EXERCISE SESSION 2A FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 3-4

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Exercises done during the lecture of March 12, 2014

Exercise 2

Is it true that the parameterized curve $\gamma(t) := (t^2, t^4)$ in $\mathbb{R}^2$ describes the parabola $y = x^2$? Motivate your answer. It this parameterized curve regular?

Let us fix any inverval I of the real line and any curve $\gamma : I \to \mathbb{R}^2$. Moreover, let us fix any set S in $\mathbb{R}^2$. To say that "γ describes the set S" means that

$$\mathrm{Im}(\gamma) = S,$$

where

$$\mathrm{Im}(\gamma) := \{\gamma(t) \text{ s.t. } t \in I\}.$$

So in our case we have to prove or disprove that we have an identity of the following 2 sets in $\mathbb{R}^2$:

$$\{(t^2, t^4) \text{ s.t. } t \in \mathbb{R}\} \stackrel{?}{=} \{(x, y) \in \mathbb{R}^2 \text{ s.t. } y = x^2\}. \tag{0.1}$$

(Note: we write $\stackrel{?}{=}$ all the times that we would like to prove something, but we have not the proof yet (or we don't know if the statement is true or not). This serves you as a reminder: you CANNOT use any equality with a sign $\stackrel{?}{=}$ in order to prove something in the next lines. Actually, what we want to prove is exactly that "$\stackrel{?}{=}$" is "$=$" or to prove that "$\stackrel{?}{=}$" is "$\neq$" and we cannot use it as an hypothesis!)

First of all, let us take any point $(t^2, t^4) \in \mathbb{R}^2$ (with coordinates (x, y)) for some $t \in \mathbb{R}$. Then it satisfies $y = x^2$ since $t^4 = (t^2)^2$. So we have proved that

$$\{(t^2, t^4) \text{ s.t. } t \in \mathbb{R}\} \subseteq \{(x, y) \in \mathbb{R}^2 \text{ s.t. } y = x^2\}. \tag{0.2}$$

Conversely, let us fix any point $(x, y) \in \mathbb{R}^2$, such that $y = x^2$. If x is non-negative, we set $t := \sqrt{x}$. Then we have:

$$(t^2, t^4) = ((\sqrt{x})^2, (\sqrt{x})^4) = (x, x^2) = (x, y).$$

So if x is non-negative, then the point (x, y) can be obtained as a point of the form $\gamma(t)$. HOWEVER, WHAT DOES IT HAPPEN IF x is negative? In this case, we would like to find a point $t \in \mathbb{R}$, such that

$$(t^2, t^4) \stackrel{?}{=} (x, y).$$

Since t is a real number, then t^2 is always non-negative. Therefore, there is no such t. For example, the point $(-1, 1)$ belongs to the set on the right of (0.2) but not to the set on the left of (0.2).

So the inclusion opposite to (0.2) does NOT hold. Therefore the 2 sets in (0.1) are different.

To be more precise: the parabola $y = x^2$ consists of 2 parts: the "right part" (the one for $x \geq 0$) and the "left part") (the one for $x < 0$); the first part coincides with $\mathrm{Im}(\gamma)$ (and not any point of the second part belongs to $\mathrm{Im}(\gamma)$). So the answer to the first question is NO.

The second question of the Exercise asks if the curve γ is regular. In order to answer to this question, we recall the definition of tangent vector of γ for a given point $t \in \mathbb{R}$. This is the vector

$$\dot{\gamma}(t) := \left(\frac{d\gamma_1(s)}{ds}\bigg|_t, \frac{d\gamma_2(s)}{ds}\bigg|_t \right) \in \mathbb{R}^2.$$

In our case, $\gamma_1(s) = s^2$ and $\gamma_2(s) = s^4$, so for any $t \in \mathbb{R}$ we have

$$\dot{\gamma}(t) = (2t, 4t^3) \in \mathbb{R}^2.$$

So for $t = 0$ we have

$$\dot{\gamma}(0) = (0, 0) \in \mathbb{R}^2.$$

Therefore, the curve γ is NOT regular (we recall that "regular" means that for every point t (in the domain of definition of γ) we have $\gamma(t) \neq 0$).

