EXERCISE SESSION 5B FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

MATTEO TOMMASINI

Homework sheet 9-10
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Exercises NOT done during the lecture of April 30, 2014

Exercise 2

Let T be an isometry of the hyperbolic plane $\mathcal{H}^2$. Prove that three points P, Q, and R of $\mathcal{H}^2$ are collinear (i.e. lie on the same hyperbolic line) if and only if the points $T(P)$, $T(Q)$ and $T(R)$ are collinear. [Hint: use Corollary 4.2.1 from Lecture notes Part III]

Let us suppose that P, Q and R are collinear. Then by Corollary 4.2.1 we have

$$d(Q,R) = d(P,Q) + d(P,R). \tag{0.1}$$

Since T is an isometry, this implies that

$$d(T(Q),T(R)) = d(T(P),T(Q)) + d(T(P),T(R)). \tag{0.2}$$

Again by Corollary 4.2.1, this implies that $T(P)$, $T(Q)$ and $T(R)$ are collinear.

Conversely, if $T(P)$, $T(Q)$ and $T(R)$ are collinear, then by Corollary 4.2.1 we have that (0.2) holds. Since T is an isometry, then this implies that (0.1) is satisfied, and we conclude using again Corollary 4.2.1.

Exercise 3

Let α, β, γ be the sides (lengths) of a hyperbolic triangle $\triangle PQR$ and let a, b, c be the opposite angles. Prove the ***hyperbolic sine rule***

$$\frac{\sinh\alpha}{\sin a} = \frac{\sinh\beta}{\sin b} = \frac{\sinh\gamma}{\sin c}.$$

The proof follows the same lines of the proof of the sine rule for spherical triangles (see Exercise 7 in Homework Sheet 5-6). The only significant differences are due to the fact that we need to use also the known formulae for cosh and sinh instead of only those for cos and sin. For clarity, we write the proof below.

By Theorem 4.2 in Lecture Notes Part III, we know that

$$\cosh\alpha = \cosh\beta\cosh\gamma - \cos a\sinh\beta\sinh\gamma.$$

Therefore,

$$\cos a = \frac{\cosh\beta\cosh\gamma - \cosh\alpha}{\sinh\beta\sinh\gamma}.$$

So

$$(\sin a)^2 = 1-(\cos a)^2 = 1 - \frac{(\cosh\beta)^2(\cosh\gamma)^2 + (\cosh\alpha)^2 - 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2} =$$
$$= \frac{(\sinh\beta)^2(\sinh\gamma)^2 - (\cosh\beta)^2(\cosh\gamma)^2 - (\cosh\alpha)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2}.$$

Now for each $x \in \mathbb{R}$ we have

$$(\sinh x)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - e^{-2x} - 2}{4} =$$
$$= \frac{e^{2x} - e^{-2x} + 1}{4} - 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = (\cosh x)^2 - 1,$$

hence by replacing above in $(\sinh\beta)^2$ and $(\sinh\gamma)^2$, we get:

$$(\sin a)^2 = \frac{1}{(\sinh\beta)^2(\sinh\gamma)^2} \cdot \Big[1 - (\cosh\beta)^2 - (\cosh\gamma)^2 + (\cosh\beta)^2(\cosh\gamma)^2 +$$
$$-(\cosh\beta)^2(\cosh\gamma)^2 - (\cosh\alpha)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma\Big] =$$
$$= \frac{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2}.$$

By construction a is an angle in $]0,\pi[$, so $\sin a \neq 0$. Therefore,

$$\frac{(\sinh\alpha)^2}{(\sin a)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}.$$

Exactly in the same way we can prove that

$$\frac{(\sinh\beta)^2}{(\sin b)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}$$

and

$$\frac{(\sinh\gamma)^2}{(\sin c)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}.$$

Therefore, we conclude that

$$\frac{(\sinh\alpha)^2}{(\sin a)^2} = \frac{(\sinh\beta)^2}{(\sin b)^2} = \frac{(\sinh\gamma)^2}{(\sin c)^2}.$$

Now α is the hyperbolic length between two distinct vertices of the triangle $\triangle PQR$, hence $\alpha > 0$, so $\sinh\alpha > 0$. Moreover, by construction a is an angle in $]0,\pi[$, therefore $\sin a > 0$. So from the previous identity we get:

$$\frac{\sinh\alpha}{\sin a} = \frac{\sinh\beta}{\sin b} = \frac{\sinh\gamma}{\sin c}.$$

