Symbolic Register Automata

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Abstract. Symbolic Finite Automata and Register Automata are two orthogonal extensions of finite automata motivated by real-world problems where data may have unbounded domains. These automata address a demand for a model over large or infinite alphabets, respectively. Both automata models have interesting applications and have been successful in their own right. In this paper, we introduce Symbolic Register Automata, a new powerful model that combines features from both symbolic and register automata, with a view on applications that were previously out of reach. We study their properties and provide algorithms for emptiness, inclusion and equivalence checking, together with experimental results.

1 Introduction

Finite automata are a ubiquitous formalism that is simple enough to model many real-life systems and phenomena. They enjoy a large variety of theoretical properties that in turn play a role in practical applications. For example, finite automata are closed under Boolean operations, and have decidable emptiness and equivalence checking procedures. Unfortunately, finite automata have a fundamental limitation: they can only operate over finite (and typically small) alphabets. Two *orthogonal* families of automata models have been proposed to overcome this: *symbolic automata* and *register automata*. In this paper, we show that these two models can be combined yielding a new powerful model that can cover interesting applications previously out of reach for existing models.

Symbolic finite automata (SFA) allow transitions to carry predicates over rich first-order alphabet theories, such as linear arithmetic, and therefore extend classic automata to operate over infinite alphabets [8]. For example, an SFAs can define the language of all lists of integers in which the first and last elements are positive integer numbers. Despite their increased expressiveness, SFAs enjoy the same closure and decidability properties of finite automata—e.g., closure under Boolean operations and decidable equivalence and emptiness.

Register automata (RA) support infinite alphabets by allowing input characters to be stored in registers during the computation and to be compared against existing values that are already stored in the registers [13]. For example, an RA can define the language of all lists of integers in which all numbers appearing in

even positions are the same. RAs do not have some of the properties of finite automata (e.g., they cannot be determinized), but they still enjoy many useful properties that have made them a popular model in static analysis, software verification, and program monitoring [11].

In this paper, we combine the best features of these two models—first order alphabet theories and registers—into a new powerful model, *symbolic register automata* (SRA). We show that this extension enjoys the same properties as register automata, and we provide new applications enabled by this new model. For example, an SRA can define the language of all lists of integers in which the first and last elements are positive rational numbers and all numbers appearing in even positions are the same. This language is not recognizable by either an SFA nor by an RA. While other attempts at combining symbolic automata and registers have resulted in undecidable models with limited closure properties [7], we show that SRAs enjoy the same closure and decidability properties of (non-symbolic) register automata.

Finally, we propose a new application enabled by SRAs and implement our model in an open-source automata library. We show that SRAs are the ideal model for representing regular expressions with capturing groups and backreferences (e.g., `(\d)[a-z]*\1`). We compile a set of benchmarks from existing regular expressions with back-references and show that SRAs are an effective model for such expression and existing models such as SFAs and RAs are not.

In summary, our contributions are:

- Symbolic Register Automata (SRA): a new automaton model that can handle complex alphabet theories while allowing symbols at arbitrary positions in the input string to be compared using equality (§ 3).
- A thorough study of the properties of SRAs. We show that SRAs are closed under intersection, union and (deterministic) complementation, and provide algorithms for emptiness and forward (bi)simulation (§ 4).
- A study of the effectiveness of our SRA implementation on handling regular expressions with back-references (§ 5).

2 Motivating example

In this section, we illustrate the capabilities of symbolic register automata using a simple example. Consider the regular expression r_p shown in Figure 1b. This expression, given a sequence of product descriptions, checks whether the products have the same code and lot number. The reader might not be familiar with some of the unusual syntax of this expression. In particular, r_p uses two back-references `\1` and `\2`. The semantics of this construct is that the string matched by the regular expression for `\1` (resp. `\2`) should be exactly the string that matched the subregular expression r appearing between the first (resp. second) two parenthesis, in this case `(.{3})` (resp. `(.)`). Back-references allow regular expressions to check whether the encountered text is the same or is different from a string/character that appeared earlier in the input (see Figures 1b and 1c for examples of positive and negative matches).
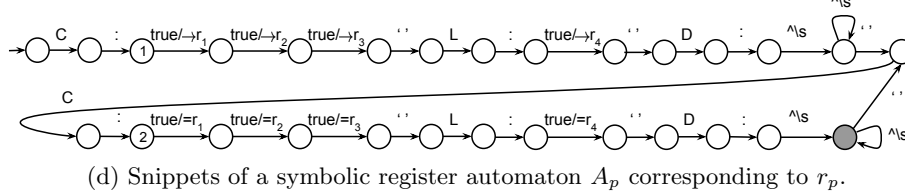
```
C:(.{3}) L:(.) D:[^\s]+( C:\1 L:\2 D:[^\s]+)+
```
(a) Regular expression r_p (with back-reference).

```
C:X4a L:4 D:bottle C:X4a L:4 D:jar      C:X4a L:4 D:bottle C:X5a L:4 D:jar
```
(b) Example text matched by r_p.   (c) Example text *not* matched by r_p.

(d) Snippets of a symbolic register automaton A_p corresponding to r_p.

Fig. 1: Regular expression for matching pairs of products with same code and lot number—i.e., the characters of C and L are the same in the two products.

Representing this complex regular expression using an automaton model requires addressing several challenges. The expression r_p:

1. operates over large input alphabets consisting of upwards of 2^{16} characters;
2. uses complex character classes (e.g., \s) to describe different sets of characters in the input;
3. adopts back-references to detect repeated strings in the input.

Existing automata models do not address one or more of these challenges. Finite automata require one transition for each character in the input alphabet and blow-up when representing large alphabets. Symbolic finite automata (SFA) allow transitions to carry predicates over rich structured first-order alphabet theories and can describe, for example, character classes [8]. However, SFAs cannot directly check whether a character or a string is repeated in the input. An SFA for describing the regular expression r_p would have to store the characters after C: directly in the states to later check whether they match the ones of the second product. Hence, the smallest SFA for this example would require billions of states! Register automata (RA) and their variants can store characters in registers during the computation and compare characters against values already stored in the registers [13]. Hence, RAs can check whether the two products have the same code. However, RAs only operate over unstructured infinite alphabets and cannot check, for example, that a character belongs to a given class.

The model we propose in this paper, *symbolic register automata* (SRA), combines the best features of SFAs and RAs—first-order alphabet theories and registers—and can address all the three aforementioned challenges. Figure 1d shows a snippet of a symbolic register automaton A_p corresponding to r_p. Each transition in A_p is labeled with a predicate that describes what characters can trigger the transition. For example, ^\s denotes that the transition can be triggered by any non-space character, L denotes that the transition can be triggered by the character L, and true denotes that the transition can be triggered by any character. Transitions of the form $\varphi/\to r_i$ denote that, if a character x satisfies the predicate φ, the character is then stored in the register r_i. For example, the transition out of state 1 reads any character and stores it in register r_1. Finally,

transitions of the form $\varphi/= r_i$ are triggered if a character x satisfies the predicate φ and x is the same character as the one stored in r_i. For example, the transition out of state 2 can only be triggered by the same character that was stored in r_1 when reading the transition out state 1—i.e., the first characters in the product codes should be the same.

SRAs are a natural model for describing regular expressions like r_p. The SRA A_p has fewer than 50 states (vs. more than 100 billion for SFAs) and can, for example, be used to check whether an input string matches the given regular expression (e.g., monitoring). More interestingly, in this paper we study the closure and decidability properties of SRAs and provide an implementation for our model. For example, consider the following regular expression r_{pC} that only checks whether the product codes are the same, but not the lot numbers:

```
C:(.{3}) L:. D:[^\s]+( C:\1 L:. D:[^\s]+)+
```

The set of strings accepted by r_{pC} is a superset of the set of strings accepted by r_p. In this paper, we present simulation and bisimulation algorithms that can check this property. Our implementation can show that r_p subsumes r_{pC} in 25 seconds and we could not find other tools that can prove the same property.

3 Symbolic Register Automata

In this section we introduce some preliminary notions, we define symbolic register automata and a variant that will be useful in proving decidability properties.

Preliminaries. An *effective Boolean algebra* $\mathcal{A}$ is a tuple $(\mathcal{D}, \Psi, [\![\_]\!], \bot, \top, \wedge, \vee, \neg)$, where: $\mathcal{D}$ is a set of domain elements; Ψ is a set of predicates closed under the Boolean connectives and $\bot, \top \in \Psi$. The denotation function $[\![\_]\!]: \; : \Psi \to 2^{\mathcal{D}}$ is such that $[\![\bot]\!] = \emptyset$ and $[\![\top]\!] = \mathcal{D}$, for all $\varphi, \psi \in \Psi$, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$, and $[\![\neg\varphi]\!] = \mathcal{D} \setminus [\![\varphi]\!]$. For $\varphi \in \Psi$, we write $\mathsf{isSat}(\varphi)$ whenever $[\![\varphi]\!] \neq \emptyset$ and say that φ is *satisfiable*. $\mathcal{A}$ is *decidable* if isSat is decidable. For each $a \in \mathcal{D}$, we assume predicates $\mathsf{atom}(a)$ such that $[\![\mathsf{atom}(a)]\!] = \{a\}$.

Example 1. The theory of integer linear arithmetic forms an effective BA where Ψ contains formulas $\varphi(x)$ in the theory with one fixed integer variables. For example, $\mathsf{div}_\mathsf{k} := (x \mod k) = 0$ denotes the set of all numbers divisible by k.

Notation. Given a set S, we write $\mathcal{P}(S)$ for its powerset. Given a function $f: A \to B$, we write $f[a \mapsto b]$ for the function such that $f[a \mapsto b](a) = b$ and $f[a \mapsto b](x) = f(x)$, for $x \neq a$. The *pre-image* of f is the function $f^{-1}: \mathcal{P}(A) \to \mathcal{P}(B)$ given by $f^{-1}(S) = \{b \mid \exists a \in S: b = f(a)\}$; for readability, we will write $f^{-1}(x)$ when $S = \{x\}$. Given a relation $\mathcal{R} \subseteq A \times B$, we write $a\mathcal{R}b$ for $(a, b) \in \mathcal{R}$.

Model definition. Symbolic register automata have transitions of the form:

$$p \xrightarrow{\varphi/E,I,U} q$$

where p and q are states and E, I, U are subsets of a fixed set of registers R. The intended interpretation of the above transition is: an input character a can be read in state q if (i) $a \in [\![\varphi]\!]$, (ii) the content of all the registers in E is *equal* to a, and (iii) the content of all the registers in I is *different* from a. If the transition succeeds then a is stored into all the registers U and the automaton moves to q.

Example 2. The transition labels in Figure 1d have been conveniently simplified to ease intuition. These labels correspond to full SRA labels as follows:

$$\varphi/{\to}r \implies \varphi/\emptyset, \emptyset, \{r\} \qquad \varphi/{=}r \implies \varphi/\{r\}, \emptyset, \emptyset \qquad \varphi \implies \varphi/\emptyset, \emptyset, \emptyset \ .$$

Given a set of registers R, the transitions of an SRA have labels over the following set: $L_R = \Psi \times \{(E, I, U) \in \mathcal{P}(R) \times \mathcal{P}(R) \times \mathcal{P}(R) \mid E \cap I \neq \emptyset\}$. The condition $E \cap I \neq \emptyset$ guarantees that register constraints are always satisfiable.

Definition 1 (Symbolic Register Automaton). *A symbolic register automaton (SRA) is a 5-tuple $(R, Q, q_0, v_0, F, \Delta)$, where R is a finite set of registers, Q is a finite set of states, $q_0 \in Q$ is the initial state, $v_0 \colon R \to D_A \cup \{\sharp\}$ is the initial register assignment (if $v_0(r) = \sharp$, the register r is considered empty), $F \subseteq Q$ is a finite set of final states, and $\Delta \subseteq Q \times L_R \times Q$ is the transition relation. Transitions $(p, (\varphi, \ell), q) \in \Delta$ will be written as $p \xrightarrow{\varphi/\ell} q$.*

An SRA can be seen as a finite description of a (possibly infinite) labeled transition system (LTS), where states have been assigned concrete register values, and transitions read a single symbol from the potentially infinite alphabet. This so-called *configuration LTS* will be used in defining the semantics of SRAs.

Definition 2 (Configuration LTS). *Given an SRA $\mathcal{S}$, the configuration LTS $\mathsf{CLTS}(\mathcal{S})$ is defined as follows. A configuration is a pair (p, v) where $p \in Q$ is a state in $\mathcal{S}$ and a $v \colon R \to \mathcal{D} \cup \{\sharp\}$ is register assignment; (q_0, v_0) is called the initial configuration; every (q, v) such that $q \in F$ is a final configuration. The set of transitions between configurations is defined as follows:*

$$\frac{p \xrightarrow{\varphi/E, I, U} q \in \Delta \qquad v^{-1}(a) \propto E, I}{(p, v) \xrightarrow{a} (q, v[U \mapsto a]) \in \mathsf{CLTS}(\mathcal{S})}$$

where $S \propto E, I$ whenever $E \subseteq S$ and $S \cap I = \emptyset$.

