EXERCISE SESSION 3A FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 5-6
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Exercises done during the lecture of March 26, 2014

Exercise 1

Show that the action of the orthogonal group $O(3)$ on $\mathbb{R}^3$ preserves the sphere

$$S^2 := \left\{(x,y,z) \in \mathbb{R}^3 \text{ s.t. } x^2+y^2+z^2=r^2\right\} \subset \mathbb{R}^3$$

centred at the origin (with radius $|r|$).

Let us fix any matrix A in $O(3)$. We have to prove that A sends every point of S^2 again to a point of S^2. So let us fix any point $p=(x,y,z)$ in S^2. The matrix A sends such a point to the point p' given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} := A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So we need to prove that this point belongs to S^2. We have:

$$\begin{aligned} x'^2+y'^2+z'^2 &=< (x',y',z'),(x',y',z') >= \\ &=< A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} >= \\ &= \begin{pmatrix} x & y & z \end{pmatrix} \cdot A^T \cdot A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \\ &= \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \\ &= x^2+y^2+z^2 = r^2. \end{aligned}$$

In the previous lines we have used the fact that $A^T \cdot A = \mathrm{Id}$ because $A \in O(3)$.

NOTE: in all the next Exercise we assume that the radius of the sphere is 1 for simplicity.

Exercise 2

Let $\mathbf{p}$ *and* $\mathbf{q}$ *be arbitrary points of the sphere* S^2*. Show that there exists an element* A *of the orthogonal group* $O(3)$ *which, when acting on* $\mathbb{R}^3$*:*

- *sends* $\mathbf{p}$ *to the point* $(0,0,1) \in S^2$
- *sends a big circle through* $\mathbf{p}$ *and* $\mathbf{q}$ *to (a circle in) the plane* $y = 0$*.*

In order to find A, we proceed in several steps as follows.

(Step a) Let us suppose that $\mathbf{p} = (p_1, p_2, p_3)$. If $p_3 = 1$ or $p_3 = -1$, then we have necessarily that $p_1 = p_2 = 0$ (since $\mathbf{p}$ belongs to S^2). In this case we set $R := \mathrm{Id} \in O(3) \subset \mathrm{GL}(3)$ and we pass to step (b). If $p_3 \in [-1, 1[$, we get that $(p_1, p_2) \in \mathbb{R}^2 \smallsetminus \{0\}$. We set $\rho := \sqrt{p_1^2 + p_2^2} \in \mathbb{R}_{>0}$. By a rotation of a suitable angle θ, the vector (p_1, p_2) can be sent to the point $(\rho, 0)$. In other terms, there is θ such that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}$$

(to be more precise, θ is computed as the opposite of the angle formed by the vectors $(1,0)$ and (p_1, p_2)). Then we set

$$R := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

and we have easily that

$$R \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \\ p_3 \end{pmatrix}.$$

So in both subcases of (a) we have proved that there is a matrix $R \in O(3)$, such that

$$R \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \\ p_3 \end{pmatrix}$$

for some $\rho \in \mathbb{R}_{\geq 0}$. For simplicity we set $\mathbf{p}' := (\rho, 0, p_3)$. Clearly $\mathbf{p}'$ belongs to S^2 since $R \in O(3)$ (see Theorem 4.2 in Lecture Notes Part II).

(Step b) Since $\mathbf{p}' := (\rho, 0, p_3)$ belongs to S^2, then $\rho^2 + p_3^2 = 1$, so the point $(\rho, p_3) \in \mathbb{R}^2$ belongs to the unit circumference. Therefore, there is a rotation of an angle σ sending the vector (ρ, p_3) to the vector $(0, 1)$. In other terms, we have

$$\begin{pmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{pmatrix} \cdot \begin{pmatrix} \rho \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we set

$$S := \begin{pmatrix} \cos\sigma & 0 & -\sin\sigma \\ 0 & 1 & 0 \\ \sin\sigma & 0 & \cos\sigma \end{pmatrix} \in O(3)$$

and we have easily that

$$S \cdot \begin{pmatrix} \rho \\ 0 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So until now we have proved that there is a pair of orthogonal matrices R, S, such that

$$S \cdot R \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{0.1}$$

(Step c) Let us denote by $\overline{\mathbf{q}} = (\overline{q}_1, \overline{q}_2, \overline{q}_3)$ the image of the point $\mathbf{q}$ via the map $S \cdot R$. In other terms, let us suppose that

$$S \cdot R \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix}.$$

Since $S \cdot R$ belongs to $O(3)$ and since $\mathbf{q}$ belongs to S^2, then also $\overline{\mathbf{q}}$ belongs to S^2 (see Exercise 1). If $\overline{q}_3 = 1$, then $\overline{q}_1 = \overline{q}_2 = 0$ (because $\overline{\mathbf{q}}$ belongs to S^2). Therefore $\overline{\mathbf{q}}$ coincides with $(0, 0, 1)$, and we set $T := \mathrm{Id} \in O(3)$. If $\overline{q}_3 \in [-1, 1[$, then we get that $(\overline{q}_1, \overline{q}_2) \in \mathbb{R}^2 \smallsetminus \{0\}$. We set $\mu := \sqrt{\overline{q}_1^2 + \overline{q}_2^2} \in \mathbb{R}_{>0}$. By a rotation of a suitable angle α, the vector $(\overline{q}_1, \overline{q}_2)$ can be sent to the point $(\mu, 0)$. In other terms, α is such that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$$

