EXERCISE SESSION 4B FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 7-8

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Exercises NOT done during the lecture of April 9, 2014

Exercise 3

Prove Proposition 2.4.1 from Lecture Notes Part III.

The proof is similar to the proof of Proposition 2.2.1 from Lecture Notes Part III. We do it anyway below in full details. We need to prove that there is a one-to-one correspondance

$$\{\text{elements } A \text{ in } O(1,n)\} \longleftrightarrow \{\text{Lorentz basis in } \mathbb{R}^{n+1}\}.$$

We recall that $O(1,n)$ is defined as the symmetry group of the Lorentz bilinear form ϕ_L on $\mathbb{R}^{n+1}$ associated to the symmetric non-degenerate matrix

$$\Phi_L := \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In other terms

$$O(1,n) := \{A \in GL(n+1,\mathbb{R}) \text{ s.t. } A \cdot \Phi_L \cdot A^T = \Phi_L\}.$$

Step a. Let us denote by $\mathbf{e}_0 := (1,0,\cdots,0),\cdots,\mathbf{e}_n := (0,\cdots,0,1)$ the standard basis of $\mathbb{R}^{n+1}$. Then we have:

$$\phi_L(\mathbf{e}_i,\mathbf{e}_j) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \in \{1,\cdots,n\} \\ -1 & \text{for } i = j = 0 \end{cases} \tag{0.1}$$

(see page 7 in Lecture Notes III). Let us fix any $A \in O(1,n)$ and let us define a set of $n+1$ vectors as follows:

$$\mathbf{f}_i := \sum_{k=0}^{n} A_{ik}\mathbf{e}_k \quad \text{for } i = 0,\cdots,n. \tag{0.2}$$

Then for each $i, j = 0, \cdots, n$ we have

$$\phi_L(\mathbf{f}_i, \mathbf{f}_j) = \sum_{k,l=0}^{n} A_{ik}\phi_L(\mathbf{e}_k, \mathbf{e}_l)A_{jl} =$$

$$= A_{i0}\phi_L(\mathbf{e}_0, \mathbf{e}_0)A_{j0} + \sum_{k=1}^{n} A_{ik}\phi_L(\mathbf{e}_k, \mathbf{e}_k)A_{jk} + \sum_{k,l=0,\cdots n \text{ s.t } k\neq l} A_{ik}\phi_L(\mathbf{e}_k, \mathbf{e}_l)A_{jl} =$$

$$= -A_{i0}A_{j0} + \sum_{k=1}^{n} A_{ik}A_{jk}$$

This last term is equal to $(A \cdot \Phi_L \cdot A^T)_{i,j}$, i.e. the (i,j)-th element of the matrix $A \cdot \Phi_L \cdot A^T$. Since A belongs to $O(1,n)$, this matrix is Φ_L. Therefore, we have

$$\phi_L(\mathbf{f}_i, \mathbf{f}_j) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \in \{1, \cdots, n\} \\ -1 & \text{for } i = j = 0. \end{cases}$$

This means that the set $\mathbf{f}_0, \cdots, \mathbf{f}_n$ is a Lorentz basis for $\mathbb{R}^{n+1}$.

Step b. Conversely, let us fix any set $\mathbf{f}_0, \cdots, \mathbf{f}_n$ that is a Lorentz basis for $\mathbb{R}^{n+1}$. We have to associate to such a basis a matrix A in $O(1,n)$. In order to do that, let us consider the standard basis $\mathbf{e}_0, \cdots, \mathbf{e}_n$ of $\mathbb{R}^{n+1}$. Since it is a basis, we can decompose each vector $\mathbf{f}_i$ in such a basis, so we can write

$$\mathbf{f}_i = \sum_{k=0}^{n} A_{ik}\mathbf{e}_i \text{ for each } i = 0, \cdots, n, \tag{0.3}$$

i.e.

