EXERCISE SESSION 1A FOR THE COURSE "GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE"

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Homework sheet 1-2

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Exercises done during the lecture of February 26, 2014

Exercise 1

Prove Theorem 2.2.1 from Lecture Notes Part I.

(i) For each X, Y, Z in $\mathbb{R}^n$ we have:

$$\begin{aligned} < X, Y + Z >= \textstyle\sum_{i=1}^{n}(X_i(Y_i + Z_i)) = \sum_{i=1}^{n}(X_iY_i + X_iZ_i) = \\ = \textstyle\sum_{i=1}^{n} X_iY_i + \sum_{i=1}^{n} X_iZ_i =< X, Y > + < X, Z > . \end{aligned}$$

(ii) For each X, Y in $\mathbb{R}^n$ and for each $\lambda \in \mathbb{R}$ we have:

$$< X, \lambda Y >= \sum_{i=1}^{n} \lambda X_iY_i = \lambda \sum_{i=1}^{n} X_iY_i = \lambda < X, Y > .$$

(iii) For each X, Y in $\mathbb{R}^n$ we have

$$< X, Y >= \sum_{i=1}^{n} X_iY_i = \sum_{i=1}^{n} Y_iX_i =< Y, X > .$$

(iv) Let us suppose that $< X, Y >= 0$ for each $X \in \mathbb{R}^n$. Then in particular we have $0 =< Y, Y >= \sum_{i=1}^{n} Y_i^2$. This is a sum of non-zero real numbers; since it is zero, then each term of that sum is zero. Hence, $Y_i^2 = 0$ for every $i = 1, \cdots, n$, so $Y_i = 0$ for each $i = 1, \cdots, n$, i.e. Y is the zero vector of $\mathbb{R}^n$. Another way of proving the same result is the following: for each $i = 1, \cdots, n$ we choose X to be equal to the i-th vector of the canonical base of $\mathbb{R}^n$, i.e. $X = e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is in the i-th place. Then we have $0 =< X, Y >= Y_i$.

Exercise 2

Prove Theorem 2.3.1 from Lecture Notes Part I.

(i) Let us fix any vector X in $\mathbb{R}^n$; then

$$|X| = \sqrt{< X, X >} = \sqrt{\sum_{i=1}^{n} X_i^2}.$$

Since each X_i belongs to $\mathbb{R}$, then the argument of the square root is a non-negative number, so $|X|$ is a well defined quantity in $\mathbb{R}_{\geq 0}$.

(ii) Let us suppose that X in $\mathbb{R}^n$ is such that $|X| = 0$; then this implies that

$$0 = |X|^2 =< X, X >= \sum_{i=1}^{n} X_i^2;$$

therefore each X_i^2 is equal to zero, hence $X_i = 0$ for each $i = 1, \cdots, n$, i.e. X is the zero vector of $\mathbb{R}^n$.

(iii) Let us fix any X in $\mathbb{R}^n$ and any λ in $\mathbb{R}$. Then

$$|\lambda X| = \sqrt{< \lambda X, \lambda X >} = \sqrt{\lambda^2 \sum_{i=1}^{n} X_i^2} = |\lambda| \sqrt{\sum_{i=1}^{n} X_i^2} = |\lambda||X|.$$

Exercise 3

Do Exercise 2.2.2 from Lecture Notes Part I.

Let us denote by $e_1, \cdots, e_n$ the standard basis for $\mathbb{R}^n$, i.e. $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is in the i-th position. In other terms $e_i = (e_{i,1}, \cdots, e_{i,n})$, where $e_{i,k} = 0$ if $i \neq k$ and $e_{i,i} = 1$. Then we have:

$$< e_i, e_i >= \sum_{k=1}^{n} e_{i,k} e_{i,k} = 0 + \cdots + 0 + 1 + 0 + \cdots + 0 = 1.$$

