EXERCISE SESSION 5A FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

MATTEO TOMMASINI

Homework sheet 9-10

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Exercises done during the lecture of April 30, 2014

Exercise 1

Given three points P, Q and R of the sphere S^2, what calculation can you perform to determine whether or not P, Q and R are collinear, i.e. belong to the same spherical line? Apply it to the points

$$P = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad Q = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right), \quad PR\left(0, \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

P, Q and R are collinear if and only if they belong to the same plane passing through the origin of $\mathbb{R}^3$, i.e. if and only if they are determined by 3 vectors that are linearly dependent. A way of checking this condition is to put the 3 vectors in a square matrix and to verify whether the determinant of such a matrix is zero or not. In our case, we have to compute:

$$\det \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{5} & 0 \\ 0 & 2/\sqrt{5} & -2/\sqrt{5} \\ 1/\sqrt{2} & 0 & 1/\sqrt{5} \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} \det \begin{pmatrix} 2/\sqrt{5} & -2/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{pmatrix} - \frac{1}{\sqrt{5}} \det \begin{pmatrix} 0 & -2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{2}{5} - \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{10}} = 0.$$

Therefore, the vectors associated to the points P, Q and R are linearly dependent, hence the point P, Q and R are collinear in $\mathcal{H}^2$.

Exercise 4

Let us consider $\mathbb{R}^3$ with coordinates T, X, Y and $\mathbb{C}$ with coordinates $z = x + iy$. Consider the map ϕ defined by

$$\begin{array}{rccc} \phi: & \mathbb{R}^3 \smallsetminus \{T = X\} & \longrightarrow & \mathbb{C} \\ & (T, X, Y) & \longrightarrow & \frac{-Y+i}{T-X} \end{array}$$

Show that if $(T, X, Y) \in \mathcal{H}^2$ then $T - X > 0$ and the map ϕ restricted to $\mathcal{H}^2$,

$$\begin{array}{rccc} \phi: & \mathcal{H}^2 & \longrightarrow & \mathbb{C} \\ & (T,X,Y) & \longrightarrow & \frac{-Y+i}{T-X} \end{array}$$

is well-defined. Show that $\phi(\mathcal{H}^2) \subseteq \mathbb{H}^2$, *where*

$$\mathbb{H}^2 := \{z = x + iy \in \mathbb{C} \;\mid\; y > 0\}$$

is the upper-half plane in $\mathbb{C}$.

The fact that $T - X > 0$ follows directly from Lemma 6.1.1 in Lecture Notes Part III, so we refer to that Lemma for a proof. Since $T - X$ is positive, then $1/(T - X) > 0$, so ϕ has values in $\mathbb{H}^2$ because

$$\phi(T,X,Y) = \frac{-Y}{T-X} + i\frac{1}{T-X}.$$

Exercise 5

Consider the map ψ *defined by*

$$\begin{array}{rccc} \psi: & \mathbb{H}^2 & \longrightarrow & \mathbb{R}^3 \\ & z = x+iy & \longrightarrow & \left(\dfrac{1+x^2+y^2}{2y}, \dfrac{-1+x^2+y^2}{2y}, \dfrac{-x}{y}\right). \end{array}$$

Show that the image of this map lies in the subspace $\mathcal{H}^2 \subset \mathbb{R}^3$.

Let us fix any point $z = x + iy \in \mathbb{H}^2$. We have to prove that $\psi(z)$ belongs to $\mathcal{H}^2$. In other words, we have to prove that if we set

$$T := \frac{1+x^2+y^2}{2y}, \quad X := \frac{-1+x^2+y^2}{2y} \quad \text{and} \quad Y := \frac{-x}{y},$$

then $-T^2 + X^2 + Y^2 = -1$. This is actually the case, indeed,

$$\begin{aligned} & -T^2 + X^2 + Y^2 = \\ &= -\frac{1+x^4+y^4+2x^2+2y^2+2x^2y^2}{4y^2} + \frac{1+x^4+y^4-2x^2-2y^2+2x^2y^2}{4y^2} + \frac{x^2}{y^2} = \\ &= \frac{-1-x^4-y^4-2x^2-2y^2-2x^2y^2+1+x^4+y^4-2x^2-2y^2+2x^2y^2+4x^2}{4y^2} = \\ &= \frac{-4y^2}{4y^2} = -1. \end{aligned}$$

Exercise 6

Show that the above maps ϕ and ψ are inverse bijections between $\mathcal{H}^2$ and $\mathbb{H}^2$.

First of all, we show that $\psi \circ \phi$ is the identity of $\mathcal{H}^2$. So let us fix any point $(T, X, Y) \in \mathcal{H}^2$; then we have

$$\psi \circ \phi(T, X, Y) = \psi\left(\frac{-Y}{T-X} + i\frac{1}{T-X}\right) =$$

$$= \left(\frac{1 + \frac{Y^2}{(T-X)^2} + \frac{1}{(T-X)^2}}{2 \cdot \frac{1}{T-X}}, \frac{-1 + \frac{Y^2}{(T-X)^2} + \frac{1}{(T-X)^2}}{2 \cdot \frac{1}{T-X}}, \frac{Y}{T-X} \cdot (T-X)\right) =$$

$$= \left(\left(1 + \frac{Y^2+1}{(T-X)^2}\right) \cdot \frac{T-X}{2}, \left(-1 + \frac{Y^2+1}{(T-X)^2}\right) \cdot \frac{T-X}{2}, Y\right) \overset{(*)}{=}$$

$$\overset{(*)}{=} \left(\left(1 + \frac{T^2-X^2}{(T-X)^2}\right) \cdot \frac{T-X}{2}, \left(-1 + \frac{T^2-X^2}{(T-X)^2}\right) \cdot \frac{T-X}{2}, Y\right) =$$

$$= \left(\frac{T-X+T+X}{T-X} \cdot \frac{T-X}{2}, \frac{-T+X+T+X}{T-X} \cdot \frac{T-X}{2}, Y\right) = (T, X, Y)$$

(here $(*)$ comes from the fact that (T, X, Y) belongs to $\mathcal{H}^2$). So we have proved that $\psi \circ \phi$ is the identity of $\mathcal{H}^2$. Conversely, let us prove that $\phi \circ \psi$ is the identity of $\mathbb{H}^2$. So let us fix any point $z = x + iy \in \mathbb{H}^2$; then we have:

$$\phi \circ \psi(x + iy) = \phi\left(\frac{1 + x^2 + y^2}{2y}, \frac{-1 + x^2 + y^2}{2y}, \frac{-x}{y}\right).$$

Now if we set

$$T := \frac{1 + x^2 + y^2}{2y}, \quad X := \frac{-1 + x^2 + y^2}{2y} \quad \text{and} \quad Y := \frac{-x}{y},$$

then

$$T - X = \frac{1 + x^2 + y^2 + 1 - x^2 - y^2}{2y} = \frac{2}{2y} = \frac{1}{y}.$$

Therefore,

$$\phi \circ \psi(x + iy) = \phi(T, X, Y) = \frac{-Y + i}{T - X} = \frac{x/y + i}{1/y} = \left(\frac{x}{y} + i\right) \cdot y = x + iy.$$

So we have proved that $\phi \circ \psi$ is the identity of $\mathbb{H}^2$.

Exercice 7

During the lecture we considered only the case (A2). You can find both that case and the remaining cases (A1), (B1) and (B2) in the pdf of Exercise Session 5B.

E-mail address: matteo.tommasini2@gmail.com, matteo.tommasini@uni.lu

Mathematics Research Unit
University of Luxembourg
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg