EXERCISE SESSION 3B FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 5-6
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Exercises NOT done during the lecture of March 26, 2014

Exercise 3

Do Exercise 1.3.2 from Lecture Notes Part II.

Let $\mathbf{p}$ and $\mathbf{q}$ be arbitrary points of $S^2 \subset \mathbb{R}^3$. Acting on $\mathbb{R}^3$ by an element of an isometry group $O(3) \subset E(3)$ we can always assume that $\mathbf{p}$ has coordinates $(0,0,1)$ and a that big circle through $\mathbf{p}$ and $\mathbf{q}$ lies in the plane $y = 0$ (see Exercise 2 above).

In the Lecture Notes (§ 1.3) we considered the case when the x-coordinate of the point $\mathbf{q}$ is non-negative.

Now we consider the remaining case, i.e. the case when the x-coordinate of the point $\mathbf{q}$ is negative. Then the point $\mathbf{q}$ is given by

$$\mathbf{q} = (-\sin d(\mathbf{p},\mathbf{q}), 0, \cos d(\mathbf{p},\mathbf{q}))\,,$$

where $d(\mathbf{p},\mathbf{q}) = \Theta(\mathbf{p},\mathbf{q})$ belongs to the interval $[0,\pi]$ (since this angle is computed along the *shorter* arc of the spherical line connecting $\mathbf{p}$ to $\mathbf{q}$). Therefore we get a parameterized curve in $\mathbb{R}^3$,

$$\begin{array}{rccc} \gamma: & [0, d(\mathbf{p},\mathbf{q})] & \longrightarrow & \mathbb{R}^3 \\ & t & \longrightarrow & (-\sin t, 0, \cos t) \end{array}$$

whose image equals the shorter arc of the spherical line connecting $\mathbf{p}$ to $\mathbf{q}$. The lenght of this curve is again equal to $d(\mathbf{p},\mathbf{q})$ as in the case described in Lecture Notes Part II.

Exercise 5

Let $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{p}', \boldsymbol{q}'$ be any two pairs of points of S^2, such that $d(\boldsymbol{p},\boldsymbol{q}) = d(\boldsymbol{p}',\boldsymbol{q}')$. Show that there is a spherical isometry $T: S^2 \to S^2$ such that $T(\boldsymbol{p}) = \boldsymbol{p}'$ and $T(\boldsymbol{q}) = \boldsymbol{q}'$.

For simplicity, we set $\rho := d(\mathbf{p},\mathbf{q}) = d(\mathbf{p}',\mathbf{q}')$.

Using Exercise 2, there is an isometry A of S^2, such that

$$A(\mathbf{p}) = (0,0,1) \quad \text{and} \quad A(\mathbf{q}) = (a,0,c) \tag{0.1}$$

for a suitable pair of scalars a, c.

Moreover, using again Exercise 2 (applied to the pair $(\mathbf{p}', \mathbf{q}')$ instead of the pair $(\mathbf{p}, \mathbf{q})$), there is an isometry B of S^2, such that

$$B(\mathbf{p}') = (0,0,1) \quad \text{and} \quad B(\mathbf{q}') = (a',0,c') \tag{0.2}$$

for a suitable pair of scalars a', c'.

Since both A and B are isometries, we get:

$$\begin{aligned} d((0,0,1),(a,0,c)) &= d(A(\mathbf{p}),A(\mathbf{q})) = d(\mathbf{p},\mathbf{q}) = \rho = \\ &= d(\mathbf{p}',\mathbf{q}') = d(A(\mathbf{p}'),A(\mathbf{q}')) = d((0,0,1),(a',0,c')). \end{aligned}$$

Now since $d((0,0,1),(a,0,c)) = \rho$ and since the distance used here is the spherical distance, this means that

$$\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \cos\rho \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \sin\rho \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\rho \\ 0 \\ \cos\rho \end{pmatrix}$$

or

$$\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -\sin\rho \\ 0 \\ \cos\rho \end{pmatrix}$$

(see § 1.3 in Lecture Notes Part II). In the same way we get also that

$$\begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} = \begin{pmatrix} \sin\rho \\ 0 \\ \cos\rho \end{pmatrix}.$$

or

$$\begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} = \begin{pmatrix} -\sin\rho \\ 0 \\ \cos\rho \end{pmatrix}.$$

