EXERCISE SESSION 7A FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Exercises done during the lecture of May 28, 2014

All the exercise of this session can be found in Exercise Sheets 1 – 10 and were not considered in full details in the previous exercise session (the solutions can also be found in the pdf's from 1B to 5B).

Exercise 8 (Exercise Sheet 1 – 2)

Prove that a matrix $A \in SO(2)$ has a (non-zero) eigenvector if and only if $A = \pm \mathrm{Id}$.

Let us suppose that X is an eigenvector for A, with eigenvalue λ. Since $A \in SO(2) \subset O(2)$, we have:

$$\begin{aligned}\lambda^2|X|^2 &= \lambda^2 < X, X >= \lambda < \lambda X, X >=< \lambda X, \lambda X >=< A \cdot X, A \cdot X >= \\ &= (A \cdot X)^T \cdot (A \cdot X) = (X^T \cdot A^T) \cdot (A \cdot X) = X^T \cdot (A^T \cdot (A \cdot X)) = \\ &= X^T \cdot ((A^T \cdot A) \cdot X) = X^T \cdot (\mathrm{Id} \cdot X) = X^T \cdot X =< X, X >= |X|^2.\end{aligned}$$

Since X is an eigenvector for A, then it is non-zero, so $|X|^2 \neq 0$. So from the previous identity we get $\lambda^2 = 1$, i.e. λ is either equal to 1 or to -1.

We already know (see Exercise 5 in Exercise Sheet 1A or Exercise 4.2.4 in Lecture Notes) that any matrix A in $O(2)$ is either a rotation

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.1}$$

(around the origin of $\mathbb{R}^2$) of an angle θ or a reflection

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.2}$$

(around the x-axis in $\mathbb{R}^2$), followed by a rotation $R(\theta)$.

By hypothesis, $A \in SO(2)$, i.e. $\det A = 1$, so we are necessarily in the first case, i.e. $A = R(\theta)$ for some $\theta \in [0, 2\pi[$ (we cannot be in the second case since in that case $\det A = -1$). Given any non-zero vector $X \in \mathbb{R}^2$, we can always write it as

$$X = \begin{pmatrix} \rho \cos \alpha \\ \rho \sin \alpha \end{pmatrix}$$

for a unique choice of $\alpha \in [0, 2\pi[$ and $\rho \in \mathbb{R}_{>0}$ (polar coordinates). Then

$$A \cdot X = R(\theta) \cdot X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta+\alpha) \\ \rho\sin(\theta+\alpha) \end{pmatrix}.$$

Therefore, if X is a (non-zero) eigenvector for A, with eigenvalue λ, then we must have:

$$\begin{pmatrix} \lambda\rho\cos\alpha \\ \lambda\rho\sin\alpha \end{pmatrix} = \lambda \cdot X = A \cdot X = \begin{pmatrix} \rho\cos(\theta+\alpha) \\ \rho\sin(\theta+\alpha) \end{pmatrix}.$$

Since $\rho \neq 0$, this implies that

$$\lambda \cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad \lambda \sin\alpha = \sin(\theta+\alpha).$$

We have already proved that λ is equal to 1 or -1. Therefore, we have 2 cases as follows.

- If $\lambda = 1$, then we conclude that

$$\cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad \sin\alpha = \sin(\theta+\alpha).$$

  i.e. $\theta + \alpha = \alpha + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore, $\theta = 2k\pi$ and

$$A = \begin{pmatrix} \cos(2k\pi) & -\sin(2k\pi) \\ \sin(2k\pi) & \cos(2k\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \operatorname{Id}.$$

- If $\lambda = -1$, then we conclude that

$$-\cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad -\sin\alpha = \sin(\theta+\alpha).$$

  i.e. $\theta + \alpha = \alpha + (2k+1)\pi$ for some $k \in \mathbb{Z}$. Therefore, $\theta = 2(k+1)\pi$ and

$$A = \begin{pmatrix} \cos((2k+1)\pi) & -\sin((2k+1)\pi) \\ \sin((2k+1)\pi) & \cos((2k+1)\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\operatorname{Id}.$$

Exercise 9 (Exercise Sheet 1 – 2)

Let $T : \mathbb{E}^2 \to \mathbb{E}^2$ be an isometry such that $T^n = \operatorname{Id}$. If n is an odd integer, what can be said about T? Explain.

