi_1 \circ \beta(x))u \quad \text{and} \quad \psi_1(x)u = \psi_1(\beta(\beta^{-1}(x))u = u\psi_1(\beta^{-1}(x))$$

for all $x \in SA$, which implies that u belongs to the C*-algebra

$$N = \{y \in B_\infty \mid yD + Dy \subseteq D\}.$$

Now, the universal property of the multiplier algebra yields a $*$-homomorphism

$$m_\bullet : N \to \mathcal{M}(D), \quad m_y d = yd,\, dm_y = dy.$$

Since we showed that uu^* and u^*u act as a unit on D, we may conclude that m_u is a unitary in $\mathcal{M}(D)$. Thus, by the universal property of the full crossed product there exists a $*$-homomorphism

$$\psi_0 : SA \rtimes_\beta \mathbb{Z} \to \mathcal{M}(D)$$

such that $\psi_0(xv^n) = \psi_1(x)m_u^n = \psi_1(x)u^n$ and $\psi_0(xv^{-n}) = \psi_1(x)m_{u^*}^n = \psi_1(x)(u^*)^n$ for all $x \in SA$, where $v \in \mathcal{M}(SA \rtimes_\beta \mathbb{Z})$ is the canonical unitary implementing β in the crossed product. In particular, the relations satisfied by ψ_0 on the generators of $SA \rtimes_\beta \mathbb{Z}$ show that it factors through B_∞, and also that $\psi_0 \restriction_{SA} = \psi_1$. Therefore, we may (and will) view ψ_0 as a $*$-homomorphism with range in B_∞. Note that ψ_0 is nuclear by Lemma 2.5.17. To simplify notation later, let us use the notation $E := \mathcal{C}_0(\mathbb{R}) \rtimes_\alpha \mathbb{Z}$, and denote by $\theta : E \otimes A \to SA \rtimes_\beta \mathbb{Z}$ the natural isomorphism from Remark 2.4.14. We know that $\mathcal{C}_0(\mathbb{R}) \rtimes_\alpha \mathbb{Z} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}$ contains a full projection p. We use this projection to define a $*$-homomorphism

$$\psi : A \to B_\infty, \quad \psi(a) = \psi_0(\theta(p \otimes a)).$$

From the relation that ψ_1 satisfies on ideals, we know that if $J \in \mathcal{I}(A)$ is generated by a positive element $a \in A_+$, and $f \in \mathcal{C}_0(\mathbb{R})$ is any non-zero positive function, then

$$\psi_0(\theta((fw^n) \otimes a)) = \psi_1(f \otimes a)u^n \in \mathcal{I}(\iota_\infty) \circ \Phi(\overline{AaA})$$

and also

$$\mathcal{I}(\iota_\infty) \circ \Phi(J) = \mathcal{I}(\psi_1)(\overline{SA(f \otimes a)SA}) \subseteq \mathcal{I}(\psi_0) \circ \mathcal{I}(\theta)(\overline{E \otimes A(f \otimes a)E \otimes A})$$

$$\subseteq \mathcal{I}(\psi_0) \circ \mathcal{I}(\theta)(\overline{E \otimes A(p \otimes a) E \otimes A}) = \mathcal{I}(\psi)(J).$$

where to pass from the first to the second line we used that p is full. Morever, ψ is nuclear because ψ_0 is. We would now like to apply Theorem 2.2.39. Hence, consider a sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\eta(n) \to \infty$ as $n \to \infty$. First of all, let us show that $\mathcal{I}(\psi)$ and $\mathcal{I}(\eta^* \circ \psi)$ coincide on every ideal $J \in \mathcal{I}(A)$. Since Φ is a **Cu**-morphism, it sends $\mathcal{I}(A) = \mathcal{I}_\sigma(A)$ to $\mathcal{I}_\sigma(B)$. Hence, one may find a positive element $b \in B_+$ such that $\Phi(J) = \overline{BbB}$. Then,

$$\mathcal{I}(\psi)(J) = \mathcal{I}(\iota_\infty) \circ \Phi(J) = \overline{B_\infty b B_\infty}.$$

Next, by functoriality of $\mathcal{I}(-)$, we have that $\mathcal{I}(\eta^* \circ \psi) = \mathcal{I}(\eta^*) \circ \mathcal{I}(\psi)$, and therefore

$$\mathcal{I}(\eta^* \circ \psi)(J) = \mathcal{I}(\eta^*)(\overline{B_\infty b B_\infty}) = \overline{B_\infty \eta^*(b) B_\infty} = \overline{B_\infty b B_\infty},$$

which means that $\mathcal{I}(\eta^* \circ \psi) = \mathcal{I}(\psi)$. By Theorem 3.1.5, this implies that ψ and $\eta^* \circ \psi$ weakly approximately dominate each other. Since B is $\mathcal{O}_\infty$-stable, we conclude with Lemma 3.1.8 that they also weakly approximately 1-dominate each other. Now we apply Lemma 3.1.16 to find a separable C*-subalgebra C_1 of B_∞ that contains the ranges of ψ and $\eta^* \circ \psi$, and such that, when corestricted to C_1, ψ and $\eta^* \circ \psi$ remain nuclear and they weakly approximately 1-dominate each other. As we argued at the beginning of the proof, B can also be assumed to be $\mathcal{O}_2$-stable, and hence we may find with Theorem 2.6.8 a unital copy of $\mathcal{O}_2$ inside $F(C_1, B_\infty)$. Consider the following commutative diagram

$$\begin{tikzcd}
C_1 \arrow[rr, hook] \arrow[dr, hook, "\mathrm{id}_{C_1} \otimes \mathbf{1}"'] & & B_\infty \\
& C_1 \otimes_{\max} F(C_1, B_\infty) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where ρ is the canonical $*$-homomorphism given by

$$\rho\left(c \otimes (x + B_\infty \cap C_1^\perp)\right) = cx + B_\infty \cap C_1^\perp$$

for all $c \in C_1$ and $x \in B_\infty \cap C_1'$. Now, the C*-algebra given by $C_0 := \rho(C_1 \otimes \mathcal{O}_2) \subseteq B_\infty$ is separable, $\mathcal{O}_2$-stable, and ψ, $\eta^* \circ \psi$ remain nuclear and weakly approximately 1-dominate each other when viewed as maps with range in C_0. We then argue in a similar way, and use Lemma 2.8.8(ii) to get a commutative diagram

$$\begin{tikzcd}
C_0 \arrow[rr, hook] \arrow[dr, hook, "\kappa"'] & & B_\infty \\
& C_0 \otimes \mathcal{K} \arrow[ur, "\iota"'] &
\end{tikzcd}$$

where κ is the embedding given by

$$\kappa(c) = c \otimes e_{1,1},$$

where $e_{1,1} \in \mathcal{K}$ is picked from a set of matrix units generating $\mathcal{K}$, and ι satisfies $\iota(c \otimes e_{1,1}) = c$ for all $c \in C_0$. Then, we may pick as new codomain for ψ and $\eta^* \circ \psi$ the separable, $\mathcal{O}_2$-stable, stable C*-algebra $C := \iota(C_0 \otimes \mathcal{K})$. Since ψ and $\eta^* \circ \psi$ weakly approximately 1-dominate each other when corestricted to C, and are nuclear, it follows by Theorem 3.1.5 that they also induce the same **Cu**-morphism on ideals with this codomain, and therefore by the stable version of Theorem 3.1.13 that they are properly approximately unitarily equivalent. Since $C^\dagger \subseteq B_\infty^\dagger$, we may also conclude that ψ and $\eta^* \circ \psi$ are properly approximately unitarily equivalent when considered as maps with their original codomain B_∞. Finally, this proves that the assumptions of Theorem 2.2.39 are satisfied, and thus there exists a $*$-homomorphism $\varphi : A \to B$ such that $\iota_\infty \circ \varphi$ is properly unitarily equivalent to ψ. Therefore, there exists a unitary $U \in \mathcal{U}(\mathbf{1} + B_\infty)$ such that

$$\iota_\infty \circ \varphi(a) = U\psi(a)U^* = U\psi(a)U^* \in B_\infty, \quad \text{for all } a \in A.$$

Recall that nuclearity of ψ is equivalent to nuclear liftability by Proposition 2.1.34. Fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ with dense union, and lift ψ to a sequence of nuclear c.p.c. maps $(\psi_n)_{n\in\mathbb{N}} : A \to \ell^\infty(\mathbb{N}, B)$, and U to a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(\mathbf{1} + B)$. We may assume that $(\psi_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ satisfy the condition

$$\lim_{n\to\infty} \max_{a\in\mathcal{F}_n} \|\varphi(a) - u_n\psi_n(a)u_n^*\| \leq 2^{-n}$$

for all $n \in \mathbb{N}$. In particular, by nuclearity of $\mathrm{Ad}(u_n) \circ \psi_n$ for all $n \in \mathbb{N}$, one may conclude that φ is also nuclear. Moreover,

$$\mathcal{I}(\iota_\infty) \circ \mathcal{I}(\varphi) = \mathcal{I}(\iota_\infty \circ \varphi) = \mathcal{I}(\psi) = \mathcal{I}(\iota_\infty) \circ \Psi,$$

and injectivity of the map $\mathcal{I}(\iota_\infty)$ implies that $\mathcal{I}(\varphi) = \Phi$. The proof is hence concluded. □

Remark 3.1.19. Note that Theorem 3.1.18 is a generalisation of the celebrated $\mathcal{O}_2$-embedding theorem (see Theorem 2.1.47). In fact, if one chooses $B = \mathcal{O}_2$, and $\Phi : \mathcal{I}(A) \to \mathcal{I}(\mathcal{O}_2)$ given by $\Phi(I) = \mathcal{O}_2$ for all non-zero ideals $I \in \mathcal{I}(A)$, then the theorem produces a $*$-homomorphism from any separable exact C*-algebra A

$$\varphi : A \to \mathcal{O}_2$$

such that $\varphi(a) \in \mathcal{O}_2$ is full for any non-zero $a \in A$. In particular, this means that if $\varphi(a) = 0$ then $a = 0$, and therefore φ is an embedding.

Finally, we can put together a classification of maps as anticipated at the beginning of this section.

Corollary 3.1.20 (see [64, Corollaries 6.11+6.12])**.** *Let A be a separable, exact* C^**-algebra, B a separable, $\mathcal{O}_2$-stable* C^**-algebra. If B is moreover stable, the functor $\mathcal{I}(-)$ induces a one-to-one correspondence*

$$\frac{\left\{\begin{array}{c}\textit{nuclear } *\textit{-homomorphisms}\\ \varphi : A \to B\end{array}\right\}}{\approx_{\mathrm{pu}}} \xrightarrow{\mathcal{I}(-)} \left\{\begin{array}{c}\mathbf{Cu}\textit{-morphisms}\\ \mathcal{I}(A) \to \mathcal{I}(B)\end{array}\right\}$$

If A and B are both unital, the functor $\mathcal{I}(-)$ induces a one-to-one correspondence

$$\frac{\left\{\begin{array}{c}\textit{unital nuclear } *\textit{-homomorphisms}\\ \varphi : A \to B\end{array}\right\}}{\approx_{\mathrm{pu}}} \xrightarrow{\mathcal{I}(-)} \left\{\begin{array}{c}\mathbf{Cu}\textit{-morphisms}\\ \Phi : \mathcal{I}(A) \to \mathcal{I}(B)\\ \textit{such that } \Phi(A) = B\end{array}\right\}$$

As explained before, one can now prove Theorem 3.1.1 via Elliott's two-sided intertwining argument, Theorem 2.2.30.

3.2 Ad hoc classification results

The main classification theorem will be proved at the very end of this chapter. However, we would like to emphasise that a few special cases of our main result can be retrieved with known techniques. To be precise, this is possible when G is $\mathbb{Z}$, $\mathbb{R}$, or compact. This was pointed out by Szabó and the author in [156], and will be outlined here. We give relatively condensed proofs as all these results will in any case follow as byproducts of Theorem 3.5.5. First, we point out an important observation about existence of actions on $\mathcal{O}_2$.

Remark 3.2.1. Assume G is exact. In light of Remark 2.7.9, there exists an amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable action $\delta : G \curvearrowright \mathcal{O}_2$, which is unique up to strong cocycle conjugacy by the dynamical Kirchberg–Phillips theorem [67]. In particular, $\delta \simeq_{\mathrm{cc}} \delta \otimes \delta$. By using the uniqueness part of the dynamical Kirchberg–Phillips theorem [67, Theorem 5.7] (or the uniqueness result from the next section, Theorem 3.3.15) the δ-to-$(\delta \otimes \delta)$ equivariant first factor embedding $\mathrm{id}_{\mathcal{O}_2} \otimes \mathbf{1}_{\mathcal{O}_2}$ is approximately unitarily equivalent to a cocycle conjugacy, and δ is thus strongly self-absorbing.

Proposition 3.2.2. *Assume G is compact. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions with the Rokhlin property on separable, nuclear, $\mathcal{O}_2$-stable* C^**-algebras that are either both unital or both stable. Then for every conjugacy*

$f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ *there exists a conjugacy* $\varphi : (A, \alpha) \to (B, \beta)$ *such that* $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Prim}(A)$.

Proof. Let f be a conjugacy between $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$ and $\beta^\sharp : G \curvearrowright \mathrm{Prim}(B)$, and Φ the induced equivariant order isomorphism between ideal lattices. One can use Theorem 3.1.1 to obtain an isomorphism $\varphi : A \to B$ that lifts f and Φ. Consider now the action $\beta' : G \curvearrowright B$ given by $\beta'_g = \varphi \circ \alpha_g \circ \varphi^{-1}$ for all $g \in G$. Since Φ is equivariant with respect to $\alpha^\sharp$ and $\beta^\sharp$, then β_{co} and α_{co} induce the same **Cu**-morphism on the ideal lattices. Hence, $\mathcal{I}(\beta'_{\mathrm{co}}) = \mathcal{I}(\beta_{\mathrm{co}})$, which by the uniqueness theorem, Theorem 3.1.13, implies that β'_{co} is approximately unitarily equivalent to β_{co}. It follows that β' is conjugate to β via an approximately inner automorphism of B by Theorem 2.6.21. This in turn implies that there exists a conjugacy between α and β that induces f. $\square$

Proposition 3.2.3. *Let* $\alpha : \mathbb{R} \curvearrowright A$ *and* $\beta : \mathbb{R} \curvearrowright B$ *be Rokhlin flows on separable, nuclear,* $\mathcal{O}_2$*-stable* C^**-algebras that are either both unital or both stable. Then for every conjugacy* $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ *there exists a cocycle conjugacy* $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ *such that* $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Prim}(A)$.

Proof. This follows from Szabó's classification of Rokhlin flows on $\mathcal{O}_\infty$-stable C^*-algebras, [191, Theorem B], which lifts to a strong classification theorem as stated here as observed in [191, Remark 5.16]. $\square$

Proposition 3.2.4. *Let* $\alpha : \mathbb{Z} \curvearrowright A$ *and* $\beta : \mathbb{Z} \curvearrowright B$ *be actions with the Rokhlin property on separable, nuclear,* $\mathcal{O}_2$*-stable* C^**-algebras that are either both unital or both stable. Then for every conjugacy* $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ *there exists a cocycle conjugacy* $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ *such that* $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Prim}(A)$.

Proof. This can be achieved directly using the Evans–Kishimoto intertwining method [57]. The form of this result that we need here can be found in the lecture notes of a mini-course given by Szabó at NCGOA in 2018; see [193]. In particular, if A is a separable $\mathcal{O}_2$-stable C^*-algebra, [193, Theorem 3.1] states that any two automorphisms of A are approximately unitarily equivalent if and only if the induced $\mathbb{Z}$-actions are cocycle conjugate via an approximately inner automorphism of A. Let α and β denote both the $\mathbb{Z}$-actions in the statement, and the automorphisms that induce them. Hence, let $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ be a conjugacy as in the statement, and $\varphi : A \to B$ a (not necessarily equivariant) isomorphism inducing it, which exists by Theorem 3.1.1. Then, consider the automorphism given by $\beta' := \varphi \circ \alpha \circ \varphi^{-1}$, whose induced $\mathbb{Z}$-action is clearly conjugate to α. Then, since $\mathcal{I}(\beta') = \mathcal{I}(\beta)$, we have by

Theorem 3.1.13 that β and β' are approximately unitarily equivalent, and hence the associated actions are conjugate by the version of the Evans–Kishimoto intertwining stated above. ☐

3.3 Uniqueness results

In this section, we work our way towards the uniqueness result underpinning the classification in Section 3.5. We will prove this result in a few steps. Inspired by the non-dynamical setting, and by Gabe–Szabó's framework to prove the dynamical Kirchberg–Phillips theorem [67, 68], we introduce the following.

Definition 3.3.1 (see [68, Definition 3.3]). Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two cocycle morphisms. We say that $(\varphi, \mathbb{u})$ *approximately 1-dominates* $(\psi, \mathbb{v})$, if for every compact subset $K \subseteq G$, finite subset $\mathcal{F} \subset A$, $\varepsilon > 0$, and contraction $b \in B$, there exists $c \in B$ such that

$$\max_{g \in K} \|b^* \mathbb{v}_g \beta_g(b) - c^* \mathbb{u}_g \beta_g(c)\| \leq \varepsilon, \tag{e1}$$

and

$$\max_{a \in \mathcal{F}} \|b^* \psi(a) b - c^* \varphi(a) c\| \leq \varepsilon. \tag{e2}$$

Remark 3.3.2. Note that, in Definition 3.3.1, one may always choose c as a contraction: Let a quadruple $(K, \mathcal{F}, \varepsilon, b)$ as above be given. Choose $\eta > 0$ such that $(1 + \max_{a \in \mathcal{F}} \|a\|)\eta \leq \varepsilon/2$. If $c \in B$ is chosen to satisfy conditions (e1) and (e2) for the quadruple $(K \cup \{1_G\}, \mathcal{F}, \eta, b)$, then the first condition at $g = 1_G$ entails that $\|c\|^2 \leq \|b^* b\| + \|b^* b - c^* c\| \leq 1 + \eta$. Then $c_0 = (1 + \eta)^{-1/2} c \in B$ is a contraction satisfying

$$\|b^* \mathbb{v}_g \beta_g(b) - c_0^* \mathbb{u}_g \beta_g(c_0)\| \leq \eta + \|c_0^* \mathbb{u}_g \beta_g(c_0)((1 + \eta) - 1)\| \leq 2\eta \leq \varepsilon$$

for all $g \in K$, and similarly

$$\begin{aligned} \|b^* \psi(a) b - c_0^* \varphi(a) c_0\| &\leq \eta + \|c_0^* \varphi(a) c_0((1 + \eta) - 1)\| \\ &\leq (1 + \max_{a \in \mathcal{F}} \|a\|)\eta \leq \varepsilon \end{aligned}$$

for all $a \in \mathcal{F}$.

Notation 3.3.3. Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two cocycle morphisms. We say that φ *approximately 1-dominates* ψ *as ordinary* $*$*-homomorphisms* if $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$ in the sense of Definition 3.3.1 when viewed as proper cocycle morphisms with respect to the action of $G = \{1\}$.

We now compare weak approximate 1-domination and approximate 1-domination for $*$-homomorphisms. It turns out that weak approximate 1-domination is in fact a weaker notion, which justifies the prefix *weak*.

Lemma 3.3.4. *Let A and B be* C**-algebras, and $\varphi, \psi : A \to B$ two $*$-homomorphisms. If φ approximately 1-dominates ψ as $*$-homomorphisms, then φ weakly approximately 1-dominates ψ.*

Proof. Fix a finite set $\mathcal{F} \subseteq A$, and $\varepsilon > 0$. Then find a positive contraction b in B such that $b\psi(a)b =_{\varepsilon/2} \psi(a)$ for all $a \in \mathcal{F}$, and use approximate 1-domination to find a $c \in B$ such that

$$\|c^*c - b^2\| \leq \varepsilon \quad \text{and} \quad \max_{a\in\mathcal{F}} \|b\psi(a)b - c^*\varphi(a)c\| \leq \varepsilon/2.$$

It follows that

$$\max_{a\in\mathcal{F}} \|\psi(a) - c^*\varphi(a)c\| \leq \varepsilon/2 + \max_{a\in\mathcal{F}} \|b\psi(a)b - c^*\varphi(a)c\| \leq \varepsilon.$$

Hence, the element c witnesses that φ weakly approximately 1-dominates ψ. $\square$

In the following lemma, we provide sufficient conditions under which weak approximate 1-domination and approximate 1-domination for $*$-homomorphisms are equivalent.

Lemma 3.3.5. *Let A be a* C**-algebra, B a stable* C**-algebra, and $\varphi, \psi : A \to B_\infty$ two $*$-homomorphisms. Then, φ weakly approximately 1-dominates ψ if and only if φ approximately 1-dominates ψ as $*$-homomorphisms.*

Proof. The "if" statement is always true. To prove that φ approximately 1-dominates ψ, fix a finite set $\mathcal{F} \subseteq A$, $\varepsilon > 0$, and a contraction $b \in B_\infty$. By assumption, there exists $c \in B_\infty$ such that

$$\|c^*\varphi(a)c - \psi(a)\| \leq \varepsilon$$

for all $a \in \mathcal{F}$. Moreover, by Remark 3.1.3 we can assume that c is a contraction. Next, let D be a separable C*-subalgebra of B_∞ containing $\{\varphi(\mathcal{F}), \psi(\mathcal{F}), b, c\}$. Invoking Lemma 2.8.8(ii), we have a $*$-homomorphism $\rho : D \otimes \mathcal{K} \to B_\infty$ such that $\rho(d \otimes e_{1,1}) = d$ for all $d \in D$, making the following diagram commutative,

$$\begin{tikzcd} D \arrow[rr, hook] \arrow[dr, hook, "\mathrm{id}_D \otimes e_{1,1}"'] & & B_\infty \\ & D \otimes \mathcal{K} \arrow[ur, "\rho"'] & \end{tikzcd}$$

By Lemma 2.8.8(i), one may find two sequences of isometries $r_{1,n}, r_{2,n} \in \mathcal{M}(D \otimes \mathcal{K})$ such that

$$r_{1,n}r_{1,n}^* + r_{2,n}r_{2,n}^* = \mathbf{1} \text{ for all } n \in \mathbb{N},$$

$$r_{1,n} \xrightarrow{n\to\infty} \mathbf{1} \text{ strictly},$$

and define a sequence of isometries $t_n \in \mathcal{M}(D \otimes \mathcal{K})$ given by

$$t_n = r_{1,n}r_{1,n}^*(c \otimes e_{1,1}) + r_{2,n}(\mathbf{1} - (c \otimes e_{1,1})^* r_{1,n}r_{1,n}^*(c \otimes e_{1,1}))^{1/2}.$$

Since $r_{1,n} \to \mathbf{1}$ strictly, it follows that there exists $N \in \mathbb{N}$ such that $t_N^*(\varphi(a) \otimes e_{1,1})t_N =_\varepsilon c^*\varphi(a)c \otimes e_{1,1}$ for all $a \in \mathcal{F}$. Let $\hat{c} = \rho(t_N(b \otimes e_{1,1})) \in B_\infty$. We have that

$$\hat{c}^*\varphi(a)\hat{c} = \rho((b \otimes e_{1,1})^* t_N^*(\varphi(a) \otimes e_{1,1})t_N(b \otimes e_{1,1})) =_\varepsilon b^*\psi(a)b$$

for all $a \in \mathcal{F}$, which implies condition (e2) of Definition 3.3.1. Additionally, we have that $b^*b = \hat{c}^*\hat{c}$, which in particular implies condition (e1) of Definition 3.3.1, and therefore that φ approximately 1-dominates ψ. □

Hence, we directly get the following result as a byproduct of the lemma above and Theorem 3.1.5.

Theorem 3.3.6. *Let A be an exact* C*-*algebra, and B a stable, $\mathcal{O}_\infty$-stable* C*-*algebra. If $\varphi, \psi : A \to B_\infty$ are nuclear $*$-homomorphisms such that $\mathcal{I}(\varphi) \geq \mathcal{I}(\psi)$, then φ approximately 1-dominates ψ as $*$-homomorphisms.*

Proof. By Theorem 3.1.5 and Lemma 3.1.8, one has that φ weakly approximately dominates ψ, and hence the statement follows from Lemma 3.3.5. □

Approximate 1-domination will be key throughout this section. We start with a few lemmas about its permanence properties and consequences.

Lemma 3.3.7. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on* C*-*algebras. Let $\varphi, \psi : (A, \alpha) \to (B, \beta)$ be equivariant $*$-homomorphisms. Then the following statements hold true.*

(i) *Assume that A is separable. If $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$, there exists a separable, β-invariant* C*-*subalgebra $D \subseteq B$ such that $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$ when corestricted to D.*

(ii) *Let $\kappa : (B, \beta) \hookrightarrow (B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_\mathcal{K})$ be the equivariant inclusion given by $\kappa(b) = b \otimes e_{1,1}$. Then, if $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$, $\kappa \circ (\varphi, \mathbf{1})$ approximately 1-dominates $\kappa \circ (\psi, \mathbf{1})$.*

(iii) $(\varphi, \mathbf{1})$ *approximately 1-dominates* $(\psi, \mathbf{1})$ *if and only if* $\iota_\infty \circ (\varphi, \mathbf{1})$ *approximately 1-dominates* $\iota_\infty \circ (\psi, \mathbf{1})$.[9]

Proof. We start by showing (i). Denote by D_0 the separable, β-invariant C*-subalgebra of B generated by the union of the image of φ and ψ. Fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ such that $\overline{\bigcup_{n\in\mathbb{N}} \mathcal{F}_n} = A$, and an increasing sequence of compact subsets $K_n \subseteq G$ such that $\bigcup_{n\in\mathbb{N}} K_n^\circ = G$.[10] Choose a countable approximate unit of contractions $(d_i^{(0)})_{i\in\mathbb{N}} \subseteq D_0$ for D_0. By assumption, there exists a contraction $c_i^{(0)} \in B$ (cf. Remark 3.3.2) such that

$$\max_{g\in K_0} \|(d_i^{(0)})^*\beta_g(d_i^{(0)}) - (c_i^{(0)})^*\beta_g(c_i^{(0)})\| \leq 1,$$

$$\max_{a\in\mathcal{F}_0} \|(d_i^{(0)})^*\psi(a)d_i^{(0)} - (c_i^{(0)})^*\varphi(a)c_i^{(0)}\| \leq 1$$

for all $i \in \mathbb{N}$. Let D_1 be the separable, β-invariant C*-subalgebra of B generated by D_0 and $(c_i^{(0)})_{i\in\mathbb{N}}$. Inductively define $D_{n+1} \subseteq B$ for $n \geq 1$ in the following way. Suppose $D_n \subseteq B$ is a separable, β-invariant C*-subalgebra of B containing D_{n-1}, and pick a countable approximate unit of contractions $(d_i^{(n)})_{i\in\mathbb{N}} \subseteq D_n$ for D_n. By assumption, there exists a contraction $c_i^{(n)} \in B$ that satisfies

$$\max_{g\in K_n} \|(d_i^{(n)})^*\beta_g(d_i^{(n)}) - (c_i^{(n)})^*\beta_g(c_i^{(n)})\| \leq 2^{-n},$$

$$\max_{a\in\mathcal{F}_n} \|(d_i^{(n)})^*\psi(a)d_i^{(n)} - (c_i^{(n)})^*\varphi(a)c_i^{(n)}\| \leq 2^{-n}$$

for all $i \in \mathbb{N}$. Let D_{n+1} be the separable, β-invariant C*-subalgebra of B generated by D_n and $(c_i^{(n)})_{i\in\mathbb{N}}$. Finally, set $D = \overline{\bigcup_{n\in\mathbb{N}} D_n}$. Let us check that D is the algebra we are looking for. Fix a finite subset $\mathcal{F} \subseteq A$, a compact subset $K \subseteq G$, $\varepsilon > 0$ and a contraction $b \in D$. If we choose $n \in \mathbb{N}$ large enough, we have that $2^{-n} \leq \varepsilon$, $K \subseteq K_n^\circ$, and there exists some $k_n \in \mathbb{N}$ such that $d_{k_n}^{(n)} b =_\varepsilon b$. Since the increasing union of the $\mathcal{F}_n$ was dense in A, we may assume (upon making n larger if necessary) that for every $x \in \mathcal{F}$ there is $a \in \mathcal{F}_n$ with $x =_\varepsilon a$. By construction, the contraction $c = c_{k_n}^{(n)} \in D$ satisfies

$$\max_{g\in K_n} \|d_{k_n}^{(n)*}\beta_g(d_{k_n}^{(n)}) - c^*\beta_g(c)\| \leq \varepsilon,$$

$$\max_{a\in\mathcal{F}_n} \|d_{k_n}^{(n)*}\psi(a)d_{k_n}^{(n)} - c^*\varphi(a)c\| \leq \varepsilon.$$

[9] Here, ι_∞ must be viewed as a map into the β-continuous sequence algebra $B_{\infty,\beta}$.
[10] Here, for any set K, K° denotes the interior of K.

Therefore, we end up with

$$\max_{g\in K}\|b^*\beta_g(b)-(cb)^*\beta_g(cb)\|\leq 2\varepsilon+\max_{g\in K}\|b^*(d_{k_n}^{(n)*}\beta_g(d_{k_n}^{(n)})-c^*\beta_g(c))\beta_g(b)\|\leq 3\varepsilon$$

and

$$\begin{aligned}&\max_{a\in\mathcal{F}}\|b^*\psi(a)b-(cb)^*\varphi(a)cb\|\\ \leq\ & 2\varepsilon+\max_{a\in\mathcal{F}_n}\|b^*\psi(a)b-(cb)^*\varphi(a)cb\|\\ \leq\ & 4\varepsilon+\max_{a\in\mathcal{F}_n}\|b^*(d_{k_n}^{(n)*}\psi(a)d_{k_n}^{(n)}-c^*\varphi(a)c)b\| \ \leq\ 5\varepsilon.\end{aligned}$$

Up to rescaling ε, this shows that $(\varphi,\mathbf{1})$ approximately 1-dominates $(\psi,\mathbf{1})$ when corestricted to D.

Now we show (ii). Fix $\varepsilon>0$, a finite subset $\mathcal{F}\subseteq A$, a compact subset $K\subseteq G$, and a contraction $d\in B\otimes\mathcal{K}$. Let $b\in B$ and $e\in\mathcal{K}$ be contractions such that $(b\otimes e)d=_\varepsilon d$ and $e^*e_{1,1}e=e_{1,1}$. Since $(\varphi,\mathbf{1})$ approximately 1-dominates $(\psi,\mathbf{1})$, there exists a contraction $c\in B$ such that

$$\max_{g\in K}\|b^*\beta_g(b)-c^*\beta_g(c)\|\leq\varepsilon,$$

$$\max_{a\in\mathcal{F}}\|b^*\psi(a)b-c^*\varphi(a)c\|\leq\varepsilon.$$

It follows that

$$\max_{g\in K}\|d^*(\beta\otimes\mathrm{id}_{\mathcal{K}})_g(d)-d^*(c\otimes e)^*(\beta\otimes\mathrm{id}_{\mathcal{K}})_g((c\otimes e)d)\|\leq 3\varepsilon,$$

and

$$\max_{a\in\mathcal{F}}\|d^*(\psi(a)\otimes e_{1,1})d-d^*(c\otimes e)^*(\varphi(a)\otimes e_{1,1})(c\otimes e)d\|\leq\varepsilon+2\varepsilon\max_{a\in\mathcal{F}}\|a\|.$$

Up to suitably rescaling ε, this shows that $\kappa\circ(\varphi,\mathbf{1})$ approximately 1-dominates $\kappa\circ(\psi,\mathbf{1})$.

To show (iii), assume that $(\varphi,\mathbf{1})$ approximately 1-dominates $(\psi,\mathbf{1})$. Let $b\in B_{\infty,\beta}$ be a contraction, which we may represent by a sequence of contractions $(b_n)_{n\in\mathbb{N}}\in\ell^\infty_\beta(\mathbb{N},B)$. Choose an increasing sequence of finite subsets $\mathcal{F}_n\subseteq A$ with dense union and increasing compact subsets $K_n\subseteq G$ with $G=\bigcup_{n\geq 1}K_n^\circ$. For every $n\geq 1$, we may find a contraction $c_n\in B$ satisfying

$$\max_{g\in K_n}\|b_n^*\beta_g(b_n)-c_n^*\beta_g(c_n)\|\leq 2^{-n},$$

$$\max_{a\in\mathcal{F}_n}\|b_n^*\psi(a)b_n-c_n^*\varphi(a)c_n\|\leq 2^{-n}.$$

Denote by $c \in B_\infty$ the element represented by the sequence $(c_n)_n$. Then clearly $b^*\beta_{\infty,g}(b) = c^*\beta_{\infty,g}(c)$ and $b^*\psi(a)b = c^*\varphi(a)c$ for all $g \in G$ and $a \in A$. We observe for every $g \in G$ that

$$\begin{aligned}\|\beta_{\infty,g}(c) - c\|^2 &= \|\beta_{\infty,g}(c^*c) + c^*c - \beta_{\infty,g}(c)^*c - c^*\beta_{\infty,g}(c)\| \\ &= \|\beta_{\infty,g}(b^*b) + b^*b - \beta_{\infty,g}(b)^*b - b^*\beta_{\infty,g}(b)\| \\ &= \|\beta_{\infty,g}(b) - b\|^2.\end{aligned}$$

It follows that $c \in B_{\infty,\beta}$, and $\iota_\infty \circ (\varphi, \mathbf{1})$ approximately 1-dominates $\iota_\infty \circ (\psi, \mathbf{1})$.

For the other direction, suppose that $\iota_\infty \circ (\varphi, \mathbf{1})$ approximately 1-dominates $\iota_\infty \circ (\psi, \mathbf{1})$. It follows that for any contraction $b \in B$, finite subset $\mathcal{F} \subseteq A$, compact subset $K \subseteq G$, and $\varepsilon > 0$, we may find $c \in B_{\infty,\beta}$ with representing sequence $(c_n)_{n\in\mathbb{N}} \in \ell^\infty_\beta(\mathbb{N}, B)$, such that

$$\begin{aligned}\max_{g\in K} \limsup_{n\to\infty} \|b^*\beta_g(b) - c_n^*\beta_g(c_n)\| &\leq \frac{\varepsilon}{2}, \\ \max_{a\in\mathcal{F}} \limsup_{n\to\infty} \|b^*\psi(a)b - c_n^*\varphi(a)c_n\| &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Hence, one may find a large enough $n \in \mathbb{N}$ such that

$$\begin{aligned}\max_{g\in K} \|b^*\beta_g(b) - c_n^*\beta_g(c_n)\| &\leq \varepsilon, \\ \max_{a\in\mathcal{F}} \|b^*\psi(a)b - c_n^*\varphi(a)c_n\| &\leq \varepsilon,\end{aligned}$$

and therefore $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$. $\square$

Lemma 3.3.8 (see [67, Lemma 2.4])**.** *Let $\alpha : G \curvearrowright A$ an action on a separable* C^**-algebra, and $\beta : G \curvearrowright B$ a strongly stable action on a* C^**-algebra. Let*

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$$

be proper cocycle morphisms, and suppose that there exists a sequence of contractions $s_n \in B$ such that for every $a \in A$ and every compact subset $K \subseteq G$,

(i) $\|s_n^*\varphi(a)s_n - \psi(a)\| \to 0$,

(ii) $\max_{g\in K} \|s_n^*\mathbb{u}_g\beta_g(s_n) - \mathbb{v}_g s_n^* s_n\| \to 0$,

(iii) $\max_{g\in K} \|s_n^*s_n - \beta_g(s_n^*s_n)\| \to 0$,

(iv) $\max_{g\in K} \|(\mathbf{1} - s_n^* s_n)(\mathbb{v}_g - \mathbf{1})\| \to 0$.

Then, there exists a sequence of isometries $S_n \in \mathcal{M}^\beta(B)$ such that

$$\lim_{n\to\infty} \|S_n^*\varphi(a)S_n - \psi(a)\| = 0, \quad \textit{and} \quad \lim_{n\to\infty} \max_{g\in K} \|\mathbb{u}_g\beta_g(S_n) - S_n\mathbb{v}_g\| = 0$$

for all $a \in A$ and for every compact subset $K \subseteq G$.

Proof. By Lemma 3.1.10, the element $v \in B_\infty$ induced by the sequence $(s_n)_n \subseteq B$ that, thanks to (i), satisfies

$$\varphi(a)v = v\psi(a), \quad \text{and} \quad v^*v\psi(a) = \psi(a)$$

for all $a \in A$ Hence, we have that

$$\|\varphi(a)s_n - s_n\psi(a)\| \to 0, \quad \text{and} \quad \|s_n^*s_n\psi(a) - \psi(a)\| \to 0 \tag{3.2}$$

for all $a \in A$. Moreover, for every compact subset $K \subseteq G$, we have that

$$\begin{aligned}
&\limsup_{n\to\infty} \max_{g\in K} \|s_n\mathbb{v}_g - \mathbb{u}_g\beta_g(s_n)\|^2 \\
&= \limsup_{n\to\infty} \max_{g\in K} \|(s_n\mathbb{v}_g - \mathbb{u}_g\beta_g(s_n))^*(s_n\mathbb{v}_g - \mathbb{u}_g\beta_g(s_n))\| \\
&= \limsup_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g^*s_n^*s_n\mathbb{v}_g - \beta_g(s_n)^*\mathbb{u}_g^*s_n\mathbb{v}_g - \mathbb{v}_g^*s_n^*\mathbb{u}_g\beta_g(s_n) + \beta_g(s_n^*s_n)\| \\
\text{(by (ii)+(iii))} &= \limsup_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g^*s_n^*s_n\mathbb{v}_g - s_n^*s_n\| \\
&= \limsup_{n\to\infty} \max_{g\in K} \|(\mathbb{v}_g - \mathbf{1})s_n^*s_n + \mathbb{v}_g^*s_n^*s_n(\mathbb{v}_g - \mathbf{1})\| \\
\text{(by (iv))} &= \limsup_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g^* - \mathbf{1} + \mathbb{v}_g^*(\mathbb{v}_g - \mathbf{1})\| \\
&= 0.
\end{aligned}$$

Now, since β is strongly stable, by Lemma 2.8.8(i) we may fix two sequences of $\mathcal{O}_2$-isometries $r_{1,n}, r_{2,n} \in \mathcal{M}(B)^\beta$ such that $r_{1,n}$ converges to $\mathbf{1}$ strictly. After passing to a subsequence of $(r_{1,n})_n$, if necessary, we also assume that

$$\lim_{n\to\infty} \|(\mathbf{1} - r_{1,n})s_n\| = 0. \tag{3.3}$$

Then, consider the sequence given by

$$S_n = r_{1,n}s_n + r_{2,n}(\mathbf{1} - s_n^*s_n)^{1/2} \in \mathcal{M}^\beta(B).$$

To check that each S_n is an isometry one needs to use that $r_{1,n}$ and $r_{2,n}$ have orthogonal range projections. Since $s_n^*s_n$ is an approximate unit, and $r_{1,n}$ satisfies (3.3), we may conclude that $\lim_n b(S_n - s_n) = 0$ for all $b \in B$, and $\lim_n (S_n - s_n)\psi(a) = 0$ for all $a \in A$. In particular,

$$\lim_{n\to\infty} \|\varphi(a)S_n - S_n\psi(a)\| = \lim_{n\to\infty} \|\varphi(a)s_n - s_n\psi(a)\| = 0$$

for all $a \in A$. Moreover, by using (iii), we have that $\lim_n \|(\mathbf{1} - s_n s_n^*)^{1/2}\| = \lim_n \|(\mathbf{1} - \beta_g(s_n s_n^*))^{1/2}\|$ uniformly on compact sets, which yields that

$$\lim_{n\to\infty} \max_{g\in K} \|S_n - \beta_g(S_n)\| = \lim_{n\to\infty} \max_{g\in K} \|s_n - \beta_g(s_n)\|$$

for every compact subset $K \subseteq G$. Since $\mathbb{u}$ has range in $\mathcal{U}(\mathbf{1} + B)$, we also have that

$$\lim_{n\to\infty} \max_{g\in K} \|(\mathbb{u}_g - \mathbf{1})\beta_g(S_n)\| = \lim_{n\to\infty} \max_{g\in K} \|(\mathbb{u}_g - \mathbf{1})\beta_g(s_n)\|$$

for every compact subset $K \subseteq G$. We may then also use (iv) to get that

$$\lim_{n\to\infty} \max_{g\in K} \|S_n(\mathbb{v}_g - \mathbf{1})\| = \lim_{n\to\infty} \max_{g\in K} \|s_n(\mathbb{v}_g - \mathbf{1})\|$$

for every compact subset $K \subseteq G$. Finally, putting everything together, we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \max_{g\in K} &\|S_n\mathbb{v}_g - \mathbb{u}_g\beta_g(S_n)\| \\
&= \lim_{n\to\infty} \max_{g\in K} \|S_n - \beta_g(S_n) + S_n(\mathbb{v}_g - \mathbf{1}) - (\mathbb{u}_g - \mathbf{1})\beta_g(S_n)\| \\
&= \lim_{n\to\infty} \max_{g\in K} \|s_n - \beta_g(s_n) + s_n(\mathbb{v}_g - \mathbf{1}) - (\mathbb{u}_g - \mathbf{1})\beta_g(s_n)\| \\
&= \lim_{n\to\infty} \max_{g\in K} \|s_n\mathbb{v}_g - \mathbb{u}_g\beta_g(s_n)\| = 0
\end{aligned}$$

for every compact subset $K \subseteq G$. Together with (3.2), this finishes the proof. $\square$

Lemma 3.3.9 (see [67, Lemma 2.5]). *Let $\alpha : G \curvearrowright A$ be an action on a separable* C**-algebra, and $\beta : G \curvearrowright B$ an action on a σ-unital* C**-algebra. Let*

$$(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$$

be cocycle morphisms and assume that $(\varphi, \mathbb{u})$ approximately 1-dominates $(\psi, \mathbb{v})$. Then, there exists a sequence of contractions $s_n \in B$ such that $s_n^*s_n$ is a (not necessarily increasing) β-invariant approximate unit, and

$$\lim_{n\to\infty} \|s_n^*\varphi(a)s_n - \psi(a)\| = 0, \quad \textit{and} \quad \lim_{n\to\infty} \max_{g\in K} \|s_n^*\mathbb{u}_g\beta_g(s_n) - \mathbb{v}_g s_n^* s_n\| = 0$$

for all $a \in A$ and for every compact subset $K \subseteq G$.