Roughly speaking, the idea is that for $t = 0$ the curve has an "inversion point", i.e. for $t < 0$ the point $\gamma(t)$ is on the "right part" of the parabola and approaches the point $(0, 0)$ when t goes to 0. Then instead of continuing on the "left side" of the parabola, for $t > 0$ the point $\gamma(t)$ returns on the "right part" of the parabola.

Exercise 3

Calculate the tangent vector to the curve $\gamma(t) := (\cos t, \sin t)$ at $t = \pi$.

Since $\gamma(t) = (\cos t, \sin t)$, then for each point $t \in \mathbb{R}$ we have

$$\dot{\gamma}(t) := \left(\left.\frac{d(\cos s)}{ds}\right|_t, \left.\frac{d(\sin s)}{ds}\right|_t \right) = (-\sin t, \cos t).$$

In particular, for $t = \pi$ we have:

$$\dot{\gamma}(\pi) = (-\sin \pi, \cos \pi) = (0, -1).$$

Exercise 4

Show that the following curves in $\mathbb{R}^3$ are unit-speed:

(i) $\gamma(t) := \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$;
(ii) $\gamma(t) := \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

Let us start with (i): first of all, the natural domain of definition of γ is the interval

$$I := [-1, 1] = \{t \text{ s.t. } -1 \leq t \leq 1\}$$

(if we ask γ to be smooth, then the domain must be equal to $]-1, 1[$).

Indeed, for $t < -1$ the quantity $(1+t)^{3/2}$ is not defined; for $t > 1$, the quantity $(1-t)^{3/2}$ is not defined. So $\gamma : I \to \mathbb{R}^3$. For every $t \in I$ we have to compute the tangent vector of γ at t. This is the vector in $\mathbb{R}^3$ given as follows:

$$\dot{\gamma}(t) := \left(\left.\frac{d\gamma_1(s)}{ds}\right|_t, \left.\frac{d\gamma_2(s)}{ds}\right|_t, \left.\frac{d\gamma_3(s)}{ds}\right|_t \right) =$$
$$= \left(\frac{1}{3}\cdot\frac{3}{2}(1+t)^{1/2}, -\frac{1}{3}\cdot\frac{3}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Therefore, for every $t \in I$ we have

$$|\dot{\gamma}(t)| = \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}} = \sqrt{1} = 1.$$

So the curve γ in (i) is a unit-speed curve.

Let us consider now the curve γ in (ii). In this case, the curve is defined for every $t \in \mathbb{R}$. For every such t, we have:

$$\dot{\gamma}(t) := \left(\left.\frac{d\gamma_1(s)}{ds}\right|_t, \left.\frac{d\gamma_2(s)}{ds}\right|_t, \left.\frac{d\gamma_3(s)}{ds}\right|_t \right) = \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right).$$

Therefore, for every $t \in \mathbb{R}$ we have:

$$|\dot{\gamma}(t)| = \sqrt{\frac{16}{25}(\sin t)^2 + (\cos t)^2 + \frac{9}{25}(\sin t)^2} = \sqrt{(\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1.$$

So also the curve γ in (ii) is a unit-speed curve.

