Macroeconometrics
Problem Set 3

Nunzio Fallico 3072744
Chiara Locatelli 3070485
Matteo Ticli 3077833

December 11, 2022
Università Bocconi

Exercise 1

We are given the following ECM:

$$\Delta y_t = \begin{bmatrix} -0.6 & 0.5 \\ 0.5 & -\frac{5}{12} \end{bmatrix} y_{t-1} - \begin{bmatrix} 0.4 & 0.5 \\ 0 & 0 \end{bmatrix} \Delta y_{t-1} + \varepsilon_t$$

In order to find the number of cointegrated relations, we need to calculate $rank(\phi_1)$. $\phi(1) = \begin{bmatrix} -0.6 & 0.5 \\ 0.5 & -\frac{5}{12} \end{bmatrix}$ is the coefficient matrix of the y_{t-1} term. Usually, $rank(\phi(1)) = k$, where $k < n$. In this case, given the bidimensional specification of the model, there are two possible scenarios:

- $rank(\phi(1)) = 1$: One cointegrated relation. y_{t-1} is non stationary because $\phi(1)$ is not invertible.
- $rank(\phi(1)) = 2$: Two cointegrated relations. y_{t-1} is stationary due to ϕ_1 invertibility.

Recall that $rank(\phi) \neq 0$ since $\phi(1)$ is not a null matrix.

In order to calculate $rank(\phi(1))$, we simply take the determinant of $\phi(1)$:

$$det(\phi(1)) = 0.6 * \frac{5}{12} - 0.5 * 0.5 = 0$$

Thus, we conclude that $k = rank(\phi(1)) = 1$. There is only one cointegreated relation specified within the system presented.

Exercise 2

To find a basis of the cointegrated space, we need to solve $\phi(1) = BA'$ thus, we have to solve the following system:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \phi(1) = \begin{bmatrix} -0.6 & 0.5 \\ 0.5 & -\frac{5}{12} \end{bmatrix}$$

Which can be written as:

$$\begin{bmatrix} a_1 b_1 & a_2 b_1 \\ a_1 b_2 & a_2 b_2 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.5 \\ 0.5 & -\frac{5}{12} \end{bmatrix}$$

The system has the following form:

$$\begin{cases} a_1b_1 = -0.6 \\ a_2b_1 = 0.5 \\ a_1b_2 = 0.5 \\ a_2b_2 = -\frac{5}{12} \end{cases}$$

As the lecture notes suggest in the first paragraph of page 89, we can impose a condition on one parameter of the vector A' and then solve for the others. We set $a_1 = 1$:

$$\begin{cases} a_1 = 1 \\ b_1 = -0.6 \\ a_2 = \frac{0.5}{-0.6} = -\frac{5}{6} \\ b2 = \frac{-5}{12} * \frac{-6}{5} = 0.5 \end{cases}$$

Therefore $\phi(1) = \begin{bmatrix} -0.6 \\ 0.5 \end{bmatrix} \begin{bmatrix} 1 & -\frac{5}{6} \end{bmatrix}$. The basis of the cointegrated space is : $\begin{bmatrix} 1 & -\frac{5}{6} \end{bmatrix}$.

Exercise 3

Starting from the theoretical VAR in first difference:

$$\Delta y_t = \phi_1 y_{t-1} - \zeta_1 \Delta y_{t-1} + \varepsilon_t$$

We present explicitly the theoretical form of the model:

$$y_t - I y_{t-1} = \phi_1 y_{t-1} - \zeta_1 \left(y_{t-1} - y_{t-2} \right) + \varepsilon_t$$

$$y_t = I y_{t-1} + \phi_1 y_{t-1} - \zeta_1 y_{t-1} + \zeta_1 y_{t-2} + \varepsilon_t$$

We simply substitute the values of the respective coefficient matrices:

$$y_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y_{t-1} + \begin{bmatrix} -0.6 & 0.5 \\ 0.5 & -\frac{5}{12} \end{bmatrix} y_{t-1} - \begin{bmatrix} 0.4 & 0.5 \\ 0 & 0 \end{bmatrix} y_{t-1} + \begin{bmatrix} 0.4 & 0.5 \\ 0 & 0 \end{bmatrix} y_{t-2} + \varepsilon_t$$

Simple algebra yields:

$$y_t = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{7}{12} \end{bmatrix} y_{t-1} + \begin{bmatrix} 0.4 & 0.5 \\ 0 & 0 \end{bmatrix} y_{t-2} + \varepsilon_t$$

Which is the VAR in levels representation of the ECM given.

