1 The SINC approach

We start with the following definition of a Fourier pair

$$g(x) = \bar{\mathcal{F}}[\hat{g}(\omega)] = \int_{\mathbb{R}} e^{-i2\pi x\omega}\hat{g}(\omega)d\omega, \qquad \bar{\mathcal{F}} : \text{inverse Fourier operator} \tag{1}$$

$$\hat{g}(\omega) = \mathcal{F}[g(x)] = \int_{\mathbb{R}} e^{+i2\pi x\omega}g(x)dx, \qquad \mathcal{F} : \text{forward Fourier operator} \tag{2}$$

and exploit the usual decomposition of a European put into Cash or Nothing (CoN) plus Asset or Nothing (AoN) options. So, assuming zero interest rate and dividend yield, i.e. $r = 0$ and $q = 0$:

$$\begin{aligned}
\mathbb{E}[(K - S_T)^+] &= \mathbb{E}[(K - S_T)\mathbb{1}_{S_T<K}] \\
&= K\mathbb{E}[\mathbb{1}_{S_T<K}] - \mathbb{E}[S_T\mathbb{1}_{S_T<K}] \\
&= K\mathbb{E}[\mathbb{1}_{s_T<k}] - S_0\mathbb{E}[e^{s_T}\mathbb{1}_{s_T<k}]
\end{aligned} \tag{3}$$

where

- S_t: underlying spot price at time t
- T: maturity of the option
- K: exercise price (strike)
- $k = \log\left(\frac{K}{S_0}\right) \qquad s_T = \log\left(\frac{S_T}{S_0}\right)$.

We write $\theta(x)$ for the Heaviside step function and recognize that

$$\theta(x) = \bar{\mathcal{F}}[\delta^-(\omega)] = \int e^{-i2\pi\omega x}\delta^-(\omega)d\omega,$$

where $\delta^-(\omega) = \frac{i}{2\pi}\frac{1}{\omega+i\varepsilon}$ [technical result: by contour integration].

Hence, if we think each of the expectations on the rhs of Equation (3) in terms of the PDF of the log-return s_T and the payoff of the option, we have that

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{s_T<k}] &= \int f(s_T)\theta(k - s_T)ds_T = (f \star \theta)(k) = \bar{\mathcal{F}}\big[\mathcal{F}[f(k)]\mathcal{F}[\theta(k)]\big] = \bar{\mathcal{F}}[\hat{f}(\omega)\delta^-(\omega)] \\
&= \frac{i}{2\pi}\int e^{-i2\pi k\omega}\hat{f}(\omega)\frac{1}{\omega + i\varepsilon}d\omega\,,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\mathbb{E}[e^{s_T}\mathbb{1}_{s_T<k}] &= \int e^{s_T}f(s_T)\theta(k - s_T)ds_T = \int h(s_T)\theta(k - s_T)ds_T \qquad h(s_T) = e^{s_T}f(s_T) \\
&= \frac{i}{2\pi}\int e^{-i2\pi k\omega}\hat{h}(\omega)\frac{1}{\omega + i\varepsilon}d\omega \\
&\overset{\star}{=} \frac{i}{2\pi}\int e^{-i2\pi k\omega}\hat{f}\left(\omega - \frac{i}{2\pi}\right)\frac{1}{\omega + i\varepsilon}d\omega.
\end{aligned} \tag{5}$$

$$\star\ \hat{h}(\omega) = \int_{\mathbb{R}} e^{i2\pi s_T\omega}h(s_T)ds_T = \int_{\mathbb{R}} e^{i2\pi s_T\omega}e^{s_T}f(s_T)ds_T = \int_{\mathbb{R}} e^{i2\pi s_T(\omega - \frac{i}{2\pi})}f(s_T)ds_T = \hat{f}\left(\omega - \frac{i}{2\pi}\right)$$

by simple means of the convolution theorem and the definition of a Fourier transform (FT).