Exercise 7

Define $\mathbb{H}^2$-lines *to be of two kinds: either vertical Euclidean half-lines,*

$$L_1 = \{x + iy \in \mathbb{H} \mid x = b\}$$

for a real constant b*, or half-circles*

$$L_2 = \{x + iy \in \mathbb{H} \mid (x-a)^2 + y^2 = c^2\}$$

with centre at $a + 0i$ *on the real axis* $\{y = 0\}$ *and radius* $c > 0$*. Show that the image of any hyperbolic line* $L \subset \mathcal{H}^2$ *under the map* ϕ *is an* $\mathbb{H}^2$*-line.*

An hyperbolic line is the intersection of $\mathcal{H}^2$ with a plane $\Pi = \{t_0T + x_0X + y_0Y = 0\}$ passing through the origin of $\mathbb{R}^3$. In particular, $(t_0, x_0, y_0) \in \mathbb{R}^3 \smallsetminus \{(0,0,0)\}$ and $\Pi \cap \mathcal{H}^2 \neq \varnothing$. We consider separately the following cases

(A) $y_0 = 0$;
(B) $y_0 \neq 0$.

(A) Case $y_0 = 0$. In this case we have $\Pi = \{t_0T = -x_0X\}$ (with $(t_0, x_0) \neq 0)$). Then we have to consider 2 subcases as follows.

(A1) Subcase $x_0 \neq 0$. In this case we can write

$$\Pi = \left\{X = -\frac{t_0}{x_0}T\right\} \tag{0.3}$$

and we claim that

$$x_0^2 - t_0^2 > 0. \tag{0.4}$$

Indeed, since $\Pi \cap \mathcal{H}^2 = \varnothing$, then there is a point $P = (\overline{T}, \overline{X}, \overline{Y})$ in $\Pi \cap \mathcal{H}^2$. For such a point, we have

$$-1 = -\overline{T}^2 + \overline{X}^2 + \overline{Y}^2 = -\overline{T}^2 + \frac{t_0^2}{x_0^2}\overline{T}^2 + \overline{Y}^2 \geq \frac{t_0^2 - x_0^2}{x_0^2}\overline{T}^2.$$

Since $x_0^2 > 0$, this implies that

$$0 > -x_0^2 \geq (t_0^2 - x_0^2)\overline{T}^2,$$

which implies (0.4). For any pair $(t_0, x_0) \in \mathbb{R}^2$ such that (0.4) holds, the set $\Pi \cap \mathcal{H}^2$ is non-empty. Indeed, it suffices to consider the point

$$P := \left(\frac{|x_0|}{\sqrt{x_0^2 - t_0^2}}, -\frac{t_0|x_0|}{x_0\sqrt{x_0^2 - t_0^2}}, 0 \right)$$

(a direct check proves that $P \in \Pi \cap \mathcal{H}^2$). So we have proved that for each pair $(t_0, x_0) \in \mathbb{R}^2$ such that (0.4) holds, the set $\Pi \cap \mathcal{H}^2$ (with Π as in (0.3)) is an hyperbolic line. Now we have to prove that ϕ maps such an hyperbolic line to an $\mathbb{H}^2$-line. For any point $(T, X, Y) \in \Pi \cap \mathcal{H}^2$ we have

$$\phi(T, X, Y) = \phi\left(T, -\frac{t_0}{x_0}T, Y\right) = \frac{-Y + i}{T + \frac{t_0}{x_0}T} = -\frac{x_0 Y}{(x_0 + t_0)T} + i\frac{x_0}{(x_0 + t_0)T}.$$

Now (T, X, Y) belongs to $\Pi \cap \mathcal{H}^2$, so

$$Y^2 + 1 = T^2 - X^2 = T^2 - \frac{t_0^2}{x_0^2}T^2 = \frac{(x_0^2 - t_0^2)T^2}{x_0^2}.$$