So we consider 2 cases as follows

(a) If the signs of $\sin\rho$ are the same for $(a,0,c)$ and $(a',0,c')$ (i.e. both positive or both negative), then we define

$$T := B^{-1} \cdot A : \; S^2 \longrightarrow S^2$$

and we have:

$$T(\mathbf{p}) \overset{(0.1)}{=} B^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.2)}{=} \mathbf{p}'$$

and

$$T(\mathbf{q}) \overset{(0.1)}{=} B^{-1} \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} \overset{(0.2)}{=} \mathbf{q}'.$$

(b) If the signs of $\sin \rho$ are the different for $(a, 0, c)$ and $(a', 0, c')$ (i.e. one positive and the other one negative, or conversely), then we have $a = -a'$ and $c = c'$. So we define

$$I := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

and we have

$$I \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix}. \tag{0.3}$$

Then we define

$$T := B^{-1} \cdot I \cdot A : \; S^2 \longrightarrow S^2$$

and we have:

$$T(\mathbf{p}) \overset{(0.1)}{=} B^{-1} \cdot I \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.2)}{=} \mathbf{p}'$$

and

$$T(\mathbf{q}) \overset{(0.1)}{=} B^{-1} \cdot I \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \overset{(0.3)}{=} B^{-1} \cdot \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} \overset{(0.2)}{=} \mathbf{q}'.$$

So we have found T as in the claim. In both cases, T is obtained as composition of matrices in $O(3)$, so it belongs to $O(3)$, hence it is an isometry of S^2 as required.

Extra exercise

In Exercise 5 we have proved that given $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{p}', \boldsymbol{q}'$ in S^2 such that $d(\boldsymbol{p}, \boldsymbol{q}) = d(\boldsymbol{p}', \boldsymbol{q}')$, there is a spherical isometry $T : S^2 \to S^2$, such that $T(\boldsymbol{p}) = \boldsymbol{p}'$ and $T(\boldsymbol{q}) = \boldsymbol{q}'$. Now we ask: how many such T's we have?

Let us fix any pair T, S with those properties. Then we we define $C := S^{-1} \cdot T \in O(3)$ and we have easily:

$$C \cdot \mathbf{p} = \mathbf{p} \quad \text{and} \quad C \cdot \mathbf{q} = \mathbf{q}.$$

Now we ask how many such C's we have. In order to do that, we distinguish some cases:

Case (a). If $\mathbf{p} = \mathbf{q}$, then we are simply considering all the matrices $C \in O(3)$ such that $C(\mathbf{p}) = \mathbf{p}$. There are infinitely many such matrices, namely:

- all the matrices of rotation around the axis I joining $\mathbf{p}$ and its antipodal $-\mathbf{p}$;
- all the matrices of rotation around the axis I, composed with symmetries with respect to any plane contaning the axis I.

To be more precise, if we apply Exercise 2 for the point $\mathbf{p}$ and for any other point $\mathbf{q} \in S^2$, then there is a matrix $A \in O(3)$, such that $A(\mathbf{p}) = N$ (the north pole). Then

$$A \cdot C \cdot A^{-1} \cdot N = A \cdot C \cdot \mathbf{p} = A \cdot \mathbf{p} = N.$$

Then we have that $A \cdot C \cdot A^{-1}$ is either of the form

$$M(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or of the form

$$M'(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\theta \in [0, 2\pi[$. Conversely, for every $\theta \in [0, 2\pi[$ we have $M(\theta) \cdot N = N$ and $M'(\theta) \cdot N = N$. Therefore, for every matrix of the form

$$C := A^{-1} \cdot M(\theta) \cdot A$$

we have $C(\mathbf{p}) = \mathbf{p}$; moreover, for every matrix of the form

$$C := A^{-1} \cdot M'(\theta) \cdot A$$

we have $C(\mathbf{p}) = \mathbf{p}$. We recall that $C = S^{-1} \cdot T$. So we have proved that (when $\mathbf{p} = \mathbf{q}$), if we choose any 2 sperical isometries T, S, such that $T(\mathbf{p}) = \mathbf{p}' = S(\mathbf{p})$ (and obviously $T(\mathbf{q}) = \mathbf{q}' = S(\mathbf{q})$), then either

