EXERCISE SESSION 1B FOR THE COURSE “GÉOMÉTRIE EUCLIDIENNE, NON EUCLIDIENNE ET PROJECTIVE”

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Homework sheet 1-2
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Exercises NOT done during the lecture of February 26, 2014

Exercise 6

Find 2 different isometries $T, T' : \mathbb{E}^2 \to \mathbb{E}^2$ taking, in some affine coordinates ϕ, the point $(0,0)$ to $(1,2)$ and the point $(0,\sqrt{2})$ to $(2,3)$. Write each of them as $X \mapsto A \cdot X + B$ for some 2×2 matrix A and for some vector $B \in \mathbb{R}^2$.

Let us fix any affine coordiinate $\phi : \mathbb{E}^2 \to \mathbb{R}^2$ and let us set

$$\overline{T} := \phi \circ T \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

and

$$\overline{T}' := \phi \circ T' \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

We are trying to find isometries T and T' such that both $\overline{T}$ and $\overline{T}'$ send the point $(0,0)$ to $(1,2)$; moreover, we want that both $\overline{T}$ and $\overline{T}'$ send the point $(0,\sqrt{2})$ to $(2,3)$.

By Theorem 5.4 in Lecture Notes part I, we have that $\overline{T}$ have the form $X \mapsto A \cdot X + B$ for some matrix $A \in O(2)$ and some vector $B \in \mathbb{R}^2$; moreover, $\overline{T}'$ have the form $X \mapsto A' \cdot X + B'$ for some matrix $A' \in O(2)$ and some vector $B' \in \mathbb{R}^2$.

Since $\overline{T}$ must send $(0,0)$ to $(1,2)$, we have

$$A \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so $B = (1,2)$; in the same way, we get that $B' = (1,2)$.

Since $\overline{T}$ must send $(0,\sqrt{2})$ to $(2,3)$, we get that

$$A \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so we must have:

$$A \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.1}$$

By the classification of matrices in $O(2)$ (see Exercise 4.2.4 in Lecture Notes part I), we have that necessarily A is a matrix of the form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.2}$$

or of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = R(\theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.3}$$

for some $\theta \in [0, 2\pi[$. So we consider the 2 cases separately.

(i) Let us suppose that A is of the form (0.2). Then from (0.1) we get:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} -\sqrt{2}\sin\theta \\ \sqrt{2}\cos\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This implies that

$$\sin\theta = -1/\sqrt{2} \quad \text{and} \quad \cos\theta = 1/\sqrt{2},$$

therefore $\theta = 7\pi/4$.

(ii) Let us suppose that A is of the form (0.3). Then from (0.1) we get:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} \sqrt{2}\sin\theta \\ -\sqrt{2}\cos\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This implies that

$$\sin\theta = 1/\sqrt{2} \quad \text{and} \quad \cos\theta = -1/\sqrt{2},$$

therefore $\theta = 3\pi/4$.

Therefore, we have found 2 candidates for θ, given by $\theta = 7\pi/4$ and $\theta = 3\pi/4$. By replacing in A, we get 2 isometries of the form $AX + B$, namely:

$$X \mapsto \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot X + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \forall\, X \in \mathbb{R}^2 \tag{0.4}$$

and

$$X \mapsto \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot X + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \forall\, X \in \mathbb{R}^2. \tag{0.5}$$

We can proceed exactly in the same way for the isometry $\overline{T}'$. Therefore, if we want that the isometries T and T' are different, we must choose for $\overline{T}$ the expression (0.4) and for $\overline{T}$ the expression (0.5), or conversely. So the only possible candidates for T and T' are given as follows

$$T(p) := \phi^{-1}\left(\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \cdot X + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \circ \phi(p) \quad \forall\, p \in \mathbb{E}^2$$

and

$$T'(p) := \phi^{-1}\left(\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \cdot X + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \circ \phi(p) \quad \forall\, p \in \mathbb{E}^2$$

(or conversely). Now a direct check (replacing in p the points $\phi^{-1}(0,0)$ and $\phi^{-1}(0,\sqrt{2})$) proves that both T and T' are solutions of this Exercise.

Exercise 7

Let $p_0, p_1, p_2 \in \mathbb{E}^2$ and $p'_0, p'_1, p'_2 \in \mathbb{E}^2$ be two triples of distinct non-collinear points such that $d(p_i, p_j) = d(p'_i, p'_j)$ for all i, j. Prove that there exists a unique isometry $T : \mathbb{E}^2 \to \mathbb{E}^2$ taking p_i to p'_i for $i = 0, 1, 2$.