Therefore, if we set $z = x + iy := \phi(T, X, Y)$, we have

$$\begin{aligned} x^2 + y^2 &= \frac{x_0^2 Y^2}{(x_0 + t_0)^2 T^2} + \frac{x_0^2}{(x_0 + t_0)^2 T^2} = \frac{x_0^2 (Y^2 + 1)}{(x_0 + t_0)^2 T^2} = \\ &= \frac{x_0^2 (x_0^2 - t_0^2) T^2}{x_0^2 (x_0 + t_0)^2 T^2} = \frac{x_0 - t_0}{x_0 + t_0}. \end{aligned} \tag{0.5}$$

Since $z = x + iy$ belongs to $\mathbb{H}^2$, then $x^2 + y^2 > 0$, so the quantity

$$\frac{x_0 - t_0}{x_0 + t_0}$$

is strictly positive. Therefore, it makes sense to consider the quantity

$$c := \sqrt{\frac{x_0 - t_0}{x_0 + t_0}} \in \mathbb{R}_{>0}.$$

Then using (0.5) we get that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the circle

$$\{z = x + iy \in \mathbb{C} \text{ s.t. } x^2 + y^2 = c^2\}.$$

By Exercise 4 we know that the image of ϕ is contained in the upper half-plane $\mathbb{H}^2$, so we conclude that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the $\mathbb{H}^2$-line

$$\{z = x + iy \in \mathbb{C} \text{ s.t. } x^2 + y^2 = c^2 \text{ and } y > 0\}$$

(i.e. an half-circle of the form L_2 with $a := 0$).

(A2) Subcase $t_0 \neq 0$. In this case we can write

$$\Pi = \left\{ T = -\frac{x_0}{t_0} X \right\} \tag{0.6}$$

and we claim that

$$x_0^2 - t_0^2 > 0. \tag{0.7}$$

Indeed, since $\Pi \cap \mathcal{H}^2 = \varnothing$, then there is a point $P = (\overline{T}, \overline{X}, \overline{Y})$ in $\Pi \cap \mathcal{H}^2$. For such a point, we have

$$-1 = -\overline{T}^2 + \overline{X}^2 + \overline{Y}^2 = -\frac{x_0^2}{t_0^2}\overline{X}^2 + \overline{X}^2 + \overline{Y}^2 \geq \frac{t_0^2 - x_0^2}{t_0^2}\overline{X}^2.$$

Since $x_0^2 > 0$, this implies that

$$0 > -t_0^2 \geq (t_0^2 - x_0^2)\overline{X}^2,$$

which implies (0.7). For any pair $(t_0, x_0) \in \mathbb{R}^2$ such that (0.7) holds, the set $\Pi \cap \mathcal{H}^2$ is non-empty. Indeed, it suffices to consider the point

$$P := \left(\frac{|x_0|}{\sqrt{x_0^2 - t_0^2}}, -\frac{t_0|x_0|}{x_0\sqrt{x_0^2 - t_0^2}}, 0 \right)$$

(a direct check proves that $P \in \Pi \cap \mathcal{H}^2$). So we have proved that for each pair $(t_0, x_0) \in \mathbb{R}^2$ such that (0.7) holds, the set $\Pi \cap \mathcal{H}^2$ (with Π as in (0.6)) is an hyperbolic line. Now we have to prove that ϕ maps such an hyperbolic line to an $\mathbb{H}^2$-line. For any point $(T, X, Y) \in \Pi \cap \mathcal{H}^2$ we have

$$\phi(T, X, Y) = \phi\left(-\frac{x_0}{t_0} X, X, Y \right) = \frac{-Y + i}{-\frac{x_0}{t_0} X - X} = \frac{t_0 Y}{(x_0 + t_0) X} - i \frac{t_0}{(x_0 + t_0) X}.$$

Now (T, X, Y) belongs to $\Pi \cap \mathcal{H}^2$, so

$$Y^2 + 1 = T^2 - X^2 = \frac{x_0^2}{t_0^2} X^2 - X^2 = \frac{(x_0^2 - t_0^2) X^2}{t_0^2}.$$

Therefore, if we set $z = x + iy := \phi(T, X, Y)$, we have

$$\begin{aligned} x^2 + y^2 &= \frac{t_0^2 Y^2}{(x_0 + t_0)^2 X^2} + \frac{t_0^2}{(x_0 + t_0)^2 X^2} = \frac{t_0^2 (Y^2 + 1)}{(x_0 + t_0)^2 X^2} = \\ &= \frac{t_0^2 (x_0^2 - t_0^2) X^2}{t_0^2 (x_0 + t_0)^2 X^2} = \frac{x_0 - t_0}{x_0 + t_0}. \end{aligned} \tag{0.8}$$