A *run* of the SRA $\mathcal{S}$ is a sequence of transitions in $\mathsf{CLTS}(\mathcal{S})$ starting from the initial configuration. A configuration is *reachable* whenever there is a run ending up in that configuration. The *language* of an SRA $\mathcal{S}$ is defined as

$$\mathscr{L}(\mathcal{S}) := \{a_1 \ldots a_n \in \mathcal{D}^n \mid \exists (q_0, v_0) \xrightarrow{a_1} \ldots \xrightarrow{a_n} (q_n, v_n) \in \mathsf{CLTS}(\mathcal{S}), q_n \in F\}$$

An SRA $\mathcal{S}$ is *deterministic* if its configuration LTS is; namely, for every word $w \in \mathcal{D}^\star$ there is at most one run in $\mathsf{CLTS}(\mathcal{S})$ spelling w. Determinism is important for some application contexts, e.g., for runtime monitoring. Since SRAs subsume RA, nondeterministic SRAs are strictly more expressive than deterministic ones, and language equivalence is undecidable for nondeterministic SRAs [21].

We now introduce the notions of *simulation* and *bisimulation* for SRAs, which capture whether one SRA behaves "at least as" or "exactly as" another one.

Definition 3 ((Bi)simulation for SRAs). *A simulation $\mathcal{R}$ on SRAs $\mathcal{S}_1$ and $\mathcal{S}_2$ is a binary relation $\mathcal{R}$ on configurations such that $(p_1, v_1)\mathcal{R}(p_2, v_2)$ implies:*

- *if $p_1 \in F_1$ then $p_2 \in F_2$;*
- *for each transition $(p_1, v_1) \xrightarrow{a} (q_1, w_1)$ in $\mathsf{CLTS}(\mathcal{S}_1)$, there exists a transition $(p_2, v_2) \xrightarrow{a} (q_2, w_2)$ in $\mathsf{CLTS}(\mathcal{S}_2)$ such that $(q_1, w_1)\mathcal{R}(q_2, w_2)$.*

A simulation $\mathcal{R}$ is a bisimulation *if $\mathcal{R}^{-1}$ is a also a simulation. We write $\mathcal{S}_1 \prec \mathcal{S}_2$ (resp. $\mathcal{S}_1 \sim \mathcal{S}_2$) whenever there is a simulation (resp. bisimulation) $\mathcal{R}$ such that $(q_{01}, v_{01})\mathcal{R}(q_{02}, v_{02})$, where (q_{0i}, v_{0i}) is the initial configuration of $\mathcal{S}_i$, for $i = 1, 2$.*

We say that an SRA is *complete* whenever for every configuration (p, v) and $a \in \mathcal{D}$ there is a transition $(p, v) \xrightarrow{a} (q, w)$ in $\mathsf{CLTS}(\mathcal{S})$. The following results connect similarity and language inclusion.

Proposition 1. *If $\mathcal{S}_1 \prec \mathcal{S}_2$ then $\mathscr{L}(\mathcal{S}_1) \subseteq \mathscr{L}(\mathcal{S}_2)$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are deterministic and complete, then the other direction also holds.*

Remark 1. RAs and SFAs can be encoded as SRAs on the same state-space:

- An RA is encoded as an SRA with all transition guards $\top$;
- an SFA can be encoded as an SRA with $R = \emptyset$, with each SFA transition $p \xrightarrow{\varphi} q$ encoded as $p \xrightarrow{\varphi/\emptyset, \emptyset, \emptyset} q$. Note that the absence of registers implies that the CLTS always has finitely many configurations.

SRAs are *strictly more expressive* than both RAs and SFAs. For instance, the language $\{n_0 n_1 \ldots n_k \mid n_0 = n_k, \mathsf{even}(n_i), n_i \in \mathbb{Z}, i = 1, \ldots, k\}$ of finite sequences of even integers where the first and last one coincide, can be recognised by an SRA, but not by an RA or by an SFA.

Single-valued variant. In some of the decision procedures presented in the next sections, it is convenient to restrict register assignments to ones which are *injective* on non-empty registers, namely are functions $v \colon R \to \mathcal{D} \cup \{\sharp\}$ such that $v(r) = v(s)$ and $v(r) \neq \sharp$ implies $r = s$. This is for instance the approach taken for RAs in the seminal papers [21, 13], where it is shown that this restriction does not affect expressivity. This is the case for SRAs as well. We say that an SRA is *single-assignment* if its initial assignment v_0 is injective on non-empty registers. For single-assignment SRAs, we only allow two kinds of transitions:

Read transitions: $p \xrightarrow{\varphi/r^=} q$ triggers when $a \in \llbracket \varphi \rrbracket$ and a is stored in r.

Fresh transitions: $p \xrightarrow{\varphi/r^{\bullet}} q$ triggers when the input $a \in \llbracket \varphi \rrbracket$ and a is *fresh*, i.e., is not stored in any register.

SRAs and their single-valued variants have the same expressive power. Translating single-valued SRAs to ordinary ones is straightforward:

$$p \xrightarrow{\varphi/r^=} q \implies p \xrightarrow{\varphi/\{r\}, \emptyset, \emptyset} q \qquad p \xrightarrow{\varphi/r^{\bullet}} q \implies p \xrightarrow{\varphi/\emptyset, R, \{r\}} q$$

The opposite translation requires a state-space blow up, because we need to encode register equalities in the states.

Theorem 1. *Given an SRA $\mathcal{S}$ with n states and r registers, there is a single-valued SRA $\mathcal{S}'$ with $\mathcal{O}(nr^r)$ states and $r+1$ registers such that $\mathcal{S} \sim \mathcal{S}'$. Moreover, the translation preservers determinism.*

4 Closure and Decidability properties

In this section we first study closure under Boolean operations of SRAs and then decision procedures for single-valued SRAs.

Closure properties. SRAs are closed under intersection and union. Intersection is given by a standard product construction whereas union is obtained by adding a new initial state that mimics the initial states of both automata.

Proposition 2 (Closure under intersection and union). *Given SRAs $\mathcal{S}_1$ and $\mathcal{S}_2$, there are SRAs $\mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_2$ such that $\mathscr{L}(\mathcal{S}_1 \cap \mathcal{S}_2) = \mathscr{L}(\mathcal{S}_1) \cap \mathscr{L}(\mathcal{S}_2)$ and $\mathscr{L}(\mathcal{S}_1 \cup \mathcal{S}_2) = \mathscr{L}(\mathcal{S}_1) \cup \mathscr{L}(\mathcal{S}_2)$.*

SRAs in general are not closed under complementation, because RAs are not. However, we still have closure under complementation for a subclass of SRAs.
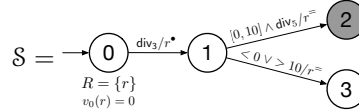
Proposition 3. *Let $\mathcal{S}$ be a complete and deterministic SRA, and let $\overline{\mathcal{S}}$ be the SRA defined as $\mathcal{S}$, except that its final states are $Q \setminus F$. Then $\mathscr{L}(\overline{\mathcal{S}}) = \mathcal{D}^{\star} \setminus \mathscr{L}(\mathcal{S})$.*

It is worth noting that given a deterministic SRA we can define its *completion* by adding transitions so that every value $a \in \mathcal{D}$ can be read from any state.

Decidability properties. We consider single-valued SRAs (hereafter we omit "single-valued" for simplicity) and provide algorithms for checking determinism and emptiness for an SRA, and (bi)similarity of two SRAs. Our algorithms leverage *symbolic* techniques that use the finite syntax of SRAs to indirectly operate over the underlying configuration LTS, which can be infinite.

While our techniques are inspired by analogous ones for non-symbolic RAs, SRAs present an additional challenge: they can have arbitrary predicates on transitions. As a consequence, the values that each transition can read, and thus which configurations it can reach, depend on the history of past transitions and their predicates. This problem emerges when checking reachability and similarity, because a transition may be *disabled* by particular register values, and so lead to unsound conclusions, a problem that does not exist in register automata.

Example 3. Consider the SRA below, defined over the BA of integers.

$$\mathcal{S} = \quad \boxed{0} \xrightarrow{\mathsf{div}_3/r^\bullet} \boxed{1} \begin{array}{c} \xrightarrow{[0,10] \wedge \mathsf{div}_5/r^=} \boxed{2} \\ \xrightarrow{< 0 \vee > 10/r^\approx} \boxed{3} \end{array}$$
$$R = \{r\}$$
$$v_0(r) = 0$$

All predicates on transitions are satisfiable, yet $\mathscr{L}(\mathcal{S}) = \emptyset$. To go from 0 to 1, $\mathcal{S}$ must read a value n such that $\mathsf{div}_3(n)$ and $n \neq 0$ and then n is stored into r. The transition from 1 to 2 can only happen if the content of r also satisfies $\mathsf{div}_5(n)$ and $n \in [0, 10]$. However, there is no n satisfying $\mathsf{div}_3(n) \wedge n \neq 0 \wedge \mathsf{div}_5(n) \wedge n \in [0, 10]$, hence the transition from 1 to 2 never happens.

To handle the complexity caused by predicates, we introduce a way of *reducing* an SRA to an equivalent one that *stores additional information about input predicates.* We first introduce some notation and terminology.

A register abstraction θ for $\mathcal{S}$, used to "keep track" of the domain of registers, is a family of predicates indexed by the registers R of $\mathcal{S}$. Given a register assignment v, we write $v \models \theta$ whenever $v(r) \in [\![\theta_r]\!]$ for $v(r) \neq \sharp$, and $\theta_r = \bot$ otherwise. Hereafter we shall only consider "meaningful" register abstractions, for which there is at least one assignment v such that $v \models \theta$.

With the contextual information about register domains given by θ, we say that a transition $p \xrightarrow{\varphi/\ell} q \in \Delta$ is *enabled by* θ whenever it has at least an instance $(p, v) \xrightarrow{a} (q, w)$ in $\mathsf{CLTS}(\mathcal{S})$, for all $v \models \theta$. Enabled transitions are important when reasoning about reachability and similarity.

Checking whether a transition has at least one realizable instance in the CLTS is difficult in practice, especially when $\ell = r^\bullet$, because it amounts to checking whether $[\![\varphi]\!] \setminus \mathsf{img}(v) \neq \emptyset$, for all injective $v \models \theta$.

To make the check for enabledness practical we will use minterms. For a set of predicates Φ, a *minterm* is a minimal satisfiable Boolean combination of all predicates that occur in Φ. Minterms are the analogue atoms in a complete atomic Boolean algebra. E.g. the set of predicates $\Phi = \{x > 2, x < 5\}$ over the theory of linear integer arithmetic has minterms $\mathsf{mint}(\Phi) = \{x > 2 \wedge x < 5, \neg x > 2 \wedge x < 5, \ x > 2 \wedge \neg x < 5\}$. Given $\psi \in \mathsf{mint}(\Phi)$ and $\varphi \in \Phi$, we will write $\varphi \sqsubset \psi$ whenever ϕ appears non-negated in φ, for instance $(x > 2) \sqsubset (x > 2 \wedge \neg x < 5)$.

Enabledness check is now simplified as formalized in the following Lemma.

Lemma 1. *Let θ be a register abstraction such that θ_r is a minterm, for all $r \in R$. If φ is a minterm, then $p \xrightarrow{\varphi/\ell} q$ is enabled by θ if and only if:*
(1) if $\ell = r^=$, then $\varphi = \theta_r$; *(2) if $\ell = r^\bullet$, then $|[\![\varphi]\!]| > \mathscr{E}(\theta, \varphi)$,*
where $\mathscr{E}(\theta, \varphi) = |\{r \in R \mid \theta_r = \varphi\}|$ is the # of registers with values from $[\![\varphi]\!]$.

The intuition behind 1 is that, for the transition to read a symbol stored in r and satisfying φ, the symbol must also satisfy θ_r, the range of r. Because φ and θ_r are minterms, this can only happen when $\varphi = \theta_r$. For 2, the enabling condition $[\![\varphi]\!] \setminus \mathsf{img}(v) \neq \emptyset$, for all injective $v \models \theta$, holds if and only if registers storing values from φ are fewer then the cardinality of φ. Those registers are exactly those $r \in R$ such that $\theta_r = \varphi$, because no other register domain can overlap with φ, by the minterm property. We note that both conditions can be effectively checked. The first one is a simple predicate-equivalence check, while the second one amounts to checking whether φ holds for at least a certain number k of distinct elements. This can be achieved by checking satisfiability of $\varphi \wedge \neg \mathsf{atom}(a_1) \wedge \cdots \wedge \neg \mathsf{atom}(a_{k-1})$, for $a_1, \ldots, a_{k-1}$ distinct elements of $[\![\varphi]\!]$.

Remark 2. The reader may wonder if the restriction to single-valued SRAs is necessary for checking whether transitions are enabled. In the case of a generic SRA, this check would require an additional step: for each state p, we would have to keep track of all possible equations among registers. In fact, register equalities determine whether (i) register constraints of an outgoing transition are satisfiable; (ii) how many elements of the guard we need for the transition to happen, analogously to condition 2 of Lemma 1. Generating such equations is the key idea behind Theorem 1, and corresponds precisely to turning the SRA into a single-valued one.

Given any SRA, we can use the notion of register abstraction to build an equivalent *normalized* SRA, where (i) states keep track of how the domains of registers change along transitions, (ii) transitions are obtained by breaking the one of the original SRA into minterms and discarding the ones that are disabled according to Lemma 1. In the following we write $\mathsf{mint}(\mathcal{S})$ for the minterms for the set of predicates $\{\varphi \mid p \xrightarrow{\varphi/\ell} q \in \Delta\} \cup \{\mathsf{atom}(v_0(r)) \mid v_0(r) \in \mathcal{D}, r \in R\}$. Observe that atomic predicates are always minterms, hence we will use them to define the initial register abstraction.