We choose any choice of affine coordinates $\phi : \mathbb{E}^2 \to \mathbb{R}^2$ and we set:

$$\phi(p_i) := \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad \forall\, i = 0,1,2 \quad \text{and} \quad \phi(p'_i) := \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \quad \forall\, i = 0,1,2.$$

By Theorem 5.4 in Lecture Notes, the induced isometry

$$\overline{T} := \phi \circ T \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is given by

$$\overline{T}(X) = A \cdot X + B \quad \forall\, X \in \mathbb{R}^2$$

for a unique matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$$

and for a unique vector

$$B = \begin{pmatrix} e \\ f \end{pmatrix} \in \mathbb{R}^2.$$

Since p_0, p_1 and p_2 are non-collinear in $\mathbb{E}^2$, then by Proposition 5.1.1 in Lecture Notes, the points $\phi(p_0), \phi(p_1)$ and $\phi(p_2)$ are non-collinear in $\mathbb{R}^2$. Then by § 3.2 in Lecture Notes, the vectors

$$\phi(p_1) - \phi(p_0) = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} \quad \text{and} \quad \phi(p_2) - \phi(p_0) = \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} \tag{0.6}$$

are linearly independent. So the matrix

$$C := \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix}$$

is invertible ($\det C \neq 0$). For the same reason, also the matrix

$$D := \begin{pmatrix} x'_1 - x'_0 & x'_2 - x'_0 \\ y'_1 - y'_0 & y'_2 - y'_0 \end{pmatrix}$$

is invertible. Now we know that $T(p_0) = p'_0$, $\phi(p_0) = (x_0, y_0)$ and $\phi(p'_0) = (x'_0, y'_0)$. Therefore,

$$\overline{T} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \overline{T} \circ \phi(p_0) = \phi \circ T \circ \phi^{-1} \circ \phi(p_0) = \phi \circ T(p_0) = \phi(p'_0) = \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix}.$$

Since $\overline{T}(X) = A \cdot X + B$ for all $X \in \mathbb{R}^2$, the previous identity implies that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix}. \tag{0.7}$$

Analogously, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix}.$$

Then by linearity we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} - \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} - \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} = \begin{pmatrix} x'_1 - x'_0 \\ y'_1 - y'_0 \end{pmatrix}. \tag{0.8}$$

Analogously, we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} = \begin{pmatrix} x_2' - x_0' \\ y_2' - y_0' \end{pmatrix}. \tag{0.9}$$

From (0.8) and (0.9) we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} = \begin{pmatrix} x_1' - x_0' & x_2' - x_0' \\ y_1' - y_0' & y_2' - y_0' \end{pmatrix}.$$

In other terms, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot C = D.$$

Now C is invertible (already proved above), and its inverse is given by

$$C^{-1} = \begin{pmatrix} y_2 - y_0 & x_0 - x_2 \\ y_0 - y_1 & x_1 - x_0 \end{pmatrix}$$

This implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = C^{-1} \cdot D = \begin{pmatrix} y_2 - y_0 & x_0 - x_2 \\ y_0 - y_1 & x_1 - x_0 \end{pmatrix} \cdot \begin{pmatrix} x_1' - x_0' & x_2' - x_0' \\ y_1' - y_0' & y_2' - y_0' \end{pmatrix}.$$

So all the scalars a, b, c, d are uniquely determined by the previous formula (we are not writing them down explicitly since we don't need them).

Now from (0.7) we have that

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{0.10}$$

In the right hand side of this identity all the scalars are known. Therefore, also e and f are completely determined. So we have proved that there is AT MOST one isometry T that sends each p_i to p_i' for $i = 0, 1, 2$.

So now we define:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} y_2 - y_0 & x_0 - x_2 \\ y_0 - y_1 & x_1 - x_0 \end{pmatrix} \cdot \begin{pmatrix} x_1' - x_0' & x_2' - x_0' \\ y_1' - y_0' & y_2' - y_0' \end{pmatrix}. \tag{0.11}$$

and

$$\begin{pmatrix} e \\ f \end{pmatrix} := \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{0.12}$$

Moreover, we set

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} e \\ f \end{pmatrix}$$

We need to prove that the 6 scalars $a, \cdots, e$ that we have just defined are such that the such that $AX + B$ is the isometry that we need. First of all, we remark that (0.11) implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} = \begin{pmatrix} x'_1 - x'_0 & x'_2 - x'_0 \\ y'_1 - y'_0 & y'_2 - y'_0 \end{pmatrix}.$$

Therefore,

$$A \cdot \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} x'_1 - x'_0 \\ y'_1 - y'_0 \end{pmatrix} \quad \text{and} \quad A \cdot \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} = \begin{pmatrix} x'_2 - x'_0 \\ y'_2 - y'_0 \end{pmatrix}. \tag{0.13}$$

Now let us prove that $T(p_i) = p'_i$ for all $i = 0, 1, 2$. For $i = 0$ we proceed as follows:

$$\overline{T} \circ \phi(p_0) = \overline{T} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \overset{(0.12)}{=}$$

$$\overset{(0.12)}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} = \phi(p'_0). \tag{0.14}$$

Now by construction $\overline{T} = \phi \circ T \circ \phi^{-1}$, so $T = \phi^{-1} \circ \overline{T} \circ \phi$. Therefore,

$$T(p_0) = \phi^{-1} \circ \overline{T} \circ \phi(p_0) \overset{(0.14)}{=} \phi^{-1} \circ \phi(p'_0) = p'_0.$$

Now let us prove that $T(p_1) = p'_1$:

$$\overline{T} \circ \phi(p_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \overset{(0.12)}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} + \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} \overset{(0.13)}{=} \begin{pmatrix} x'_1 - x'_0 \\ y'_1 - y'_0 \end{pmatrix} + \begin{pmatrix} x'_0 \\ y'_0 \end{pmatrix} = \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} = \phi(p'_1).$$

Therefore, as before we conclude that $T(p_1) = p'_1$. Analogously, we prove that $T(p_2) = p'_2$.

The last thing that we need to prove is that T defined like that is an isometry, i.e. that the matrix A defined above belongs to $O(2)$. In order to prove this, we recall that by hypothesis $d(p_i, p_j) = d(p'_i, p'_j)$ for all i, j. This means that the triangles determined by p_0, p_1, p_2 and by p'_0, p'_1, p'_2 have the same edges, so they are equivalent triangles. In particular, their angles are the same. So the angle θ formed by the points p_1, p_0, p_2 (in this order) is equal to the angle θ' formed by the points p'_1, p'_0, p'_2 (in this order). By definition (see Lecture Notes), we have

$$\theta = \arccos\left(\frac{< \phi(p_1) - \phi(p_0), \phi(p_2) - \phi(p_0) >}{|\phi(p_1) - \phi(p_0)||\phi(p_2) - \phi(p_0)|}\right)$$

and

$$\theta' = \arcos\left(\frac{< \phi(p'_1) - \phi(p'_0), \phi(p'_2) - \phi(p'_0) >}{|\phi(p'_1) - \phi(p'_0)||\phi(p'_2) - \phi(p'_0)|}\right).$$

So

$$< \phi(p_1) - \phi(p_0), \phi(p_2) - \phi(p_0) >= (\cos\theta)|\phi(p_1) - \phi(p_0)||\phi(p_2) - \phi(p_0)|$$

and

$$< \phi(p'_1) - \phi(p'_0), \phi(p'_2) - \phi(p'_0) >= (\cos\theta')|\phi(p'_1) - \phi(p'_0)||\phi(p'_2) - \phi(p'_0)|.$$

Now let us fix any vector $V \in \mathbb{R}^2$. We already proved above that the vectors in (0.6) are linearly independent, so there is a (unique) pair of real numbers λ_1, λ_2, such that

$$V = \lambda_1 \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix}.$$

Then we have

$$\begin{gathered} |V|^2 =< V, V >=< \lambda_1 \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix}, \lambda_1 \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} >= \\ = \lambda_1^2 < \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}, \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix} > + \lambda_2^2 < \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix}, \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} > + \\ + 2\lambda_1\lambda_2 < \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}, \begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \end{pmatrix} >= \\ = \lambda_1^2|\phi(p_1) - \phi(p_0)|^2 + \lambda_2^2|\phi(p_2) - \phi(p_0)|^2 + 2\lambda_1\lambda_2(\cos\theta)|\phi(p_1) - \phi(p_0)||\phi(p_2) - \phi(p_0)|. \end{gathered}$$

Since ϕ is an isometry, then $|\phi(p_1) - \phi(p_0)| = d(p_1, p_0)$ and $|\phi(p_2) - \phi(p_0)| = d(p_2, p_0)$. So the previous quantity is equal to

$$\lambda_1^2(d(p_1, p_0))^2 + \lambda_2^2(d(p_2, p_0))^2 + 2\lambda_1\lambda_2(\cos\theta)d(p_1, p_0)d(p_2, p_0).$$

Now $\theta = \theta'$, $d(p_1, p_0) = d(p'_1, p'_0)$ and $d(p_2, p_0) = d(p'_2, p'_0)$, so the previous quantity is equal to

$$\begin{gathered}\lambda_1^2(d(p_1',p_0'))^2+\lambda_2^2(d(p_2',p_0'))^2+2\lambda_1\lambda_2(\cos\theta')d(p_1'-p_0')d(p_2',p_0')=\\
=\lambda_1^2|\phi(p_1')-\phi(p_0')|^2+\lambda_2^2|\phi(p_2')-\phi(p_0')|^2+2\lambda_1\lambda_2(\cos\theta')|\phi(p_1')-\phi(p_0')||\phi(p_2')-\phi(p_0')|=\\
=\lambda_1^2<\begin{pmatrix}x_1'-x_0'\\y_1'-y_0'\end{pmatrix},\begin{pmatrix}x_1'-x_0'\\y_1'-y_0'\end{pmatrix}>+\lambda_2^2<\begin{pmatrix}x_2'-x_0'\\y_2'-y_0'\end{pmatrix},\begin{pmatrix}x_2'-x_0'\\y_2'-y_0'\end{pmatrix}>+\\
+2\lambda_1\lambda_2<\begin{pmatrix}x_1'-x_0'\\y_1'-y_0'\end{pmatrix},\begin{pmatrix}x_2'-x_0'\\y_2'-y_0'\end{pmatrix}>=\\
=<\lambda_1\begin{pmatrix}x_1'-x_0'\\y_1'-y_0'\end{pmatrix}+\lambda_2\begin{pmatrix}x_2'-x_0'\\y_2'-y_0'\end{pmatrix},\lambda_1\begin{pmatrix}x_1'-x_0'\\y_1'-y_0'\end{pmatrix}+\lambda_2\begin{pmatrix}x_2'-x_0'\\y_2'-y_0'\end{pmatrix}>=\\
=<\lambda_1A\cdot\begin{pmatrix}x_1-x_0\\y_1-y_0\end{pmatrix}+\lambda_2A\cdot\begin{pmatrix}x_2-x_0\\y_2-y_0\end{pmatrix},\lambda_1A\cdot\begin{pmatrix}x_1-x_0\\y_1-y_0\end{pmatrix}+\lambda_2A\cdot\begin{pmatrix}x_2-x_0\\y_2-y_0\end{pmatrix}>=\\
=<A\left(\lambda_1\begin{pmatrix}x_1-x_0\\y_1-y_0\end{pmatrix}+\lambda_2\begin{pmatrix}x_2-x_0\\y_2-y_0\end{pmatrix}\right),A\left(\lambda_1\begin{pmatrix}x_1-x_0\\y_1-y_0\end{pmatrix}+\lambda_2\begin{pmatrix}x_2-x_0\\y_2-y_0\end{pmatrix}\right)>=\\
=<A\cdot V,A\cdot V>=|A\cdot V|^2.\end{gathered}$$

So we have proved that for every vector V in $\mathbb{R}^2$ we have $|V|^2=|A\cdot V|^2$. Therefore, for every vector V in $\mathbb{R}^2$ we have $|V|=|A\cdot V|$. So using Proposition 4.3.1 in Lecture Notes, we conclude that A belongs to $O(2)$.

So we have proved that (having fixed a choice of affine coordinates ϕ), T can be written uniquely as

$$T(p)=\phi^{-1}(A\cdot\phi(p)+B)\quad\forall\,p\in\mathbb{E}^2,$$

where A is a matrix in $O(2)$ and $B\in\mathbb{R}^2$. In particular, by Theorem 5.4 in Lecture Notes, we get that T is an isometry.

Exercise 8

Prove that a matrix $A\in SO(2)$ has a (non-zero) eigenvector if and only if $A=\pm\,\mathrm{Id}$.

NOTE: The original statement in the Exercise Sheet is wrong because it says "$A\in O(2)$" instead of "$A\in SO(2)$". After this Exercise we discuss what happens in the case when A belongs to $O(2)$ but not to $SO(2)$ (i.e. the case when A has determinant -1).