Theorem 1 (Convolution theorem for Fourier transforms)**.** *Consider two functions f and g with Fourier transforms $\hat{f}$ and $\hat{g}$, respectively. Then*

$$\mathcal{F}[(f \star g)(\omega)] = \mathcal{F}[f(x)]\mathcal{F}[g(x)],$$

where

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(u)g(x-u)du$$

Proof. simple computations:

$$\begin{aligned}
\mathcal{F}[(f \star g)(\omega)] &= \int_{-\infty}^{+\infty} e^{i2\pi\omega x}(f \star g)(x)dx = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(u)g(x-u)du \right) e^{i2\pi\omega x}dx \\
&= \int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} g(x-u)e^{i2\pi\omega x}dx \right) du \\
&= \int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} g(v)e^{i2\pi\omega(v+u)}dv \right) du \qquad\qquad v = x-u \\
&= \int_{-\infty}^{+\infty} f(u)e^{i2\pi\omega u}du \int_{-\infty}^{+\infty} g(v)e^{i2\pi\omega v}dv = \hat{f}(\omega)\hat{g}(\omega) = \mathcal{F}[f(x)]\mathcal{F}[g(x)]
\end{aligned}$$

□

Observe that a change of measure is implicit in the expectation defining the AoN put, which fact requires that $\mathbb{E}[e^{s_T}] = 1$. This actually justifies our choice for $r = 0$ and $q = 0$.

In this language, and later in the notes, $f(s_T)$ stands for the density of s_T and $\hat{f}(\omega)$ represents the associated characteristic function. Therefore, for any TOL > 0, we can always find $X_c \in \mathbb{R}^+$ such that

$$\left| 1 - \int_{-X_c}^{+X_c} f(s_T)ds_T \right| < \text{TOL},$$

and the Shannon-Nyquist Sampling Theorem guarantees that the Fourier transform of the truncated density $f(s_T)\mathbb{1}_{-X_c \leq s_T \leq X_c}$ can be fully recovered given a discrete set of points.

Theorem 2 (Shannon-Nyquist Sampling Theorem)**.** *A function containing no frequency higher than ωHz, is completely determined by sampling at 2ωHz, i.e. every $\Delta t = \frac{1}{2\omega}$s.*

SNST introduces sufficient conditions for a (minimum) sampling rate to guarantee a discrete sequence of measures is able to resolve all of its frequency content and perfectly reproduces the original function.
Assumption The input signal has finite bandwidth in the frequency domain.
`Inverse problem` What if the original function is limited in the direct space and one wants to reconstruct its Fourier transform?

The Shannon-Nyquist Sampling Theorem reversed
Let us consider a function $c(x)$ whose domain is centered around the origin, i.e. $c(x)$: $[-X_c, X_c] \to \mathbb{R}$. Its transform is defined as

$$\hat{c}(\omega) = \int_{-X_c}^{+X_c} e^{i2\pi\omega x} c(x) dx,$$

and the theory of Fourier series produces [see Appendix A.1]

$$c(x) = \frac{1}{2X_c} \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) e^{-i2\pi\omega_n x}.$$

An immediate consequence is that

$$\begin{aligned}
\hat{c}(\omega) &= \frac{1}{2X_c} \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) \int_{-X_c}^{X_c} e^{i2\pi(\omega-\omega_n)x} dx = \frac{1}{2X_c} \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) \frac{e^{i2\pi(\omega-\omega_n)X_c} - e^{-i2\pi(\omega-\omega_n)X_c}}{i2\pi(\omega-\omega_n)} \\
&= \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) \frac{\sin[2\pi(\omega-\omega_n)X_c]}{2\pi(\omega-\omega_n)X_c} \\
&= \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) \operatorname{sinc}[2\pi(\omega-\omega_n)X_c]
\end{aligned}$$

where the sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

The previous result shows that

$$e^{-i2\pi k\omega} \widehat{f\mathbb{1}_{-X_c \le s_T \le X_c}}(\omega) = \sum_{n=-\infty}^{+\infty} e^{-i2\pi k\omega_n} \widehat{f\mathbb{1}_{-X_c \le s_T \le X_c}}(\omega_n) \operatorname{sinc}[2\pi X_c(\omega-\omega_n)]. \tag{6}$$