Let us fix any affine coordinate system $\phi : \mathbb{E}^2 \to \mathbb{R}^2$. By Theorem 5.4 we get that the induced isometry

$$\overline{T} := \phi \circ T \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is of the form

$$\overline{T}(X) = A \cdot X + B \quad \forall\, X \in \mathbb{R}^2$$

for some matrix $A \in O(2)$ and some vector $B \in \mathbb{R}^2$. Now let us fix any vector X in $\mathbb{R}^2$; then we have:

$$\phi \circ T^2 \circ \phi^{-1}(X) = (\phi \circ T \circ \phi^{-1}) \circ (\phi \circ T \circ \phi^{-1}) = \overline{T}^2(X) =$$
$$= \overline{T} \circ \overline{T}(X) = \overline{T}(A \cdot X + B) = A(A \cdot X + B) + B = A \cdot A \cdot X + A \cdot B + B.$$

So we have $\phi \circ T^2 \circ \phi^{-1}(X) = A^2 \cdot X + B_2$ (where $B_2 = A \cdot B + B$). By induction, we conclude that

$$\phi \circ T^n \circ \phi^{-1}(X) = A^n \cdot X + B_n \quad \forall\, X \in \mathbb{R}^2$$

for some B_n in $\mathbb{R}^2$. Since $T^n = \mathrm{Id}$, then this implies that

$$A^n \cdot X + B_n = \phi \circ \mathrm{Id} \circ \phi^{-1}(X) = X \quad \forall\, X \in \mathbb{R}^2,$$

so $B_n = 0 \in \mathbb{R}^2$ and $A^n = \mathrm{Id}$. Since n is odd, then this implies that $\det A = 1$.

We recall the classification for 2×2 orthogonal matrices that we saw already (see Exercise 5 in Exercise Sheet 1A or Exercise 4.2.4 in Lecture Notes): any 2×2 orthogonal matrix is either of the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.3}$$

or of the form

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.4}$$

for some $\theta \in [0, 2\pi[$. The first matrix represents a rotation in $\mathbb{R}^2$ of an angle θ (and has determinant 1), while the second one represents a reflection in $\mathbb{R}^2$, followed by a rotation of θ (and has determinant -1).

So we conclude that A is a rotation around the origin of $\mathbb{R}^2$, of a certain angle θ. Therefore, A^2 is the rotation around the origin of an angle 2θ; by induction A^n is the rotation around the origin of an angle $n\theta$. Since A^n is equal to Id, this means that $n\theta$ is an integer multiple of 2π, so θ is equal to $2k\pi/n$ for some $k \in \mathbb{Z}$. So we have:

$$\overline{T}(X) = \begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix} \cdot X + B \quad \forall\, X \in \mathbb{R}^2.$$

and

$$T(p) = \phi^{-1} \circ \overline{T} \circ \phi(p) = \phi^{-1} \left[\begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix} \cdot \phi(p) + B \right] \quad \forall\, p \in \mathbb{E}^2.$$

With the information contained in the statement of the Exercise, we cannot say anything more about the vector $B \in \mathbb{R}^2$ and/or the integer k.

Exercise 10 (Exercise Sheet 1 – 2)

Let $A \in O(n)$ and let $X, Y \in \mathbb{R}^n$ be two eigenvectors with distinct eigenvalues $\lambda \neq \mu$. Prove that X and Y are orthogonal, i.e. $< X, Y >= 0$.