Exercise 5

Find the length of the following curve in $\mathbb{R}^3$*:*

$$\begin{array}{rccc} \gamma : & [-1,1] & \longrightarrow & \mathbb{R}^3 \\ & t & \longrightarrow & \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right). \end{array}$$

In Exercise 4 we have already proved that γ is a unit-speed curve, i.e. we have

$$|\dot{\gamma}(t)| = 1 \quad \text{for all } \ t \in I = [-1,1].$$

Therefore, the length of the curve γ is given as follows:

$$S(\gamma) = \int_{-1}^{1} |\dot{\gamma}(t)| dt = \int_{-1}^{1} 1 dt = 2.$$

Exercise 6

Find the length of the following curve in $\mathbb{R}^2$*:*

$$\begin{array}{rccc} \gamma : & [0,1] & \longrightarrow & \mathbb{R}^3 \\ & t & \longrightarrow & \left(t, \frac{1}{2}t^2, \frac{2\sqrt{2}}{3}t^{3/2}\right). \end{array}$$

For each $t \in [0,1]$ we have:

$$\dot{\gamma}(t) := \left(\left.\frac{d\gamma_1(s)}{ds}\right|_t, \left.\frac{d\gamma_2(s)}{ds}\right|_t, \left.\frac{d\gamma_3(s)}{ds}\right|_t \right) = \left(1, t, \sqrt{2}\, t^{1/2}\right).$$

Therefore, for each $t \in [0,1]$ we have:

$$|\dot{\gamma}(t)| = \sqrt{1 + t^2 + 2t} = \sqrt{(1+t)^2} = |1+t|.$$

Therefore, the length of the curve γ is given as follows:

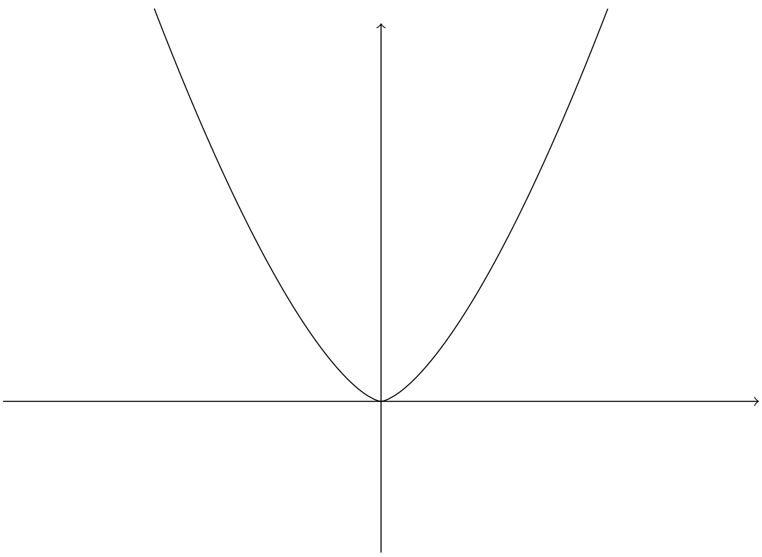
$$S(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt = \int_0^1 |1+t| dt = \int_0^1 (1+t) dt.$$

Since a primitive for $1+t$ is given by $t + t^2/2$, we have

$$S(\gamma) = (t + t^2/2)\big|_{t=1} - (t + t^2/2)\big|_{t=0} = 1 + \frac{1}{2} = \frac{3}{2}.$$

Exercise 8

Find a parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ whose image coincides with the subset $y^3 - x^2 = 0$ of $\mathbb{R}^2$.



We have to find $\gamma(t)$ such that

$$\mathrm{Im}(\gamma) = \{(x, y) \in \mathbb{R}^2 \text{ s. t. } y^3 = x^2\}.$$

First of all, we define

$$\gamma(t) := (t^3, t^2) \quad \text{for all } t \in \mathbb{R}.$$

Clearly $(\gamma_1(t))^2 = t^6 = (\gamma_2(t))^3$, so we have

$$\mathrm{Im}(\gamma) \subseteq \{(x, y) \in \mathbb{R}^2 \text{ s. t. } y^3 = x^2\}.$$