Now, we are in the position to elaborate on the two instruments separately, but only keep track of the CoN put for making things concise: deriving the AoN put price is perfectly equivalent in this setting and going through each of its steps wouldn't add anything new. We plug Shannon's representation (6) into the CoN Equation (4) and write

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{s_T<k}] &\simeq \mathbb{E}[\mathbb{1}_{s_T<k} \mathbb{1}_{-X_c \le s_T \le X_c}] \\
&= \frac{i}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-i2\pi k\omega_n} \widehat{f\mathbb{1}_{-X_c \le s_T \le X_c}}(\omega_n) \int \frac{\operatorname{sinc}[2\pi X_c(\omega-\omega_n)]}{\omega + i\varepsilon} d\omega \\
&= \frac{i}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-i2\pi k\omega_n} \widehat{f\mathbb{1}_{-X_c \le s_T \le X_c}}(\omega_n) \left[-i\pi \mathbb{1}_{n=0} + \frac{1-(-1)^n}{n} \mathbb{1}_{n\neq 0} \right]
\end{aligned} \tag{7}$$

where the last equality comes from contour integration - again.

Truncating the infinite sum in (7) to a finite number of terms

$$\mathbb{E}[\mathbb{1}_{s_T<k}] \simeq \frac{i}{2\pi} \sum_{n=-N/2}^{+N/2} e^{-i2\pi k\omega_n} \widehat{f\mathbb{1}_{-X_c\leq s_T\leq X_c}}(\omega_n)\Big[-i\pi\mathbb{1}_{n=0} + \frac{1-(-1)^n}{n}\mathbb{1}_{n\neq 0}\Big]$$

and replacing the characteristic function of the truncated density $\widehat{f\mathbb{1}_{-X_c\leq s_T\leq X_c}}(\omega_n)$ - that we do not know - with the one from the full p.d.f. gives

$$\mathbb{E}[\mathbb{1}_{s_T<k}] \simeq \frac{i}{2\pi} \sum_{n=-N/2}^{+N/2} e^{-i2\pi k\omega_n} \hat{f}(\omega_n)\Big[-i\pi\mathbb{1}_{n=0} + \frac{1-(-1)^n}{n}\mathbb{1}_{n\neq 0}\Big].$$