Proof. Fix a countable increasing approximate unit $e_n \in B$, and consider (e_n, e_n) as a countable increasing approximate unit of $B \oplus B$, which we equip with the action $\beta \oplus \beta^{\mathbb{v}}$. By Lemma 2.2.12 applied to the approximate unit (e_n, e_n) of $B \oplus B$ with action $\beta \oplus \beta^{\mathbb{v}}$, we may extract an approximate unit $E_n \in B \oplus B$ from the convex hull of $\{(e_n, e_n)\}_{n\in\mathbb{N}}$ that is aymptotically $\beta \oplus \beta^{\mathbb{v}}$-invariant. By choosing either the first or second direct summand of E_n (which are identical) for each $n \in \mathbb{N}$, we have a countable approximate unit $b_n \in B$ that is at the same time asymptoptically β-invariant and asymptotically $\beta^{\mathbb{v}}$-invariant. This means that, in particular, b_n approximately commutes with the range of $\mathbb{v}$ uniformly over compact sets as $n \to \infty$ because

$$\begin{aligned}\lim_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g b_n - b_n \mathbb{v}_g\| &= \lim_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g b_n \mathbb{v}_g^* - b_n\| \\ &= \lim_{n\to\infty} \max_{g\in K} \|\mathbb{v}_g \beta_g(b_n) \mathbb{v}_g^* - b_n\| = 0\end{aligned}$$

for all compact subsets $K \subseteq G$. Fix now an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ with dense union and an increasing sequence of compact subsets $K_n \subseteq G$ whose union is G. Then, using the definition of approximate 1-domination with $b = b_n$, we may find a sequence of contractions $s_n \in B$ such that

$$\lim_{n\to\infty} \|\psi(a) - s_n^* \varphi(a) s_n\| = \lim_{n\to\infty} \|b_n \psi(a) b_n - s_n^* \varphi(a) s_n\| = 0,$$

$$\lim_{n\to\infty} \max_{g\in K} \|b_n \mathbb{v}_g \beta_g(b_n) - s_n^* \mathbb{u}_g \beta_g(s_n)\| = 0$$

for all $a \in A$ and every compact subset $K \subseteq G$. Observe that applying the latter limit with $g = 1_G$, and using that b_n approximately commutes with the range of $\mathbb{v}$, we have that

$$\lim_{n\to\infty} \|s_n^* s_n - b_n^2\| = 0,$$

and hence, in particular $(s_n)_n$ has the desired properties. □

By combining Lemma 3.3.8 and Lemma 3.3.9, one straight away obtains the following useful corollary.

Corollary 3.3.10 (see [67, Corollary 2.6])**.** *Let $\alpha : G \curvearrowright A$ an action on a separable* C*-*algebra and $\beta : G \curvearrowright B$ a strongly stable action on a σ-unital* C*-*algebra. Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two proper cocycle morphisms such that $(\varphi, \mathbb{u})$ approximately 1-dominates $(\psi, \mathbb{v})$. Then, there exists a sequence of isometries $s_n \in \mathcal{M}^\beta(B)$ such that*

$$\lim_{n\to\infty} \|\varphi(a) s_n - s_n \psi(a)\| = 0, \ \textit{and} \ \lim_{n\to\infty} \max_{g\in K} \|\mathbb{u}_g \beta_g(s_n) - s_n \mathbb{v}_g\| = 0$$

for all $a \in A$ and every compact set $K \subseteq G$.

The following lemma is, together with results from [68], the driving force behind the dynamical Kirchberg–Phillips theorem [67]. In the context of this thesis, it will be crucial in the uniqueness result, Theorem 3.3.15, and remains an indispensable technical device in the overarching structure of this work. Given its importance for the overall results obtained here, we give a proof.

Lemma 3.3.11 (see [67, Lemma 3.16]). *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions on separable* C^**-algebras, and assume that β is amenable and isometrically shift-absorbing. Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two proper cocycle morphisms. If φ approximately 1-dominates ψ as ordinary $*$-homomorphisms, then $(\varphi, \mathbb{u})$ approximately 1-dominates $(\psi, \mathbb{v})$.*

Proof. Fix a Haar measure μ on G. Let $K \subseteq G$ be a compact set containing 1_G, $\mathcal{F}$ a compact set in the unit ball of A, $b \in B$ a contraction, and $\varepsilon > 0$. Since amenability of β implies that $\beta^{\mathbb{v}}$ is also amenable, we may find a continuous $\zeta \in \mathcal{C}_c(G, B)$ with compact support $R = \overline{\mathrm{supp}(\zeta)}$, $\|\zeta\|_2 \leq 1$ and such that

$$\max_{a \in \mathcal{F}} \|(\langle \zeta, \psi(a)\zeta \rangle - \psi(a))b\| \leq \varepsilon \tag{3.4}$$

$$\max_{g \in K} \|(\mathbf{1} - \langle \zeta, \bar{\beta}^{\mathbb{v}}_g(\zeta) \rangle)\mathbb{v}_g \beta_g(b)\| \leq \varepsilon \tag{3.5}$$

By Lemma 2.2.12, we may find an approximate unit of B that is asymptotically $\beta^{\mathbb{v}}$-invariant, and also approximately central with respect to B, and pick some contraction $e \in B$ out of it that satisfies the following properties,

$$\max_{g \in R} \|\beta^{\mathbb{v}}_g(e) - e\| \leq \varepsilon, \tag{3.6}$$

$$\max_{g \in K} \|\langle \zeta, e^2 \mathbb{v}_g \bar{\beta}_g(\zeta) \rangle - e\langle \zeta, \mathbb{v}_g \bar{\beta}_g(\zeta) \rangle e\| \leq \varepsilon, \tag{3.7}$$

$$\max_{a \in \mathcal{F}} \|\langle \zeta, e\psi(a)e\zeta \rangle - e\langle \zeta, \psi(a)\zeta \rangle e\| \leq \varepsilon, \tag{3.8}$$

$$\max_{g \in K} \|(\mathbf{1} - e)\beta_g(b)\| \leq \varepsilon. \tag{3.9}$$

By assumption, φ approximately 1-dominates ψ as $*$-homomorphisms. Hence, we may find a contraction $c \in B$ with the property that

$$\|e^2 - c^*c\| \leq \varepsilon, \tag{3.10}$$

$$\max_{a \in \mathcal{F}} \max_{g \in R} \|e\psi(\alpha_{g^{-1}}(a))e - c^*\varphi(\alpha_{g^{-1}}(a))c\| \leq \varepsilon. \tag{3.11}$$

As a word of caution, we remark that we can find an element $c \in B$ as in 3.11 because of the following reasons. By continuity of α and of the $*$-homomorphisms

φ, ψ, compactness of $R \times \mathcal{F} \subseteq G \times A$ can be used to ensure that there are finitely-many elements $(g_i, a_i) \in R \times \mathcal{F}$, for $i = 1, \dots, m$, that satisfy

$$\min_{1 \le i \le m} \Big(\|\psi(\alpha_{g_i^{-1}}(a_i)) - \psi(\alpha_{g^{-1}}(a))\| + \|\varphi(\alpha_{g_i^{-1}}(a_i)) - \varphi(\alpha_{g^{-1}}(a))\| \Big) \le \varepsilon/3,$$

for all pairs $(g, a) \in R \times \mathcal{F}$. Then, by approximate 1-domination, one may find an element $c \in B$ that satisfies 3.10 and

$$\max_{1 \le i \le m} \|e\psi(\alpha_{g_i^{-1}}(a_i))e - c^*\varphi(\alpha_{g_i^{-1}}(a_i))c\| \le \varepsilon/3.$$

Hence, the triangle inequality ensures that 3.11 holds. Using the characterisation of isometric shift-absorption from Proposition 2.7.8, there exists an equivariant linear B-bimodule map

$$\theta : (L^2_\infty(G, B), \bar{\beta}) \to (B_{\infty,\beta}, \beta_\infty)$$

such that $\theta(\xi)^*\theta(\eta) = \langle \xi, \eta \rangle_B$ for all $\xi, \eta \in L^2_\infty(G, B)$. Then consider the function $\xi \in \mathcal{C}_c(G, B)$ given by

$$\xi(g) = \mathbb{u}_g \beta_g(c) \mathbb{v}_g^* \zeta(g), \quad \text{for } g \in G.$$

Then, using that θ is an equivariant B-bimodule linear map, we may compute the following

$$\begin{aligned}
&\theta(\xi)^* \mathbb{u}_g \beta_g(\theta(\xi)) \\
&\quad = \theta(\xi)^*(\theta(\mathbb{u}_g \bar{\beta}_g(\xi))) \\
&\quad = \langle \xi, \mathbb{u}_g \bar{\beta}_g(\xi) \rangle \\
&\quad = \int_G \zeta(h)^* \mathbb{v}_h \beta_h(c^*) \mathbb{u}_h^* \mathbb{u}_g \beta_g(\mathbb{u}_{g^{-1}h} \beta_{g^{-1}h}(c) \mathbb{v}_{g^{-1}h}^* \zeta(g^{-1}h)) \, d\mu(h) \\
&\quad = \int_G \zeta(h)^* \mathbb{v}_h \beta_h(c^* c) \beta_g(\mathbb{v}_{g^{-1}h})^* \bar{\beta}_g(\zeta)(h) \, d\mu(h) \\
&\quad = \int_G \zeta(h)^* \beta_h^{\mathbb{v}}(c^* c) \mathbb{v}_g \bar{\beta}_g(\zeta)(h) \, d\mu(h) \\
\text{(by 3.6+3.7)} &\quad =_{2\varepsilon} \int_G \zeta(h)^* e^2 \mathbb{v}_g \bar{\beta}_g(\zeta)(h) \, d\mu(h) \\
&\quad = \langle \zeta, e^2 \mathbb{v}_g \bar{\beta}_g(\zeta) \rangle \\
\text{(by 3.7)} &\quad =_{\varepsilon} e \langle \zeta, \mathbb{v}_g \bar{\beta}_g(\zeta) \rangle e.
\end{aligned}$$

We use this to infer the following,

$$\begin{aligned} &\theta(\xi b)^*\mathbb{u}_g\beta_g(\theta(\xi b)) \\ &\quad =_{3\varepsilon}\ b^*e\langle\zeta, \mathbb{v}_g\bar{\beta}_g(\zeta)\rangle e\beta_g(b) \\ \text{(by 3.9)}\ &\quad =_{2\varepsilon}\ b^*\langle\zeta, \bar{\beta}^{\mathbb{v}}_g(\zeta)\rangle\mathbb{v}_g\beta_g(b) \\ \text{(by 3.5)}\ &\quad =_{\varepsilon}\ b^*\mathbb{v}_g\beta_g(b). \end{aligned}$$

In a similar fashion, we carry out the following calculation for every $a \in \mathcal{F}$,

$$\begin{aligned} &\theta(\xi b)^*\varphi(a)\theta(\xi b) \\ &\quad = b^*\theta(\xi)^*\theta(\varphi(a)\xi)b \\ &\quad = b^*\int_G \zeta(h)^*\mathbb{v}_h\beta_h(c)^*\mathbb{u}_h^*\varphi(a)\mathbb{u}_h\beta_h(c)\mathbb{v}_h^*\zeta(h)\,d\mu(h)\cdot b \\ &\quad = b^*\int_G \zeta(h)^*\mathbb{v}_h\beta_h(c)^*\beta_h(\varphi(\alpha_{h^{-1}}(a)))\beta_h(c)\mathbb{v}_h^*\zeta(h)\,d\mu(h)\cdot b \\ &\quad = b^*\int_G \zeta(h)^*\beta^{\mathbb{v}}_h(c^*\varphi(\alpha_{h^{-1}}(a))c)\zeta(h)\,d\mu(h)\cdot b \\ \text{(by 3.11)}\ &\quad =_{\varepsilon}\ b^*\int_G \zeta(h)^*\beta^{\mathbb{v}}_h(e\psi(\alpha_{h^{-1}}(a))e)\zeta(h)\,d\mu(h)\cdot b \\ \text{(by 3.6)}\ &\quad =_{2\varepsilon}\ b^*\int_G \zeta(h)^*e\beta^{\mathbb{v}}_h(\psi(\alpha_{h^{-1}}(a)))e\zeta(h)\,d\mu(h)\cdot b \\ &\quad = b^*\langle\zeta, e\psi(a)e\zeta\rangle b \\ \text{(by 3.8)}\ &\quad =_{\varepsilon}\ b^*e\langle\zeta, \psi(a)\zeta\rangle eb \\ \text{(by 3.4)}\ &\quad =_{\varepsilon}\ b^*\psi(a)b. \end{aligned}$$

We may now lift the element $\theta(\xi b) \in B_{\infty,\beta}$ to a sequence of contractions in B. Thanks to the computations above, we may find a contraction $v \in B$ such that

$$\max_{g\in K}\|b^*\mathbb{v}_g\beta_g(b) - v^*\mathbb{u}_g\beta_g(v)\| \leq 7\varepsilon,$$

$$\max_{a\in\mathcal{F}}\|b^*\psi(a)b - v^*\varphi(a)v\| \leq 8\varepsilon.$$

In particular, this shows that $(\varphi, \mathbb{u})$ approximately 1-dominates $(\psi, \mathbb{v})$. $\square$

Definition 3.3.12. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on C*-algebras. Suppose there exist a pair of isometries $s_1, s_2 \in \mathcal{M}(B)^\beta$ satisfying $s_1 s_1^* + s_2 s_2^* = \mathbf{1}$. One may define the *Cuntz sum* of elements $a, b \in \mathcal{M}(B)$ as

$$a \oplus_{s_1,s_2} b = s_1 a s_1^* + s_2 b s_2^* \in \mathcal{M}(B).$$

Let $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ be two (proper) cocycle morphisms. The Cuntz sum of $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ is the (proper) cocycle morphism $(\varphi \oplus_{s_1,s_2} \psi, \mathbb{u} \oplus_{s_1,s_2} \mathbb{v}) : (A, \alpha) \to (B, \beta)$ given pointwise by $(\varphi \oplus_{s_1,s_2} \psi)(a) = \varphi(a) \oplus_{s_1,s_2} \psi(a)$, and $(\mathbb{u} \oplus_{s_1,s_2} \mathbb{v})_g = \mathbb{u}_g \oplus_{s_1,s_2} \mathbb{v}_g$.

Note that whenever $a, b \in B$, their Cuntz sum lies in B as well. Moreover, $a \oplus_{s_1,s_2} b$ is unitarily equivalent to $b \oplus_{s_1,s_2} a$ via the unitary $U = s_1 s_2^* + s_2 s_1^*$, and if one picks another pair of isometries $t_1, t_2 \in \mathcal{M}(B)^\beta$ such that $t_1 t_1^* + t_2 t_2^* = \mathbf{1}$, then $a \oplus_{s_1,s_2} b$ is unitarily equivalent to $a \oplus_{t_1,t_2} b$ via the unitary $V = s_1 t_1^* + s_2 t_2^* \in \mathcal{M}(B)^\beta$. Given $a, b \in \mathcal{M}(B)$, we write "$a \oplus b$" as a shorthand notation for "$a \oplus_{s_1,s_2} b$".

The following technical lemma was derived and used in various forms, e.g., [173, Lemma 6.3.7] and [123, Lemma 2.4], and appeared in this form in [68, Lemma 3.8].

Lemma 3.3.13. *Let B be a* C*-*algebra, and $\sigma \in \mathrm{Aut}(B)$ an automorphism, which extends to an automorophism of $\mathcal{M}(B)$ canonically. Let $b_1, b_2 \in \mathcal{M}(B)$ be two elements, $s \in \mathcal{M}(B)$ an isometry, and $r_1, r_2 \in \mathcal{M}(B)^\sigma$ a pair of $\mathcal{O}_2$-isometries that $*$-commute with b_1. Then the element given by*

$$u = (r_1 r_1^* + r_2 s r_2^*) s^* + r_2(\mathbf{1} - s s^*) \in \mathcal{M}(B)$$

is a unitary, and satisfies

$$\|u b_2 \sigma(u)^* - (b_1 \oplus_{r_1,r_2} b_2)\| \leq 5 \cdot \max\left\{\|s b_1 - b_2 \sigma(s)\|, \|\sigma(s) b_1^* - b_2^* s\|\right\}.$$

Moreover, if $s b_1 - b_2 \sigma(s) \in B$ and $\sigma(s) b_1^* - b_2^* s \in B$, then

$$u b_2 \sigma(u)^* - (b_1 \oplus_{r_1,r_2} b_2) \in B.$$

Proof. To see that u is a unitary, we evaluate

$$u^* u = s(r_1 r_1^* + r_2 r_2^*) s^* + \mathbf{1} - s s^* = \mathbf{1}$$

and similarly,

$$u u^* = r_1 r_1^* + r_2 s s^* r_2^* + r_2(\mathbf{1} - s s^*) r_2^* = \mathbf{1}.$$

If $\varepsilon := \max\left\{\|sb_1 - b_2\sigma(s)\|, \|\sigma(s)b_1^* - b_2^* s\|\right\}$, then we have that

$$\begin{aligned}
&(b_1 \oplus_{r_1,r_2} b_2)\sigma(u) \\
&= (r_1 b_1 r_1^* + r_2 b_2 r_2^*)\sigma(u) \\
&= r_1 b_1 r_1^* \sigma(s)^* + r_2 b_2 r_2^* \sigma(u) \\
&= r_1 r_1^* b_1 \sigma(s)^* + r_2 b_2(\mathbf{1} - \sigma(s)\sigma(s)^*) + r_2 b_2 \sigma(s) r_2^* \sigma(s)^* \\
&=_{2\varepsilon} r_1 r_1^* b_1 \sigma(s)^* + r_2(b_2 - sb_1\sigma(s)^*) + r_2 s b_1 r_2^* \sigma(s)^* \\
&= r_1 r_1^* b_1 \sigma(s)^* + r_2(b_2 - sb_1\sigma(s)^*) + r_2 s r_2^* b_1 \sigma(s)^* \\
&=_{2\varepsilon} r_1 r_1^* b_1 \sigma(s)^* + r_2(\mathbf{1} - ss^*)b_2 + r_2 s r_2^* s^* b_2 \\
&=_{\varepsilon} (r_1 r_1^* s^* + r_2(\mathbf{1} - ss^*) + r_2 s r_2^* s^*)b_2 = ub_2.
\end{aligned}$$

By multiplying $\sigma(s)^*$ on the right, we obtain the statement. The moreover part of the statement can be deduced from the same calculation because when $sb_1 - b_2\sigma(s) \in B$ and $\sigma(s)b_1^* - b_2^* s \in B$ also the intermediate differences that appear here are elements of B. $\square$

Lemma 3.3.14. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable* C**-algebras, and $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ proper cocycle morphisms. Suppose that β is strongly stable and equivariantly $\mathcal{O}_2$-stable. If $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ approximately 1-dominate each other, then $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent.*

Proof. Since β is equivariantly $\mathcal{O}_2$-stable, Theorem 2.6.27 implies that there exists a proper cocycle conjugacy

$$(\theta, \mathbb{x}) : (B, \beta) \to (B \otimes \mathcal{O}_2, \beta \otimes \mathrm{id}_{\mathcal{O}_2})$$

that is properly approximately unitarily equivalent to the equivariant first factor embedding $\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2}$. By transitivity of proper approximate unitary equivalence, it suffices to show that the proper cocycle morphisms given by

$$(\varphi_1, \mathbb{u}^{(1)}) := (\varphi \otimes \mathbf{1}_{\mathcal{O}_2}, \mathbb{u} \otimes \mathbf{1}_{\mathcal{O}_2}), \text{ and}$$

$$(\psi_1, \mathbb{v}^{(1)}) := (\psi \otimes \mathbf{1}_{\mathcal{O}_2}, \mathbb{v} \otimes \mathbf{1}_{\mathcal{O}_2})$$

are properly approximately unitarily equivalent. We start by showing that they are approximately unitarily equivalent with unitaries in $\mathcal{M}^\beta(B \otimes \mathcal{O}_2)$.

Since $(\varphi, \mathbb{u})$ approximately 1-dominates $(\psi, \mathbb{v})$, we know from Corollary 3.3.10 that there exist isometries $s_n \in \mathcal{M}^\beta(B)$ such that

$$\lim_{n\to\infty} \|\varphi(a)s_n - s_n\psi(a)\| = 0,$$

$$\lim_{n\to\infty} \max_{g\in K} \|\mathbb{u}_g\beta_g(s_n) - s_n\mathbb{v}_g\| = 0$$

for all $a \in A$ and for all compact sets $K \subseteq G$. We now fix $\mathcal{O}_2$-isometries $r_i \in \mathcal{O}_2$ for $i = 1, 2$, and define the following elements,

$$S_n = s_n \otimes \mathbf{1}_{\mathcal{O}_2} \in \mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B \otimes \mathcal{O}_2), \qquad n \in \mathbb{N},$$

$$R_i = \mathbf{1}_{\mathcal{M}(B)} \otimes r_i \in \mathcal{M}(B \otimes \mathcal{O}_2)^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}, \qquad i = 1, 2.$$

Henceforth, all Cuntz sums will be defined via the $\mathcal{O}_2$-isometries R_i, which commute with the image of $\varphi_1, \psi_1, \mathbb{u}^{(1)}$ and $\mathbb{v}^{(1)}$. By Lemma 3.3.13, for each $n \in \mathbb{N}$, the elements W_n given by

$$W_n = (R_1R_1^* + R_2S_nR_2^*)S_n^* + R_2(\mathbf{1} - S_nS_n^*) \in \mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B)$$

are unitaries that, thanks to the properties of the sequence $(s_n)_n$, satisfy

$$\lim_{n\to\infty} \|W_n\varphi_1(a)W_n^* - (\psi_1 \oplus \varphi_1)(a)\| = 0,$$

$$\lim_{n\to\infty} \max_{g\in K} \|\mathbb{V}_g \oplus \mathbb{U}_g - W_n\mathbb{U}_g(\beta_g \otimes \mathrm{id}_{\mathcal{O}_2})(W_n)^*\| = 0$$

for all $a \in A$ and for all compact sets $K \subseteq G$. Starting from the assumption that $(\psi, \mathbb{v})$ approximately 1-dominates $(\varphi, \mathbb{u})$, the analogous argument where the roles of $(\psi, \mathbb{v})$ and $(\varphi, \mathbb{u})$ are switched yields that $(\psi_1, \mathbb{v}^{(1)})$ is approximately unitarily equivalent to $(\varphi_1 \oplus \psi_1, \mathbb{u}^{(1)} \oplus \mathbb{v}^{(1)})$ via a sequence of unitaries in $\mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B \otimes \mathcal{O}_2)$. Since $(\varphi_1 \oplus \psi_1, \mathbb{u}^{(1)} \oplus \mathbb{v}^{(1)})$ and $(\psi_1 \oplus \varphi_1, \mathbb{v}^{(1)} \oplus \mathbb{u}^{(1)})$ are unitarily equivalent via the unitary $R_2R_1^* + R_1R_2^* \in \mathcal{M}(B \otimes \mathcal{O}_2)^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}$, by transitivity we have that $(\varphi_1, \mathbb{u}^{(1)})$ is approximately unitarily equivalent to $(\psi_1, \mathbb{v}^{(1)})$ via a sequence of unitaries U_n in $\mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B \otimes \mathcal{O}_2)$. In the rest of the proof, we show that it is possible to lift $(U_n)_n$ to a sequence of unitaries in the smallest (proper) unitization at the cost of composing the cocycle morphisms with $\mathrm{id} \otimes \mathbf{1}_{\mathcal{O}_2}$ once more. First of all, observe that the unitaries given by

$$u_n := U_n \otimes \mathbf{1}_{\mathcal{O}_2} \in \mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B \otimes \mathcal{O}_2) \otimes \mathcal{O}_2$$

implement approximate unitary equivalence between

$$(\varphi_2, \mathbb{u}^{(2)}) := (\varphi_1 \otimes \mathbf{1}_{\mathcal{O}_2}, \mathbb{u}^{(1)} \otimes \mathbf{1}_{\mathcal{O}_2}) \text{ and } (\psi_2, \mathbb{v}^{(2)}) := (\psi_1 \otimes \mathbf{1}_{\mathcal{O}_2}, \mathbb{v}^{(1)} \otimes \mathbf{1}_{\mathcal{O}_2}).$$

By the same argument used in the first paragraph of the proof, and the fact that $\mathcal{O}_2$ absorbs itself, in order to show the statement it suffices to prove that $(\varphi_2, \mathbb{u}^{(2)})$ and $(\psi_2, \mathbb{v}^{(2)})$ are properly approximately unitarily equivalent. By Lemma 2.1.66, for all $n \in \mathbb{N}$, u_n is a unitary in

$$\mathcal{U}_0(\mathcal{M}^{\beta\otimes\mathrm{id}_{\mathcal{O}_2}}(B \otimes \mathcal{O}_2) \otimes \mathcal{O}_2) \subseteq \mathcal{U}_0(\mathcal{M}(B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2)).$$

Hence by Lemma 2.2.29, $(\mathrm{Ad}(u_n), u_n\beta_\bullet(u_n)^*)$ is properly approximately unitarily equivalent to the identity map for every $n \in \mathbb{N}$. Thus, fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ with dense union, and an increasing sequence of compact subsets $K_n \subseteq G$ such that $\bigcup_n K_n^\circ = G$. As a consequence of the observation above, for each $n \in \mathbb{N}$, we may find a unitary $v_n \in \mathcal{U}(\mathbf{1}+B\otimes\mathcal{O}_2\otimes\mathcal{O}_2)$ such that

$$\max_{a\in\mathcal{F}_n} \|v_n\varphi_2(a)v_n^* - u_n\varphi_2(a)u_n^*\| \leq 2^{-n},$$

$$\max_{g\in K_n} \|v_n\mathbb{u}_g^{(2)}\beta_g(v_n)^* - u_n\mathbb{u}_g^{(2)}\beta_g(u_n)^*\| \leq 2^{-n}.$$

As a consequence, $(v_n)_n$ witnesses proper approximate unitary equivalence between $(\varphi_2, \mathbb{u}^{(2)})$ and $(\psi_2, \mathbb{v}^{(2)})$ as shown below. For each $a \in A$, compact set $K \subseteq G$ and $\varepsilon > 0$, one may find some $n \in \mathbb{N}$ and $a_n \in \mathcal{F}_n$ such that $a =_{\varepsilon/4} a_n$, $K \subseteq K_n$, $2^{-n} \leq \varepsilon/4$ and also

$$u_n\varphi_2(a_n)u_n^* =_{\varepsilon/4} \psi_2(a_n) =_{\varepsilon/4} \psi_2(a) \quad \text{and} \quad u_n\mathbb{u}_g^{(2)}\beta_g(u_n)^* =_{3\varepsilon/2} \mathbb{v}_g^{(2)}.$$

Then one has that

$$v_n\varphi_2(a)v_n^* =_{\varepsilon/4} v_n\varphi_2(a_n)v_n^* =_{\varepsilon/4} u_n\varphi_2(a_n)u_n^* =_{\varepsilon/4} \psi_2(a_n) =_{\varepsilon/4} \psi_2(a),$$

and for every $g \in K$,

$$v_n\mathbb{u}_g^{(2)}\beta_g(v_n)^* =_{\varepsilon/4} u_n\mathbb{u}_g^{(2)}\beta_g(u_n)^* =_{3\varepsilon/2} \mathbb{v}_g^{(2)}.$$

Thus, $(\varphi_2, \mathbb{u}^{(2)})$ and $(\psi_2, \mathbb{v}^{(2)})$ are properly approximately unitarily equivalence, and the proof is concluded. □

We are ready to prove the main result of this section:

Theorem 3.3.15 (Uniqueness)**.** *Let $\alpha : G \curvearrowright A$ be an action on a separable, exact C^*-algebra, and $\beta : G \curvearrowright B$ an amenable, equivariantly $\mathcal{O}_2$-stable, strongly stable, and isometrically shift-absorbing action on a separable C^*-algebra. If $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ are two proper cocycle morphisms with φ and ψ nuclear, then $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$ if and only if $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent.*

Proof. If $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent, then it follows straightaway that $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$.

Assume now that $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$. Note that, since φ and ψ are nuclear, Theorem 3.3.6 ensures that $\iota_\infty \circ \varphi$ and $\iota_\infty \circ \psi$ approximately 1-dominate each other as $*$-homomorphisms. Thus, we may conclude with part (iii) of Lemma 3.3.7 that φ and ψ approximately 1-dominate each other as $*$-homomorphisms, and with Lemma 3.3.11 that $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ approximately 1-dominate each other. One can therefore conclude the proof with Lemma 3.3.14. □

We derive a unital version of the uniqueness theorem when G is exact. Note that if G is not exact, then an action β as in the following corollary cannot exist as a consequence of Corollary 2.5.32.

Corollary 3.3.16. *Assume G is exact. Let $\alpha : G \curvearrowright A$ be an action on a separable, exact, unital* C^**-algebra, and $\beta : G \curvearrowright B$ an amenable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a separable, unital* C^**-algebra. If $(\varphi, \mathbb{u}), (\psi, \mathbb{v}) : (A, \alpha) \to (B, \beta)$ are two proper cocycle morphisms with φ and ψ nuclear and unital, then $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$ if and only if $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent.*

Proof. Let us denote by B^s the C^*-algebra $B \otimes \mathcal{K}$ and by β^s the action $\beta \otimes \mathrm{id}_{\mathcal{K}} : G \curvearrowright B \otimes \mathcal{K}$. Consider the canonical inclusion

$$\kappa : (B, \beta) \hookrightarrow (B^s, \beta^s), \quad \kappa(b) = b \otimes e_{1,1},$$

which is β-to-β^s-equivariant. We apply Theorem 3.3.15 to $\kappa \circ (\varphi, \mathbb{u})$ and $\kappa \circ (\psi, \mathbb{v})$,[11] and conclude that they are properly approximately unitarily equivalent. Let us denote by $(v_n)_n \subseteq \mathcal{U}(\mathbf{1} + B^s)$ the sequence of unitaries witnessing the equivalence. Then we have that

$$\lim_{n\to\infty} \|[v_n, \mathbf{1}_B \otimes e_{1,1}]\| = \lim_{n\to\infty} \|v_n \kappa(\varphi(\mathbf{1}_A)) - \kappa(\psi(\mathbf{1}_A))v_n\| = 0.$$

Now, for each $n \in \mathbb{N}$, the element $(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1})$ is invertible in the unital hereditary C^*-subalgebra of B^s generated by $(\mathbf{1}_B \otimes e_{1,1})$, and the element

$$(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1}) \cdot |(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1})|^{-1}$$

is a unitary in this hereditary subalgebra (see, e.g., [172, Proposition 2.1.8]), where we used the *absolute value* notation:

$$|(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1})| = ((\mathbf{1}_B \otimes e_{1,1})v_n^*(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1}))^{1/2}.$$

[11] Recall from Definition 2.2.18 that $\kappa \circ (\varphi, \mathbb{u}) = (\kappa \circ \varphi, \kappa^\dagger(\mathbb{u}_\bullet))$.

Then, the new sequence of unitaries given by

$$u_n = \kappa^{-1}\Big((\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1}) \cdot |(\mathbf{1}_B \otimes e_{1,1})v_n(\mathbf{1}_B \otimes e_{1,1})|^{-1}\Big) \in \mathcal{U}(B)$$

satisfies

$$\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0,$$

$$\lim_{n\to\infty} \max_{g\in K} \|u_n\mathbb{u}_g\beta_g(u_n)^* - \mathbb{v}_g\| = 0$$

for all $a \in A$ and compact sets $K \subseteq G$. As a result, $(\varphi, \mathbb{u})$ and $(\psi, \mathbb{v})$ are properly approximately unitarily equivalent. □

3.4 Existence results

In this section, we establish a number of technical lemmas that build up to the existence theorem, Theorem 3.4.17, which underpins the final classification result in Section 3.5. Furthermore, some of the intermediate results that are established on the way can be of independent interest, and may have applications elsewhere.

First, we establish some groundwork that will be used throughout this section.

Definition 3.4.1 (see [65, Definition 3.1])**.** Let A and B be two C*-algebras. A c.p. map $\varphi : A \to \mathcal{M}(B)$ is said to be *weakly nuclear* if the c.p. map $b^*\varphi(-)b : A \to B$ is nuclear for all $b \in B$.

It turns out that weakly nuclear maps can be used to characterise exactness.

Proposition 3.4.2 (see [65, Proposition 3.2])**.** *A* C**-algebra A is exact if and only if, for any σ-unital* C**-algebra B and any weakly nuclear c.p. map $\varphi : A \to \mathcal{M}(B)$, one has that φ is nuclear.*

Lemma 3.4.3. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on* C**-algebras. For any weakly nuclear $*$-homomorphism $\psi : A \to \mathcal{M}(B)$, the $*$-homomorphism $\hat{\psi} : A \to \mathcal{M}(\mathcal{C}_0(G, B))$ defined in Notation 2.4.6 is weakly nuclear.*

Proof. For each $f \in \mathcal{C}_0(G, B)$, we know that

$$\begin{aligned} \mathrm{ev}_g(f^*\hat{\psi}(-)f) &= f(g)^*(\beta_g\psi(\alpha_{g^{-1}}(-)))f(g) \\ &= \beta_g\Big(\beta_g^{-1}(f(g))^*\psi(\alpha_{g^{-1}}(-))\beta_g^{-1}(f(g))\Big) \end{aligned}$$

is nuclear for all $g \in G$. Hence, we may apply Lemma 2.1.31 to conclude that $f^*\hat{\psi}(-)f$ is nuclear. This implies that $\hat{\psi}$ is weakly nuclear. $\square$

Lemma 3.4.4. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable* C*-*algebras. If $\psi : A \to \mathcal{M}(B)$ is a $*$-homomorphism such that $\mathcal{I}_B(\psi) : \mathcal{I}(A) \to \mathcal{I}(B)$ is equivariant with respect to $\alpha^\sharp$ and $\beta^\sharp$,*[12] *then the $*$-homomorphism $\hat{\psi} : A \to \mathcal{M}(\mathcal{C}_0(G, B))$ defined in Notation 2.4.6 satisfies*

$$\mathcal{I}_{\mathcal{C}_0(G,B)}(\hat{\psi})(I) = \mathcal{C}_0(G, \mathcal{I}_B(\psi)(I))$$

for all $I \in \mathcal{I}(A)$.

Proof. Combining Remarks 2.3.35 and 2.3.29 we can write the ideal generated by $\hat{\psi}(I)$ in $\mathcal{C}_0(G, B)$ as

$$\bigcap \left\{\ker(\mathrm{ev}_g \otimes \pi) \mid \pi\left(\overline{B(\beta_g \circ \psi \circ \alpha_{g^{-1}}(I))B}\right) = 0\right\},$$

where $g \in G$ and π ranges over all non-zero irreducible representations of B. Since we assumed $\mathcal{I}_B(\varphi)$ to be equivariant, we have that

$$\overline{B(\beta_g \circ \psi \circ \alpha_{g^{-1}}(I))B} = \beta_g^\sharp\left(\overline{B\psi(\alpha_{g^{-1}}^\sharp(I))B}\right) = \overline{B\psi(I)B}$$

for all $g \in G$. These observations ensure that

$$\begin{aligned}
\mathcal{I}_{\mathcal{C}_0(G,B)}(\hat{\psi})(I) &= \bigcap \left\{\ker(\mathrm{ev}_g \otimes \pi) \mid \pi\left(\overline{B(\beta_g \circ \psi \circ \alpha_{g^{-1}}(I))B}\right) = 0\right\} \\
&= \bigcap \left\{\ker(\mathrm{ev}_g \otimes \pi) \mid \pi(\mathcal{I}_B(\psi)(I)) = 0\right\} \\
&= \mathcal{C}_0\left(G, \bigcap \left\{\ker(\pi) \mid \pi(\mathcal{I}_B(\psi)(I)) = 0\right\}\right) \\
&= \mathcal{C}_0(G, \mathcal{I}_B(\psi)(I)),
\end{aligned}$$

where the intersections are taken over all $g \in G$ and all non-zero irreducible representations π of B, and the last identity follows from the fact that the ideal $\mathcal{I}_B(\psi)(I)$ corresponds to the open set

$$U = \{\ker(\pi) \mid \pi(\mathcal{I}_B(\psi)(I)) \neq 0\} \subseteq \mathrm{Prim}(B)$$

and the bijection in Remark 2.3.29 gives

$$\mathcal{I}_B(\psi)(I) = \bigcap_{\mathrm{Prim}(B)\setminus U} \ker(\pi) = \bigcap \{\ker(\pi) \mid \pi(\mathcal{I}_B(\psi)(I)) = 0\}.$$

$\square$

[12] Recall from Notation 2.3.18 that $\mathcal{I}_B(\psi)(I) = \overline{B\psi(I)B}$ for all $I \in \mathcal{I}(A)$.

Now we prove an equivariant analogue of [64, Corollary 5.7]. We record two versions of the lemma. First a more general result, which may be of independent interest, and then a second and more restrictive variant that is specifically tailored to our need.

Lemma 3.4.5. *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be a pair of actions on separable* C**-algebras. For any $*$-homomorphism $\psi : A \to \mathcal{M}(B)$ such that $\mathcal{I}_B(\psi) : \mathcal{I}(A) \to \mathcal{I}(B)$ is equivariant with respect to $\alpha^\sharp$ and $\beta^\sharp$, there exists an equivariant $*$-homomorphism*

$$\kappa : (A, \alpha) \to (\mathcal{M}(\mathcal{K}(\mathcal{H}_G) \otimes B), \mathrm{Ad}(\lambda) \otimes \beta)$$

such that

$$\mathcal{I}_{\mathcal{K}(\mathcal{H}_G)\otimes B}(\kappa)(I) = \mathcal{K}(\mathcal{H}_G) \otimes \mathcal{I}_B(\psi)(I) \tag{3.12}$$

for all $I \in \mathcal{I}(A)$.

If A is moreover exact, and ψ is weakly nuclear, then κ is nuclear.

Proof. Let us denote by $\pi : \mathcal{C}_0(G) \hookrightarrow \mathcal{B}(L^2(G))$ the representation of $\mathcal{C}_0(G)$ as multiplication operators on $L^2(G)$. Note that the $*$-homomorphism

$$\mathcal{C}_b^s(G, \mathcal{M}(B)) = \mathcal{M}(\mathcal{C}_0(G) \otimes B) \xhookrightarrow{\pi \otimes \mathrm{id}_B} \mathcal{M}(\mathcal{K} \otimes B)$$

is equivariant with respect to $\bar{\beta}$ and $\mathrm{Ad}(\lambda) \otimes \beta$.

We may now start the construction of an α-to-$(\mathrm{Ad}(\lambda) \otimes \beta)$ equivariant map $A \to \mathcal{M}(\mathcal{K}(\mathcal{H}_G) \otimes B)$ satisfying (3.12). Consider the α-to-$\bar{\beta}$-equivariant $*$-homomorphism $\hat{\psi} : A \to \mathcal{M}(\mathcal{C}_0(G, B))$ given as in Notation 2.4.6. Define an α-to-$(\mathrm{Ad}(\lambda) \otimes \beta)$-equivariant $*$-homomorphism given by

$$\kappa = (\pi \otimes \mathrm{id}_B) \circ \hat{\psi} : A \to \mathcal{M}(\mathcal{K}(\mathcal{H}_G) \otimes B).$$

By Lemma 2.3.20, we have that for any ideal $I \in \mathcal{I}(A)$,

$$\mathcal{I}_{\mathcal{K}(\mathcal{H}_G)\otimes B}(\kappa)(I) = \mathcal{I}_{\mathcal{K}(\mathcal{H}_G)\otimes B}(\pi \otimes \mathrm{id}_B) \circ \mathcal{I}_{\mathcal{C}_0(G,B)}(\hat{\psi})(I).$$

Since by Lemma 3.4.4 we also have that

$$\mathcal{I}_{\mathcal{C}_0(G,B)}(\hat{\psi})(I) = \mathcal{C}_0(G, \overline{B\psi(I)B}),$$

we may conclude that

$$\begin{aligned}\mathcal{I}_{\mathcal{K}(\mathcal{H}_G)\otimes B}(\kappa)(I) &= \mathcal{I}_{\mathcal{K}(\mathcal{H}_G)\otimes B}(\pi \otimes \mathrm{id}_B)(\mathcal{C}_0(G, \mathcal{I}_B(\psi)(I))) \\ &= \mathcal{K}(\mathcal{H}_G) \otimes \mathcal{I}_B(\psi)(I).\end{aligned}$$

For the moreover part, assume that ψ is weakly nuclear and A exact. By Lemma 3.4.3, it follows that $\hat{\psi}$ is weakly nuclear. We infer from Proposition 3.4.2 that $\hat{\psi}$ is nuclear if and only if it is weakly nuclear. In particular, this implies that κ is nuclear. $\square$

The following result is an adaptation of the previous lemma to a more specific setting that will ultimately lead us to the existence theorem at the end of this section. It is important to note that the central ingredient of our proof is the existence result in the non-dynamical setting, Theorem 3.1.18.