$$S^{-1} \cdot T = A^{-1} \cdot M(\theta) \cdot A$$

or

$$S^{-1} \cdot T = A^{-1} \cdot M'(\theta) \cdot A$$

for some $\theta \in [0, 2\pi[$. Equivalently,

$$T = S \cdot A^{-1} \cdot M(\theta) \cdot A$$

or

$$T = S \cdot A^{-1} \cdot M'(\theta) \cdot A$$

for some $\theta \in [0, 2\pi[$. This give a complete solution to the Extra Exercise (in case (a)), provided that we already know an isometry S such that $S(\mathbf{p}) = \mathbf{p}'$.

Case (b). Let us consider the case when $\mathbf{p} = -\mathbf{q}$. We recall that we are considering all the matrices $C \in O(3)$ such that $C(\mathbf{p}) = \mathbf{p}$ and $C(\mathbf{q}) = \mathbf{q}$. Since $\mathbf{q} = -\mathbf{p}$, then $C(\mathbf{q}) = \mathbf{q}$ is a consequence of $C(\mathbf{p}) = \mathbf{p}$ (and the fact that C is a linear map). Therefore, we are exactly considering all the the matrices $C \in O(3)$ such that $C(\mathbf{p}) = \mathbf{p}$. Therefore, we get exactly the same result of case (a) above.

Case (c). Let us consider the case when $\mathbf{p} \neq \mathbf{q}$ and $\mathbf{p} \neq -\mathbf{q}$ (i.e. $\mathbf{p}$ and $\mathbf{q}$ are *disjoint and non-antipodal*). This means that the vectors (associated to the points) $\mathbf{p}$ and $\mathbf{q}$ are linearly independent in $\mathbb{R}^3$. Since $C(\mathbf{p}) = \mathbf{p}$ and $C(\mathbf{q}) = \mathbf{q}$, this means that the invertible matrix C is fixing all the points of the plane H generated by $\mathbf{p}$ and $\mathbf{q}$ (in the ambient space $\mathbb{R}^2$). Now let us denote by I the line passing through the origin and orthogonal to the plane H. This line intersects the sphere S^2 in 2 antipodal points; we denote one of them by $\mathbf{t}$, so the other one is simply $-\mathbf{t}$. Now since I is orthogonal to the plane H and since $\mathbf{p}$ belongs to $H \cap S^2$, this implies that the spherical distance $d(\mathbf{p}, \mathbf{t})$ is equal to $\pi/2$. For the same reason, $d(\mathbf{q}, \mathbf{t}) = \pi/2$. Since C is a spherical isometry, then we have

$$d(\mathbf{p}, C \cdot \mathbf{t}) = d(C \cdot \mathbf{p}, C \cdot \mathbf{t}) = d(\mathbf{p}, \mathbf{t}) = \pi/2$$

and

$$d(\mathbf{q}, C \cdot \mathbf{t}) = d(C \cdot \mathbf{q}, C \cdot \mathbf{t}) = d(\mathbf{q}, \mathbf{t}) = \pi/2.$$

This means that the point $C \cdot \mathbf{t}$ lies in the set

$$U := \{\mathbf{u} \in S^2 \text{ s.t. } d(\mathbf{p}, \mathbf{u}) = \pi/2 = d(\mathbf{q}, \mathbf{u})\}.$$

This set is the intersection of

$$L_1 := \{\mathbf{u} \in S^2 \text{ s.t. } d(\mathbf{p}, \mathbf{u}) = \pi/2\}$$

and

$$L_2 := \{\mathbf{u} \in S^2 \text{ s.t. } d(\mathbf{q}, \mathbf{u}) = \pi/2\}.$$