We can now rearrange terms a very convenient way:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{s_T<k}] &= \frac{1}{2} + \frac{i}{2\pi}\sum_{n=1}^{+N/2} e^{-i2\pi k\omega_n}\hat{f}(\omega_n)\frac{1-(-1)^n}{n} + \frac{i}{2\pi}\sum_{n=-N/2}^{-1} e^{-i2\pi k\omega_n}\hat{f}(\omega_n)\frac{1-(-1)^n}{n} \\
&= \frac{1}{2} + \frac{i}{2\pi}\sum_{n=1}^{+N/2} e^{-i2\pi k\omega_n}\hat{f}(\omega_n)\frac{1-(-1)^n}{n} - \frac{i}{2\pi}\sum_{n=1}^{+N/2} e^{i2\pi k\omega_n}\hat{f}^\dagger(\omega_n)\frac{1-(-1)^n}{n} \\
&= \frac{1}{2} + \frac{i}{2\pi}\sum_{n=1}^{+N/2} \frac{1-(-1)^n}{n}\Big[e^{-i2\pi k\omega_n}\hat{f}(\omega_n) - e^{i2\pi k\omega_n}\hat{f}^\dagger(\omega_n)\Big] \\
&= \frac{1}{2} + \frac{i}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[e^{-i2\pi k\omega_{2n-1}}\hat{f}(\omega_{2n-1}) - e^{i2\pi k\omega_{2n-1}}\hat{f}^\dagger(\omega_{2n-1})\Big] \\
&= \frac{1}{2} + \frac{i}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[\big(\cos(2\pi k\omega_{2n-1}) - i\sin(2\pi k\omega_{2n-1})\big)\hat{f}(\omega_{2n-1}) \\
&\qquad\qquad - \big(\cos(2\pi k\omega_{2n-1}) + i\sin(2\pi k\omega_{2n-1})\big)\hat{f}^\dagger(\omega_{2n-1})\Big] \\
&= \frac{1}{2} + \frac{i}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[\cos(2\pi k\omega_{2n-1})\big(\hat{f}(\omega_{2n-1}) - \hat{f}^\dagger(\omega_{2n-1})\big) \\
&\qquad\qquad - i\sin(2\pi k\omega_{2n-1})\big(\hat{f}(\omega_{2n-1}) + \hat{f}^\dagger(\omega_{2n-1})\big)\Big] \\
&= \frac{1}{2} + \frac{i}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[\cos(2\pi k\omega_{2n-1})2i\Im\big[\hat{f}(\omega_{2n-1})\big] - \sin(2\pi k\omega_{2n-1})2i\Re\big[\hat{f}(\omega_{2n-1})\big]\Big] \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[\sin(2\pi k\omega_{2n-1})\Re\big[\hat{f}(\omega_{2n-1})\big] - \cos(2\pi k\omega_{2n-1})\Im\big[\hat{f}(\omega_{2n-1})\big]\Big]. \qquad (8)
\end{aligned}$$

Remark 1. *Out of the $N+1$ terms that we included in the expansions, only $N/4$ survive. They correspond to the positive odd frequencies.*

Similarly, for the AoN option:

$$\mathbb{E}[e^{s_T}\mathbb{1}_{s_T<k}] \simeq \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{+N/4} \frac{1}{2n-1}\Big[\sin(2\pi k\omega_{2n-1})\Re\big[\hat{f}(\omega_{2n-1} - \frac{i}{2\pi})\big] - \cos(2\pi k\omega_{2n-1})\Im\big[\hat{f}(\omega_{2n-1} - \frac{i}{2\pi})\big]\Big]. \qquad (9)$$

Finally, combining Eq.s (8) and (9) into the European put option price

$$\begin{aligned}\mathbb{E}[(K-S_T)^+] \simeq \frac{1}{2}(K-S_0) \\ &+ \frac{2}{\pi}\sum_{n=1}^{N/4}\frac{1}{2n-1}\Big[\sin(2\pi k\omega_{2n-1})\Re\big[K\hat{f}(\omega_{2n-1}) - S_0\hat{f}(\omega_{2n-1}-\frac{i}{2\pi})\big] \\ &- \cos(2\pi k\omega_{2n-1})\Im\big[K\hat{f}(\omega_{2n-1}) - S_0\hat{f}(\omega_{2n-1}-\frac{i}{2\pi})\big]\Big]. \end{aligned} \tag{10}$$

Algorithm 1 FT pricing with the SINC approach

$k = \log(K/S_0)$
$N_{strike} = \text{length}(k)$

$n = 1:2:\frac{N}{2}$
$\omega_n = \frac{n}{2X_c}$

$sn = \text{zeros}(N_{strike}, \frac{N}{4})$
$cs = \text{zeros}(N_{strike}, \frac{N}{4})$

for $i = 1 : N_{strike}$ **do**
  $sn(i,:) = \sin(2\pi k(i)\omega_n)$
  $cs(i,:) = \cos(2\pi k(i)\omega_n)$
end for

$f_1 = \text{cf}(\text{model}, \text{params}, 2\pi\omega_n + 0, T)$
$f_2 = \text{cf}(\text{model}, \text{params}, 2\pi\omega_n - i, T)$

