EXERCISE SESSION 6B FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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# Solutions for the test of May 6, 2014

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# Exercises NOT done during the lecture of May 14, 2014

## Exercise 2

*Find the length of the following curve in* $\mathbb{R}^3$

$$\begin{array}{rccc} \gamma: & [2,7] & \longrightarrow & \mathbb{R}^3 \\ & t & \mapsto & \left(t, \frac{1}{2}t^2, \frac{2\sqrt{2}}{3}t^{3/2}\right). \end{array}$$

**NOTE: this Exercise coincides with Exercise 5 in Exercise Sheet 3 – 4, except for the domain of definition of $\gamma$ (see Exercise Solutions 2A).**

For each $t \in [2,7]$ we have

$$\dot{\gamma}(t) = \left(1, t, t^{1/2}\right).$$

Therefore, for any such $t$ we have

$$|\dot{\gamma}(t)| = \sqrt{1 + t^2 + 2t} = |1 + t|.$$

(note that in the previous line we had to put an absolute value!). Since $t \in [2,7]$, we conclude that

$$|\dot{\gamma}(t)| = 1 + t \quad \forall\ t \in [2,7].$$

Therefore,

$$S(\gamma) = \int_2^7 |\dot{\gamma}(t)| dt = \int_2^7 (1+t) dt = \left(t + \frac{t^2}{2}\right)\Big|_{t=7} - \left(t + \frac{t^2}{2}\right)\Big|_{t=2} = 7 + \frac{49}{2} - 2 - 2 = \frac{55}{2}.$$

## Exercise 3

*Find a parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ whose image lies in the subset $\{y^4 - x^7 = 0\}$ of $\mathbb{R}^2$.*

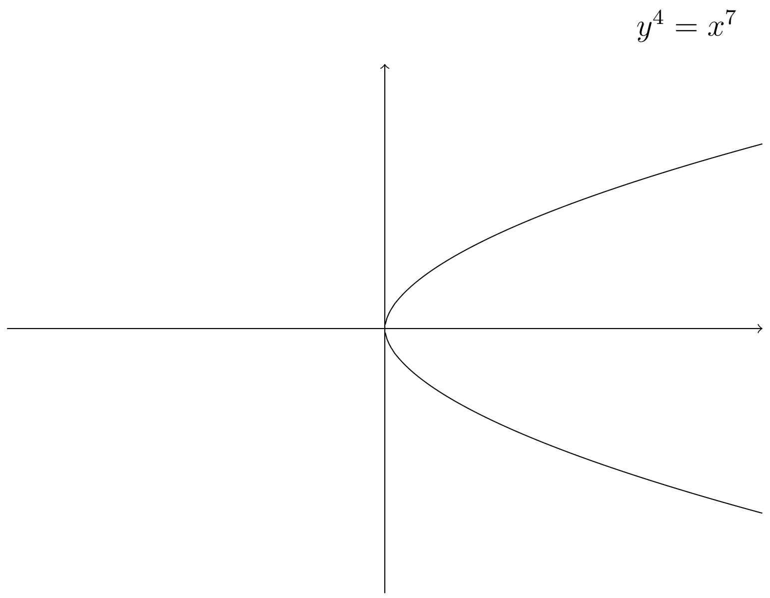
**NOTE: this Exercise coincides with Exercise 9 in Exercise Sheet 3 – 4, in the special case when $m = 4$ and $n = 7$ (i.e. the case $m$ even, $n$ odd) (see Exercise Solutions 2A)**.

We define

$$\gamma(t) := \left(t^4, t^7\right) \qquad \text{for all } t \in \mathbb{R}.$$

Since $(t^7)^4 = (t^4)^7$, we get that $\gamma(t)$ has values in the set $\{y^4 - x^7 = 0\}$. Moreover, $\gamma(t)$ is smooth, so we are done (this suffices as a solution for the exercise).