Let us suppose that X is an eigenvector for A, with eigenvalue λ. Since $A\in SO(2)\subset O(2)$, we have:

$$\begin{aligned}\lambda^2|X|^2 = \lambda^2 < X, X >= \lambda < \lambda X, X >=< \lambda X, \lambda X >=< A \cdot X, A \cdot X >= \\ = (A \cdot X)^T \cdot (A \cdot X) = (X^T \cdot A^T) \cdot (A \cdot X) = X^T \cdot (A^T \cdot (A \cdot X)) = \\ = X^T \cdot ((A^T \cdot A) \cdot X) = X^T \cdot (\mathrm{Id} \cdot X) = X^T \cdot X =< X, X >= |X|^2.\end{aligned}$$

Since X is an eigenvector for A, then it is non-zero, so $|X|^2 \neq 0$. So from the previous identity we get $\lambda^2 = 1$, i.e. λ is either equal to 1 or to -1.

We already know (see Exercise 5 in Exercise Sheet 1A or Exercise 4.2.4 in Lecture Notes) that any matrix A in $O(2)$ is either a rotation $R(\theta)$ (around the origin of $\mathbb{R}^2$) of an angle θ (as in (0.2)) or a reflection

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.15}$$

(around the x-axis in $\mathbb{R}^2$), followed by a rotation $R(\theta)$ (as in (0.3)).

By hypothesis, $A \in SO(2)$, i.e. $\det A = 1$, so we are necessarily in the first case, i.e. $A = R(\theta)$ for some $\theta \in [0, 2\pi[$ as in (0.2) (we cannot be in the second case since in that case $\det A = -1$). Given any non-zero vector $X \in \mathbb{R}^2$, we can always write it as

$$X = \begin{pmatrix} \rho \cos \alpha \\ \rho \sin \alpha \end{pmatrix}$$

for a unique choice of $\alpha \in [0, 2\pi[$ and $\rho \in \mathbb{R}_{>0}$ (polar coordinates). Then

$$A \cdot X = R(\theta) \cdot X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta+\alpha) \\ \rho\sin(\theta+\alpha) \end{pmatrix}.$$

Therefore, if X is a (non-zero) eigenvector for A, with eigenvalue λ, then we must have:

$$\begin{pmatrix} \lambda\rho\cos\alpha \\ \lambda\rho\sin\alpha \end{pmatrix} = \lambda \cdot X = A \cdot X = \begin{pmatrix} \rho\cos(\theta+\alpha) \\ \rho\sin(\theta+\alpha) \end{pmatrix}.$$

Since $\rho \neq 0$, this implies that

$$\lambda\cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad \lambda\sin\alpha = \sin(\theta+\alpha).$$

We have already proved that λ is equal to 1 or -1. Therefore, we have 2 cases as follows.

- If $\lambda = 1$, then we conclude that

$$\cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad \sin\alpha = \sin(\theta+\alpha).$$

  i.e. $\theta + \alpha = \alpha + 2k\pi$ for some $k \in \mathbb{Z}$. Therefore, $\theta = 2k\pi$ and

$$A = \begin{pmatrix} \cos(2k\pi) & -\sin(2k\pi) \\ \sin(2k\pi) & \cos(2k\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}.$$

- If $\lambda = -1$, then we conclude that

$$-\cos\alpha = \cos(\theta+\alpha) \quad \text{and} \quad -\sin\alpha = \sin(\theta+\alpha).$$

i.e. $\theta + \alpha = \alpha + (2k+1)\pi$ for some $k \in \mathbb{Z}$. Therefore, $\theta = 2(k+1)\pi$ and

$$A = \begin{pmatrix} \cos((2k+1)\pi) & -\sin((2k+1)\pi) \\ \sin((2k+1)\pi) & \cos((2k+1)\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\operatorname{Id}.$$

Extra exercise

The statement of the previous exercise was false as it was written on the Exercise Sheet; the corrected version is stated above. Given that, the extra exercise is the following:

Describe what happens if $A \in O(2)$ but $A \notin SO(2)$. Are there any (non-zero) eigenvectors for A? What is their form?