Definition 4 (Normalized SRA). *Given an SRA $\mathcal{S}$, its normalisation $\mathsf{N}(\mathcal{S})$ is the SRA $(R, \mathsf{N}(Q), \mathsf{N}(q_0), v_0, \mathsf{N}(F), \mathsf{N}(\Delta))$ where:*

- $\mathsf{N}(Q) = \{\theta \mid \theta \text{ is a register abstraction over } \mathsf{mint}(\mathcal{S}) \cup \{\bot\}\} \times Q$;
- $\mathsf{N}(q_0) = \theta_0 \rhd q_0$, *where* $(\theta_0)_r = \mathsf{atom}(v_0(r))$ *if* $v_0(r) \in \mathcal{D}$, *and* $(\theta_0)_r = \bot$ *if* $v_0(r) = \sharp$;
- $\mathsf{N}(F) = \{\theta \rhd p \in \mathsf{N}(Q) \mid p \in F\}$
- $\mathsf{N}(\Delta) = \{\theta \rhd p \xrightarrow{\theta_r/r^=} \theta \rhd q \mid p \xrightarrow{\varphi/r^=} q \in \Delta, \varphi \sqsubseteq \theta_r\} \cup$

  $\{\theta \rhd p \xrightarrow{\psi/r^\bullet} \theta[r \mapsto \psi] \rhd q \mid p \xrightarrow{\varphi/r^\bullet} q \in \Delta, \varphi \sqsubseteq \psi, \|[\![\psi]\!]\| > \mathscr{E}(\theta, \psi)\}$

The resulting $\mathsf{N}(\mathcal{S})$ enjoys the desired property: each transition from $\theta \rhd p$ is enabled by θ, by construction. $\mathsf{N}(\mathcal{S})$ is always *finite*. In fact, suppose $\mathcal{S}$ has n states and m transitions. Then it has at most m predicates, and $|\mathsf{mint}(\mathcal{S})|$ is $\mathcal{O}(2^m)$, because each minterm is a boolean combination of all the m predicates. Therefore $\mathsf{N}(\mathcal{S})$ has $\mathcal{O}(n2^m)$ states and $\mathcal{O}(m2^m)$ transitions.

Example 4. We now show the normalized version of Example 3. The first step is computing the set $\mathsf{mint}(\mathcal{S})$ of minterms for $\mathcal{S}$, i.e., the satisfiable Boolean combinations of $\{\mathsf{atom}(0), \mathsf{div}_3, [0, 10] \wedge \mathsf{div}_5, < 0 \vee > 10\}$. For simplicity, we represent minterms as bitvectors where a 0 component means that the corresponding predicate is negated, e.g., $[1, 1, 1, 0]$ stands for the minterm $\mathsf{atom}(0) \wedge ([0, 10] \wedge \mathsf{div}_3) \wedge \mathsf{div}_5 \wedge \neg(< 0 \vee > 10)$. Minterms and the resulting SRA $\mathsf{N}(\mathcal{S})$ are shown below.
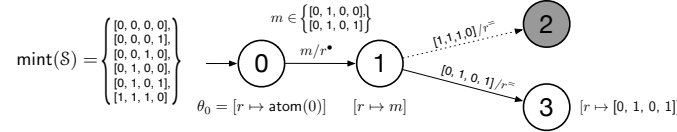
On each transition we show how it is broken down to minterms, and for each state we show the register abstraction (note that state 1 becomes two states in $\mathsf{N}(\mathcal{S})$). The transition from 1 to 2 is *not* part of $\mathsf{N}(\mathcal{S})$ – this is why it is dotted. In fact, in every register abstraction $[r \mapsto m]$ reachable at state 1, the component for the transition guard $[0, 10] \wedge \mathsf{div}_5$ in the minterm m (3rd component) is 0, i.e., $([0, 10] \wedge \mathsf{div}_5) \not\sqsubseteq m$. Intuitively, this means that r will never be assigned a value that satisfies $[0, 10] \wedge \mathsf{div}_5$. As a consequence, the construction of Definition 4 will not add a transition from 1 to 2.

Finally, we show that the normalized SRA behaves exactly as the original one.

Proposition 4. $(p, v) \sim (\theta \rhd p, v)$, *for all $p \in Q$ and $v \models \theta$. Hence, $\mathcal{S} \sim \mathsf{N}(\mathcal{S})$.*

Emptiness and Determinism. The transitions of $N(\mathcal{S})$ are always enabled by construction, therefore every path in $N(\mathcal{S})$ always corresponds to a run in $\mathsf{CLTS}(N(\mathcal{S}))$.

Lemma 2. *The state $\theta \rhd p$ is reachable in $N(\mathcal{S})$ if and only if there is a reachable configuration $(\theta \rhd p, v)$ in $\mathsf{CLTS}(N(\mathcal{S}))$ such that $v \models \theta$. Moreover, if $(\theta \rhd p, v)$ is reachable, then all configurations $(\theta \rhd p, w)$ such that $w \models \theta$ are reachable.*

Therefore, using Proposition 4, we can reduce the reachability and emptiness problems of $\mathcal{S}$ to that of $N(\mathcal{S})$.

Theorem 2 (Emptiness). *There is an algorithm to decide reachability of any configuration of $\mathcal{S}$, hence whether $\mathscr{L}(\mathcal{S}) = \emptyset$.*

Proof. Let (p, v) a configuration of $\mathcal{S}$. To decide whether it is reachable in $\mathsf{CLTS}(\mathcal{S})$, we can perform a visit of $N(\mathcal{S})$ from its initial state, stopping when a state $\theta \rhd p$ such that $v \models \theta$ is reached. If we are just looking for a final state, we can stop at any state such that $p \in F$. In fact, by Lemma 2, there is a run in $\mathsf{CLTS}(\mathcal{S})$ ending in (p, v) if and only if there is a run in $\mathsf{CLTS}(N(\mathcal{S}))$ ending in $(\theta \rhd p, v)$ such that $v \models \theta$. By Proposition 4, the latter holds if and only if there is a path in $N(\mathcal{S})$ ending in $\theta \rhd p$. This algorithm has complexity $\mathcal{O}((n + m)2^m)$, i.e., the complexity of a standard visit of $N(\mathcal{S})$. $\qquad\square$

Now that we characterized what transitions are reachable, we define what it means for a normalized SRA to be deterministic and we show that determinism is preserved by the translation from SRA.

Proposition 5 (Determinism). $N(\mathcal{S})$ *is* deterministic *if and only if for all reachable transitions $p \xrightarrow{\varphi_1/\ell_1} q_1, p \xrightarrow{\varphi_2/\ell_2} q_2 \in N(\Delta)$ the following holds: if (1) $\ell_1 = \ell_2$ and $q_1 \neq q_2$, or; (2) $\ell_1 = r^{\bullet}$, $\ell_2 = s^{\bullet}$, and $r \neq s$; then $\varphi_1 \neq \varphi_2$.*

One can check determinism of an SRA by looking at its normalized version.

Proposition 6. $\mathcal{S}$ *is deterministic if and only if* $N(\mathcal{S})$ *is deterministic.*

Similarity and bisimilarity. We now introduce a symbolic technique to decide similarity and bisimilarity of SRAs. The basic idea is similar to *symbolic (bi)simulation* [21, 15] for RAs. Recall that RAs are SRAs whose transition guards are all $\top$. Given two RAs $\mathcal{S}_1$ and $\mathcal{S}_2$ a symbolic simulation between them is defined over their state spaces Q_1 and Q_2, not on their configurations. For this to work, one needs to add an extra piece of information about how registers of the two states are related. More precisely, a symbolic simulation is a relation on triples (p_1, p_2, σ), where $p_1 \in Q_1, p_2 \in Q_2$ and $\sigma \subseteq R_1 \times R_2$ is a *partial injective* function. This function encodes constraints between registers: $(r, s) \in \sigma$ is an equality constraint between $r \in R_1$ and $s \in R_2$, and $(r, s) \notin \sigma$ is an inequality constraint. Intuitively, (p_1, p_2, σ) says that all configurations (p_1, v_1) and (p_2, v_2) such that v_1 and v_2 satisfy σ – e.g., $v_1(r) = v_2(s)$ whenever $(r, s) \in \sigma$ – are in the simulation relation $(p_1, v_1) \prec (p_2, v_2)$. In the following we will use $v_1 \bowtie v_2$ to denote the function encoding constraints among v_1 and v_2, explicitly: $\sigma(r) = s$ if and only if $v_1(r) = v_2(s)$ and $v_1(r) \neq \sharp$.

Definition 5 (Symbolic (bi)similarity [21]). *A symbolic simulation is a relation $\mathcal{R} \subseteq Q_1 \times Q_1 \times \mathcal{P}(R_1 \times R_2)$ such that if $(p_1, p_2, \sigma) \in \mathcal{R}$, then $p_1 \in F_1$ implies $p_2 \in F_2$, and if $p_1 \xrightarrow{\ell} q_1 \in \Delta_1$[3] then:*

1. *if $\ell = r^=$:*
   (a) *if $r \in \mathsf{dom}(\sigma)$, then there is $p_2 \xrightarrow{\sigma(r)^=} q_2 \in \Delta_2$ such that $(q_1, q_2, \sigma) \in \mathcal{R}$.*
   (b) *if $r \notin \mathsf{dom}(\sigma)$ then there is $p_2 \xrightarrow{s^{\bullet}} q_2 \in \Delta_2$ s.t. $(q_1, q_2, \sigma[r \mapsto s]) \in \mathcal{R}$.*
2. *if $\ell = r^{\bullet}$:*
   (a) *for all $s \in R_2 \setminus \mathsf{img}(\sigma)$, there is $p_2 \xrightarrow{s^=} q_2 \in \Delta_2$ such that $(q_1, q_2, \sigma[r \mapsto s]) \in \mathcal{R}$, and;*
   (b) *there is $p_2 \xrightarrow{s^{\bullet}} q_2 \in \Delta_2$ such that $(q_1, q_2, \sigma[r \mapsto s]) \in \mathcal{R}$.*

Here $\sigma[r \mapsto s]$ stands for $\sigma \setminus (\sigma^{-1}(s), s) \cup (r, s)$, which ensures that σ stays injective when updated.

Given a symbolic simulation $\mathcal{R}$, its inverse is defined as $\mathcal{R}^{-1} = \{t^{-1} \mid t \in \mathcal{R}\}$, where $(p_1, p_2, \sigma)^{-1} = (p_2, p_1, \sigma^{-1})$. A symbolic bisimulation $\mathcal{R}$ is a relation such that both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are symbolic simulations.

Case 1 deals with cases when p_1 can perform a transition that reads the register r. If $r \in \mathsf{dom}(\sigma)$, meaning that r and $\sigma(r) \in R_2$ contain the same value, then p_2 must be able to read $\sigma(r)$ as well. If $r \notin \mathsf{dom}(\sigma)$, then the content of r is fresh w.r.t. p_2, so p_2 must be able to read any fresh value — in particular the content of r. Case 2 deals with the cases when p_1 reads a fresh value. It ensures that p_2 is able to read all possible values that are fresh for p_1, be them already in some register s – i.e., $s \in R_2 \setminus \mathsf{img}(\sigma)$, case 2(a) – or fresh for p_2 as well – case 2(b). In all these cases, σ must be updated to reflect the new equalities among registers.

Keeping track of equalities among registers is enough for RAs, because the actual content of registers does not determine the capability of a transition to fire (RA transitions have implicit $\top$ guards). As seen in Example 3, this is no longer the case for SRAs: a transition may or may not happen depending on the register assignment being compatible with the transition guard.

As in the case of reachability, normalized SRAs provide the solution to this problem. We will reduce the problem of checking (bi)similarity of $\mathcal{S}_1$ and $\mathcal{S}_2$ to that of checking symbolic (bi)similarity on $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$, with minor modifications to the definition. To do this, we need to assume that minterms for both $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ are computed over the union of predicates of $\mathcal{S}_1$ and $\mathcal{S}_2$.

Definition 6 (N-simulation). *A N-simulation on $\mathcal{S}_1$ and $\mathcal{S}_2$ is a relation $\mathcal{R} \subseteq \mathsf{N}(Q_1) \times \mathsf{N}(Q_2) \times \mathcal{P}(R_1 \times R_2)$, defined as in Definition 5, with the following modifications:*

(i) *we require that $\theta_1 \rhd p_1 \xrightarrow{\varphi_1/\ell_1} \theta_1' \rhd q_1 \in \mathsf{N}(\Delta_1)$ must be matched by transitions $\theta_2 \rhd p_2 \xrightarrow{\varphi_2/\ell_2} \theta_2' \rhd q_2 \in \mathsf{N}(\Delta_2)$ such that $\varphi_2 = \varphi_1$.*

(ii) *we modify case 2 as follows (changes are underlined):*

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[3] We will keep the $\top$ guard implicit for succinctness.