Geometrically, the set $\{y^4 - x^7 = 0\}$ is symmetric with respect to the $x$ axis (i.e. by replacing $y$ with $-y$):




Then the image of $\gamma$ covers all the set $\{y^4 - x^7 = 0\}$. Indeed, let us fix any point $(x, y)$ in $\{y^4 - x^7 = 0\}$. Since 4 is even, then $y^4$ is non-negative, so $x^7$ is non-negative. Since 7 is odd, this implies that $x$ is non-negative. So it makes sense to consider $\sqrt[4]{x}$. Since $y^4 = x^7$, we have:

$$y = \sqrt[4]{x^7} = (\sqrt[4]{x})^7 \quad \text{if } y \geq 0$$

(note that $\sqrt[4]{x^7} = (\sqrt[4]{x})^7$ because $x$ is non-negative) and

$$y = -\sqrt[4]{x^7} = -(\sqrt[4]{x})^7 = (-\sqrt[4]{x})^7 \quad \text{if } y < 0$$

(because 7 is odd). So we have:

$$(x,y) = \left((\sqrt[4]{x})^4, (\sqrt[4]{x})^7\right) = \gamma(\sqrt[4]{x}) \quad \text{if } y \geq 0.$$

Moreover, since 4 is even, $(-\sqrt[4]{x})^4 = (\sqrt[4]{x})^4 = x$, so

$$(x,y) = \left((-\sqrt[4]{x})^4, (-\sqrt[4]{x})^7\right) = \gamma(-\sqrt[4]{x}) \quad \text{if } y < 0.$$

So this proves that the set $\{y^4 - x^7 = 0\}$ is completely covered by the image of $\gamma$.

## Exercise 6

*(i) Give a definition of a metric space.*
*(ii) Give a definition of the distance function $d$ in the spherical geometry.*
*(iii) Recall that for any spherical triangle $\triangle\mathbf{pqr}$ one has*

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\Xi, \tag{0.1}$$

*where*

$$\alpha := d(\mathbf{q},\mathbf{r}), \qquad \beta := d(\mathbf{p},\mathbf{q}), \qquad \gamma := d(\mathbf{p},\mathbf{r})$$

*and $\Xi$ is the dihedral angle at $\mathbf{p}$. Use this formula to prove that $S^2$ equipped with the distance function $d$ is a metric space.*

**NOTE: the answer of (i) below is simply Definition 2.5 in Lecture Notes Part I.**

(i) A metric space is a pair $(M,d)$, where $M$ is a set and $d$ is a function

$$\begin{array}{rccc} d: & M \times M & \longrightarrow & \mathbb{R}_{\geq 0} \\ & (p,q) & \mapsto & d(p,q) \end{array}$$

satisfying the following properties:
(1) $d(p,q) = 0$ if and only if $p = q$;
(2) $d(p,q) = d(q,p)$ for any $p,q \in M$;
(3) $d(p,q) \leq d(p,r) + d(q,r)$ for any $p,q,r \in M$.
The function $d$ is called the distance function (or the metric) on $M$.

**NOTE: the answer of (ii) below is simply Definition 1.2 in Lecture Notes Part II.**

(ii) The spherical distance $d(\mathbf{p},\mathbf{q})$ between 2 points $\mathbf{p},\mathbf{q}$ in a sphere is the distance between $\mathbf{p}$ and $\mathbf{q}$ measured along the shorter arc of a great circle through $p$ and $q$. In particular,

$$d(\mathbf{p},\mathbf{q}) = r \cdot \theta(\mathbf{p},\mathbf{q}),$$

where $r$ is the radius of the sphere and $\theta(\mathbf{p},\mathbf{q})$ is the angle $\angle\mathbf{p}0\mathbf{q}$ at 0 between the vector joining 0 and $\mathbf{p}$ and the vector joining 0 and $\mathbf{q}$. Here the angle is always interpreted as the absolute value in the range $[0, \pi]$. In particular, if the sphere is the (canonical) unit sphere, we have

$$d(\mathbf{p}, \mathbf{q}) = \theta(\mathbf{p}, \mathbf{q}).$$

**NOTE: the triangle inequality in (iii) below basically coincides (with more details) with the proof of the triangle inequality in Lecture Notes Part II (see Theorem 3.1)**.