Lemma 3.4.6. *Let $\alpha : G \curvearrowright A$ be an action on a separable, exact* C^**-algebra, and $\beta : G \curvearrowright B$ an action on a separable* C^**-algebra. Then, for every equivariant* **Cu**-*morphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$, there exists an equivariant, nuclear $*$-homomorphism*

$$\kappa : (A, \alpha) \to (\mathcal{M}(\mathcal{K}(\mathcal{H}_G^\infty) \otimes B), \mathrm{Ad}(\lambda^\infty) \otimes \beta)$$

such that

$$\mathcal{I}_{\mathcal{K}(\mathcal{H}_G^\infty)\otimes B}(\kappa)(I) = \mathcal{K}(\mathcal{H}_G^\infty) \otimes \Phi(I) \tag{3.13}$$

for all $I \in \mathcal{I}(A)$.

Proof. By Theorem 3.1.18 there exists a nuclear $*$-homomorphism $\psi : A \to \mathcal{O}_\infty \otimes B$ such that $\overline{(\mathcal{O}_\infty \otimes B)\psi(I)(\mathcal{O}_\infty \otimes B)} = \mathcal{O}_\infty \otimes \Phi(I)$ for all $I \in \mathcal{I}(A)$. Hence, we may apply Lemma 3.4.5 and get an α-to-$(\mathrm{Ad}(\lambda) \otimes \mathrm{id}_{\mathcal{O}_\infty} \otimes \beta)$-equivariant $*$-homomorphism $\kappa_1 : A \to \mathcal{M}(\mathcal{K}(\mathcal{H}_G) \otimes \mathcal{O}_\infty \otimes B)$ that satisfies property (3.12) with B replaced by $\mathcal{O}_\infty \otimes B$.

We may now pick a unital copy of $\mathcal{O}_\infty$ inside $\mathcal{B}(\ell^2(\mathbb{N}))$, and denote by ι the unital inclusion $\mathcal{O}_\infty \hookrightarrow \mathcal{B}(\ell^2(\mathbb{N}))$. Then, we have that

$$\mathcal{K}(\mathcal{H}_G) \otimes \mathcal{O}_\infty \xhookrightarrow{\mathrm{id}_{\mathcal{K}(\mathcal{H}_G)}\otimes\iota} \mathcal{K}(\mathcal{H}_G) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \subseteq \mathcal{B}(\mathcal{H}_G^\infty).$$

Observe that $\mathrm{id}\otimes\iota$ is equivariant with respect to $\mathrm{Ad}(\lambda)\otimes\mathrm{id}_{\mathcal{O}_\infty}$ and $\mathrm{Ad}(\lambda^\infty)$. The map given by $\kappa = (\mathrm{id}_{\mathcal{K}(\mathcal{H}_G)} \otimes \iota \otimes \mathrm{id}_B) \circ \kappa_1$ is an α-to-$(\mathrm{Ad}(\lambda^\infty) \otimes \beta)$-equivariant $*$-homomorphism satisfying (3.13).

Since A is exact and ψ is weakly nuclear, it follows from the moreover part of Lemma 3.4.5 that κ_1 is nuclear. This implies that κ is nuclear as well. $\square$

Remark 3.4.7. It was observed in [67, Remark 4.5] that there exists an equivariant embedding $\iota : (\mathcal{K}(\mathcal{H}_G^\infty), \mathrm{Ad}(\lambda^\infty)) \hookrightarrow (\mathcal{O}_\infty, \gamma)$, given on rank one operators[13] by $\iota(E_{\xi,\eta}) = \mathfrak{s}(\xi)\mathfrak{s}(\eta)^*$ for all ξ, η non-zero vectors in $\mathcal{H}_G^\infty$.

[13] A rank one operator $E_{\xi,\eta}$ for $\xi, \eta \in \mathcal{H}_G^\infty \setminus \{0\}$, is defined by $E_{\xi,\eta}(\zeta) = \eta \cdot \langle \xi, \zeta \rangle$ for all $\zeta \in \mathcal{H}_G^\infty$.

In the following lemma we establish a general property of separable C*-algebras that shows that it is enough to know a fullness-type condition on a countable set in order for it to be true on the entire set of positive elements. We note that the countable subset defined in the lemma is independent of the C*-algebra B and the $*$-homomorphism $\varphi : A \to B$.

Lemma 3.4.8. *Let A be a* C*-*algebra, $S \subset A_+$ a set of positive elements that admits a countable dense subset $Q_0 \subseteq S$, and*

$$Q = \left\{ \frac{(q-\varepsilon)_+}{\|(q-\varepsilon)_+\|} \in A_+ \mid q \in Q_0,\ \varepsilon \in \mathbb{Q},\ 0 < \varepsilon < \|q\| \right\}.$$

Consider the following statements for any C*-*algebra B,* **Cu**-*morphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$, and $*$-homomorphism $\varphi : A \to B$.*

(i) *$\varphi(q)$ is a full element of $\Phi(\overline{AqA})$ for all $q \in Q$,*

(ii) *$\varphi(a)$ is a full element of $\Phi(\overline{AaA})$ for all $a \in S$.*

Then, (i)$\Longrightarrow$(ii).

Proof. Fix a C*-algebra B, a **Cu**-morphism $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$, and a $*$-homomorphism $\varphi : A \to B$, and suppose that $\varphi(q)$ is a full element of $\Phi(\overline{AqA})$ for all $q \in Q$. Pick a positive element $a \in S$, and a sequence $(q_n)_{n\in\mathbb{N}}$ in Q_0 such that $\lim_{n\to\infty} q_n = a$. For any $\delta \in \mathbb{Q}_{>0}$, we may find $\varepsilon \in \mathbb{Q}_{>0}$ with $\delta > \varepsilon$, and by Lemma 2.1.53(ii) there exists $n_\delta \in \mathbb{N}$ for which

$$(a-\delta)_+ \in \overline{A(q_n-\varepsilon)_+A} \subseteq \overline{AaA}$$

for all $n \geq n_\delta$. By assumption, $\frac{1}{\|(q_n-\varepsilon)_+\|}\varphi((q_n-\varepsilon)_+)$ is a full element of $\Phi\left(\overline{A\frac{(q_n-\varepsilon)_+}{\|(q_n-\varepsilon)_+\|}A}\right) = \Phi(\overline{A(q_n-\varepsilon)_+A})$ for all $n \in \mathbb{N}$, hence $\varphi((q_n-\varepsilon)_+)$ is also a full element of $\Phi(\overline{A(q_n-\varepsilon)_+A})$ for all $n \in \mathbb{N}$. Therefore, we have that

$$\Phi(\overline{A(a-\delta)_+A}) \subseteq \Phi(\overline{A(q_n-\varepsilon)_+A}) = \mathcal{I}_\sigma(\varphi)(\overline{A(q_n-\varepsilon)_+A}) \subseteq \mathcal{I}_\sigma(\varphi)(\overline{AaA})$$

for all $n \geq n_\delta$. As a consequence, the ideal generated by $\varphi(a)$ in B contains $\Phi(\overline{A(a-\delta)_+A})$ for any given $\delta \in \mathbb{Q}_{>0}$. Moreover, we know that $\overline{B\varphi(a)B}$ contains $\Phi(\overline{AaA}) = \overline{\bigcup_{\delta\in\mathbb{Q}_{>0}} \Phi(\overline{A(a-\delta)_+A})}$ as well, where in the last identity we used that Φ preserves countable increasing suprema. Note that

$$\varphi((a-\delta)_+) \in \mathcal{I}_\sigma(\varphi)(\overline{A(q_n-\varepsilon)_+A}) = \Phi(\overline{A(q_n-\varepsilon)_+A}) \subseteq \Phi(\overline{AaA})$$

for all $n \geq n_\delta$. This implies that $(\varphi(a)-\delta)_+$ belongs to $\Phi(\overline{AaA})$ for all $\delta \in \mathbb{Q}_{>0}$, and therefore

$$\overline{B\varphi(a)B} = \overline{\bigcup_{\delta\in\mathbb{Q}_{>0}} \overline{B(\varphi(a)-\delta)_+B}} \subseteq \Phi(\overline{AaA}).$$

The two observations together give that $\overline{B\varphi(a)B} = \Phi(\overline{AaA})$. In particular, $\varphi(a)$ is a full element of $\Phi(\overline{AaA})$. □

Warning 3.4.9. Let A and B be C*-algebras, and suppose that B is equipped with an action $\beta : G \curvearrowright B$. We adopt the convention, introduced for convenience in the setting we work in, that a c.p.c. map $\varphi : A \to B_{\infty,\beta}$ is nuclearly liftable if it is nuclearly liftable as a map with range in B_∞. When A is separable and exact, then this is equivalent to φ being nuclear *as a map with range in* B_∞. Clearly, this is *not* equivalent to $\varphi : A \to B_{\infty,\beta}$ being nuclear.

Lemma 3.4.10. *Let $\alpha : G \curvearrowright A$ be an action on a separable, exact* C*-*algebra, and $\beta : G \curvearrowright B$ an equivariantly $\mathcal{O}_2$-stable and isometrically shift-absorbing action on a separable* C*-*algebra. Let $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$ be an equivariant* **Cu**-*morphism. Assume that there exists a nuclearly liftable, equivariant $*$-homomorphism $\varphi : A \to (B \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)_{\infty,\beta\otimes\mathrm{Ad}(\lambda^\infty)\otimes\mathrm{id}_{\mathcal{O}_2}}$ with the property that $\varphi(a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)$ for all $a \in A_+$.*[14] *Then, there exists a nuclearly liftable, equivariant $*$-homomorphism $\psi : A \to B_{\infty,\beta}$ such that $\psi(a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$ for all $a \in A_+$.*

Proof. Since A is separable, we may fix a countable dense subset $Q_0 \subseteq A_+$, and consider the set $Q \subseteq A_+$ of norm one elements given as in Lemma 3.4.8. Since Q is countable, we can write it as $Q = \{q_i\}_{i\in\mathbb{N}}$. For every $i \in \mathbb{N}$, if $\Phi(\overline{Aq_iA}) \neq \{0\}$ we choose a strictly positive norm-one element $h_i^0 \in \Phi(\overline{Aq_iA})$, otherwise we set $h_i^0 = 0$. Moreover, we fix a rank-one projection $e \in \mathcal{K}(\mathcal{H}_G^\infty)$, so that $h_i = h_i^0 \otimes e \otimes 1_{\mathcal{O}_2}$ is a full element of $\Phi(\overline{Aq_iA}) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2$, and thus also full in $\mathcal{I}(\iota_\infty)(\Phi(\overline{Aq_iA}) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)$ for all $i \in \mathbb{N}$. Moreover, for any given $i \in \mathbb{N}$, the ideal $\mathcal{I}(\iota_\infty)(\Phi(\overline{Aq_iA}) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)$ is purely infinite by Theorem 2.1.63 and Proposition 2.1.61. Therefore, $\varphi(q_i)$ and h_i are Cuntz equivalent, and by Lemma 2.1.53(ii) we can find $K_i \in \mathbb{N}$ and elements $c_{i,k}$, $d_{i,k}$ in $\mathcal{I}(\iota_\infty)(\Phi(\overline{Aq_iA}) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)$ such that

$$(h_i - 2^{-k})_+ = c_{i,k}^*\varphi(q_i)c_{i,k} \quad \text{and} \quad (\varphi(q_i) - 2^{-k})_+ = d_{i,k}^*(h_i - 2^{-K_i})_+ d_{i,k}$$

for all $i, k \in \mathbb{N}$.

[14] Recall from Definition 2.2.33 that ι_∞ denotes the embedding into the sequence algebra, not its continuous part.

Recall from Remark 3.4.7 that there exists a full equivariant embedding

$$\iota : (\mathcal{K}(\mathcal{H}_G^\infty), \mathrm{Ad}(\lambda^\infty)) \hookrightarrow (\mathcal{O}_\infty, \gamma).$$

Since β is isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable, one can use Proposition 2.7.8 and Theorem 2.6.8 to obtain an equivariant $*$-homomorphism $\eta : (\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \mathrm{id}_{\mathcal{O}_2}) \hookrightarrow (F_{\infty,\beta}(B), \tilde{\beta}_\infty)$. Note that we have a canonical commutative diagram of equivariant $*$-homomorphisms given by

$$\begin{tikzcd}
(B,\beta) \arrow[rr, hook] \arrow[dr, hook, "\mathrm{id}_B \otimes \mathbf{1}"'] & & (B_{\infty,\beta}, \beta_\infty) \\
& (B \otimes_{\max} F_{\infty,\beta}(B), \beta_\infty \otimes \tilde{\beta}_\infty) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where ρ is the natural $*$-homomorphism defined as $\rho(b \otimes (x + B_{\infty,\beta} \cap B^\perp)) = bx$, for all $b \in B$, $x \in B_{\infty,\beta} \cap B'$. In the following paragraph, let us denote by D the image of $B \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2$ under ρ. We define the equivariant $*$-homomorphism

$$\theta : (B \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2, \beta \otimes \mathrm{Ad}(\lambda^\infty) \otimes \mathrm{id}_{\mathcal{O}_2}) \to (B_{\infty,\beta}, \beta_\infty),$$

$$\theta = \rho \circ (\mathrm{id}_B \otimes \eta) \circ (\mathrm{id}_B \otimes \iota \otimes \mathrm{id}_{\mathcal{O}_2}).$$

Let us show that $\theta(I \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)$ generates the ideal $\mathcal{I}(\iota_\infty)(I)$ in B_∞ for all $I \in \mathcal{I}(B)$.

$$\begin{aligned}
\mathcal{I}(\theta)(I \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2) &= \mathcal{I}(\rho) \circ \mathcal{I}(\mathrm{id}_B \otimes \eta) \circ \mathcal{I}(\mathrm{id}_B \otimes \iota \otimes \mathrm{id}_{\mathcal{O}_2})(I \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2) \\
&= \mathcal{I}(\rho) \circ \mathcal{I}(\mathrm{id}_B \otimes \eta)(I \otimes \mathcal{O}_\infty \otimes \mathcal{O}_2) \\
&= \mathcal{I}(\rho)(I \otimes F_{\infty,\beta}(B)) = \mathcal{I}(\iota_\infty)(I).
\end{aligned}$$

In particular, for each $i \in \mathbb{N}$, we have that either $\Phi(\overline{Aq_iA}) = 0$ or $\theta(h_i)$ and h_i^0 are full, norm one elements of $\mathcal{I}(\iota_\infty)(\Phi(\overline{Aq_iA}))$. By Theorem 2.1.63 and Proposition 2.1.61 the latter C*-algebra is purely infinite. Hence, by using Lemma 2.1.53(ii), we can find $L_i \in \mathbb{N}$ and elements $y_{i,k}$, $z_{i,k}$ in B_∞ satisfying

$$(h_i^0 - 2^{-k})_+ = y_{i,k}^* \theta((h_i - 2^{-L_i})_+) y_{i,k} \quad \text{and} \quad (\theta(h_i) - 2^{-k})_+ = z_{i,k}^* h_i^0 z_{i,k}$$

for all $i, k \in \mathbb{N}$. Now, we consider

$$\theta_\infty : (B \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2)_\infty \to (B_\infty)_\infty,$$

the componentwise application of θ. The composite $*$-homomorphism

$$\Theta = \theta_\infty \circ \varphi : A \to (B_{\infty,\beta})_{\infty,\beta_\infty}$$

is nuclearly liftable and equivariant with respect to α and $(\beta_\infty)_\infty$. Set $e_{i,k} = \theta_\infty(c_{i,L_i})y_{i,k}$ and $f_{i,k} = z_{i,K_i}\theta_\infty(d_{i,k})$ as elements of $(B_\infty)_\infty$. Then, we have that

$$(h_i^0 - 2^{-k})_+ = e_{i,k}^*\Theta(q_i)e_{i,k} \qquad \text{and} \qquad (\Theta(q_i) - 2^{-k})_+ = f_{i,k}^* h_i^0 f_{i,k}$$

for all $i,k \in \mathbb{N}$.

Let $(\Theta_t)_{t\in\mathbb{N}} : A \to \prod_{\mathbb{N}} B_\infty$ be a sequence of c.p.c. nuclear maps lifting Θ, and for each pair $i,k \in \mathbb{N}$, let $(e_{i,k}^{(t)})_{t\in\mathbb{N}}$ and $(f_{i,k}^{(t)})_{t\in\mathbb{N}}$ be elements of $\ell^\infty(\mathbb{N}, B_\infty)$ representing $e_{i,k}$ and $f_{i,k}$, respectively. We may assume $\|e_{i,k}^{(t)}\| \leq \|e_{i,k}\|$ and $\|f_{i,k}^{(t)}\| \leq \|f_{i,k}\|$ for all $t \geq 1$. It follows that

$$\limsup_{t\to\infty}\Big(\|(h_i^0 - 2^{-k})_+ - (e_{i,k}^{(t)})^*\Theta_t(q_i)e_{i,k}^{(t)}\| + \|(\Theta_t(q_i) - 2^{-k})_+ - (f_{i,k}^{(t)})^* h_i^0 f_{i,k}^{(t)}\|\Big) = 0$$

for all $i,k \in \mathbb{N}$. Moreover, we may assume that

$$\limsup_{t\to\infty}\|\Theta_t(ab) - \Theta_t(a)\Theta_t(b)\| = 0,$$

$$\limsup_{t\to\infty}\|\Theta_t(a)^* - \Theta_t(a^*)\| = 0,$$

$$\limsup_{t\to\infty}\max_{g\in K}\|\beta_{\infty,g}\circ\Theta_t(a) - \Theta_t\circ\alpha_g(a)\| = 0$$

for all $a,b \in A$ and compact subsets $K \subseteq G$. Note that the latter condition can be assumed as sequences lift to β_∞-continuous sequences, as recalled in Definition 2.2.33. Fix an increasing sequence of finite subsets $\mathcal{F}_n \subseteq A$ such that $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n = A$, and an increasing sequence of compact subsets $K_n \subseteq G$ such that $\bigcup_{n\in\mathbb{N}} K_n = G$. For each $t \in \mathbb{N}$, let $(\Theta_t^{(j)})_{j\in\mathbb{N}} : A \to \prod_{\mathbb{N}} B$ be a sequence of c.p.c. nuclear maps representing Θ_t, and for each $i,k,t \in \mathbb{N}$, let $(e_{i,k}^{(t,j)})_{j\in\mathbb{N}}$ and $(f_{i,k}^{(t,j)})_{j\in\mathbb{N}}$ elements of $\ell^\infty(\mathbb{N}, B)$ representing $e_{i,k}^{(t)}$ and $f_{i,k}^{(t)}$, respectively. We may assume $\|e_{i,k}^{(t,j)}\| \leq \|e_{i,k}\|$ and $\|f_{i,k}^{(t,j)}\| \leq \|f_{i,k}\|$ for all $t,j \geq 1$. Then, inductively on $n \in \mathbb{N}$, we may find a strictly increasing sequence $(t_n)_n \subseteq \mathbb{N}$ such that

$$\max_{1\leq i,k\leq n}\limsup_{j\to\infty}\|(e_{i,k}^{(t_n,j)})^*\Theta_{t_n}^{(j)}(q_i)e_{i,k}^{(t_n,j)} - (h_i^0 - 2^{-k})_+\| \leq 2^{-2n},$$

$$\max_{1\leq i,k\leq n}\limsup_{j\to\infty}\|(\Theta_{t_n}^{(j)}(q_i) - 2^{-k})_+ - (f_{i,k}^{(t_n,j)})^* h_i^0 f_{i,k}^{(t_n,j)}\| \leq 2^{-2n}$$

for all $n \in \mathbb{N}$. Therefore, for each index t_n, we may find increasing $j_n \in \mathbb{N}$ such that when $j \geq j_n$ one has that

$$\max_{1\leq i,k\leq n}\|(e_{i,k}^{(t_n,j)})^*\Theta_{t_n}^{(j)}(q_i)e_{i,k}^{(t_n,j)} - (h_i^0 - 2^{-k})_+\| \leq 2^{-n},$$

$$\max_{1\leq i,k\leq n} \|(\Theta_{t_n}^{(j)}(q_i) - 2^{-k})_+ - (f_{i,k}^{(t_n,j)})^* h_i^0 f_{i,k}^{(t_n,j)}\| \leq 2^{-n}.$$

Moreover, we may assume that the conditions

$$\max_{a,b\in\mathcal{F}_n} \|\Theta_{t_n}^{(j)}(ab) - \Theta_{t_n}^{(j)}(a)\Theta_{t_n}^{(j)}(b)\| \leq 2^{-n},$$

$$\max_{a,b\in\mathcal{F}_n} \|\Theta_{t_n}^{(j)}(a)^* - \Theta_{t_n}^{(j)}(a^*)\| \leq 2^{-n},$$

$$\max_{a\in\mathcal{F}_n, g\in K_n} \|\beta_g \circ \Theta_{t_n}^{(j)}(a) - \Theta_{t_n}^{(j)} \circ \alpha_g(a)\| \leq 2^{-n},$$

hold when $n \in \mathbb{N}$ and $j \geq j_n$. Now, observe that the diagonal sequence of c.p.c. nuclear maps $\{\Theta_{t_n}^{(j_n)}\}_{n\in\mathbb{N}} : A \to \prod_{\mathbb{N}} B$ produces a nuclearly liftable equivariant $*$-homomorphism $\psi : A \to B_{\infty,\beta}$. Consider the elements $\bar{e}_{i,k} = [(e_{i,k}^{(t_n,j_n)})_n] \in B_\infty$ and $\bar{f}_{i,k} = [(f_{i,k}^{(t_n,j_n)})_n] \in B_\infty$. From the approximate conditions above we may then conclude

$$\bar{e}_{i,k}^* \psi(q_i)\bar{e}_{i,k} = (h_i^0 - 2^{-k})_+ \quad \text{and} \quad \bar{f}_{i,k}^* h_i^0 \bar{f}_{i,k} = (\psi(q_i) - 2^{-k})_+.$$

Since we may let $k \to \infty$, this implies that h_i^0 and $\psi(q_i)$ are Cuntz equivalent in B_∞. In particular, $\psi(q_i)$ belongs to and is a full element in $\mathcal{I}(\iota_\infty)(\Phi(\overline{Aq_iA}))$, for all $i \in \mathbb{N}$. Since we chose Q at the start of the proof to satisfy the conclusion of Lemma 3.4.8 for S consisting of all positive contractions in A, it follows that $\psi(a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$ for all $a \in A_+$. $\square$

Remark 3.4.11 (see [67, Remark 4.6]). In Remark 2.4.14, we have produced a $\mathbb{Z}$-action σ on $\mathcal{C}_0(\mathbb{R})$ such that $\mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}$. Now, for any C*-algebra A, denote by τ the automorphism of its suspension algebra $SA = \mathcal{C}_0(\mathbb{R}) \otimes A$ given by $\sigma \otimes \mathrm{id}_A$. We then have the following isomorphism,

$$SA \rtimes_\tau \mathbb{Z} \cong A \otimes (\mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z}) \cong A \otimes \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}.$$

For an action $\alpha : G \curvearrowright A$, we use the notation $S\alpha := \mathrm{id}_{\mathcal{C}_0(\mathbb{R})} \otimes \alpha : G \curvearrowright SA$. Since $S\alpha : G \curvearrowright SA$ pointwise commutes with the action $\tau : \mathbb{Z} \curvearrowright SA$, the natural isomorphism in the previous paragraph shows that $\alpha \otimes \mathrm{id} : G \curvearrowright A \otimes \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}$ corresponds to the unique action $S\alpha \rtimes \mathbb{Z} : G \curvearrowright SA \rtimes_\tau \mathbb{Z}$ that extends $S\alpha$ by acting trivially on the copy of $\mathbb{Z}$.

Lemma 3.4.12. *Let $\alpha : G \curvearrowright A$ be an action on a separable, exact* C*-*algebra, and $\beta : G \curvearrowright B$ a strongly stable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a separable* C*-*algebra. Then, for every equivariant* **Cu**-*morphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$, there exists a nuclearly liftable, equivariant $*$-homomorphism*

$$\psi : (SA, S\alpha) \to (B_{\infty,\beta}, \beta_\infty)$$

such that, for each positive $a \in A$ and positive non-zero $f \in \mathcal{C}_0(\mathbb{R})$, $\psi(f \otimes a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$.[15]

Proof. In the rest of the proof, we use the notation B^s for $B \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2$, β^s for the action $\beta \otimes \mathrm{Ad}(\lambda^\infty) \otimes \mathrm{id}_{\mathcal{O}_2}$ on B^s, and $\Phi^s(I)$ for the ideal given by $\Phi(I) \otimes \mathcal{K}(\mathcal{H}_G^\infty) \otimes \mathcal{O}_2$ for all $I \in \mathcal{I}(A)$. Moreover, we will identify $\mathcal{K}$ with the C*-algebra of compact operators on $\mathcal{H}_G^\infty$.

Let $\rho_\mu : SA \to A$ be the right slice map associated to a faithful positive linear functional μ on $\mathcal{C}_0(\mathbb{R})$. Then, the induced map $\mathcal{I}(\rho_\mu) : \mathcal{I}(SA) \to \mathcal{I}(A)$ has the property that for any positive non-zero function $f \in \mathcal{C}_0(\mathbb{R})$ and positive element $a \in A$, $\mathcal{I}(\rho_\mu)(\overline{SA(f \otimes a)SA}) = \overline{AaA}$. Denote by $\mathcal{S}$ the set of positive elements of SA of the form $f \otimes a$ for positive non-zero functions $f \in \mathcal{C}_0(\mathbb{R})$ and positive elements $a \in A$. In order to prove the lemma, it is enough to show that there exists an equivariant $*$-homomorphism

$$\psi : (SA, S\alpha) \to (B^s_{\infty,\beta^s}, \beta^s_\infty)$$

that is nuclearly liftable, and with the property that $\psi(f \otimes a)$ is a full element of

$$\mathcal{I}(\iota_\infty)(\Phi^s(\mathcal{I}(\rho_\mu)(\overline{SA(f \otimes a)SA}))) = \mathcal{I}(\iota_\infty)(\Phi^s(\overline{AaA}))$$

for all $(f \otimes a) \in \mathcal{S}$, and then apply Lemma 3.4.10.

From Lemma 3.4.6, there exists a nuclear, equivariant $*$-homomorphism $\kappa : (A, \alpha) \to (\mathcal{M}(B \otimes \mathcal{K}), \beta \otimes \mathrm{Ad}(\lambda^\infty))$ satisfying

$$\overline{(B \otimes \mathcal{K})\kappa(I)(B \otimes \mathcal{K})} = \Phi(I) \otimes \mathcal{K}$$

for all $I \in \mathcal{I}(A)$. We use Kasparov's lemma, Lemma 2.2.12, to infer the existence of an approximate unit $(b_n)_{n \in \mathbb{N}}$ in $B \otimes \mathcal{K}$ such that

$$\lim_{n\to\infty} \max_{g \in K} \|(\beta \otimes \mathrm{Ad}(\lambda^\infty))_g(b_n) - b_n\| = 0, \quad \lim_{n\to\infty} \|\left[b_n, \kappa(A)\right]\| = 0$$

for every compact set $K \subseteq G$. The approximate unit $(b_n)_n$ induces an element $b \in (B \otimes \mathcal{K})_{\infty, \beta \otimes \mathrm{Ad}(\lambda^\infty)}$ that commutes with $\kappa(A)$ and is fixed under $(\beta \otimes \mathrm{Ad}(\lambda^\infty))_\infty$. Note that we may pick a positive contraction $h \in \mathcal{O}_2 \subseteq (\mathcal{O}_2)_\infty$ with full spectrum $[0,1]$ by, e.g., embedding a copy of $\mathcal{C}[0,1]$ into $\mathcal{O}_2$ (which exists by Theorem 2.1.47). Then, the elementary tensor $b \otimes h$ is an element in B^s_{∞,β^s} that commutes with $\kappa(A) \otimes \mathbf{1}_{\mathcal{O}_2}$. Pick a homeomorphism $\theta : (0,1) \to \mathbb{R}$, for instance $\theta(t) = \log(\frac{1}{t} - 1)$. We use $b \otimes h$ to define the $*$-homomorphism given by

$$\psi : (SA, S\alpha) \to (B^s_{\infty,\beta^s}, \beta^s_\infty), \quad \psi(f \otimes a) = (f \circ \theta)(b \otimes h) \cdot (\kappa(a) \otimes \mathbf{1}_{\mathcal{O}_2})$$

[15] Recall that ι_∞ denotes the canonical embedding into the sequence algebra, not its continuous part.

for all $f \in \mathcal{C}_0(\mathbb{R})$ and $a \in A$, which is equivariant because $(f \circ \theta)(b \otimes h) \in B^s_{\infty,\beta^s}$ is fixed by β^s_∞, and κ is equivariant. Moreover, ψ is nuclear by Lemma 2.1.25.

For any positive element $a \in A$ and positive non-zero function $f \in \mathcal{C}_0(\mathbb{R})$, we want to show that $\psi(f \otimes a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi^s(\overline{AaA}))$. In order to show that $\psi(f \otimes a)$ is contained in $\mathcal{I}(\iota_\infty)(\Phi^s(\overline{AaA}))$, it is sufficient to prove that, for any given $\varepsilon > 0$, the element $\psi(f \otimes (a-\varepsilon)_+)$ belongs to $\mathcal{I}(\iota_\infty)(\Phi^s(\overline{AaA}))$. Then, fix some $\varepsilon > 0$. Recall that **Cu**-morphisms preserve compact containment and that $\overline{A(a-\varepsilon)_+A} \Subset \overline{AaA}$ from Remark 2.3.9(i). Then, we have that $\Phi^s(\overline{A(a-\varepsilon)_+A}) \Subset \Phi^s(\overline{AaA})$. Subsequently, we use Remark 2.3.9(ii) to infer that

$$\begin{aligned}\psi(f \otimes (a-\varepsilon)_+) &= (f \circ \theta)(b \otimes h)\kappa(a-\varepsilon)_+ \\ &\in \mathcal{I}(\iota_\infty)\left(\overline{B^s(\kappa \otimes \mathbf{1}_{\mathcal{O}_2})(\overline{A(a-\varepsilon)_+A})B^s}\right) \\ &\subseteq \mathcal{I}(\iota_\infty)\left(\Phi^s(\overline{AaA})\right).\end{aligned}$$

Finally, we want to show that $\psi(f \otimes a)$ is also a full element. Recall that, since B is separable, $\Phi(\overline{AaA}) \otimes \mathcal{K}$ contains a full, positive element, which we denote by e. Moreover, since $\mathcal{O}_2$ is simple, f is non-zero and h has full spectrum $[0,1]$, $e \otimes (f \circ \theta)(h)$ is full in $\Phi^s(\overline{AaA})$, and thus also in $\mathcal{I}(\iota_\infty)(\Phi^s(\overline{AaA}))$. It is therefore sufficient to show that, for any $\varepsilon > 0$, the element $(e-\varepsilon)_+ \otimes (f \circ \theta)(h)$ belongs to the ideal generated by $\psi(f \otimes a)$. We thus fix an arbitrary $\varepsilon > 0$. Observe that $b \otimes h$ is in $B^s_\infty \cap (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})'$, and that $b \otimes 1_{\mathcal{O}_2} + (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})^\perp$ is the unit of $F(B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2}, B^s_\infty)$. In particular,

$$b \otimes h + (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})^\perp = 1 \otimes h + (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})^\perp,$$

which furthermore implies that

$$(f \circ \theta)(b \otimes h) + (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})^\perp = 1 \otimes (f \circ \theta)(h) + (B \otimes \mathcal{K} \otimes 1_{\mathcal{O}_2})^\perp.$$

Hence, for any $c \in B \otimes \mathcal{K}$,

$$(f \circ \theta)(b \otimes h)(c \otimes 1_{\mathcal{O}_2}) = (c \otimes 1_{\mathcal{O}_2})(f \circ \theta)(b \otimes h) = (c \otimes 1_{\mathcal{O}_2})(1 \otimes (f \circ \theta)(h)). \quad (3.14)$$

Since $e \in \overline{(B \otimes \mathcal{K})\kappa(a)(B \otimes \mathcal{K})}$, by Lemma 2.1.53(i) there exist some $n \in \mathbb{N}$, and $c_1, \ldots, c_n \in B \otimes \mathcal{K}$ such that $(e-\varepsilon)_+ = \sum_{i=1}^n c_i^* \kappa(a) c_i$, and

$$\begin{aligned}(e-\varepsilon)_+ \otimes (f \circ \theta)(h) &= \sum_{i=1}^n (c_i^* \otimes 1_{\mathcal{O}_2})(\kappa(a) \otimes 1_{\mathcal{O}_2})(c_i \otimes 1_{\mathcal{O}_2})(1 \otimes (f \circ \theta)(h)) \\ &= \sum_{i=1}^n (c_i^* \otimes 1_{\mathcal{O}_2})\kappa(a)(f \circ \theta)(b \otimes h)(c_i \otimes 1_{\mathcal{O}_2})\end{aligned}$$

$$= \sum_{i=1}^{n} (c_i^* \otimes 1_{\mathcal{O}_2}) \psi(f \otimes a)(c_i \otimes 1_{\mathcal{O}_2}),$$

where in the second passage we used (3.14). This finishes the proof. □

For the reader's convenience, we recall and prove Lemma 3.4.14, which is Gabe and Szabó's [67, Lemma 4.2]. First, we need a preliminary lemma.

Lemma 3.4.13 (see [67, Lemma 4.1])**.** *Let $\alpha : G \curvearrowright A$ be an action on a separable C^*-algebra, and $\beta : G \curvearrowright B$ a strongly stable action on a C^*-algebra. Let $\varphi, \psi : (A, \alpha) \to (B_{\infty,\beta}, \beta_\infty)$ be equivariant $*$-homomorphisms, and suppose that for every contraction $d \in A$, there exists a contraction $s \in B_\infty$ such that*

$$s^* \beta_{\infty,g}(s) = \psi(d^* \alpha_g(d)), \quad \textit{for all } g \in G,$$

$$s^* \varphi(a) s = \psi(d^* a d), \quad \textit{for all } a \in A.$$

Then, $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$.

Proof. Fix a contraction $b \in B_{\infty,\beta}$. By Lemma 2.2.12, we may find an approximately α-invariant approximate unit $(e_n)_n$ of A. By assumption, for each e_n, there exists a contraction $s_n \in B_\infty$ such

$$s_n^* \beta_{\infty,g}(s_n) = \psi(e_n^* \alpha_g(e_n)), \text{for all } g \in G,$$

$$s_n^* \varphi(a) s_n = \psi(e_n^* a e_n), \text{for all } a \in A.$$

In particular, the second condition implies that $s_n^* \varphi(a) s_n \to \psi(a)$ for all $a \in A$ because $(e_n)_n$ is an approximate unit. We also note that the first condition, combined with the fact that $(e_n)_n$ is approximately fixed by α, implies that

$$\begin{aligned}
\|s_n - \beta_{\infty,g}(s_n)\|^2 &= \|(s_n - \beta_{\infty,g}(s_n))^*(s_n - \beta_{\infty,g}(s_n))\| \\
&= \|s_n^* s_n + \beta_{\infty,g}(s_n^* s_n) - s_n^* \beta_{\infty,g}(s_n) - \beta_{\infty,g}(s_n)^* s_n\| \\
&= \|\psi(e_n^* e_n) + \beta_{\infty,g}(\psi(e_n^* e_n)) - \psi(e_n^* \alpha_g(e_n)) - \psi(\alpha_g(e_n)^* e_n)\| \\
&= \|\psi(e_n^* e_n) + \psi(\alpha_g(e_n^* e_n)) - \psi(e_n^* \alpha_g(e_n)) - \psi(\alpha_g(e_n)^* e_n)\| \\
&\leq \|e_n^* e_n + \alpha_g(e_n^* e_n) - e_n^* \alpha_g(e_n) - \alpha_g(e_n)^* e_n\| \to 0.
\end{aligned}$$

We may imply from this that $s_n \in B_{\infty,\beta}$, and therefore that the limit holds also uniformly over compact subsets of G. By Lemma 2.8.8, we may find a separable, β_∞-invariant C^*-subalgebra $D \subseteq B_{\infty,\beta}$ containing b, $\varphi(A)$, $\psi(A)$, and $\{s_n\}_{n\in\mathbb{N}}$ such that $\beta \restriction_D$ is strongly stable. Hence, this shows that the sequence of contractions $(s_n)_n$ satisfies the assumptions of Lemma 3.3.8 with trivial cocycles. In particular, this yields a sequence of isometries $S_n \in \mathcal{M}^{\beta_\infty}(D)$ such that

$$S_n^*\varphi(a)S_n \to \psi(a), \text{ and } \max_{g\in K} \|S_n - \beta_{\infty,g}(S_n)\| \to 0,$$

for all $a \in A$ and for every compact subset $K \subseteq G$. Set $c_n = S_n b \in D$, and note that

$$c_n^*\varphi(a)c_n \to b^*\psi(a)b, \text{ and } \max_{g\in K} \|c_n^*\beta_{\infty,g}(c_n) - b^*\beta_{\infty,g}(b)\| \to 0,$$

for all $a \in A$ and for every compact subset $K \subseteq G$. Since b was arbitrarily chosen, we conclude that $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$. □

Lemma 3.4.14 (see [67, Lemma 4.2])**.** *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C*-*algebra, and $\beta : G \curvearrowright B$ a strongly stable, isometrically shift-absorbing action on a separable* C*-*algebra. Let $\varphi, \psi : A \to B_{\infty,\beta}$ be nuclearly liftable,*[16] *equivariant $*$-homomorphisms. If φ approximately 1-dominates ψ as ordinary $*$-homomorphisms, then $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$.*

Proof. One may notice that the proof is in large part similar to that of Lemma 3.3.11, with a few important changes. First, let μ be a Haar measure on G. Next, fix a contraction $d \in A$, a set $K \subseteq G$ with $1_G \in K$, a compact set $\mathcal{F}$ in the unit ball of A, and $\varepsilon > 0$. Since α is assumed to be amenable, we may find a function $\zeta \in \mathcal{C}_c(G, A)$ with $\|\zeta\|_2 \leq 1$ with compact support $R = \mathrm{supp}(\zeta)$ such that

$$\max_{a\in\mathcal{F}} \|\langle\zeta, a\zeta\rangle - a\| \leq \varepsilon, \tag{3.15}$$

$$\max_{g\in K} \|d^*(\mathbf{1} - \langle\zeta, \bar{\alpha}_g(\zeta)\rangle)\| \leq \varepsilon. \tag{3.16}$$

By Lemma 2.2.12, there exists an approximately central, approximately α-invariant, approximate unit for A. We may therefore extract a positive contraction from this approximate unit with the following properties,

$$\max_{g\in R} \|\alpha_g(e) - e\| \leq \varepsilon, \tag{3.17}$$

$$\max_{g\in K} \|\langle\zeta, e^2\bar{\alpha}_g(\zeta)\rangle - e\langle\zeta, \bar{\alpha}_g(\zeta)\rangle e\| \leq \varepsilon, \tag{3.18}$$

[16] One could equivalently ask that φ and ψ are nuclear as maps with range in B_∞ in the context of this lemma as a consequence of Proposition 2.1.34.

$$\max_{a\in\mathcal{F}} \|\langle\zeta, eae\zeta\rangle - e\langle\zeta, a\zeta\rangle e\| \le \varepsilon, \tag{3.19}$$

$$\max_{g\in K} \|(\mathbf{1}-e)\alpha_g(d)\| \le \varepsilon. \tag{3.20}$$

By assumption, φ and ψ are nuclearly liftable, and thus we may pick nuclear c.p.c. lifts

$$(\varphi_k)_k, (\psi_k)_k : A \to \ell^\infty_\beta(\mathbb{N}, B)$$

for φ and ψ, respectively. (Note that these lifts can be chosen in $\ell^\infty_\beta(\mathbb{N}, B)$ because φ and ψ have range in the continuous part $B_{\infty,\beta}$ of B_∞.) Now, set

$$\mathcal{G}_0 = \{d, d^*, e, e^2\} \cup \mathcal{F} \cup \zeta(G) \cup \{\langle\zeta, a\bar{\alpha}_g(\zeta)\rangle \mid a \in \mathcal{F}, g \in K\}$$

and

$$\mathcal{G}_1 = \bigcup_{g\in K\cup R\cup R^{-1}} \alpha_g(\mathcal{G}_0).$$

Moreover, we will denote by $\mathcal{G}$ the set of products of five or fewer elements from $\mathcal{G}_1$, and $M = 1 + \mu(R)(\|\zeta\|_\infty + \|\zeta\|_\infty^2)$ (Note that $\mathcal{G} \subseteq A$ is compact.) We use the fact that both $(\varphi_k)_k$ and $(\psi_k)_k$ are pointwise G-equicontinuous (as they have range in $\ell^\infty_\beta(\mathbb{N}, B)$), and they are approximately equivariant approximate $*$-homomorphisms (as they lift φ and ψ, respectively) to find $n \in \mathbb{N}$ such that

$$\max_{a\in\mathcal{G}} \max_{g\in K\cup R\cup R^{-1}} \sup_{k\ge n} \|\beta_g(\varphi_k(a)) - \varphi_k(\alpha_g(a))\| \le \varepsilon/M, \tag{3.21}$$

$$\max_{a,b\in\mathcal{G}} \sup_{k\ge n} \|\varphi_k(ab) - \varphi_k(a)\varphi_k(b)\| \le \varepsilon, \tag{3.22}$$

and the analogous inequalities for $(\psi_k)_k$. It follows that, for every $f \in \mathcal{C}_c(G, A)$ and $F \in \mathcal{C}_b(G, A)$ with $f(G), F(G) \subseteq \mathcal{G}$, $\mu(\overline{\mathrm{supp}(f)}) \le \mu(R)$, and

$$\|f\|_\infty \le \|\zeta\|_\infty, \text{ and } \|F\|_\infty \le \min\{1, \|\zeta\|_\infty\},$$

one has that

$$\max_{g\in K} \sup_{k\ge n} \|\beta_g \circ \psi_k \circ f - \psi_k \circ \alpha_g \circ f\|_2 \le \varepsilon, \tag{3.23}$$

$$\sup_{k\ge n} \|\psi_k \circ (F \cdot f) - (\psi_k \circ F) \cdot (\psi_k \circ f)\|_2 \le \varepsilon, \tag{3.24}$$

where we used that $\|f\|_2 \le \mu(\overline{\mathrm{supp}(f)})\|f\|_\infty$ for all $f \in \mathcal{C}_c(G)$. Now we use that φ approximately 1-dominates ψ when viewed as maps with range in B_∞ to find a sequence of contractions $(c_k)_k \in \ell^\infty(\mathbb{N}, B)$ such that, after possibly increasing $n \in \mathbb{N}$ from before,

$$\max_{a\in\mathcal{G}} \sup_{k\ge n} \|c_k^*\varphi_k(a)c_k - \psi_k(a)\| \le \varepsilon. \tag{3.25}$$