$ad1 = \text{zeros}(1, N_{strike})$
$ad2 = \text{zeros}(1, N_{strike})$

for $i = 1 : N_{strike}$ **do**
  $ad1(i) = \text{sum}((sn(i,:)\Re[f_1] - cs(i,:)\Im[f_1])/n)$
  $ad2(i) = \text{sum}((sn(i,:)\Re[f_2] - cs(i,:)\Im[f_2])/n)$
end for

$conP = K(0.5 + \frac{2}{\pi}ad1)$
$aonP = S_0(0.5 + \frac{2}{\pi}ad2)$

$put = conP - aonP$

Observe how simple it is to price with the SINC approach: the pseudo-code above only involves trigonometric functions to be properly combined with the real and imaginary parts of the Fourier transform of the density. Of course, the method is insensitive to changing the stochastic model underneath pricing, but computing the characteristic function (and/or evaluating it numerically) is not always easy.

It is important to notice that the c.f. of the asset log-price does not depend on the strike of the option. Computing one smile therefore reduces to a loop over sines and cosines only.

1.1 The FFT form of SINC

One merit of the SINC approach is that it is easily adapted to the stiff structure of the FFT algorithm; the computational speed of the Fast Fourier Transform is crucial for any concrete application within the calibration process and the extension comes with almost no effort in our setting.

So, say one wants to price a discrete grid of strikes $k_m = m\frac{2X_c}{N} \quad -N/2 \leq m < N/2$ and assume interior points are reached by linear interpolation from bucket to bucket.

Digital put prices at the aforementioned vector of strikes are now calculated as follows

$$
\begin{aligned}
\mathbb{E}[e^{as_T}\mathbb{1}_{s_T<k_m}] &\simeq \frac{i}{2\pi}\sum_{n=-N/2}^{+N/2} e^{-i2\pi k_m\omega_n}\hat{f}\Big(\omega_n - a\frac{i}{2\pi}\Big)\Big[-i\pi\mathbb{1}_{n=0} + \frac{1-(-1)^n}{n}\mathbb{1}_{n\neq 0}\Big] \\
&= \frac{i}{2\pi}\sum_{n=-N/2}^{+N/2} e^{-i\frac{2\pi}{N}mn}\hat{f}\Big(\omega_n - a\frac{i}{2\pi}\Big)\Big[-i\pi\mathbb{1}_{n=0} + \frac{1-(-1)^n}{n}\mathbb{1}_{n\neq 0}\Big] \\
&= \frac{1}{2} + \frac{i}{2\pi}\sum_{n=1}^{+N/2} e^{-i\frac{2\pi}{N}mn}\hat{f}\Big(\omega_n - a\frac{i}{2\pi}\Big)\frac{1-(-1)^n}{n} + \\
&\quad + \frac{i}{2\pi}\sum_{n=-N/2}^{-1} e^{-i\frac{2\pi}{N}mn}\hat{f}\Big(\omega_n - a\frac{i}{2\pi}\Big)\frac{1-(-1)^n}{n} \\
&= \frac{1}{2} + \frac{i}{2\pi}\sum_{n=1}^{+N/2} e^{-i\frac{2\pi}{N}mn}\hat{f}\Big(\omega_n - a\frac{i}{2\pi}\Big)\frac{(1-(-1)^n)}{n} + \\
&\quad + \frac{i}{2\pi}\sum_{n=N/2}^{N-1} e^{-i\frac{2\pi}{N}mn}\hat{f}\Big(\omega_{n-N} - a\frac{i}{2\pi}\Big)\frac{1-(-1)^{n-N}}{n-N} \\
&= \frac{i}{2\pi}\sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}mn}q_n
\end{aligned}
\tag{11}
$$

where

$$
q_n = \begin{cases} \frac{\pi}{i} & n = 0 \\ \hat{f}(\omega_n - a\frac{i}{2\pi})\frac{1-(-1)^n}{n} & n \in [1, \frac{N}{2}) \\ 0 & n = \frac{N}{2} \\ \hat{f}(\omega_{n-N} - a\frac{i}{2\pi})\frac{1-(-1)^{n-N}}{n-N} & n \in (\frac{N}{2}, N-1] \end{cases}
\tag{12}
$$

and a takes value 0 or 1 for CoN and AoN options, respectively.