It is easy to see that both L_1 and L_2 are lines on S^2 (i.e. maximal circles on the sphere). Indeed, if you denote by I_1 the line in $\mathbb{R}^3$ joining $\mathbf{p}$ with $-\mathbf{p}$ and by H_1 the plane passing through the origin of $\mathbb{R}^3$ and orthogonal to the line I_1, then you can prove easily (do it as an homework) that

$$L_1 = H_1 \cap S^2.$$

Analogously,

$$L_2 = H_2 \cap S^2,$$

where H_2 is the plane passing through the origin of $\mathbb{R}^3$ and orthogonal to the line I_2 joining $\mathbf{q}$ with $-\mathbf{q}$. We are in the case when $\mathbf{p}$ is different from $\mathbf{q}$ and $-\mathbf{q}$. Therefore the lines I_1 and I_2 are distinct (the first one intersects the sphere only in $\mathbf{p}$ and $-\mathbf{p}$, the second one intersects the sphere only in $\mathbf{q}$ and $-\mathbf{q}$, and $\mathbf{p} \neq \pm\mathbf{q}$). Therefore, also the planes H_1 and H_2 are different. They are vector subspaces of dimension 2 of $\mathbb{R}^3$, so they intersect on a line I_3 (passing through the origin). Therefore,

$$U = L_1 \cap L_2 = (H_1 \cap S^2) \cap (H_2 \cap S^2) = (H_1 \cap H_2) \cap S^2 = I_3 \cap S^2.$$

So we get that U is the intersection of the sphere with a line passing through the origin, so it consists only of a couple of antipodal points. But we already know that $C \cdot \mathbf{t}$ belongs to U, hence we have proved that

$$U = \{C \cdot \mathbf{t}, -C \cdot \mathbf{t}\}.$$

But we recall also that we have $d(\mathbf{p}, \mathbf{t}) = \pi/2$ and $d(\mathbf{q}, \mathbf{t}) = \pi/2$. Therefore, we have

$$\mathbf{t} \in U = \{\mathbf{u} \in S^2 \text{ s.t. } d(\mathbf{p}, \mathbf{u}) = \pi/2 = d(\mathbf{q}, \mathbf{u})\}.$$

Therefore we have necessarily

$$\mathbf{t} = C \cdot \mathbf{t} \tag{0.4}$$

or

$$\mathbf{t} = -C \cdot \mathbf{t} \tag{0.5}$$

(by the way, this also proves that the lines I and I_3 used before coincide). Let us consider the cases (0.4) and (0.5) subcases separately:

Subcase (1). In the first case, C is a matrix in $O(3) \subset M(3 \times 3, \mathbb{R})$, such that

$$C \cdot \mathbf{p} = \mathbf{p}, \quad C \cdot \mathbf{q} = \mathbf{q} \quad \text{and} \quad C \cdot \mathbf{t} = \mathbf{t}.$$

We remark that the triple $\mathbf{p}, \mathbf{q}, \mathbf{t}$ is a triple of linearly independent vectors of $\mathbb{R}^2$. Indeed:

- $\mathbf{p} \in S^2$, so it is not the zero vector.
- $\mathbf{q} \neq \lambda\mathbf{p}$ for each $\lambda \in \mathbb{R}$. Indeed, if there is λ such that $\mathbf{q} = \lambda\mathbf{p}$, then we have $1 = |\mathbf{q}| = |\lambda\mathbf{p}| = |\lambda||\mathbf{p}| = |\lambda|$, so $\lambda = +1$ or $\lambda = -1$. In the first case, we would have $\mathbf{p} = \mathbf{q}$, in the second case $\mathbf{p} = -\mathbf{q}$, but we are in case (c), so neither of such cases can happen. So the pair $\mathbf{p}, \mathbf{q}$ is made of linearly independent vectors.
- $\mathbf{t}$ is linearly independent from $\mathbf{p}$ and $\mathbf{q}$ because it does not belong to the plane H generated by $\mathbf{p}$ and $\mathbf{q}$ (actually, it belongs to the intersection of S^2 with the line orthogonal to H). Therefore, the triple $\mathbf{p}, \mathbf{q}, \mathbf{t}$ is made of linearly independent vectors.