(iii) First of all, conditions (1) and (2) above are obvious by definition of spherical distance. So we need only to prove (3), i.e. the triangle inequality. So let us fix any triple of points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ in $S^2$. If $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ belong to the same maximal circle of $S^2$, then we have

$$d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p}, \mathbf{r}) + d(\mathbf{q}, \mathbf{r}).$$

So we have only to consider the case when $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are not on the same maximal circle. In this case we consider the triangle $\triangle\mathbf{pqr}$ and we denote by $\alpha, \beta$ and $\gamma$ its angles. Since $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are not on the same maximal circle, then $\alpha, \beta, \gamma \in ]0, \pi[$. By standard trigonometry we have

$$\cos(\beta + \gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma.$$

Using formula (0.1), we get

$$\begin{aligned} \cos(\beta + \gamma) &= \cos\alpha - \sin\beta\sin\gamma\cos\Xi - \sin\beta\sin\gamma = \\ &= \cos\alpha - \sin\beta\sin\gamma(1 + \cos\Xi). \end{aligned} \tag{0.2}$$

Now $\alpha, \beta$ and $\gamma$ are angles in $]0, \pi[$ by construction, so $\sin\beta$ and $\sin\gamma$ are positive. Moreover, $\cos\Xi \in [-1, 1]$ for any $\Xi \in \mathbb{R}$. Therefore, $1 + \cos\Xi$ is non-negative. So from (0.2) we conclude

$$\cos\alpha \geq \cos(\beta + \gamma). \tag{0.3}$$

Now the function $x \mapsto \cos x$ is strictly decreasing in the interval $[0, \pi]$, so (0.3) implies that

$$\alpha \leq \beta + \gamma.$$

In other terms (using Exercise 6(ii) above), we have

$$d(\mathbf{p}, \mathbf{q}) \leq d(\mathbf{p}, \mathbf{r}) + d(\mathbf{q}, \mathbf{r}) \quad \forall\, \mathbf{p}, \mathbf{q}, \mathbf{r} \in S^2.$$

So we have proved also condition (3) of metric spaces. So $S^2$ with the spherical distance is a metric space.

## Exercise 8

*(i) Give the definition of a spherical line.*
*(ii) Give the definition of a spherical angle between spherical lines.*
*(iii) Let $\triangle\mathbf{pqr}$ be a spherical triangle. Show that the sum of its spherical angles is given by*

$$\angle\mathbf{p} + \angle\mathbf{q} + \angle\mathbf{r} = \pi + \operatorname{area}\triangle\mathbf{pqr}.$$

(i) A spherical line is the intersection of the sphere with a plane $\Pi$ of $\mathbb{R}^3$, passing through the origin of $\mathbb{R}^3$ (this is simply Definition 1.1.1 in Lecture Notes Part II).

(ii) (This is mainly the combination of Definitions 2.2 and 2.3 in Lecture Notes Part II.) Let us fix any 2 spherical distinct lines $L_1$ and $L_2$ (as in Exercise 7) and let us denote by $P$ one of their 2 intersection points. Then the spherical angle $\Xi$ at $P$ in $S^2$ is defined as the dihedral angle between $\Pi_1$ and $\Pi_2$. This dihedral angle is computed as the planar angle determined by $\Pi_1$, $\Pi_2$ and by the unique plane $\Pi$ passing through 0 and orthogonal to the line $R$ joining $P$ and $-P$.

**NOTE: the proof of (iii) below is simply the proof of Proposition 4.3.2 in Lecture Notes Part II**.

(iii) Let us denote by $\Sigma_{\mathbf{p}}$ the part of the sphere contained between the planes $0\mathbf{pq}$ and $0\mathbf{pr}$; it is a union of two surfaces, and we have

$$\operatorname{area}\Sigma_{\mathbf{p}} = \frac{2\angle\mathbf{p}}{2\pi}\operatorname{area}S^2,$$

where $\angle\mathbf{p}$ is the dihedral angle between the planes $0\mathbf{pq}$ and $0\mathbf{pr}$ (that is, the spherical angle at $\mathbf{p}$). Similarly, one defines $\Sigma_{\mathbf{q}}$ as the part of the sphere contained between the planes $0\mathbf{qp}$ and $0\mathbf{qr}$, with

$$\operatorname{area}\Sigma_{\mathbf{q}} = \frac{2\angle\mathbf{q}}{2\pi}\operatorname{area}S^2,$$

and $\Sigma_{\mathbf{r}}$ as the part of the sphere contained between the planes $0\mathbf{rp}$ and $0\mathbf{rq}$, with

$$\operatorname{area}\Sigma_{\mathbf{r}} = \frac{2\angle\mathbf{r}}{2\pi}\operatorname{area}S^2.$$

