01b_interpolation_properties

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1 Properties of the Interpolating Polynomial

We fix the interpolation nodes $q_i, i = 0, \dots, n$, and an interval $[a, b]$ containing all the interpolation nodes. The process of interpolation maps the function f to a polynomial p in $\mathcal{P}^n$. This defines a mapping $\mathcal{L}^n$ from the space $C^0([a, b])$ of all continuous functions on $[a, b]$ to itself. The map $\mathcal{L}^n$ is linear and it is a **projection** on the subspace $\mathcal{P}^n$ of polynomials of degree n or less.

1.1 Interpolation error for smooth functions

If f is $n+1$ times continuously differentiable on a closed interval I and $p := \mathcal{L}^n f$ is the polynomial of degree at most n that interpolates f at the distinct points $\{q_i\}_{i=0}^n$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$e(x) := f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} w(x) \qquad w(x) := \prod_{i=0}^{n} (x - q_i).$$

The above error bound suggests choosing the interpolation points such that the product $|\prod_{i=0}^{n}(x - q_i)|$, is as small as possible.

1.1.1 Proof

Fix $x \in [a, b]$, and consider the function

$$G(t) = (f(x) - p(x))w(t) - (f(t) - p(t))w(x)$$

The function $G(t)$ has exactly $n+2$ zeros in the interval $[a, b]$ (the set of zeros of $w(t)$, plus the point x). By Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$\frac{d^{(n+1)}G(\xi)}{dt^{(n+1)}} = 0$$

The function $G^{(n+1)}$ is given by

$$G^{(n+1)}(t) = (f(x) - p(x))(n+1)! - f(t)^{(n+1)} w(x)$$

and the thesis follow, by computing it in ξ.

1.2 Error estimates in the L^∞ norm

Given the theorem above, we then conclude that

$$\|f - \mathcal{L}^n f\|_\infty \leq \|f^{(n+1)}\|_\infty \left\| \frac{w}{(n+1)!} \right\|_\infty$$

The estimate above can be improved if the function f is *analytically extendible* in a oval $O(a, b, R)$ with $R > 0$.

In this case, it is possible to show that the $(n+1)$ derivative of f is controlled by f itself as

$$\|f^{(n+1)}\|_{\infty,\overline{O(a,b,R)}} \leq \frac{(n+1)!}{R^{n+1}} \|f\|_{\infty,\overline{O(a,b,R)}}$$

and therefore we have:

Theorem

$$\|f - \mathcal{L}^n f\|_\infty \leq \|f\|_{\infty,\overline{O(a,b,R)}} \left(\frac{|b-a| = 1}{R} \right)^{n+1}$$

The counter example of Runge:

$$f(x) = \frac{1}{1+x^2}$$

is an example of function which is $C^\infty(R)$, but it is not analytically extendible in the entire complex plane. In fact, the radius of the oval where f is analytically extendible is 1 (the numerator goes to zero when $z = \pm i$).

In this case, if the interpolation interval is larger than $R = 1$, then only the first result is valid.

1.3 Relation to best approximation

If p is the best possible approximation of f in $\mathcal{P}^n$, i.e.,

$$E(f) = \|f - p\|_\infty \leq \inf_{q \in \mathcal{P}^n} \|f - q\|_\infty,$$

then the relation with the Lagrangian interpolation is given by

$$\begin{aligned}
\|f - \mathcal{L}^n f\|_\infty &= \|f - p + p - \mathcal{L}^n f\|_\infty \\
&= \|f - p + \mathcal{L}^n (f - p)\|_\infty \\
&\leq (1 + \|\mathcal{L}^n\|) \|f - p\|_\infty
\end{aligned}$$

where the norm $\|\mathcal{L}^n\|$ is defined as

$$\|\mathcal{L}^n\| := \sup_{f\in C^0([0,1]), \quad \|f\|_\infty\leq 1} \|\mathcal{L}^n f\|_\infty \leq \|\sum_{i=0}^{n} |l_i|\|_\infty := \|\Lambda(\{q\}_{i=0}^n)\|_\infty$$

and

$$\Lambda(q) := \sum_{i=0}^{n} |l_i|$$

is often called the **Lebesgue function**.

In other words, the interpolation polynomial is at most a factor $(\|\mathcal{L}^n\| \;+\; 1)$ worse than the best possible approximation. This suggests that we should look for a set of interpolation nodes that makes $\|\mathcal{L}^n\|$ small.

The **Chebyshev nodes** are the roots of the Chebyshev polynomials of the first kind. They are explicitly defined as

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right) \;,\; k = 1, \ldots, n$$

and they minimize the operatorial norm $\|\mathcal{L}^n\|$ on the interval $[-1,1]$. In particular, we have for Chebyshev nodes on $[-1,1]$:

$$\|\mathcal{L}^n\| \;\; \leq \frac{2}{\pi}\log(n+1)+1.$$

Compare this to equidistant nodes:

$$\|\mathcal{L}^n\| \;\; \leq \frac{2^{n+1}}{cn\log(n)}$$

We conclude that Chebyshev nodes are a good choice for polynomial interpolation, as the growth in n is exponential for equidistant nodes. Notice however that $\|\mathcal{L}^n\|$ is still diverging when $n\to\infty$.

1.4 Convergence properties

It is natural to ask, for which classes of functions and for which interpolation nodes the sequence of interpolating polynomials converges to the interpolated function as $n\to\infty$? Convergence may be understood in different ways, e.g. pointwise, uniform or in some integral norm.