$$\mathbf{f}_i = \begin{pmatrix} A_{i0} \\ \cdots \\ A_{in} \end{pmatrix} \text{ for each } i = 0, \cdots, n.$$

Let us consider the $(n+1) \times (n+1)$ matrix $A := (A_{ij})_{i,j}$ (where both i and j vary between 0 and n). Then for each $t = 0, \cdots n$ we have

$$A^T \cdot \mathbf{e}_t = (A_{ji}) \cdot \mathbf{e}_t = \begin{pmatrix} A_{t0} \\ \cdots \\ A_{tn} \end{pmatrix} = \mathbf{f}_t.$$

So for each $i, j = 0, \cdots, n$ we have

$$(A \cdot \Phi_L \cdot A^T)_{i,j} = \mathbf{e}_i^T \cdot A \cdot \Phi_L \cdot A^T \cdot \mathbf{e}_j = (A^T \cdot \mathbf{e}_i)^T \cdot \Phi_L \cdot (A^T \cdot \mathbf{e}_j) =$$
$$= \mathbf{f}_i^T \cdot \Phi_L \cdot \mathbf{f}_j = \phi_L(\mathbf{f}_i, \mathbf{f}_j) = \left\{ \begin{array}{cc} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \in \{1, \cdots, n\} \\ -1 & \text{for } i = j = 0 \end{array} \right\} = (\Phi_L)_{i,j}.$$

This proves that $A \cdot \Phi_L \cdot A^T = \Phi_L$, i.e. we have proved that the matrix A belongs to $O(1, n)$.

Step 3. Until now we have proved how we can associate to each $A \in O(1, n)$ a Lorentz basis of $\mathbb{R}^{n+1}$ and conversely. Now we have to show that the 2 procedures are inverse of one another. This fact is obvious since our 2 constructions are based on (0.2) and (0.3).

Exercise 5

Let us consider any $s \in \mathbb{R}$ and let us define the following matrices

$$A_1(s) := \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \quad \text{and} \quad A_2(s) := \begin{pmatrix} \cosh s & \sinh s \\ -\sinh s & -\cosh s \end{pmatrix}.$$

By writing down explicitly the conditions for a 2×2 matrix to be Lorentz, show that any such matrix is a matrix of the form $A_i(s)$ or of the form $-A_i(s)$ for some $i = 1, 2$ and for some $s \in \mathbb{R}$.

Let us choose any matrix $A \in GL(2, \mathbb{R})$ such that

$$A \cdot \Phi_L \cdot A^T = \Phi_L. \tag{0.4}$$

In this case,

$$\Phi_L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then from (0.4) we have:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} =$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -a & -c \\ b & d \end{pmatrix} =$$
$$= \begin{pmatrix} -a^2 + b^2 & -ac + bd \\ -ac + bd & -c^2 + d^2 \end{pmatrix}.$$

So we have

$$\begin{cases} -a^2+b^2=-1 \\ -c^2+d^2=1 \\ -ac+bd=0. \end{cases} \tag{0.5}$$

If we multiply the first line by c^2 and the second line by b^2, we get

$$-a^2c^2+b^2c^2=-c^2 \quad \text{and} \quad -b^2c^2+b^2d^2=b^2.$$

Summing such relations we get:

$$-a^2c^2+b^2d^2=b^2-c^2. \tag{0.6}$$

Moreover, by the third line of (0.5) we have $a^2c^2=b^2d^2$. By replacing in (0.6) we get

$$b^2=c^2. \tag{0.7}$$

Moreover, using the first and second lines of (0.5) and (0.7), we have

$$d^2=1+c^2=1+b^2=a^2$$

So until now we have proved that $c^2=b^2$ and $d^2=a^2$. By the first line of (0.5) we have $a^2=1+b^2\geq 1$, so $a\geq 1$ or $a\leq -1$. We consider such 2 cases separately.