2(a)' *for all $s \in R_2 \setminus \text{img}(\sigma)$ such that $\varphi_1 = (\theta_2)_s$, there is $\theta_2 \triangleright p_2 \xrightarrow{\varphi_1/s^=}$ $\theta_2' \triangleright q_2 \in \mathsf{N}(\Delta_2)$ such that $(\theta_1' \triangleright q_1, \theta_2' \triangleright q_2, \sigma[r \mapsto s]) \in \mathcal{R}$, and;*

2(b)' *if $\mathscr{E}(\theta_1, \varphi_1) + \mathscr{E}(\theta_2, \varphi_1) < |[\![\varphi_1]\!]|$, then there is $\theta_2 \triangleright p_2 \xrightarrow{\varphi_1/s^{\bullet}} \theta_2' \triangleright q_2 \in$ $\mathsf{N}(\Delta_2)$ such that $(\theta_1' \triangleright q_1, \theta_2' \triangleright q_2, \sigma[r \mapsto s]) \in \mathcal{R}$.*

A N-bisimulation $\mathcal{R}$ is a relation such that both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are N-simulations. We write $\mathcal{S}_1 \overset{\mathsf{N}}{\prec} \mathcal{S}_2$ (resp. $\mathcal{S}_1 \overset{\mathsf{N}}{\sim} \mathcal{S}_2$) if there is a N-simulation (resp. bisimulation) $\mathcal{R}$ such that $(\mathsf{N}(q_{01}), \mathsf{N}(q_{02}), v_{01} \bowtie v_{02}) \in \mathcal{R}$.

The intuition behind this definition is as follows. Recall that, in a normalized SRA, transitions are defined over minterms, which cannot be further broken down, and are mutually disjoint. Therefore two transitions can read the same values if and only if they have the same minterm guard. Thus condition (i) makes sure that matching transitions can read exactly the same set of values. Analogously, condition (ii) restricts how a fresh transition of $\mathsf{N}(\mathcal{S}_1)$ must be matched by one of $\mathsf{N}(\mathcal{S}_2)$: 2(a)' only considers transitions of $\mathsf{N}(\mathcal{S}_2)$ reading registers $s \in R_2$ such that $\varphi_1 = (\theta_2)_s$ because, by definition of normalized SRA, $\theta_2 \triangleright p_2$ has no such transition if this condition is not met. Condition 2(b)' amounts to requiring a fresh transition of $\mathsf{N}(\mathcal{S}_2)$ that is enabled by both θ_1 and θ_2 (see Lemma 1), i.e., that can read a symbol that is fresh w.r.t. both $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$.

N-simulation is sound and complete for standard simulation.

Theorem 3. *$\mathcal{S}_1 \prec \mathcal{S}_2$ if and only if $\mathcal{S}_1 \overset{\mathsf{N}}{\prec} \mathcal{S}_2$.*

As a consequence, we can decide similarity of SRAs via their normalized versions. Note that a N-simulation is a relation over a finite set, namely $\mathsf{N}(Q_1) \times \mathsf{N}(Q_2) \times \mathcal{P}(R_1 \times R_2)$, therefore N-similarity can be decided in finite time.

In case $\mathcal{S}_1$ and $\mathcal{S}_2$ are deterministic, we can explicitly give an algorithm to decide (bi)similarity.

Theorem 4. *Given two deterministic SRAs $\mathcal{S}_1$ and $\mathcal{S}_2$, there are algorithms to decide $\mathcal{S}_1 \prec \mathcal{S}_2$ and $\mathcal{S}_1 \sim \mathcal{S}_2$.*

Proof. We can decide $\mathcal{S}_1 \prec \mathcal{S}_2$ by iteratively building a relation $\mathcal{R}$ on triples that is a simulation on $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$. The algorithm initializes $\mathcal{R}$ with $(\mathsf{N}(q_{01}), \mathsf{N}(q_{02}), v_{01} \bowtie v_{02})$, as this triple is required to be in $\mathcal{R}$ by Definition 6. At each iteration, it considers a candidate triple t and checks whether it satisfies the conditions for N-simulation. If yes, it adds t to $\mathcal{R}$ and computes the next set of candidate triples, i.e., those which are required to belong to the simulation relation, and adds them to the list of triples still to be processed. If no, the algorithm returns $\mathcal{S}_1 \not\prec \mathcal{S}_2$. Crucially, since $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ are deterministic, there is no ambiguity in computing these new triples. The algorithm terminates returning $\mathcal{S}_1 \prec \mathcal{S}_2$ when no triples are left to process. To decide $\mathcal{S}_1 \sim \mathcal{S}_2$, at each iteration we need to check that both t and t^{-1} satisfy the conditions for N-simulation.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ have n_1, n_2 states, m_1, m_2 transitions, and r_1, r_2 registers respectively, their normalized versions have $\mathcal{O}(n_1 2^{m_1})$ and $\mathcal{O}(n_2 2^{m_2})$ states. Since each triple is processed exactly once, and triples are from the finite set $\mathsf{N}(Q_1) \times \mathsf{N}(Q_2) \times \mathcal{P}(R_1 \times R_2)$, the algorithm iterates $\mathcal{O}(n_1 n_2 2^{m_1 m_2 r_1 r_2})$ times.

5 Evaluation

We implemented SRAs in the open source Java library SVPALib [20]. In our implementation, constructions are computed lazily when possible (e.g., the normalized SRA for emptiness and (bi)similarity checks). All experiments were performed on a machine with 3.5 GHz Intel Core i7 CPU with 16GB of RAM (JVM 8GB) and we use a timeout value of 300s. The goal of our evaluation is to answer the following research questions:

Q1: Are SRAs more succinct than existing models when processing strings over large but finite alphabets? (§ 5.1)

Q2: What is the performance of membership for deterministic SRA? (§ 5.2)

Q3: Are SRAs decision procedures practical? (§ 5.3)

Benchmarks. We focus on regular expressions with back-references, therefore all our benchmarks operate over the Boolean algebra of Unicode characters with interval—i.e., the set of characters is the set of all 2^{16} UTF-16 characters and the predicates are union of intervals (e.g., [a-zA-Z]).[4] Our benchmark set contains 19 SRAs that represent variants of regular expressions with back-references obtained from the regular-expression crowd-sourcing website RegExLib [18]. The expressions check whether inputs have, for example, matching first/last name initials or both (Name-F, Name-L and Name), correct Product Codes/Lot number of total length n (Pr-Cn, Pr-CLn), matching XML tags (XML),

	SRA				**SFA**	
	states	tr	reg	|reg|	states	tr
IP2	44	46	3	10	4,013	4,312
IP3	44	46	4	10	39,113	42,112
IP4	44	46	5	10	372,113	402,112
IP6	44	46	7	10	—	—
IP9	44	46	10	10	—	—
Name-F	7	10	2	26	201	300
Name-L	7	10	2	26	129	180
Name	7	10	3	26	3,201	4,500
XML	12	16	4	52	—	—
Pr-C2	26	28	3	2^{16}	—	—
Pr-C3	28	30	4	2^{16}	—	—
Pr-C4	30	32	5	2^{16}	—	—
Pr-C6	34	36	7	2^{16}	—	—
Pr-C9	40	42	10	2^{16}	—	—
Pr-CL2	26	28	3	2^{16}	—	—
Pr-CL3	28	30	4	2^{16}	—	—
Pr-CL4	30	32	5	2^{16}	—	—
Pr-CL6	34	36	7	2^{16}	—	—
Pr-CL9	40	42	10	2^{16}	—	—

Fig. 2: Size of SRAs vs SFAs. (—) denotes the SFA didn't fit in memory. |reg| denotes how many different characters a register stored.

and IP addresses that match for n positions (IPn). We also create variants of the product benchmark presented in Section 2 where we vary the numbers of characters in the code and lot number. All the SRAs are deterministic.

5.1 Succinctness of SRAs vs SFAs

In this experiment, we relate the size of SRAs over finite alphabets to the size of the smallest equivalent SFAs. For each SRA, we construct the equivalent SFA

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[4] Our experiments are over finite alphabets, but our Boolean algebra can be made infinite by allowing the alphabet to be positive integers and allowing intervals to contain ∞ as an upper bound. This modification does not affect the running time of our procedures, therefore we do not report it.

by equipping the state space with the values stored in the registers at each step (this construction effectively builds the configuration LTS). Figure 2 shows the results. As expected, SFAs tend to blow up in size when the SRA contains multiple registers or complex register values. In cases where the register values range over small sets (e.g., `[0-9]`) it is often feasible to build an SFA equivalent to the SRA, but the construction always yields very large automata. In cases where the registers can assume many values (e.g., 2^{16}) SFAs become prohibitively large and do not fit in memory. To answer **Q1**, even for finite alphabets, **it is not feasible to compile SRAs to SFAs**. Hence, SRAs are a succinct model.

5.2 Performance of membership checking

In this experiment, we measure the performance of SRA membership. For each benchmark, we generate inputs of length varying between approximately 100 and 10^8 characters and measure the time taken to check membership. Figure 3 shows the results (all SRAs have similar performance and the lines overlap). As expected, the time taken to check membership grows linearly in the size of the input (x-axis log scale). Remarkably, even though our implementation does not employ particular input processing optimizations, it can still check membership for strings with millions of characters in less than 100 seconds. To answer **Q2**, **deterministic SRAs can be efficiently executed on large inputs**.
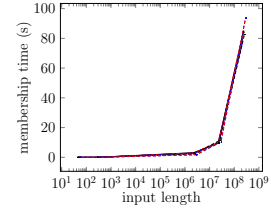

Fig. 3: SRA membership performance.

5.3 Performance of decision procedures

In this experiment, we measure the performance of SRAs simulation and bisimulation algorithms. Since all our SRAs are deterministic, these two checks correspond to language equivalence and inclusion. We select pairs of benchmarks for which the above tests are meaningful (e.g., variants of the problem discussed at the end of Sec. 2). The results are shown in Figure 4. As expected, due to the translation to single-valued SRAs, our decision procedures do not scale well in the number of registers. This is already the case for classic register automata and it is not a surprising result. However, our technique

SRA S_1	SRA S_2	$L(S_1)=\emptyset$	$L(S_1)=L(S_1)$	$L(S_1\subseteq S_2)$
Pr-C2	Pr-CL2	0.125s	0.905s	3.426s
Pr-C3	Pr-CL3	1.294s	5.558s	24.688s
Pr-C4	Pr-CL4	13.577s	55.595s	—
Pr-C6	Pr-CL6	—	—	—
Pr-CL2	Pr-C2	1.067s	0.952s	0.889s
Pr-CL3	Pr-C3	10.998s	11.104s	11.811s
Pr-CL4	Pr-C4	—	—	—
Pr-CL6	Pr-C6	—	—	—
IP-2	IP-3	0.125s	0.408s	1.845s
IP-3	IP-4	1.288s	2.953s	21.627s
IP-4	IP-6	18.440s	42.727s	—
IP-6	IP-9	—	—	—

Fig. 4: Performance of decision procedures.

can still check equivalence and inclusion for regular expressions that no existing tool can handle. To answer **Q3**, **bisimulation and simulation algorithms for SRAs only scale to small numbers of registers**.

6 Conclusions

In this paper we have presented *Symbolic Register Automata*, a novel class of automata that can handle complex alphabet theories while allowing symbol comparisons for equality. SRAs encompass – and are strictly more powerful – than both Register and Symbolic Automata. We have shown that they enjoy the same closure and decidability properties of the former, despite the presence of arbitrary guards on transitions, which are not allowed by RAs. Via a comprehensive set of experiments, we have concluded that SRAs are vastly more succinct than SFAs and membership is efficient on large inputs. Decision procedures do not scale well in the number of registers, which is already the case for basic RAs.

Related work. RAs where first introduced in [13]. There is an extensive literature on register automata, their formal languages and decidability properties [9, 17, 19, 4, 16], including variants with *global freshness* [21, 15] and totally ordered data [2, 10]. SRAs are based on the original model of [13], but are much more expressive, due to the presence of guards from an arbitrary decidable theory.

In recent work, variants over richer theories have appeared. In [6] RA over rationals were introduced. They allow for a restricted form of linear arithmetic among registers (RAs with arbitrary linear arithmetic subsume two-counter automata, hence are undecidable). SRAs do not allow for operations on registers, but encompass a wider range of theories without any loss in decidability. Moreover, [6] does not study Boolean closure properties. In [5, 12], RAs allowing guards over a range of theories – including (in)equality, total orders and increments/sums – are studied. Their focus is different than ours as they are interested primarily in *active learning* techniques, and several restrictions are placed on models for the purpose of the learning process.

Symbolic automata were first introduced in [22] and many variants of them have been proposed [8]. The one that is closer to SRAs is Symbolic Extended Finite Automata (SEFA) [7]. SEFAs are SFAs in which transition can read more than one character at a time. A transition of arity k reads k symbols which are consumed if they satisfy the predicate $\varphi(x_1, \ldots, x_k)$. SEFA allow arbitrary k-ary predicates over the input theory, which results in most problems being undecidable (e.g., equivalence and intersection emptiness) and in the model not being closed under Boolean operations. Even when deterministic, SEFAs are not closed under union and intersection. In terms of expressiveness, SRAs and SEFAs are incomparable. SRAs can only use equality, but can compare symbols at arbitrary points in the input while SFAs can only compare symbols within a constant window, but using arbitrary predicates.

Future work. In [16] a polynomial algorithm for checking language equivalence of deterministic RAs is presented. This crucially relies on closure properties of symbolic bisimilarity, some of which are lost for SRAs. We plan to investigate whether this algorithm can be adapted to our setting. Extending SRAs with more complex comparison operators other than equality (e.g., a total order $<$) is an interesting research question, but most extensions of the model quickly lead to undecidability. We also plan to study active automata learning for SRAs, building on techniques for SFAs [1], RAs [3, 5, 12] and nominal automata [14].

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A Proofs

Proof (of Proposition 2). Intersection and union are defined as follows:

1. $S_1 \cap S_2 = (R_\cap, Q_\cap, q_{0\cap}, v_{0\cap}, F_\cap, \Delta_\cap)$ where $R_\cap = R_1 \uplus R_2$; $Q_\cap = Q_1 \times Q_2$; $q_{0\cap} = (q_{01}, q_{02})$; $v_{0\cap} = [v_{01}, v_{02}]$; $F_\cap = F_1 \times F_2$; and

$$\Delta_\cap = \{(p_1, p_2) \xrightarrow{\varphi_1 \wedge \varphi_2 / \ell_1 \cup \ell_2} (q_1, q_2) \mid \forall i = 1, 2 \colon p_i \xrightarrow{\varphi_i / \ell_i} q_i \in \Delta_i\}$$

   where $(E_1, I_1, U_1) \cup (E_2, I_2, U_2) = E_1 \cup E_2, I_1 \cup I_2, U_1 \cup U_2$.

2. $S_1 \cup S_2 = (R_\cup, Q_\cup, q_{0\cup}, v_{0\cup}, F_\cup, \Delta_\cup)$ where $R_\cup = R_1 \uplus R_2$; $Q_\cup = Q_1 \cup Q_2 \cup \{\hat{q}_0\}$; $q_{0\cup} = \hat{q}_0$; $v_{0\cup} = [v_{01}, v_{02}]$; $F_\cup = F_1 \cup F_2 \cup I$, where $I = \{\hat{q}_0\}$ if $q_{0i} \in F_i$, for some $i \in \{1, 2\}$, and $I = \emptyset$ otherwise. Transitions are given by

$$\Delta_\cup = \bigcup_{i \in \{1,2\}} \Delta_i \cup \{\hat{q}_0 \xrightarrow{\varphi / E, I, U} q \mid q_{0i} \xrightarrow{\varphi / E, I, U} q \in \Delta_i\}$$

Proof (of Theorem 1).

For notational convenience, we introduce an additional label for single-valued SRAs: $p \xrightarrow{\varphi / \bullet} q$ whenever a fresh symbol is read, but not stored anywhere. This can be translated back to ordinary single-valued transitions by adding a new register $\hat{r}$, replacing $p \xrightarrow{\varphi / \bullet} q$ with a pair of parallel transitions $p \mathrel{\substack{\xrightarrow{\varphi / \hat{r}^\bullet} \\ \xrightarrow{\varphi / \hat{r}}}} q$,

and by adding a transition $p \xrightarrow{\varphi / \hat{r}} q$ for every transition $p \xrightarrow{\varphi / r^\bullet} q$.

The SRA $S' = (R, Q', q_0', v_0', F', \Delta')$ is defined as follows:

- $Q' = \{(q, f) \mid q \in Q, f \colon R \to R\}$;
- v_0' is any function $R \to \mathcal{D} \cup \{\sharp\}$ injective on non-empty registers such that $\mathsf{img}(v_0') = \mathsf{img}(v_0)$;
- $q_0' = (q_0, f_0)$, where f_0 is such that $v_0 = v_0' \circ f_0$;
- $F' = \{(q, f) \in Q' \mid q \in F\}$;
- Δ' is generated by the following rules

$$(\text{REG}) \quad \frac{p \xrightarrow{\varphi / E, I, U} q \quad S \propto E, I \quad f^{-1}(r) = S}{(p, f) \xrightarrow{\varphi / r^=} (q, f[U \mapsto r])}$$

$$(\text{FRESH}) \quad \frac{p \xrightarrow{\varphi / E, I, U} q \quad \emptyset \propto E, I \quad f^{-1}(r) \subseteq U \quad U \neq \emptyset}{(p, f) \xrightarrow{\varphi / r^\bullet} (q, f[U \mapsto r])}$$

$$(\text{NOP}) \quad \frac{p \xrightarrow{\varphi / E, I, \emptyset} q \quad \emptyset \propto E, I}{(p, f) \xrightarrow{\varphi / \bullet} (q, f)}$$

Moreover, for (FRESH), we assume that the choice of r only depends on f and U, i.e., given any two transitions $p_1 \xrightarrow{\varphi_1 / E_1, I_1, U} q_1$ and $p_2 \xrightarrow{\varphi_2 / E_2, I_2, U} q_2$ of

S, the rule (FRESH) produces a unique pair $(p_1, f) \xrightarrow{\varphi_1/r^\bullet} (q_1, f[U \mapsto r])$ and $(p_2, f) \xrightarrow{\varphi_2/r^\bullet} (q_2, f[U \mapsto r])$. This can be achieved by ordering the registers and picking the least r such that $f^{-1}(r) \subseteq U$.

We now prove that the following relation is a bisimulation between S and S':

$$\mathcal{R} = \{((p, v), ((p, f), w) \mid v = w \circ f\}$$

We first prove the following, which will be useful in all the cases below.

Lemma 3. *Let $X \subseteq R$ and let $v = w \circ f$, with v, w, f as above. For any $r \in R$ such that $f^{-1}(r) \subseteq X$, and any $a \in \mathcal{D}$, we have*

$$v[X \mapsto a] = w[r \mapsto a] \circ f[X \mapsto r]$$

Proof. We proceed by cases. Let $s \in X$. Then we have $v[X \mapsto a](s) = a$ and $(w[r \mapsto a] \circ f[X \mapsto r])(s) = w[r \mapsto a](r) = a$. If $s \notin X$, then we must have $f(s) \neq r$, because $f^{-1}(r) \subseteq X$, so $(w[r \mapsto a] \circ f[X \mapsto r])(s) = w[r \mapsto a](f(s)) = w(f(s)) = v(s) = v[X \mapsto a](s)$, by the hypothesis $v = w \circ f$.

Clearly we have $((q_0, v_0), ((p, f_0), v_0')) \in \mathcal{R}$, by construction. We first prove that $\mathcal{R}$ is a simulation. Suppose we have $(p, v) \xrightarrow{a} (q, v') \in \mathsf{CLTS}(S)$, instance of $p \xrightarrow{\varphi/E, I, U} q$, so $v^{-1}(a) \propto E, I$ and $v' = v[U \mapsto a]$. We need to find a matching transition in $\mathsf{CLTS}(S')$ from any $((p, f), w)$ such that w is injective on non-empty registers and $v = w \circ f$. We have two cases:

- Suppose $a \in \mathsf{img}(v)$, and let $S = v^{-1}(a)$. Since $v = w \circ f$, there must be $r \in R$ (unique, by injectivity of w) such that $f(S) = \{r\}$. Therefore, recalling the assumption $S \propto E, I$, by (REG) there is $p \xrightarrow{\varphi/r^=} q$ and, since $w(r) = (w \circ f)(E) = v(E) = a$, there is $((p, f), w) \xrightarrow{a} ((q, f[U \mapsto r]), w) \in \mathsf{CLTS}(S')$. We have

$$
\begin{aligned}
v[U \mapsto a] &= v[U \cup S \mapsto a] && (v(S) = a) \\
&= (w \circ f)[E \cup S \mapsto a] && (v = w \circ f) \\
&= w[r \mapsto a] \circ f[S \cup U \mapsto r] && \text{(Lemma 3 and } f^{-1}(r) = S \subseteq S \cup U) \\
&= w \circ f[U \mapsto r] && (a = w(r), f^{-1}(r) = S)
\end{aligned}
$$

  from which we get $(q, v[U \mapsto a]) \mathcal{R}((q, f[U \mapsto r]), w)$.
- Suppose $a \in \llbracket \varphi \rrbracket \setminus \mathsf{img}(v)$. Then we have $\emptyset \propto E, I$. If $a \in \mathsf{img}(w)$, with $r = w(a)$, we must have $f^{-1}(r) = \emptyset$, therefore $f^{-1}(r) \propto E, I$. We can now use (REG) to obtain $(p, f) \xrightarrow{\varphi/r^=} (q, f[U \mapsto r]) \in \Delta'$. This transition has an instance $((p, f), w) \xrightarrow{a} ((q, f[U \mapsto r]), w)$ in $\mathsf{CLTS}(S')$. We have $v[U \mapsto a] = w \circ f[U \mapsto r]$, because $w(r) = a$, so $(q, v[U \mapsto a]) \mathcal{R}((q, f[U \mapsto r]), w)$. If $a \notin \mathsf{img}(w)$, we can apply either (REG) or (FRESH), depending on U:

- Suppose $U \neq \emptyset$, and consider $r \in R$ such that that $f^{-1}(r) \subseteq U$. Such r must exist: if f is injective then it is obvious (this crucially depends on $U \neq \emptyset$), otherwise there is r such that $f^{-1}(r) = \emptyset \subseteq U$. Therefore by (FRESH) there is a transition $(p, f) \xrightarrow{\varphi/r^{\bullet}} (q, f[U \mapsto r]) \in \Delta'$, which has an instance $((p, f), w) \xrightarrow{a} ((q, f[U \mapsto r]), w[r \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S}')$. By Lemma 3 and $f^{-1}(r) \subseteq U$ (see previous point), we have $v[U \mapsto a] = w[r \mapsto a] \circ f[U \mapsto r]$, from which $(q, v[U \mapsto a])\mathcal{R}((q, f[U \mapsto r]), w[r \mapsto a])$ follows.
- Suppose $U = \emptyset$. Then by (NOP) there is $(p, f) \xrightarrow{\varphi/\bullet} (q, f) \in \Delta'$. This transition has an instance $((p, f), w) \xrightarrow{a} ((q, f), w)$ in $\mathsf{CLTS}(\mathcal{S}')$. By assumption $v = w \circ f$, so $(q, v)\mathcal{R}((q, f), w)$.

Now we prove that $\mathcal{R}^{-1}$ is a simulation. Suppose we have $((p, f), w) \xrightarrow{a} ((q, f'), w')$ in $\mathsf{CLTS}(\mathcal{S}')$. We need to find a matching transition in $\mathsf{CLTS}(\mathcal{S})$ from any (p, v) such that $v = w \circ f$. We proceed by cases:

- Suppose $a \in \mathsf{img}(w)$, with $w(r) = a$. Then there must be $(p, f) \xrightarrow{\varphi/r^{=}} (q, f')$ in Δ' and we have $w' = w$. This transition is given by (REG), so we have $p \xrightarrow{\varphi/E, I, U} q \in \Delta$, $f' = f[U \mapsto r]$ and $f^{-1}(r) = S \propto E, I$. By injectivity of w and $v = w \circ f$, we must have $v^{-1}(a) = S$. Therefore there is $(p, v) \xrightarrow{a} (q, v[U \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S})$. We have $v[U \mapsto a] = w \circ f[U \mapsto r]$, so $((q, f[U \mapsto r]), w)\mathcal{R}^{-1}(q, v[U \mapsto a])$.
- Suppose $a \in \llbracket \varphi \rrbracket \setminus \mathsf{img}(w)$. Then we must have $a \notin \mathsf{img}(v)$, because $\mathsf{img}(v) \subseteq \mathsf{img}(w)$. We have two cases:
  - Suppose $(p, f) \xrightarrow{\varphi/r^{\bullet}} (q, f') \in \Delta'$. Then we have $w' = w[r \mapsto a]$. By (FRESH), there is $p \xrightarrow{\varphi/E, I, U} q \in \Delta$. Because $a \notin \mathsf{img}(v)$ implies $v^{-1}(a) = \emptyset$, and we have $\emptyset \propto E, I$, there is $(p, v) \xrightarrow{a} (q, v[U \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S})$. By $f^{-1}(r) \subseteq U$ and Lemma 3, there is $v[U \mapsto a] = w[r \mapsto a] \circ f[U \mapsto r]$, so $((q, f[U \mapsto r]), w[r \mapsto a])\mathcal{R}^{-1}(q, v[U \mapsto a])$.
  - Suppose $(p, f) \xrightarrow{\varphi/\bullet} (q, f) \in \Delta'$. Then by (NOP) there must be $p \xrightarrow{\varphi/E, I, \emptyset} q \in \Delta$. Because $a \notin \mathsf{img}(v)$, we have $v^{-1}(a) = \emptyset$, and because $\emptyset \propto E, I$ there is $(p, v) \xrightarrow{a} (q, v)$ in $\mathsf{CLTS}(\mathcal{S})$. By the assumption $v = w \circ f$, we get $((q, f), v)\mathcal{R}^{-1}(q, w)$.