Exercise 4

We present here the theoretical impulse responses of the model solved above.

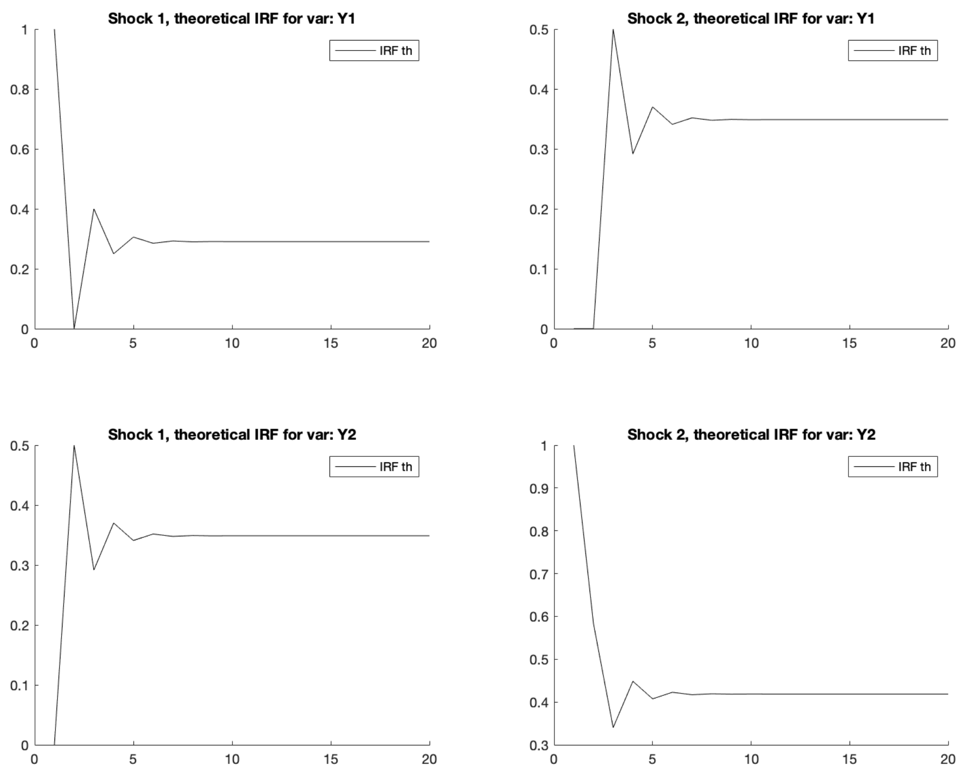

Figure 1: Theoretical Impulse Responses

Exercise 5

In order to sample from the theoretical model presented before, we have to create a simulation path using the Monte Carlo simulation approach. We simply generate the underlying random vector starting from the null condition. We append at each step of the simulation, the corresponding values that are obtained from the theoretical form in levels of the model. We simulate more observations than required for each sample in the simulation, so that we can discard some of them and keep the required 250.

This simulation procedure is used for all the sub-points of exercise 5. Even though we do not have real data, we are able to generate and infinite amount of them starting from the theoretical specification of a model (VEC in this case). In this way we can overcome the data availability problem and perform econometric procedures as if we had real data.