By the previous classification result (see (0.2) and (0.3)), we know that $A = R(\theta) \cdot I$ for some $\theta \in [0, 2\pi[$ (here $R(\theta)$ is as in (0.2) and I is as in (0.15)). As in the previous Exercise, let us suppose that A has a (non-zero) eigenvector $X = (\rho\cos\alpha, \rho\sin\alpha)$ (with eigenvalue λ) for some $\rho \in \mathbb{R}_{>0}$ and some $\alpha \in [0, 2\pi[$. Then

$$I \cdot X = \begin{pmatrix} \rho\cos\alpha \\ -\rho\sin\alpha \end{pmatrix} = \begin{pmatrix} \rho\cos(-\alpha) \\ \rho\sin(-\alpha) \end{pmatrix},$$

so

$$A \cdot X = R(\theta) \cdot I \cdot X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \rho\cos(-\alpha) \\ \rho\sin(-\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta-\alpha) \\ \rho\sin(\theta-\alpha) \end{pmatrix}.$$

Therefore, if X is a (non-zero) eigenvector for A, with eigenvalue λ, then we must have:

$$\begin{pmatrix} \lambda\rho\cos\alpha \\ \lambda\rho\sin\alpha \end{pmatrix} = \lambda \cdot X = A \cdot X = \begin{pmatrix} \rho\cos(\theta-\alpha) \\ \rho\sin(\theta-\alpha) \end{pmatrix}.$$

Since $\rho \neq 0$, this implies that

$$\lambda\cos\alpha = \cos(\theta-\alpha) \quad \text{and} \quad \lambda\sin\alpha = \sin(\theta-\alpha).$$

As in the previous Exercise, we have that any eigenvalue λ for $A \in O(2)$ is necessarily equal to 1 or -1. Therefore, we have to consider 2 cases as follows.

- If $\lambda = 1$, then we conclude that

$$\cos\alpha = \cos(\theta-\alpha) \quad \text{and} \quad \sin\alpha = \sin(\theta-\alpha).$$

This implies that $\alpha = \theta - \alpha + 2k\pi$ for some $k \in \mathbb{Z}$, i.e. $2\alpha = \theta + 2k\pi$. This implies either that $\alpha = \theta/2 + k\pi$ or that $\alpha = \theta/2 + k\pi + \pi$. Both cases can be considered together by saying that $\alpha = \theta/2 + k\pi$ for some $k \in \mathbb{Z}$. If k is even, then we have:

$$X = \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + k\pi) \\ \rho\sin(\theta/2 + k\pi) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2) \\ \rho\sin(\theta/2) \end{pmatrix}. \tag{0.16}$$

If k is odd, then we have:

$$X = \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + k\pi) \\ \rho\sin(\theta/2 + k\pi) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + \pi) \\ \rho\sin(\theta/2 + \pi) \end{pmatrix} = \begin{pmatrix} -\rho\cos(\theta/2) \\ -\rho\sin(\theta/2) \end{pmatrix}. \tag{0.17}$$

This is simply the opposite of the vector (0.16). We already knew a priori that such a vector is an eigenvector for A. Indeed, if X is an eigenvector for a matrix A (with eigenvalue λ, then $A \cdot (-X) = -A \cdot X = -\lambda X = \lambda(-X)$, so $-X$ is also an eigenvector for A (with the same eigenvalue λ). More generally, each vector of the form (0.16) or (0.17) is simply a scalar multiple of the vector $X := (\cos(\theta/2), \sin(\theta/2))$.

- If $\lambda = -1$, then we conclude that

$$-\cos\alpha = \cos(\theta - \alpha) \quad \text{and} \quad -\sin\alpha = \sin(\theta - \alpha).$$

This implies that $\alpha = \theta - \alpha + (2k+1)\pi$ for some $k \in \mathbb{Z}$, i.e. $2\alpha = \theta + (2k+1)\pi$ for some $k \in \mathbb{Z}$. This implies either that $\alpha = \theta/2 + ((2k+1)\pi)/2 = \theta/2 + k\pi + \pi/2$ or that $\alpha = \theta/2 + ((2k+1)\pi)/2 + \pi = \theta/2 + (k+1)\pi + \pi/2$. Both cases can be considered together by saying that $\alpha = \theta/2 + k\pi + \pi/2$ for some $k \in \mathbb{Z}$. If k is even, then we have:

$$X = \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + k\pi + \pi/2) \\ \rho\sin(\theta/2 + k\pi + \pi/2) \end{pmatrix} =$$

$$= \begin{pmatrix} \rho\cos(\theta/2 + \pi/2) \\ \rho\sin(\theta/2 + \pi/2) \end{pmatrix} = \begin{pmatrix} -\rho\sin(\theta/2) \\ \rho\cos(\theta/2) \end{pmatrix}. \tag{0.18}$$

