EXERCISE SESSION 4A FOR THE COURSE “GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE”

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Homework sheet 7-8
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Exercises done during the lecture of April 9, 2014

Exercise 1

Show that any quadratic form q on a vector space V satisfies

$$q(\lambda\mathbf{u}+\mu\mathbf{v}) = \lambda^2 q(\mathbf{u}) + 2\lambda\mu\phi_q(\mathbf{u},\mathbf{v}) + \mu^2 q(\mathbf{v})$$

for all $\mathbf{u},\mathbf{v}\in V$, $\lambda,\mu\in\mathbb{R}$.

By Lecture Notes Part III, page 2, we have a bijection between quadratic forms and bilinear forms on a vector space V, given by $\phi\mapsto q_\phi$ and $q\mapsto\phi_q$. In particular, we have $q_{\phi_q}=q$, i.e. we have

$$q(\mathbf{z}) = \phi_q(\mathbf{z},\mathbf{z}) \quad \text{for all } \mathbf{z}\in V.$$

Since ϕ_q is a symmetric bilinear form, we have

$$\begin{aligned} q(\lambda\mathbf{u}+\mu\mathbf{v}) &= \phi_q(\lambda\mathbf{u}+\mu\mathbf{v},\lambda\mathbf{u}+\mu\mathbf{v}) = \\ &= \phi_q(\lambda\mathbf{u},\lambda\mathbf{u}) + \phi_q(\lambda\mathbf{u},\mu\mathbf{v}) + \phi_q(\mu\mathbf{v},\lambda\mathbf{u}) + \phi_q(\mu\mathbf{v},\mu\mathbf{v}) = \\ &= \lambda^2\phi_q(\mathbf{u},\mathbf{u}) + \lambda\mu\phi_q(\mathbf{u},\mathbf{v}) + \lambda\mu\phi_q(\mathbf{v},\mathbf{u}) + \phi_q(\mathbf{v},\mathbf{v}) = \\ &= \lambda^2 q(\mathbf{u}) + \lambda\mu\phi_q(\mathbf{u},\mathbf{v}) + \lambda\mu\phi_q(\mathbf{u},\mathbf{v}) + q(\mathbf{v}) = \\ &= \lambda^2 q(\mathbf{u}) + 2\lambda\mu\phi_q(\mathbf{u},\mathbf{v}) + \mu^2 q(\mathbf{v}). \end{aligned}$$

Exercise 2

Do Exercise 1.4.1 from Lecture Notes Part III.

We fix any vector space V of dimension n and any *symmetric* bilinear form ϕ on V. We denote by $\mathbf{e}_1,\cdots,\mathbf{e}_n$ any basis for V. For each $i,j=1,\cdots,n$ we set $\Phi_{i,j}:=\phi(\mathbf{e}_i,\mathbf{e}_j)$ and we consider the $n\times n$ matrix $\Phi:=(\Phi_{i,j})_{i,j}$. This matrix is symmetric because ϕ is a symmetric form. We recall that ϕ is called a scalar product if and only if the matrix Φ is non-degenerate (i.e. if its determinant is non-zero - note that this does NOT depend on the basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$ chosen above). We fix also any vector $\mathbf{u} \in V$; then Exercise 1.4.1 asks to prove that the following facts are equivalent

(a) ϕ is a scalar product (i.e. Φ is non-degenerate);
(b) given any vector $\mathbf{u} \in V$, the following property holds

$$\text{if } \phi(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in V, \text{ then } \mathbf{u} = 0. \tag{0.1}$$

Let us suppose that Φ is non-degenerate and let us prove (b), so let us fix any vector $\mathbf{u} \in V$ such that $\phi(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v} \in V$. Since Φ is the matrix associated to the bilinear form ϕ, then for each pair of vectors

$$\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in V$$

we have

$$\phi(\mathbf{w}, \mathbf{z}) = \mathbf{w}^T \cdot \Phi \cdot \mathbf{z}.$$

Then we set $\mathbf{v} := \Phi \cdot \mathbf{u}$; then we have

$$0 = \phi(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{v}, \mathbf{u}) = \mathbf{v}^T \cdot \Phi \cdot \mathbf{u} = \mathbf{v}^T \cdot \mathbf{v} =< \mathbf{v}, \mathbf{v} >= |\mathbf{v}|^2.$$

This implies that $\mathbf{v} = 0$, i.e. we have $\Phi \cdot \mathbf{u} = 0$. Since Φ is non-degenerate, this implies that $\mathbf{u} = 0$ (for example, multiply by Φ^{-1}, which exists since Φ is non degenerate). So we have proved that (a) implies (b).

Conversely, let us prove that (b) implies (a). We need to prove that Φ is non-degenerate. One of the various equivalent conditions that we can prove is that

$$\text{given any } \mathbf{w} \in V, \text{ if } \Phi \cdot \mathbf{w} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ then } \mathbf{w} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{0.2}$$

So let us fix any vector $\mathbf{w}$ in V and let us suppose that $\Phi \cdot \mathbf{w} = 0$. Then we have

$$0 = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{v}^T \cdot \Phi \cdot \mathbf{w} = \Phi(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{w}, \mathbf{v}) \text{ for each } \mathbf{v} \in V.$$

By (0.1) we conclude that $\mathbf{w} = (0, \cdots, 0)$. This means that Φ is non-degenerate, i.e. that ϕ is a scalar product.