Since for all $k \in \mathbb{N}$, ψ_k is a c.p.c. map, we may infer from Kadison's Cauchy–Schwarz inequality that, for every $f \in \mathcal{C}_c(G, A)$,

$$\|\psi_k \circ f\|_2^2 = \left\| \int_G \psi_k(f(h))^* \psi_k(f(h)) \, d\mu(h) \right\| \leq \|\psi_k(\langle f, f \rangle)\| \leq \|f\|_2^2.$$

It is now the right moment to use that β is isometric shift-absorbing, and the characterisation of Proposition 2.7.8, to find an equivariant linear B-bimodule map

$$\theta : (L^2(G, B), \bar{\beta}) \to (B_{\infty,\beta}, \beta_\infty)$$

satisfying $\theta(\xi)^*\theta(\eta) = \langle \xi, \eta \rangle_B$ for all $\xi, \eta \in L^2(G, B)$. We may therefore define compactly supported functions $\xi_k \in \mathcal{C}_c(G, B)$ given by

$$\xi_k(g) = \beta_g(\varphi_k(e)c_k)\psi_k(\zeta(g)), \quad \text{for all } g \in G.$$

We carry out the following computations using the conditions we found above. For every $g \in K$ and $k \geq n$, one has that

$$\begin{aligned}
&\theta(\xi_k)^* \beta_{\infty,g}(\theta(\xi_k)) \\
&= \theta(\xi_k)^* \theta(\bar{\beta}_{\infty,g}(\xi_k)) \\
&= \langle \xi_k, \bar{\beta}_{\infty,g}(\xi_k) \rangle \\
&= \int_G \psi_k(\zeta(h))^* \beta_h(c_k^* \varphi_k(e)) \beta_g(\beta_{g^{-1}h}(\varphi_k(e)c_k)\psi_k(\zeta(g^{-1}h))) \, d\mu(h) \\
&= \int_G \psi_k(\zeta(h))^* \beta_h(c_k^* \varphi_k(e)^2 c_k) \beta_g(\psi_k(\zeta(g^{-1}h))) \, d\mu(h) \\
&\overset{(3.23)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^* \beta_h(c_k^* \varphi_k(e)^2 c_k) \psi_k(\bar{\alpha}_g(\zeta)(h)) \, d\mu(h) \\
&\overset{(3.22)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^* \beta_h(c_k^* \varphi_k(e^2) c_k) \psi_k(\bar{\alpha}_g(\zeta)(h)) \, d\mu(h) \\
&\overset{(3.25)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^* \beta_h(\psi_k(e^2)) \psi_k(\bar{\alpha}_g(\zeta)(h)) \, d\mu(h) \\
&\overset{(3.21)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^* \psi_k(\alpha_h(e^2)) \psi_k(\bar{\alpha}_g(\zeta)(h)) \, d\mu(h) \\
&\overset{(3.24)}{=}_{2\varepsilon} \int_G \psi_k \left(\zeta(h))^* \alpha_h(e^2) \bar{\alpha}_g(\zeta)(h) \right) \, d\mu(h)
\end{aligned}$$

$$
\begin{aligned}
&= \psi_k\left(\int_G \zeta(h))^* \alpha_h(e^2)\bar{\alpha}_g(\zeta)(h)\, d\mu(h)\right) \\
&\overset{(3.17)}{=}_{2\varepsilon} \psi_k\left(\int_G \zeta(h))^* e^2\bar{\alpha}_g(\zeta)(h)\, d\mu(h)\right) \\
&= \psi_k(\langle \zeta, e^2\bar{\alpha}_g(\zeta)\rangle) \\
&\overset{(3.18)}{=}_{\varepsilon} \psi_k(e\langle \zeta, \bar{\alpha}_g(\zeta)\rangle e).
\end{aligned}
$$

In particular, we use this to infer that, for any $k \geq n$ and $g \in K$,

$$
\begin{aligned}
&\psi_k(d)^*\theta(\xi_k)^*\beta_{\infty,g}(\theta(\xi_k)\psi_k(d)) \\
&\quad = \psi_k(d)^*\theta(\xi_k)^*\beta_{\infty,g}(\theta(\xi_k))\beta_g(\psi_k(d)) \\
&\quad =_{9\varepsilon} \psi_k(d)^*\psi_k(e\langle \zeta, \bar{\alpha}_g(\zeta)\rangle e)\beta_g(\psi_k(d)) \\
&\quad \overset{(3.21)}{=}_{\varepsilon} \psi_k(d)^*\psi_k(e\langle \zeta, \bar{\alpha}_g(\zeta)\rangle e)\psi_k(\alpha_g(d)) \\
&\quad \overset{(3.22)}{=}_{2\varepsilon} \psi_k(d^*e\langle \zeta, \bar{\alpha}_g(\zeta)\rangle e\alpha_g(d)) \\
&\quad \overset{(3.20)}{=}_{2\varepsilon} \psi_k(d^*\langle \zeta, \bar{\alpha}_g(\zeta)\rangle \alpha_g(d)) \\
&\quad \overset{(3.16)}{=}_{\varepsilon} \psi_k(d^*\alpha_g(d)).
\end{aligned}
$$

Now, we may also compute for all $k \geq n$ and $a \in \mathcal{F}$,

$$
\begin{aligned}
&\theta(\xi_k)^*\varphi_k(a)\theta(\xi_k) \\
&\quad = \theta(\xi_k)^*\theta(\varphi_k(a)\xi_k) \\
&\quad = \int_G \psi_k(\zeta(h))^*\beta_h(c_k^*\varphi_k(e)\beta_{h^{-1}}(\varphi_k(a))\varphi_k(e)c_k)\psi_k(\zeta(h))\, d\mu(h) \\
&\quad \overset{(3.21)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^*\beta_h(c_k^*\varphi_k(e)\varphi_k(\alpha_{h^{-1}}(a))\varphi_k(e)c_k)\psi_k(\zeta(h))\, d\mu(h) \\
&\quad \overset{(3.22)}{=}_{2\varepsilon} \int_G \psi_k(\zeta(h))^*\beta_h(c_k^*\varphi_k(e\alpha_{h^{-1}}(a)e)c_k)\psi_k(\zeta(h))\, d\mu(h) \\
&\quad \overset{(3.25)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^*\beta_h(\psi_k(e\alpha_{h^{-1}}(a)e))\psi_k(\zeta(h))\, d\mu(h)
\end{aligned}
$$

$$\begin{aligned}
&\overset{(3.17)}{=}_{2\varepsilon} \int_G \psi_k(\zeta(h))^* \beta_h(\psi_k(\alpha_{h^{-1}}(eae)))\psi_k(\zeta(h))\, d\mu(h) \\
&\overset{(3.21)}{=}_{\varepsilon} \int_G \psi_k(\zeta(h))^* \psi_k(eae)\psi_k(\zeta(h))\, d\mu(h) \\
&\overset{(3.24)}{=}_{2\varepsilon} \int_G \psi_k(\zeta(h)^* eae\zeta(h))\, d\mu(h) \\
&= \psi_k(\langle \zeta, eae\zeta\rangle) \\
&\overset{(3.19)}{=}_{\varepsilon} \psi_k(e\langle \zeta, a\zeta\rangle e).
\end{aligned}$$

Thus we may infer that

$$\begin{aligned}
&\psi_k(d)^*\theta(\xi_k)^*\varphi_k(a)\theta(\xi_k)\psi_k(d) \\
&\qquad =_{10\varepsilon} \psi_k(d)^*\psi_k(e\langle \zeta, a\zeta\rangle e)\psi_k(d) \\
&\qquad \overset{(3.22)}{=}_{2\varepsilon} \psi_k(d^* e\langle \zeta, a\zeta\rangle ed) \\
&\qquad \overset{(3.20)}{=}_{2\varepsilon} \psi_k(d^* \langle \zeta, a\zeta\rangle d) \\
&\qquad \overset{(3.15)}{=}_{\varepsilon} \psi_k(d^* ad).
\end{aligned}$$

Hence, we may lift every element $\theta(\xi_k)\psi_k(d)$, for $k \geq n$, to a contraction in $\ell^\infty_\beta(\mathbb{N}, B)$ and extract a contraction $z_k \in B$ such that

$$\sup_{k\geq n} \max_{g\in K} \|z_k^*\beta_g(z_k) - \psi_k(d^*\alpha_g(d))\| \leq 16\varepsilon,$$

$$\sup_{k\geq n} \max_{a\in \mathcal{F}} \|z_k^*\varphi_k(a)z_k - \psi_k(d^*ad)\| \leq 16\varepsilon.$$

Finally, by using a standard diagonal argument over the triples $(\varepsilon, \mathcal{F}, K) = (\varepsilon_n, \mathcal{F}_n, K)$ with the property that $\varepsilon_n \to 0$, $\mathcal{F}_n$ are increasing with dense union in A, and K_n are increasing with $\bigcup_n K_n = G$, and leaving d fixed, we may find a contraction $s \in B_\infty$ satisfying the following conditions,

$$s^*\varphi(a)s = \psi(d^*ad), \text{for all } a \in A,$$

$$s^*\beta_{\infty,\beta}(s) = \psi(d^*\alpha_g(d)), \text{for all } g \in G.$$

We may therefore apply Lemma 3.4.13, and obtain that $(\varphi, \mathbf{1})$ approximately 1-dominates $(\psi, \mathbf{1})$. □

The following can be considered as an ideal-related version of [67, Lemma 4.3].

Lemma 3.4.15. *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C**-algebra, and $\beta : G \curvearrowright B$ an isometrically shift-absorbing, strongly stable, equivariantly $\mathcal{O}_2$-stable action on a separable* C**-algebra. Let $\varphi, \psi : (A, \alpha) \to (B_{\infty,\beta}, \beta_\infty)$ be two nuclearly liftable, equivariant $*$-homomorphisms. If $\mathcal{I}(\varphi) = \mathcal{I}(\psi)$ as maps into $\mathcal{I}(B_\infty)$, then $(\varphi, \mathbf{1})$ and $(\psi, \mathbf{1})$ are properly unitarily equivalent.*

Proof. It follows from Theorem 3.3.6 that φ and ψ approximately 1-dominate each other. Hence, we conclude with Lemma 3.4.14 that $(\varphi, \mathbf{1})$ and $(\psi, \mathbf{1})$ approximately 1-dominate each other. By Lemma 3.3.7(i), there exists a separable, β_∞-invariant C*-subalgebra D_0 of $B_{\infty,\beta}$ containing the image of φ and ψ, and such that $(\varphi, \mathbf{1})$ and $(\psi, \mathbf{1})$ approximately 1-dominate each other when corestricted to D_0. From Lemma 3.3.7(ii), we have that $\kappa \circ (\varphi, \mathbf{1})$ and $\kappa \circ (\psi, \mathbf{1})$ approximately 1-dominate each other, where $\kappa : (D_0, \beta_\infty) \to (D_0 \otimes \mathcal{K}, \beta_\infty \otimes \mathrm{id}_\mathcal{K})$ is the equivariant inclusion given by $\kappa(d) = d \otimes e_{1,1}$ for all $d \in D_0$. Now, using that β is strongly stable, we apply Lemma 2.8.8 to infer that there exists an equivariant $*$-homomorphism $\chi : (D_0 \otimes \mathcal{K}, \beta_\infty \otimes \mathrm{id}_\mathcal{K}) \to (B_{\infty,\beta}, \beta_\infty)$ such that $\chi(d \otimes e_{1,1}) = d$ for all $d \in D_0$. These maps then fit into the following commutative diagram,

$$\begin{tikzcd}
(D_0, \beta_\infty) \arrow[rr] \arrow[dr, "\kappa"'] & & (B_{\infty,\beta}, \beta_\infty) \\
& (D_0 \otimes \mathcal{K}, \beta_\infty \otimes \mathrm{id}_\mathcal{K}) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

Let D_1 denote the image of $D_0 \otimes \mathcal{K}$ in $B_{\infty,\beta}$ under χ. It follows that $\beta_\infty|_{D_1}$ is strongly stable, and $(\varphi, \mathbf{1})$, $(\psi, \mathbf{1})$ approximately 1-dominate each other when corestricted to $D_1 = \chi(D_0 \otimes \mathcal{K})$. Moreover, as $\beta \simeq_{\mathrm{cc}} \beta \otimes \mathrm{id}_{\mathcal{O}_2}$, it follows from Theorem 2.6.8 that there exists a unital embedding of $\mathcal{O}_2$ into $F(D_1, B_{\infty,\beta})^{\tilde{\beta}_\infty}$. Consider the canonical commutative diagram of equivariant $*$-homomorphisms given by

$$\begin{tikzcd}
(D_1, \beta_\infty) \arrow[rr] \arrow[dr, "\mathrm{id}_{D_1} \otimes \mathbf{1}"'] & & (B_{\infty,\beta}, \beta_\infty) \\
& (D_1 \otimes_{\max} F(D_1, B_{\infty,\beta}), \beta_\infty \otimes \tilde{\beta}_\infty) \arrow[ur, "\rho"'] &
\end{tikzcd}$$

where ρ is the natural $*$-homomorphism defined as $\rho(d \otimes (x + B_{\infty,\beta} \cap D_1^\perp)) = dx$, for all $d \in D_1$, $x \in B_{\infty,\beta} \cap D_1'$. Denote by D the image of $D_1 \otimes \mathcal{O}_2$ under ρ. Since $\beta_\infty|_D$ is strongly stable and equivariantly $\mathcal{O}_2$-stable, and $(\varphi, \mathbf{1})$, $(\psi, \mathbf{1})$ approximately 1-dominate each other when corestricted to D, may apply Lemma

3.3.14 and conclude that $(\varphi, \mathbf{1})$ and $(\psi, \mathbf{1})$ are properly approximately unitarily equivalent when corestricted to D. Since the cocycles are trivial, the sequence $u_n \in \mathcal{U}(\mathbf{1} + D)$ implementing the equivalence is approximately $\beta_\infty|_D$-invariant, namely

$$\max_{g \in K} \|\beta_{\infty,g}(u_n) - u_n\| = 0, \quad \text{for all compact subsets } K \subseteq G.$$

By performing a standard diagonal sequence argument inside B_∞, one can show that φ and ψ are properly unitarily equivalent via a β_∞-invariant unitary, and therefore $(\varphi, \mathbf{1})$ and $(\psi, \mathbf{1})$ are properly unitarily equivalent. $\square$

Lemma 3.4.16. *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C**-algebra, and $\beta : G \curvearrowright B$ a strongly stable, equivariantly $\mathcal{O}_2$-stable and isometrically shift-absorbing action on a separable* C**-algebra. Then, for every equivariant* **Cu***-morphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$ there exists an equivariant, nuclearly liftable $*$-homomorphism*

$$\psi : (A, \alpha) \to (B_{\infty,\beta}, \beta_\infty)$$

that satisfies $\mathcal{I}(\psi) = \mathcal{I}(\iota_\infty) \circ \Phi$ as a map with range in B_∞.

Proof. Consider the automorphisms σ and $\tau = \sigma \otimes \mathrm{id}_A$ of $\mathcal{C}_0(\mathbb{R})$ and SA, respectively, as in Remark 3.4.11. The assumptions of Lemma 3.4.12 are fullfilled, hence there exists a nuclearly liftable, equivariant $*$-homomorphism

$$\psi_1 : (SA, S\alpha) \to (B_{\infty,\beta}, \beta_\infty)$$

with the property that for each positive element $a \in A$ and positive non-zero function $f \in \mathcal{C}_0(\mathbb{R})$, $\psi_1(f \otimes a)$ is a full element of $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$. Since also $\psi_1 \circ \tau(f \otimes a) = \psi_1(\sigma(f) \otimes a)$ is full in $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$, it follows that $\mathcal{I}(\psi_1)$ and $\mathcal{I}(\psi_1 \circ \tau)$, where ψ_1 is viewed as a map with range in B_∞, agree on ideals generated by elements of the form $f \otimes a$, with f and a as before. Furthermore, since $\mathcal{I}(\psi_1)(I)$ is generated by $\psi_1(I_+)$ for any ideal $I \in \mathcal{I}(SA)$, and $\mathcal{I}(\psi_1)$ preserves suprema, it follows that $\mathcal{I}(SA)$ is completely determined by ideals generated by $f \otimes a$ for positive non-zero $f \in \mathcal{C}_0(\mathbb{R})$ and $a \in A_+$. Hence, we may conclude that $\mathcal{I}(\psi_1) = \mathcal{I}(\psi_1 \circ \tau)$.

We can therefore apply Lemma 3.4.15 to the equivariant $*$-homomorphisms ψ_1 and $\psi_1 \circ \tau$, and obtain that $(\psi_1, \mathbf{1})$ and $(\psi_1 \circ \tau, \mathbf{1})$ are properly unitarily equivalent via a unitary $u \in \mathcal{U}(\mathbf{1} + B_{\infty,\beta}^{\beta_\infty})$.

From the universal property of crossed products, there exists a $*$-homomorphism

$$\psi_0 : SA \rtimes_\tau \mathbb{Z} \to B_{\infty,\beta}, \quad \psi_0|_{SA} = \psi_1,$$

which is also nuclear as a map $SA \rtimes_\tau \mathbb{Z} \to B_\infty$ by Lemma 2.5.17. Let us show that ψ_0 is $(S\alpha \rtimes \mathbb{Z})$-to-$\beta_\infty$-equivariant[17]. Let v be the canonical unitary for the crossed product $SA \rtimes_\tau \mathbb{Z}$, which is sent to u by ψ_0. Then we have that

$$\begin{aligned}\psi_0 \circ (S\alpha \rtimes \mathbb{Z})_g(xv^n) &= \psi_0(S\alpha_g(x)v^n)\\ &= \psi_1(S\alpha_g(x))u^n\\ &= \beta_{\infty,g}(\psi_1(x)u^n)\\ &= \beta_{\infty,g} \circ \psi_0(xv^n)\end{aligned}$$

for all $x \in SA$, $n \in \mathbb{Z}$ and $g \in G$.

Recall from Remark 3.4.11 that there exists a natural $*$-isomorphism

$$\theta : (\mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z}) \otimes A \to SA \rtimes_\tau \mathbb{Z},$$

and the action $\mathrm{id} \otimes \alpha$ on the left corresponds to $S\alpha \rtimes \mathbb{Z}$, which acts trivially on the copy of $\mathbb{Z}$, on the right. For a full projection $p \in \mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z} \cong \mathcal{C}(\mathbb{T}) \otimes \mathcal{K}$, the $*$-homomorphism

$$\psi : A \to B_{\infty,\beta}, \quad a \mapsto \psi_0(\theta(p \otimes a)),$$

is nuclear as a map with range in B_∞, and equivariant with respect to α on the left and β_∞ on the right as $A \hookrightarrow \mathcal{C}(\mathbb{T}) \otimes \mathcal{K} \otimes A$ is α-to-$\mathrm{id} \otimes \alpha$ equivariant, and

$$\begin{aligned}\psi \circ \alpha_g(a) &= \psi_0(\theta(p \otimes \alpha_g(a)))\\ &= \psi_0(\theta \circ S\alpha_g(p \otimes a))\\ &= \psi_0((S\alpha \rtimes \mathbb{Z})_g \circ \theta(p \otimes a))\\ &= \beta_{\infty,g} \circ \psi(a)\end{aligned}$$

for all $a \in A$, and $g \in G$.

We want to show that $\mathcal{I}(\psi) = \mathcal{I}(\iota_\infty) \circ \Phi$. Equivalently, we want to prove that for any positive element $a \in A$, its image $\psi(a)$ is contained in $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$ and it is full.

Pick a positive element $a \in A$, together with a positive non-zero function $f \in \mathcal{C}_0(\mathbb{R})$. Denote by w the canonical unitary in $\mathcal{M}(\mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z})$. It follows that for every $n \in \mathbb{Z}$ we have

$$\psi_0(\theta(fw^n \otimes a)) = \psi_1(f \otimes a)u^n \in \mathcal{I}(\iota_\infty)(\Phi(\overline{AaA})).$$

[17]Recall from Remark 3.4.11 that $S\alpha \rtimes \mathbb{Z} : G \curvearrowright SA \rtimes_\tau \mathbb{Z}$ is the action that extends $S\alpha$ by acting trivially on the copy of $\mathbb{Z}$.

Hence, we may conclude that

$$\psi(a) = \psi_0(\theta(p \otimes a)) \in \mathcal{I}(\iota_\infty)(\Phi(\overline{AaA})).$$

Since $\psi_1(f \otimes a)$ is full in $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$, and p is full in $\mathcal{C}_0(\mathbb{R}) \rtimes_\sigma \mathbb{Z}$, we get the following inclusion,

$$\begin{aligned}\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA})) &\subseteq \overline{B_\infty \psi_1(f \otimes a) B_\infty} \\ &= \overline{B_\infty \psi_0(\theta(f \otimes a)) B_\infty} \\ &\subseteq \overline{B_\infty \psi_0(\theta(p \otimes a)) B_\infty} \\ &= \overline{B_\infty \psi(a) B_\infty}.\end{aligned}$$

Thus, $\psi(a)$ is full in $\mathcal{I}(\iota_\infty)(\Phi(\overline{AaA}))$. □

We are now able to prove the existence result that will ultimately lead to our classification theorem.

Theorem 3.4.17 (Existence)**.** *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C**-algebra, and $\beta : G \curvearrowright B$ an equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a separable* C**-algebra. Then, for every equivariant* **Cu***-morphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$, there exists a nuclear proper cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ such that $\mathcal{I}(\varphi) = \Phi$.*

Proof. By Theorem 2.6.27 there exists a proper cocycle conjugacy

$$(\theta, \mathbb{x}) : (B, \beta) \to (B \otimes \mathcal{O}_2, \beta \otimes \mathrm{id}_{\mathcal{O}_2})$$

that is properly approximately unitarily equivalent to the first factor embedding $\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2}$ with trivial cocycle. Then, for any embedding $\iota : \mathcal{K} \hookrightarrow \mathcal{O}_2$, which exists by Theorem 2.1.47, one may define a proper cocycle embedding $(B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}}) \xhookrightarrow{(\theta,\mathbb{x})^{-1} \circ (\mathrm{id} \otimes \iota)} (B, \beta)$ inducing the map $I \otimes \mathcal{K} \mapsto I$ for all $I \in \mathcal{I}(B)$ on the ideal lattice. For this reason, it is sufficient to prove the statement with the additional assumption that β is strongly stable. In fact, this would allow us to find a proper cocycle morphism $(A, \alpha) \to (B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_{\mathcal{K}})$ inducing $J \mapsto \Phi(J) \otimes \mathcal{K}$ for all $J \in \mathcal{I}(A)$ on the ideal lattice, which proves the general statement when composed with $(\theta, \mathbb{x})^{-1} \circ (\mathrm{id} \otimes \iota)$. Therefore, let us assume that β is strongly stable

Since the hypotheses of Lemma 3.4.16 are satisfied, there exists an equivariant, nuclearly liftable $*$-homomorphism

$$\psi : (A, \alpha) \to (B_{\infty,\beta}, \beta_\infty),$$

that satisfies $\mathcal{I}(\psi) = \mathcal{I}(\iota_\infty) \circ \Phi$ as a map with range in B_∞.

To obtain a proper cocycle morphism with range in B, we want to appeal to Theorem 2.2.39. Fix a sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \eta(n) = \infty$. One has that

$$\begin{aligned} \mathcal{I}(\eta^* \circ \psi) &= \mathcal{I}(\eta^*) \circ \mathcal{I}(\psi) \\ &= \mathcal{I}(\eta^*) \circ \mathcal{I}(\iota_\infty) \circ \Phi \\ &= \mathcal{I}(\iota_\infty) \circ \Phi \\ &= \mathcal{I}(\psi). \end{aligned}$$

By Lemma 3.4.15, we get that $(\eta^* \circ \psi, \mathbf{1})$ and $(\psi, \mathbf{1})$ are properly unitarily equivalent. It follows from Theorem 2.2.39 that there exists a proper cocycle morphism $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ such that $\iota_\infty \circ (\varphi, \mathbb{u})$ is properly unitarily equivalent to $(\psi, \mathbf{1})$. When considered as maps with range B_∞, we have that

$$\mathcal{I}(\iota_\infty) \circ \mathcal{I}(\varphi) = \mathcal{I}(\iota_\infty \circ \varphi) = \mathcal{I}(\psi) = \mathcal{I}(\iota_\infty) \circ \Phi,$$

and since $\mathcal{I}(\iota_\infty)$ is injective, it follows that $\mathcal{I}(\varphi) = \Phi$. In particular, $\iota_\infty \circ \varphi$ is nuclearly liftable, and thus φ is nuclear. □

Remark 3.4.18. If in Theorem 3.4.17 one assumes G to be exact and removes the amenability assumption on α, then the same conclusion holds true. Let us show why in the following paragraph. Since G is exact, by Remark 2.7.9 there exists an amenable action $\delta : G \curvearrowright \mathcal{O}_2$. Then from Theorem 3.4.17 we get a proper cocycle morphism $(\varphi_0, \mathbb{u}_0) : (A \otimes \mathcal{O}_2, \alpha \otimes \delta) \to (B, \beta)$ such that $\mathcal{I}(\varphi_0)(I \otimes \mathcal{O}_2) = \Phi(I)$ for all $I \in \mathcal{I}(A)$. Hence, one may define $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ to be the composition of $(\varphi_0, \mathbb{u}_0)$ with $\mathrm{id}_A \otimes \mathbf{1}_{\mathcal{O}_2}$. The proper cocycle morphism $(\varphi, \mathbb{u})$ clearly satisfies $\mathcal{I}(\varphi) = \Phi$.

3.5 The classification theorem

The uniqueness and existence results obtained in the previous sections yield a one-to-one correspondence between equivariant **Cu**-morphisms and proper cocycle morphisms identified up to proper approximate unitary equivalence.

Corollary 3.5.1. *Let $\alpha : G \curvearrowright A$ be an amenable action on a separable, exact* C*-*algebra, and $\beta : G \curvearrowright B$ an amenable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing, strongly stable action on a separable* C*-*algebra. Then, there* *exists a canonical one-to-one correspondence*

$$\frac{\left\{\begin{matrix}\text{nuclear proper cocycle morphisms}\\ (A,\alpha)\to(B,\beta)\end{matrix}\right\}}{\text{proper approximate unitary equivalence}} \to \left\{\begin{matrix}\text{equivariant } \mathbf{Cu}\text{-morphisms}\\ (\mathcal{I}(A),\alpha^\sharp)\to(\mathcal{I}(B),\beta^\sharp)\end{matrix}\right\}$$

Proof. Let $\Phi : (\mathcal{I}(A),\alpha^\sharp)\to(\mathcal{I}(B),\beta^\sharp)$ be an equivariant $\mathbf{Cu}$-morphism. By Theorem 3.4.17, there exists a proper cocycle morphism $(\varphi,\mathbb{u}) : (A,\alpha)\to(B,\beta)$ such that $\mathcal{I}(\varphi)=\Phi$. Suppose that there exists another proper cocycle morphism $(\psi,\mathbb{v}) : (A,\alpha)\to(B,\beta)$ such that $\mathcal{I}(\psi)=\Phi$. Then by Theorem 3.3.15 we have that $(\varphi,\mathbb{u})$ and $(\psi,\mathbb{v})$ are properly approximately unitarily equivalent. This finishes the proof. $\square$

The analogous conclusion holds in the unital setting.

Corollary 3.5.2. *Assume G to be exact. Let $\alpha : G \curvearrowright A$ be an action on a separable, exact, unital* C**-algebra, and $\beta : G\curvearrowright B$ an amenable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing action on a separable, unital* C**-algebra. Then, there exists a canonical one-to-one correspondence*

$$\frac{\left\{\begin{matrix}\text{unital nuclear proper cocycle morphisms}\\ (A,\alpha)\to(B,\beta)\end{matrix}\right\}}{\text{proper approximate unitary equivalence}} \to \left\{\begin{matrix}\text{equivariant } \mathbf{Cu}\text{-morphisms}\\ \Phi : (\mathcal{I}(A),\alpha^\sharp)\to(\mathcal{I}(B),\beta^\sharp)\\ \text{such that } \Phi(A)=B\end{matrix}\right\}$$

Proof. From Theorem 3.4.17 combined with Remark 3.4.18 there exists a proper cocycle morphism $(\psi,\mathbb{v}) : (A,\alpha)\to(B,\beta)$ inducing Φ. Since $\Phi(A)=B$ and $B\cong B\otimes\mathcal{O}_2$, we have that $\psi(\mathbf{1}_A)$ is a full projection in B. Moreover, $\psi(\mathbf{1}_A)$ is properly infinite by Definition 2.1.58. Now, again using $\mathcal{O}_2$-stability of B, Theorem 2.6.8 yields a unital embedding $\mathcal{O}_2\hookrightarrow B_\infty\cap\psi(A)'$. We may lift a pair of $\mathcal{O}_2$-isometries in $B_\infty\cap\psi(A)'$ to sequences of isometries $r_{1,n}, r_{2,n}\in B$ such that

$$\|r_{1,n}r_{1,n}^* + r_{2,n}r_{2,n}^* - \mathbf{1}_B\| \xrightarrow{n\to\infty} 0,$$
$$\|[r_{i,n},\psi(\mathbf{1}_A)]\| \xrightarrow{n\to\infty} 0 \quad \text{for } i=1,2.$$

Thanks to these approximate conditions, $\psi(\mathbf{1}_A)r_{1,n}\psi(\mathbf{1}_A)$ is close to being an isometry in $D:=\psi(\mathbf{1}_A)B\psi(\mathbf{1}_A)$, and the same is true for $\psi(\mathbf{1}_A)r_{2,n}\psi(\mathbf{1}_A)$. Therefore, for $\varepsilon<1/5$, we may find some $n\in\mathbb{N}$ and isometries $s_1,s_2\in D$ such that

$$s_1 =_\varepsilon \psi(\mathbf{1}_A)r_{1,n}\psi(\mathbf{1}_A), \quad s_2 =_\varepsilon \psi(\mathbf{1}_A)r_{1,n}\psi(\mathbf{1}_A)$$

and we may also assume that

$$\psi(\mathbf{1}_A)r_{1,n}\psi(\mathbf{1}_A)r_{1,n}^*\psi(\mathbf{1}_A) =_\varepsilon \psi(\mathbf{1}_A)r_{1,n}r_{1,n}^*\psi(\mathbf{1}_A),$$

$$\psi(\mathbf{1}_A)r_{2,n}\psi(\mathbf{1}_A)r_{2,n}^*\psi(\mathbf{1}_A) =_\varepsilon \psi(\mathbf{1}_A)r_{2,n}r_{2,n}^*\psi(\mathbf{1}_A),$$

$$r_{1,n}r_{1,n}^* + r_{2,n}r_{2,n}^* =_\varepsilon \mathbf{1}_B.$$

One may then infer that

$$\|s_1s_1^* + s_2s_2^* - \psi(\mathbf{1}_A)\| \leq 5\varepsilon < 1.$$

This implies that, in particular, there exists a unitary u in D such that $(us_1)(us_1)^* = \psi(\mathbf{1}_A) - s_2s_2^*$. Choose $s_1' = us_1$, and note that s_1' and s_2 are isometries in D with $s_1'(s_1')^* + s_2s_2^* = \psi(\mathbf{1}_A)$. Hence, it follows that $\psi(\mathbf{1}_A)$ is a properly infinite, full projection of B with $[\psi(\mathbf{1}_A)]_0 = 0$. Since B has a unital embedding of $\mathcal{O}_2$, we also have that $[\mathbf{1}_B]_0 = 0$. Therefore, by Lemma 2.1.44 and the fact checked above that $[\psi(\mathbf{1}_A)]_0 = [\mathbf{1}_B]_0$, we conclude that $\psi(\mathbf{1}_A)$ is Murray–von Neumann equivalent to $\mathbf{1}_B$. Hence, there exists an isometry $v \in B$ such that $vv^* = \psi(\mathbf{1}_A)$ We then define a unital $*$-homomorphism

$$\varphi : A \to B, \quad \varphi(a) = v^*\psi(a)v,$$

and a norm-continuous map

$$\mathbb{u} : G \to B, \quad \mathbb{u}_g = v^*\mathbb{v}_g\beta_g(v).$$

Note that, for each $g \in G$, $\mathbb{u}_g \in \mathcal{U}(B)$ because

$$\begin{aligned}\mathbb{u}_g^*\mathbb{u}_g &= \beta_g(v)^*\mathbb{v}_g^*vv^*\mathbb{v}_g\beta_g(v) = \beta_g(v)^*\mathbb{v}_g^*\psi(\mathbf{1}_A)\mathbb{v}_g\beta_g(v)\\ &= \beta_g(v)^*\mathbb{v}_g^*\mathbb{v}_g\beta_g(\psi(\mathbf{1}_A))\mathbb{v}_g^*\mathbb{v}_g\beta_g(v) = \beta_g(v^*\psi(\mathbf{1}_A)v)\\ &= \varphi(\mathbf{1}_A) = \mathbf{1}_B.\end{aligned}$$

where we used that $\beta_g^{\mathbb{v}} \circ \psi(\mathbf{1}_A) = \psi(\mathbf{1}_A)$, and we also have that

$$\begin{aligned}\mathbb{u}_g\mathbb{u}_g^* &= v^*\mathbb{v}_g\beta_g(v)\beta_g(v)^*\mathbb{v}_g^*v = v^*\beta_g^{\mathbb{v}}(vv^*)v\\ &= v^*\beta^{\mathbb{v}}(\psi(\mathbf{1}_A))v = v^*\psi(\mathbf{1}_A)v\\ &= \varphi(\mathbf{1}_A) = \mathbf{1}_B.\end{aligned}$$

Hence, we may view $\mathbb{u}$ as a continuous map $G \to \mathcal{U}(B) \cong \mathcal{U}(\mathbf{1} + B)$. Let us show that $\mathbb{u}$ is a β-cocycle,

$$\mathbb{u}_{gh} = v^*\mathbb{v}_{gh}\beta_{gh}(v) = v^*\mathbb{v}_g\beta_g(\mathbb{v}_h\beta_h(v))$$

$$= v^*\mathbb{v}_g\beta_g(vv^*\mathbb{v}_h\beta_h(v)) = \mathbb{u}_g\beta_g(\mathbb{u}_h)$$

for all $g, h \in G$, where we used that $\mathbb{v}_h\beta_h(vv^*) = vv^*\mathbb{v}_h$, which follows from $\beta_h^{\mathbb{v}} \circ \psi(\mathbf{1}_A) = \psi(\mathbf{1}_A)$. It follows that $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is a unital proper cocycle morphism inducing Φ. By Corollary 3.3.16 we have that $(\varphi, \mathbb{u})$ is the unique unital proper cocycle morphism inducing Φ up to proper approximate unitary equivalence. $\square$

Definition 3.5.3 (see [67, Definition 5.9] and [185, Definition 2.4(iii)])**.** Let $\beta : G \curvearrowright B$ be an action on a C*-algebra. A norm-continuous β-cocycle $\mathbb{u} : G \to \mathcal{U}(\mathbf{1} + B)$ is said to be an *approximate coboundary* if there exists a sequence of unitaries $v_n \in \mathcal{U}(\mathbf{1} + B)$ such that

$$\lim_{n\to\infty} \max_{g\in K} \|\mathbb{u}_g - v_n\beta_g(v_n)^*\| = 0$$

for every compact set $K \subseteq G$.

If $\alpha : G \curvearrowright A$ is an action on a C*-algebra, then a proper cocycle conjugacy $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ is a *strong cocycle conjugacy* if $\mathbb{u}$ is an approximate coboundary.

Remark 3.5.4. Let $\beta : G \curvearrowright B$ be an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action on a separable C*-algebra, and $\mathbb{u} : G \to \mathcal{U}(\mathbf{1}+B)$ a norm-continuous β-cocycle. Moreover, consider the following assumptions:

(i) β is strongly stable;

(ii) B is unital, and G exact.

Assume (i) for the rest of this paragraph. Note that, if D denotes the zero algebra, every proper cocycle morphism $(D, \mathrm{id}_D) \to (B, \beta)$ induces the same **Cu**-morphism between ideal lattices. Hence, Theorem 3.3.15 implies that that $(0, \mathbb{u})$ is properly approximately unitarily equivalent to $(0, \mathbf{1})$, where 0 denotes the zero map. In other words, there exists a sequence of unitaries $v_n \in \mathcal{U}(\mathbf{1}+B)$ such that $\max_{g\in K} \|\mathbb{u}_g - v_n\beta_g(v_n)^*\| \xrightarrow{n\to\infty} 0$ for all compact sets $K \subseteq G$.

Assume now (ii). Observe that any unital proper cocycle morphism $(\mathbb{C}, \mathrm{id}_\mathbb{C}) \to (B, \beta)$ induces the same **Cu**-morphism between ideal lattices. Thus, by Corollary 3.3.16, one obtains that $(\iota, \mathbb{u})$ is properly approximately unitarily equivalent to $(\iota, \mathbf{1})$, where $\iota : \mathbb{C} \hookrightarrow B$ is the canonical unital map.

Consequently, if $\alpha : G \curvearrowright A$ is an action on a C*-algebra, and $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ a proper cocycle conjugacy, the previous paragraph implies that whenever (i) or (ii) holds true, $\mathbb{u}$ is an approximate coboundary and thus $(\varphi, \mathbb{u})$ is a strong cocycle conjugacy.

The following theorem represents the main application of the previous sections, and should be considered as a generalization of Kirchberg's $\mathcal{O}_2$-stable classification theorem as appeared in Gabe's work (see Theorem 3.1.1) to the framework of C*-dynamical systems.

Theorem 3.5.5 (Classification). *Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be amenable, equivariantly $\mathcal{O}_2$-stable, isometrically shift-absorbing actions on separable, nuclear* C*-*algebras. Then, the following statements hold true.*

(i) *If both A and B are stable, then for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ there exists a cocycle conjugacy $(\varphi, \mathbbm{u}) : (A, \alpha) \to (B, \beta)$ such that $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.*

(ii) *If both α and β are strongly stable, then for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ there exists a strong cocycle conjugacy $(\varphi, \mathbbm{u}) : (A, \alpha) \to (B, \beta)$ such that $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.*

(iii) *If G is exact, and both A and B are unital, then for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ there exists a strong cocycle conjugacy $(\varphi, \mathbbm{u}) : (A, \alpha) \to (B, \beta)$ such that $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.*

Proof. We first observe that (i) follows from (ii). From Proposition 2.8.7 we know that when both A and B are stable, there exist cocycle conjugacies

$$(\theta_\alpha, \mathbbm{x}_\alpha) : (A, \alpha) \to (A \otimes \mathcal{K}, \alpha \otimes \mathrm{id}_\mathcal{K}), \quad (\theta_\beta, \mathbbm{x}_\beta) : (B, \beta) \to (B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_\mathcal{K}).$$

Hence, for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$, we may apply (ii) to the conjugacy given by

$$(\mathrm{Prim}(A \otimes \mathcal{K}), \alpha \otimes \mathrm{id}_\mathcal{K}) \to (\mathrm{Prim}(B \otimes \mathcal{K}), \beta \otimes \mathrm{id}_\mathcal{K})$$

$$\mathfrak{p} \otimes \mathcal{K} \mapsto (\theta_\beta \circ f \circ \theta_\alpha^{-1}(\mathfrak{p})) \otimes \mathcal{K}$$

and get a strong cocycle conjugacy $(\varphi_0, \mathbbm{u}_0) : (A \otimes \mathcal{K}, \alpha \otimes \mathrm{id}_\mathcal{K}) \to (B \otimes \mathcal{K}, \beta \otimes \mathrm{id}_\mathcal{K})$ such that $\varphi_0(\mathfrak{p} \otimes \mathcal{K}) = (\theta_\beta \circ f \circ \theta_\alpha^{-1}(\mathfrak{p})) \otimes \mathcal{K}$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$. Then, the cocycle conjugacy given as

$$(\varphi, \mathbbm{u}) = (\theta_\beta, \mathbbm{x}_\beta)^{-1} \circ (\varphi_0, \mathbbm{u}_0) \circ (\theta_\alpha, \mathbbm{x}_\alpha) : (A, \alpha) \to (B, \beta)$$

satisfies $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.