If $i \neq j$, then for each $k = 1, \cdots, n$ the quantity $e_{i,k} e_{j,k}$ is equal to 0. So we have:

$$< e_i, e_j >= \sum_{k=1}^{n} e_{i,k} e_{j,k} = 0 + \cdots + 0 = 0.$$

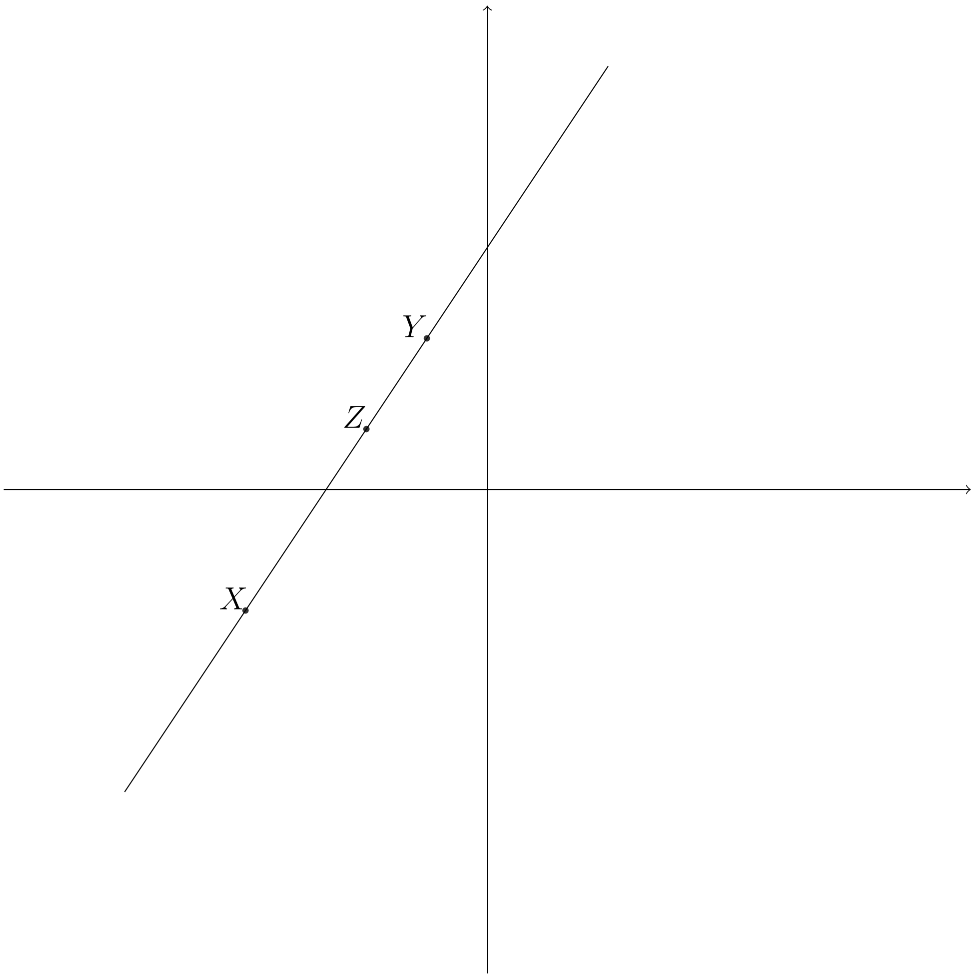
Exercise 4

Prove Proposition 3.2.1 from Lecture Notes Part I.

We fix any 3 *distinct* points X, Y, Z in $\mathbb{R}^n$. We need to prove that they are collinear (i.e. they are on the same affine line of $\mathbb{R}^n$) *if and only if* (after a permutation of X, Y, Z if necessary) we have

$$|X - Y| + |Y - Z| = |X - Z|. \tag{0.1}$$

Before starting to prove this fact, let us see why in general we need to write in the statement "after a permutation of X, Y, Z if necessary". If we suppose that X, Y, Z are in the following relative positions on an affine line (drawn in $\mathbb{R}^2$ for simplicity):