1.**Case** $a\geq 1$. In this case if makes sense to define the real number

$$s:=\operatorname{arcosh} a$$

(we recall that for each $x\in\mathbb{R}$ we have $\cosh x\geq 1$, so the inverse function arcosh can be applied only to values ≥ 1). Then we have

$$a=\cosh s$$

hence

$$b^2=a^2-1=(\cosh s)^2-1\overset{(*)}{=}(\sinh s)^2.$$

We recall that $(*)$ comes from the fact that

$$\cosh s=\frac{e^x+e^{-x}}{2} \quad \text{and} \quad \sinh s=\frac{e^x-e^{-x}}{2}.$$

So we consider 2 subcases as follow.:

1.1.**Subcase** $b=\sinh s$. We have $d^2=a^2=(\cosh s)^2$, so either $d=\cosh s$ or $d=-\cosh s$.

1.1.1.**Sub-subcase** $d=\cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = -(\cosh s)c + (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = \sinh s$. So we have found the matrix

$$A_1(s) := \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}.$$

1.1.2.**Sub-subcase** $d = -\cosh s$ from the third line of (0.5) we have

$$0 = -ac + bd = -(\cosh s)c - (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = -\sinh s$. So we have found the matrix

$$A_2(s) := \begin{pmatrix} \cosh s & \sinh s \\ -\sinh s & -\cosh s \end{pmatrix}.$$

1.2.**Subcase** $b = -\sinh s$. Again we consider separately the case $d = \cosh s$ and the case $d = -\cosh s$.

1.2.1.**Sub-subcase** $d = \cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = -(\cosh s)c - (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = -\sinh s$. So we have found the matrix

$$\begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} = A_1(-s).$$

Here we have used the fact that $\cosh(-s) = \cosh s$ and $\sinh(-s) = \sinh s$ for each $s \in \mathbb{R}$.
1.2.2.**Sub-subcase** $d = -\cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = -(\cosh s)c + (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = \sinh s$. So we have found the matrix

$$\begin{pmatrix} \cosh s & -\sinh s \\ \sinh s & -\cosh s \end{pmatrix} = A_2(-s).$$

2.**Case** $a \leq -1$. In this case it makes sense to define

$$s := \operatorname{arcosh}(-a) \in \mathbb{R}$$

(this makes sense since $-a \geq 1$). Then we have

$$a = -\cosh s.$$

As in case 1, we have $b^2 = (\sinh s)^2$.

2.1.**Subcase** $b = \sinh s$. Again we consider separately the case $d = \cosh s$ and the case $d = -\cosh s$.

2.1.1.**Sub-subcase** $d = \cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = (\cosh s)c + (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = -\sinh s$. So we have found the matrix

$$\begin{pmatrix} -\cosh s & \sinh s \\ -\sinh s & \cosh s \end{pmatrix} = -\begin{pmatrix} \cosh s & -\sinh s \\ \sinh s & -\cosh s \end{pmatrix} = -A_2(-s).$$

2.1.2.**Sub-subcase** $d = -\cosh s$ from the third line of (0.5) we have

$$0 = -ac + bd = (\cosh s)c - (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = \sinh s$. So we have found the matrix

$$\begin{pmatrix} -\cosh s & \sinh s \\ \sinh s & -\cosh s \end{pmatrix} = -\begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} = -A_1(-s).$$

2.2.**Subcase** $b = -\sinh s$. Again we consider separately the case $d = \cosh s$ and the case $d = -\cosh s$.