If k is odd, then we have:

$$X = \begin{pmatrix} \rho\cos(\alpha) \\ \rho\sin(\alpha) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + k\pi + \pi/2) \\ \rho\sin(\theta/2 + k\pi + \pi/2) \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta/2 + \pi + \pi/2) \\ \rho\sin(\theta/2 + \pi + \pi/2) \end{pmatrix} =$$

$$= \begin{pmatrix} -\rho\cos(\theta/2 + \pi/2) \\ -\rho\sin(\theta/2 + \pi/2) \end{pmatrix} = \begin{pmatrix} \rho\sin(\theta/2) \\ -\rho\cos(\theta/2) \end{pmatrix}. \tag{0.19}$$

This is simply the opposite of the vector (0.18). More generally, each vector of the form (0.19) or (0.18) is simply a scalar multiple of the vector $X := (\sin(\theta/2), -\cos(\theta/2))$.

So until now we have proved the following result: *for any matrix of the form*

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

the only possible eigenvectors are (scalar multiples of) vectors of the form

$$X(\theta) := (\cos(\theta/2), \sin(\theta/2)) \quad \text{or} \quad Y(\theta) := (\sin(\theta/2), -\cos(\theta/2)).$$

Now a direct check proves that actually both $X(\theta)$ and $Y(\theta)$ are eigenvectors for A (with eigenvalues $+1$ and -1) respectively. We remark that $X(\theta)$ and $Y(\theta)$ are orthogonal, indeed:

$$\begin{aligned} < X(\theta), Y(\theta) >=< (\cos(\theta/2), \sin(\theta/2)), (\sin(\theta/2), -\cos(\theta/2)) >= \\ = \cos(\theta/2)\sin(\theta/2) - \sin(\theta/2)\cos(\theta/2) = 0 \end{aligned}$$

(we will see in Exercise 10 below that this is simply a special case of a more general result on orthogonal matrices). We denote by $L(\theta)$ the line (passing through the origin) generated by the vector $X(\theta)$ and by $M(\theta)$ the line generated by the vector $Y(\theta)$. So each matrix A as before fixes all the points of $L(\theta)$ and sends each vector of $M(\theta)$ to its opposite. Hence A is simply a *reflection around the line* $L(\theta)$.

Exercise 9

Let $T : \mathbb{E}^2 \to \mathbb{E}^2$ *be an isometry such that* $T^n = \mathrm{Id}$. *If* n *is an odd integer, what can be said about* T*? Explain.*

Let us fix any affine coordinate system $\phi : \mathbb{E}^2 \to \mathbb{R}^2$. By Theorem 5.4 we get that the induced isometry

$$\overline{T} := \phi \circ T \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is of the form

$$\overline{T}(X) = A \cdot X + B \quad \forall X \in \mathbb{R}^2$$

for some matrix $A \in O(2)$ and some vector $B \in \mathbb{R}^2$. Now let us fix any vector X in $\mathbb{R}^2$; then we have:

$$\begin{aligned} \phi \circ T^2 \circ \phi^{-1}(X) = (\phi \circ T \circ \phi^{-1}) \circ (\phi \circ T \circ \phi^{-1}) = \overline{T}^2(X) = \\ = \overline{T} \circ \overline{T}(X) = \overline{T}(A \cdot X + B) = A(A \cdot X + B) + B = A \cdot A \cdot X + A \cdot B + B. \end{aligned}$$

So we have $\phi \circ T^2 \circ \phi^{-1}(X) = A^2 \cdot X + B_2$ (where $B_2 = A \cdot B + B$). By induction, we conclude that

$$\phi \circ T^n \circ \phi^{-1}(X) = A^n \cdot X + B_n \quad \forall\, X \in \mathbb{R}^2$$

for some B_n in $\mathbb{R}^2$. Since $T^n = \mathrm{Id}$, then this implies that

$$A^n \cdot X + B_n = \phi \circ \mathrm{Id} \circ \phi^{-1}(X) = X \quad \forall\, X \in \mathbb{R}^2,$$

so $B_n = 0 \in \mathbb{R}^2$ and $A^n = \mathrm{Id}$.

Now we recall that given any two square matrices B, C of the same dimension, we have $\det(B \cdot C) = \det(B) \cdot \det(C)$. Moreover, for any square matrix B we have $\det(B^T) = \det B$.