Point a

Once we have generated the data, we can estimate the VAR in levels using and we than compute the required statistics of the impulse response function of such estimated model. The result of the procedure can be seen in the following plot:

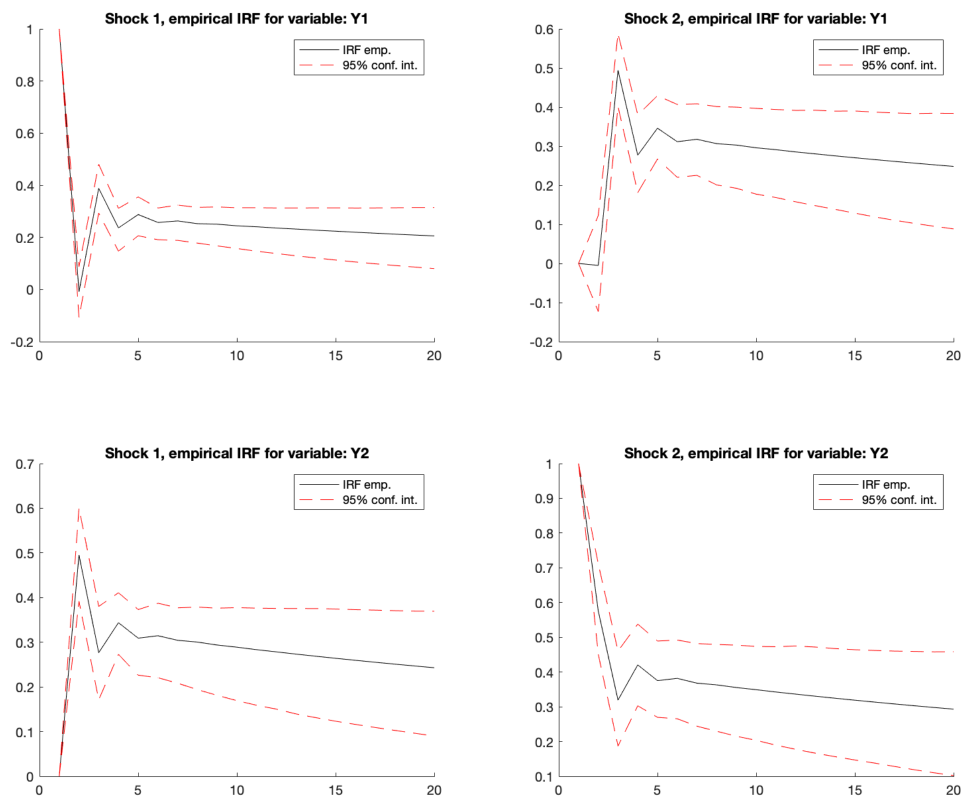

Figure 2: Empirical Impulse Responses - VAR in levels

The plot represents the mean of the 1000 generated samples, each with 250 observations in time, with associated 95% confidence interval. The plot itself is very close to the theoretical impulse response functions but only for early (short) periods of time. Afterwards, the model decays and it is not suitable anymore to represent the theoretical impulse responses.

Point b

We are required to estimate the VAR in first difference. Using the same generated data, as previously explained, we can estimate the model in first differencing. We simply subtract observation t-1 form observation t and store the resulting difference in an empty array. We estimate a VAR model from the data in first difference using only one lag. We then compute the statistics of the impulse response function of this model specification.