2.2.1.**Sub-subcase** $d = \cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = (\cosh s)c - (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = \sinh s$. So we have found the matrix

$$\begin{pmatrix} -\cosh s & -\sinh s \\ \sinh s & \cosh s \end{pmatrix} = -\begin{pmatrix} \cosh s & \sinh s \\ -\sinh s & -\cosh s \end{pmatrix} = -A_2(s).$$

2.2.2.**Sub-subcase** $d = -\cosh s$. From the third line of (0.5) we have

$$0 = -ac + bd = (\cosh s)c + (\sinh s)(\cosh s).$$

Since $\cosh s$ is non-zero for each $s \in \mathbb{R}$, this implies that $c = -\sinh s$. So we have found the matrix

$$\begin{pmatrix} -\cosh s & -\sinh s \\ -\sinh s & -\cosh s \end{pmatrix} = -\begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} = -A_1(s).$$

Until now we have only proved the following result:

"If $A \in GL(2,\mathbb{R})$ is such that $A \cdot \Phi_L \cdot A^T = \Phi_L$, then A is equal to $A_i(s)$ or $-A_i(s)$ for some $i = 1,2$ and for some $s \in \mathbb{R}$".

Conversely, a direct check proves that if A is any matrix of the form $A_i(s)$ or $-A_i(s)$ for some $i = 1,2$ and for some $s \in \mathbb{R}$, then A is a Lorentz matrix.

Exercise 6

Let $\mathbb{R}^3$ be equipped with the Lorentz quadratic form q_L. Which of the following vectors are time-like, space-like or light-like?

(i) $\mathbf{u} = (1,1,1)$
(ii) $\mathbf{v} = (4,-3,1)$
(iii) $\mathbf{w} = (5,4,3)$

We recall that $q_L(t,x,y) = -t^2 + x^2 + y^2$. Therefore,

$$\begin{aligned} q_L(\mathbf{u}) &= -1+1+1 = 1, \\ q_L(\mathbf{v}) &= -16+9+1 = -6, \\ q_L(\mathbf{w}) &= -25+16+9 = 0. \end{aligned}$$

So $\mathbf{u}$ is space-like, $\mathbf{v}$ is time-like and $\mathbf{w}$ is light-like (null).

Exercise 7

Let $P = (1,0,0) \in \mathcal{H}^2$; show how to parametrise the circle with centre at P and of radius $r \geq 0$ in $\mathcal{H}^2 \subset \mathbb{R}^3$. Deduce that a circle of radius r has circumference $2\pi \sinh r$.

Given **any** point $P \in \mathcal{H}^2$ and a radius r, we define the circle of radius r centered at P as the set

$$C(P,r) := \{Q \in \mathcal{H}^2 \text{ s.t. } d(P,Q) = r\}$$

where $d(-,-)$ is the hyperbolic distance. We recall (see page 9 Lecture Notes Part III) that any point Q in $\mathcal{H}^2 \subset \mathbb{R}^3$ has coordinates (t,x,y) such that $t^2 - x^2 - y^2 = 1$ and $t > 0$. Moreover, we recall (see page 10 Lecture Notes Part III) that given a point $P \in \mathcal{H}^2$ with coordinates (t',x',y'), we have that

$$d(P,Q) = \log\left(tt' - xx' - yy' + \sqrt{(tt' - xx' - yy')^2 - 1}\right).$$

In the case under exam, P is the point $(1,0,0)$ (i.e. $t' = 1, x' = 0$ and $y' = 0$), so

$$d(P,Q) = \log\left(t + \sqrt{t^2 - 1}\right).$$

So $Q = (t,x,y) \in \mathcal{H}^2$ belongs to $C(P,r)$ if and only if

$$e^r = t + \sqrt{t^2 - 1}.$$

From this we get:

$$e^{-r} = \frac{1}{t + \sqrt{t^2 - 1}},$$

hence

$$\cosh r = \frac{e^r + e^{-r}}{2} = \frac{1}{2}\left(t + \sqrt{t^2 - 1} + \frac{1}{t + \sqrt{t^2 - 1}}\right) =$$
$$= \frac{1}{2} \cdot \frac{t^2 + t^2 - 1 + 2t\sqrt{t^2 - 1} + 1}{t + \sqrt{t^2 - 1}} = \frac{1}{2} \cdot \frac{2t^2 + 2t\sqrt{t^2 - 1}}{t + \sqrt{t^2 - 1}} = t.$$

