EXERCISE SESSION 6A FOR THE COURSE “GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE”

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Solutions for the test of May 6, 2014

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Exercises done during the lecture of May 14, 2014

Exercise 1

(i) Give a definition of the isometry group of the Euclidean plane.
(ii) Find all isometries of the Euclidean plane $\mathbb{E}^2 \to \mathbb{E}^2$ which, in some affine coordinates, send the point $(0, \sqrt{2})$ to the point $(2, 3)$.

(i) The group of isometries of $\mathbb{E}^2$ is the set of all bijective maps $T : \mathbb{E}^2 \to \mathbb{E}^2$, such that $d(T(p), T(q)) = d(p, q)$ for every pair of points $p, q \in \mathbb{E}^2$. Such a set is equipped with a structure of group with multiplication given by composition of maps, identity element given by the identity of $\mathbb{E}^2$ and inverse of any T given by the (set theoretical) inverse T^{-1}. Having fixed a coordinate system on $\mathbb{E}^2$ (i.e. a bijection $\phi : \mathbb{E}^2 \to \mathbb{R}^2$ preserving distances), the group of isometries of $\mathbb{E}^2$ can be identified with the group of isometries of $\mathbb{R}^2$. This last group is given by the group of all applications $F : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$F(X) := A \cdot X + B \quad \forall\, X \in \mathbb{R}^2$$

for some vector $B = (b_1, b_2) \in \mathbb{R}^2$ and for some matrix $A \in O(2)$, i.e. of the form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.1}$$

or of the form

$$S(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = R(\theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.2}$$

for some $\theta \in [0, 2\pi[$ (see Exercise 4.2.4 in Lecture Notes Part I).

NOTE: The solution of (ii) below is similar to the solution of Exercise 6 in Exercise Sheet 1 – 2 (see Exercise solutions 1B)

(ii) Let us fix any affine coordinate $\phi : \mathbb{E}^2 \to \mathbb{R}^2$ and let us set

$$\overline{T} := \phi \circ T \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

We are trying to find an isometry T such that $\overline{T}$ sends the point $(0, \sqrt{2})$ to $(2, 3)$.

By Theorem 5.4 in Lecture Notes Part I, we get that $\overline{T}$ has the form $X \mapsto A \cdot X + B$ for some matrix $A \in O(2)$ and some vector $B = (b_1, b_2) \in \mathbb{R}^2$. Since $\overline{T}$ must send $(0, \sqrt{2})$ to $(2, 3)$, we get that

$$A \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so we must have:

$$A \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 - b_1 \\ 3 - b_2 \end{pmatrix}. \tag{0.3}$$

If A is a matrix of the form (0.1), then from (0.3) we get:

$$\begin{pmatrix} -\sqrt{2}\sin\theta \\ \sqrt{2}\cos\theta \end{pmatrix} = \begin{pmatrix} 2 - b_1 \\ 3 - b_2 \end{pmatrix}.$$

This implies that

$$\sin\theta = \frac{b_1 - 2}{\sqrt{2}} \quad \text{and} \quad \cos\theta = \frac{3 - b_2}{\sqrt{2}}. \tag{0.4}$$

Since we want to find a θ such that (0.4) holds, we have necessarily that

$$1 = (\sin\theta)^2 + (\cos\theta)^2 = \frac{1}{2}(b_1^2 + 4 - 4b_1 + 9 + b_2^2 - 6b_2).$$

This is equivalent to imposing that

$$b_1^2 + b_2^2 - 4b_1 - 6b_2 + 11 = 0. \tag{0.5}$$

Under this condition, the point

$$\left(\frac{b_1 - 2}{\sqrt{2}}, \frac{3 - b_2}{\sqrt{2}}\right) \in \mathbb{R}^2$$

is a point on S^1, hence there is a unique $\theta \in [0, 2\pi[$, such that (0.4) holds. So if A is of the form (0.1), then $\overline{T}$ sends the point $(0, \sqrt{2})$ to $(2, 3)$ if and only if (0.5) holds. In this case, θ is uniquely determined by (0.4).