The situation is rather bad for equidistant nodes, in that uniform convergence is not even guaranteed for infinitely differentiable functions. One classical example, due to Carl Runge, is the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-5,5]$. The interpolation error $\|f-\mathcal{L}^n f\|_\infty$ grows without bound as $n\to\infty$. Another example is the function $f(x) = |x|$ on the interval $[-1,1]$, for which the interpolating polynomials do not even converge pointwise except at the three points $x = +1, -1, 0$.

One might think that better convergence properties may be obtained by choosing different interpolation nodes. The following result seems to give a rather encouraging answer:

Theorem. For any function $f(x)$ continuous on an interval $[a, b]$ there exists a table of nodes for which the sequence of interpolating polynomials $p_n(x) = \mathcal{L}^n f$ converges to $f(x)$ uniformly on $[a, b]$.

Proof. It's clear that the sequence of polynomials of best approximation $p_n(x)$ converges to $f(x)$ uniformly (due to Weierstrass approximation theorem, shown later on). Now we have only to show that each $p_n(x)$ may be obtained by means of interpolation on certain nodes. But this is true due to a special property of polynomials of best approximation known from the **Chebyshev alternation theorem**. Specifically, we know that such polynomials should intersect $f(x)$ at least $n + 1$ times. Choosing the points of intersection as interpolation nodes we obtain the interpolating polynomial coinciding with the best approximation polynomial.

The defect of this method, however, is that interpolation nodes should be calculated anew for each new function $f(x)$, but the algorithm is hard to be implemented numerically. Does there exist a single table of nodes for which the sequence of interpolating polynomials converge to any continuous function $f(x)$? The answer is unfortunately negative:

Theorem. For any table of nodes there is a continuous function $f(x)$ on an interval $[a, b]$ for which the sequence of interpolating polynomials diverges on $[a, b]$.

The proof essentially uses the lower bound estimation of the Lebesgue constant, which we defined above to be the operator norm of $\mathcal{L}^n$. Now we seek a table of nodes for which

$$\lim_{n\to\infty} \mathcal{L}^n f = f, \qquad \forall f \in C^0([a, b]).$$

Due to the Banach–Steinhaus theorem, this is only possible when norms of $\mathcal{L}^n$ are uniformly bounded, which cannot be true since we know that

$$\mathcal{L}^n \geq \frac{2}{\pi} \log(n + 1) + C.$$

For example, if equidistant points are chosen as interpolation nodes, the function from Runge's phenomenon demonstrates divergence of such interpolation. Note that this function is not only continuous but even infinitely times differentiable on R. For the Chebyshev nodes, however, such an example is much harder to find due to the following result:

Theorem. For every absolutely continuous function on $[a, b]$, the sequence of interpolating polynomials constructed on Chebyshev nodes converges to $f(x)$ uniformly.

1.5 Weierstrass approximation theorem

Let B_n be a sequence of linear positive operators such that B_n converges uniformly to f, $\forall f \in \mathcal{P}^2$.

Then $B_n f$ converges uniformly to f for all $f \in C^0([a, b])$.

Proof

The idea of the proof is that for any $f \in C^0([a, b])$, for any $\varepsilon > 0$, and for each $x_0 \in [a, b]$, we can find a quadratic function $q > f$ s.t. $|q(x_0) - f(x_0)| < \varepsilon$.

Then there exists a $\bar{n}$ such that for $n > \bar{n}$, $|(B_n q)(x_0) - q(x_0)| < \varepsilon$, that is, $|(B_n q)(x_0) - f(x_0)| < 2\varepsilon$. If we can show that this is valid uniformly with respect to x_0, we have done.

Since f is continuous on $[a,b]$ compact, f is uniformly continuous:

$$\forall \varepsilon > 0, \quad \exists \delta \text{ s.t. } |f(x_1) - f(x_2)| < \varepsilon \text{ if } |x_1 - x_2| \leq \delta.$$

Then, chosen x_0, set

$$q^{\pm}(x) := f(x_0) \pm \left(\varepsilon + \frac{2\|f\|_\infty}{\delta^2}(x - x_0)^2 \right)$$

Such $q^{\pm}$ can be expressed as $ax^2 + bx + c$ where $|a|, |b|, |c| \leq M$ with M depending on $\|f\|_\infty, \varepsilon, \delta$, but **not** on x_0.

We have, by construction that $q^-(x) \leq f(x) \leq q^+(x)$ for all $x \in [a,b]$.

Choose N large enough so that

$$\|B_n x^i - x^i\|_\infty \leq \frac{\varepsilon}{M} \qquad i = 0, 1, 2$$

In this way we have

$$\|B_n q^{\pm} - q^{\pm}\|_\infty \leq 3\varepsilon$$

At this point it is easy to show that

$$\begin{aligned} (B_n f)(x_0) \leq (B_n q^+)(x_0) \leq q^+(x_0) + 3\varepsilon = f(x_0) + 4\varepsilon \\ (B_n f)(x_0) \geq (B_n q^-)(x_0) \leq q^-(x_0) - 3\varepsilon = f(x_0) - 4\varepsilon \end{aligned}$$

that is

$$-4\varepsilon \leq (B_n f)(x_0) - f(x_0) \leq 4\varepsilon \qquad \forall x_0 \qquad \Rightarrow \qquad \|B_n f - f\|_\infty \leq 4\varepsilon$$