Given any matrix $B \in O(k)$, we have $\mathrm{Id} = B^T \cdot B$. Hence,

$$1 = \det \mathrm{Id} = \det(B^T) \cdot \det(B) = \det(B) \cdot \det(B) = (\det(B))^2, \tag{0.20}$$

so any orthogonal matrix has either determinant 1 or determinant -1. We recall also that by induction, we have that $\det(C^n) = (\det C)^n$ for any square matrix C. Hence,

$$(\det A)^n = \det(A^n) = \det \mathrm{Id} = 1. \tag{0.21}$$

Since n is odd, then this implies that $\det A = 1$.

We recall the classification for 2×2 orthogonal matrices that we saw before (see Exercise 5 in Exercise Sheet 1A or Exercise 4.2.4 in Lecture Notes): any 2×2 orthogonal matrix is either of the form

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.22}$$

or of the form

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.23}$$

for some $\theta \in [0, 2\pi[$. The first matrix represents a rotation in $\mathbb{R}^2$ of an angle θ (and has determinant 1), while the second one represents a reflection in $\mathbb{R}^2$, followed by a rotation of θ (and has determinant -1).

So we conclude that A is a rotation around the origin of $\mathbb{R}^2$, of a certain angle θ. Therefore, A^2 is the rotation around the origin of an angle 2θ; by induction A^n is the rotation around the origin of an angle $n\theta$. Since A^n is equal to Id, this means that $n\theta$ is an integer multiple of 2π, so θ is equal to $2k\pi/n$ for some $k \in \mathbb{Z}$. So we have:

$$\overline{T}(X) = \begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix} \cdot X + B \quad \forall\, X \in \mathbb{R}^2.$$

and

$$T(p) = \phi^{-1} \circ \overline{T} \circ \phi(p) = \phi^{-1} \left[\begin{pmatrix} \cos(2k\pi/n) & -\sin(2k\pi/n) \\ \sin(2k\pi/n) & \cos(2k\pi/n) \end{pmatrix} \cdot \phi(p) + B \right] \quad \forall\, p \in \mathbb{E}^2.$$

With the information contained in the statement of the Exercise, we cannot say anything more about the vector $B \in \mathbb{R}^2$ and/or the integer k.

Exercise 10

Let $A \in O(n)$ and let $X, Y \in \mathbb{R}^n$ be two eigenvectors with distinct eigenvalues $\lambda \neq \mu$. Prove that X and Y are orthogonal, i.e. $< X, Y >= 0$.

Since X is an eigenvector for A, with eigenvalue λ, then we have:

$$\begin{aligned}\lambda^2|X|^2 = \lambda^2 < X, X >= \lambda < \lambda X, X >=< \lambda X, \lambda X >=< A \cdot X, A \cdot X >= \\ = (A \cdot X)^T \cdot (A \cdot X) = (X^T \cdot A^T) \cdot (A \cdot X) = X^T \cdot (A^T \cdot (A \cdot X)) = \\ = X^T \cdot ((A^T \cdot A) \cdot X) = X^T \cdot (\mathrm{Id} \cdot X) = X^T \cdot X =< X, X >= |X|^2.\end{aligned}$$

Since X is an eigenvector for A, then it is non-zero, so $|X|^2 \neq 0$. So from the previous identity we get $\lambda^2 = 1$. Hence, we have proved that each eigenvector λ for an orthogonal matrix is either equal to 1 or to -1. By hypothesis, Y is another eigenvector for A with eigenvalue μ, and $\lambda \neq \mu$. Since μ is an eigenvalue for A, then also μ is equal to 1 or -1. So, up to permuting the roles of (X, λ) and (Y, μ), we can always assume that $\lambda = 1$ and $\mu = -1$. Now we have:

$$\begin{aligned}- < X, Y >=< X, -Y >=< \lambda X, \mu Y >=< A \cdot X, A \cdot Y >= \\ = (A \cdot X)^T \cdot (A \cdot Y) = (X^T \cdot A^T) \cdot (A \cdot Y) = X^T \cdot (A^T \cdot (A \cdot Y)) = \\ = X^T \cdot ((A^T \cdot A) \cdot Y) = X^T \cdot (\mathrm{Id} \cdot Y) = X^T \cdot Y =< X, Y > .\end{aligned}$$

So $- < X, Y >=< X, Y >$. This implies that $< X, Y >= 0$, i.e. that X and Y are orthogonal. In other terms, any 2 different eigenspaces of an orthogonal matrix are orthogonal. A special case of this phenomenon is given by the vectors $X(\theta)$ and $Y(\theta)$ obtained in the Extra Exercise above.

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