If A is a matrix of the form (0.2), then from (0.3) we get:

$$\begin{pmatrix} \sqrt{2}\sin\theta \\ -\sqrt{2}\cos\theta \end{pmatrix} = \begin{pmatrix} 2 - b_1 \\ 3 - b_2 \end{pmatrix}.$$

This implies that

$$\sin\theta = \frac{2-b_1}{\sqrt{2}} \quad \text{and} \quad \cos\theta = \frac{b_2-3}{\sqrt{2}}. \tag{0.6}$$

Proceeding as before, we get that (0.6) holds if and only if (0.5) holds. In this case, θ is uniquely determined by (0.6). Now if θ is the solution of (0.4) for a fixed pair (b_1, b_2), then the solution of (0.6) (for the same pair (b_1, b_2)) is simply given by $-\theta$.

So we conclude that the only isometries $\overline{T}$ of $\mathbb{R}^2$ sending $(0, \sqrt{2})$ to $(2, 3)$ are exactly all those of the form

$$\overline{T}(X) = R(\theta) \cdot X + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \forall\, X \in \mathbb{R}^2 \tag{0.7}$$

and those of the form

$$\overline{T}(X) = S(\theta) \cdot X + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \forall\, X \in \mathbb{R}^2 \tag{0.8}$$

for all those values of $(b_1, b_2) \in \mathbb{R}^2$ such that

$$b_1^2 + b_2^2 - 4b_1 - 6b_2 + 11 = 0.$$

For any fixed value of (b_1, b_2) satisfying this condition, the value of θ above is completely determined by (0.4) in case (0.7) and as $-\theta$ in case (0.8).

Exercise 5

Let $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{p}', \boldsymbol{q}'$ be any two pairs of points of S^2 such that $d(\boldsymbol{p}, \boldsymbol{q}) = d(\boldsymbol{p}', \boldsymbol{q}')$. Show that there is a spherical isometry $T : S^2 \to S^2$ such that $T(\boldsymbol{p}) = \boldsymbol{p}'$ and $T(\boldsymbol{q}) = \boldsymbol{q}'$.

NOTE: this Exercise coincides with Exercise 5 in Exercise Sheet 5 – 6 (see Exercise Solutions 3B).

For simplicity, we set $\rho := d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p}', \mathbf{q}')$.

Using Exercise 2 in Exercise Sheet 5 – 6, there is an isometry A of S^2, such that

$$A(\mathbf{p}) = (0, 0, 1) \quad \text{and} \quad A(\mathbf{q}) = (a, 0, c) \tag{0.9}$$

for a suitable pair of scalars a, c.

Moreover, using again Exercise 2 in Exercise Sheet 5 – 6 (applied to the pair $(\mathbf{p}', \mathbf{q}')$ instead of the pair $(\mathbf{p}, \mathbf{q})$), there is an isometry B of S^2, such that

$$B(\mathbf{p}') = (0,0,1) \quad \text{and} \quad B(\mathbf{q}') = (a',0,c') \tag{0.10}$$

for a suitable pair of scalars a', c'.

Since both A and B are isometries, we get:

$$\begin{aligned} d((0,0,1),(a,0,c)) &= d(A(\mathbf{p}),A(\mathbf{q})) = d(\mathbf{p},\mathbf{q}) = \rho = \\ &= d(\mathbf{p}',\mathbf{q}') = d(A(\mathbf{p}'),A(\mathbf{q}')) = d((0,0,1),(a',0,c')). \end{aligned}$$

Now since $d((0,0,1),(a,0,c)) = \rho$ and since the distance used here is the spherical distance, this means that

$$\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \cos\rho \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \sin\rho \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\rho \\ 0 \\ \cos\rho \end{pmatrix}$$

or

$$\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -\sin\rho \\ 0 \\ \cos\rho \end{pmatrix}$$

(see § 1.3 in Lecture Notes Part II). In the same way we get also that

$$\begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} = \begin{pmatrix} \sin\rho \\ 0 \\ \cos\rho \end{pmatrix}$$

or

$$\begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} = \begin{pmatrix} -\sin\rho \\ 0 \\ \cos\rho \end{pmatrix}.$$

So we consider 2 cases as follows

(a) If the signs of $\sin\rho$ are the same for $(a,0,c)$ and $(a',0,c')$ (i.e. both positive or both negative), then we define

$$T := B^{-1} \cdot A : \, S^2 \longrightarrow S^2$$

and we have:

$$T(\mathbf{p}) \overset{(0.9)}{=} B^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.10)}{=} \mathbf{p}'$$

and

$$T(\mathbf{q}) \overset{(0.9)}{=} B^{-1} \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} \overset{(0.10)}{=} \mathbf{q}'.$$