Now the surfaces $\Sigma_{\mathbf{p}}, \Sigma_{\mathbf{q}}$ and $\Sigma_{\mathbf{r}}$ overlap over the triangle $\triangle\mathbf{pqr}$ and its antipodal triangle $\triangle\mathbf{p}'\mathbf{q}'\mathbf{r}'$. Therefore, we have

$$\begin{aligned}\operatorname{area}\Sigma_{\mathbf{p}} + \operatorname{area}\Sigma_{\mathbf{q}} + \operatorname{area}\Sigma_{\mathbf{r}} &= (\operatorname{area}S^2 - 2\operatorname{area}\triangle\mathbf{pqr}) + 6\operatorname{area}\triangle\mathbf{pqr} = \\ &= \operatorname{area}S^2 + 4\operatorname{area}\triangle\mathbf{pqr}.\end{aligned}$$

Here we used also the fact that points in $\triangle\mathbf{pqr}$ and in $\triangle\mathbf{p}'\mathbf{q}'\mathbf{r}'$ are covered 3 times, while the rest of $S^2$ only once. Then we conclude that

$$\text{area}\, S^2 + 4\, \text{area}\, \triangle\mathbf{pqr} = \left(\frac{2\angle\mathbf{p}}{2\pi} + \frac{2\angle\mathbf{q}}{2\pi} + \frac{2\angle\mathbf{r}}{2\pi}\right) \text{area}\, S^2.$$

Now area $S^2 = 4\pi r^2 = 4\pi$ (since we are working with the unit sphere). So we obtain that

$$\angle\mathbf{p} + \angle\mathbf{q} + \angle\mathbf{r} = \pi + \text{area}\, \triangle\mathbf{pqr}.$$

## Exercise 9

*Give a definition of:*
*(i) the Lorentz quadratic form,*
*(ii) the Lorentz group,*
*(iii) the hyperbolic plane $\mathcal{H}^2$,*
*(iv) the hyperbolic line in $\mathcal{H}^2$,*
*(v) the hyperbolic distance function on $\mathcal{H}^2$.*

**NOTE: all the definitions below are taken from Lecture Notes Part III.**

(i) Given any $n \geq 1$, the Lorentz quadratic form on $\mathbb{R}^{n+1}$ is the function

$$\begin{array}{cccc} q_L : & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R} \\ & \mathbf{x} = (x_0, \cdots, x_n) & \mapsto & q_L(\mathbf{x}) := -x_0^2 + x_1^2 + \cdots + x_n^2. \end{array}$$

(ii) The Lorentz group is the symmetry group of the Lorentz bilinear form $\phi_L$ on $\mathbb{R}^{n+1}$

$$\begin{array}{cccc} \phi_L : & \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R} \\ & (\mathbf{x} = (x_0, \cdots, x_n), \mathbf{y} = (y_0, \cdots, y_n)) & \mapsto & \phi_L(\mathbf{x}, \mathbf{y}) := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n. \end{array}$$

This group is denoted by $O(1, n)$. This group consists of all the elements $A \in \mathrm{GL}(n+1, \mathbb{R})$ which satisfy the equation

$$A \cdot \Phi_L \cdot A^T = \Phi_L,$$

where $\Phi_L$ is the $(n+1) \times (n+1)$ matrix defined as follows

$$\Phi_L := \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(iii) The hyperbolic plane $\mathcal{H}^2$ is the subset of $\mathbb{R}^3$ (with coordinates $(t, x, y)$) defined as follows:

$$\mathcal{H}^2 := \left\{(t, x, y) \in \mathbb{R}^3 \text{ s.t. } -t^2 + x^2 + y^2 = -1 \text{ and } t > 0\right\}.$$

(iv) An hyperbolic line in $\mathcal{H}^2$ is the hyperbola $L$ obtained as the intersection

$$L = \mathcal{H}^2 \cap \Pi$$

of $\mathcal{H}^2$ with a plane $\Pi$ passing through the origin of $\mathbb{R}^3$.