Since $z = x + iy$ belongs to $\mathbb{H}^2$, then $x^2 + y^2 > 0$, so the quantity

$$\frac{x_0 - t_0}{x_0 + t_0}$$

is strictly positive. Therefore, it makes sense to consider the quantity

$$c := \sqrt{\frac{x_0 - t_0}{x_0 + t_0}} \in \mathbb{R}_{>0}.$$

Then using (0.8) we get that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the circle

$$\{z = x + iy \in \mathbb{C} \text{ s.t. } x^2 + y^2 = c^2\}$$

By Exercise 4 we know that the image of ϕ is contained in the upper half-plane $\mathbb{H}^2$, so we conclude that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the $\mathbb{H}^2$-line

$$\{z = x + iy \in \mathbb{C} \text{ s.t. } x^2 + y^2 = c^2 \text{ and } y > 0\}.$$

(B) Case $y_0 \neq 0$. In this case we have $\Pi = \{t_0 T + x_0 X = Y\}$ (with a priori any choice of $(t_0, x_0) \in \mathbb{R}^2$). We obtain this expression by replacing t_0 with $-t_0/y_0$ and x_0 with $-x_0/y_0$ in the expression of Π at the beginning of this exercise). Then we consider 2 subcases as follows.

(B1) Subcase $t_0 + x_0 = 0$. In this case we have $\Pi = \{t_0 T - t_0 X = Y\}$ (for any $t_0 \in \mathbb{R}$). We claim that $\Pi \cap \mathcal{H}^2$ is an hyperbolic line, i.e. that $\Pi \cap \mathcal{H}^2 \neq \varnothing$. Indeed, if we consider the point

$$P := (\overline{T}, \overline{X}, \overline{Y}) := \left(1 + \frac{t_0^2}{2}, \frac{t_0^2}{2}, t_0\right),$$

we have that $P \in \Pi$; moreover $\overline{T} > 0$ and

$$-\overline{T}^2 + \overline{X}^2 + \overline{Y}^2 = -1 - \frac{t_0^4}{4} - t_0^2 + \frac{t_0^4}{4} + t_0^2 = -1.$$

So the point P belongs to $\Pi \cap \mathcal{H}^2$. Now let us apply ϕ to the hyperbolic line $\Pi \cap \mathcal{H}^2$. For any point (T, X, Y) in such a set we have

$$\phi(T, X, Y) = \frac{-Y}{T - X} + i\frac{1}{T - X} = \frac{t_0 X - t_0 T}{T - X} + i\frac{1}{T - X} = -t_0 + i\frac{1}{T - X}.$$

So we have proved that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the vertical line of $\mathbb{C}$:

$$\{z = x + iy \text{ s.t. } x = -t_0\}.$$

By Exercise 4, the image of ϕ is contained in $\mathbb{H}^2$, so the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the vertical semi-line of $\mathbb{H}^2$:

$$L = \{z = x + iy \text{ s.t. } x = -t_0 \text{ and } y > 0\}$$

(i.e. an $\mathbb{H}^2$-line of the form L_1 with $b := -t_0$).

(B2) Subcase $t_0 + x_0 \neq 0$. In this case, we claim that

$$x_0^2 - t_0^2 + 1 > 0. \tag{0.9}$$

By contradiction, let us suppose that

$$x_0^2 - t_0^2 + 1 \leq 0. \tag{0.10}$$

Since $\Pi \cap \mathcal{H}^2$ is an hyperbolic line, then $\Pi \cap \mathcal{H}^2$ is non-emtpy so there must be a point $P = (\overline{T}, \overline{X}, \overline{Y}) \in \Pi \cap \mathcal{H}^2$. For such a point, we must have

$$\begin{aligned} 0 = 1 - \overline{T}^2 + \overline{X}^2 + \overline{Y}^2 = 1 - \overline{T}^2 + \overline{X}^2 + (t_0\overline{T} + x_0\overline{X})^2 = \\ = (1 + x_0^2)\overline{X}^2 + 2t_0x_0\overline{T}\,\overline{X} + 1 - \overline{T}^2 + t_0^2\overline{T}^2. \end{aligned}$$