Therefore, in subcase (1) we have that C fixes a triple of linearly independent vectors. Then C is the identity matrix. Indeed, for each vector $v \in \mathbb{R}^3$, there is a triple $a, b, c \in \mathbb{R}$ such that $v = a\mathbf{p} + b\mathbf{q} + c\mathbf{t}$ and

$$C \cdot v = C \cdot (a\mathbf{p} + b\mathbf{q} + c\mathbf{t}) = C \cdot (a\mathbf{p}) + C \cdot (b\mathbf{q}) + C \cdot (c\mathbf{t}) = aC \cdot \mathbf{p} + bC \cdot \mathbf{q} + cC \cdot \mathbf{t} = a\mathbf{p} + b\mathbf{q} + c\mathbf{t} = v.$$

Subcase (2). In this case we assume that $\mathbf{t} = -C \cdot \mathbf{t}$. As before, we have that the triple $\mathbf{p}, \mathbf{q}, \mathbf{t}$ is made of linearly independent vectors. So for each vector $v \in \mathbb{R}^3$, there is a triple $a, b, c \in \mathbb{R}$ such that $v = a\mathbf{p} + b\mathbf{q} + c\mathbf{t}$ and

$$\begin{aligned} C \cdot v = C \cdot (a\mathbf{p} + b\mathbf{q} + c\mathbf{t}) &= C \cdot (a\mathbf{p}) + C \cdot (b\mathbf{q}) + C \cdot (c\mathbf{t}) = aC \cdot \mathbf{p} + bC \cdot \mathbf{q} + cC \cdot \mathbf{t} = \\ &= aC \cdot \mathbf{p} + bC \cdot \mathbf{q} - cC \cdot \mathbf{t} = a\mathbf{p} + b\mathbf{q} - c\mathbf{t}. \end{aligned} \tag{0.6}$$

In other terms, C is the symmetry with respect to the plane H generated by $\mathbf{p}$ and $\mathbf{q}$.

Putting together subcases (1) and (2) we conclude that in case (c) the matrix $S^{-1} \cdot T$ is either equal to Id or to the matrix C acting on each vector v of $\mathbb{R}^3$ as described in (0.6). In other terms, either $T = S$ or $T = S \cdot C$ for the matrix C of (0.6). So in case (c) we have only 2 matrices sending $\mathbf{p}$ to $\mathbf{p}'$ and $\mathbf{q}$ to $\mathbf{q}'$ (one is S, the other one is $S \cdot C$ for C as in (0.6)).

Exercise 6

(a) *Define (by analogy with Euclidean geometry) the notions of spherical circle and spherical disc with centre $\mathbf{p} \in S^2$ and radius ρ.*
(b) *Prove that a spherical circle with radius $\rho < \pi$ has circumference $2\pi \sin \rho$.*
(c) *Prove that a spherical disc of radius $\rho < \pi$ has area $2\pi(1 - \cos \rho)$. [Hint: for (c), integrate (b).]*

(a) Let us fix any point $\mathbf{p} \in S^2$ and any radius $\rho \geq 0$. Then we define the spherical circle centered in $\mathbf{p}$ with radius ρ as the set:

$$C(\mathbf{p}, \rho) := \left\{\mathbf{q} \in S^2 \text{ s.t. } d(\mathbf{p}, \mathbf{q}) = \rho\right\}$$

(here d stands for the spherical distance). Note: the only "interesting" circles are those for $\rho < \pi$. Indeed, for $\rho = \pi$ the circle $C(\mathbf{p}, \pi)$ consists of a single point, namely the antipodal of $\mathbf{p}$. Moreover, since any 2 points in S^2 are joint by a great circle, then their distance is at most π (and exactly π if and only if they are antipodal). Therefore, if $\rho > \pi$, then the circle $C(\mathbf{p}, \rho)$ is empty.

Moreover, we define a spherical disc centered in $\mathbf{p}$ with radius ρ as the set:

$$D(\mathbf{p}, \rho) := \left\{ \mathbf{q} \in S^2 \text{ s.t. } d(\mathbf{p}, \mathbf{q}) \leq \rho \right\}.$$

As for the circle, the only "interesting" discs are those for $\rho < \pi$. For $\rho \geq \pi$ we have $D(\mathbf{p}, \rho) = S^2$ because each point $\mathbf{q}$ in S^2 is at distance at most π from $\mathbf{p}$.