In order to conclude, we need also to prove that the opposite inclusion holds. So let us fix any pair (x, y) such that $y^3 = x^2$. Since x^2 is non-negative, then y^3 is non-negative, hence y is non-negative. So it makes sense to consider $(\sqrt{y})^3 = \sqrt{y^3} \in \mathbb{R}$. Since $y^3 = x^2$, we have:

$$x = \sqrt{y^3} = (\sqrt{y})^3 \quad \text{if } x \geq 0$$

and

$$x = -\sqrt{y^3} = -\sqrt{y}^3 = (-\sqrt{y})^3 \quad \text{if } x < 0.$$

So we have:

$$(x,y) = ((\sqrt{y})^3, (\sqrt{y})^2) = \gamma(\sqrt{y}) \quad \text{if } x \geq 0$$

and

$$(x,y) = ((-\sqrt{y})^3, (\sqrt{y})^2) = \gamma(-\sqrt{y}) \quad \text{if } x < 0.$$

So this proves that every point (x,y) such that $y^3 = x^2$ is of the form $\gamma(t)$ for some point $t \in \mathbb{R}$. So we have proved that γ is such that

$$\mathrm{Im}(\gamma) = \{(x,y) \in \mathbb{R}^2 \text{ s. t. } y^3 = x^2\}.$$

Exercise 9

Find a parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ whose image lies in the set

$$\{(x,y) \in \mathbb{R}^2 \text{ s.t. } y^m - x^n = 0\}, \tag{0.3}$$

where m and n are arbitrary natural numbers.

Taking into account the previous Exercise above, we define

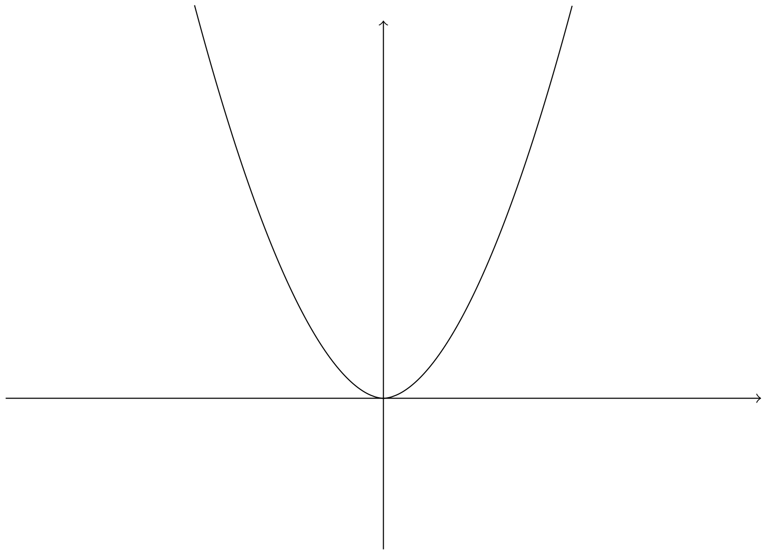
$$\gamma(t) := (t^m, t^n) \qquad \text{for all } t \in \mathbb{R}.$$

Since $(t^n)^m = (t^m)^n$, we get that $\gamma(t)$ has values in the set (0.3). Moreover, $\gamma(t)$ is smooth, so we are done.

NOTE: in general the image of γ does not cover the whole set (0.3). To be more precise, let us consider the following cases:

Case m odd and n even. The set is symmetric with respect to the y axis (i.e. by replacing x with $-x$):

$y^m = x^n$ for m odd and n even



In this case the image of γ covers all the set (0.3). Indeed, let us fix any point (x, y) in (0.3). Since n is even, then x^n is non-negative, so y^m is non-negative. Since m is odd, this implies that y is non-negative. So it makes sense to consider $\sqrt[n]{y}$. Since $y^m = x^n$, we have:

$$x = \sqrt[n]{y^m} = (\sqrt[n]{y})^m \quad \text{if } x \geq 0$$

(note that $\sqrt[n]{y^m} = (\sqrt[n]{y})^m$ because y is non-negative) and

$$x = -\sqrt[n]{y^m} = -(\sqrt[n]{y})^m = (-\sqrt[n]{y})^m \quad \text{if } x < 0$$