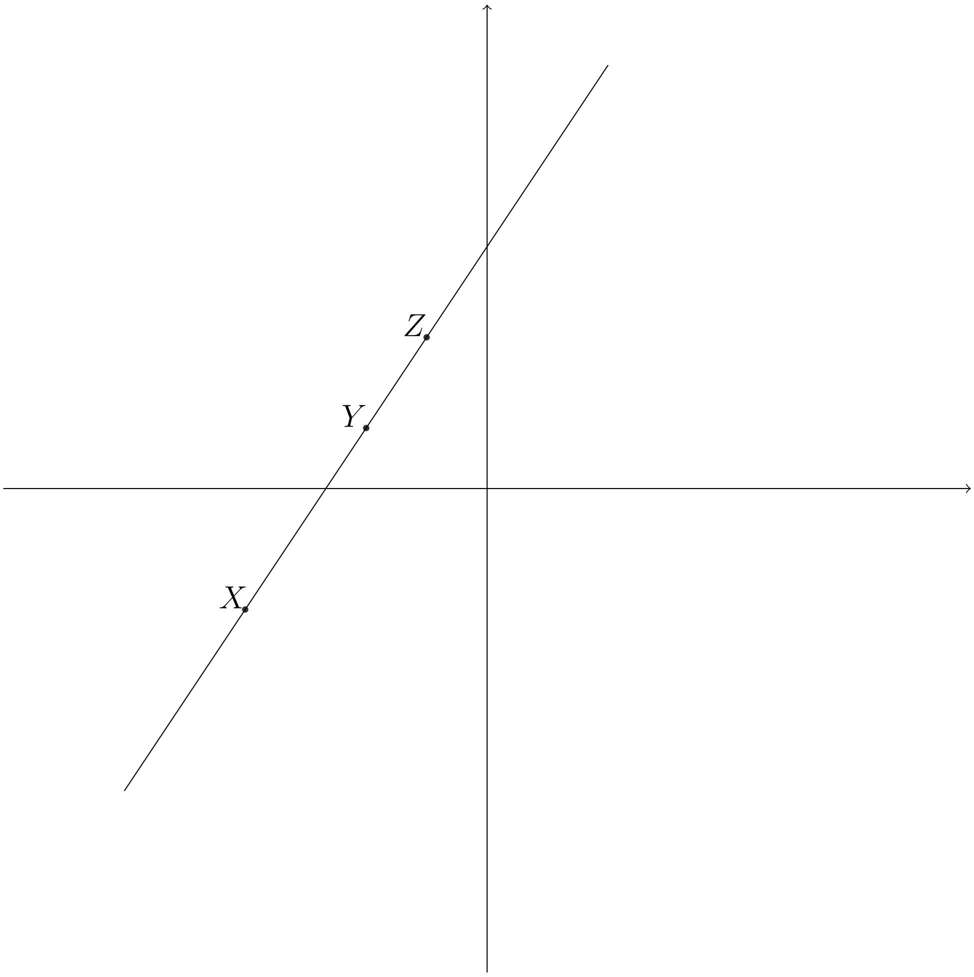
then we have

$$|X - Z| + |Z - Y| = |X - Y|$$

instead of formula (0.1). However, up to relabeling Y as Z and Z as Y, we get formula (0.1).

Now let us prove Proposition 3.2.1.

- Let us suppose that X, Y and Z are collinear. Up to permuting the labels of the points X, Y and Z, we can assume that the 3 points are in the following relative positions:



(0.2)

By § 3.2 in Lecture Notes Part I, we have that

$$Z = X + t(Y - X) \tag{0.3}$$

for some $t \in \mathbb{R}$ (this is the same as saying that Z belongs to the line passing from X and having as "direction" the vector $Y - X$, i.e. the line passing from X and Y). Since Z is distinct from X by hypothesis, then $t \neq 0$; since Z is distinct from Y, then $t \neq 1$. Moreover, if we are in the case of diagram (0.2), then $t > 1$. In particular, we have $|1 - t| = t - 1$ and $|-t| = t$. Now by (0.3) we have:

$$\begin{gathered} |X - Y| + |Y - Z| = |Y - X| + |Y - (X + t(Y - X))| = \\ = |Y - X| + |Y - X - t(Y - X)| = \\ = |Y - X| + |(1 - t)(Y - X)| = |Y - X| + |1 - t||Y - X| = \\ = |Y - X| + (t - 1)|Y - X| = t|Y - X|. \end{gathered} \tag{0.4}$$

Again by (0.3) we have:

$$|X-Z| = |X-X-t(Y-X)| = |-t(Y-X)| = |-t||Y-X| = t|Y-X|. \tag{0.5}$$

By comparing (0.4) and (0.5), we conclude that formula (0.1) holds.

- Conversely, let us suppose that X, Y, Z are such that formula (0.1) holds. We recall that by Remark 2.4.2 in Lecture Notes Part I we have the so-called "triangular inequality":

$$|X-Y| \leq |X-Z| + |Y-Z| \quad \forall\, X, Y, Z \in \mathbb{R}^n;$$

such an inequality is an equality if and only if $Z = X + \lambda(Y-X)$ for some real number $0 \leq \lambda \leq 1$. By interchanging the roles of Y and Z, we get that

$$|X-Z| \leq |X-Y| + |Z-Y| \quad \forall\, X, Y, Z \in \mathbb{R}^n;$$

such an inequality is an equality if and only if $Y = X + \lambda(Z-X)$ for some real number $0 \leq \lambda \leq 1$. Since (0.1) holds, then we are exactly in the conditions to apply such a result, i.e. we have that

$$Y = X + \lambda(Z-X)$$

for some real number $0 \leq \lambda \leq 1$. This means that Y belongs to the line passing through X and Z, i.e. X, Y and Z are collinear.