In other terms, there mus exist a pair $(\overline{T}, \overline{X})$ in $\mathbb{R}^2$ (with $\overline{T} > 0$), such that the equation

$$(1 + x_0^2)\overline{X}^2 + 2t_0x_0\overline{T}\,\overline{X} + 1 - \overline{T}^2 + t_0^2\overline{T}^2 = 0 \tag{0.11}$$

is satisfied. We consider this as a polynomial equation in $\overline{X}$. In order to have a (real) solution in $\overline{X}$, we must have that the determinant Δ (with respect to $\overline{X}$) is non-negative. Now Δ (with respect to $\overline{X}$) is computed as

$$\begin{aligned} \Delta = 4t_0^2x_0^2\overline{T}^2 - 4(1 + x_0^2)(1 - \overline{T}^2 + t_0^2\overline{T}^2) = \\ = 4\left[t_0^2x_0^2\overline{T}^2 - 1 + \overline{T}^2 - t_0^2\overline{T}^2 - x_0^2 + x_0^2\overline{T}^2 - t_0^2x_0^2\overline{T}^2\right] = \\ = 4\left[\overline{T}^2(1 - t_0^2 + x_0^2) - (x_0^2 + 1)\right]. \end{aligned}$$

Using (0.10), we have

$$\Delta \leq -4(x_0^2 + 1) \leq -4 < 0.$$

This proves that (0.11) has no (real) solutions, so $\Pi \cap \mathcal{H}^2 = \varnothing$. Therefore we get a contradiction. So (0.10) cannot be true, so we conclude that necessarily (0.9) holds.

Now $t_0 + x_0 \neq 0$; moreover we have (0.9). So it makes sense to set

$$a := \frac{-1}{t_0 + x_0} \in \mathbb{R} \quad \text{and} \quad c := \frac{\sqrt{x_0^2 - t_0^2 + 1}}{|t_0 + x_0|} \in \mathbb{R}_{>0}.$$

Then we consider the following point

$$P := (\overline{T}, \overline{X}, \overline{Y}) := \left(\frac{a^2+c^2+1}{2c}, \frac{a^2+c^2-1}{2c}, -\frac{a}{c}\right) = \tag{0.12}$$

$$= \left(\frac{1+x_0^2+t_0x_0}{|t_0+x_0|\sqrt{x_0^2-t_0^2+1}}, \frac{1-t_0^2-t_0x_0}{|t_0+x_0|\sqrt{x_0^2-t_0^2+1}}, \frac{t_0+x_0}{|t_0+x_0|\sqrt{x_0^2-t_0^2+1}}\right). \tag{0.13}$$

Using formula (0.12) we get easily that P belongs to $\mathcal{H}^2$; using formula (0.13) we get that P belongs to $\Pi = \{t_0T + x_0X = Y\}$. So $\Pi \cap \mathcal{H}^2 \neq \varnothing$, so it is actually an hyperbolic line. Now let us fix any point $(T, X, Y) \in \Pi \cap \mathcal{H}^2$. Then we have

$$\phi(T, X, Y) = \frac{-Y}{T-X} + i\frac{1}{T-X}.$$

For simplicity, we set $z = x + iy := \phi(T, X, Y)$. So we have

$$\begin{aligned}
(x-a)^2 + y^2 &= \left(-\frac{Y}{T-X} - a\right)^2 + \frac{1}{(T-X)^2} = \\
&= \frac{Y^2}{(T-X)^2} + a^2 + \frac{2aY}{T-X} + \frac{1}{(T-X)^2} \overset{(*)}{=} \\
&\overset{(*)}{=} \frac{T^2-X^2-1}{(T-X)^2} + \frac{1}{(t_0+x_0)^2} - \frac{2(t_0T+x_0X)}{(t_0+x_0)(T-X)} + \frac{1}{(T-X)^2}
\end{aligned}$$