We now prove the non-trivial implication in (ii). Assume that both α and β are strongly stable, and that $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ is a conjugacy. By Remark 2.3.40 there exists an equivariant order isomorphism $\Phi : (\mathcal{I}(A), \alpha^\sharp) \to (\mathcal{I}(B), \beta^\sharp)$ such that $\Phi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$. From Corollary 3.5.1

there exist proper cocycle morphisms $(\varphi_0, \mathbb{u}_0) : (A, \alpha) \to (B, \beta)$ and $(\psi_0, \mathbb{v}_0) : (B, \beta) \to (A, \alpha)$ such that $\mathcal{I}(\varphi_0) = \Phi$ and $\mathcal{I}(\psi_0) = \Phi^{-1}$. Since we have that

$$\mathcal{I}(\psi_0) \circ \mathcal{I}(\varphi_0) = \mathcal{I}(\mathrm{id}_A), \quad \mathcal{I}(\varphi_0) \circ \mathcal{I}(\psi_0) = \mathcal{I}(\mathrm{id}_B),$$

Corollary 3.5.1 guarantees that $(\psi_0, \mathbb{v}_0) \circ (\varphi_0, \mathbb{u}_0)$ is properly approximately unitarily equivalent to id_A, and $(\varphi_0, \mathbb{u}_0) \circ (\psi_0, \mathbb{v}_0)$ is properly approximately unitarily equivalent to id_B. It is therefore possible to apply Theorem 2.2.30 to get a proper cocycle conjugacy $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ such that $(\varphi, \mathbb{u})$ is properly approximately unitarily equivalent to $(\varphi_0, \mathbb{u}_0)$, and therefore $\mathcal{I}(\varphi) = \Phi$. By Remark 3.5.4, $(\varphi, \mathbb{u})$ is a strong cocycle conjugacy with the property that $\varphi(\mathfrak{p}) = \Phi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.

The non-trivial implication in (iii) can be proved analogously as (ii) by virtue of Corollary 3.5.2, which has to be used in place of Corollary 3.5.1. □

Remark 3.5.6. We emphasize that our classification of cocycle morphisms is carried out up to proper approximate unitary equivalence. By contrast, in our article [156], the classification was established using strong asymptotic unitary equivalence, which is, a priori, a stronger notion. Crucially, this distinction does not affect the final classification result for actions, except in the formulation of results involving strong cocycle conjugacy. In fact, in the main classification theorem of [156], the sequence of unitaries implementing the approximate coboundary in the definition of strong cocycle conjugacy can be replaced by a continuous path of unitaries starting at **1**. This yields an even stronger notion, often referred to as *very strong cocycle conjugacy.*

The following is a refined version of Theorem 3.5.5 in case G is compact.

Corollary 3.5.7. *Let G be a second-countable, compact group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable actions on separable, nuclear* C^**-algebras. Then, the following statements hold true.*

(i) *If both A and B are stable, then for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ there exists a conjugacy $\varphi : (A, \alpha) \to (B, \beta)$ such that $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.*

(ii) *If both A and B are unital, then for every conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to (\mathrm{Prim}(B), \beta^\sharp)$ there exists a conjugacy $\varphi : (A, \alpha) \to (B, \beta)$ such that $\varphi(\mathfrak{p}) = f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$.*

Proof. Observe that, by Remark 2.8.9, if A and B are stable, α and β are strongly stable. Now, in both cases (i) and (ii), a conjugacy $f : (\mathrm{Prim}(A), \alpha^\sharp) \to$

$(\mathrm{Prim}(B), \beta^\sharp)$ lifts to a strong cocycle conjugacy $(\varphi, \mathbb{u}) : (A, \alpha) \to (B, \beta)$ by Theorem 3.5.5. By [67, Corollary 5.11], $(\varphi, \mathbb{u})$ is properly unitarily equivalent to a conjugacy, which concludes the proof. $\square$

Theorem 3.5.8. *Suppose G is exact. Let $\beta : G \curvearrowright B$ be an action on a separable, nuclear C^*-algebra. Let $\delta : G \curvearrowright \mathcal{O}_2$ be an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action. Then, β is amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable if and only if β is cocycle conjugate to $\beta \otimes \delta$.*

Proof. Since δ is strongly self-absorbing, the property of absorbing δ is invariant under stable cocycle conjugacy; see [8, Theorem 4.30]. The same is the case for equivariant $\mathcal{O}_2$-stability. Since furthermore amenability and isometric shift-absorption are properties of the induced action on $F_{\infty,\beta}(B)$, which is also an invariant under stable cocycle conjugacy, we may assume without any loss of generality that β is strongly stable.

Assume that β is cocycle conjugate to $\beta \otimes \delta$. Since amenability, isometric shift-absorption and equivariant $\mathcal{O}_2$-stability are preserved under cocycle conjugacy and tensor products, it follows that β inherits these properties from $\beta \otimes \delta$.

Conversely, assume β is amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable. Consider the equivariant first factor embedding $\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2} : (B, \beta) \to (B \otimes \mathcal{O}_2, \beta \otimes \delta)$. Since $\mathcal{I}(\mathrm{id}_B \otimes \mathbf{1}_{\mathcal{O}_2}) : (\mathcal{I}(B), \beta^\sharp) \to (\mathcal{I}(B \otimes \mathcal{O}_2, (\beta \otimes \delta)^\sharp))$ is an equivariant order isomorphism, Theorem 3.5.5 implies that there exists a cocycle conjugacy $(\theta, \mathbb{x}) : (B, \beta) \to (B \otimes \mathcal{O}_2, \beta \otimes \delta)$. $\square$

Remark 3.5.9. We argue that the classification results given in Section 3.2 can be deduced as special cases of Theorem 3.5.5 and Corollary 3.5.7.

Let G be compact, and δ denote an action as in Remark 3.2.1. Since δ necessarily absorbs every G-action on $\mathcal{O}_2$, it also absorbs the Rokhlin action $G \curvearrowright \mathcal{O}_2$ from Example 2.6.24, and thus has the Rokhlin property as well. On the other hand, any action with the Rokhlin property on a separable, $\mathcal{O}_2$-stable C^*-algebra is δ-absorbing by [74, Theorem 4.50]. Hence, Corollary 3.5.7 becomes equivalent to Proposition 3.2.2.

Let us now look at the case $G = \mathbb{R}$. As recalled in Remark 2.7.5, any flow on a separable $\mathcal{O}_\infty$-stable C^*-algebra is isometrically shift-absorbing if and only if it is a Rokhlin flow. It is a consequence of a more general result of Szabó, [190, Theorem 4.4], that every Rokhlin flow on an $\mathcal{O}_2$-stable C^*-algebra is furthermore equivariantly $\mathcal{O}_2$-stable. Hence, Theorem 3.5.5 in this case reduces to Proposition 3.2.3.

Finally, let us review the special case $G = \mathbb{Z}$. Recall from Remark 3.2.1 that $\delta : \mathbb{Z} \curvearrowright \mathcal{O}_2$ is the unique outer action up to strong cocycle conjugacy, and is well known to have the Rokhlin property; see [149, Theorem 1]. Since this action is strongly self-absorbing, the main result of [190] implies that every $\mathbb{Z}$-action with the Rokhlin property on any $\mathcal{O}_2$-stable C*-algebra absorbs δ. In light of Theorem 3.5.8, it is clear that our main result, Theorem 3.5.5, is about automorphisms with the Rokhlin property on nuclear $\mathcal{O}_2$-stable C*-algebras, and therefore it is equivalent to Proposition 3.2.4.

Chapter 4

Range of the invariant

In this chapter, which is based on the published research article [155], we will extend on previous work from Chapter 3, and establish a number of results of independent interest. The main theorem proved here establishes that every continuous action on the primitive ideal space of a separable nuclear C*-algebra lifts to a C*-action. This implies, in particular, that the classification invariant for equivariantly $\mathcal{O}_2$-stable actions from Chapter 3 can always be realised.

4.1 Harnisch–Kirchberg systems

The notation present in this section follows that of [91], with the exception of Definition 4.1.7, where we use the name *Harnisch–Kirchberg system* for a dynamical object that is called σ-modular in [91]. We recapitulate a number of results from the same work, and complement them with some new useful observations.

Notation 4.1.1 (cf. [91, Section 4])**.** Let B be a C*-algebra, $\sigma \in \mathrm{Aut}(B)$ an automorphism, and $D \subseteq B$ a C*-subalgebra. For every $k \in \mathbb{Z}$ and $\ell \geq k$, one defines the vector space

$$D_{k,\ell} = \sigma^{k-1}(D) + \cdots + \sigma^{\ell-1}(D).$$

One can moreover form the following vector spaces,

$$D_{-\infty,k} = \overline{\bigcup_{n\in\mathbb{N}} D_{k-n,k}}, \quad D_{k,\infty} = \overline{\bigcup_{n\in\mathbb{N}} D_{k,k+n}}, \quad D_{\sigma} = \overline{\bigcup_{n\in\mathbb{N}} D_{-n,n}}.$$

Lemma 4.1.2 (cf. [91, Lemma 4.7(i)]). *Let B be a* C^**-algebra,* $\sigma \in \mathrm{Aut}(B)$*, and $D \subseteq B$ a* C^**-subalgebra. For every $i, k, \ell \in \mathbb{Z}$ with $k \leq \ell$, one has that*

$$\sigma^i(D_{k,\ell}) = D_{k+i,\ell+i}, \qquad \sigma^i(D_{-\infty,\ell}) = D_{-\infty,\ell+i},$$

$$\sigma^i(D_{k,\infty}) = D_{k+i,\infty}, \qquad \sigma^i(D_\sigma) = D_\sigma.$$

Proof. The following identities are true by definition,

$$\begin{aligned}\sigma^i(D_{k,\ell}) &= \sigma^i(\sigma^{k-1}(D) + \cdots + \sigma^{\ell-1}(D)) \\ &= \sigma^{k+i-1}(D) + \cdots + \sigma^{\ell+i-1}(D) \\ &= D_{k+i,\ell+i}\end{aligned}$$

for all $k \leq \ell$ and i in $\mathbb{Z}$. This implies that the statements for $D_{-\infty,\ell}$, $D_{k,\infty}$ and D_σ also hold true as they are inductive limits of $\{D_{\ell-n,\ell}\}_{n\in\mathbb{N}}$, $\{D_{k,k+n}\}_{n\in\mathbb{N}}$ and $\{D_{-n,n}\}_{n\in\mathbb{N}}$, respectively. $\square$

The following lemma is well known when the C^*-algebra D in the statement is an ideal of B; see for instance [111, Corollary I.5.6].

Lemma 4.1.3 (cf. [91, Lemma 4.8]). *Let C, D be* C^**-subalgebras of a* C^**-algebra B such that $CD \subseteq D$. Then the $*$-subalgebra of B given by $C+D$ is a* C^**-algebra and there exists an isomorphism*

$$\frac{C}{C \cap D} \cong \frac{C+D}{D}.$$

Proof. We know that D is a closed two-sided ideal of $C + D$ as $CD \subseteq D$, and $DC = (C^*D^*)^* = (CD)^* \subseteq D^* = D$. Moreover, $C \cap D$ is a closed two-sided ideal of C. The map $\varphi : C/(C \cap D) \to (C + D)/D$ given by $\varphi(x + (C \cap D)) = x + D$ for all $x \in C$, is a well-defined $*$-homomorphism of $*$-algebras. Moreover, it is injective for if $x + D = y + D$ for some $x, y \in C$, we have that $(x - y) \in C \cap D$, and thus $y + (C \cap D) = y + (x - y) + (C \cap D) = x + (C \cap D)$. It is surjective because for any element $x + D$ of $(C + D)/D$, there exists $x' \in C$ such that $x + D = x' + D$ with $x' \in C$, and $\varphi(x' + (C \cap D)) = x' + D$. Since $C/(C \cap D)$ is a C^*-algebra, so is $\varphi(C/(C \cap D)) = (C + D)/D$. Hence, being the preimage of the quotient map $(C + D) \to (C + D)/D$, $C + D$ is norm-closed, and thus a C^*-algebra. $\square$

Lemma 4.1.4 (see [111, Lemma III.4.1]). *Let $A = \overline{\bigcup_n A_n}$ be an inductive limit* C^**-algebra, where $A_n \subseteq A$ is an incresing sequence of* C^**-algebras. Then, for* *every ideal $I \in \mathcal{I}(A)$, one has that*

$$I = \overline{\bigcup_n (I \cap A_n)} = \overline{I \cap \bigcup_n A_n}.$$

Proof. By Lemma 4.1.3, there exists an isomorphism $\frac{A_n}{A_n \cap I} \cong \frac{A_n + I}{I}$. In particular, one has that for any $a_n \in A_n$,

$$\inf_{x \in A_n \cap I} \|a_n + x\| = \inf_{y \in I} \|a_n + y\|.$$

Now, for any element $y \in I$ and $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ and $a_n \in A_n$ such that $y =_\varepsilon a_n$. It follows from the norm-identity above that one can find some $x \in A_n \cap I$ such that $x =_\varepsilon a_n$, and therefore $y =_{2\varepsilon} x$. Since $y \in I$ and $\varepsilon > 0$ were arbitrary, we may conclude that $I = \overline{\bigcup_n (I \cap A_n)}$, where the latter is clearly the same as $\overline{I \cap \bigcup_n A_n}$. □

Lemma 4.1.5 (cf. [91, Lemma 4.7(ii)])**.** *Let B be a C^*-algebra, $\sigma \in \mathrm{Aut}(B)$, and $D \subseteq B$ a C^*-subalgebra such that $D\sigma(D) = \sigma(D)$. Then, for all $n > 0$ and integers $k \leq \ell$ and $i \leq j$, one has that*

$$D\sigma^n(D) = \sigma^n(D), \quad \textit{and} \quad D_{k,\ell} D_{i,j} \subseteq D_{\max(k,i),\max(\ell,j)}.$$

Moreover, $D_{k,\ell}, D_{-\infty,\ell}$, and $D_{k,\infty}$ are C^*-subalgebras of B, and D_σ is the smallest σ-invariant C^*-subalgebra of B containing D.

Proof. First, observe that if $n = 1$, $D\sigma^n(D) = D\sigma(D) = \sigma(D)$ by assumption. Assume that $D\sigma^n(D) = \sigma^n(D)$ for some $n > 1$, and note that

$$D\sigma^{n+1}(D) = D\sigma^n(D\sigma(D)) = D\sigma^n(D)\sigma^{n+1}(D) = \sigma^n(D\sigma(D)) = \sigma^{n+1}(D).$$

By induction on $n \in \mathbb{N}$ we get that the first condition holds. Observe that whenever $i \leq j$ are integers, by the previous part of the proof we have that

$$\sigma^i(D)\sigma^j(D) = \sigma^i(D\sigma^{j-i}(D)) = \sigma^i(\sigma^{j-i}(D)) = \sigma^j(D).$$

Then, since we know that $\sigma^j(D)\sigma^i(D) = \left(\sigma^i(D)\sigma^j(D)\right)^* = \sigma^j(D)$, this proves the second condition.

For the moreover part, note that each $D_{k,k} = \sigma^{k-1}(D)$, for $k \in \mathbb{Z}$, is a C^*-algebra as σ is an automorphism. It follows from Lemma 4.1.3 that $D_{k,k+1} = D_{k,k} + \sigma^k(D)$ is a C^*-algebra as well because $D_{k,k}\sigma^k(D) = \sigma^{k-1}(D\sigma(D)) \subseteq \sigma^k(D)$. Suppose that $D_{k,\ell}$ is a C^*-algebra, where $\ell > k + 1$. It follows from the first part of the lemma, which we have just proven, that $D_{k,\ell}\sigma^\ell(D) \subseteq \sigma^\ell(D)$, and thus $D_{k,\ell+1} = D_{k,\ell} + \sigma^{\ell}(D)$ is a C*-algebra by Lemma 4.1.3. By induction on $\ell > k+1$, we have that $D_{k,\ell}$ is a C*-algebra for all pairs of integers $k \leq \ell$. As a consequence, $D_{k,\infty}$ and $D_{-\infty,\ell}$ are C*-algebras for all $k, \ell \in \mathbb{Z}$. Since D_σ is generated by all $\sigma^k(D)$, for $k \in \mathbb{Z}$, it is the smallest σ-invariant C*-subalgebra of B containing D. □

Lemma 4.1.6 (cf. [91, Lemma 4.9+4.10])**.** *Let B be a* C*-*algebra, $\sigma \in \mathrm{Aut}(B)$, and $D \subseteq B$ a* C*-*subalgebra such that $D\sigma(D) = \sigma(D)$. Then, for all integers $k_1 \leq k_2 \leq \ell$, $D_{k_2,\ell}$ is an ideal of $D_{k_1,\ell}$, and $D_{\ell,\infty}$ is an ideal of D_σ.*

If moreover $D \cap \sigma(D) = \{0\}$ and $D \cap \sigma(D)^\perp = \{0\}$,[1] *then $D_{k_2,\ell}$ is an essential ideal of $D_{k_1,\ell}$, and $D_{\ell,\infty}$ is an essential ideal of D_σ.*

Proof. The first part follows from Lemma 4.1.5.

The assumption asking that $D \cap \sigma(D)^\perp = \{0\}$ is equivalent to saying that $D_{2,2} = \sigma(D)$ is an essential ideal of $D_{1,2} = D + \sigma(D)$. Suppose that $D_{n,n}$ is an essential ideal of $D_{1,n}$. Let $x \in D_{1,n+1}$ with decomposition $x = y + z$, where $y \in D_{1,n}$ and $z \in D_{n+1,n+1}$, and assume that $xD_{n+1,n+1} = \{0\}$. Then, for every element $d \in D_{n,n}$, we have from Lemma 4.1.5 that $dx = dy + dz \in D_{n,n} + D_{n+1,n+1} = \sigma^{n-1}(D_{1,2})$. Since $(D + \sigma(D)) \cap \sigma(D)^\perp = \{0\}$, we have that $dxD_{n+1,n+1} = \{0\}$ implies $dx = 0$. Hence, $dy = -dz \in \sigma^{n-1}(D \cap \sigma(D)) = \{0\}$, so that $D_{n,n}y = 0$ and since we assumed that $D_{n,n}$ is essential in $D_{1,n}$, it must be $y = 0$. Therefore, $x = z \in D_{n+1,n+1}$ with $xD_{n+1,n+1} = 0$, which implies that $x = 0$. Arguing by induction on $n \geq 1$, we have shown that $D_{n,n}$ is an essential ideal of $D_{1,n}$ for all $n \geq 1$.

Now, for the general case, pick integers $k_1 \leq k_2 \leq \ell$. To show that $D_{k_2,\ell}$ is an essential ideal of $D_{k_1,\ell}$, first observe that $D_{\ell,\ell} = \sigma^{k_1-1}(D_{\ell+1-k_1,\ell+1-k_1})$ is essential in $D_{k_1,\ell} = \sigma^{k_1-1}(D_{1,\ell+1-k_1})$. Moreover, we have already observed that $D_{k_2,\ell}$ is an ideal of $D_{k_1,\ell}$. Since $D_{k_2,\ell}$ contains $D_{\ell,\ell}$, it follows that it is an essential ideal of $D_{k_1,\ell}$.

By definition, D_σ is the inductive limit C*-algebra of $\{D_{-n,n}\}_{n\in\mathbb{N}}$ with canonical inclusions. Hence, if $J \subseteq D_\sigma$ is a non-zero ideal, by Lemma 4.1.4 J is the inductive limit of the inductive system given by $J_n = J \cap D_{-n,n}$ with canonical inclusions. Fix some $\ell \in \mathbb{Z}$. One has that there exists $n_0 \in \mathbb{N}$ with $n_0 \geq \ell$, and such that $J_{n_0} \neq \{0\}$. Since D_{n_0,n_0} is an essential ideal of D_{-n_0,n_0}, it follows that $J_{n_0} \cap D_{n_0,n_0} \neq \{0\}$. Finally, one may conclude that $D_{\ell,\infty}$ is an essential ideal of D_σ as $J \cap D_{\ell,\infty} \supseteq J_{n_0} \cap D_{n_0,n_0}$. □

[1] Here, one should formally write $(D + \sigma(D)) \cap \sigma(D)^\perp$. However, since the only element of $\sigma(D)$ that annihilates $\sigma(D)$ itself is 0, we abuse notation, and write $D \cap \sigma(D)^\perp$ instead.

The following definition was introduced in [155] with the sole purpose of streamlining a number of statements in that article. It does make statements lighter, and thus we keep using it here.

Definition 4.1.7 (cf. [91, Definition 4.2]). Let B be a C*-algebra equipped with an automorphism $\sigma \in \mathrm{Aut}(B)$, and $D \subseteq B$ a C*-subalgebra. We say that the triple (B, σ, D) is a *Harnisch–Kirchberg system* if the following conditions hold,

(i) $D\sigma(D) = \sigma(D)$,

(ii) $D \cap \sigma(D) = \{0\}$,

(iii) $D \cap \sigma(D)^{\perp} = \{0\}$.

Note that, if (B, σ, D) is a triple with the properties of Definition 4.1.7, then Harnisch and Kirchberg say that D is σ-modular (see [91, Definition 4.2]). We avoid using this terminology.

Lemma 4.1.8 (cf. [91, Lemma 4.11]). *Let (B, σ, D) be a Harnisch–Kirchberg system. Then, for all integers $k_1 \leq k_2 < k_3$, we have that $D_{k_1,k_2} \cap D_{k_2+1,k_3} = \{0\}$.*

Proof. Firs, observe that it is enough to show that $D_{k_2,k_2} \cap D_{k_2+1,k_3} = \{0\}$. Indeed, if $D_{k_1,k_2} \cap D_{k_2+1,k_3}$ is a non-zero ideal of D_{k_1,k_2}, we can see with Lemma 4.1.6 that D_{k_2,k_2} is an essential ideal of D_{k_1,k_2}, and hence $D_{k_2,k_2} \cap D_{k_2+1,k_3} \neq \{0\}$, which means that

$$D_{k_2,k_2} \cap D_{k_2+1,k_3} = \{0\} \quad \Longrightarrow \quad D_{k_1,k_2} \cap D_{k_2+1,k_3} = \{0\}.$$

Moreover, after applying σ a suitable number of times, it is sufficient to show that $D \cap D_{2,n} = \{0\}$ for all $n \geq 2$.

By assumption, $D \cap D_{2,2} = D \cap \sigma(D) = \{0\}$. Suppose that $D \cap D_{2,n} = \{0\}$ for some $n > 2$. Assume, towards a contradiction, that there exists a non-zero element $d \in D \cap D_{2,n+1}$, and find $v \in D_{2,n}$ and $w \in D_{n+1,n+1}$ such that $d = v + w$. It follows that $w \neq 0$ for if $w = 0$, then $d = v \in D \cap D_{2,n} = \{0\}$. Now, we have shown that $w = d - v \in D_{1,n} \cap D_{n+1,n+1}$ is a non-zero element, and therefore $D_{1,n} \cap D_{n+1,n+1}$ is a non-zero ideal of $D_{1,n}$. Since $D_{n,n}$ is an essential ideal of $D_{1,n}$ by Lemma 4.1.6, we have that $D_{n,n} \cap D_{1,n} \cap D_{n+1,n+1} \neq 0$, and thus $\sigma^n(D \cap \sigma(D)) \neq 0$, a contradiction. We argue by induction on n that $D \cap D_{2,n} = \{0\}$ for all $n \geq 2$, which finishes the proof. $\square$

Lemma 4.1.9. *Let (B, σ, D) be a Harnisch–Kirchberg system. For any given integer $k \in \mathbb{Z}$, an approximate unit of $D_{k,k}$ is also an approximate unit of $D_{k,\infty}$.*

Proof. By assumption $D\sigma(D) = \sigma(D)$, hence $D_{k,k}D_{k+1,k+1} = D_{k+1,k+1}$. Assume that $\{u_\lambda\}_\lambda \subseteq D_{k,k}$ is an approximate unit for $D_{k,k}$. We have that for any $x \in D_{k+1,k+1}$, there exist $y \in D_{k,k}$ and $z \in D_{k+1,k+1}$ such that $x = yz$. Therefore, $\lim_\lambda u_\lambda x = \lim_\lambda (u_\lambda y)z = yz = x$. Assume now that $\{u_\lambda\}_\lambda \subseteq D_{k,k}$ is an approximate unit for $D_{k,\ell}$, for some $\ell > k$. It follows in particular that $\{u_\lambda\}_\lambda$ is an approximate unit for $D_{\ell,\ell}$, and, from the previous part of the proof, of $D_{\ell+1,\ell+1}$. This implies that $\{u_\lambda\}_\lambda$ is an approximate unit for $D_{k,\ell+1}$. By induction on $\ell \geq k$, $\{u_\lambda\}_\lambda$ is an approximate unit for $D_{k,\ell}$ for all $\ell \geq k$, and thus for $D_{k,\infty}$. □

Proposition 4.1.10. *Let (B, σ, D) be a Harnisch–Kirchberg system, where D is σ-unital. If D is stable, then so are D_σ and $D_\sigma \rtimes_\sigma \mathbb{Z}$.*

Proof. By combining Lemma 4.1.9 and Proposition 2.8.4 we obtain that $D_{1,\infty}$ is stable. Hence, D_σ can be expressed as the inductive limit of stable, σ-unital C*-algebras, thus implying with Proposition 2.8.4 that it is stable. Moreover, it follows from Proposition 2.8.4 that $D_\sigma \rtimes_\sigma \mathbb{Z}$ is stable as well. □

4.1.1 Ideal structure of Harnisch–Kirchberg systems

One of the key features of C*-algebras associated to Harnisch–Kirchberg systems is that they admit a complete characterisation of their the ideal structure. The main result of this section, Theorem 4.1.23, first appeared in [91]. Nevertheless, the author [155] presented a new, arguably shorter, proof (see Remark 4.1.16).

Notation 4.1.11. Let (B, σ, D) be a Harnisch–Kirchberg system. We denote by $\mathcal{I}(D_\sigma)^\sigma$ the sublattice of $\mathcal{I}(D_\sigma)$ of σ-invariant ideals, i.e., $I \in \mathcal{I}(D_\sigma)$ such that $\sigma(I) = I$.

Lemma 4.1.12. *Let (B, σ, D) be a Harnisch–Kirchberg system. Then D detects ideals in $\mathcal{I}(D_\sigma)^\sigma$.*

Proof. We use the characterisation given in Lemma 2.3.6. Suppose that ρ is a $*$-representation of D_σ whose restriction to D is faithful and $\ker(\rho) \in \mathcal{I}(D_\sigma)^\sigma$. Observe that we have the following identification,

$$\sigma(\ker(\rho)) = \{\sigma(x) \in D_\sigma \mid \rho(x) = 0\} = \{x \in D_\sigma \mid \rho(\sigma^{-1}(x)) = 0\} = \ker(\rho \circ \sigma^{-1}).$$

Then, since $\ker(\rho)$ is σ-invariant, we have that $\ker(\rho) = \sigma^{-k}(\ker(\rho)) = \ker(\rho \circ \sigma^k)$ for all $k \in \mathbb{Z}$, and consequently ρ is faithful on $D_{k,k} = \sigma^k(D)$ for all $k \in \mathbb{Z}$. Now, if $\ker(\rho) \cap D_{-k,k} \neq 0$, then $\ker(\rho) \cap D_{-k,k} \cap D_{k,k} \neq 0$ because $D_{k,k}$ is an essential ideal of $D_{-k,k}$ by Lemma 4.1.6, which contradicts faithfulness of $\rho|_{D_{k,k}}$. Hence, it must be that $\ker(\rho) \cap D_{-k,k} = 0$, and thus ρ is faithful on D_σ. □

Remark 4.1.13. One should note that if $\sigma \in \mathrm{Aut}(A)$, the associated action $\mathbb{Z} \curvearrowright A$ is aperiodic whenever for all $n > 0$, the automorphism σ^n satisfies Kishimoto's condition (see Definition 2.4.17). In fact, if σ^n with $n > 0$ satisfies Kishimoto's condition, then so does σ^{-n}.

Lemma 4.1.14. *Let (B, σ, D) be a Harnisch–Kirchberg system. Then the action $\mathbb{Z} \curvearrowright D_\sigma$ induced by σ is aperiodic. In particular, D_σ detects ideals in $D_\sigma \rtimes_\sigma \mathbb{Z}$.*

Proof. By Remark 4.1.13, it is sufficient to show that σ^n satisfies Kishimoto's condition for all $n > 0$. We first show that this follows from assuming that Kishimoto's condition holds for hereditary subalgebras of the form $\overline{cD_\sigma c}$ for $c \in D_{-i,i}$ and $i \in \mathbb{N}$. Fix $\varepsilon > 0$, $n > 0$, $x \in D_\sigma$, and a non-zero hereditary C*-subalgebra $H \subseteq D_\sigma$. Pick a positive element $b \in H$ with norm one. We may find some $\delta_0 > 0$, an index $i > 0$ and a positive element $b_i \in (D_{-i,i})_+$ such that $b =_{\delta_0} b_i$ and $(b_i - \delta_0)_+ \neq 0$. By Lemma 2.1.53(i), we have that $(b_i - \delta_0)_+ \precsim b$. Set $c_0 := \frac{(b_i-\delta_0)_+}{\|(b_i-\delta_0)_+\|}$, which is a norm-one element of $D_{-i,i}$, and note that one still has that $c_0 \precsim b$. Thus, for any $0 < \delta_1 < \frac{1}{4}$, we may find with Lemma 2.1.53(iii) some $\delta_2 > 0$ and some $r \in D_\sigma$ such that $f_{\delta_1}(c_0) = r^* f_{\delta_2}(b) r$. Observe that $c := f_{\delta_1}(c_0)$ is a norm-one element of $D_{-i,i}$, and hence the identity above also implies that $f_{\delta_2}(b) \neq 0$. Let

$$g : [0,1] \to [0,1] \quad g(t) := \begin{cases} 0 & \text{if } t \leq 2\delta_1 \\ \frac{t-2\delta_1}{2\delta_1} & \text{if } 2\delta_1 < t < 4\delta_1 \\ 1 & \text{if } 4\delta_1 \leq t \leq 1 \end{cases}$$

and note that the element $d := g(c_0)$ satisfies $dc = d = cd$ by functional calculus. Moreover, d is a positive norm-one element of $D_{-i,i}$ as well. Set $s := (f_{\delta_2}(b))^{1/2} r$, and note that

$$c = s^* s \quad \text{and} \quad ss^* \in H,$$

which also implies that s has norm one as $1 = \|c\| = \|s^*s\| = \|s\|^2$. Now, let $\psi : \overline{dD_\sigma d} \to H$ be the completely positive map given by $\psi(a) = sas^*$. The range in H is justified by the fact that for each $a \in \overline{dD_\sigma d}$

$$\psi(a) = sas^* = s(cac)s^* = ss^*(sas^*)ss^* \in H,$$

where we used that c acts like a unit on $\overline{dD_\sigma d}$. Note that ψ is multiplicative (and hence a $*$-homomorphism) because

$$\psi(a)\psi(b) = (sas^*)(sbs^*) = sacbs^* = sabs^* = \psi(ab)$$

for all $a, b \in \overline{cD_\sigma c}$. Moreover, it is isometric because for all $a \in \overline{cD_\sigma c}$

$$\|a\| = \|cac\| = \|s^*sas^*s\| \leq \|sas^*\| = \|\psi(a)\|.$$

Finally, set $x' = s^* x\sigma^n(s)$, and find a positive, norm-one element $a \in \overline{cD_\sigma c}$ such that $\|ax'\sigma^n(a)\| < \varepsilon$. We may then infer that

$$\|\psi(a)x\sigma^n(\psi(a))\| = \|sas^*x\sigma^n(\psi(sas^*))\| \leq \|as^*x\sigma^n(sa)\| = \|ax'\sigma^n(a)\| < \varepsilon,$$

which means that the norm-one, positive element $\psi(a) \in H$ witnesses Kishimoto's condition for H.

It is therefore sufficient to prove that σ^n satisfies Kishimoto's condition under the additional assumption that $H = \overline{cD_\sigma c}$, where $c \in D_{-i,i}$ for some $i \in \mathbb{N}$. So fix such an H and $\varepsilon > 0$, $n > 0$, $x \in D_\sigma$. We may moreover assume that x is an element of $D_{-k,k}$ for some $k \in \mathbb{N}$, and then by an approximation argument get the conclusion for all elements of D_σ. Note that $\overline{cD_{i,i}c} \neq \{0\}$ as $D_{i,i}$ is an essential ideal in $D_{-i,i}$ by Lemma 4.1.6. Hence, we may find a non-zero, positive element $h \in \overline{cD_{i,i}c} \subseteq D_{i,i}$. Pick an approximate unit of positive contractions $\{e_\lambda\}_\lambda \subseteq D_{i,i}$ for $D_{i,i}$, and let $E_\lambda \in D_{i,i+1}$ be given by $E_\lambda = e_\lambda - \sigma(e_\lambda)$ for all λ. Note that by Lemma 4.1.9 $\{e_\lambda\}_\lambda$ and $\{\sigma(e_\lambda)\}_\lambda$ are approximate units for $D_{i,\infty}$ and $D_{i+1,\infty}$, respectively, and hence it follows that for any element $y \in D_{i+1,\infty}$, the net $\{E_\lambda y\}_\lambda$ converges to zero. On the other hand, the net $\{E_\lambda h\}_\lambda \subseteq D_{i,i+1}$ cannot converge to zero for if $h = \lim_\lambda e_\lambda h = \lim_\lambda \sigma(e_\lambda)h \in D_{i,i} \cap D_{i+1,i+1}$, then $h = 0$, a contradiction. Then, after passing to a subnet, we may assume that there exists $\delta > 0$ such that $\|E_\lambda h\| \geq \delta$ for all λ. Now, $hx\sigma^n(h)$ belongs to $D_{i+n,\infty}$ by Lemmas 4.1.2 and 4.1.5. Hence, we may find an index λ_0 for which $\|E_{\lambda_0} hx\sigma^n(h)\| < \frac{\varepsilon\delta^2}{2}$. Finally, we claim that $a \in H_+$ witnesses that Kishimoto's condition is satisfied, where

$$a = \frac{1}{\|E_{\lambda_0}h\|^2}(E_{\lambda_0}h)^* E_{\lambda_0}h = \frac{1}{\|E_{\lambda_0}h\|^2} hE_{\lambda_0}^2 h.$$

Let us show the claim:

$$\begin{aligned}
\|ax\sigma^n(a)\| &= \frac{1}{\|E_{\lambda_0}h\|^4}\|hE_{\lambda_0}^2 hx\sigma^n(hE_{\lambda_0}^2 h)\| \\
&\leq \frac{2}{\|E_{\lambda_0}h\|^2}\|E_{\lambda_0}hx\sigma^n(h)\| \\
&< \frac{\varepsilon\delta^2}{\|E_{\lambda_0}h\|^2} \leq \varepsilon.
\end{aligned}$$

The "in particular" part follows from Theorem 2.4.19. ☐

Corollary 4.1.15 (cf. [91, Proposition 4.5]). *Let (B, σ, D) be a Harnisch–Kirchberg system. Then D detects ideals in $D_\sigma \rtimes_\sigma \mathbb{Z}$.*

Proof. It follows from Lemma 4.1.14 that D_σ detects ideals in $D_\sigma \rtimes_\sigma \mathbb{Z}$. Hence, for any non-zero ideal I in $D_\sigma \rtimes_\sigma \mathbb{Z}$, the intersection $I \cap D_\sigma$ is non-zero. Moreover, $I \cap D_\sigma$ is a σ-invariant ideal of D_σ because $\sigma(I \cap D_\sigma) = \sigma(I) \cap D_\sigma = I \cap D_\sigma$, where the last identity follows from the fact that $\sigma(I) = uIu^* = I$, where $u \in \mathcal{M}(D_\sigma \rtimes_\sigma \mathbb{Z})$ is the canonical unitary implementing σ. Consequently, we have that $I \cap D$ is non-zero as well by Lemma 4.1.12, and thus D detects ideals in $D_\sigma \rtimes_\sigma \mathbb{Z}$. □

Remark 4.1.16. We would like to point out that, in Harnisch and Kirchberg's proof of Corollary 4.1.15, they first work in the universal enveloping von Neumann algebra of D_σ, and then develop the necessary tools to lift these results to D_σ itself. On the contrary, we work directly in D_σ, which is possible by our Lemma 4.1.14 showing that σ is an aperiodic action, and Lemma 2.4.19. Our new approach has the key advantage of being more straightforward and, in many ways, easier to carry out.

Definition 4.1.17 (cf. [91, Definition 4.14])**.** Let (B, σ, D) be a Harnisch–Kirchberg system. A closed two-sided ideal I of D is said to be

- *semi-invariant* if $I\sigma(D) \subseteq \sigma(I)$,
- *cancellative* if for all $a, b \in D$ the condition $(a + \sigma(b))\sigma(D) \subseteq \sigma(I)$ implies that $a \in I$.

The set of all semi-invariant, cancellative ideals of D is denoted by $\mathcal{I}(D)^{\mathrm{sic}}$.

Note that the condition $I\sigma(D) \subseteq \sigma(I)$ is equivalent to saying that $I\sigma^n(D) \subseteq \sigma^n(I)$ for all $n \in \mathbb{N}$. This follows from the fact that $D\sigma(D) = \sigma(D)$.

Remark 4.1.18. Every σ-invariant ideal I of D_σ, induces a semi-invariant ideal $I \cap D$ of D because $(I \cap D)\sigma(D) = (\sigma(I) \cap D)\sigma(D) \subseteq \sigma(I \cap D)$.

Note that when $I \in \mathcal{I}(D)$ is a semi-invariant ideal, we have that $I_{k,\ell}$, $I_{-\infty,\ell}$, $I_{k,\infty}$, and I_σ are C*-algebras for all $k, \ell \in \mathbb{Z}$ with $k \leq \ell$ by Lemma 4.1.3. Moreover, they are ideals of $D_{k,\ell}$, $D_{-\infty,\ell}$, $D_{k,\infty}$ and D_σ, respectively. To see this, we show that $\sigma^i(I)\sigma^j(D) \subseteq \sigma^{\max(i,j)}(I)$ for all $i, j \in \mathbb{Z}$. Pick $i, j \in \mathbb{Z}$ such that $i \geq j$,

$$\begin{aligned}\sigma^i(I)\sigma^j(D) &= \sigma^j(\sigma^{i-j}(I)D) = \sigma^j(\sigma^{i-j}(ID)D)\\ &\subseteq \sigma^j(\sigma^{i-j}(ID)) = \sigma^i(I).\end{aligned}$$

While if $i < j$ we have

$$\sigma^i(I)\sigma^j(D) = \sigma^i(I\sigma^{j-i}(D)) \subseteq \sigma^i(\sigma^{j-i}(I)) = \sigma^j(I),$$

where we used that $I\sigma^n(D) \subseteq \sigma^n(I)$ for all $n \in \mathbb{N}$. Together with the fact that $\sigma^i(D)\sigma^j(I) = (\sigma^j(I)\sigma^i(D))^*$, we have shown that $I_{k,\ell}$ is a closed two-sided ideal in $D_{k,\ell}$ for all $k \leq \ell$, and by the inductive limit decomposition the same holds for $I_{-\infty,\ell}$, $I_{k,\infty}$ and I_σ. Note, moreover, that I_σ is clearly σ-invariant.

Lemma 4.1.19 (cf. [91, Lemma 4.15(v)])**.** *Let (B, σ, D) be a Harnisch–Kirchberg system. Every semi-invariant ideal $J \in \mathcal{I}(D)$ satisfies $J_\sigma \cap D = J$.*

Proof. It is evident that $J \subseteq J_\sigma \cap D$. We want to show the opposite inclusion.

We start by showing that $J_{-k,k} \cap D = J$ for all $k \geq 1$. Fix an element $x \in J_{-k,k} \cap D$, and find $y \in J_{-k,0}$, $z \in J_{1,k}$ such that $x = y + z$. In particular, it follows that $y = x - z$ belongs to $D_{1,k}$, but $J_{-k,0} \cap D_{1,k} = \{0\}$ by Lemma 4.1.8, and hence $y = 0$ and $x = z \in J_{1,k} \cap D$. Now, using the same trick, we find $v \in J$ and $w \in J_{2,k}$ such that $x = v + w$, which gives that $w = (x - v) \in D$, but $D \cap J_{2,k} = \{0\}$ by Lemma 4.1.8 and therefore $w = 0$ and $x = v \in J$.

Consider now the quotient C*-algebra D_σ / J_σ, and note that for every $x \in J_\sigma \cap D$, one has $\|x + J_\sigma\| = 0$. Hence,

$$\|x + J\| = \lim_{k\to\infty} \|x + J_{-k,k} \cap D\| \leq \lim_{k\to\infty} \|x + J_{-k,k}\| = \|x + J_\sigma\| = 0,$$

which shows that $x \in J$, and finishes our proof. □

Notation 4.1.20. Let (B, σ, D) be a Harnisch–Kirchberg system, and J a semi-invariant ideal of D. Then we denote by $\pi_J : D_\sigma \to D_\sigma / J_\sigma$ the quotient map, and use the notation $B_J = D_\sigma / J_\sigma$, $D_J = \pi_J(D)$. Moreover, we denote by σ_J the automorphism of B_J given by $\sigma_J \circ \pi_J = \pi_J \circ \sigma$.