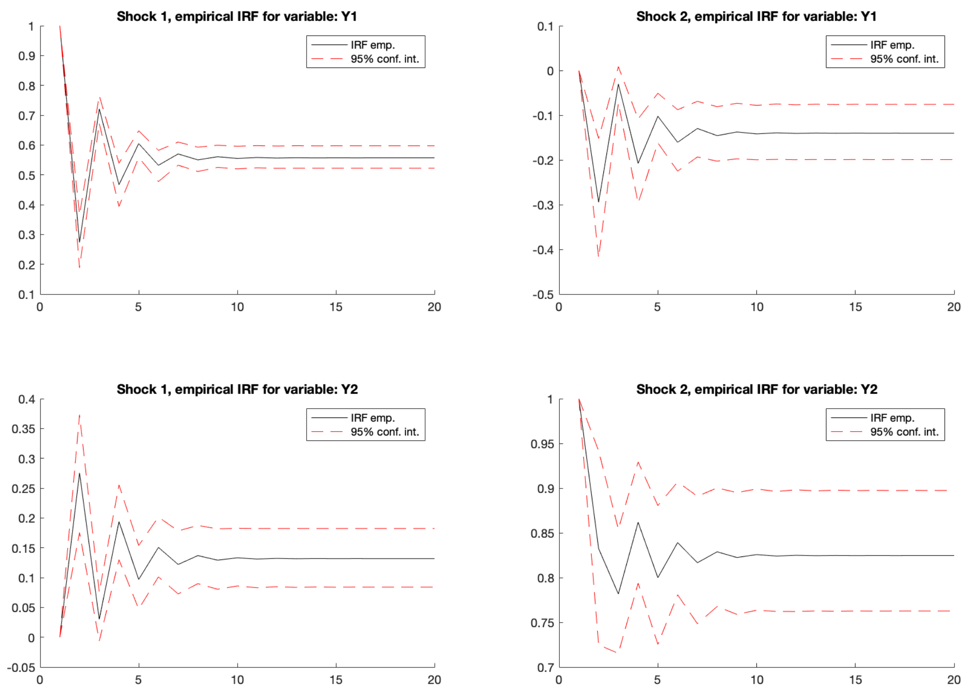

Figure 3: Empirical Impulse Responses - VAR in first difference, one lag

As we can see, the impulse responses are substantially different from the theoretical one and the empirical in levels, this shows that the model in first difference is mis-specified.

Point c

We proceed in estimating a VAR with 4 lags using the data in first difference used for the previous model estimation. The newly estimated VAR can be checked in the following plot:

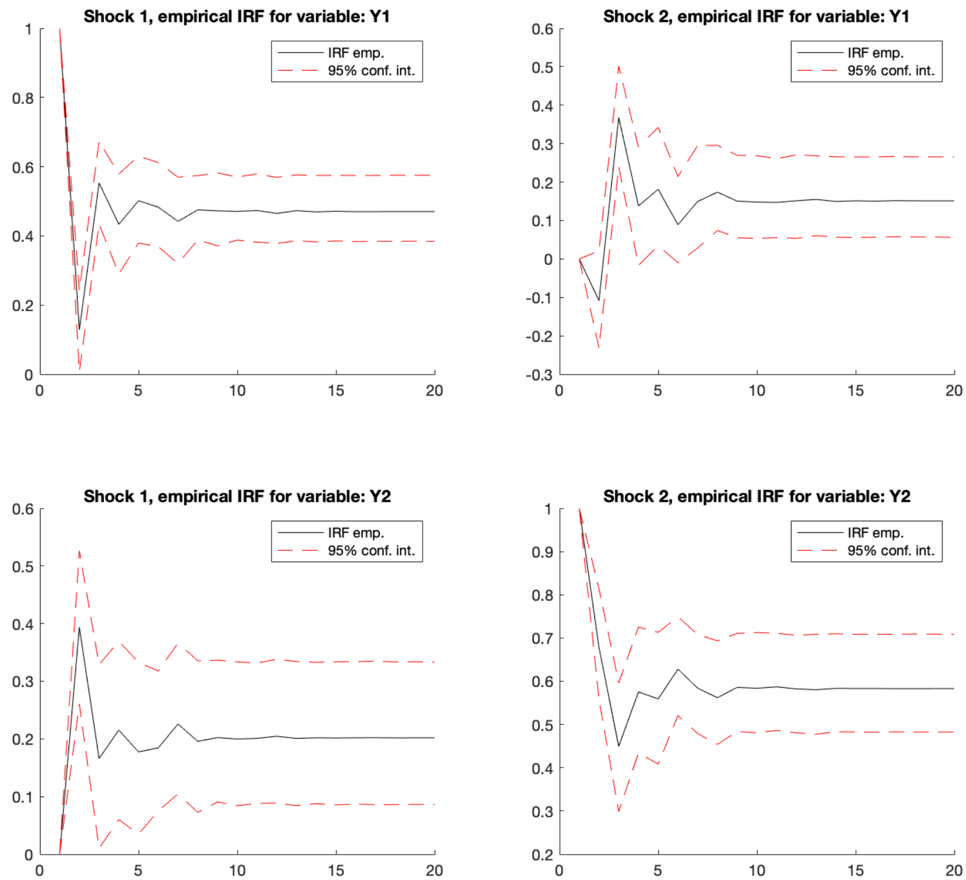

Figure 4: Empirical Impulse Responses - VAR in first difference, four lags

In this peculiar case, as the lagged terms increase to four, more terms are added to specify the VAR in first difference (a model mis-specified as we have seen in Point b) the less the model will be mis-specified. If we proceed with this fashion, the model in first difference with an infinite number of lags will be correctly specified as it would simply tend to be a VAR in levels which is correctly specified as we have seen in Point a.

Point d

We finally implement the Johansen procedure of the simulated data. We convert the VEC representation to a VAR representation. We estimate the VAR on each sample of the simulated data and we compute the impulse response function. We store the data and compute the standard statistics such as mean and 95% confidence interval. The resulting impulse responses can be checked below:

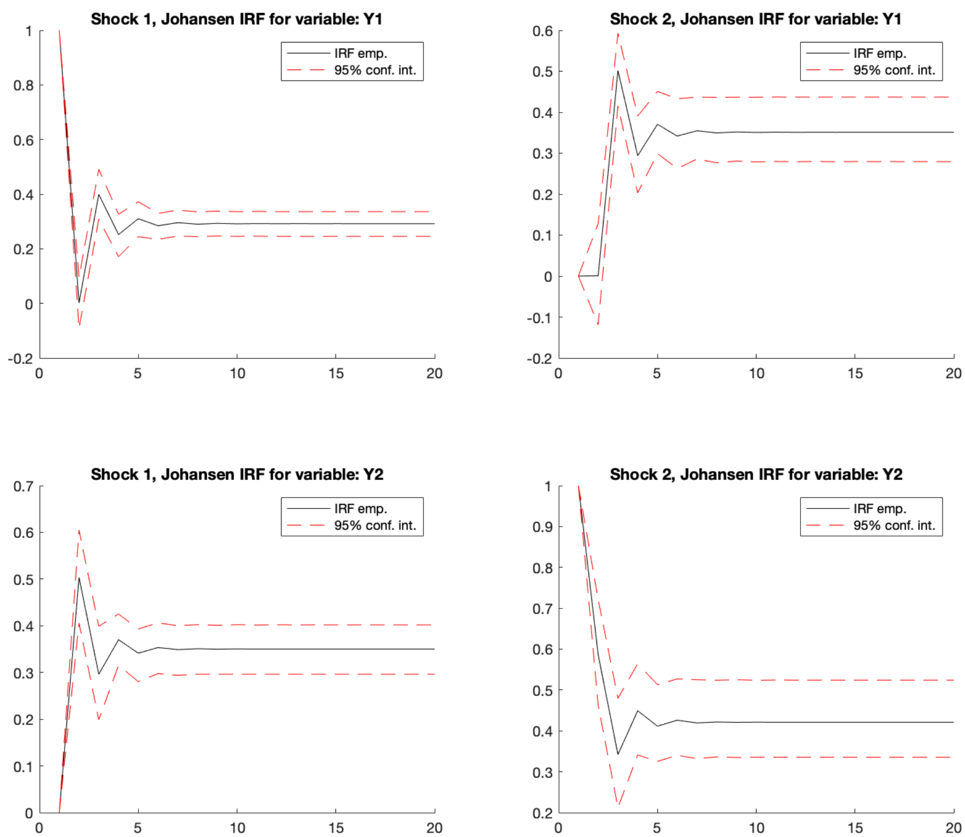

Figure 5: Empirical Impulse Responses - Johansen Procedure

The Johansen procedure allows to correctly specify the theoretical model using empirical data. In fact, the resulting impulse responses' trajectories are the same as the theoretical model's one.