Now we show that, if $\mathcal{S}$ is deterministic, so is $\mathcal{S}'$. We proceed by contradiction. Suppose $\mathcal{S}'$ is not deterministic, i.e., there is a reachable configuration $((p, f), w)$ in $\mathsf{CLTS}(\mathcal{S}')$ and two transitions $((p, f), w) \xrightarrow{a} ((q_1, f_1), w_1)$ and $((p, f), w) \xrightarrow{a} ((q_2, f_2), w_2)$ such that $((q_1, f_1), w_1) \neq ((q_2, f_2), w_2)$. Then these transitions must be instances of $(p, f) \xrightarrow{\varphi_i/\ell_i} (q_i, f_i) \in \Delta'$, for $i = 1, 2$. We have three cases:

- if $\ell_1 = r^{=}$, then we must also have $\ell_2 = r^{=}$ (and vice versa), because $a = w(r)$, and $w_1 = w_2 = w$. Since $\mathcal{S} \sim \mathcal{S}'$, there must be a reachable configuration (p, v) in $\mathsf{CLTS}(\mathcal{S})$ such that $v = w \circ f$, and two transitions

$(p, v) \xrightarrow{a} (q_i, v_i)$ such that $v_i = w \circ f_i$, for $i = 1, 2$. Now, we must have either $q_1 \neq q_2$ or $v_1 \neq v_2$. In fact, suppose $q_1 = q_2$ and $v_1 = v_2$, then we would have

$$v_1 = w \circ f_1 \implies w \circ f_1 = w \circ f_2 \qquad (v_2 = w \circ f_2 \wedge v_1 = v_2)$$
$$\implies f_1 = f_2 \qquad\qquad\qquad (\text{injectivity of } w)$$

which implies $((q_1, f_1), w_1) = ((q_2, f_2), w_2)$, contradicting our initial assumption on non-determinism of $\mathcal{S}$. Therefore $(q_1, v_1) \neq (q_2, v_2)$, and $\mathcal{S}$ is non-deterministic as well.

- if $\ell_1 = r^\bullet$, then we have $\ell_2 \in \{\bullet, s^\bullet\}$ and $w_1 = w[r \mapsto a]$, $w_2 \in \{w_2, w[s \mapsto a]\}$, because $a \notin \mathsf{img}(w)$. Suppose $\ell_2 = \bullet$. By the definition of Δ', the two transitions of $\mathcal{S}'$ we are considering must correspond to transitions $p \xrightarrow{\varphi_i / E_i, I_i, U_i} q_i \in \Delta$ such that $\emptyset \propto E_i, I_i$ where U_2 can be empty, and $f_i = f[U_i \mapsto r]$, for $i = 1, 2$. Let (p, v) be a reachable configuration in $\mathsf{CLTS}(\mathcal{S})$ such that $v = w \circ f$, which must exist by $\mathcal{S} \sim \mathcal{S}'$. Then those transitions in Δ have instances $(p, v) \xrightarrow{a} (q_i, v[U_i \mapsto a])$, for $i = 1, 2$. We distinguish the two cases:

  - If $\ell_2 = \bullet$, then $v[U_2 \mapsto a] = v \neq v[U_1 \mapsto a]$, because $a \notin \mathsf{img}(w)$ and $v = w \circ f$ imply $a \notin \mathsf{img}(v)$, and because $U_1 \neq \emptyset$, by the premise of (FRESH). Therefore we have $(q_1, v[U_1 \mapsto a]) \neq (q_2, v)$, from which non-determinism of $\mathcal{S}$ follows.

  - If $\ell_2 = s^\bullet$, then we must have either $q_1 \neq q_2$ or $v[U_1 \mapsto a] \neq v[U_2 \mapsto a]$. In fact, suppose $q_1 = q_2$ and $v[U_1 \mapsto a] = v[U_2 \mapsto a]$. Because $a \notin \mathsf{img}(v)$, the latter equality implies $U_1 = U_2$, thus $r = s$, because we assumed that (FRESH) picks a unique r for each pair $(f, U_1) (= (f, U_2))$. It follows that $f_1 = f_2$ and $w_1 = w_2$, from which we get the contradiction $((q_1, f_1), w_1) = ((q_2, f_2), w_2)$.

- the case $\ell_1 = \bullet$ is symmetrical to the one above. $\qquad\qquad\square$

Proof (of Lemma 1).

1. $p \xrightarrow{\varphi / r^=} q$ is enabled if and only if $v(r) \in \llbracket \varphi \rrbracket$. Since both θ_r and φ are minterms, this holds if and only if $\theta_r = \varphi$.

2. Suppose the transition is enabled, that is, for any (p, v) such that $v \models \theta$ there is $(p, v) \xrightarrow{a} (q, v[r \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S})$, with $a \in \llbracket \varphi \rrbracket \setminus \mathsf{img}(v)$. Since v is injective on non-empty registers and $\varphi \neq \bot$, by definition of minterm, v picks $|\{r \in R \mid \theta_r = \varphi\}|$ distinct elements from $\llbracket \varphi \rrbracket$. Therefore, for a to exist, $\llbracket \varphi \rrbracket$ must have at least $|\{r \in R \mid \theta_r = \varphi\}| + 1$ elements.
   Viceversa, suppose $|\llbracket \varphi \rrbracket| > |\{r \in R \mid \theta_r = \varphi\}| = k$, and take any $v \models \theta$. By injectivity of v on non-empty registers, we have $|\mathsf{img}(v) \cap \llbracket \varphi \rrbracket| = k$. Since $|\llbracket \varphi \rrbracket| > k$, there exists $a \in \llbracket \varphi \rrbracket \setminus \mathsf{img}(v)$, therefore there is $(p, v) \xrightarrow{a} (p, v[r \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S})$.
   $\qquad\qquad\square$

Proof (of Proposition 4). We will prove that

$$\mathcal{R} = \{((p, v), (\theta \rhd p, v) \mid v \models \theta\}$$

is a bisimulation. We first prove that it is a simulation. Consider $(p, v) \in \mathsf{CLTS}(\mathcal{S})$ and any transition $(p, v) \xrightarrow{a} (q, w)$. We will prove that this transition can be simulated by one from $((p, v), \theta)$ such that $v \models \theta$. We proceed by cases on which transition in Δ has generated $(p, v) \xrightarrow{a} (q, w)$:

  – $p \xrightarrow{\varphi/r^=} q$: then $w = v$, $a \in [\![\varphi]\!]$ and $v(r) = a$ and $a \in [\![\varphi]\!]$. By $v \models \theta$ we also have $a \in [\![\theta_r]\!]$, therefore $\varphi \sqsubset \theta_r$, which implies $\theta \rhd p \xrightarrow{\theta_r/r^=} (q, \theta) \in \mathsf{N}(\Delta)$. By Lemma 1(1) this transition is enabled, therefore there is $(\theta \rhd p, v) \xrightarrow{a} ((q, \theta), v)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$, which is the required simulating transition. In fact, $v \models \theta$ implies $(q, v)\mathcal{R}((q, \theta), v)$.

  – $p \xrightarrow{\varphi/r^\bullet} q$: then $w = v[r \mapsto a]$ and $a \in [\![\varphi]\!] \setminus \mathsf{img}(v)$. Take the minterm $\psi \in \mathsf{mint}(\mathcal{S})$ such that $a \in [\![\psi]\!]$ (there is only one, by definition of minterm). We will prove that $\theta \rhd p \xrightarrow{\psi/r^\bullet} (q, \theta[r \mapsto \psi]) \in \mathsf{N}(\Delta)$. Clearly we have $\psi \sqsubset \varphi$, as $a \in [\![\varphi]\!]$. We have to check $|[\![\psi]\!]| > |\{r \in R \mid \theta_r = \psi\}|$: the proof is the same as Lemma 1(2), showing by contradiction that $(p, v) \xrightarrow{a} (q, v[r \mapsto a])$ such that $a \in [\![\psi]\!]$ cannot be generated from $p \xrightarrow{\varphi/r^\bullet} q$ whenever $|[\![\psi]\!]| \leq |\{r \in R \mid \theta_r = \psi\}|$. Therefore $\theta \rhd p \xrightarrow{\psi/r^\bullet} (q, \theta[r \mapsto \psi]) \in \mathsf{N}(\Delta)$ and, by Lemma 1(2), it is enabled, i.e., there is $(\theta \rhd p, v) \xrightarrow{b} ((p, \theta[r \mapsto \psi]), v[r \mapsto b])$ such that $b \in [\![\psi]\!] \setminus \mathsf{img}(v)$. Notice that there is such a transition for any $b \in [\![\psi]\!] \setminus \mathsf{img}(v)$, in particular for $b = a$ we get the required simulating transition. In fact, because of the assumptions $v \models \theta$ and $a \in [\![\psi]\!]$, we have $v[r \mapsto a] \models \theta[r \mapsto \psi]$ from which, by definition of $\mathcal{R}$, we get $(p, v[r \mapsto a])\mathcal{R}((p, \theta[r \mapsto \psi]), v[r \mapsto a])$.

We now prove that $\mathcal{R}^{-1}$ is a simulation. Consider $(\theta \rhd p, v) \in \mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ such that $v \models \theta$ and any transition $(\theta \rhd p, v) \xrightarrow{a} ((q, \gamma), w)$. We will prove that this transition can be simulated by one from (p, v). We proceed by cases on which transition in $\mathsf{N}(\Delta)$ has generated $(\theta \rhd p, v) \xrightarrow{a} ((q, \gamma), w)$:

  – $\theta \rhd p \xrightarrow{\theta_r/r^=} (q, \theta)$: then $\theta = \gamma$, $w = v$, $a \in [\![\theta_r]\!]$, $v(r) = a$. By definition of $\mathsf{N}(\Delta)$, there is $p \xrightarrow{\varphi/r^=} q \in \Delta$ such that $\varphi \sqsubset \theta_r$, which implies $[\![\theta_r]\!] \subseteq [\![\varphi]\!]$. Therefore $a \in [\![\varphi]\!]$, and this transition from Δ instantiates to $(p, v) \xrightarrow{a} (q, v) \in \mathsf{CLTS}(\mathcal{S})$. This is the required simulating transition, because $v \models \theta$ implies $((q, \theta), v)\mathcal{R}^{-1}(q, v)$.

  – $\theta \rhd p \xrightarrow{\psi/r^\bullet} (q, \theta[r \mapsto \psi])$: then $\gamma = \theta[r \mapsto \psi]$, $w = v[r \mapsto a]$ and $a \in [\![\psi]\!] \setminus \mathsf{img}(v)$. By definition of $\mathsf{N}(\Delta)$, there is $p \xrightarrow{\varphi/r^\bullet} q \in \Delta$ such that $\varphi \sqsubset \psi$. Therefore we have $[\![\psi]\!] \subseteq [\![\varphi]\!]$, which implies $a \in [\![\varphi]\!] \setminus \mathsf{img}(v)$, so there is $(p, v) \xrightarrow{a} (q, v[r \mapsto a])$ in $\mathsf{CLTS}(\mathcal{S})$ corresponding to $p \xrightarrow{\varphi/r^\bullet} q$. This is the required simulating transition. In fact, because of the assumptions $v \models \theta$ and $a \in [\![\psi]\!]$, we have $v[r \mapsto a] \models \theta[r \mapsto \psi]$ from which, by definition of $\mathcal{R}$, we get $((p, \theta[r \mapsto \psi]), v[r \mapsto a])\mathcal{R}^{-1}(p, v[r \mapsto a])$.

$\square$

Proof (of Proposition 4). For the first part of the claim, we shall prove that there is a path $\theta_0 \rhd q_0 \xrightarrow{\varphi_1/\ell_1} \theta_1 \rhd q_1 \xrightarrow{\varphi_2/\ell_2} \ldots \xrightarrow{\varphi_n/\ell_n} \theta_n \rhd q_n$ in $\mathsf{N}(\mathcal{S})$ if and only if there is a run $(\theta_0 \rhd q_0, v_0) \xrightarrow{a_1} (\theta_1 \rhd q_1, v_1) \xrightarrow{a_2} \ldots \xrightarrow{a_n} (\theta_n \rhd q_n, v_n)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ such that $v_n \models \theta_n$. The right-to-left part is obvious, because every run in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ is instance of a path of $\mathsf{N}(\mathcal{S})$. For the other direction, we will proceed by the length n of the paths. It is obvious for $n = 0$. For $n > 0$, suppose it holds for a path ending in $\theta_n \rhd q_n$, so there is a run ending in $(\theta_n \rhd q_n, v_n)$ for all $v_n \models \theta_n$. Suppose there is

$$\theta_n \rhd q_n \xrightarrow{\varphi_{n+1}/\ell_{n+1}} \theta_{n+1} \rhd q_{n+1} \in \mathsf{N}(\Delta). \qquad (1)$$

Because $v_n \models \theta_n$ by induction hypothesis and (1) is enabled by θ_n, there is $(\theta_n \rhd q_n, v_n) \xrightarrow{a} (\theta_{n+1} \rhd q_{n+1}, v_{n+1})$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$. We either have $\theta_{n+1} = \theta_n$ and $v_{n+1} = v_n$, if $\ell_{n+1} \in R$, or $\theta_{n+1} = \theta_n[r \mapsto \varphi_{n+1}]$ and $v_{n+1} = v_n[a \mapsto r]$, if $\ell_{n+1} = r^\bullet$, with $a \in [\![\varphi_{n+1}]\!]$. In both cases we have $v_{n+1} \models \theta_{n+1}$, which concludes the fart of the claim.