(to be more precise, α is computed as the opposite of the angle formed by the vectors $(1, 0)$ and $(\overline{q}_1, \overline{q}_2)$). Then we set

$$T := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

and we have easily that

$$T \cdot \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} = \begin{pmatrix} \mu \\ 0 \\ \overline{q}_3 \end{pmatrix}.$$

In both subcases, we have found T such that

$$T \cdot \overline{\mathbf{q}} = \begin{pmatrix} \mu \\ 0 \\ \overline{q}_3 \end{pmatrix} \tag{0.2}$$

for some $\mu \in \mathbb{R}_{\geq 0}$. Then we set

$$A := T \cdot S \cdot R.$$

Such a matrix belongs to $O(3)$ because $O(3)$ is a group (see Exercise Session 1A). Now we have

$$T \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{0.3}$$

Therefore,

$$A \cdot \mathbf{p} \overset{(0.1)}{=} T \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.3)}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Moreover,

$$A \cdot \mathbf{q} = T \cdot \begin{pmatrix} \overline{q}_1 \\ \overline{q}_2 \\ \overline{q}_3 \end{pmatrix} \overset{(0.2)}{=} \begin{pmatrix} \mu \\ 0 \\ \overline{q}_3 \end{pmatrix}.$$

Therefore, a big circle through $A(\mathbf{p})$ and $A(\mathbf{q})$ lies in the plane $y = 0$.

Exercise 4

Let $I_{\boldsymbol{p}}$ be the set of spherical lines through a point $\boldsymbol{p} \in S^2$. Show that spherical isometries act transitively on this set, i.e. for any two lines $L_1, L_2 \in I_{\boldsymbol{p}}$ there is a spherical isometry $T : S^2 \to S^2$ such that $T(L_1) = L_2$.

By Exercise 2, there is a spherical isometry A of S^2 such that

$$A \cdot \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{0.4}$$

By Theorem 4.2 in Lecture Notes Part II, $A(L_1)$ and $A(L_2)$ are again spherical lines in S^2, i.e. great circles. In the case under exam, both lines are meridian (i.e. spherical lines joining the north and the south pole of S^2).

Let us denote by $\mathbf{q}_1$ the intersection of $A(L_1)$ with the equator line E (i.e. the spherical line obtained by intersecating S^2 with the plane $z = 0$). Let us denote by $\mathbf{q}_2$ the intersection of $A(L_2)$ with E.

Now $\mathbf{q}_1$ is necessarily of the form $(a_1, b_1, 0)$ for a pair of scalars (a_1, b_1) and analogously $\mathbf{q}_2$ is necessarily of the form $(a_2, b_2, 0)$ for a pair of scalars (a_2, b_2). Since $\mathbf{q}_1$ belongs to S^2, then $a_1^2 + b_1^2 = 1$, so the point (a_1, b_1) belongs to the unit circumference in $\mathbb{R}^2$. Analogously, the point (a_2, b_2) belongs to the unit circumference in $\mathbb{R}^2$. Therefore, there is a rotation of an angle θ mapping (a_1, b_1) to (a_2, b_2), i.e. there is θ such that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Then we set

$$R := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

and we have

$$R \cdot \mathbf{q}_1 = R \cdot \begin{pmatrix} a_1 \\ b_1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ 0 \end{pmatrix} = \mathbf{q}_2. \tag{0.5}$$

Moreover,

$$R \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{0.6}$$

Now let us set

$$\mathbf{p}_1 := A^{-1} \cdot \mathbf{q}_1 \in L_1 \quad \text{and} \quad \mathbf{p}_2 := A^{-1} \cdot \mathbf{q}_2 \in L_2; \tag{0.7}$$

note that $\mathbf{p}_1$ belongs to L_1 because $A \cdot \mathbf{p}_1 = \mathbf{q}_1 \in A(L_1)$ and A is a bijection; analogously for $\mathbf{p}_1$. Moreover, let us set

$$T := A^{-1} \cdot R \cdot A : S^2 \longrightarrow S^2.$$

Then we have

$$T \cdot \mathbf{p} \overset{(0.4)}{=} A^{-1} \cdot R \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.6)}{=} A^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.4)}{=} \mathbf{p}$$

and

$$T \cdot \mathbf{p}_1 \overset{(0.7)}{=} A^{-1} \cdot R \cdot \mathbf{q}_1 \overset{(0.5)}{=} A^{-1}\mathbf{q}_2 \overset{(0.7)}{=} \mathbf{p}_2.$$

Now by construction $\mathbf{q}_1 = (a_1, b_1, 0) \neq (0, 0, 1)$. Since A is a bijection (because it is a spherical isometry), then

$$\mathbf{p}_1 \overset{(0.7)}{=} A^{-1} \cdot \begin{pmatrix} a_1 \\ b_1 \\ 0 \end{pmatrix} \neq A^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.4)}{=} \mathbf{p}.$$

So the *distinct points* $\mathbf{p}$ and $\mathbf{p}_1$ (both belonging to L_1) completely determine the line L_1. In the same way, the points $\mathbf{p}$ and $\mathbf{p}_2$ are distinct and they completely determine the line L_2.

Now T is a spherical isometry (because composition of spherical isometries) that sends $\mathbf{p}$ to itself and $\mathbf{p}_1$ to $\mathbf{p}_2$. Therefore, it sends the entire line passing through $\mathbf{p}$ and $\mathbf{p}_1$ to the line passing through $\mathbf{p}$ and $\mathbf{p}_2$. In other terms, T sends the line L_1 to the line L_2.

Exercise 6

During the lecture we only did part of point (a) of this Exercise. You can find the solutions of (a), (b) and (c) in the sheets of Exercise Session 3B

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