(in $(*)$ we used the fact that $Y^2 = T^2 - X^2 - 1$ because $(T, X, Y) \in \mathcal{H}^2$ and $Y = t_0T + x_0X$ because (T, X, Y) belongs to Π). Then we get:

$$\begin{aligned}
(x-a)^2 + y^2 &= \frac{T^2-X^2}{(T-X)^2} + \frac{1}{(t_0+x_0)^2} - \frac{2(t_0T+x_0X)}{(t_0+x_0)(T-X)} = \\
&= \frac{T+X}{T-X} + \frac{1}{(t_0+x_0)^2} - \frac{2(t_0T+x_0X)}{(t_0+x_0)(T-X)} = \\
&= \frac{1}{(t_0+x_0)^2(T-X)}\Big[(T+X)(t_0^2+2t_0x_0+x_0^2) + T - X - 2(t_0+x_0)(t_0T+x_0X)\Big] = \\
&= \frac{1}{(t_0+x_0)^2(T-X)}\Big[t_0^2T + 2t_0x_0T + x_0^2T + t_0^2X + 2t_0x_0X + \\
&\quad + x_0^2X + T - X - 2t_0^2T - 2t_0x_0X - 2t_0x_0T - 2x_0^2X\Big] = \\
&= \frac{1}{(t_0+x_0)^2(T-X)}\Big[-t_0^2T + x_0^2T + t_0^2X + T - X - x_0^2X\Big] = \\
&= \frac{1}{(t_0+x_0)^2(T-X)} \cdot (T-X)(x_0^2 - t_0^2 + 1) = \frac{x_0^2-t_0^2+1}{(t_0+x_0)^2} = c^2.
\end{aligned}$$

Therefore, we have proved that the image of $\Pi \cap \mathcal{H}^2$ via ϕ is contained in the circle

$$\{z = x + iy \text{ s.t. } (x-a)^2 + y^2 = c^2\}.$$

By Exercise 4 we know that the image of ϕ is contained in $\mathbb{H}^2$, so we have proved that ϕ maps $\Pi \cap \mathcal{H}^2$ in the $\mathbb{H}^2$-line

$$L_2 := \{z = x + iy \text{ s.t. } (x-a)^2 + y^2 = c^2 \text{ and } y > 0\}.$$

Exercise 8

Show that the image of any $\mathbb{H}^2$-line under ψ is a hyperbolic line $L \subset \mathcal{H}^2$.

First of all, let us conside a vertical semi-line L_1 as above. In this case, any point of L_1 is of the form $z = b + iy$ for some fixed $b \in \mathbb{R}$ and for each $y > 0$. Therefore,

$$\psi(z) = \psi(b+iy) = \left(\frac{1+b^2+y^2}{2y}, \frac{-1+b^2+y^2}{2y}, -\frac{b}{y}\right).$$

Now we have

$$b \cdot \frac{1+b^2+y^2}{2y} - b \cdot \frac{-1+b^2+y^2}{2y} + \left(-\frac{b}{y}\right) = \frac{b+b^3+by^2+b-b^3-by^2-2b}{2y} = 0,$$

so each $\psi(z)$ belongs to the plane $\Pi_b := \{bT - bX + Y = 0\} = \{Y = -bT + bX\}$. Moreover, it belongs to $\mathcal{H}^2$ by Exercise 5, so it belongs to $\Pi_b \cap \mathcal{H}^2$. So we have proved that ψ maps the line L_1 (with fixed choice of the parameter b) to the hyperbolic line $\Pi_b \cap \mathcal{H}^2$.

Now let us consider the case when the $\mathbb{H}^2$-line that we choose is an half-circle L_2 as above. In this case, for each point $x + iy \in L_2$ we have $x^2 + y^2 = c^2 - a^2 + 2ax$. Therefore,

$$\psi(x+iy) = \left(\frac{1+c^2-a^2+2ax}{2y}, \frac{-1+c^2-a^2+2ax}{2y}, -\frac{x}{y}\right). \tag{0.14}$$