For the second part, we will show that if there is a path of length n reaching $(\theta \rhd p, v)$ then there is on of the same length reaching $(\theta \rhd p, w)$, for all $w \models \theta$. We proceed again by induction on n. It is obvious for $n = 0$, because only $(\theta_0 \rhd q_0, v_0)$ is reachable. Suppose it holds for $n > 0$, and suppose $(\theta_{n+1} \rhd q_{n+1}, v_{n+1})$ can be reached via a path of length $n + 1$. Then we have to show that all $(\theta_{n+1} \rhd q_{n+1}, w_{n+1})$ such that $w_n \models \theta_n$ are reachable. Suppose $(\theta_n \rhd q_n, v_n) \xrightarrow{a} (\theta_{n+1} \rhd q_{n+1}, v_{n+1})$ is the last transition in the run to $(\theta_{n+1} \rhd q_{n+1}, v_{n+1})$. Then this is instance of a transition of $\mathsf{N}(\mathcal{S})$ of the form (1). We proceed by cases on ℓ_{n+1}:

- if $\ell_{n+1} = r^=$, then $\theta_{n+1} = \theta_n$ and $v_{n+1} = v_n$. By inductive hypothesis, we have that all $(\theta_n \rhd q_n, w_n)$ such that $w_n \models \theta_n$ are reachable. Since (1) is enabled by θ_n, it follows that there are $(\theta_n \rhd q_n, w_n) \xrightarrow{w_n(r)} (\theta_n \rhd q_{n+1}, w_n)$, for all $w_n \models \theta_n$, which concludes the proof for this case.
- if $\ell_{n+1} = r^\bullet$, then $\theta_{n+1} = \theta_n[r \mapsto \varphi_{n+1}]$ and $v_{n+1}(r) = a$. Consider any $w_{n+1} \models \theta_{n+1}$, with $w_{n+1} \neq v_{n+1}$. We will show that there must be $b \in [\![(\theta_n)_r]\!]$ such that $w_{n+1}[r \mapsto b]$ is injective, which implies $w_{n+1}[r \mapsto b] \models \theta_n$. The claim would then follow: in fact, $(\theta_n \rhd q_n, w_{n+1}[r \mapsto b])$ would be reachable by the inductive hypothesis, and since (1) is enabled by θ_n, there is an instance of (1) going to $(\theta_{n+1} \rhd q_{n+1}, w_{n+1})$. Suppose there is no such b. Then $[\![(\theta_n)_r]\!]$ is already completely contained in the image of $w_{n+1} \upharpoonright (R \setminus \{r\})$. Since w_{n+1} is injective, and θ_n and θ_{n+1} are made of minterms, we have

$$|\{r' \in R \setminus \{r\} \mid (\theta_{n+1})_{r'} = (\theta_n)_r\}| = |[\![(\theta_n)_r]\!]|$$

Now, $(\theta_n)_{r'} = (\theta_{n+1})_{r'}$ for $r' \in R \setminus \{r\}$, so from the equation above we get that θ_n maps $[\![(\theta_n)_r]\!] + 1$ registers to $(\theta_n)_r$. This contradicts reachability of $(\theta_n \rhd q_n, v_n)$, because there cannot be any injective v_n such that $v_n \models \theta_n$. □

Proof (of Proposition 5). For the right-to-left implication, suppose $\mathsf{N}(\mathcal{S})$ is not deterministic. Then there are two reachable transitions $(\theta \rhd p, v) \xrightarrow{a} (\theta_1 \rhd q_1, v_1)$

and $(\theta \triangleright p, v) \xrightarrow{a} (\theta_2 \triangleright q_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ such that $(\theta_1 \triangleright q_1, v_1) \neq (\theta_2 \triangleright q_2, v_2)$. Suppose these two transitions are respectively instances of $\theta \triangleright p \xrightarrow{\varphi_1/\ell_1} \theta_1 \triangleright q_1$ and $\theta \triangleright p \xrightarrow{\varphi_2/\ell_2} \theta_2 \triangleright q_2$. Then we have the following cases, corresponding to the ones in the statement:

1. Suppose $v_1 = v_2$, then we must have $\ell_1 = \ell_2$ and $\theta_1 \triangleright q_1 \neq \theta_2 \triangleright q_2$, because we assumed $(\theta_1 \triangleright q_1, v_1) \neq (\theta_2 \triangleright q_2, v_2)$. Since $a \in [\![\varphi_1 \wedge \varphi_2]\!]$ and φ_1, φ_2 are minterms, we obtain $\varphi_1 = \varphi_2$.
2. Suppose $v_1 \neq v_2$, then we must have $\ell_1 = r^\bullet$ and $\ell_2 = s^\bullet$, for $r \neq s$, because assignments are only changed by transitions reading fresh symbols. Again, we have $a \in [\![\varphi_1 \wedge \varphi_2]\!]$, thus $\varphi_1 = \varphi_2$.

For the other direction, we assume that there are two reachable transitions $\theta \triangleright p \xrightarrow{\varphi_1/\ell_1} \theta_1 \triangleright q_1, \theta \triangleright p \xrightarrow{\varphi_2/\ell_2} \theta_2 \triangleright q_2 \in \mathsf{N}(\Delta)$ that violate the condition, and we show that $\mathsf{N}(\mathcal{S})$ is not deterministic. We proceed by cases:

- $\ell_1 = \ell_2$, $q_1 \neq q_2$ and $\varphi_1 = \varphi_2$. Then $\theta_1 = \theta_2$. Take a reachable configuration $(\theta \triangleright p, v)$, which exists because $\theta \triangleright p$ is reachable, by Proposition 4. Then the two transitions can be instantiated to $(\theta \triangleright p, v) \xrightarrow{v(r)} (\theta \triangleright q_1, v)$ and $(\theta \triangleright p, v) \xrightarrow{v(r)} (\theta \triangleright q_2, v)$. Nondeterminism follows from $q_1 \neq q_2$.
- $\ell_1 = r^\bullet$, $\ell_2 = s^\bullet$, $r \neq s$ and $\varphi_1 = \varphi_2$. Take a reachable configuration $(\theta \triangleright p, v)$, then since the two transitions are enabled by construction, there is $a \in [\![\varphi_1]\!] \setminus \mathsf{img}(v)$. Therefore the two transitions can be instantiated to $(\theta \triangleright p, v) \xrightarrow{a} (\theta_1 \triangleright q_1, v[r \mapsto a])$ and $(\theta \triangleright p, v) \xrightarrow{a} (\theta_2 \triangleright q_2, v[s \mapsto a])$. By $r \neq s$, we have $v[r \mapsto a] \neq v[s \mapsto a]$, from which nondeterminism follows.

$\square$

Proof (of Proposition 6). Suppose $\mathcal{S}$ is not deterministic, then there is a reachable configuration (p, v) in $\mathsf{CLTS}(\mathcal{S})$ and two transitions $(p, v) \xrightarrow{a} (q_1, v_1)$ and $(p, v) \xrightarrow{a} (q_2, v_2)$, with $(q_1, v_1) \neq (q_2, v_2)$. By Proposition 4, these transitions exist if an only if there are two transitions $(\theta \triangleright p, v) \xrightarrow{a} (\theta_1 \triangleright q_1, v_1)$, $(\theta \triangleright p, v) \xrightarrow{a} (\theta_2 \triangleright q_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ i.e., if and only if $\mathsf{N}(\mathcal{S})$ is not deterministic. $\square$

A.1 Proof of Theorem 3

Given any two SRAs $\mathcal{S}_1$ and $\mathcal{S}_2$, by Theorem 4 we have $\mathcal{S}_1 \prec \mathcal{S}_2$ if and only if $\mathsf{N}(\mathcal{S}_1) \prec \mathsf{N}(\mathcal{S}_2)$. We will now show that $\mathsf{N}(\mathcal{S}_1) \prec \mathsf{N}(\mathcal{S}_2)$ if and only if $\mathcal{S}_1 \overset{\mathsf{N}}{\prec} \mathcal{S}_2$. We show the two directions separately in the following lemmata.

Lemma 4. *Let $\mathcal{R}$ be a N-simulation such that*

$$(\mathsf{N}(q_{01}), \mathsf{N}(q_{02}), v_{01} \bowtie v_{02}) \in \mathcal{R}.$$

Then the following relation:

$$\mathcal{R}' = \{((\theta_1 \triangleright p_1, v_1), (\theta_2 \triangleright p_2, v_2)) \,|\, (\theta_1 \triangleright p_1, \theta_2 \triangleright p_2, \sigma) \in \mathcal{R},$$
$$v_1 \models \theta_1, v_2 \models \theta_2, v_1, v_2 \models \sigma\}$$

is a simulation on $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ such that $((\mathsf{N}(q_{01}), v_{01}), (\mathsf{N}(q_{02}), v_{02})) \in \mathcal{R}'$.

Proof. First of all, by definition of $\mathcal{R}'$, $(\mathsf{N}(q_{01}), \mathsf{N}(q_{02}), v_{01} \bowtie v_{02}) \in \mathcal{R}$ implies $((\mathsf{N}(q_{01}), v_0), (\mathsf{N}(q_{02}), \sigma_0)) \in \mathcal{R}'$. The other conditions follow by definition of θ_{01}, θ_{02} and $v_{01} \bowtie v_{02}$.

We will now prove that $\mathcal{R}'$ is a simulation. Suppose $((\theta_1 \rhd p_1, v_1), (\theta_2 \rhd p_2, v_2)) \in \mathcal{R}'$, hence $(\theta_1 \rhd p_1, \theta_2 \rhd p_2, \sigma) \in \mathcal{R}$. By Definition 5, we must have that $\theta_1 \rhd p_1 \in \mathsf{N}(F_1)$ implies $\theta_2 \rhd p_2 \in \mathsf{N}(F_2)$. Now, suppose $(\theta_1 \rhd p_1, v_1) \xrightarrow{a} (\theta_1' \rhd q_1, v_1')$. We have to prove that there is $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2' \rhd q_2, v_2')$ such that $((\theta_1' \rhd q_1, v_1'), (\theta_2' \rhd q_2, v_2')) \in \mathcal{R}'$. Let $(\theta_1 \rhd p_1, v_1) \xrightarrow{a} (\theta_1' \rhd q_1, v_1')$ be instance of $\theta_1 \rhd p_1 \xrightarrow{\varphi_1/\ell_1} \theta_1' \rhd q_1$. We proceed by cases on ℓ_1:

1. $\ell_1 = r^=$: Then $\theta_1' = \theta_1$, $v_1' = v_1$, $a = v_1(r)$ and we have two cases.
   (a) If $r \in \mathsf{dom}(\sigma)$, then by definition of $\mathcal{R}$ there is $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/\sigma(r)^=} \theta_2 \rhd q_2 \in \mathsf{N}(\Delta_2)$ such that $(\theta_1 \rhd q_1, \theta_2 \rhd q_2, \sigma) \in \mathcal{R}$. This transition is enabled by θ_2, so by $v_2 \models \theta_2$ and $a = v_2(\sigma(r))$, by definition of σ, we get the existence of $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2 \rhd p_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_2))$. This is the required simulating transition. In fact, from $(\theta_1 \rhd q_1, \theta_2 \rhd q_2, \sigma) \in \mathcal{R}$, and the assumptions $v_1 \models \theta_1$ and $v_2 \models \theta_2$, we obtain $((\theta_1 \rhd q_1, v_1), (\theta_2 \rhd q_2, v_2)) \in \mathcal{R}'$, by definition of $\mathcal{R}'$.

   (b) If $r \notin \mathsf{dom}(\sigma)$, then by definition of $\mathcal{R}$ there is $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/s^\bullet} \theta_2[r \mapsto \varphi_1] \rhd q_2 \in \mathsf{N}(\Delta_2)$ such that $(\theta_1 \rhd q_1, \theta_2[r \mapsto \varphi_1] \rhd q_2, \sigma[r \mapsto s]) \in \mathcal{R}$. Since $r \notin \mathsf{dom}(\sigma)$ and $a \in [\![\varphi_1]\!]$, we have $a \in [\![\varphi_1]\!] \setminus \mathsf{img}(v_2)$. Therefore there is $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2[r \mapsto \varphi_1] \rhd q_2, v_2[s \mapsto a]) \in \mathsf{N}(\Delta_2)$. This is the required simulating transition. In fact, we have $v_1 \models \theta_1$ and, from $v_2 \models \theta_2$ and $a \in [\![\varphi_1]\!]$, we get $v_2[s \mapsto a] \models \theta_2[s \mapsto \varphi_1]$. Moreover, we have $(\theta_1 \rhd q_1, \theta_2[r \mapsto \varphi_1] \rhd q_2, \sigma[r \mapsto s]) \in \mathcal{R}$, and $\sigma[r \mapsto s]$ encodes the new equality $v_1(r) = v_2[r \mapsto v_1(r)](s)$. Therefore, by definition of $\mathcal{R}'$, we obtain $((\theta_1 \rhd q_1, v_1), (\theta_2[s \mapsto \varphi_1] \rhd q_2, v_2[s \mapsto a])) \in \mathcal{R}'$.