Exercise 5

Do Exercises 4.2.2, 4.2.3 and 4.2.4 from Lecture 2.

Exercise 4.2.2. *Consider the subset*

$$O(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) \text{ s.t. } A^T \cdot A = \mathrm{Id}\},$$

where A^T means the transpose of the matrix A and $\cdot$ stands for the multiplication between matrices. Prove that $O(n)$ is a group with respect to ordinary multiplication of matrices.

- The group law $\cdot$ is *associative* because it is simply the usual multiplication of matrices.
- Clearly Id belongs to $O(n)$, so $O(n)$ contains a *neutral element*.
- Let us fix any 2 matrices A, B in $O(n)$ and let us prove that *their product belongs to $O(n)$*. By hypothesis, we have

$$A^T \cdot A = \mathrm{Id} \quad \text{and} \quad B^T \cdot B = \mathrm{Id}\,.$$

We recall that given any 2 matrices A, B of dimension $n \times n$, $(A \cdot B)^T = B^T \cdot A^T$. Then by associativity we get:

$$\begin{aligned}(A \cdot B)^T \cdot (A \cdot B) = (B^T \cdot A^T) \cdot (A \cdot B) = B^T \cdot (A^T \cdot (A \cdot B)) = \\ = B^T \cdot ((A^T \cdot A) \cdot B) = B^T \cdot (\mathrm{Id} \cdot B) = B^T \cdot B = \mathrm{Id},\end{aligned}$$

so we have proved that $A \cdot B$ belongs to $O(n)$.

- We need to prove that for any $A \in O(n)$, there is an object $A^{-1} \in O(n)$, such that

$$A^{-1} \cdot A = \mathrm{Id} \quad \text{and} \quad A \cdot A^{-1} = \mathrm{Id} \tag{0.6}$$

(note: since the multiplication between matrices is NOT commutative, we need to prove separately both identities!). Since the group law of $O(n)$ is the multiplication between matrices, then the inverse of A in $O(n)$, *if it exists* is simply the inverse of A in $GL(n)$. By definition of $O(n)$, we have that A is invertible, so it makes sense of considering its inverse matrix A^{-1}. With such a choice of inverse, clearly (0.6) holds. BUT we still need to prove that A^{-1} belongs to $O(n)$. So we need to prove that

$$(A^{-1})^T \cdot A^{-1} = \mathrm{Id}\,.$$

In order to prove that, we proceed as follows: since A belongs to $O(n)$, then we have:

$$\mathrm{Id} = A^T \cdot A.$$

Then by multiplying on the right by A^{-1} both sides of such an identity, we get:

$$A^{-1} = (A^T \cdot A) \cdot A^{-1} = A^T \cdot (A \cdot A^{-1}) = A^T \cdot \mathrm{Id} = A^T.$$

Now we recall that for every (non-necessarily invertible) matrix A, we have $(A^T)^T = A$. Then from the previous identity we get:

$$(A^{-1})^T = (A^T)^T = A.$$

If we multiply on the right the previous identity by A^{-1}, we get:

$$(A^{-1})^T \cdot A^{-1} = \mathrm{Id}\,.$$

So we are done.

As a remark on this exercise, we note that we have proved that for each A in $O(n)$ we have $A^T = A^{-1}$.

Exercise 4.2.3. *Show that the set* $\mathrm{ISO}(n)$ *of isometries of the euclidean space* $\mathbb{E}^n$ *is a group with respect to composition of maps.*

- The composition of set maps is *associative.*
- Clearly the identity map $\mathrm{id} : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry, so we take this as *neutral element* of $\mathrm{ISO}(n)$.