(b) After applying Exercise 1, we can assume that $\mathbf{p}$ is the north pole $(0, 0, 1)$. Then let us consider any rotation of S^2 (or $\mathbb{R}^3$ equivalently) along the z-axis, i.e. any isometry of S^2 given by a matrix A as follows:

$$A := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.7}$$

for any angle $\theta \in [0, 2\pi[$ (compare this matrix with the matrices used in Exercise session 1A). A direct check proves that this matrix belongs to $O(3)$ (i.e. $A^T \cdot A = \mathrm{Id}$), so by Theorem 4.2 in Lecture Notes Part II we have that A is an isometry of S^2. Moreover, A fixes the point $\mathbf{p} = (0, 0, 1)$. Since A is an isometry of S^2, for each point $\mathbf{q} \in C(\mathbf{p}, \rho)$ we have

$$d(\mathbf{p}, A(\mathbf{q})) = d(A(\mathbf{p}), A(\mathbf{q})) = d(\mathbf{p}, \mathbf{q}) = \rho,$$

so A sends each point of $C(\mathbf{p}, \rho)$ again to a point of $C(\mathbf{p}, \rho)$. Moreover, given any 2 points $\mathbf{q}, \mathbf{q}'$ in $C(\mathbf{p}, \rho)$, we have $d(\mathbf{p}, \mathbf{q}) = \rho = d(\mathbf{p}, \mathbf{q}')$. So using Exercise 5 (for the particular case when $\mathbf{p} = \mathbf{p}'$), we get that there is an isometry A of S^2 such that $A(\mathbf{p}) = \mathbf{p}$ and $A(\mathbf{q}) = \mathbf{q}'$. It is not difficult to prove that A is necessarily of the form (0.7) for some $\theta \in [0, 2\pi[$. Then we conclude that the spherical circle $C(\mathbf{p}, \rho)$ coincides with the intersection of S^2 with an horizontal plane H in $\mathbb{R}^3$ (not necessarily passing through the origin).

In other terms, the *spherical* circle $C(\mathbf{p}, \rho)$ is an *euclidean* circle in the plane $H \simeq \mathbb{R}^2$. In order to compute its length, we need to understand what is its radius. Since the set $C(\mathbf{p}, \rho)$ is invariant by applying rotations of the form (0.7), then $C(\mathbf{p}, \rho)$ contains a point $\mathbf{q}$ of the form $(a, 0, c)$ for some $a, c \in \mathbb{R}$. In other terms, $\mathbf{q}$ is obtained as a linear combination of $\mathbf{p} = \mathbf{e}_3$ and of the vector $\mathbf{e}_1$ of $\mathbb{R}^3$. Moreover, since $d(\mathbf{p}, \mathbf{q}) = \rho$, as in Lecture Notes Part II § 1.3, we have that either

$$\mathbf{q} = (\cos\rho)\mathbf{e}_3 + (\sin\rho)\mathbf{e}_1 = \begin{pmatrix} \sin\rho \\ 0 \\ \cos\rho \end{pmatrix}.$$

or

$$\mathbf{q} = \begin{pmatrix} -\sin\rho \\ 0 \\ \cos\rho \end{pmatrix}.$$

In both cases, this implies that the horizontal plane H is ginve by the following equation

$$z = \cos\rho.$$

Since $C(\mathbf{p},\rho)$ is equal to $S^2 \cap H$, we have

$$\begin{aligned} C(\mathbf{p},\rho) &= \left\{(x,y,\cos\rho) \text{ s.t. } x^2+y^2+(\cos\rho)^2 = 1\right\} = \\ &= \left\{(x,y,\cos\rho) \text{ s.t. } x^2+y^2 = (\sin\rho)^2\right\}. \end{aligned}$$

Now $\rho \in [0,\pi]$ (by the general constructions in spherical geometry, see § 1.3 in Lecture Notes Part II), hence $\sin\rho \geq 0$, so we can also write:

$$C(\mathbf{p},\rho) = \left\{(x,y,\cos\rho) \text{ s.t. } \sqrt{x^2+y^2} = \sin\rho\right\}.$$

Therefore, $C(\mathbf{p},\rho)$ is a circumference in $\mathbb{R}^3$ (lying in the plane H, with center given by $(0,0,\cos\rho)$) and with radius $\sin\rho$. Therefore, the length of $C(\mathbf{p},\rho)$ is simply given by $2\pi\sin\rho$.