Exercise 4

Let $\boldsymbol{f}_1 = (2/3, 1/3, 2/3)$ and $\boldsymbol{f}_2 = (1/3, 2/3, -2/3)$ be two vectors in $\mathbb{R}^3$; find all vectors $\boldsymbol{f}_3 \in \mathbb{R}^3$ for which $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ is an orthonormal basis.

Step a. First of all, we have to verify that $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent. This is obvious since they are not scalar multiples of one another. Secondly, we have to verify that $\mathbf{f}_1$ and $\mathbf{f}_2$ are unit vectors. This is easy to see, indeed:

$$|\mathbf{f}_1|^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

and

$$|\mathbf{f}_2|^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.$$

So the set $\{\mathbf{f}_1, \mathbf{f}_2\}$ is an orthonormal basis of a linear subspace W of $\mathbb{R}^3$ (W is simply the span of $\mathbf{f}_1$ and $\mathbf{f}_2$). Then we apply the procedure described in Proposition 2.1.1 Lecture Notes Part III (Gram-Schmidt algorithm) in order to find a vector $\mathbf{f}_3$ such that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Since W has dimension 2, then we can choose a non-zero vector in $\mathbb{R}^3$, not belonging to W. For example, let us consider the vector

$$\mathbf{v}_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We claim that $\mathbf{v}_3$ does not belong to W. Indeed, by contradiction let us suppose that there is a pair $\lambda, \mu \in \mathbb{R}$, such that

$$\mathbf{v}_3 = \lambda \mathbf{f}_1 + \mu \mathbf{f}_2.$$

This implies that λ and μ must satisfy the following set of identities

$$\begin{cases} \lambda\frac{2}{3} + \mu\frac{1}{3} = 1 \\ \\ \lambda\frac{1}{3} + \mu\frac{2}{3} = 0 \\ \\ \lambda\frac{2}{3} - \mu\frac{2}{3} = 0. \end{cases}$$

But this set of equations has no solution. Therefore, we get a contradiction. So we have proved that $\mathbf{v}_3$ does not belong to W. So we can apply the procedure described in Proposition 2.1.1 (for $i = 2$) and we have:

$$\mathbf{f}_3 := \frac{\mathbf{v}_3 - \phi_E(\mathbf{v}_3, \mathbf{f}_1)\mathbf{f}_1 - \phi_E(\mathbf{v}_3, \mathbf{f}_2)\mathbf{f}_2}{|\mathbf{v}_3 - \phi_E(\mathbf{v}_3, \mathbf{f}_1)\mathbf{f}_1 - \phi_E(\mathbf{v}_3, \mathbf{f}_2)\mathbf{f}_2|}.$$

Now

$$\mathbf{v}_3 - \phi_E(\mathbf{v}_3, \mathbf{f}_1)\mathbf{f}_1 - \phi_E(\mathbf{v}_3, \mathbf{f}_2)\mathbf{f}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4/9 \\ 2/9 \\ 4/9 \end{pmatrix} - \begin{pmatrix} 1/9 \\ 2/9 \\ -2/9 \end{pmatrix} = \begin{pmatrix} 4/9 \\ -4/9 \\ -2/9 \end{pmatrix}$$

The norm of such a vector is $\sqrt{16/81 + 16/81 + 4/81} = \sqrt{36/81} = /3$. Therefore, we have

$$\mathbf{f}_3 = \frac{3}{2} \begin{pmatrix} 4/9 \\ -4/9 \\ -2/9 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}.$$

Even if we know a priori that the set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, it is always better to check it (in order to verify that we were not wrong with some computations). A direct check proves that everything works as expected.

Step 2. For the moment we have only found **one** vector $\mathbf{f}_3$ that solves Exercise 4. However, you are asked to find **all** vectors $\mathbf{f}_3$ such that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

So let us suppose that there is any other vector $\mathbf{f}_3'$ such that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3'\}$ is an orthonormal basis. In particular, $\mathbf{f}_3'$ is orthogonal to $\mathbf{f}_1$ and to $\mathbf{f}_2$, so it is orthogonal to all the vectors of W. In other terms, $\mathbf{f}_3'$ belongs to the vector subspace $W^\perp \subset \mathbb{R}^3$. Since we are working with the euclidean scalar product ϕ_E, then we have a decomposition $\mathbb{R}^3 = W \oplus W^\perp$; in particular, $W^\perp$ has necessarily dimension 1. By construction, we know that $\mathbf{f}_3$ is orthogonal to $\mathbf{f}_1$ and $\mathbf{f}_2$, so $\mathbf{f}_3$ belongs to $W^\perp$. Since $W^\perp$ has dimension 1 and since $\mathbf{f}_3 \neq (0,0,0)$, we have that $W^\perp$ is generated by $\mathbf{f}_3$. So necessarily $\mathbf{f}_3' = \lambda \mathbf{f}_3$ for some $\lambda \in \mathbb{R}$. We want that the set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3'\}$ is an orthonormal basis, hence we have to impose that $\mathbf{f}_3'$ has norm 1. Since $\mathbf{f}_3$ has norm 1, this implies that $\lambda = 1$ or $\lambda = -1$. So the only 2 vectors extending $\{\mathbf{f}_1, \mathbf{f}_2\}$ to an orhonormal basis of $\mathbb{R}^3$ are $\mathbf{f}_3$ and $-\mathbf{f}_3$.

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