Therefore,

$$\sqrt{t^2 - 1} = \sqrt{(\cosh r)^2 - 1} = \sqrt{(\sinh r)^2} = |\sinh r| = \sinh r,$$

where we obtain the last identity using the fact that $r \geq 0$. Therefore, when $P = (1, 0, 0)$ we have

$$C(P, r) = \{(t, x, y) \text{ s.t. } t^2 - x^2 - y^2 = 1, t = \cosh r \text{ and } \sqrt{t^2 - 1} = \sinh r\} =$$
$$= \{(\cosh r, x, y) \text{ s.t. } \sqrt{x^2 + y^2} = \sinh r\}.$$

In other terms, the circle with radius $r \geq 0$ centered in $P = (1, 0, 0)$ is simply the intersection of $\mathcal{H}^2$ with the affine plane $\{t = \cosh r\}$; moreover, such a circle is simply a circle in the euclidean geometry, with center in $(\cosh r, 0, 0)$ and radius equal to $\sinh r$. Therefore, its circumference is equal to $2\pi \sinh r$.

Exercise 8

Let

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} \text{ s.t. } y > 0\}$$

be the upper half-plane in the complex plane. Define $\mathbb{H}$-lines to be of two kinds: either vertical Euclidean half-lines

$$L_1 = \{x + iy \in \mathbb{H} \text{ s.t. } x = b\} = \{b + iy \text{ s.t. } y > 0\} \tag{0.8}$$

for any real constant b, or half-circles

$$L_2 = \{x + iy \in \mathbb{H} \text{ s.t.} (x - a)^2 + y^2 = c^2\} \tag{0.9}$$

with centre at $a + 0i$ on the real axis $\{y = 0\}$, for any $a \in \mathbb{R}$ and for any $c \in \mathbb{R}_{\geq 0}$. Show, algebraically or by drawing pictures, that:

(a) *two distinct $\mathbb{H}$-lines meet in at most one point;*
(b) *every pair of distinct points of $\mathbb{H}$ lies on a unique $\mathbb{H}$-line.*

(c) *Given an $\mathbb{H}$-line L and a point $P \in \mathbb{H}$ not on L, there are more than two $\mathbb{H}$-lines through P which are parallel to L (i.e. have no intersection with L).*

(a). Given 2 distinct vertical lines L_1 and L_1', we have that $L_1 \cap L_1' = \varnothing$. Given a vertical line L_1 as in (0.8) and a half-circle as in (0.9), their intersection is the set of points $x + iy$ (with $x, y \in \mathbb{R}$)), such that $x = b$,

$$(b-a)^2 + y^2 = c^2 \quad \text{and} \quad y > 0.$$

If $c^2 - (b-a)^2 > 0$, then there is exactly one such y, namely

$$y = \sqrt{c^2 - (b-a)^2};$$

if $c^2 - (b-a)^2 \leq 0$ there is no solution, hence $L_1 \cap L_2 = \varnothing$. Lastly, we have to consider the case when we are intersecting 2 half-circles. So let us fix L_2 as in (0.9) and

$$L_2' = \{x + iy \in \mathbb{H} \text{ s.t.} (x-a')^2 + y^2 = c'^2\}$$

for any pair pair (a', c') such that $L_2 \neq L_2'$ (this is equivalent to impose that $(a, c) \neq (a', c')$ - here both points belong to $\mathbb{R} \times \mathbb{R}_{\geq 0}$). We know that the intersection of 2 distinct circles consists of at most 2 points, that are symmetric with respect to the (euclidean) line joining the centers of the 2 circles. If the intersection is empty or consists of only 1 point, we are done. Otherwise, let us suppose that the intersection of the circles consists of exactly 2 points. Since the line joining their centers is the axis $\{y = 0\}$, then this means that at exactly one of such points has $y > 0$. The other one belongs to the intersection of the circles, but NOT to the intersection of the half-circles. This suffices to conclude.