Lemma 4.1.21 (cf. [91, Lemma 4.15(iii)+(iv)])**.** *Let (B, σ, D) be a Harnisch–Kirchberg system, and $J \in \mathcal{I}(D)^{\mathrm{sic}}$. Then the triple (B_J, σ_J, D_J) is a Harnisch–Kirchberg system.*

Proof. Note that $D_J\sigma_J(D_J) = \pi_J(D\sigma(D)) = D_J$. Moreover, $D_J \cap \sigma_J(D_J) = \pi_J(D \cap \sigma(D)) = \{0\}$ because $D \cap \sigma(D) = \{0\}$. We want to that show $D_J \cap \sigma_J(D_J)^\perp = \{0\}$. Hence, pick $a, b \in D$, and suppose that

$$(\pi_J(a) + \sigma_J(\pi_J(b)))\sigma_J(D_J) = \{0\}.$$

Since we have that $\pi_J(J_\sigma) = \{0\}$, it follows that

$$(a + \sigma(b))\sigma(D) \subseteq J_\sigma \cap \sigma(D) = \sigma(J),$$

where we used Lemma 4.1.19 for the last identification. By assumption J is cancellative, and therefore $a \in J$, which means that $\pi_J(a) = 0$. As a result, we have that $\pi_J(b)\pi_J(D) = \{0\}$, which means that $\pi_J(b) = 0$ as well, and therefore (B_J, σ_J, D_J) is a Harnisch–Kirchberg system. □

Proposition 4.1.22 (cf. [91, Proposition 4.16])**.** *Let (B, σ, D) be a Harnisch–Kirchberg system. If I is an ideal of $D_\sigma \rtimes_\sigma \mathbb{Z}$ such that $I \cap D$ is cancellative, then I is the natural image of $(I \cap D)_\sigma \rtimes_\sigma \mathbb{Z}$ in $D_\sigma \rtimes_\sigma \mathbb{Z}$.*

Proof. We know that $I \cap D_\sigma$ is a σ-invariant ideal of D_σ, and therefore $I \cap D$ is a semi-invariant ideal of D by Remark 4.1.18. Let us denote $I \cap D$ by J for the rest of the proof. By Lemma 4.1.21, the triple (B_J, σ_J, D_J) associated to the ideal J as in Notation 4.1.20 is a Harnisch–Kirchberg system.

The kernel of the canonical surjection

$$\pi_J \rtimes \mathbb{Z} : D_\sigma \rtimes_\sigma \mathbb{Z} \to (D_J)_{\sigma_J} \rtimes_{\sigma_J} \mathbb{Z}, \quad \pi_J \rtimes \mathbb{Z}|_{D_\sigma} = \pi_J$$

is the natural image of $J_\sigma \rtimes_\sigma \mathbb{Z}$ in $D_\sigma \rtimes_\sigma \mathbb{Z}$, and it is contained in I because $J_\sigma = (I \cap D)_\sigma \subseteq I$. We want to show that the other inclusion holds as well, i.e., $I \subseteq J_\sigma \rtimes_\sigma \mathbb{Z}$. Consider $I_J = (\pi_J \rtimes \mathbb{Z})(I)$, which is an ideal of $(D_J)_{\sigma_J} \rtimes_{\sigma_J} \mathbb{Z}$ that satisfies $I_J \cap D_J = \pi_J(J) = \{0\}$. By Corollary 4.1.15 it follows that $I_J = \{0\}$, and thus $I \subseteq J_\sigma \rtimes_\sigma \mathbb{Z}$. ☐

The theorem that follows establishes that, under suitable assumptions, the set of semi-invariant, cancellative ideals of a Harnisch–Kirchberg system corresponds to the ideal lattice of a crossed product C*-algebra arising from the same Harnisch–Kirchberg system.

Theorem 4.1.23 (cf. [91, Corollary 4.18])**.** *Let (B, σ, D) be a Harnisch–Kirchberg system, and assume that every semi-invariant ideal of D has cancellation. Then the order morphisms $\Delta : \mathcal{I}(D)^{\mathrm{sic}} \to D_\sigma \rtimes_\sigma \mathbb{Z}$ and $\nabla : D_\sigma \rtimes_\sigma \mathbb{Z} \to \mathcal{I}(D)^{\mathrm{sic}}$ given by*

$$\Delta(J) = J_\sigma \rtimes_\sigma \mathbb{Z}, \quad \nabla(I) = I \cap D$$

for all $J \in \mathcal{I}(D)^{\mathrm{sic}}$ and $I \in D_\sigma \rtimes_\sigma \mathbb{Z}$, are mutually inverse order isomorphisms.

Proof. Fix an ideal $J \in \mathcal{I}^{\mathrm{sic}}(D)$. By Lemma 4.1.19, we have that $J_\sigma \cap D = J$. As a consequence,

$$\nabla\big(\Delta(J)\big) = (J_\sigma \rtimes_\sigma \mathbb{Z}) \cap D = J.$$

Now pick a non-zero ideal $I \subseteq D_\sigma \rtimes_\sigma \mathbb{Z}$. By Corollary 4.1.15, D detects ideals in $D_\sigma \rtimes_\sigma \mathbb{Z}$, and hence $I \cap D$ is a non-zero ideal of D. Moreover, $I \cap D$ is semi-invariant by Remark 4.1.18, and therefore also cancellative by assumption. From Proposition 4.1.22, we have that I coincides with the natural image of $(I \cap D)_\sigma \rtimes_\sigma \mathbb{Z}$ in $D_\sigma \rtimes_\sigma \mathbb{Z}$, thus showing that

$$\Delta\big(\nabla(I)\big) = (I \cap D)_\sigma \rtimes_\sigma \mathbb{Z} = I.$$

☐

4.1.2 Harnisch–Kirchberg system associated to a $*$-homomorphism

The aim of this final section is to build a Harnisch–Kirchberg system from a C*-algebra A and use the order isomorphism on ideals from the previous section to realise the ideal lattice of A as that arising from the system. It turns out that it is possible to construct such a Harnisch–Kirchberg system whenever one has an injective, non-degenerate $*$-homomorphism $\varphi : A \to \mathcal{M}(A)$ such that $\varphi(A) \cap A = \{0\}$. This construction first appeared in [91].

Notation 4.1.24 (cf. [91, Section 4.3])**.** Let A be a C*-algebra, and $\varphi : A \to \mathcal{M}(A)$ an injective, non-degenerate $*$-homomorphism. Then φ extends uniquely to an injective unital $*$-homomorphism $\mathcal{M}(A) \to \mathcal{M}(A)$ that is strictly continuous on the unit ball by Proposition 2.1.6. Slightly abusing notation, we denote this extension by φ. Define $\varphi_\infty : \mathcal{M}(A) \to \mathcal{M}(A)_\infty$ to be the $*$-homomorphism given by

$$\varphi_\infty(a) = \big(\varphi^n(a)\big)_{n\in\mathbb{N}} + c_0\left(\mathbb{N}, \mathcal{M}(A)\right)$$

for all $a \in \mathcal{M}(A)$, where φ^n denotes the n-fold application of φ. Moreover, we will denote by σ the forward shift automorphism on $\mathcal{M}(A)_\infty$, i.e.,

$$\sigma\left((x_n)_{n\in\mathbb{N}} + c_0(\mathbb{N}, \mathcal{M}(A))\right) = (x_{n-1})_{n\in\mathbb{N}} + c_0(\mathbb{N}, \mathcal{M}(A))$$

for all $(x_n)_{n\in\mathbb{N}} \in \ell^\infty\left(\mathbb{N}, \mathcal{M}(A)\right)$, with the convention that $x_n = 0$ for $n < 0$.

Let us also recall from Section 2.1.3 that $\overline{(\varphi(A)A)} = \varphi(A)A$. Since φ is non-degenerate, this furthermore implies that $\varphi(A)A = A$, and will be used later.

Lemma 4.1.25 (cf. [91, Lemma 4.23(i)])**.** *Let A be a* C**-algebra, and $\varphi : A \to \mathcal{M}(A)$ an injective, non-degenerate $*$-homomorphism such that $\varphi(A) \cap A = \{0\}$. Then the triple $(\mathcal{M}(A)_\infty, \sigma, \varphi_\infty(A))$ given as in Notation 4.1.24 is a Harnisch–Kirchberg system.*

Proof. It follows by the definition of σ that $\varphi_\infty \circ \varphi = \sigma^{-1} \circ \varphi_\infty$. We will use this fact multiple times below without further mentioning it. Since $\varphi(A)A = A$, we have that

$$\varphi_\infty(A) = \varphi_\infty(\varphi(A)A) = \sigma^{-1}(\varphi_\infty(A))\varphi_\infty(A), \tag{4.1}$$

which is equivalent to the first condition in Definition 4.1.7.

The second condition is $\sigma(\varphi_\infty(A)) \cap \varphi_\infty(A) = \{0\}$. One may obtain this relation by applying σ to the following identity,

$$\varphi_\infty(A) \cap \sigma^{-1}(\varphi_\infty(A)) = \varphi_\infty(A \cap \varphi(A)) = \{0\}.$$

Now, note that since A is an essential ideal in $\mathcal{M}(A)$, then it is also an essential ideal of $A + \varphi(A)$. Then, $\varphi_\infty(A)$ is an essential ideal of $\varphi_\infty(A + \varphi(A)) = \varphi_\infty(A) + \sigma^{-1}(\varphi_\infty(A))$, which implies that $\sigma(\varphi_\infty(A))$ is an essential ideal of $\varphi_\infty(A) + \sigma(\varphi_\infty(A))$, and therefore that the triple given by

$$(\mathcal{M}(A)_\infty, \sigma, \varphi_\infty(A))$$

is a Harnisch–Kirchberg system. $\square$

Remark 4.1.26. Let A and $\varphi : A \to \mathcal{M}(A)$ be as in Lemma 4.1.25. It was observed in [91, Corollary 4.26(ii)] that the Cuntz–Pimsner algebra $\mathcal{O}_\mathcal{H}$ [162] associated to $\varphi : A \to \mathcal{M}(A) = \mathcal{B}_A(\mathcal{H})$, where $\mathcal{H}$ is the trivial right Hilbert C*-module over A, is the full hereditary C*-subalgebra of $(\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}$ generated by $\varphi_\infty(A)$. Therefore, if A is moreover σ-unital and stable, it follows from Brown's stable isomorphism theorem [22, Theorem 2.8] and Proposition 4.1.10 that $\mathcal{O}_\mathcal{H}$ and $(\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}$ are isomorphic. In our approach, we do not use this fact, and work directly in $(\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}$, which makes the overall construction conceptually clearer and more direct.

Notation 4.1.27. Let A be a C*-algebra, and $\varphi : A \to \mathcal{M}(A)$ a $*$-homomorphism. Denote by $\mathcal{I}(A)^\varphi \subseteq \mathcal{I}(A)$ the set consisting of all ideals $J \in \mathcal{I}(A)$ such that $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$, where $\mathcal{M}(A, J)$ was defined in Definition 2.3.3.

The following lemma provides necessary and sufficient conditions for an ideal of A to generate a semi-invariant or cancellative ideal (see Definition 4.1.17) in the Harnisch–Kirchberg system associated to a map $\varphi : A \to \mathcal{M}(A)$.

Lemma 4.1.28 (cf. [91, Lemma 4.23(ii)+(iii)])**.** *Let A be a C*-algebra, $\varphi : A \to \mathcal{M}(A)$ an injective, non-degenerate $*$-homomorphism such that $\varphi(A) \cap A = \{0\}$, and $(\mathcal{M}(A)_\infty, \sigma, \varphi(A)_\infty)$ the associated triple as in Notation 4.1.24. Then, for any (closed and two-sided) ideal $J \subseteq A$, the following statements hold true.*

(i) *$\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$ if and only if $\varphi(J)A \subseteq J$ if and only if $\varphi_\infty(J)$ is a semi-invariant ideal of $\varphi_\infty(A)$;*

(ii) *$\varphi(A) \cap (A + \mathcal{M}(A, J)) \subseteq \varphi(J)$ if and only if $\varphi_\infty(J)$ is a cancellative ideal of $\varphi_\infty(A)$.*

In particular, $\varphi_\infty : \mathcal{I}(A) \to \mathcal{I}(\varphi_\infty(A))$ restricts to a bijection between $\mathcal{I}(A)^\varphi$ and $\mathcal{I}(\varphi_\infty(A))^{\mathrm{sic}}$.

Proof. Recall that $(\mathcal{M}(A)_\infty, \sigma, \varphi(A)_\infty)$ is a Harnisch–Kirchberg system by Lemma 4.1.25. In order to prove (i), observe that the identity $\varphi_\infty \circ \varphi = \sigma^{-1} \circ \varphi_\infty$

implies that

$$\sigma^{-1}(\varphi_\infty(J))\varphi_\infty(A) = \varphi_\infty(\varphi(J)A).$$

Moreover, by definition we have that

$$\varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J)) = \{a \in A \mid \varphi(a)A \subseteq J\}.$$

Hence, we have that $J \subseteq \varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J))$ if and only if $\varphi(J)A \subseteq J$ if and only if

$$\varphi_\infty(J)\sigma(\varphi_\infty(A)) = \sigma(\varphi_\infty(\varphi(J)A)) \subseteq \sigma(\varphi_\infty(J)),$$

which concludes part (i).

In order to prove (ii), note that for any $a \in A$, one has that $\varphi(a) \in \mathcal{M}(A, J) + A$ precisely when there exists $b \in A$ such that $(\varphi(a) + b)A \subseteq J$. By applying $\sigma \circ \varphi_\infty$ to left and right hand sides, this condition becomes equivalent to

$$(\varphi_\infty(a) + \sigma(\varphi_\infty(b)))\sigma(\varphi_\infty(A)) \subseteq \sigma(\varphi_\infty(J)).$$

It follows that $\varphi_\infty(J)$ is cancellative if and only if $J \supseteq \varphi^{-1}(\varphi(A) \cap (A + \mathcal{M}(A, J)))$.

In particular, $\varphi_\infty(J)$ is semi-invariant and cancellative precisely when

$$\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J) \subseteq \varphi(A) \cap (A + \mathcal{M}(A, J)) \subseteq \varphi(J),$$

or, equivalently, $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$. □

We are now able to establish, under suitable assumptions on a map $\varphi : A \to \mathcal{M}(A)$, that $\mathcal{I}^\varphi(A)$ is naturally isomorphic to the lattice of ideals of $D_\sigma \rtimes_\sigma \mathbb{Z}$, where D_σ comes from the Harnisch–Kirchberg system associated to φ.

Theorem 4.1.29 (cf. [91, Corollary 4.18])**.** *Let A be a* C*-*algebra, and $\varphi : A \to \mathcal{M}(A)$ an injective, non-degenerate $*$-homomorphism such that $\varphi(A) \cap A = \{0\}$. If every ideal $J \in \mathcal{I}(A)$ such that $\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$ automatically satisfies $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$, then the order preserving map given by*

$$\Psi : \mathcal{I}(A)^\varphi \to \mathcal{I}((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}), \quad J \mapsto (\varphi_\infty(J))_\sigma \rtimes_\sigma \mathbb{Z}$$

is an order isomorphism with inverse

$$\Psi^{-1} : \mathcal{I}((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}) \to \mathcal{I}(A)^\varphi, \quad I \mapsto \varphi_\infty^{-1}(I \cap \varphi_\infty(A))$$

Proof. To simplify notation, let us denote $\varphi_\infty(A)$ by D. From Theorem 4.1.23 we get an order isomorphism $\Delta : \mathcal{I}(D)^{\mathrm{sic}} \to \mathcal{I}(D_\sigma \rtimes_\sigma \mathbb{Z})$ given by $\Delta(J) = J_\sigma \rtimes_\sigma \mathbb{Z}$ and inverse $\Delta^{-1} : \mathcal{I}(D_\sigma \rtimes_\sigma \mathbb{Z}) \to \mathcal{I}(D)^{\mathrm{sic}}$, where $\Delta^{-1}(I) = I \cap D$. Moreover, by Lemma 4.1.28, we have that $\Psi = \Delta \circ \varphi_\infty$ and its inverse have the desired properties. $\square$

In the last part of this section, we establish that for a separable, stable C*-algebra B and a locally compact, second-countable group G, one can always find a $*$-homomorphism from $A = \mathcal{C}_0(G) \otimes B$ into its multiplier algebra giving rise to a Harnisch–Kirchberg system with certain properties that will be used in the next section. Let us first introduce a few preliminaries.

Remark 4.1.30. Let B be a stable C*-algebra, and fix a sequence of isometries $s_n \in \mathcal{M}(B)$ such that $\sum_{n\in\mathbb{N}} s_n s_n^* = \mathbf{1}$ strictly. Let $\kappa : B \otimes \mathcal{K} \to B$ the $*$-isomorphism that sends $b \otimes e_{i,j}$ to $s_i b s_j^*$ for all $b \in B$, and a set of matrix units $e_{i,j}$ generating $\mathcal{K}$. One may extend this to an isomorphism $\kappa : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{M}(B)$. In the context of this thesis, we use the notation $x^\infty = \kappa(x \otimes \mathbf{1})$, and say that x^∞ is the infinite repeat of x. Equivalently, one can define $x^\infty = \sum_{n\in\mathbb{N}} s_n x s_n^*$ with convergence in the strict topology. If $A \subseteq \mathcal{M}(B)$ is a C*-subalgebra, we denote by A^∞ the set of infinite repeats of elements of A. Note that $I^\infty \subseteq \mathcal{M}(B, I)$. Observe that the injective, non-degenerate $*$-homomorphism $\zeta : \mathcal{K} \xhookrightarrow{\mathbf{1}\otimes \mathrm{id}} \mathcal{M}(B \otimes \mathcal{K}) \xrightarrow{\kappa} \mathcal{M}(B)$ is given on matrix units by $\zeta(e_{i,j}) = s_i s_j^*$, and its image lies in $\mathcal{M}(B) \cap (\mathcal{M}(B)^\infty)'$. It follows that ζ extends uniquely to an injective unital $*$-homomorphism $\mathcal{M}(\mathcal{K}) \to \mathcal{M}(B) \cap (\mathcal{M}(B)^\infty)'$ that is strictly continuous on the unit ball by Proposition 2.1.6.

The following result can be considered as a special version of [91, Corollary 2.18] (see also [120, Theorem 5.7]). We give a hands-on construction of a map with desired properties in relation to the ideal lattice of a C*-algebra. In the aforementioned results the analogous map is realised starting from a more general type of lattice, and the techniques needed are drastically different, and more abstract.

Theorem 4.1.31. *Let B be a separable, stable* C**-algebra, and denote the* C**-algebra $\mathcal{C}_0(G) \otimes B$ by A. There exists a non-degenerate, injective $*$-homomorphism $\varphi : A \to \mathcal{M}(A)$ with $\varphi(A) \cap A = \{0\}$, and such that the following are equivalent for any $J \in \mathcal{I}(A)$,*

(i) $\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$,

(ii) $J = \mathcal{C}_0(G) \otimes I$ *for some* $I \in \mathcal{I}(B)$,

(iii) $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$.

Proof. In this proof, we denote by $\mathcal{K}$ the compact operators on $\mathcal{H}_G^\infty = \ell^2(\mathbb{N})\hat{\otimes}L^2(G)$. Define a non-degenerate, injective $*$-homomorphism given by

$$\psi : \mathcal{C}_0(G) \overset{\pi}{\hookrightarrow} \mathcal{M}(\mathcal{K}) \overset{\zeta}{\to} \mathcal{M}(B) \cap (\mathcal{M}(B)^\infty)',$$

where ζ is an isomorphism as in Remark 4.1.30, and $\pi : \mathcal{C}_0(G) \to \mathcal{M}(\mathcal{K})$ is the $*$-representation that lets elements of $\mathcal{C}_0(G)$ act as multiplication operators on $L^2(G)$ and trivially on $\ell^2(\mathbb{N})$. One can therefore define the following non-degenerate $*$-homomorphism

$$\varphi : \mathcal{C}_0(G) \otimes B \to \mathcal{M}(\mathcal{C}_0(G) \otimes B), \quad f \otimes b \mapsto \mathbf{1} \otimes \psi(f) \cdot b^\infty.$$

To see that φ is faithful, it suffices to observe that the image of ψ under the isomorphism $\mathcal{M}(B \otimes \mathcal{K}) \cong \mathcal{M}(B)$ in Remark 4.1.30 lies in $\mathcal{M}(\mathcal{K}) \otimes \mathbf{1}_{\mathcal{M}(B)}$, while that of $\mathcal{M}(B)^\infty$ lies in $\mathbf{1}_{\mathcal{M}(\mathcal{K})} \otimes \mathcal{M}(B)$. Moreover, $\varphi(A) \cap A = \{0\}$ because $\varphi(A) \subseteq \mathbf{1} \otimes \mathcal{M}(B)$ and $(\mathbf{1} \otimes \mathcal{M}(B)) \cap A = \{0\}$.

Since (iii)$\Rightarrow$(i) is always true, we only need to prove that (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

We first show that for any ideal $I_1 \in \mathcal{I}(\mathcal{C}_0(G))$ and $I_2 \in \mathcal{I}(B)$, one has that $\overline{A\varphi(I_1 \otimes I_2)A} = \mathcal{C}_0(G) \otimes I_2$. This follows from the fact that $\overline{A\varphi(x_1 \otimes x_2)A} = \mathcal{C}_0(G) \otimes (\overline{B\psi(x_1)x_2^\infty B})$, and that $\overline{B\psi(x_1)x_2^\infty B} = \overline{Bx_2B}$ for all non-zero $x_1 \in I_1$ and $x_2 \in I_2$. The second observation holds true because the image of the ideal $\overline{B\psi(x_1)x_2^\infty B}$ under the isomorphism $\mathcal{M}(B \otimes \mathcal{K}) \cong \mathcal{M}(B)$ of Remark 4.1.30 is $\overline{Bx_2B} \otimes \mathcal{K}$.

We want to show that (i)$\Rightarrow$(ii). Pick an ideal $J \in \mathcal{I}(A)$ satisfying $\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$. For each group element $g \in G$, let I_g be the ideal of B given by $\mathcal{I}(\mathrm{ev}_g)(J)$. Note that $J \subseteq \mathcal{C}_0(G) \otimes \overline{\sum_{g\in G} I_g}$, and we want to prove that the opposite inclusion holds as well. Since B is separable and $\mathcal{C}_0(G)$ nuclear, by Lemma 2.3.2 for each $g \in G$ one may find a full element b of $I_g \in \mathcal{I}(B)$, and a non-zero function $f \in \mathcal{C}_0(G)$ such that $f \otimes b \in J$. One may then conclude that $\mathcal{C}_0(G) \otimes I_g = \overline{A\varphi(f \otimes b)A} \subseteq J$, where the last inclusion follows from the fact that $J \subseteq \varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J))$. As a consequence, one gets that $J = \mathcal{C}_0(G) \otimes I$, where $I = \overline{\sum_{g\in G} I_g}$.

In order to show that (ii)$\Rightarrow$(iii), pick an ideal of the form $J = \mathcal{C}_0(G) \otimes I$ for some $I \in \mathcal{I}(B)$. First, we observe that

$$\varphi(J)A = \mathcal{C}_0(G) \otimes (\psi(\mathcal{C}_0(G)) \cdot I^\infty \cdot B) \subseteq J,$$

and hence $J \subseteq \varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J))$. Pick now an element $(f \otimes b) \in \varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J))$ for some $f \in \mathcal{C}_0(G)$ and $b \in B$. The condition $\varphi(f \otimes b)A \subseteq J$ implies that $\psi(f)b^\infty B \subseteq I$, and by the claim proved before, that $\overline{BbB} \subseteq I$. In particular, $b \in I$. Consequently, $(f \otimes b) \in J$, and therefore $J = \varphi^{-1}(\varphi(A) \cap \mathcal{M}(A, J))$. This implies that J satisfies $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$. □

4.2 The invariant is realised

Now that we have gathered all necessary results, we consider a continuous topological dynamical system on the primitive spectrum of a separable nuclear stable $\mathcal{O}_2$-stable C*-algebra A, namely $\gamma : G \curvearrowright \mathrm{Prim}(A)$, and lift it to a C*-action. By combining this theorem with classification results in Chapter 3, we obtain an explicit bijection between continuous actions on primitive ideal spaces, identified up to conjugacy, and amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable actions up to cocycle conjugacy (or conjugacy when G is compact).

Lemma 4.2.1. *Let A be a* C*-*algebra, and $\varphi_1, \varphi_2 : A \to \mathcal{M}(A)$ two $*$-homomorphisms. If $\mathcal{I}_A(\varphi_1)(J) = \mathcal{I}_A(\varphi_2)(J)$ for an ideal $J \in \mathcal{I}(A)$, then $\varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J)) = \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$.*

Proof. By assumption, $\overline{A\varphi_1(J)A} = \overline{A\varphi_2(J)A}$ for an ideal $J \in \mathcal{I}(A)$. Pick an element $a \in \varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J))$. We have that $\varphi_2(a)A \subseteq \overline{A\varphi_2(a)A} = \overline{A\varphi_1(a)A} \subseteq J$, which means that $a \in \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$. By switching the role of φ_1 and φ_2, one shows that if $a \in \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$ then it must be $a \in \varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J))$. In particular, $\varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J)) = \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$. $\square$

Lemma 4.2.2. *Let B be a* C*-*algebra, $A = \mathcal{C}_0(G) \otimes B$, and $\varphi_1, \varphi_2 : A \to \mathcal{M}(A)$ two $*$-homomorphisms such that $\varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J)) = \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$ for all $J \in \mathcal{I}(A)$. Consider the following properties for an ideal $J \in \mathcal{I}(A)$, and $i = 1, 2$*

(i) *$\varphi_i(J) \subseteq \varphi_i(A) \cap \mathcal{M}(A, J)$,*

(ii) *$J = \mathcal{C}_0(G) \otimes I$ for some $I \in \mathcal{I}(B)$,*

(iii) *$\varphi_i(J) = \varphi_i(A) \cap \mathcal{M}(A, J)$.*

If (i)⇔(ii)⇔(iii) *for $i = 1$, the same holds for $i = 2$.*

Proof. Assume that (i)⇔(ii)⇔(iii) holds true for $i = 1$, and suppose that (i) holds for $i = 2$, i.e., $\varphi_2(J) \subseteq \varphi_2(A) \cap \mathcal{M}(A, J)$ for some ideal $J \in \mathcal{I}(A)$. One can re-write this condition as $J \subseteq \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J)) = \varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J))$, and thus $\varphi_1(J) \subseteq \varphi_1(A) \cap \mathcal{M}(A, J)$. Since, by assumption, this is equivalent to $J = \mathcal{C}_0(G) \otimes I$ for some $I \in \mathcal{I}(B)$, and to $J = \varphi_1^{-1}(\varphi_1(A) \cap \mathcal{M}(A, J)) = \varphi_2^{-1}(\varphi_2(A) \cap \mathcal{M}(A, J))$, we have that (i)⇔(ii)⇔(iii) holds also when $i = 2$. $\square$

The following result is a generalisation of Theorem 4.1.31 to the dynamical setting. In particular, it provides sufficient conditions for the resulting map $\varphi : A \to \mathcal{M}(A)$ to be equivariant with respect to a given action.

Theorem 4.2.3 (cf. [120, Theorem 5.8])**.** *Let B be a separable, stable* C*-*algebra, and denote the* C*-*algebra $\mathcal{C}_0(G) \otimes B$ by A. Let $\alpha : G \curvearrowright A$ be an action that is conjugate to $\mathrm{Ad}(\lambda^\infty) \otimes \alpha : G \curvearrowright \mathcal{K}(\mathcal{H}_G^\infty) \otimes A$, and such that $\alpha^\sharp = \tau \times \gamma : G \curvearrowright G \times \mathrm{Prim}(B)$, where $\tau : G \curvearrowright G$ and $\gamma : G \curvearrowright \mathrm{Prim}(B)$ are continuous actions.*[2] *Then, there exists an α-to-α-equivariant, non-degenerate, injective $*$-homomorphism $\varphi : A \to \mathcal{M}(A)$ with $\varphi(A) \cap A = \{0\}$, and such that the following are equivalent for any ideal $J \in \mathcal{I}(A)$,*

(i) $\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$,

(ii) $J = \mathcal{C}_0(G) \otimes I$ *for some* $I \in \mathcal{I}(B)$,

(iii) $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$.

Proof. By Theorem 4.1.31, there exists an injective, non-degenerate $*$-homomorphism $\varphi_0 : A \to \mathcal{M}(A)$ such that $\varphi_0(A) \cap A = \{0\}$, and such that conditions (i), (ii), (iii) above are equivalent for any ideal $J \in \mathcal{I}(A)$. Consider the injective, non-degenerate $*$-homomorphism induced from φ_0 as in Notation 2.4.6, namely

$$\hat{\varphi}_0 : A \to \mathcal{C}_b^s(G, \mathcal{M}(A)) = \mathcal{M}(\mathcal{C}_0(G) \otimes A), \quad \hat{\varphi}_0(a)(g) = \alpha_g(\varphi_0(\alpha_{g^{-1}}(a)))$$

for all $g \in G$, and $a \in A$. Moreover, we have by definition that $\hat{\varphi}_0(A) \cap (\mathcal{C}_0(G) \otimes A) = \{0\}$. If we equip $\mathcal{C}_b^s(G, \mathcal{M}(A))$ with the G-action $\bar{\alpha}$ given by

$$\bar{\alpha}_g(f)(h) = \alpha_g(f(g^{-1}h))$$

for all $g, h \in G$, and $f \in \mathcal{C}_b^s(G, \mathcal{M}(A))$, it follows that $\hat{\varphi}_0$ is α-to-$\bar{\alpha}$-equivariant. By Remark 2.3.35, any ideal $J \in \mathrm{Prim}(A)$ can be written as $J = \mathcal{C}_0(G) \otimes I + \mathcal{C}_0(G \setminus h) \otimes B$ for some $I \in \mathrm{Prim}(B)$ and $h \in G$. By definition of φ_0 (see proof of Theorem 4.1.31) we have that $\overline{A\varphi_0(J)A} = \mathcal{C}_0(G) \otimes I$, and therefore that

$$\overline{A\varphi_0(\alpha_g^\sharp(J))A} = \mathcal{C}_0(G) \otimes \gamma_g(I) = \alpha_g^\sharp\left(\overline{A\varphi_0(J)A}\right)$$

for all $g \in G$. Pick now an ideal $J \in \mathcal{I}(A)$. We know that J can be written as $J = \bigcap_{\lambda \in \Lambda} J_\lambda$, where $J_\lambda \in \mathrm{Prim}(A)$ for all $\lambda \in \Lambda$. Then,

$$\alpha_g^\sharp(\overline{A\varphi_0(J)A}) = \bigcap_\lambda \alpha_g^\sharp\left(\overline{A\varphi_0(J_\lambda)A}\right) = \bigcap_\lambda \overline{A\varphi_0(\alpha_g^\sharp(J_\lambda))A} = \overline{A\varphi_0(\alpha_g^\sharp(J))A}$$

[2]By Remark 2.3.35, $\mathrm{Prim}(A) \cong G \times \mathrm{Prim}(B)$. The action $\tau \times \gamma$ is the continuous action given by acting componentwise on the product $G \times \mathrm{Prim}(B)$.

for all $g \in G$. As a consequence, the map $\mathcal{I}_A(\varphi_0) : \mathcal{I}(A) \to \mathcal{I}(A)$ is $\alpha^\sharp$-to-$\alpha^\sharp$ equivariant, and we may apply Lemma 3.4.4 to conclude that

$$\mathcal{I}_{\mathcal{C}_0(G)\otimes A}(\hat{\varphi}_0) = \mathcal{C}_0(G) \otimes \mathcal{I}_A(\varphi_0).$$

By assumption, there exists a conjugacy $\eta : (\mathcal{M}(\mathcal{K} \otimes A), \mathrm{Ad}(\lambda^\infty) \otimes \alpha) \to (\mathcal{M}(A), \alpha)$, where $\mathcal{K}$ is tacitly assumed to be $\mathcal{K}(\mathcal{H}_G^\infty)$. Consider the $*$-homomorphism $\varphi : A \to \mathcal{M}(A)$ given by the composition of the following maps,

$$A \xhookrightarrow{\hat{\varphi}_0} \mathcal{C}_b^s(G, \mathcal{M}(A)) = \mathcal{M}(\mathcal{C}_0(G, A)) \xhookrightarrow{\pi \otimes \mathrm{id}_A} \mathcal{M}(\mathcal{K} \otimes A) \xrightarrow{\eta} \mathcal{M}(A),$$

where $\pi : \mathcal{C}_0(G) \to \mathcal{M}(\mathcal{K})$ is the $*$-representation as multiplication operators, and $\pi \otimes \mathrm{id}_A$ is $\bar{\alpha}$-$(\mathrm{Ad}(\lambda^\infty) \otimes \alpha)$ equivariant. Note that φ is an injective, non-degenerate, α-to-α equivariant $*$-homomorphism because it is a composition of injective, non-degenerate, equivariant $*$-homomorphisms. Moreover, one can compute that $\varphi(A) \cap A = \{0\}$ holds true ultimately because $\varphi_0(A) \cap A = \{0\}$.

Now, we want to prove that (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) holds true for the map φ. By combining Lemma 4.2.1 and Lemma 4.2.2, it is sufficient to show that $\mathcal{I}_A(\varphi)(J) = \mathcal{I}_A(\varphi_0)(J)$ for every ideal $J \in \mathcal{I}(A)$. By Lemma 2.3.20, we have that

$$\begin{aligned}
\mathcal{I}_A(\varphi) &= \mathcal{I}_A(\eta) \circ \mathcal{I}_{\mathcal{K}\otimes A}(\pi \otimes \mathrm{id}_A) \circ \mathcal{I}_{\mathcal{C}_0(G)\otimes A}(\hat{\varphi}_0) \\
&= \mathcal{I}_A(\eta) \circ \mathcal{I}_{\mathcal{K}\otimes A}(\pi \otimes \mathrm{id}_A) \big((\mathcal{C}_0(G) \otimes \mathcal{I}_A(\varphi_0))\big) \\
&= \mathcal{I}_A(\eta) \big((\mathcal{K} \otimes \mathcal{I}_A(\varphi_0))\big) \\
&= \mathcal{I}_A(\varphi_0),
\end{aligned}$$

and thus the proof is concluded. □

Notation 4.2.4. Let A be a C*-algebra equipped with an action $\alpha : G \curvearrowright A$, and an automorphism σ such that $\alpha_g \circ \sigma = \sigma \circ \alpha_g$ for all $g \in G$. We denote by $\alpha \rtimes \mathbb{Z}$ the G-action on $A \rtimes_\sigma \mathbb{Z}$ that extends α and acts trivially on the canonical unitary representing $\mathbb{Z}$.

We are ready to prove the main result of this chapter.

Theorem 4.2.5. *Let B be a separable, nuclear, stable and $\mathcal{O}_2$-stable* C*-*algebra, and $\gamma : G \curvearrowright \mathrm{Prim}(B)$ a continuous action on the primitive spectrum of B. Then there exists an action $\beta : G \curvearrowright B$ such that $\beta^\sharp = \gamma$.*

Proof. Denote the C*-algebra $\mathcal{C}_0(G) \otimes B$ by A, and let $\mathcal{K}$ be concretely represented as $\mathcal{K}(\mathcal{H}_G^\infty)$. Since γ is continuous between $G \times \mathrm{Prim}(B)$ and $\mathrm{Prim}(B)$, the map given by

$$G \times \mathrm{Prim}(B) \to G \times \mathrm{Prim}(B), \quad (g, \mathfrak{p}) \mapsto (g, \gamma_g(\mathfrak{p}))$$

for all $g \in G$, and $\mathfrak{p} \in \mathrm{Prim}(B)$, is a homeomorphism. By composing this map with the canonical homeomorphism between $G \times \mathrm{Prim}(B)$ and $\mathrm{Prim}(A)$, we obtain a homeomorphism $\Theta : \mathrm{Prim}(A) \to \mathrm{Prim}(A)$. By Theorem 3.1.1 Θ is induced by an automorphism $\theta \in \mathrm{Aut}(A)$.

Since B is stable by assumption, there exist an isomorphism $\kappa : \mathcal{M}(B \otimes \mathcal{K}) \hookrightarrow \mathcal{M}(B)$ and an embedding $\zeta : \mathcal{M}(\mathcal{K}) \to \mathcal{M}(B)$ as in Remark 4.1.30 such that $\kappa(\mathbf{1} \otimes \lambda_g^\infty) = \zeta(\lambda_g^\infty)$. Denote by $\mathtt{rt}$ the G-action on $\mathcal{C}_0(G)$ induced by the right translation on G, i.e., $\mathtt{rt}_g(f)(h) = f(hg)$ for all $f \in \mathcal{C}_0(G)$, $g, h \in G$. It is now possible to define an action $\alpha : G \curvearrowright A$ as follows,

$$\alpha_g = \theta^{-1} \circ \Big(\mathtt{rt}_g \otimes \mathrm{Ad}(\zeta(\lambda_g^\infty))\Big) \circ \theta$$

for all $g \in G$. It follows that α is conjugate to $\mathrm{Ad}(\lambda^\infty) \otimes \alpha : G \curvearrowright \mathcal{K} \otimes A$ by Remark 2.5.22.

Note that the action induced by α on $\mathrm{Prim}(A)$ is given by

$$\alpha_g^\sharp(\mathcal{C}_0(G) \otimes I + \mathcal{C}_0(G \setminus h) \otimes B) = \mathcal{C}_0(G) \otimes \gamma_g(I) + \mathcal{C}_0(G \setminus hg^{-1}) \otimes B,$$

for all $g, h \in G$ and $I \in \mathrm{Prim}(B)$, where we used Remark 2.3.35. Therefore, $\alpha^\sharp$ is $\mathtt{rt}^\sharp \times \gamma$ on $\mathrm{Prim}(A)$. We may apply Theorem 4.2.3 to obtain an α-equivariant, injective, non-degenerate $*$-homomorphism $\varphi : A \to \mathcal{M}(A)$ such that $\varphi(A) \cap A = \{0\}$ and with the following property. Any ideal $J \in \mathcal{I}(A)$ such that $\varphi(J) \subseteq \varphi(A) \cap \mathcal{M}(A, J)$ already satisfies $\varphi(J) = \varphi(A) \cap \mathcal{M}(A, J)$ and is of the form $J = \mathcal{C}_0(G) \otimes I$ for some $I \in \mathcal{I}(B)$. In particular, we have a natural order isomorphism $\Phi : \mathcal{I}(B) \to \mathcal{I}(A)^\varphi$, given by $\Phi(I) = \mathcal{C}_0(G) \otimes I$ for all $I \in \mathcal{I}(B)$.

By Theorem 4.1.29, there exists a natural order isomorphism $\Psi : \mathcal{I}(A)^\varphi \to \mathcal{I}((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z})$ given by $\Psi(J) = (\varphi_\infty(J))_\sigma \rtimes_\sigma \mathbb{Z}$ for all $J \in \mathcal{I}(A)^\varphi$, with inverse $\Psi^{-1}(I) = \varphi_\infty^{-1}(I \cap \varphi_\infty(A))$ for all $I \in \mathcal{I}(\varphi_\infty(A)_\sigma \rtimes_\sigma \mathbb{Z})$. Hence, we have an order isomorphism $\Psi \circ \Phi : \mathcal{I}(B) \to \mathcal{I}((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z})$. Note that the crossed product $(\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}$ is separable from separability of A, nuclear because arises from an amenable group action on a nuclear C*-algebra, and stable by Proposition 4.1.10. Hence, one can apply Theorem 3.1.1 to obtain a $*$-isomorphism $\eta : B \to \big((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}\big) \otimes \mathcal{O}_2$ inducing $\mathcal{I}(\mathrm{id} \otimes \mathbf{1}_{\mathcal{O}_2}) \circ \Psi \circ \Phi$.[3]

[3]Here, $\mathcal{I}(\mathrm{id} \otimes \mathbf{1}_{\mathcal{O}_2})$ denotes the order isomorphism $\mathcal{I}(((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z})) \to \mathcal{I}(((\varphi_\infty(A))_\sigma \rtimes_\sigma \mathbb{Z}) \otimes \mathcal{O}_2)$ that sends an ideal I to $I \otimes \mathcal{O}_2$.

Denote by $\beta : G \curvearrowright B$ the action given by $\beta_g = \eta^{-1} \circ ((\alpha_\infty \rtimes \mathbb{Z}) \otimes \mathrm{id}_{\mathcal{O}_2})_g \circ \eta$ for all $g \in G$. Then β satisfies the following identity,

$$\begin{aligned}\beta_g^\sharp(I) &= \Phi^{-1} \circ \Psi^{-1} \circ (\alpha \rtimes \mathbb{Z})_g^\sharp \left(\big(\varphi_\infty(\mathcal{C}_0(G) \otimes I)\big)_\sigma \rtimes_\sigma \mathbb{Z} \right) \\ &= \Phi^{-1} \circ \Psi^{-1} \left(\big(\varphi_\infty(\mathcal{C}_0(G) \otimes \gamma_g(I))\big)_\sigma \rtimes_\sigma \mathbb{Z} \right) \\ &= \Phi^{-1}(\mathcal{C}_0(G) \otimes \gamma_g(I)) = \gamma_g(I)\end{aligned}$$

for all $g \in G$, and $I \in \mathcal{I}(B)$. It follows that $\beta^\sharp = \gamma$ as actions on $\mathrm{Prim}(B)$, which concludes the proof. $\square$

As an application of Theorem 4.2.5 and the classification theorem from Chapter 3, we derive the following one-to-one correspondences between C*-dynamics (up to cocycle conjugacy) and induced topological dynamics on ideals (up to cojugacy).