(because m is odd). So we have:

$$(x, y) = ((\sqrt[n]{y})^m, (\sqrt[n]{y})^n) = \gamma(\sqrt[n]{y}) \quad \text{if } x \geq 0$$

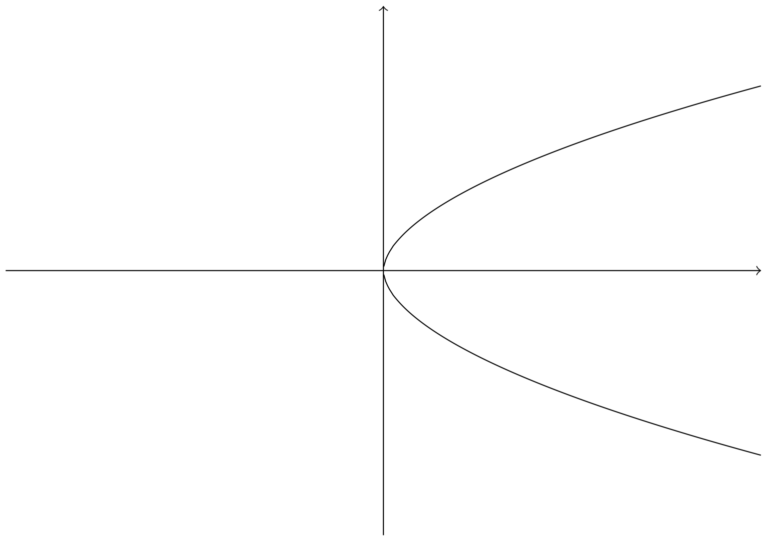
Moreover, since n is even, we have $(-\sqrt{y})^n = (\sqrt{y})^n = y$, so

$$(x, y) = ((-\sqrt[n]{y})^m, (-\sqrt[n]{y})^n) = \gamma(-\sqrt[n]{y}) \quad \text{if } x < 0.$$

So this proves that the set (0.3) is completely covered by the image of γ.

Case m even and n odd. The set is symmetric with respect to the x axis (i.e. by replacing y with $-y$):

$y^m = x^n \quad$ for m even and n odd



Also in this case the image of γ covers all the set (0.3). Indeed, let us fix any point (x, y) in (0.3). Since m is even, then y^m is non-negative, so x^n is non-negative. Since n is odd, this implies that x is non-negative. So it makes sense to consider $\sqrt[m]{x}$. Since $y^m = x^n$, we have:

$$y = \sqrt[m]{x^n} = (\sqrt[m]{x})^n \quad \text{if } y \geq 0$$

(note that $\sqrt[m]{x^n} = (\sqrt[m]{x})^n$ because x is non-negative) and

$$y = -\sqrt[m]{x^n} = -(\sqrt[m]{x})^n = (-\sqrt[m]{x})^n \quad \text{if } y < 0$$

(because n is odd). So we have:

$$(x, y) = \left((\sqrt[m]{x})^m, (\sqrt[m]{x})^n\right) = \gamma(\sqrt[m]{x}) \quad \text{if } y \geq 0.$$

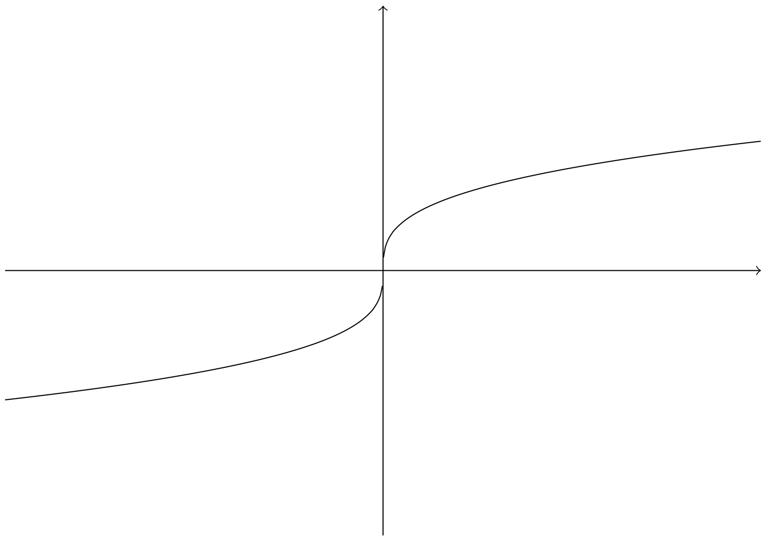
Moreover, since m is even, $(-\sqrt[m]{x})^m = (\sqrt[m]{x})^m = x$, so