(c) We can compute the area of $D(\mathbf{p},\rho)$ by a multiple integral as follows:

$$\int_{D(\mathbf{p},\rho)} \mathrm{ds}\,\mathrm{dt} = \int_0^\rho \left(\int_{C(\mathbf{p},\sigma)} \mathrm{dl}\right) \mathrm{d}\,\sigma =$$
$$= \int_0^\rho 2\pi\sin\sigma\,\mathrm{d}\,\sigma = 2\pi\int_0^\rho \sin\sigma\,\mathrm{d}\,\sigma = 2\pi(-\cos(\rho)+\cos(0)) = 2\pi(1-\cos\rho).$$

Exercise 7

Let $\triangle\boldsymbol{pqr}$ be a spherical triangle with spherical angles $\angle\boldsymbol{p}$, $\angle\boldsymbol{q}$, $\angle\boldsymbol{r}$ and with spherical lengths of opposite edges equal, respectively, to α, β and γ. Using the main formula

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\angle\mathbf{p}, \tag{0.8}$$

prove the ***sine rule for spherical triangles****:*

$$\frac{\sin\alpha}{\sin\angle\mathbf{p}} = \frac{\sin\beta}{\sin\angle\mathbf{q}} = \frac{\sin\gamma}{\sin\angle\mathbf{r}}$$

From (0.8) we ge that

$$\sin\beta\sin\gamma\cos\angle\mathbf{p}=\cos\alpha-\cos\beta\cos\gamma. \tag{0.9}$$

Therefore,

$$\begin{aligned}(\sin\beta)^2(\sin\gamma)^2(\sin\angle\mathbf{p})^2 &= (\sin\beta)^2(\sin\gamma)^2(1-(\cos\angle\mathbf{p})^2)=\\ &=(\sin\beta)^2(\sin\gamma)^2-(\sin\beta\sin\gamma\cos\angle\mathbf{p})^2 \overset{(0.9)}{=}\\ &\overset{(0.9)}{=} 1-(\cos\beta)^2-(\cos\gamma)^2+(\cos\beta)^2(\cos\gamma)^2+\\ &-\Big((\cos\alpha)^2+(\cos\beta)^2(\cos\gamma)^2-2\cos\alpha\cos\beta\cos\gamma\Big)=\\ &=1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma.\end{aligned}$$

So

$$\frac{1}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma},$$

hence

$$\frac{(\sin\alpha)^2}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma}. \tag{0.10}$$

Note that in the previous computations $\angle\mathbf{p}$ is necessarily in $]0,\pi[$. Indeed, each angle we are considering is an angle in $[0,\pi]$. Moreover, if $\angle\mathbf{p}$ is zero or π, this means that the triangle is degenerate, so necessarily $\angle\mathbf{p}\in]0,\pi[$. So $\sin\angle\mathbf{p}\neq 0$, hence we could divide by $(\sin\angle\mathbf{p})^2$ above.

Exactly in the same way, we are able to prove also

$$\frac{(\sin\beta)^2}{(\sin\angle\mathbf{q})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma} \tag{0.11}$$

and

$$\frac{(\sin\gamma)^2}{(\sin\angle\mathbf{r})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma}. \tag{0.12}$$

Since the right hand sides of (0.10), (0.11) and (0.12) are equal, we get that

$$\frac{(\sin\alpha)^2}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\beta)^2}{(\sin\angle\mathbf{q})^2}=\frac{(\sin\gamma)^2}{(\sin\angle\mathbf{r})^2}.$$

Now we recall that the angles that we are considering are all in the interval $[0, \pi]$ (see § 1.2 in Lecture Notes Part II for the angles of the form α, β, γ and § 2.3 for the angles of the form $\angle\mathbf{p}, \angle\mathbf{q}, \angle\mathbf{r}$). Therefore, each sin in the previous identities has non-negative value (actually, it has values in $[0, 1]$). Therefore, by taking square roots we get:

$$\frac{\sin\alpha}{\sin\angle\mathbf{p}} = \frac{\sin\beta}{\sin\angle\mathbf{q}} = \frac{\sin\gamma}{\sin\angle\mathbf{r}}.$$

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