Since X is an eigenvector for A, with eigenvalue λ, then we have:

$$\begin{aligned} \lambda^2 |X|^2 = \lambda^2 < X, X >= \lambda < \lambda X, X >=< \lambda X, \lambda X >=< A \cdot X, A \cdot X >= \\ = (A \cdot X)^T \cdot (A \cdot X) = (X^T \cdot A^T) \cdot (A \cdot X) = X^T \cdot (A^T \cdot (A \cdot X)) = \\ = X^T \cdot ((A^T \cdot A) \cdot X) = X^T \cdot (\mathrm{Id} \cdot X) = X^T \cdot X =< X, X >= |X|^2. \end{aligned}$$

Since X is an eigenvector for A, then it is non-zero, so $|X|^2 \neq 0$. So from the previous identity we get $\lambda^2 = 1$. Hence, we have proved that each eigenvector λ for an orthogonal matrix is either equal to 1 or to -1. By hypothesis, Y is another eigenvector for A with eigenvalue μ, and $\lambda \neq \mu$. Since μ is an eigenvalue for A, then also μ is equal to 1 or -1. So, up to permuting the roles of (X, λ) and (Y, μ), we can always assume that $\lambda = 1$ and $\mu = -1$. Now we have:

$$\begin{aligned} - < X, Y >=< X, -Y >=< \lambda X, \mu Y >=< A \cdot X, A \cdot Y >= \\ = (A \cdot X)^T \cdot (A \cdot Y) = (X^T \cdot A^T) \cdot (A \cdot Y) = X^T \cdot (A^T \cdot (A \cdot Y)) = \\ = X^T \cdot ((A^T \cdot A) \cdot Y) = X^T \cdot (\mathrm{Id} \cdot Y) = X^T \cdot Y =< X, Y > . \end{aligned}$$

So $- < X, Y >=< X, Y >$. This implies that $< X, Y >= 0$, i.e. that X and Y are orthogonal. In other terms, any 2 different eigenspaces of an orthogonal matrix are orthogonal.

Exercise 7 (Exercise sheet 5 – 6)

Let $\triangle \boldsymbol{pqr}$ be a spherical triangle with spherical angles $\angle \boldsymbol{p}$, $\angle \boldsymbol{q}$, $\angle \boldsymbol{r}$ and with spherical lengths of opposite edges equal, respectively, to α, β and γ. Using the main formula

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\angle\mathbf{p}, \tag{0.5}$$

*prove the **sine rule for spherical triangles**:*

$$\frac{\sin\alpha}{\sin\angle\mathbf{p}} = \frac{\sin\beta}{\sin\angle\mathbf{q}} = \frac{\sin\gamma}{\sin\angle\mathbf{r}}$$

From (0.5) we ge that

$$\sin\beta\sin\gamma\cos\angle\mathbf{p}=\cos\alpha-\cos\beta\cos\gamma. \tag{0.6}$$

Therefore,

$$\begin{aligned}(\sin\beta)^2(\sin\gamma)^2(\sin\angle\mathbf{p})^2 &= (\sin\beta)^2(\sin\gamma)^2(1-(\cos\angle\mathbf{p})^2) = \\ &= (\sin\beta)^2(\sin\gamma)^2-(\sin\beta\sin\gamma\cos\angle\mathbf{p})^2 \overset{(0.6)}{=} \\ &\overset{(0.6)}{=} 1-(\cos\beta)^2-(\cos\gamma)^2+(\cos\beta)^2(\cos\gamma)^2+ \\ &-\Big((\cos\alpha)^2+(\cos\beta)^2(\cos\gamma)^2-2\cos\alpha\cos\beta\cos\gamma\Big) = \\ &= 1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma.\end{aligned}$$

So

$$\frac{1}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma},$$

hence

$$\frac{(\sin\alpha)^2}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma}. \tag{0.7}$$

Note that in the previous computations $\angle\mathbf{p}$ is necessarily in $]0,\pi[$. Indeed, each angle we are considering is an angle in $[0,\pi]$. Moreover, if $\angle\mathbf{p}$ is zero or π, this means that the triangle is degenerate, so necessarily $\angle\mathbf{p}\in]0,\pi[$. So $\sin\angle\mathbf{p}\neq 0$, hence we could divide by $(\sin\angle\mathbf{p})^2$ above.