$$(x, y) = \left((-\sqrt[m]{x})^m, (-\sqrt[m]{x})^n\right) = \gamma(-\sqrt[m]{x}) \quad \text{if } y < 0.$$

So this proves that the set (0.3) is completely covered by the image of γ.

Case m odd and n odd. The set is symmetric with respect to the origin (i.e. by replacing the pair (x, y) with $(-x, -y)$):

$y^m = x^n \quad$ for m odd and n odd



Also in this case the image of γ covers all the set (0.3). Indeed, let us fix any point (x, y) in (0.3). Since both m and n are odd, we can take both m-th roots and n-th roots without having to check if x or y are non-negative. So for example it makes sense to consider $\sqrt[m]{x}$ and we have we have:

$$y = \sqrt[m]{x^n} = (\sqrt[m]{x})^n$$

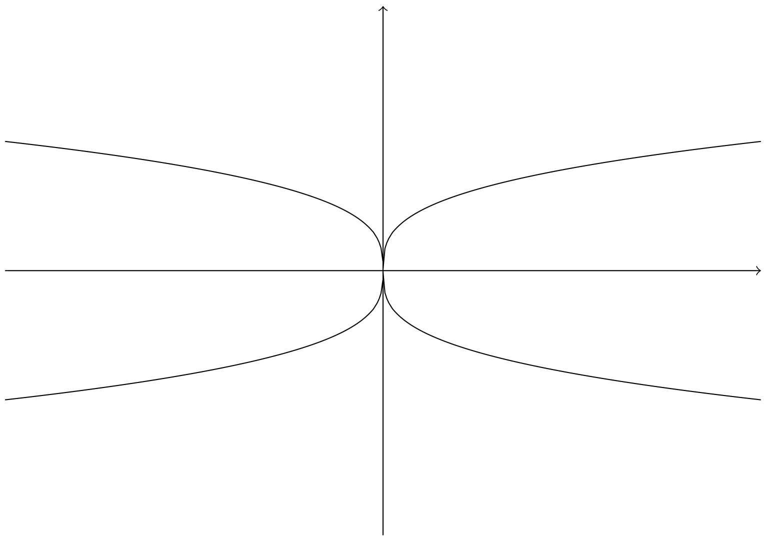
So we have:

$$(x, y) = \big((\sqrt[m]{x})^n, (\sqrt[m]{x})^m\big) = \gamma(\sqrt[m]{x}).$$

This proves that the set (0.3) is completely covered by the image of γ.

Case m even and n even. The set is symmetric with respect to the x axis and to the y axiss (i.e. both by replacing x with $-x$ and by replacing y with $-y$), hence the set is symmetric also with respect to the origin of $\mathbb{R}^2$:

$y^m = x^n \quad$ for m odd and n odd



In this case the curve γ covers only the part of (0.3) that lies in the first quadrant because both t^m and t^n are non-negative (since both m and n are even natural numbers). In this case, we are able to cover separately any of the parts of (0.3) in another quadrant by chosing a different curve. For example, if we want to cover the part of (0.3) in the second quadrant, we can consider the curve

$$\eta(t) := (-t^m, t^n) \quad \text{for all } t \in \mathbb{R}.$$

Anyway, there is no curve that can cover the whole set (0.3) if both m and n are even.

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