(b). Let us fix a pair of distinct points $P_1 = x_1 + iy_1$ and $P_2 = x_2 + iy_2$. If $x_1 = x_2$, then P_1 and P_2 are joined by the vertical line

$$L := \{x_1 + iy \text{ s.t. } y > 0\}.$$

Moreover, this is the only vertical line joining P_1 and P_2; in addition, since $P_1 \neq P_2$, there is no half-circle joining such a pair of points.

Now let us consider the remaining case, i.e. the case when $x_1 \neq x_2$. In this case there is no vertical line joining P_1 and P_2. Then we consider the (euclidean) segment S joining P_1 and P_2 and its medium point M. From M we draw the (euclidean) line T perpendicular to S. Since $x_1 \neq x_2$, then S is not vertical, so T is not an horizontal line. So it intersects the axis $\{y = 0\}$ in exactly one point $a + 0i$ (for some $a \in \mathbb{R}$). Now T is the axis of the segment S, so each point R in T has the same distance from P_1 and P_2. In particular,

$$d(a + 0i, P_1) = d(a + oi, P_2).$$

We denote such a distance by c for simplicity. Then the circumference C with center in $a + 0i$ and radius c passes through the points P_1 and P_2. Since P_1 and P_2 belong to $\mathbb{H}$, then the points P_1 and P_2 belong to the half-circle

$$L_2 := \{x + iy \in \mathbb{H} \text{ s.t. } (x-a)^2 + y^2 = c^2\} = \mathbb{H} \cap C.$$

So we have proved that there is a half-circle passing through P_1 and P_2. In order to conclude, we need to prove that this is the only such half-circle passing through P_1 and P_2. Let us suppose that there is another such semicircle L'_2, with center A in the axis $\{y = 0\}$ and radius r. Then we have that the center A of L'_2 must be at the same distance from P_1 and P_2, hence it must belong to the euclidean line T. Moreover, we are considering only half-circles centered at points in the axis $\{y = 0\}$. So the center A must belong to $T \cap \{y = 0\}$, so it must coincide with $a + 0i$. In this case, the radius r coincides with the radius c, hence L'_2 is equal to L_2. So we have proved that also in this case there is only an $\mathbb{H}$-line passing through the pair of distinct points P_1 and P_2.

(c). We have to consider 2 cases separately.

Case 1. We suppose that L is a vertical line

$$L = \{x + iy \in \mathbb{H} \text{ s.t. } x = b\} = \{b + iy \text{ s.t. } y > 0\}.$$

We fix any point $P = x_0 + iy_0$, with $y_0 > 0$ and $x_0 \neq b$. For simplicity, we suppose that $b < x_0$ (the same proof holds with minor modifications in the other case). We consider any point $Q := b' + 0i$, with

$$b < b' < x_0 \tag{0.10}$$

i.e. any point between P and the intersection of L with the axis $\{y = 0\}$. Then as we did in (b) we construct the unique $\mathbb{H}$-line L_Q passing through P and Q (note: in (b) the point Q belonged to $\mathbb{H}$ and not to the axis $\{y = 0\}$, but this does not give additional problems in the present construction). Now it is clear (either you draw a picture or you do some basic computations) that $L_Q \cap L = \varnothing$. This holds for every point $Q = b' + 0i$ such that (0.10) holds, i.e. we have obtained infinitely many $\mathbb{H}$-lines passing through P that are parallel to L. In this case there is also an extra line parallel to L, namely the vertical line passing through P.