Exactly in the same way, we are able to prove also that

$$\frac{(\sin\beta)^2}{(\sin\angle\mathbf{q})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma} \tag{0.8}$$

and

$$\frac{(\sin\gamma)^2}{(\sin\angle\mathbf{r})^2}=\frac{(\sin\alpha)^2(\sin\beta)^2(\sin\gamma)^2}{1-(\cos\alpha)^2-(\cos\beta)^2-(\cos\gamma)^2+2\cos\alpha\cos\beta\cos\gamma}. \tag{0.9}$$

Since the right hand sides of (0.7), (0.8) and (0.9) are equal, we get that

$$\frac{(\sin\alpha)^2}{(\sin\angle\mathbf{p})^2}=\frac{(\sin\beta)^2}{(\sin\angle\mathbf{q})^2}=\frac{(\sin\gamma)^2}{(\sin\angle\mathbf{r})^2}.$$

Now we recall that the angles that we are considering are all in the interval $[0,\pi]$ (see § 1.2 in Lecture Notes Part II for the angles of the form α,β,γ and § 2.3 for the angles of the form $\angle\mathbf{p}, \angle\mathbf{q}, \angle\mathbf{r}$). Therefore, each sin in the previous identities has non-negative value (actually, it has values in $[0,1]$). Therefore, by taking square roots we get:

$$\frac{\sin\alpha}{\sin\angle\mathbf{p}} = \frac{\sin\beta}{\sin\angle\mathbf{q}} = \frac{\sin\gamma}{\sin\angle\mathbf{r}}.$$

Exercise 8 (Exercise Sheet 7 – 8)

Let

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} \text{ s.t. } y > 0\}$$

be the upper half-plane in the complex plane. Define $\mathbb{H}$-lines to be of two kinds: either vertical Euclidean half-lines

$$L_1 = \{x + iy \in \mathbb{H} \text{ s.t. } x = b\} = \{b + iy \text{ s.t. } y > 0\} \tag{0.10}$$

for any real constant b, or half-circles

$$L_2 = \{x + iy \in \mathbb{H} \text{ s.t.} (x-a)^2 + y^2 = c^2\} \tag{0.11}$$

with centre at $a + 0i$ on the real axis $\{y = 0\}$, for any $a \in \mathbb{R}$ and for any $c \in \mathbb{R}_{\geq 0}$. Show, algebraically or by drawing pictures, that:

(a) *two distinct $\mathbb{H}$-lines meet in at most one point*;
(b) *every pair of distinct points of $\mathbb{H}$ lies on a unique $\mathbb{H}$-line.*
(c) *Given an $\mathbb{H}$-line L and a point $P \in \mathbb{H}$ not on L, there are more than two $\mathbb{H}$-lines through P which are parallel to L (i.e. have no intersection with L).*

(a). Given 2 distinct vertical lines L_1 and L'_1, we have that $L_1 \cap L'_1 = \varnothing$. Given a vertical line L_1 as in (0.10) and a half-circle as in (0.11), their intersection is the set of points $x+iy$ (with $x, y \in \mathbb{R}$)), such that $x = b$,

$$(b-a)^2 + y^2 = c^2 \quad \text{and} \quad y > 0.$$

If $c^2 - (b-a)^2 > 0$, then there is exactly one such y, namely

$$y = \sqrt{c^2 - (b-a)^2};$$

if $c^2 - (b-a)^2 \leq 0$ there is no solution, hence $L_1 \cap L_2 = \varnothing$. Lastly, we have to consider the case when we are intersecting 2 half-circles. So let us fix L_2 as in (0.11) and

$$L'_2 = \{x + iy \in \mathbb{H} \text{ s.t.} (x-a')^2 + y^2 = c'^2\}$$

for any pair pair (a', c') such that $L_2 \neq L'_2$ (this is equivalent to impose that $(a,c) \neq (a',c')$ - here both points belong to $\mathbb{R} \times \mathbb{R}_{\geq 0}$). We know that the intersection of 2 distinct circles consists of at most 2 points, that are symmetric with respect to the (euclidean) line joining the centers of the 2 circles. If the intersection is empty or consists of only 1 point, we are done. Otherwise, let us suppose that the intersection of the circles consists of exactly 2 points. Since the line joining their centers is the axis $\{y = 0\}$, then this means that at exactly one of such points has $y > 0$. The other one belongs to the intersection of the circles, but NOT to the intersection of the half-circles. This suffices to conclude.