Then we have

$$\begin{aligned}
&(c^2-a^2-1) \cdot \frac{1+c^2-a^2+2ax}{2y} + (a^2-c^2-1) \cdot \frac{-1+c^2-a^2+2ax}{2y} - 2a \cdot \left(-\frac{x}{y}\right) = \\
&= \frac{1}{2y}\left[(c^2-a^2-1)(1+c^2-a^2+2ax) + (a^2-c^2-1)(-1+c^2-a^2+2ax) + 4ax\right] = \\
&\quad = \frac{1}{2y}\big[c^2+c^4-a^2c^2+2ac^2x-a^2-a^2c^2+a^4-2a^3x-1-c^2+a^2-2ax+ \\
&\quad -a^2+a^2c^2-a^4+2a^3x+c^2-c^4+a^2c^2-2ac^2x+1-c^2+a^2-2ax+4ax\big] = 0.
\end{aligned}$$

Therefore, we have proved that for each point z in L_2 (with a fixed choice of a and c), we have that $\psi(z)$ belongs to the plane $\Pi_{a,c} := \{(c^2-a^2-1)T + (a^2-c^2-1)X - 2aY = 0\}$.

Therefore, we have proved that ψ maps the $\mathbb{H}^2$-line L_2 in the hyperbolic line $\Pi_{a,c} \cap \mathcal{H}^2$. Note that $\Pi_{a,c}$ is really a plane. Indeed, at least one of the 3 coefficients $c^2 - a^2 - 1, a^2 - c^2 - 1$ and $-2a$ is non-zero. Indeed, if the third one is zero (i.e. if a is zero), then at least one among $c^2 - a^2 - 1 = c^2 - 1$ and $a^2 - c^2 - 1 = -c^2 - 1$ is non-zero (otherwise, we would have $c = 0$, which is impossible by definition of L_2). This concludes the exercise.

Note: a natural question to ask is the following: where are the coefficients $c^2 - a^2 - 1, a^2 - c^2 - 1$ and $-2a$ coming from? The idea to find such coefficients is the following: you have to prove that the all the points $\psi(z)$ in (0.14) (for every $x + iy \in L_2$) belong to some plane Π of $\mathbb{R}^3$ (passing through the origin). **If it exists** (and we return later on that), such a plane will have an equation of the form

$$\Pi := \{t_0 T + x_0 X + y_0 Y = 0\}$$

for a triple $(t_0, x_0, y_0) \in \mathbb{R}^3 \smallsetminus \{(0, 0, 0)\}$ (that you don't know yet). Therefore, you set $(X, Y, Z) := \psi(z)$ (as in (0.14)) and you get that the triple (t_0, x_0, y_0) that you are looking for, must satisfy

$$t_0 \cdot \frac{1 + c^2 - a^2 + 2ax}{2y} + x_0 \cdot \frac{-1 + c^2 - a^2 + 2ax}{2y} + y_0 \cdot (-\frac{x}{y}) = 0$$

for all $x + iy \in L_2$. After multiplying by $2y$ (note that $y \neq 0$ because $z = x + iy$ belongs to $L_2 \subset \mathbb{H}^2$), this implies that

$$t_0 \cdot (1 + c^2 - a^2 + 2ax) + x_0 \cdot (-1 + c^2 - a^2 + 2ax) - 2y_0 x = 0,$$

i.e.

$$t_0(1 + c^2 - a^2) + x_0(-1 + c^2 - a^2) + (2at_0 + 2ax_0 - 2y_0)x = 0.$$

In this last identity:

- t_0, y_0 and x_0 are still undetermined;
- a and c are fixed (because we have chosen a fixed $\mathbb{H}^2$-line L_2).

Moreover, such an equation must be valid for each x such that $x + iy$ belongs to L_2. Since we have infinitely many such values of x, this implies that t_0, y_0 and x_0 are such that

$$\begin{cases} t_0(1 + c^2 - a^2) + x_0(-1 + c^2 - a^2) = 0 \\ 2at_0 + 2ax_0 - 2y_0 = 0. \end{cases}$$

Equivalently, we must have

$$\begin{cases} t_0(1 + c^2 - a^2) + x_0(c^2 - a^2 - 1) = 0 \\ y_0 = a(t_0 + x_0). \end{cases}$$

So y_0 is completely determined by chosing a value of t_0 and a value for x_0, verifying the first identity. Since we don't know if $1 + c^2 - a^2 \neq 0$ or if $c^2 - a^2 - 1 \neq 0$ (we only know that at least one of them is non-zero), then the simplest solution for the first equation is given by chosing $t_0 := c^2 - a^2 - 1$ and $x_0 := -(1 + c^2 - a^2) = a^2 - c^2 - 1$. With such a choice, we have $y_0 = -2a$. Then the only thing that one has to check is that the triple (t_0, x_0, y_0) is different from $(0, 0, 0)$ and that each point $\psi(z)$ as before satisfies the equation for the plane that we have found. These are exactly the computations done before.