2. $\ell_1 = r^\bullet$: Then $\theta_1' = \theta[r \mapsto \varphi_1]$, $v_1' = v_1[r \mapsto a]$, $a \in [\![\varphi_1]\!] \setminus \mathsf{img}(v_1)$, and we have two cases:
   (a) There is $s \in R_2$ such that $v_2(s) = a$. Since $a \notin \mathsf{img}(v_1)$, we have $s \in R_2 \setminus \mathsf{dom}(\sigma)$, so by Definition 6 there is $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/s^=} \theta_2 \rhd q_2 \in \mathsf{N}(\Delta_2)$ such that $(\theta_1[r \mapsto \varphi_1] \rhd q_1, \theta_2 \rhd q_2, \sigma[r \mapsto s]) \in \mathcal{R}$, which instantiates to $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2 \rhd q_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_2))$. The argument is then similar to the point 1.(b).
   (b) No $s \in R_2$ is such that $v_2(s) = a$. Then we must have $\mathscr{E}(\theta_1, \varphi_1) + \mathscr{E}(\theta_2, \varphi_1) < |[\![\varphi_1]\!]|$. In fact, if this is not the case, we would have

$$|(\mathsf{img}(v_1) \cup \mathsf{img}(v_2)) \cap [\![\varphi_1]\!]| = |[\![\varphi_1]\!]|$$

by injectivity of v_1 and v_2 and $v_1 \models \theta_1, v_2 \models \theta_2$. Therefore there would be no $a \in [\![\varphi_1]\!] \setminus (\mathsf{img}(v_1) \cup \mathsf{img}(v_2))$, a contradiction. Hence, by definition of $\mathcal{R}$, there is $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/s^\bullet} \theta_2[s \mapsto \varphi_1] \rhd q_2 \in \mathsf{N}(\Delta_2)$ such that

$$(\theta_1[r \mapsto \varphi_1] \rhd q_1, \theta_2[s \mapsto \varphi_1] \rhd q_2, \sigma[r \mapsto s]) \in \mathcal{R}. \tag{2}$$

Because $a \in \llbracket \varphi_1 \rrbracket \setminus \mathsf{img}(v_2)$, this transition instantiates to $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2[r \mapsto \varphi_1] \rhd p_2, v_2[s \mapsto a])$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S})_2)$. Since $v_i \models \theta_i$, $i = 1, 2$, $a \in \llbracket \varphi_1 \rrbracket$ and $v_1, v_2 \models \sigma$, we have $v_1[r \mapsto a] \models \theta_2[r \mapsto \varphi_1]$, $v_2[s \mapsto a] \models \theta_2[s \mapsto \varphi_1]$, and $v_1[r \mapsto a], v_2[s \mapsto a] \models \sigma[r \mapsto s]$. Therefore, by definition of $\mathcal{R}'$, (2) implies

$$((\theta_1[r \mapsto \varphi_1] \rhd q_1, v_1[r \mapsto a]), (\theta_2[s \mapsto \varphi_1] \rhd q_2, v_2[s \mapsto a]) \in \mathcal{R}'.$$

Lemma 5. *Let $\mathcal{R}$ be a simulation on $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ such that*

$$((\mathsf{N}(q_{01}), v_{01}), (\mathsf{N}(q_{02}), v_{02}) \in \mathcal{R}.$$

Then the following relation:

$$\mathcal{R}' = \{(\theta_1 \rhd p_1, \theta_2 \rhd p_2, \sigma) \mid \exists v_1 \models \theta_1, v_2 \models \theta_2 : ((\theta_1 \rhd p_1, v_1), (\theta_2 \rhd p_2, v_2)) \in \mathcal{R},$$
$$\sigma = v_1 \bowtie v_2\}$$

is a N-simulation on $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ such that $((\mathsf{N}(q_{01}), v_0), (\mathsf{N}(q_{02}), \sigma_0)) \in \mathcal{R}'$

Proof. First of all, $((\mathsf{N}(q_{01}), v_0), (\mathsf{N}(q_{02}), \sigma_0)) \in \mathcal{R}'$ clearly follows from

$$((\mathsf{N}(q_{01}), v_{01}), (\mathsf{N}(q_{02}), v_{02})) \in \mathcal{R}.$$

We will now prove that $\mathcal{R}'$ is a N-simulation. Suppose $(\theta_1 \rhd p_1, \theta_2 \rhd p_2, \sigma) \in \mathcal{R}'$. Then, by definition of $\mathcal{R}'$, there are $((\theta_1 \rhd p_1, v_1), (\theta_2 \rhd p_2, v_2)) \in \mathcal{R}$ such that $v_1 \models \theta_1, v_2 \models \theta_2$. Since $\mathcal{R}$ is a simulation, we must have that $\theta_1 \rhd p_1 \in \mathsf{N}(F_1)$ implies $\theta_1 \rhd p_1 \in \mathsf{N}(F_2)$. Now, suppose $\theta_1 \rhd p_1 \xrightarrow{\varphi_1/\ell_1} \theta_1' \rhd q_1 \in \mathsf{N}(\Delta_1)$. We have to exhibit transitions from $\theta_2 \rhd p_2$ that match the definition of normalised symbolic bisimulation:

1. $\ell_1 = r^=$: then $\theta_1' = \theta_1$. Since $\theta_1 \rhd p_1 \xrightarrow{\varphi_1/r^=} \theta_1 \rhd q_1$ is enabled by θ_1, by construction, and $v_1 \models \theta_1$, there is $(\theta_1 \rhd p_1, v_1) \xrightarrow{a} (\theta_1 \rhd q_1, v_1)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_1))$. Since $\mathcal{R}$ is a bisimulation, there is $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2' \rhd q_1, v_2')$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_2))$. We have two cases:
   (a) $r \in \mathsf{dom}(\sigma)$, then $v_2(\sigma(r)) = a$, and the transition in $\mathsf{CLTS}(\mathcal{S}_2)$ must be instance of some $\theta_2 \rhd p_2 \xrightarrow{\varphi_2/r^=} \theta_2 \rhd q_2 \in \mathsf{N}(\Delta_2)$. Now, since $a \in \llbracket \varphi_1 \rrbracket \cap \llbracket \varphi_2 \rrbracket$, and φ_1 and φ_2 are minterms (recall that $\mathsf{N}(\mathcal{S}_1)$ and $\mathsf{N}(\mathcal{S}_2)$ are defined over the same set of minterms), we must have $\varphi_2 = \varphi_1$. Therefore the given transition of $\mathsf{N}(\mathcal{S}_2)$ is the required simulating one. In fact, because $\mathcal{R}$ is a bisimulation, we have $((\theta_1 \rhd q_1, v_1), (\theta_2 \rhd q_2, v_2)) \in \mathcal{R}$, which implies $(\theta_1 \rhd q_1, \theta_2 \rhd q_2, \sigma) \in \mathcal{R}'$, by definition of $\mathcal{R}'$.
   (b) $r \notin \mathsf{dom}(\sigma)$, then $a \notin \mathsf{img}(v_2)$, and the transition in $\mathsf{CLTS}(\mathcal{S}_2)$ must be instance of some $\theta_2 \rhd p_2 \xrightarrow{\varphi_2/s^\bullet} \theta_2[s \mapsto \varphi_2] \rhd q_2 \in \mathsf{N}(\Delta_2)$. By the same reasoning as above, we must have $\varphi_1 = \varphi_2$. Because $\mathcal{R}$ is a bisimulation, we have $((\theta_1 \rhd q_1, v_1), (\theta_2[s \mapsto \varphi_2] \rhd q_2, v_2[s \mapsto a])) \in \mathcal{R}$, which implies $(\theta_1 \rhd q_1, \theta_2[r \mapsto \varphi_1] \rhd q_2, v_1 \bowtie v_2[s \mapsto a]) \in \mathcal{R}'$, by definition of $\mathcal{R}'$.

2. $\ell_1 = r^\bullet$: then $\theta_1' = \theta[r \mapsto \varphi_1]$. Since $\theta_1 \rhd p_1 \xrightarrow{\varphi_1/r^=} \theta[r \mapsto \varphi_1] \rhd q_1$ is enabled by $\theta[r \mapsto \varphi_1]$, by construction, and $v_1 \models \theta_1$, there is $(\theta_1 \rhd p_1, v_1) \xrightarrow{a} (\theta[r \mapsto \varphi_1] \rhd q_1, v_1[r \mapsto a])$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_1))$, for all $a \in [\![\varphi_1]\!] \setminus \mathsf{img}(v_1)$. The values of a can be of two kinds:

(a) $a = v_2(s)$, for some s. This holds if and only if $s \in R_2 \setminus \mathsf{dom}(\sigma)$ and $\varphi_1 = (\theta_2)_s$. In fact, $s \in R_2 \setminus \mathsf{dom}(\sigma)$ if and only if $v_2(s) \notin \mathsf{img}(v_1)$, by definition of σ, and $\varphi_1 = (\theta_2)_s$ if and only if $v_2(s) \in [\![\varphi_1]\!]$, because the two predicates are minterms. Therefore the condition on s above is equivalent to $v_2(s) \in [\![\varphi_1]\!] \setminus \mathsf{img}(v_1)$.

Because $\mathcal{R}$ is a simulation, there is a transition $(\theta_2 \rhd p_2, v_2) \xrightarrow{a} (\theta_2 \rhd q_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}_2))$. This transition must be instance of some $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/s^=} \theta_2 \rhd q_2 \in \mathsf{N}(\Delta_2)$. This is the required simulating transition. In fact, we have $((\theta_1[r \mapsto \varphi_1] \rhd q_1, v_1[r \mapsto a]), (\theta_2 \rhd q_2, v_2)) \in \mathcal{R}$, which implies

$$(\theta_1 \rhd q_1, \theta_2 \rhd q_2, v_1[r \mapsto a] \bowtie v_2) \in \mathcal{R}'$$

Notice that $v_1[r \mapsto a] \bowtie v_2 = \sigma[r \mapsto s]$ follows from the assumption $\sigma = v_1 \bowtie v_2$, and from $v_1[r \mapsto a](r) = v_2(s)$.

(b) $a \notin \mathsf{img}(v_2)$. Then the transition above must be an instance of $\theta_2 \rhd p_2 \xrightarrow{\varphi_1/s^\bullet} \theta_2[r \mapsto \varphi_1] \rhd q_2$. The argument proceeds similarly as the point above. $\qquad\square$

Proof (of Proposition 6).

Suppose $\mathcal{S}$ is not deterministic, then there is a reachable configuration (p, v) in $\mathsf{CLTS}(\mathcal{S})$ and two transitions $(p, v) \xrightarrow{a} (q_1, v_1)$ and $(p, v) \xrightarrow{a} (q_2, v_2)$, with $(q_1, v_1) \neq (q_2, v_2)$. By Proposition 4, these transitions exist if an only if there are two transitions $(\theta \rhd p, v) \xrightarrow{a} (\theta_1 \rhd q_1, v_1)$, $(\theta \rhd p, v) \xrightarrow{a} (\theta_2 \rhd q_2, v_2)$ in $\mathsf{CLTS}(\mathsf{N}(\mathcal{S}))$ i.e., if and only if $\mathsf{N}(\mathcal{S})$ is not deterministic. $\qquad\square$

B   Additional results

Proposition 7. *Given a deterministic single-valued SRA $\mathcal{S}$, let $\mathsf{pred} \colon Q \times (R \cup \{\bullet\}) \to \Psi$ be the function*

$$\mathsf{pred}(p, x) = \begin{cases} \bigvee\{\varphi \mid p \xrightarrow{\varphi/x} q \in \Delta\} & x \in R \\ \bigvee\{\varphi \mid p \xrightarrow{\varphi/r^\bullet} q \in \Delta\} & x = \bullet \end{cases}$$

where $\bigvee \emptyset = \bot$. Let $\mathcal{S}' = (R, Q \cup \{\mathsf{sink}\}, q_0, v_0, , F, \Delta')$ be given by

$$\Delta' = \Delta \cup \Delta_\neg \cup \Delta_{\mathsf{sink}}$$

$$\Delta_\neg = \bigcup_{p \in Q} \{p \xrightarrow{\neg\mathsf{pred}(p,r)/r} \mathsf{sink} \mid r \in R\} \cup \{p \xrightarrow{\neg\mathsf{pred}(p,\bullet)/\tilde{r}^\bullet} \mathsf{sink}\}$$

$$\Delta_{\mathsf{sink}} = \{\mathsf{sink} \xrightarrow{\top/r} \mathsf{sink} \mid r \in R\} \cup \{\mathsf{sink} \xrightarrow{\top/\tilde{r}^\bullet} \mathsf{sink}\}$$

where $\tilde{r} \in R$ is a chosen register. Then S' is complete and deterministic, and $\mathscr{L}(S') = \mathscr{L}(S)$.

Proof. We first show that S' is complete, i.e, that for any configuration (p, v) of S' and any $a \in \mathcal{D}$ there is a transition $(p, v) \xrightarrow{a} (q, w)$ in $\mathsf{CLTS}(S')$. This is clearly true for $p = \mathsf{sink}$, as its outgoing transitions can read anything. If $p \neq \mathsf{sink}$, then either there is $r \in R$ such that $v(r) =$, or $a \notin \mathsf{img}(v)$. In the first case, either $a \in [\![\mathsf{pred}(p, r)]\!]$, and so a transition in Δ can read it, or $a \in \neg[\![\mathsf{pred}(p, r)]\!]$, and so a transition in $\Delta_\neg$ can read it. The case $a \notin \mathsf{img}(v)$ is analogous.

Notice that this case analysis also tells us that the transitions in Δ, $\Delta_\neg$ and Δ_sink are mutually exclusive, from which determinism follows.

The last claim $\mathscr{L}(S) = \mathscr{L}(S')$ follows from the fact that all the additional transitions of S' go to sink, which is non-accepting. $\qquad\square$