(b). Let us fix a pair of distinct points $P_1 = x_1 + iy_1$ and $P_2 = x_2 + iy_2$. If $x_1 = x_2$, then P_1 and P_2 are joined by the vertical line

$$L := \{x_1 + iy \text{ s.t. } y > 0\}.$$

Moreover, this is the only vertical line joining P_1 and P_2; in addition, since $P_1 \neq P_2$, there is no half-circle joining such a pair of points.

Now let us consider the remaining case, i.e. the case when $x_1 \neq x_2$. In this case there is no vertical line joining P_1 and P_2. Then we consider the (euclidean) segment S joining P_1 and P_2 and its medium point M. From M we draw the (euclidean) line T perpendicular to S. Since $x_1 \neq x_2$, then S is not vertical, so T is not an horizontal line. So it intersects the axis $\{y = 0\}$ in exactly one point $a + 0i$ (for some $a \in \mathbb{R}$). Now T is the axis of the segment S, so each point R in T has the same distance from P_1 and P_2. In particular,

$$d(a + 0i, P_1) = d(a + oi, P_2).$$

We denote such a distance by c for simplicity. Then the circumference C with center in $a + 0i$ and radius c passes through the points P_1 and P_2. Since P_1 and P_2 belong to $\mathbb{H}$, then the points P_1 and P_2 belong to the half-circle

$$L_2 := \{x + iy \in \mathbb{H} \text{ s.t. } (x - a)^2 + y^2 = c^2\} = \mathbb{H} \cap C.$$

So we have proved that there is a half-circle passing through P_1 and P_2. In order to conclude, we need to prove that this is the only such half-circle passing through P_1 and P_2. Let us suppose that there is another such semicircle L_2', with center A in the axis $\{y = 0\}$ and radius r. Then we have that the center A of L_2' must be at the same distance from P_1 and P_2, hence it must belong to the euclidean line T. Moreover, we are considering only half-circles centered at points in the axis $\{y = 0\}$. So the center A must belong to $T \cap \{y = 0\}$, so it must coincide with $a + 0i$. In this case, the radius r coincides with the radius c, hence L_2' is equal to L_2. So we have proved that also in this case there is only an $\mathbb{H}$-line passing through the pair of distinct points P_1 and P_2.

(c). We have to consider 2 cases separately.

Case 1. We suppose that L is a vertical line

$$L = \{x + iy \in \mathbb{H} \text{ s.t. } x = b\} = \{b + iy \text{ s.t. } y > 0\}.$$

We fix any point $P = x_0 + iy_0$, with $y_0 > 0$ and $x_0 \neq b$. For simplicity, we suppose that $b < x_0$ (the same proof holds with minor modifications in the other case). We consider any point $Q := b' + 0i$, with

$$b < b' < x_0 \tag{0.12}$$

i.e. any point between P and the intersection of L with the axis $\{y = 0\}$. Then as we did in (b) we construct the unique $\mathbb{H}$-line L_Q passing through P and Q (note: in (b) the point Q belonged to $\mathbb{H}$ and not to the axis $\{y = 0\}$, but this does not give additional problems in the present construction). Now it is clear (either you draw a picture or you do some basic computations) that $L_Q \cap L = \varnothing$. This holds for every point $Q = b' + 0i$ such that (0.12) holds, i.e. we have obtained infinitely many $\mathbb{H}$-lines passing through P that are parallel to L. In this case there is also an extra line parallel to L, namely the vertical line passing through P.