Corollary 4.2.6. *Let A be a separable, nuclear, stable and $\mathcal{O}_2$-stable* C*-*algebra. Then the mapping given by associating $\alpha^\sharp$ to an action $\alpha : G \curvearrowright A$ gives the following bijections,*

$$\frac{\left\{\begin{matrix}\text{amenable, isometrically shift-absorbing,}\\ \text{equivariantly } \mathcal{O}_2\text{-stable actions } G \curvearrowright A\end{matrix}\right\}}{\text{cocycle conjugacy}} \longrightarrow \frac{\left\{\begin{matrix}\text{continuous actions}\\ G \curvearrowright \mathrm{Prim}(A)\end{matrix}\right\}}{\text{conjugacy}}$$

and if G is compact,

$$\frac{\left\{\begin{matrix}\text{amenable, isometrically shift-absorbing,}\\ \text{equivariantly } \mathcal{O}_2\text{-stable actions } G \curvearrowright A\end{matrix}\right\}}{\text{conjugacy}} \longrightarrow \frac{\left\{\begin{matrix}\text{continuous actions}\\ G \curvearrowright \mathrm{Prim}(A)\end{matrix}\right\}}{\text{conjugacy}}.$$

Proof. Let A be separable, nuclear, stable and $\mathcal{O}_2$-stable. By Theorem 3.5.5, an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action $\alpha : G \curvearrowright A$ is classified, up to cocycle conjugacy, by the induced action $\alpha^\sharp : G \curvearrowright \mathrm{Prim}(A)$. Hence, by associating $\alpha^\sharp$ to α, one gets that the map

$$\frac{\left\{\begin{matrix}\text{amenable, isometrically shift-absorbing,}\\ \text{equivariantly } \mathcal{O}_2\text{-stable actions } G \curvearrowright A\end{matrix}\right\}}{\text{cocycle conjugacy}} \longrightarrow \frac{\left\{\begin{matrix}\text{continuous actions}\\ G \curvearrowright \mathrm{Prim}(A)\end{matrix}\right\}}{\text{conjugacy}}$$

is injective. To see that it is also surjective, note that by Theorem 4.2.5, given a continuous action $\sigma : G \curvearrowright \mathrm{Prim}(A)$, there exists an action $\alpha : G \curvearrowright A$ such that $\alpha^\sharp = \sigma$. By Remark 2.7.9, there exists an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action $\beta : G \curvearrowright \mathcal{O}_2 \otimes \mathcal{K}$. Then $\alpha' := \alpha \otimes \beta$ is an amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable action on $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong A$ such that $(\alpha')^\sharp$ is conjugate to σ. This concludes the first part of the proof.

If G is compact, the bijection above holds true when every instance of “cocycle conjugacy” is replaced by “conjugacy” by Corollary 3.5.7. $\square$

We present an example where our result can be applied.

Example 4.2.7. Consider the left-translation action $\mathtt{lt} : G \curvearrowright G$. Then $\mathtt{lt}$ naturally extends to a continuous action $\tilde{\mathtt{lt}}$ on the non-Hausdorff compactification $\tilde{G}$ of G presented in Example 2.3.31. By the discussion in that example, there exists a separable, nuclear C*-algebra A, which we may assume to absorb $\mathcal{O}_2 \otimes \mathcal{K}$, such that $\mathrm{Prim}(A) = \tilde{G}$. Hence, we may conclude from Theorem 4.2.5 that there exists an action $\alpha : G \curvearrowright A$ such that $\alpha^\sharp = \tilde{\mathtt{lt}}$. Moreover, by Corollary 4.2.6, up to cocycle conjugacy there exists a unique action α as above that is moreover amenable, isometrically shift-absorbing and equivariantly $\mathcal{O}_2$-stable.

Chapter 5

Corona algebras and strongly self-absorbing actions

This chapter is based on the preprint [134], co-authored with Xiuyuan Li and Gábor Szabó. Let us briefly outline the contribution of each co-author to achieve the main results reported in this chapter. In Xiuyuan Li's Master's thesis [133], a dissertation that was supervised by Gábor Szabó with the help of the author, one can find a special case of the main theorem of this chapter, which yields Theorem 5.2.4 for compact groups. One may argue that the main technical result carried out in the aforementioned thesis project is Proposition 5.2.3. In order to obtain Theorem 5.2.4 in full generality, one moreover needs a second key technical result, Theorem 5.1.8, which was developed by Gabór Szabó and the author after the aforementioned project was concluded.

Going in a somewhat different direction than the previous two chapters, here we achieve dynamical analogues of certain properties of corona algebras that have played a major role in the field [60, 62, 136, 161]. One of the main applications of the theory developed here allows one to characterise — under suitable assumptions — when a C^*-dynamical system absorbs a strongly self-absorbing action via induced dynamics on the corona.

Unitarily regular actions

We start by recalling the concept of unitary regularity. It may be viewed as a dynamical analogue of the assumption on a (unital) C^*-algebra A that the quotient group $\mathcal{U}(A)/\mathcal{U}_0(A)$ is abelian. This is automatic for K_1-injective C^*-

algebras, for instance, which includes all strongly self-absorbing C*-algebras or C*-algebras absorbing them by a result of Winter [208]. We stress that unitary regularity is a relatively mild condition; in fact it is currently unknown if strongly self-absorbing actions can fail this property. Note that by [188, Proposition 2.19], all equivariantly $\mathcal{Z}$-absorbing actions on unital C*-algebras are unitarily regular, which covers many examples.

Definition 5.0.1 (see [188, Definition 2.17])**.** Let $\gamma : G \curvearrowright D$ be an action on a unital C*-algebra. One says that γ is *unitarily regular* if for every compact subset $K \subseteq G$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever u_1, u_2 are unitaries in $\mathcal{U}(D)$ that satisfy $\max_{j=1,2} \max_{g\in K} \|\gamma_g(u_j) - u_j\| \leq \delta$, there exists a norm-continuous path of unitaries $(w_t)_{t\in[0,1]} \subseteq \mathcal{U}(D)$ such that

$$\max_{g\in K} \max_{t\in[0,1]} \|\gamma_g(w_t) - w_t\| \leq \varepsilon, \quad w_0 = \mathbf{1}, \quad w_1 = u_1 u_2 u_1^* u_2^*.$$

It turns out that for strongly self-absorbing actions, unitary regularity can be characterised by a tensorial absorption property using the following auxiliary object.

Definition 5.0.2. Given a unital C*-algebra D, let us define

$$D^{(2)} = \{f \in \mathcal{C}([0,1], D \otimes_{\max} D) \mid f(0) \in D \otimes \mathbf{1}, f(1) \in \mathbf{1} \otimes D\}.$$

Consider c.p.c. order zero maps $\eta_i : D \to D^{(2)}$ for $i = 0, 1$ given by

$$\eta_i(d)(t) = \begin{cases} (1-t)(d \otimes \mathbf{1}) & i = 0, \\ t(\mathbf{1} \otimes d) & i = 1 \end{cases}$$

for all $t \in [0,1]$ and $d \in D$.

Remark 5.0.3. Let D be a unital C*-algebra. As a consequence of [96, Lemma 5.2] and [210] (see also [94, Lemma 6.6]), $D^{(2)}$ and the c.p.c. order zero maps η_0 and η_1 satisfy the following universal property. For every unital C*-algebra B and c.p.c. order zero maps $\mu_i : D \to B$ for $i = 0, 1$ with commuting ranges and such that $\mu_0(\mathbf{1}) + \mu_1(\mathbf{1}) = \mathbf{1}$, there exists a unique unital $*$-homomorphism $\varphi : D^{(2)} \to B$ such that $\varphi \circ \eta_i = \mu_i$ for $i = 0, 1$.

Let $\gamma : G \curvearrowright D$ be an action, and define $\gamma^{(2)} : G \curvearrowright D^{(2)}$ to be the restriction of the action on $\mathcal{C}([0,1], D \otimes_{\max} D)$ that acts fiberwise by $\gamma \otimes \gamma$. It follows that there exists a well-defined continuous action $\gamma^{(2)}$ satisfies the identity $\gamma_g^{(2)} \circ \eta_i = \eta_i \circ \gamma_g$ for all $g \in G$ and $i = 0, 1$. Moreover, the G-C*-algebra $(D^{(2)}, \gamma^{(2)})$ has the following universal property. Whenever (B, β) is a unital G-C*-algebra, and $\mu_i : D \to B$ for $i = 0, 1$ are equivariant c.p.c. order zero maps with commuting ranges and such that $\mu_0(\mathbf{1}) + \mu_1(\mathbf{1}) = \mathbf{1}$, then there exists a unique unital equivariant $*$-homomorphism $\varphi : (D^{(2)}, \gamma^{(2)}) \to (B, \beta)$ such that $\varphi \circ \eta_i = \mu_i$ for $i = 0, 1$.

We are ready to recall from [188] the characterisation of strongly self-absorbing actions that are additionally unitarily regular, which relies on the C^*-dynamical system $(\mathcal{D}^{(2)}, \gamma^{(2)})$.

Theorem 5.0.4 (see [188, Theorem 5.9])**.** *Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action on a separable unital C^*-algebra. Then γ is unitarily regular if and only if $\gamma^{(2)}$ is γ-absorbing.*

The following proposition follows from well-known results of Szabó [185,187,192], and characterises absorption of a unitarily regular, strongly self-absorbing action via path algebras (see Definition 2.2.36).

Proposition 5.0.5. *Let $\alpha : G \curvearrowright A$ be an action on a separable, unital C^*-algebra, and $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing, unitarily regular action on a separable, unital C^*-algebra. The following are equivalent:*

(i) *There exists an equivariant unital embedding $(\mathcal{D}, \gamma) \hookrightarrow (A_{\mathfrak{c},\alpha} \cap A', \alpha_{\mathfrak{c}})$.*

(ii) *α is γ-absorbing.*

(iii) *There exists a cocycle conjugacy*

$$(\varphi, \mathbb{u}) : (A, \alpha) \to (A \otimes \mathcal{D}, \alpha \otimes \gamma)$$

and a continuous path of unitaries $w_t : [0, \infty) \to \mathcal{U}(\mathcal{D} \otimes \mathcal{D})$ with $w_0 = \mathbf{1}$ such that

$$0 = \lim_{t\to\infty} \|\varphi(x) - w_t(x \otimes \mathbf{1})w_t^*\| + \max_{g\in K} \|\mathbb{u}_g - w_t(\gamma_g \otimes \gamma_g)(w_t)^*\|$$

for all $x \in \mathcal{D}$ and every compact set $K \subseteq G$.

Proof. (i)⇒(ii): As $A_{\mathfrak{c}} \cap A'$ maps equivariantly into $A_\infty \cap A'$ by restricting functions to $\mathbb{N} \subset \mathbb{R}$, this is a consequence of [187, Theorem 4.7].

(ii)⇒(iii): We refer to [185, Theorem 3.2], which almost yields this conclusion, but not literally as that result was stated. We note, however, that the proof of said result hinges on verifying that the equivariant embedding

$$\mathbf{1}_{\mathcal{D}} \otimes \mathrm{id}_A : (A, \alpha) \to (\mathcal{D} \otimes A, \gamma \otimes \alpha)$$

satisfies the conditions stated in [185, Proposition 3.1]. However, we note that those conditions are identical to the ones stated in [192, Proposition 4.3], which has a strong enough conclusion to yield the desired statement.

(iii)⇒(i): Let the cocycle conjugacy $(\varphi, \mathbb{u})$ and the map w be given as in the statement. Define a point-norm continuous path $(\Phi_t)_{t\in[0,\infty)}$ of unital embeddings $\mathcal{D} \to A$ given by

$$\Phi_t := \varphi^{-1} \circ \mathrm{Ad}(w_t) \circ (\mathbf{1} \otimes \mathrm{id}_{\mathcal{D}}).$$

Firstly, note that we have for all $a \in A$ and $d \in \mathcal{D}$ that

$$\lim_{t\to\infty} \|[\Phi_t(d), a]\| = \lim_{t\to\infty} \|[\Phi_t(d), \varphi^{-1}(w_t(a\otimes\mathbf{1})w_t^*)]\| = \lim_{t\to\infty} \|[\mathbf{1}\otimes d, a\otimes\mathbf{1}]\| = 0.$$

Secondly, observe that

$$\begin{array}{rcl} \alpha_g(\Phi_t(d)) & = & \alpha_g \circ \varphi^{-1} \circ \mathrm{Ad}(w_t) \circ (\mathbf{1} \otimes d) \\ & = & (\varphi \circ \alpha_{g^{-1}})^{-1} \circ \mathrm{Ad}(w_t) \circ (\mathbf{1} \otimes d) \\ & = & (\mathrm{Ad}(\mathbb{u}_{g^{-1}}) \circ (\alpha\otimes\gamma)_{g^{-1}} \circ \varphi)^{-1} \circ \mathrm{Ad}(w_t) \circ (\mathbf{1} \otimes d) \\ & = & \varphi^{-1} \circ (\alpha\otimes\gamma)_g \circ \mathrm{Ad}(\mathbb{u}_{g^{-1}}^* w_t) \circ (\mathbf{1} \otimes d) \\ & = & \varphi^{-1} \circ \mathrm{Ad}(\mathbb{u}_g(\alpha\otimes\gamma)_g(w_t)) \circ (\mathbf{1} \otimes \gamma_g(d)) \end{array}$$

for any $g \in G$ and $d \in \mathcal{D}$. Using the cocycle condition above, one may infer that

$$\lim_{t\to\infty} \max_{g\in K} \|w_t - \mathbb{u}_g(\alpha\otimes\gamma)_g(w_t)\| = 0$$

for every compact subset $K \subseteq G$, and thus that the path $(\Phi_t)_{t\in[0,\infty)}$ is asymptotically equivariant, i.e.,

$$\lim_{t\to\infty} \max_{g\in K} \|\Phi(\gamma_g(d)) - \alpha_g(\Phi_t(d))\| = 0$$

for all $d \in \mathcal{D}$ and every compact subset $K \subseteq G$. It follows from the previous two observations that $(\Phi_t)_{t\in[0,\infty)}$ induces a well-defined equivariant unital embedding

$$\Phi : (\mathcal{D}, \gamma) \to (A_{\mathfrak{c},\alpha} \cap A', \alpha_{\mathfrak{c}}).$$

This finishes the proof. □

The following notion is a straightforward generalisation of [176, Definition 1.4]:

Definition 5.0.6. Let A be a C*-algebra with a continuous action $\alpha : G \curvearrowright A$. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing action. We say that α is *separably γ-stable* if for every separable C*-subalgebra $C \subseteq A$, there exists a separable α-invariant C*-algebra $B \subseteq A$ containing C and such that $(\alpha \restriction B)$ is γ-stable.

Note that any γ-stable action on a separable C*-algebra is trivially seen to be separably γ-stable. There are many more straightforward observations one could make about this notion in analogy to [62, Section 1], but we shall omit it as it is not so relevant for the rest of this chapter.

5.1 The dynamical folding property

Definition 5.1.1. Let A be a C*-algebra. One says that A has the *folding property* if for every separable C*-algebra B, C*-subalgebra $B_0 \subseteq B$, and $*$-homomorphism $\psi : B \to A_{\mathfrak{c}}$ with $\psi(B_0) \subseteq A$, there exists a $*$-homomorphism $\varphi : B \to A$ such that $\varphi \restriction_{B_0} = \psi \restriction_{B_0}$, and $\ker\varphi = \ker\psi$.

It is known from [161, Proposition 1.4] that the corona algebra $\mathcal{Q}(A)$ of any σ-unital C*-algebra A has the folding property. The goal of this section is to show that if A is equipped with a G-action α, then $\mathcal{Q}(A)$ equipped with the (algebraic) G-action induced by α satisfies a dynamical generalisation of the folding property introduced in Definition 5.1.2. We note that the last condition about the kernels in Definition 5.1.1 was not explicitly recorded in the context of corona algebras in [161], but it can be observed as an outcome of the proof. Since we are about to generalize this property further and hence reprove it anyway, we shall not explain this in greater detail here.

One can summarize the folding property by saying that for each commuting square diagram as below, there exists a map φ with same kernel as ψ that makes the upper-left triangle commute.

$$\begin{tikzcd}
B_0 \arrow[r, "\psi\restriction_{B_0}"] \arrow[d, hook] & A \arrow[d, hook, "\iota_{\mathfrak{c}}"] \\
B \arrow[r, "\psi"'] \arrow[ur, "\varphi"] & A_{\mathfrak{c}}
\end{tikzcd}$$

The following is a dynamical generalisation of the folding property.

Definition 5.1.2. Let A be a C*-algebra with an algebraic action $\alpha : G \curvearrowright A$.[1] One says that (A, α) (or α) has the *dynamical folding property* if the following statement holds. Let B be a separable C*-algebra and $B_0, D \subseteq B$ C*-subalgebras such that D is equipped with a continuous action $\delta : G \curvearrowright D$. If $\psi : B \to A_{\mathfrak{c}}$ is a $*$-homomorphism such that

- $\psi(B_0) \subseteq A$, and
- $\psi \restriction_D$ is δ-to-$\alpha_{\mathfrak{c}}$ equivariant with range in $(A_\alpha)_{\mathfrak{c}}$,

then there exists a $*$-homomorphism $\varphi : B \to A$ such that

[1] Recall that the C*-subalgebra of A containing every element on which the action α is continuous is denoted by A_α.

- $\varphi\restriction_{B_0} = \psi\restriction_{B_0}$,
- $\varphi\restriction_D$ is δ-to-α equivariant, and
- $\ker\varphi = \ker\psi$.

One can once again summarize the above definition by saying that φ fits in the following diagram with commuting squares and equivariant restriction $\psi\restriction_D$, in such a way that the upper-left triangle commutes, $\varphi\restriction_D$ is equivariant, and $\ker\varphi = \ker\psi$.

$$\begin{tikzcd}
B_0 \arrow[r, "\psi\restriction_{B_0}"] \arrow[d, hook] & A \arrow[d, hook, "\iota_{\mathfrak{c}}"] \\
B \arrow[ur, "\varphi"] \arrow[r, "\psi"] & A_{\mathfrak{c}} \\
D \arrow[u, hook] \arrow[r, "\psi\restriction_D"] & (A_\alpha)_{\mathfrak{c}} \arrow[u, hook]
\end{tikzcd}$$

We shall now collect a few technical prerequisites to prove the main result of this section. The following result of Michael [141, Proposition 7.2] is a generalisation of the Bartle–Graves selection theorem [10, Theorem 4].

Theorem 5.1.3 (see [141, Proposition 7.2])**.** *Let X and Y be complex Banach spaces, and $q : X \to Y$ a linear and continuous surjection. Then, for any $M > 1$, there exists a (not necessarily linear) continuous map $\rho : Y \to X$ such that*

(i) $q \circ \rho = \mathrm{id}_Y$,

(ii) $\|\rho(y)\| \leq M \cdot \inf\{\|x\| \mid x \in q^{-1}(y)\}$ *for all* $y \in Y$,

(iii) $\rho(\lambda y) = \lambda\rho(y)$ *for all* $\lambda \in \mathbb{C}$ *and* $y \in Y$.

In order to apply Michael's result in an efficient manner for our purposes, we restrict ourselves to the case when X and Y are C*-algebras. In this case, q necessarily behaves like a quotient map, and therefore (ii) amounts exactly to saying that $\|\rho(y)\| \leq M\|y\|$ for all $y \in Y$. We are now ready to present an application of Theorem 5.1.3 that will play a crucial role in the main result of this section. We remark that its proof is inspired by that of [161, Lemma 1.2].

Corollary 5.1.4. *Let $q : B \to Q$ be a surjective $*$-homomorphism of* C*-*algebras. Suppose, moreover, that $B_0 \subseteq B$ and $Q_0 \subseteq Q$ are* C*-*subalgebras that satisfy $q(B_0) = Q_0$. Then, for each $M > 1$, there exists a (not necessarily linear) continuous map $\rho : Q \to B$ such that*

(i) $q \circ \rho = \mathrm{id}_Q$,

(ii) $\rho(Q_0) \subseteq B_0$,

(iii) $\|\rho(y)\| \leq 5M \cdot \|y\|$ *for every* $y \in Q$,

(iv) $\rho(\lambda y) = \lambda\rho(y)$ *for every* $\lambda \in \mathbb{C}$ *and for every* $y \in Q$.

Proof. Fix $M > 1$ and $1 < m_1 < M$, and then set $m_2 := M/m_1 > 1$, which satisfies $m_1 \cdot m_2 = M$.

First of all, we apply Theorem 5.1.3 to $q \restriction_{B_0}: B_0 \to Q_0$, and obtain a continuous map $\rho_0 : Q_0 \to B_0$ such that

- $(q \restriction_{B_0}) \circ \rho_0 = \mathrm{id}_{Q_0}$,
- $\|\rho_0(y)\| \leq m_1 \cdot \|y\|$ for all $y \in Q_0$,
- $\rho_0(\lambda y) = \lambda\rho_0(y)$ for all $y \in Q_0$ and $\lambda \in \mathbb{C}$.

Below, we argue that one can extend ρ_0 to a right-inverse of q with the desired properties.

We proceed by defining, as in [161, Lemma 1.2], a retraction $q_0 : Q \to Q_0$, namely, a continuous function $q_0 : Q \to Q_0$ such that $q_0 \restriction_{Q_0} = \mathrm{id}_{Q_0}$. By applying Theorem 5.1.3 to the (Banach space) quotient map $p : Q \to Q/Q_0$, we may find a continuous map $\kappa_0 : Q/Q_0 \to Q$ such that

- $p \circ \kappa_0 = \mathrm{id}_{Q/Q_0}$,
- $\|\kappa_0(p(y))\| \leq m_2 \cdot \|y\|$ for all $y \in Q$,
- $\kappa_0(\lambda p(y)) = \lambda\kappa_0(p(y))$ for all $y \in Q$ and $\lambda \in \mathbb{C}$.

Then, the continuous map $q_0 : Q \to Q_0$ given by

$$q_0(y) = y - \kappa_0(p(y))$$

for all $y \in Q$, is the desired retraction. Note that $\|q_0(y)\| \leq 2m_2 \cdot \|y\|$ for all $y \in Q$.

We apply once more Theorem 5.1.3, this time to $q : B \to Q$, and obtain a continuous map $\kappa : Q \to B$ such that

- $q \circ \kappa = \mathrm{id}_Q$,
- $\|\kappa(y)\| \leq m_1 \cdot \|y\|$ for all $y \in Q$,

- $\kappa(\lambda y) = \lambda\kappa(y)$ for all $y \in Q$ and $\lambda \in \mathbb{C}$.

Now, the continuous function $\rho : Q \to B$ defined by

$$\rho(y) = \kappa(y - q_0(y)) + \rho_0(q_0(y))$$

for all $y \in Q$, is the desired right-inverse of q. We check that all conditions are satisfied. First of all, note that

$$q(\rho(y)) = q(\kappa(y - q_0(y))) + q(\rho_0(q_0(y))) = y - q_0(y) + q_0(y) = y$$

for all $y \in Q$, which establishes Condition (i). Since we know that $q_0 : Q \to Q_0$ is a restraction, then $\rho \restriction_{Q_0} = \rho_0$, and Condition (ii) follows immediately. In order to check that Condition (iii) holds, note that

$$\begin{aligned}
\|\rho(y)\| &\leq \|\kappa(y - q_0(y))\| + \|\rho_0(q_0(y))\| \\
&\leq m_1 \left(\|y - q_0(y)\| + \|q_0(y)\|\right) \\
&\leq m_1 \left(\|y\| + 2\|q_0(y)\|\right) \\
&\leq m_1 \left(\|y\| + 4m_2\|y\|\right) \\
&\leq 5M\|y\|
\end{aligned}$$

for all $y \in Y$. Finally, since all maps involved preserve multiplication by scalars, ρ satisfies condition (iv). $\square$

Definition 5.1.5. Let A and B be C*-algebras. A *path of maps*, denoted by $(\sigma_t)_{t\in[0,\infty)}$, is a function $\sigma : [0,\infty) \times A \to B$ such that, for every $a \in A$ the map $[t \mapsto \sigma_t(a)]$ is continuous and bounded.

A path of maps $(\sigma_t)_{t\in[0,\infty)}$ from A to B is said to be

- *equicontinuous* if for every $a \in A$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\sigma_t(a) - \sigma_t(b)\| \leq \varepsilon$ for all $t \in [0,\infty)$ and $b \in A$ with $\|a - b\| \leq \delta$;
- an *asymptotic $*$-homomorphism* if it becomes approximately linear, $*$-preserving, and multiplicative as $t \to \infty$.

If A and B are equipped with actions $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$, then we say that a path of maps $(\sigma_t)_{t\in[0,\infty)}$ from A to B is asymptotically equivariant if

$$\lim_{t\to\infty} \max_{g\in K} \|\sigma_t(\alpha_g(a)) - \beta_g(\sigma_t(a))\| = 0$$

for all compact sets $K \subseteq G$ and $a \in A$.

Lemma 5.1.6. *Let A be a σ-unital C^*-algebra. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of pairwise commuting positive contractions in A with $\sum_{n\in\mathbb{N}} f_n^2 = \mathbf{1}$ strictly and such that $f_k f_\ell = 0$ whenever $|k-\ell| \geq 2$. Let $Y \subseteq \mathbb{N}$ be any subset. Then:*

(i) *For every bounded sequence $(a_n)_n$ in $\mathcal{M}(A)$, the series $\sum_{n\in Y} f_n a_n f_n$ converges strictly in $\mathcal{M}(A)$. The map $\Psi_Y : \ell^\infty(\mathbb{N}, \mathcal{M}(A)) \to \mathcal{M}(A)$ given by*
$$\Psi_Y((a_n)_n) = \sum_{n\in Y} f_n a_n f_n$$
is completely positive and contractive. Furthermore $\Psi_Y(c_0(\mathbb{N}, \mathcal{M}(A))) \subseteq A$.

(ii) *Suppose that $\sigma \in \mathrm{Aut}(A)$ is an automorphism such that $\sum_{n=1}^\infty \|\sigma(f_n) - f_n\| < \infty$. Then one has*
$$\sigma(\Psi_Y((a_n)_n)) - \Psi_Y((\sigma(a_n))_n) \ \in A$$
for every $(a_n)_n \in \ell^\infty(\mathbb{N}, \mathcal{M}(A))$.

Proof. (i) is a direct consequence of [136, Lemma 3.1].

For (ii), observe that every $n \geq 1$ that
$$\begin{aligned} &\sigma(f_n a_n f_n) - f_n \sigma(a_n) f_n \\ &= \sigma(f_n a_n f_n) - f_n \sigma(a_n f_n) + f_n \sigma(a_n f_n) - f_n \sigma(a_n) f_n \\ &= (\sigma(f_n) - f_n)\sigma(a_n f_n) + f_n \sigma(a_n)(\sigma(f_n) - f_n). \end{aligned}$$
By assumption, we get
$$\sum_{n\in Y} \|\sigma(f_n a_n f_n) - f_n \sigma(a_n) f_n\| \leq 2\|(a_n)_n\| \cdot \sum_{n=1}^\infty \|\sigma(f_n) - f_n\| < \infty$$
and we may conclude that
$$\sigma(\Psi_Y((a_n)_n)) - \Psi_Y((\sigma(a_n))_n) = \sum_{n\in Y} \sigma(f_n a_n f_n) - \sum_{n\in Y} f_n \sigma(a_n) f_n \ \in A.$$
□

The following fact is well-known, but we give a proof for sake of completeness.

Lemma 5.1.7. *For every $\varepsilon > 0$, there exists a constant $\delta > 0$ with the following property. Let A be any C^*-algebra, $a \in A$ a positive contraction.*

(i) *If* $x \in A$ *is any contraction with* $\|[x,a]\| < \delta$*, then* $\|[x,\sqrt{a}]\| < \varepsilon$*.*

(ii) *If* $\alpha \in \mathrm{Aut}(A)$ *is an automorphism with* $\|a - \alpha(a)\| < \delta$*, then* $\|\sqrt{a} - \alpha(\sqrt{a})\| < \varepsilon$*.*

Proof. Fix a positive contraction $a \in A_+$, a contraction $x \in A$ and $\varepsilon > 0$. If $\|[x,a]\| \leq \delta$ for some $\delta > 0$, then

$$\|[x,a^n]\| = \|xaa^{n-1} - a^{n-1}ax\| \leq \delta + \|xaa^{n-1} - a^{n-1}xa\| \leq \delta + \|[x,a^{n-1}]\|,$$

and repeating the same argument n-times yields the inequality $\|[x,a^n]\| \leq n\delta$. Hence, for any polynomial p one may find a constant C such that

$$\|[x,a]\| \leq \delta \quad \Rightarrow \quad \|[x,p(a)]\| \leq C\delta.$$

By the Stone–Weiestrass approximation theorem we may find a polynomial p such that $|\sqrt{t} - p(t)| \leq \frac{\varepsilon}{3}$ for all $t \in [0,1]$. Let C be a constant as above associated to the polynomial p, and set $\delta_0 := \frac{\varepsilon}{3C}$. Then, if $\|[x,a]\| \leq \delta_0$, we have that

$$\|[x,\sqrt{a}]\| \leq \frac{2\varepsilon}{3} + \|[x,p(a)]\| \leq \varepsilon,$$

which shows (i). For (ii), one proceeds similarly. With the same a as above, assume that $\|a - \alpha(a)\| \leq \delta$ for some $\delta > 0$, and note that

$$\|a^n - \alpha(a^n)\| \leq \delta + \|a^n - a\alpha(a^{n-1})\| \leq \delta + \|a^{n-1} - \alpha(a^{n-1})\|.$$

Applying the same argument n-times gives $\|a^n - \alpha(a^n)\| \leq n\delta$, and thus if p is the polynomial approximating $\sqrt{a}$ from before, the constant C will yield that $\|p(a) - \alpha(p(a))\| \leq C\delta$. So we may now consider δ_0 as above to conclude that

$$\|\sqrt{a} - \alpha(\sqrt{a})\| \leq \frac{2\varepsilon}{3} + \|p(a) + \alpha(p(a))\| \leq \varepsilon.$$

□

The following is the main result of this section and represents a dynamical generalisation of [161, Proposition 1.4].

Theorem 5.1.8. *Let* $\alpha : G \curvearrowright A$ *be an action on a* σ*-unital* C*-*algebra, and equip* $\mathcal{Q}(A)$ *with the algebraic action induced by* α*. Then* $(\mathcal{Q}(A),\alpha)$ *has the dynamical folding property.*

Proof. Fix a separable C*-algebra B with C*-subalgebras $B_0, D \subseteq B$ such that D is equipped with a continuous G-action δ. Let $\psi : B \to \mathcal{Q}(A)_{\mathfrak{c}}$ be a

$*$-homomorphism such that $\psi(B_0) \subset \mathcal{Q}(A)$, and $\psi \restriction_D$ is equivariant with range in $(\mathcal{Q}(A)_\alpha)_\mathfrak{c}$.

Consider the quotient map $\omega : \mathcal{M}(A) \to \mathcal{Q}(A)$, and note that $\omega(\mathcal{M}(A)_\alpha) = \mathcal{Q}(A)_\alpha$. Indeed, by Brown's theorem (see [23, Theorem 2]), one has that α-continuous elements in $\mathcal{Q}(A)$ lift to α-continuous elements in $\mathcal{M}(A)$. By Corollary 5.1.4 (applied with $M = 2$), there exists a (not necessarily linear) continuous map $\zeta : \mathcal{Q}(A) \to \mathcal{M}(A)$ such that

- $\omega \circ \zeta = \mathrm{id}_{\mathcal{Q}(A)}$,
- $\zeta(\mathcal{Q}(A)_\alpha) \subseteq \mathcal{M}(A)_\alpha$,
- $\|\zeta(y)\| \leq 10\|y\|$ for all $y \in \mathcal{Q}(A)$,
- $\zeta(\lambda y) = \lambda\zeta(y)$ for all $\lambda \in \mathbb{C}$ and $y \in \mathcal{Q}(A)$.

The map

$$\omega_* : \mathcal{C}_b([0,\infty), \mathcal{M}(A)) \to \mathcal{C}_b([0,\infty), \mathcal{Q}(A)), \quad \omega_*(f)(t) = \omega(f(t))$$

is surjective, and one can see that the continuous map given by

$$\zeta_* : \mathcal{C}_b([0,\infty), \mathcal{Q}(A)) \to \mathcal{C}_b([0,\infty), \mathcal{M}(A)), \quad \zeta_*(h)(t) = \zeta(h(t))$$

enjoys the following properties,

- $\omega_* \circ \zeta_* = \mathrm{id}_{\mathcal{C}_b([0,\infty),\mathcal{Q}(A))}$,
- $\zeta_*(\mathcal{C}_b([0,\infty), \mathcal{Q}(A)_\alpha)) \subseteq \mathcal{C}_b([0,\infty), \mathcal{M}(A)_\alpha)$,
- $\|\zeta_*(h)\| \leq 10\|h\|$ for all $h \in \mathcal{C}_b([0,\infty), \mathcal{Q}(A))$,
- $\zeta_*(\lambda h) = \lambda\zeta_*(h)$ for all $\lambda \in \mathbb{C}$ and $h \in \mathcal{C}_b([0,\infty), \mathcal{Q}(A))$.

The quotient map $\pi_\mathfrak{c} : \mathcal{C}_b([0,\infty), \mathcal{Q}(A)) \to \mathcal{Q}(A)_\mathfrak{c}$ is of course surjective. Moreover, after identifying $(\mathcal{Q}(A)_\alpha)_\mathfrak{c}$ as a C*-subalgebra of $\mathcal{Q}(A)_\mathfrak{c}$, we have that $\pi_\mathfrak{c}(\mathcal{C}_b([0,\infty), \mathcal{Q}(A)_\alpha)) = (\mathcal{Q}(A)_\alpha)_\mathfrak{c}$. Using Corollary 5.1.4 again, we get a continuous map $\rho_\mathfrak{c} : \mathcal{Q}(A)_\mathfrak{c} \to \mathcal{C}_b([0,\infty), \mathcal{Q}(A))$ such that

- $\pi_\mathfrak{c} \circ \rho_\mathfrak{c} = \mathrm{id}_{\mathcal{Q}(A)_\mathfrak{c}}$,
- $\rho_\mathfrak{c}((\mathcal{Q}(A)_\alpha)_\mathfrak{c}) \subseteq \mathcal{C}_b([0,\infty), \mathcal{Q}(A)_\alpha)$,
- $\|\rho_\mathfrak{c}(y)\| \leq 10\|y\|$ for all $y \in \mathcal{Q}(A)_\mathfrak{c}$,

- $\rho_{\mathfrak{c}}(\lambda y) = \lambda \rho_{\mathfrak{c}}(y)$ for all $\lambda \in \mathbb{C}$ and $y \in \mathcal{Q}(A)_{\mathfrak{c}}$.

Hence, we may define an equicontinuous bounded map

$$\sigma : [0,\infty) \times B \to \mathcal{M}(A), \quad (t,b) \mapsto \sigma_t(b) := \zeta\big(\rho_{\mathfrak{c}}(\psi(b))(t)\big),$$

satisfying the following properties,

(i) $\omega \circ \sigma_t(b) = \rho_{\mathfrak{c}}(\psi(b))(t)$ for all $b \in B$ and $t \in [0,\infty)$,

(ii) $\sigma_t(d) \in \mathcal{C}_b([0,\infty), \mathcal{M}(A)_\alpha)$, for all $d \in D$,

(iii) $\sigma_t(\lambda b) = \lambda \sigma_t(b)$ for all $\lambda \in \mathbb{C}$ and $b \in B$.

From conditions (i) and (ii) one derives a key property of the path of maps $(\sigma_t)_t$. Indeed, it follows that $\omega \circ \sigma_t$ is an asymptotic $*$-homomorphism from B to $\mathcal{Q}(A)$. Moreover, it is also asymptotically equivariant with respect to δ and α when restricted to $D \subseteq B$. Since $\psi(B_0)$ is in the constant part of $\mathcal{Q}(A)_{\mathfrak{c}}$, we may furthermore conclude that

$$\lim_{t\to\infty} \omega \circ \sigma_t(b) = \psi(b), \quad b \in B_0. \tag{5.1}$$

For the rest of the proof, we choose an increasing sequence of compact subsets $K_n \subseteq G$ such that $\bigcup_n K_n = G$ and an increasing sequence of finite sets $F_n \subseteq B$ such that $F := \bigcup_n F_n$ is dense in B, $F \cap B_0$ is dense in B_0, and $F \cap D$ is dense in D.

Now we proceed to find an unbounded increasing sequence $(t_n)_n$ in $[0,\infty)$ such that $\sigma_{t_n}(a)$ becomes closer and closer to $\sigma_{t_{n+1}}(a)$ when $n \to \infty$ for all $a \in B$. (Although this resembles [136, Lemma 3.3], we write out the proof for the reader's convenience.) Note first that for every element $a \in B$, the function $t \mapsto \sigma_t(a)$ is continuous. Hence, for every $n \in \mathbb{N}$, one may find a finite increasing sequence $(s_i^{(n)})_{i=0}^{k_n} \subset [n, n+1]$ with $s_0^{(n)} = n$ and $s_{k_n}^{(n)} = n+1$, such that

$$\max_{a \in F_n} \max_{s,t \in [s_i^{(n)}, s_{i+1}^{(n)}]} \|\sigma_s(a) - \sigma_t(a)\| \leq 2^{-n}.$$

Set $r_n := \sum_{j=0}^{n-1} k_j$ for $n \geq 1$, and $r_0 = 0$. Define an increasing sequence $(t_i)_{i\in\mathbb{N}}$ by pasting together the the sequences above, that is, by setting $t_0 := 0$, and $t_{i+r_n} := s_i^{(n)} \in [n, n+1]$ for $0 < i \leq k_n$ and $n \in \mathbb{N}$. By construction, one has that

$$\lim_{n\to\infty} \max_{a \in F_n} \max_{r_n \leq \ell \leq r_{n+1}} \max_{s,t \in [t_\ell, t_{\ell+1}]} \|\sigma_s(a) - \sigma_t(a)\| = 0. \tag{5.2}$$

By equicontinuity of $(\sigma_t)_t$ we may conclude that the condition holds for any $a \in B$. To simplify notation, we denote $\sigma_{t_\ell}(a)$ by $\sigma_\ell(a)$ for every $\ell \geq 1$.

We consider the increasing sequence of norm-compact subsets of $\mathcal{M}(A)$ given by

$$\begin{aligned} C_n := &\{\zeta(\psi(b)) \mid b \in F_n \cap B_0\} \\ &\cup \{\sigma_\ell(a) \mid 0 \leq \ell \leq n+1,\ a \in F_n\} \\ &\cup \{\sigma_\ell(a) - \zeta(\psi(b)) \mid 0 \leq \ell \leq n,\ a \in F_n\} \\ &\cup \{\sigma_\ell(a)\sigma_\ell(b) - \sigma_\ell(ab) \mid 0 \leq \ell \leq n,\ a, b \in F_n\} \\ &\cup \{\sigma_\ell(a) + \lambda\sigma_\ell(b) - \sigma_\ell(a+\lambda b) \mid 0 \leq \ell \leq n,\ a, b \in F_n,\ \lambda \in \mathbb{C}, |\lambda| \leq n\} \\ &\cup \{\sigma_\ell(a)^* - \sigma_\ell(a^*) \mid 0 \leq \ell \leq n,\ a \in F_n\} \\ &\cup \{\alpha_g(\sigma_\ell(d)) - \sigma_\ell(\delta_g(d)) \mid 0 \leq \ell \leq n,\ d \in D \cap F_n,\ g \in K_n\}. \end{aligned}$$

We remark that compactness of the last component in the definition of C_n follows from the fact that $\sigma_\ell(d)$ is an α-continuous element of $\mathcal{M}(A)$ by Condition (ii) for all $\ell \geq 1$ and $d \in D$.

By Lemma 2.2.12 there exists an approximate unit $(e_n)_{n\in\mathbb{N}}$ of A such that $e_0 = 0$, and $e_{n+1}e_n = e_n$, and

$$\lim_{n\to\infty} \Big(\max_{g\in K_n} \|\alpha_g(e_n) - e_n\| + \max_{x\in C_n} \|[e_n, x]\| \Big) = 0. \tag{5.3}$$

Let $x \in \mathcal{M}(A)$ be any element. Given that $(e_n)_n$ is an increasing approximate unit, we have

$$\|\omega(x)\| = \lim_{n\to\infty} \|(\mathbf{1} - e_n)^{1/2}x\| = \lim_{n\to\infty} \lim_{m\to\infty} \|(e_m - e_n)^{1/2}x\|.$$

In particular, after passing to a subsequence of $(e_n)_n$, we may ensure that

$$\lim_{n\to\infty} \max_{x\in C_n} \Big| \|(\mathbf{1} - e_{n-2})^{1/2}x\| - \|\omega(x)\| \Big| = 0 \tag{5.4}$$

and

$$\lim_{n\to\infty} \max_{x\in C_n} \Big| \|\omega(x)\| - \|(e_n - e_{n-1})^{1/2}x\| \Big| = 0. \tag{5.5}$$

For every $n \geq 1$, let us apply Lemma 5.1.7 and find a constant δ_n satisfying the condition stated there for 2^{-n} in place of ε. We may assume $\delta_{n+1} \leq \delta_n \leq 2^{-n}$ for all n. After passing to a subsequence of $(e_n)_n$ again, which preserves all the approximate properties assumed above, we may assume for all $n \geq 1$ that

$$\max_{g\in K_n} \|\alpha_g(e_n) - e_n\| \leq \delta_{n+1}/2$$

and

$$\|[e_n, x]\| \leq \frac{\delta_n}{2(1+\|x\|)}, \quad x \in C_n.$$

We shall denote $e_0 = e_{-1} = 0$ and set $f_n = (e_n - e_{n-1})^{1/2}$ for all $n \geq 0$. Due to the equation $e_n e_{n+1} = e_n$ and the choice of δ_n, we observe the following properties:

(1) $f_m f_n = 0$ if $|m-n| \geq 2$,

(2) $\max_{\ell \leq n+1} \|[f_n, \sigma_\ell(a)]\| \leq 2^{-n} \cdot 100\|a\|$ for all $n \geq 1$ and $a \in F_n$,[2]

(3) $\|[f_n, \zeta(\psi(b))]\| \leq 2^{-n} \cdot 10\|b\|$ for all $n \geq 1$ and $b \in F_n \cap B_0$,

(4) $\max_{g \in K_n} \|\alpha_g(f_n) - f_n\| \leq 2^{-n}$ for all $n \geq 1$,

(5) $\sum_{n=1}^\infty f_n^2 = \mathbf{1}$ strictly,

(6) $f_n(f_{n-1}^2 + f_n^2 + f_{n+1}^2) = f_n$ for all $n \geq 1$.