Case 2. In this case we suppose that L is an half-circle, i.e.

$$L = \{x + iy \in \mathbb{H} \text{ s.t. } (x-a)^2 + y^2 = c^2\}$$

for some $a \in \mathbb{R}$ and some $c \in \mathbb{R}_{\geq 0}$. Let us fix again a point $P = x_0 + iy_0$ with $y_0 > 0$ and such that $P \notin L$. This means that

$$(x_0 - a)^2 + y_0^2 \neq c^2$$

So we have to consider 2 subcases separately as follows:

Subcase 2.1. In this subcase we suppose that

$$(x_0 - a)^2 + y_0^2 < c^2.$$

This means that P belongs to the interior of the half-disc defined by L and by the axis $\{y = 0\}$. Now we define

$$c' := d(P, a + 0i).$$

Since P belongs to the interiour of the semidisk mentioned above, then we have

$$c' < c.$$

Then we choose any $\varepsilon \in \mathbb{R}$ satisfying the following conditions

$$0 < \varepsilon < \frac{c - c'}{2} \tag{0.11}$$

(this makes sense because $c' < c$). Having fixed ε, we define

$$c''_\varepsilon := d(P, a + \varepsilon + 0i). \tag{0.12}$$

By triangular inequality, we have

$$c''_\varepsilon \leq d(P, a + 0i) + d(a + 0i, a + \varepsilon + 0i) = c' + \varepsilon. \tag{0.13}$$

Then we consider the $\mathbb{H}$-line E_ε given by the half-circle centered in $a + \varepsilon + 0i$ and with radius c''_ε. Let us fix any point T in E_ε. Then by triangular inequality, (0.13) and (0.11) we have

$$\begin{aligned} d(T, a + 0i) &\leq d(T, a + \varepsilon + 0i) + d(a + \varepsilon + 0i, a + 0i) = \\ &= c''_\varepsilon + \varepsilon \leq c' + \varepsilon + \varepsilon = c' + 2\varepsilon < c' + 2\frac{c - c'}{2} = c. \end{aligned}$$

This means that the whole half-circle L_ε is contained in the half-disc defined by L and by the axis $\{y = 0\}$. In particular, $E_\varepsilon \cap L = \varnothing$ for all ε such that (0.11) holds. Moreover, P belongs to E_ε because of (0.12). So we have found infinitely many $\mathbb{H}$-lines as required.

Subcase 2.2. In this subcase we suppose that

$$(x_0 - a)^2 + y_0^2 > c^2.$$

This means that $P = x_0 + iy_0$ is "outside" the half-disk defined by L and by the axis $\{y = 0\}$. We define

$$c' := d(P, a + 0i).$$

Since P is outside the semidisk mentioned above, then we have

$$c' > c.$$

Then we choose any σ in $\mathbb{R}$ satisfying the following conditions

$$0 < \sigma < \frac{c' - c}{2} \tag{0.14}$$

(this makes sense because $c' > c$). Having fixed σ, we define

$$c''_\sigma := d(P, a + \sigma + 0i).$$

By triangular inequality, we have

$$c' = d(P, a + 0i) \leq d(P, a + \sigma + 0i) + d(a + \sigma + 0i, a + 0i) = c''_\sigma + \sigma. \tag{0.15}$$

Hence,

$$c''_\sigma \geq c' - \sigma. \tag{0.16}$$

Then we consider the $\mathbb{H}$-line D_σ given by the half-circle centered in $a + \sigma + 0i$ and with radius c''_σ. Then for each point T in D_σ using the triangular inequality we have

$$c''_\sigma = d(T, a + \sigma + 0i) \leq d(T, a + 0i) + d(a + 0i, a + \sigma + 0i) = d(T, a + 0i) + \sigma. \tag{0.17}$$

Therefore, using (0.17), (0.16) and (0.14) we have

$$d(T, a + 0i) \geq c''_\sigma - \sigma \geq c' - \sigma - \sigma = c' - 2\sigma > c' - 2\frac{c' - c}{2} = c.$$

This means that T is "outside" the semidisk mentioned above, so $D_\sigma \cap L = \varnothing$ for all σ as in (0.14). Moreover, $P \in D_\sigma$ because of (0.15). So we have found infinitely many $\mathbb{H}$-lines as required. Note that if $|x_0 - a| > c$, then there is an extra $\mathbb{H}$-line passing through P and not intersecting L, namely the vertical line passing through P.

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