Case 2. In this case we suppose that L is an half-circle, i.e.

$$L = \{x + iy \in \mathbb{H} \text{ s.t. } (x - a)^2 + y^2 = c^2\}$$

for some $a \in \mathbb{R}$ and some $c \in \mathbb{R}_{\geq 0}$. Let us fix again a point $P = x_0 + iy_0$ with $y_0 > 0$ and such that $P \notin L$. This means that

$$(x_0 - a)^2 + y_0^2 \neq c^2$$

So we have to consider 2 subcases separately as follows:

Subcase 2.1. In this subcase we suppose that

$$(x_0 - a)^2 + y_0^2 < c^2.$$

This means that P belongs to the interior of the half-disc defined by L and by the axis $\{y = 0\}$. Now we define

$$c' := d(P, a + 0i).$$

Since P belongs to the interiour of the semidisk mentioned above, then we have

$$c' < c.$$

Then we choose any $\varepsilon \in \mathbb{R}$ satisfying the following conditions

$$0 < \varepsilon < \frac{c - c'}{2} \tag{0.13}$$

(this makes sense because $c' < c$). Having fixed ε, we define

$$c''_\varepsilon := d(P, a + \varepsilon + 0i). \tag{0.14}$$

By triangular inequality, we have

$$c''_\varepsilon \leq d(P, a + 0i) + d(a + 0i, a + \varepsilon + 0i) = c' + \varepsilon. \tag{0.15}$$

Then we consider the $\mathbb{H}$-line E_ε given by the half-circle centered in $a+\varepsilon+0i$ and with radius c''_ε. Let us fix any point T in E_ε. Then by triangular inequality, (0.15) and (0.13) we have

$$d(T, a+0i) \leq d(T, a+\varepsilon+0i) + d(a+\varepsilon+0i, a+0i) =$$
$$= c''_\varepsilon + \varepsilon \leq c' + \varepsilon + \varepsilon = c' + 2\varepsilon < c' + 2\frac{c-c'}{2} = c.$$

This means that the whole half-circle L_ε is contained in the half-disc defined by L and by the axis $\{y=0\}$. In particular, $E_\varepsilon \cap L = \varnothing$ for all ε such that (0.13) holds. Moreover, P belongs to E_ε because of (0.14). So we have found infinitely many $\mathbb{H}$-lines as required.

Subcase 2.2. In this subcase we suppose that

$$(x_0-a)^2 + y_0^2 > c^2.$$

This means that $P = x_0 + iy_0$ is "outside" the half-disk defined by L and by the axis $\{y=0\}$. We define

$$c' := d(P, a+0i).$$

Since P is outside the semidisk mentioned above, then we have

$$c' > c.$$

Then we choose any σ in $\mathbb{R}$ satisfying the following conditions

$$0 < \sigma < \frac{c'-c}{2} \tag{0.16}$$

(this makes sense because $c' > c$). Having fixed σ, we define

$$c''_\sigma := d(P, a+\sigma+0i).$$

By triangular inequality, we have

$$c' = d(P, a+0i) \leq d(P, a+\sigma+0i) + d(a+\sigma+0i, a+0i) = c''_\sigma + \sigma. \tag{0.17}$$

Hence,

$$c''_\sigma \geq c' - \sigma. \tag{0.18}$$

Then we consider the $\mathbb{H}$-line D_σ given by the half-circle centered in $a+\sigma+0i$ and with radius c''_σ. Then for each point T in D_σ using the triangular inequality we have

$$c''_\sigma = d(T, a+\sigma+0i) \leq d(T, a+0i) + d(a+0i, a+\sigma+0i) = d(T, a+0i) + \sigma. \tag{0.19}$$

Therefore, using (0.19), (0.18) and (0.16) we have

$$d(T, a + 0i) \geq c''_\sigma - \sigma \geq c' - \sigma - \sigma = c' - 2\sigma > c' - 2\frac{c' - c}{2} = c.$$

This means that T is "outside" the semidisk mentioned above, so $D_\sigma \cap L = \varnothing$ for all σ as in (0.16). Moreover, $P \in D_\sigma$ because of (0.17). So we have found infinitely many $\mathbb{H}$-lines as required. Note that if $|x_0 - a| > c$, then there is an extra $\mathbb{H}$-line passing through P and not intersecting L, namely the vertical line passing through P.

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