EXERCISE SESSION 2B FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 3-4
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Exercises NOT done during the lecture of March 12, 2014

Exercise 1

Do Exercise 5.5.2 from Lecture Notes Part I.

Let X be any set: we need to prove that the set $Bij(X)$ of bijective maps on X is a group.

- The composition is the usual composition of set maps, hence it is associative;
- the neutral element is the identity $\mathrm{id}_X : X \to X$; indeed for any bijective map $f : X \to X$ we have $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ f = f$;
- given any bijective map $f : X \to X$, its inverse $f^{-1} : X \to X$ is again bijective (so it belongs to $Bij(X)$); moreover $f \circ f^{-1} = \mathrm{id}_X$ and $f^{-1} \circ f = \mathrm{id}_X$, so f^{-1} is the inverse of f in the set $Bij(X)$;
- given any pair of bijective maps $f : X \to X$ and $g : X \to X$, their composition $g \circ f : X \to X$ is also a bijective map, hence it belongs to $Bij(X)$.

Exercise 7

Consider the maps

$$\begin{array}{cccc} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \mapsto & (xy, x^2+z^2) \end{array} \quad \textit{and} \quad \begin{array}{cccc} g: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \mapsto & (y\sin x, y-2x). \end{array}$$

and compute the composition $g \circ f : \mathbb{R}^3 \to \mathbb{R}^2$.

The composition is given as follows:

$$g \circ f(x,y,z) = g(xy, x^2+z^2) = \left((x^2+z^2)\sin(xy), x^2+z^2-2xy\right).$$

Exercise 10

Prove the implication (1) $\Rightarrow$ (2) *in Proposition 5.3.2 from Lecture Notes Part I.*

We recall the statement of this proposition: given any map $T : \mathbb{E}^n \to \mathbb{E}^n$, the following facts are equivalent:

(1) T is given in some affine coordinate system by $T(X) = A \cdot X + B$, where A is an $n \times n$ matrix and B is a vector in $\mathbb{R}^n$.
(2) For any pair of vectors $X, Y \in \mathbb{R}^n$ and for any pair of scalars $\lambda, \mu \in \mathbb{R}$, we have

$$T(\lambda X + \mu Y) - T(0) = \lambda(T(X) - T(0)) + \mu(T(Y) - T(0))$$

(here 0 is the zero vector of $\mathbb{R}^n$).
(3) For any pair of vectors $X, Y \in \mathbb{R}^n$ and for any $\lambda \in \mathbb{R}$, we have

$$T((1-\lambda)X + \lambda Y = (1-\lambda)T(X) + \lambda T(Y).$$

We need to prove the implication (1) $\Rightarrow$ (2). So let us assume that (1) holds and let us fix any pair of vectors $X, Y \in \mathbb{R}^n$ and any pair of scalars $\lambda, \mu \in \mathbb{R}$. Then we have:

$$T(0) = A \cdot 0 + B = B.$$

So,

$$\begin{aligned} T(\lambda X + \mu Y) - T(0) = A \cdot (\lambda X + \mu Y) + B - B = \\ = A \cdot (\lambda X) + A \cdot (\mu Y) = \lambda(A \cdot X) + \mu(A \cdot Y) = \\ = \lambda(A \cdot X + B - B) + \mu(A \cdot Y + B - B) = \\ = \lambda(T(X) - T(0)) + \mu(T(Y) - T(0)). \end{aligned}$$

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