Remark 2. *Again, positive odd frequencies only play a role and digital options are priced with $N/4$ evaluations of the characteristic function each. Even frequencies are killed by the n-th power of -1 and negative odd frequencies come by conjugation from positive odd ones.*

The described procedure outputs two vectors of prices which we properly combine for getting the corresponding put option values. Then, in case quoted strikes do not belong to the grid a linear interpolation kicks in and the associated error eventually reduced by increasing the number of terms in the expansion.

A Boring computations

We report in this Appendix a few pedantic computations that would have made the derivation of the SINC formulas too heavy.

A.1 Fourier coefficients of the $c(\cdot)$ function

A simple series expansion of the target function $c(\cdot)$ yields

$$c(x) = \sum_{n=-\infty}^{+\infty} A_n e^{-i2\pi \frac{n}{2X_c} x}$$

Then

$$\int_{-X_c}^{+X_c} c(x) e^{i2\pi \frac{m}{2X_c} x} dx = \int_{-X_c}^{+X_c} \sum_{n=-\infty}^{+\infty} A_n e^{-i2\pi \frac{n-m}{2X_c} x} dx = \sum_{n=-\infty}^{+\infty} A_n \int_{-X_c}^{+X_c} e^{-i2\pi \frac{n-m}{2X_c} x} dx$$

$$\begin{aligned}
&\int_{-X_c}^{+X_c} e^{-i2\pi \frac{n-m}{2X_c} x} dx = \int_{-X_c}^{+X_c} e^{i2\pi \frac{m-n}{2X_c} x} dx \\
&= \int_{-X_c}^{+X_c} \cos\left(2\pi \frac{m-n}{2X_c} x\right) dx + i \int_{-X_c}^{+X_c} \sin\left(2\pi \frac{m-n}{2X_c} x\right) dx \\
&= \left[\frac{\sin\left(2\pi \frac{m-n}{2X_c} x\right)}{2\pi \frac{m-n}{2X_c}} \right]_{-X_c}^{+X_c} - i \left[\frac{\cos\left(2\pi \frac{m-n}{2X_c} x\right)}{2\pi \frac{m-n}{2X_c}} \right]_{-X_c}^{+X_c} \\
&= \frac{\sin(\pi(m-n)) - \sin(-\pi(m-n))}{\frac{\pi(m-n)}{X_c}} - i \frac{\cos(\pi(m-n)) - \cos(-\pi(m-n))}{\frac{\pi(m-n)}{X_c}} \\
&= 0 \qquad\qquad\qquad m \neq n \in \mathbb{Z} \\
&= 2X_c \qquad\qquad\quad\; m = n
\end{aligned}$$

$= A_m 2X_c$

from which

$$\begin{aligned}
A_n &= \frac{1}{2X_c} \int_{-X_c}^{+X_c} c(x) e^{i2\pi \frac{n}{2X_c} x} dx \\
&= \frac{1}{2X_c} \int_{-X_c}^{+X_c} c(x) e^{i2\pi \omega_n x} dx = \frac{1}{2X_c} \hat{c}(\omega_n)
\end{aligned}$$

where $\omega_n = \frac{n}{2X_c}$.

Finally

$$c(x) = \frac{1}{2X_c} \sum_{n=-\infty}^{+\infty} \hat{c}(\omega_n) e^{-i2\pi \omega_n x}$$

as we used in the main body.