- Let us fix any pair of isometries T_1, T_2 of $\mathbb{E}^n$. Then we need to prove that *their composition is an isometry.* We fix any pair of points P, Q in $\mathbb{E}^n$; then we get:

$$d(T_1 \circ T_2(P), T_1 \circ T_2(Q)) = d(T_1(P), T_1(Q)) = d(P, Q),$$

  so also $T_2 \circ T_1$ is an isometry of $\mathbb{E}^n$.
- Let us fix any isometry T of $\mathbb{E}^n$. In particular, T is a bijective map from $\mathbb{E}^n$ to itself, so it makes sense to consider its set inverse T^{-1}, that is again a bijection on $\mathbb{E}^n$. T^{-1} is the inverse of T with respect to the product chosen in $\mathrm{Iso}(n)$ (i.e. the composition of set maps). We need only to prove that T^{-1} belongs to $\mathrm{ISO}(n)$. Since T is an isometry, for every pair of points P, Q in $\mathbb{E}^n$, we have:

$$d(T^{-1}(P), T^{-1}(Q)) = d(T \circ T^{-1}(P), T \circ T^{-1}(Q)) = d(P, Q),$$

  so T^{-1} is an isometry. Hence $\mathrm{ISO}(n)$ contains the inverse of each of its elements.

As a side note on this Exercise, we remark that actually we have not used at all any property of $\mathbb{E}^n$, except the fact that it is a metric space. Therefore, exactly the same proof shows that *the set of isometries of any metric space* (X, d) *is a group.*

Exercise 4.2.4. *Show that any element $A \in O(2)$ is either of the form*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.7}$$

or of the form

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.8}$$

for some $\theta \in [0, 2\pi[$. The first matrix represents a rotation in $\mathbb{R}^2$ of an angle θ, while the second one represents a reflection in $\mathbb{R}^2$ (around the x-axis of $\mathbb{R}^2$), followed by a rotation of θ.

Let us suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some scalar a, b, c, d in $\mathbb{R}$. Since A belongs to the orthogonal group $O(2)$, then we have $\mathrm{Id} = A^T \cdot A$, i.e.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}.$$

This implies that

$$a^2 + c^2 = 1, \tag{0.9}$$

$$b^2 + d^2 = 1, \tag{0.10}$$

and

$$ab + cd = 0. \tag{0.11}$$

From (0.9) we get

$$a^2 b^2 + b^2 c^2 = b^2. \tag{0.12}$$

Moreover, from (0.11) we get:

$$a^2 b^2 = c^2 d^2. \tag{0.13}$$

Then from (0.12) and (0.13), we get:

$$c^2 d^2 + b^2 c^2 = b^2.$$

Hence

$$c^2 (b^2 + d^2) = b^2.$$

Using (0.10), this identity implies that

$$b^2 = c^2. \tag{0.14}$$

By replacing in (0.10) and comparing with (0.9), we get that:

$$d^2 = a^2. \tag{0.15}$$

Now since $a^2 + c^2 = 1$, then there is exactly one $\theta \in [0, 2\pi[$ such that $a = \cos\theta$ and $c = \sin\theta$ (this is the angle formed by the vectors $(1, 0)$ and (a, c) in $\mathbb{R}^2$).

Then we consider the following 2 cases.

(I) if $a = 0$, then from (0.15) we get $d = 0$. From (0.9) we get that $c^2 = 1$. If $c = 1$, then θ is necessarily equal to $\pi/2$. If $c = -1$, then $\theta = 3\pi/2$. By (0.14) we have $b^2 = c^2 = 1$. So we have that either $b = c$ or $b = -c$. In the first case, A is of the form

$$A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},$$

so A is of the form (0.7) (for $\theta = \pi/2$ if $c = 1$ or $\theta := 3\pi/2$ if $c = -1$). If $b = -c$, then

$$A = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix},$$

so A is of the form (0.8) (for $\theta := \pi/2$ if $c = 1$ or $\theta := 3\pi/2$ if $c = -1$).

(II) The other case that we have to consider is the case when $a \neq 0$. From (0.15) we have that either $d = a$ or $d = -a$. We consider such 2 cases separately.

(i) If $d = a$, then from (0.11) (and the fact that $a \neq 0$), we get $b = -c$. Then

$$A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

so A is of the form (0.7).

(ii) If $d = -a$, then from (0.11) (and the fact that $a \neq 0$), we get $b = c$. Then

$$A = \begin{pmatrix} a & c \\ c & -a \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

so A is of the form (0.8).

This suffices to conclude. As a side remark, we note that any matrix of the form (0.7) or of the form (0.8) is an orthogonal matrix, i.e. it belongs to $O(2)$ (the proof is straighforward: just sit down and compute using the definition of orthogonal matrix). Moreover, the matrices of the form (0.7) belong to the special orthogonal group $SO(2)$, i.e. they have all determinant 1 (see the definition of $SO(n)$ after Exercise 4.2.2 in the Lecture Notes). The matrices of the form (0.8) have all determinant -1.

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