We shall now apply Lemma 5.1.6 to this choice of the sequence $(f_n)_n$. Since $(\omega \circ \sigma_t)_t$ is approximately $*$-homomorphic and equivariant on D as $t \to \infty$, we can conclude with Lemma 5.1.6 and condition (5.4) (using $f_n = f_n(1-e_{n-2})^{1/2}$) that

(7) $\sum_{n=1}^\infty f_n \big(\sigma_n(a)\sigma_n(b) - \sigma_n(ab)\big) f_n \in A$ for all $a, b \in B$,

(8) $\sum_{n=1}^\infty f_n \big(\sigma_n(a) + \lambda\sigma_n(b) - \sigma_n(a+\lambda b)\big) f_n \in A$ for all $a, b \in B$ and $\lambda \in \mathbb{C}$,

(9) $\sum_{n=1}^\infty f_n \big(\sigma_n(a)^* - \sigma_n(a^*)\big) f_n \in A$ for all $a \in B$,

(10) $\sum_{n=1}^\infty f_n \big(\alpha_g(\sigma_n(d)) - \sigma_n(\delta_g(d))\big) f_n \in A$ for all $g \in G$ and $d \in D$.

Let $\Phi : B \to \mathcal{M}(A)$ be the map given by

$$\Phi(b) = \sum_{n=1}^\infty f_n \sigma_n(b) f_n, \quad b \in B,$$

and denote by $\varphi = \omega \circ \Phi : B \to \mathcal{Q}(A)$ its composition with the quotient map onto the corona algebra. We show that φ is a $*$-homomorphism with $\varphi\restriction_{B_0} = \psi\restriction_{B_0}$ and such that $\varphi\restriction_D$ is equivariant. We prove this by verifying that Φ has these properties modulo A. Since Φ is continuous (as a composition of continuous maps), it suffices to prove these properties on dense subsets of B, B_0 and D,

[2] Here the constant 100 comes the fact that the composition of ζ with $\rho_{\mathfrak{c}}$ increases the norm of an element in $\mathcal{Q}(A)$ by a factor of at most $10 \cdot 10$.

respectively. We start with multiplicativity. We shall use the notation $a \equiv b$ to say that two elements $a, b \in \mathcal{M}(A)$ agree modulo A. We have for every $a, b \in F$

$$\begin{aligned}\Phi(a)\Phi(b) &= \left(\sum_{n=1}^{\infty} f_n\sigma_n(a)f_n\right)\left(\sum_{n=1}^{\infty} f_n\sigma_n(b)f_n\right)\\ &\overset{(1)}{=} \sum_{n=1}^{\infty} f_n\sigma_n(a)f_n\left(\sum_{j=n-1}^{n+1} f_j\sigma_j(b)f_j\right)\\ &\overset{(2)}{\equiv} \sum_{n=1}^{\infty} f_n\big(\sigma_n(a)\sigma_{n-1}(b)f_{n-1}^2 + \sigma_n(a)\sigma_n(b)f_n^2 + \sigma_n(a)\sigma_{n+1}(b)f_{n+1}^2\big)f_n\end{aligned}$$

If we consider condition (5.2) with Lemma 5.1.6, we conclude

$$\begin{aligned}\Phi(a)\Phi(b) &\equiv \sum_{n=1}^{\infty} f_n\sigma_n(a)\sigma_n(b)(f_{n-1}^2 + f_n^2 + f_{n+1}^2)f_n\\ &\overset{(6)}{\equiv} \sum_{n=1}^{\infty} f_n\sigma_n(a)\sigma_n(b)f_n\\ &\overset{(7)}{\equiv} \sum_{n=1}^{\infty} f_n\sigma_n(ab)f_n \;=\; \Phi(ab).\end{aligned}$$

We proceed analogously to show that Φ is linear and $*$-preserving modulo A. For all $a, b \in B$ and $\lambda \in \mathbb{C}$ we have

$$\Phi(a+\lambda b) - \Phi(a) - \lambda\Phi(b) = \sum_{n=1}^{\infty} f_n\left(\sigma_n(a+\lambda b) - \sigma_n(a) - \lambda\sigma_n(b)\right)f_n \overset{(8)}{\equiv} 0$$

and

$$\Phi(a^*) - \Phi(a)^* = \sum_{n=1}^{\infty} f_n\left(\sigma_n(a^*) - \sigma_n(a)^*\right)f_n \overset{(9)}{\equiv} 0.$$

Let $b \in B_0 \cap F$. We estimate

$$\limsup_{n\to\infty} \|(\mathbf{1} - e_{n-2})^{1/2}(\sigma_n(b) - \zeta(\psi(b)))\| \overset{(5.4)}{\leq} \limsup_{n\to\infty} \|\omega(\sigma_n(b) - \zeta(\psi(b)))\| \overset{(5.1)}{=} 0.$$

If use Lemma 5.1.6 again, we hence observe that

$$\Phi(b) = \sum_{n=1}^{\infty} f_n\sigma_n(b)f_n$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} f_n \zeta(\psi(b)) f_n + \sum_{n=1}^{\infty} f_n \big(\sigma_n(b) - \zeta(\psi(b))\big) f_n \\
&\overset{(6)}{=} \sum_{n=1}^{\infty} f_n \zeta(\psi(b)) f_n + \sum_{n=1}^{\infty} f_n (\mathbf{1} - e_{n-2})^{1/2} \big(\sigma_n(b) - \zeta(\psi(b))\big) f_n \\
&\equiv \sum_{n=1}^{\infty} f_n \zeta(\psi(b)) f_n \overset{(3)}{\equiv} \sum_{n=1}^{\infty} f_n^2 \zeta(\psi(b)) \;=\; \zeta(\psi(b)).
\end{aligned}
$$

This proves $\psi(b) = \varphi(b)$ for all $b \in F \cap B_0$ and hence also for all $b \in B_0$ by continuity. Finally, let us show that $\Phi \restriction_D$ is equivariant modulo A. Let $d \in D \cap F$ and $g \in G$. Using Lemma 5.1.6 and condition (4), we compute

$$
\begin{aligned}
\Phi(\delta_g(d)) - \alpha_g(\Phi(d)) &= \sum_{n=1}^{\infty} f_n \sigma_n(\delta_g(d)) f_n - \alpha_g \left(\sum_n f_n \sigma_n(d) f_n \right) \\
&\equiv \sum_{n=1}^{\infty} f_n \sigma_n(\delta_g(d)) f_n - \sum_{n=1}^{\infty} f_n \alpha_g(\sigma_n(d)) f_n \\
&= \sum_{n=1}^{\infty} f_n \left(\sigma_n(\delta_g(d)) - \alpha_g(\sigma_n(d)) \right) f_n \overset{(10)}{\equiv} 0.
\end{aligned}
$$

Thus we have establish that φ is a $*$-homomorphism that is equivariant when restricted to D and equal to ψ when restricted to B_0.

Let us now check that $\ker\varphi = \ker\psi$. It follows from the fact that a right inverse obtained from Corollary 5.1.4 necessarily maps zero to zero that $\ker\psi \subseteq \ker\varphi$. For the opposite inclusion, choose $b \in \ker\varphi$. Firstly, we point out as a direct consequence of the above chain of computations (used to verify that φ is a $*$-homomorphism) that $0 = \varphi(b^*b)$ is represented by the multiplier $\sum_{k=1}^{\infty} f_k \sigma_k(b) \sigma_k(b)^* f_k$, hence the series defines an element in A. Thus

$$
\|f_n \sigma_n(b) \sigma_n(b)^* f_n\| \le \|(\mathbf{1} - e_{n-2}) \Big(\sum_{k=1}^{\infty} f_k \sigma_k(b) \sigma_k(b)^* f_k \Big) (\mathbf{1} - e_{n-2})\| \xrightarrow{n\to\infty} 0.
$$

Thus $\|f_n \sigma_n(b)\| \xrightarrow{n\to\infty} 0$. If we apply (5.5), we get for every $a \in F$ that

$$
\limsup_{n\to\infty} \|f_n \sigma_n(a)\| = \limsup_{n\to\infty} \|\omega(\sigma_n(a))\|.
$$

By equicontinuity and the fact that F is dense in B, this equation persists for all $a \in B$. We may therefore conclude $\|\omega(\sigma_n(b))\| \xrightarrow{n\to\infty} 0$. Given condition (5.2), this implies $\|\omega(\sigma_t(b))\| \xrightarrow{t\to\infty} 0$ or equivalently $\rho_{\mathfrak{c}}(\psi(b)) \in \mathcal{C}_0([0,\infty), \mathcal{Q}(A))$. Therefore, we have that $\psi(b) = \pi_{\mathfrak{c}}(\rho_{\mathfrak{c}}(\psi(b))) = 0$, and hence that $\ker\varphi = \ker\psi$. $\square$

Notation 5.1.9. Let $\alpha : G \curvearrowright A$ be an algebraic action on a C*-algebra. Given a C*-subalgebra $C \subseteq A$, we write

$$A \cap_\alpha C' := A \cap \Big(\bigcup_{g\in G} \alpha_g(C) \Big)'.$$

This is clearly the largest α-invariant C*-subalgebra of $A \cap C'$. Hence any α-invariant C*-subalgebra $D \subseteq A$ is contained in $A \cap C'$ if and only if it is contained in $A \cap_\alpha C'$.

Before we move on to the next section, we include two further observations, the first of which will be needed in the next section.

Lemma 5.1.10. *Let $\alpha : G \curvearrowright A$ be an algebraic action with the dynamical folding property. Let $C \subseteq A$ be a separable C*-subalgebra and $\delta : G \curvearrowright D$ a continuous action on a separable C*-algebra. Suppose that $\varphi, \psi : (D, \delta) \to (A \cap_\alpha C', \alpha)$ are two equivariant $*$-homomorphisms that are asymptotically G-unitarily equivalent, i.e., there exists a norm-continuous path $w : [0,\infty) \to \mathcal{U}\big(\mathbf{1} + (A \cap_\alpha C')_\alpha\big)$ with*

$$\psi(d) = \lim_{t\to\infty} w_t \varphi(d) w_t^*, \quad \lim_{t\to\infty} \max_{g\in K} \|w_t - \alpha_g(w_t)\| = 0$$

for all $d \in D$ and every compact set $K \subseteq G$. Then φ and ψ are G-unitarily equivalent, i.e., there exists a unitary $u \in \mathcal{U}(\mathbf{1} + (A \cap_\alpha C')^\alpha)$ such that $\psi = \mathrm{Ad}(u) \circ \varphi$.

Proof. The given unitary path induces a unitary element

$$\bar{w} \in \mathbf{1} + \big((A \cap_\alpha C')_\alpha\big)_{\mathfrak{c}}^{\alpha_{\mathfrak{c}}} \subseteq \mathbf{1} + \Big(\big((A_\alpha)_{\mathfrak{c}}\big)^{\alpha_{\mathfrak{c}}} \cap C'\Big)$$

such that $\psi = \mathrm{Ad}(\bar{w}) \circ \varphi$ as equivariant $*$-homomorphisms from D to $(A_\alpha)_{\mathfrak{c}} \cap C'$. We shall apply the dynamical folding property with the choice $B_0 = \mathrm{C}^*(\varphi(D) \cup \psi(D) \cup C)$, the algebra $\mathrm{C}^*(\bar{w} - \mathbf{1})$ in place of D equipped with the trivial G-action, and B the C*-algebra generated by both of these. This yields an equivariant $*$-homomorphism $\kappa : B \to A$ satisfying $\kappa \circ \varphi = \varphi$, $\kappa \circ \psi = \psi$ and $\kappa(c) = c$ for all $c \in C$. The element $u = \mathbf{1} + \kappa(\bar{w} - \mathbf{1})$ then defines a unitary in $\mathbf{1} + A^\alpha$ that commutes with C. It satisfies

$$\mathrm{Ad}(u) \circ \varphi = \kappa \circ \big(\mathrm{Ad}(\bar{w}) \circ \varphi\big) = \kappa \circ \psi = \psi.$$

Since $A^\alpha \cap C' = (A \cap_\alpha C')^\alpha$, this is a unitary we were looking for. $\square$

Lemma 5.1.11. *Let $\beta : G \curvearrowright B$ be an algebraic action with the dynamical folding property. Let $A \subset B$ be a β-invariant separable* C^**-subalgebra and assume that the restricted action $\alpha = \beta\restriction_A$ is continuous. Then $B \cap A^\perp$ is an algebraic G-σ-ideal in $B \cap A'$.*[3] *That is, for any separable* C^**-subalgebra $C \subset B \cap A'$, there exists a positive contraction $e \in B^\beta \cap A^\perp \cap C'$ such that $ec = c$ for all $c \in C \cap A^\perp$.*

Proof. Since the action α on A is continuous, it follows from Lemma 2.2.12 that we find an increasing approximate unit $(h_n)_{n\geq 1}$ in A such that

$$\max_{g\in K} \|h_n - \alpha_g(h_n)\| \to 0$$

for every compact set $K \subseteq G$. By linear interpolation, we may extend this to a norm-continuous family $(h_t)_{t\geq 1}$ with the same asymptotic properties. Obviously we have $h_t c = 0$ for every $t \geq 1$ and $c \in C \cap A^\perp$. We may then consider the element $h \in A_\mathfrak{c} \subset (B_\beta)_\mathfrak{c}$ defined by the path $(h_t)_{t\geq 1}$. This element is fixed by the induced action of β, commutes with C, and satisfies $ha = a$ for every $a \in A$.

We shall define D as the unital C^*-algebra generated by h in $(B_\beta)_\mathfrak{c}$, equipped with the trivial G-action. We consider B_0 to be the C^*-algebra generated by $A \cup C$ in B. By the dynamical folding property, we may find a unital $*$-homomorphism $\varphi : \mathrm{C}^*(A \cup C \cup D) \to B$ with $\varphi\restriction_{B_0} = \mathrm{id}_{B_0}$ and $\varphi(D) \subseteq B^\beta$. Set $e = \varphi(\mathbf{1} - h) \in B^\beta$. Since $B_0 \supset C$, it readily follows that e commutes with C. Furthermore, the relation $ha = a$ for all $a \in A$ implies $e \in B \cap A^\perp$. Lastly, if $c \in C \cap A^\perp$, then the relation $hc = 0$ readily implies $ec = c$.

As C was arbitrarily chosen, this finishes the proof. $\square$

5.2 Absorption properties

In this section we prove the main result of the chapter, Theorem 5.2.4, which generalizes [62, Theorem 2.5] to the dynamical setting. Recall that a unital C^*-algebra A is said to be D-saturated for some separable unital C^*-algebra D if for every separable C^*-subalgebra $C \subseteq A$, there exists a unital embedding of D into $A \cap C'$. The following generalizes this property to the dynamical setting.

Definition 5.2.1. Let $\alpha : G \curvearrowright A$ be an algebraic action on a unital C^*-algebra, and $\gamma : G \curvearrowright D$ an action on a separable unital C^*-algebra. We say that (A, α) is *γ-saturated* if for every separable C^*-subalgebra $C \subseteq A$, there exists an equivariant unital embedding $(D, \gamma) \to (A \cap_\alpha C', \alpha)$.

[3] See [117] and [188, Definition 4.1].

Lemma 5.2.2 (see [62, Lemma 2.4]). *Let A be a σ-unital C^*-algebra, and $X \subseteq \mathcal{M}(A)$ a strictly-separable subset. Then there exists a non-degenerate, separable C^*-subalgebra $B \subseteq A$ such that $X \subseteq \mathcal{M}(B)$ under the canonical inclusion $\mathcal{M}(B) \subseteq \mathcal{M}(A)$.*

Proof. Upon replacing X with $X \cup X^* \cup \{\mathbf{1}_{\mathcal{M}(A)}\}$ if necessary, we can assume that X is self-adjoint and contains the unit. Pick a countable approximate unit $(e_n)_{n\in\mathbb{N}}$ of A and let B denote the separable C^*-subalgebra of A generated by the set

$$\{xe_n \mid x \in X,\, n \in \mathbb{N}\}.$$

Let us show that B is separable. Since X contains a countable strictly-dense set $\{d_n\}_{n\in\mathbb{N}} \subseteq X$, for any element $x \in X$ and $\varepsilon > 0$, one may find some $n \in \mathbb{N}$ and d_n such that $xe_i =_\varepsilon d_n e_i$ for all $i \in \mathbb{N}$. Now pick $b \in B$ and $\varepsilon > 0$, and find $x_1, \dots, x_n \in X$ and $e_{k_1}, \dots, e_{k_n}$, where $k_i \in \mathbb{N}$ for $i = 1, \dots, n$, such that $b =_{\frac{\varepsilon}{2}} \sum_{i=1}^n x_i e_{k_i}$. Using the observation above, we may find for every $i = 1, \dots, n$ an index $\ell_i \in \mathbb{N}$ such that $x_i e_{k_i} =_{\frac{\varepsilon}{2n}} d_{\ell_i} e_{k_i}$. Hence,

$$b =_{\frac{\varepsilon}{2}} \sum_{i=1}^n x_i e_{k_i} =_{\frac{\varepsilon}{2}} \sum_{i=1}^n d_{\ell_i} e_{k_i}$$

which proves that B is separable. Since B contains $(e_n)_n$, one has right away that $B \subseteq A$ is non-degenerate. Moreover, if $x, y \in X$, then

$$x(ye_n) = \lim_{k\to\infty} xe_k ye_n \in B,$$

and hence $xB \subseteq B$, and $Bx = (x^*B)^* \subseteq B$. Hence, $X \subseteq \mathcal{M}(B)$ under the canonical inclusion $\mathcal{M}(B) \subseteq \mathcal{M}(A)$. □

The next proposition represents one of the main technical steps towards the proof of the main result of this chapter. As discussed at the beginning of the chapter, a special case of this result was originally obtained in Xiuyuan Li's Master's thesis [133] at KU Leuven. In particular, in the aforementioned Master's project, the result was established under the additional assumption that the subalgebra $C \subseteq \mathcal{Q}(A)$ in the statement is α-invariant.

Proposition 5.2.3. *Let $\alpha : G \curvearrowright A$ be an action on a σ-unital, non-unital C^*-algebra and $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing action. Suppose that α is separably γ-stable. Then for every separable C^*-subalgebra $C \subseteq \mathcal{Q}(A)$, there exists an equivariant unital $*$-homomorphism*

$$(\mathcal{D}^{(2)}, \gamma^{(2)}) \to (\mathcal{Q}(A) \cap_\alpha C', \alpha).$$

Proof. Fix a separable C*-subalgebra $C \subseteq \mathcal{Q}(A)$. Due to Lemma 5.2.2 and the fact that α is separably γ-stable, there exists a separable nondegenerate α-invariant C*-subalgebra $A_1 \subseteq A$ such that $(\alpha \restriction A_1)$ is γ-stable and under the canonical embedding $\mathcal{Q}(A_1) \subseteq \mathcal{Q}(A)$, we have $C \subseteq \mathcal{Q}(A_1)$. Considering what the claim says, it suffices to prove it for (A_1, α) in place of (A, α). In other words, we may assume without loss of generality that A is separable and that α is γ-stable.

In order to find a $*$-homomorphism as in the claim, we shall appeal to the universal property of $(\mathcal{D}^{(2)}, \gamma^{(2)})$ as specified in Remark 5.0.3. In other words, we want to find two equivariant c.p.c. order zero maps $\mu_0, \mu_1 : (\mathcal{D}, \gamma) \to (\mathcal{Q}(A) \cap_\alpha C', \alpha)$ with commuting ranges such that $\mu_0(\mathbf{1}) + \mu_1(\mathbf{1}) = \mathbf{1}$.

Pick a sequence $(c_n)_{n\geq 1}$ in the unit ball of $\mathcal{M}(A)$ whose image under the quotient map $\omega : \mathcal{M}(A) \to \mathcal{Q}(A)$ is dense in the unit ball of $C \subseteq \mathcal{Q}(A)$. Moreover, fix an increasing sequence of finite subsets F_n of the unit ball of $\mathcal{D}$ with dense union $F := \bigcup_n F_n$, and an increasing sequence of compact subsets $K_n \subseteq G$ whose union is G. For every $n \geq 1$, let us apply Lemma 5.1.7 and find a constant δ_n satisfying the condition stated there for 2^{-n} in place of ε. We may assume $\delta_{n+1} \leq \delta_n \leq 2^{-n}$ for all n. By Lemma 2.2.12 there exists an approximate unit $(e_n)_{n\geq}$ of A such that $e_{n+1}e_n = e_n$, and

$$\max_{\ell\leq n} \|[c_\ell, e_n]\| + \max_{g\in K_n} \|\alpha_g(e_n) - e_n\| \leq \delta_{n+1}/2 \quad \text{for all } n \geq 1.$$

If we define $e_{-1} = e_0 := 0$ and $f_n = (e_n - e_{n-1})^{1/2}$ for all $n \geq 0$, then it follows from the choice of the constants δ_n that

(a) $\|[f_n, c_\ell]\| \leq 2^{-n}$ for all $\ell \leq n$ and $n \geq 1$,

(b) $\max_{g\in K_n} \|\alpha_g(f_n) - f_n\| \leq 2^{-n}$ for all $n \geq 1$,

(c) $f_m f_n = 0$ if $|m - n| \geq 2$,

(d) $\sum_{n=1}^\infty f_n^2 = \mathbf{1}$ strictly.

Moreover, thanks to Theorem 2.6.10, we may find a sequence of unital $*$-homomorphisms $\varphi_n : \mathcal{D} \to \mathcal{M}(A)$ such that

(e) $\max_{d\in F_n} \|[\varphi_n(d), f_m]\| \leq 2^{-n}$ for all $m \leq n+1$ and $n \geq 1$,

(f) $\max_{d\in F_n} \|[\varphi_n(d), f_n c_\ell]\| \leq 2^{-n}$ for all $\ell \leq n$ and $n \geq 1$,

(g) $\max_{d,d'\in F_n} \|[\varphi_n(d), f_\ell \varphi_\ell(d') f_\ell]\| \leq 2^{-n}$ for all $\ell < n$ and $n \geq 1$,

(h) $\max_{d\in F_n} \max_{g\in K_n} \|f_n(\alpha_g(\varphi_n(d)) - \varphi_n(\gamma_g(d)))f_n\| \leq 2^{-n}$ for all $n \geq 1$.

Define two maps $\Psi_0, \Psi_1 : \mathcal{D} \to \mathcal{M}(A)$ given by

$$\Psi_i(d) = \sum_{n=0}^{\infty} f_{2n+i}\varphi_{2n+i}(d)f_{2n+i}, \quad d \in \mathcal{D},\ i = 0, 1.$$

By Lemma 5.1.6, we know that Ψ_0 and Ψ_1 are c.p.c. maps. The formula $\Psi_0(\mathbf{1}) + \Psi_1(\mathbf{1}) = \mathbf{1}$ evidently holds by construction due to condition (d). As before, we shall use the notation $a \equiv b$ to say that two elements $a, b \in \mathcal{M}(A)$ agree modulo A. For every $g \in G$ and $d \in \mathcal{D}$, Lemma 5.1.6 furthermore implies with condition (b) that

$$\alpha_g(\Psi_i(d)) \equiv \sum_{n=0}^{\infty} f_{2n+i}\alpha_g(\varphi_{2n+i}(d))f_{2n+i}, \quad i = 0, 1.$$

We may therefore infer with condition (h) that

$$\alpha_g(\Psi_i(d)) \equiv \Psi_i(\gamma_g(d)) \quad \text{for all } g \in G,\ d \in F \text{ and } i = 0, 1.$$

Since F is dense in the unit ball of $\mathcal{D}$, we conclude that $\alpha_g(\Psi_i(d)) \equiv \Psi_i(\gamma_g(d))$ holds for all $g \in G$ and $d \in \mathcal{D}$. In other words, the c.p.c. maps given by

$$\mu_i = \omega \circ \Psi_i : \mathcal{D} \to \mathcal{Q}(A), \quad i = 0, 1,$$

are equivariant with respect to γ and α. To show that the maps μ_i have range in $\mathcal{Q}(A) \cap C'$, pick $d \in F_n$ for some $n \geq 1$. Note that for all $j \leq n$,

$$f_n\varphi_n(d)f_nc_j \overset{\text{(f)}}{=}_{2^{-n}} f_n^2c_j\varphi_n(d) \overset{\text{(a)}}{=}_{2^{1-n}} c_jf_n^2\varphi_n(d) \overset{\text{(e)}}{=}_{2^{-n}} c_jf_n\varphi_n(d)f_n.$$

Thus, $\|[f_n\varphi_n(d)f_n, c_j]\| \leq 2^{2-n}$ for $j \leq n$ and $d \in F_n$, which implies that $[\Psi_i(d), c_j] \in A$ for all $j \geq 1$ and $d \in F$, where $i = 0, 1$. Since F and $(\omega(c_n))_n$ are dense in the unit ball of $\mathcal{D}$ and C, respectively, it follows for every $d \in \mathcal{D}$ and $c \in C$ that $[\Psi_i(d), c] \in A$, and therefore $\mu_i(\mathcal{D}) \subseteq \mathcal{Q}(A) \cap C'$ for $i = 0, 1$. Since we have already shown that μ_0 and μ_1 are equivariant, this yields $\mu_i(\mathcal{D}) \subseteq \mathcal{Q}(A) \cap_\alpha C'$ for $i = 0, 1$.

In order to show that the ranges of μ_0 and μ_1 commute, let us choose $d, d' \in F_n$. We note that

$$\begin{aligned} f_{n+1}\varphi_{n+1}(d')f_{n+1} \cdot f_n\varphi_n(d)f_n &\overset{\text{(e)}}{=}_{2^{-(n+1)}} f_{n+1}^2\varphi_{n+1}(d')f_n\varphi_n(d)f_n \\ &\overset{\text{(g)}}{=}_{2^{-(n+1)}} f_{n+1}^2f_n\varphi_n(d)f_n\varphi_{n+1}(d') \\ &\overset{\text{(e)}}{=}_{2^{1-n}} f_n\varphi_n(d)f_nf_{n+1}^2\varphi_{n+1}(d') \end{aligned}$$

$$\overset{\text{(e)}}{=}_{2^{-(n+1)}} f_n\varphi_n(d)f_nf_{n+1}\varphi_{n+1}(d')f_{n+1}.$$

In particular, this implies that

$$\|[f_{n+1}\varphi_{n+1}(d')f_{n+1}, f_n\varphi_n(d)f_n]\| \leq 2^{2-n}.$$

By using condition (c), we then have that

$$\begin{aligned}[\Psi_0(d), \Psi_1(d')] &= \left[\sum_{n=0}^{\infty} f_{2n}\varphi_{2n}(d)f_{2n}\ ,\ \sum_{n=0}^{\infty} f_{2n+1}\varphi_{2n+1}(d')f_{2n+1}\right] \\ &= \sum_{n=0}^{\infty}[f_{2n}\varphi_{2n}(d)f_{2n}, f_{2n+1}\varphi_{2n+1}(d')f_{2n+1}] \\ &\quad + \sum_{n=1}^{\infty}[f_{2n}\varphi_{2n}(d)f_{2n}, f_{2n-1}\varphi_{2n-1}(d')f_{2n-1}].\end{aligned}$$

The estimate above implies that these define norm-convergent series in A and thus $[\Psi_0(d), \Psi_1(d')] \in A$ for every $d, d' \in F$. Since F was dense in the unit ball of $\mathcal{D}$, the same holds for $d, d' \in \mathcal{D}$, and hence μ_0 and μ_1 have commuting ranges.

To see that μ_0 and μ_1 are order zero maps, we use once again condition (c) to see that, for $i = 0, 1$,

$$\Psi_i(d)\Psi_i(d') = \sum_{n=0}^{\infty} f_{2n+i}\varphi_{2n+i}(d)f_{2n+i}^2\varphi_{2n+i}(d')f_{2n+i}$$

for all $d, d' \in \mathcal{D}$. If $d, d' \in F_n$ for some $n \geq 1$, then

$$f_{2n+i}\varphi_{2n+1}(d)f_{2n+i}^2\varphi_{2n+i}(d')f_{2n+i} \overset{\text{(e)}}{=}_{2^{1-(2n+i)}} f_{2n+i}\varphi_{2n+1}(dd')f_{2n+i}^3.$$

We may conclude that

$$\Psi_i(d)\Psi_i(d') \equiv \Psi_i(ab)\Psi_i(\mathbf{1})$$

for all $d, d' \in F$, and hence also for all $d, d' \in \mathcal{D}$. To summarize, μ_0 and μ_1 are equivariant c.p.c. order zero maps with commuting ranges such that $\mu_0(\mathbf{1}) + \mu_1(\mathbf{1}) = \mathbf{1}$, and thus the universal property recalled in Remark 5.0.3 yields an equivariant unital $*$-homomorphism $(\mathcal{D}^{(2)}, \gamma^{(2)}) \to \mathcal{Q}(A) \cap_\alpha C'$. $\square$

We are now ready to prove the main theorem of this section.

Theorem 5.2.4. *Let $\alpha : G \curvearrowright A$ be an action on a σ-unital, non-unital* C^**-algebra, and $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing, unitarily regular action. If α is separably γ-stable, then $(\mathcal{Q}(A), \alpha)$ is γ-saturated.*

Proof. Fix $C \subseteq \mathcal{Q}(A)$ as in the statement. By Proposition 5.2.3, there exists an equivariant unital $*$-homomorphism

$$\kappa : (\mathcal{D}^{(2)}, \gamma^{(2)}) \to (\mathcal{Q}(A) \cap_\alpha C', \alpha).$$

It extends to an equivariant unital $*$-homomorphism

$$\kappa_{\mathfrak{c}} : \mathcal{D}_{\mathfrak{c}}^{(2)} \to (\mathcal{Q}(A)_\alpha)_{\mathfrak{c}} \quad \text{with} \quad [\kappa_{\mathfrak{c}}(\mathcal{D}^{(2)}), C] = 0$$

that sends a representative function $f : [0, \infty) \to \mathcal{D}^{(2)}$ of an element in $\mathcal{D}_{\mathfrak{c}}^{(2)}$ to $\kappa \circ f \in \mathcal{C}_b([0, \infty), \mathcal{Q}(A)_\alpha \cap C')$. This is well defined because whenever $f(t) \xrightarrow{t\to\infty} 0$ one gets $\kappa(f(t)) \xrightarrow{t\to\infty} 0$. Since the strongly self-absorbing action γ is assumed to be unitarily regular, $\gamma^{(2)}$ is γ-absorbing by Theorem 5.0.4. By Proposition 5.0.5 there exists an equivariant unital embedding

$$\theta : (\mathcal{D}, \gamma) \hookrightarrow (\mathcal{D}^{(2)}_{\mathfrak{c}, \gamma^{(2)}}, \gamma^{(2)}_{\mathfrak{c}}).$$

The resulting composition $\psi = \kappa_{\mathfrak{c}} \circ \theta$ is a unital equivariant $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(\mathcal{Q}(A)_\alpha)_{\mathfrak{c}}$ with $[\psi(\mathcal{D}), C] = 0$. We want to appeal to the dynamical folding property via Theorem 5.1.8. We make the choice $D = \psi(\mathcal{D})$ with action induced from γ, we choose $B_0 = \mathrm{C}^*(C, \mathbf{1})$, and $B = \mathrm{C}^*(B_0 \cup D) \subseteq \mathcal{Q}(A)_{\mathfrak{c}}$. This allows us to find a $*$-homomorphism $\varphi : B \to \mathcal{Q}(A)$ such that $\varphi(b) = b$ for all $b \in B_0$ and $\varphi \circ \psi$ defines an equivariant $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(\mathcal{Q}(A), \alpha)$. Since $\psi(\mathbf{1}_D) = \mathbf{1}_{\mathcal{Q}(A)} \in B_0$, it follows that $\varphi \circ \psi$ is unital. Since $C \subseteq B_0$ and $[\psi(\mathcal{D}), C] = 0$, it follows that also $[(\varphi \circ \psi)(\mathcal{D}), C] = 0$. In other words, we have found an equivariant unital $*$-homomorphism from $(\mathcal{D}, \gamma)$ to $(\mathcal{Q}(A) \cap_\alpha C', \alpha)$. ☐

We would like to end this section by pointing out that the main result also leads to a uniqueness theorem that generalizes the one observed in [59, Theorem E]. We shall deduce this as a consequence of a more general formal observation.

Theorem 5.2.5. *Let A be a unital* C^**-algebra and $\alpha : G \curvearrowright A$ an algebraic action with the dynamical folding property. Suppose that $\gamma : G \curvearrowright \mathcal{D}$ is a strongly self-absorbing, unitarily regular action such that α is γ-saturated. Let $C \subseteq A$ be a separable* C^**-subalgebra. Then all equivariant unital $*$-homomorphisms from $(\mathcal{D}, \gamma)$ to $(A \cap_\alpha C', \alpha)$ are mutually G-unitarily equivalent.*

Proof. Fix $C \subseteq A$ as in the statement. We first note that due to the assumption, there must exist an equivariant unital $*$-homomorphisms from $(\mathcal{D}, \gamma)$ to $(A \cap_\alpha C', \alpha)$ to begin with. Let ψ_1 and ψ_2 be two arbitrary ones. Consider $E = \mathrm{C}^*(\psi_1(\mathcal{D}) \cup \psi_2(\mathcal{D}))$, which is a separable α-invariant C^*-subalgebra of A. Using that α is γ-saturated, we find a third equivariant unital $*$-homomorphism $\psi_3 : (\mathcal{D}, \gamma) \to (A \cap_\alpha (C \cup E)', \alpha)$. By the universal property of the tensor product, we obtain two equivariant unital $*$-homomorphisms

$$\theta_1, \theta_2 : (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma) \to (A \cap_\alpha C', \alpha)$$

defined by $\theta_1(a \otimes b) = \psi_1(a)\psi_3(b)$ and $\theta_2(a \otimes b) = \psi_2(a)\psi_3(b)$ for all $a, b \in \mathcal{D}$. In other words, we have

$$\psi_1 = \theta_1 \circ (\mathrm{id}_\mathcal{D} \otimes \mathbf{1}),\ \psi_2 = \theta_2 \circ (\mathrm{id}_\mathcal{D} \otimes \mathbf{1}),\ \psi_3 = \theta_1 \circ (\mathbf{1} \otimes \mathrm{id}_D) = \theta_2 \circ (\mathbf{1} \otimes \mathrm{id}_D).$$

As γ is strongly self-absorbing and unitarily regular, it follows from [188, Theorem 3.15] that the two equivariant unital embeddings

$$\mathrm{id}_D \otimes \mathbf{1},\ \mathbf{1} \otimes \mathrm{id}_\mathcal{D} : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

are asymptotically G-unitarily equivalent. This immediately implies with the above equations (via transitivity) that ψ_1 and ψ_2 are asymptotically G-unitarily equivalent as maps into $A \cap_\alpha C'$. The claim thus follows from Lemma 5.1.10. $\square$

Corollary 5.2.6. *Let A be a σ-unital* C^**-algebra and $\alpha : G \curvearrowright A$ an action. We consider the induced algebraic action $\alpha : G \curvearrowright \mathcal{Q}(A)$. Suppose that $\gamma : G \curvearrowright \mathcal{D}$ is a strongly self-absorbing, unitarily regular action such that α is separably γ-stable. Let $C \subseteq \mathcal{Q}(A)$ be any separable* C^**-subalgebra. Then all equivariant unital $*$-homomorphisms from $(\mathcal{D}, \gamma)$ to $(A \cap_\alpha C', \alpha)$ are mutually G-unitarily equivalent.*

Proof. Combine Theorems 5.1.8, 5.2.4 and 5.2.5. $\square$

Remark 5.2.7. We would like to point out that our main result has a partial converse. However, proving this would amount to repeating verbatim the proof of [62, Theorem 4.4] and checking that all the involved maps are equivariant. For this reason, it is not a contribution of the author and should not be contained in this thesis.

Let us just state here that, as a byproduct of the aforementioned result, one has the following partial converse of Theorem 5.2.4. If $\alpha : G \curvearrowright A$ is an action on a separable C^*-algebra, $\gamma : G \curvearrowright \mathcal{D}$ a strongly self-absorbing action, and the algebraic action induced by $\alpha^s := \mathrm{id}_\mathcal{K} \otimes \alpha$ on $\mathcal{Q}(\mathcal{K} \otimes A)$ is γ-saturated, then α is γ-stable.

Chapter 6

Conclusion

The main problem studied in this thesis was the classification of group actions on non-simple C*-algebras, focusing on the $\mathcal{O}_2$-stable settings, where dynamics induced on ideals serves as the classification invariant.

In the first part, we developed a classification theorem — up to cocycle conjugacy — for amenable, isometrically shift-absorbing, equivariantly $\mathcal{O}_2$-stable actions of second-countable, locally compact groups on separable, nuclear C*-algebras that are either stable or unital; see Theorem 3.5.5. The topological dynamical system induced on the primitive ideal space, or equivariant ideal lattice functor into the category of abstract Cuntz semigroups, serve as the classification invariant. The theorem could already be obtained with known techniques when the acting group is $\mathbb{Z}$, $\mathbb{R}$ or compact, but even straightforward follow-up cases, such as $\mathbb{Z}^2$, are proved here for the first time. The main theorem should be considered as the full-fledged dynamical analogue of Gabe and Kirchberg's $\mathcal{O}_2$-stable classification [64, 116], and fits into the broader effort to bridge the Elliott classification program for C*-algebras and its dynamical counterpart. The methodology employed to obtain this result was inspired by non-dynamical setting [64] and by the groundbreaking dynamical Kirchberg–Phillips theorem of Gabe and Szabó [67, 68]. The key innovation of the latter, which is shared with the present thesis, is the decisive shift towards an abstract classification approach, made possible by the categorical framework introduced by Szabó in [192]. This marks a critical change of perspective from earlier dynamical classification results, which heavily relied on the specific structure of the acting group; see [100] for a comprehensive overview. More precisely, this abstract approach unfolds in two stages. First, one establishes a classification of appropriate morphisms between dynamical systems up to approximate unitary equivalence, encompassing both existence and uniqueness results. Subsequently, by employing a quintessential tool in the classification program, Elliott's two-sided intertwining argument [52], one derives a classification of the C*-dynamical systems themselves. The existence result underpinning our classification, Theorem 3.4.17, represents arguably the main technical innovation of this work, as it requires simultaneously controlling both the dynamics and the ideal structure to achieve the final outcome. Moreover, this can be thought of as an ideal-related, dynamical version of the celebrated $\mathcal{O}_2$-embedding theorem [119].

The second part of this work tackled the range-of-invariant problem corresponding to the classification theorem established in the previous part. Specifically, the question at hand reduces to whether every continuous action on the primitive ideal space of a separable, nuclear C*-algebra arises as the induced action of a C*-dynamical system. In Theorem 4.2.5, we affirmatively answer this question. This result partially overlaps with recent work by Kirchberg and Phillips [120], who solve a similar problem from the perspective of topological dynamics, i.e., simultaneously realising both the space as a primitive ideal space and the dynamics as a C*-action. However, their approach imposes additional conditions, including primeness of the resulting C*-algebra, which our result shows to be unnecessary, and relies heavily on (unpublished) work by Harnisch and Kirchberg [91], contrarily to our work. One of the advantages of the approach presented here is that we can prove Harnisch–Kirchberg's ideal correspondence [91], Theorem 4.1.23, without resorting to double dual techniques from [91], which makes the overall strategy more accessible. Moreover, our construction relies only on crossed products by $\mathbb{Z}$ where previous work invoked Cuntz–Pimsner algebras. The main application of the results derived in this part of the thesis then ensures that every admissible invariant in the classification theorem is realised by an actual C*-dynamical system.

In the last part of the thesis, we have developed a dynamical analogue of recent results concerning the structure of corona algebras and their connection to absorption phenomena for strongly self-absorbing C*-algebras. This work partially overlaps with Xiuyuan Li's Master's thesis at KU Leuven [133], which was supervised by Gábor Szabó and mentored by the author, and addressed a special case of Theorem 5.2.4. The other main technical tool of this part of the thesis, Theorem 5.1.8, and the full version of Theorem 5.2.4 was subsequently developed by the author in collaboration with Szabó.

Motivated by earlier work of Farah [60] and Farah–Szabó [62] in the non-equivariant setting, we investigated to what extent absorption properties for group actions can also be detected at the level of corona algebras. Our main result, Theorem 5.2.4, shows that if an action on a non-unital, separable C*-algebra is γ-absorbing for a strongly self-absorbing, unitarily regular action γ, then the induced action on the corona algebra is γ-saturated. Under additional assumptions, we also showed the converse, thereby characterizing absorption in terms of saturation properties of the corona. The key technical innovation underpinning this result is the dynamical folding property, which we establish in Theorem 5.1.8 for all second-countable, locally compact group actions on σ-unital C*-algebras. This property enables one to lift equivariant embeddings from path algebras into corona algebras in a way that captures both the algebraic and dynamical structure of the system. This served as a crucial ingredient in overcoming cocycle obstructions that arise when attempting to reduce approximate absorption to genuine absorption in the corona setting, particularly in the presence of non-compact group actions. The dynamical folding property not only allows us to conclude our main absorption result, but also yields a number of interesting structural consequences. For instance, it ensures that all equivariant unital embeddings of a strongly self-absorbing action into a corona algebra are G-unitarily equivalent, demonstrating a unique embedding property analogous to known results in the non-equivariant case [62]. This confirms that, in the equivariant context, corona algebras continue to retain a high degree of internal symmetry when equipped with sufficiently regular dynamics.

Together, these results contribute to the ongoing effort to extend the reach of C*-algebra theory to the dynamical setting. They also offer tools and methods that may prove useful in subsequent research questions. Most notably, any viable approach to establishing a dynamical analogue of Gabe and Kirchberg's $\mathcal{O}_\infty$-stable classification theorem [66, 116] will most certainly require the use of the ideal-related, dynamical $\mathcal{O}_2$-embedding theorem — namely, our existence result, Theorem 3.4.17 — mimicking its crucial role in the non-dynamical setting.

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I did not use generative AI assistance tools during the research/writing process of my thesis.

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