(v) Let us fix any point $P$, given by the vector $\mathbf{v} = (t_1, x_1, y_1)$, and a point $Q$, given by the vector $\mathbf{u} = (t_2, x_2, y_2)$, both in $\mathcal{H}^2$. Then the hyperbolic distance between $P$ and $Q$ is defined by

$$\begin{aligned} d(P,Q) &:= \operatorname{arcosh}(-\phi_L(\mathbf{v},\mathbf{u})) = \\ &= \operatorname{arcosh}(t_1t_2 - x_1x_2 - y_1y_2) = \\ &= \log\left(t_1t_2 - x_1x_2 - y_1y_2 + \sqrt{(t_1t_2 - x_1x_2 - y_1y_2)^2 - 1}\right). \end{aligned}$$

## Exercise 11

*Let $\alpha, \beta, \gamma$ be the sides (lengths) of an hyperbolic triangle $\triangle PQR$ and let $a, b, c$ be the opposite angles. Prove the hyperbolic sine rule:*

$$\frac{\sinh\alpha}{\sin a} = \frac{\sinh\beta}{\sin b} = \frac{\sinh\gamma}{\sin c}.$$

*Hint: Use the basic formula for an hyperbolic triangle*:

$$\cosh\alpha = \cosh\beta\cosh\gamma - \cos a \sinh\beta\sinh\gamma. \tag{0.4}$$

**NOTE: this is Exercise 3 in Exercise Sheet 9 – 10 (proof already given in Exercise Solutions 5B)**.

By (0.4) we have:

$$\cos a = \frac{\cosh\beta\cosh\gamma - \cosh\alpha}{\sinh\beta\sinh\gamma}.$$

So

$$\begin{aligned} (\sin a)^2 = 1 - (\cos a)^2 = 1 - \frac{(\cosh\beta)^2(\cosh\gamma)^2 + (\cosh\alpha)^2 - 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2} = \\ = \frac{(\sinh\beta)^2(\sinh\gamma)^2 - (\cosh\beta)^2(\cosh\gamma)^2 - (\cosh\alpha)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2}. \end{aligned}$$

Now for each $x \in \mathbb{R}$ we have

$$\begin{aligned}(\sinh x)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - e^{-2x} - 2}{4} = \\ = \frac{e^{2x} - e^{-2x} + 1}{4} - 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = (\cosh x)^2 - 1,\end{aligned}$$

hence by replacing above in $(\sinh\beta)^2$ and $(\sinh\gamma)^2$, we get:

$$\begin{aligned}(\sin a)^2 = \frac{1}{(\sinh\beta)^2(\sinh\gamma)^2} \cdot \Big[1 - (\cosh\beta)^2 - (\cosh\gamma)^2 + (\cosh\beta)^2(\cosh\gamma)^2 + \\ -(\cosh\beta)^2(\cosh\gamma)^2 - (\cosh\alpha)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma\Big] = \\ = \frac{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}{(\sinh\beta)^2(\sinh\gamma)^2}.\end{aligned}$$

By construction $a$ is an angle in $]0,\pi[$, so $\sin a \neq 0$. Therefore,

$$\frac{(\sinh\alpha)^2}{(\sin a)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}.$$

Exactly in the same way we can prove that

$$\begin{aligned}\frac{(\sinh\beta)^2}{(\sin b)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}, \\ \frac{(\sinh\gamma)^2}{(\sin c)^2} = \frac{(\sinh\alpha)^2(\sinh\beta)^2(\sinh\gamma)^2}{1 - (\cosh\alpha)^2 - (\cosh\beta)^2 - (\cosh\gamma)^2 + 2\cosh\alpha\cosh\beta\cosh\gamma}.\end{aligned}$$

Therefore, we conclude that

$$\frac{(\sinh\alpha)^2}{(\sin a)^2} = \frac{(\sinh\beta)^2}{(\sin b)^2} = \frac{(\sinh\gamma)^2}{(\sin c)^2}.$$

Now $\alpha$ is the hyperbolic length between two distinct vertices of the triangle $\triangle PQR$, hence $\alpha > 0$, so $\sinh\alpha > 0$. Moreover, by construction $a$ is an angle in $]0,\pi[$, therefore $\sin a > 0$. So from the previous identity we get:

$$\frac{\sinh\alpha}{\sin a} = \frac{\sinh\beta}{\sin b} = \frac{\sinh\gamma}{\sin c}.$$

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