(b) If the signs of $\sin\rho$ are the different for $(a, 0, c)$ and $(a', 0, c')$ (i.e. one positive and the other one negative, or conversely), then we have $a = -a'$ and $c = c'$. So we define

$$I := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(3)$$

and we have

$$I \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix}. \tag{0.11}$$

Then we define

$$T := B^{-1} \cdot I \cdot A : \, S^2 \longrightarrow S^2$$

and we have:

$$T(\mathbf{p}) \overset{(0.9)}{=} B^{-1} \cdot I \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \overset{(0.10)}{=} \mathbf{p}'$$

and

$$T(\mathbf{q}) \overset{(0.9)}{=} B^{-1} \cdot I \cdot \begin{pmatrix} a \\ 0 \\ c \end{pmatrix} \overset{(0.11)}{=} B^{-1} \cdot \begin{pmatrix} a' \\ 0 \\ c' \end{pmatrix} \overset{(0.10)}{=} \mathbf{q}'.$$

So we have found T as in the claim. In both cases, T is obtained as composition of matrices in $O(3)$, so it belongs to $O(3)$, hence it is an isometry of S^2 as required.

Exercise 7

Show that any two distinct spherical lines intersect in a pair of antipodal points.

By definition, a spherical line is the intersection of S^2 with a plane of $\mathbb{R}^3$, passing through the origin of $\mathbb{R}^3$. Now let us fix any 2 such distinct spherical lines L_1 and L_2. So there are 2 planes Π_1, Π_2 passing through the origin of $\mathbb{R}^3$, such that

$$L_1 = \Pi_1 \cap S^2 \qquad \text{and} \qquad L_2 = \Pi_2 \cap S^2.$$

Since $L_1 \neq L_2$, then $\Pi_1 \neq \Pi_2$. Therefore, $\Pi_1 \cap \Pi_2$ is a (straight) line R passing through the origin of $\mathbb{R}^3$. As such, R intersects S^2 in a pair of antipodal points P and $-P$. Therefore,

$$L_1 \cap L_2 = (\Pi_1 \cap S^2) \cap (\Pi_2 \cap S^2) = (\Pi_1 \cap \Pi_2) \cap S^2 = R \cap S^2 = \{P, -P\}.$$

Exercise 10

Prove that there is a unique hyperbolic line through any two different points in $\mathcal{H}^2$.

NOTE: this is an extended proof of Lemma 3.1.1 in Lecture Notes Part III.

Let us fix any pair of distinct points P, Q in $\mathcal{H}^2$. Let us denote by $\mathbf{v} = (t_1, x_1, y_1)$ the vector joining P and 0. Such a vector is non-zero (otherwise, P would coincide with 0, but this is impossible because $t_1 > 0$ by definition of $\mathcal{H}^2$). Analogously, let us denote by $\mathbf{u} = (t_2, x_2, y_2)$ the non-zero vector joining Q and 0. If there is $\lambda \in \mathbb{R}$ such that $\mathbf{u} = \lambda \cdot \mathbf{v}$, then we have

$$-1 = -t_2^2 + x_2^2 + y_2^2 = -(\lambda t_1)^2 + (\lambda x_1)^2 + (\lambda y_1)^2 = \lambda^2(-t_1^2 + x_1^2 + y_1^2) = -\lambda^2.$$

So we get that $\lambda = +1$ or $\lambda = -1$. The first case is impossible because $\mathbf{v}$ and $\mathbf{u}$ are different (because P and Q are distinct). Also the second case is impossible because it would imply that one between t_1 and t_2 is non-positive (but this contradicts the fact that both t_1 and t_2 are positive by definition of $\mathcal{H}^2$).

So we conclude that there is not any $\lambda \in \mathbb{R}$ such that $\mathbf{u} = \lambda \cdot \mathbf{v}$. Since $\mathbf{v} \neq (0, 0, 0)$, this implies that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. So it makes sense to consider the plane Π generated by $\mathbf{u}$ and $\mathbf{v}$. This plane passes through 0, P and Q and it is the only plane with such a property. Therefore, there is a unique hyperbolic line (namely $\mathcal{H}^2 \cap \Pi$) passing through P and Q.

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