Exercise 9

Let $z_1 = x_1 + y_1$, $z_2 = x_2 + iy_2$ be arbitrary points of the upper half-plane $\mathbb{H}^2$, and let $\mathbf{v}_i = \psi(z_i) \in \mathcal{H}^2$ be their images under the map ψ. Show, using the formulas above, that

$$-\phi_L(\mathbf{v}_1, \mathbf{v}_2) = 1 + \frac{|z_1 - z_2|^2}{2y_1y_2}.$$

Deduce that the function

$$\begin{array}{rccc} d_{\mathbb{H}^2}: & \mathbb{H}^2 \times \mathbb{H}^2 & \longrightarrow & \mathbb{R} \\ & (z_1, z_2) & \longrightarrow & d_{\mathbb{H}^2}(z_1, z_2) := \operatorname{arccosh}\left(1 + \frac{|z_1-z_2|^2}{2\operatorname{Im}(z_1)\operatorname{Im}(z_2)}\right) \end{array}$$

makes the upper-half plane $\mathbb{H}^2$ into a metric space isometric to $(\mathcal{H}^2, d)$.

Let us fix any pair of points z_1, z_2 as above. For simplicity, let us set $\mathbf{v}_i = (T_i, X_i, Y_i) := \psi(z_i)$ for $i = 1, 2$. Then we have

$$\begin{aligned} -\phi_L(\mathbf{v}_1, \mathbf{v}_2) &= T_1T_2 - X_1X_2 - Y_1Y_2 = \\ = \frac{1 + x_1^2 + y_1^2}{2y_1} \cdot \frac{1 + x_2^2 + y_2^2}{2y_2} &- \frac{-1 + x_1^2 + y_1^2}{2y_1} \cdot \frac{-1 + x_2^2 + y_2^2}{2y_2} - \left(-\frac{x_1}{y_1}\right) \cdot \left(-\frac{x_2}{y_2}\right) = \\ &= \frac{1}{4y_1y_2}\Big[1 + x_2^2 + y_2^2 + x_1^2 + x_1^2x_2^2 + x_1^2y_2^2 + y_1^2 + y_1^2x_2^2 + y_1^2y_2^2 - 1 + \\ &\quad + x_2^2 + y_2^2 + x_1^2 - x_1^2x_2^2 - x_1^2y_2^2 + y_1^2 - y_1^2x_2^2 - y_1^2y_2^2 - 4x_1x_2\Big] = \\ = \frac{1}{4y_1y_2}\left[2x_1^2 + 2x_2^2 + 2y_1^2 + 2y_2^2 - 4x_1x_2\right] &= 1 + \frac{1}{2y_1y_2}\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right] = \\ &= 1 + \frac{|z_1 - z_2|^2}{2y_1y_2}. \end{aligned} \tag{0.15}$$

By § 3.2 in Lecture Notes Part III we have

$$\begin{aligned} d_{\mathcal{H}^2}(\psi(z_1), \psi(z_2)) &= \operatorname{arccosh}(-\phi_L(\psi(z_1), \psi(z_2))) = \\ = \operatorname{arccosh}(-\phi_L(\mathbf{v}_1, \mathbf{v}_2)) &\overset{(0.15)}{=} \operatorname{arccosh}\left(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\right) = \\ = \operatorname{arccosh}\left(1 + \frac{|z_1 - z_2|^2}{2\operatorname{Im}(z_1)\operatorname{Im}(z_2)}\right) &= d_{\mathbb{H}^2}(z_1, z_2). \end{aligned}$$

Moreover, by Exercise 6 we know that ψ is a bijection from $\mathbb{H}^2$ to $\mathcal{H}^2$, so we have proved that ψ is an isometry from $(\mathbb{H}^2, d_{\mathbb{H}^2})$ to $(\mathcal{H}^2, d)$.

E-mail address: matteo.tommasini2@gmail.com, matteo.tommasini@uni.lu

Mathematics Research Unit
University of Luxembourg
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg