01_interpolation

October 16, 2019

0.1 Lagrange interpolation

Given $(n+1)$ distinct points $\{q_i\}_{i=0}^n$ in the interval $[0,1]$, we define the *Lagrange interpolation* operator $\mathcal{L}^n$ the operator

$$\mathcal{L}^n : C^0([0,1]) \mapsto \mathcal{P}^n$$

which satisfies

$$(\mathcal{L}^n f)(q_i) = f(q_i), \qquad i = 0, \dots, n.$$

This operator is used to approximate the infinitely dimensional space $C^0([0,1])$ with a finite dimensional one, $\mathcal{P}^n$, which is the space of polynomials of order n.

Such a space has dimension $n+1$, and can be constructed using linear combinations of monomials of order $\leq n$:

$$\mathcal{P}^n = \text{span}\{p_i := x^i\}_{i=0}^n$$

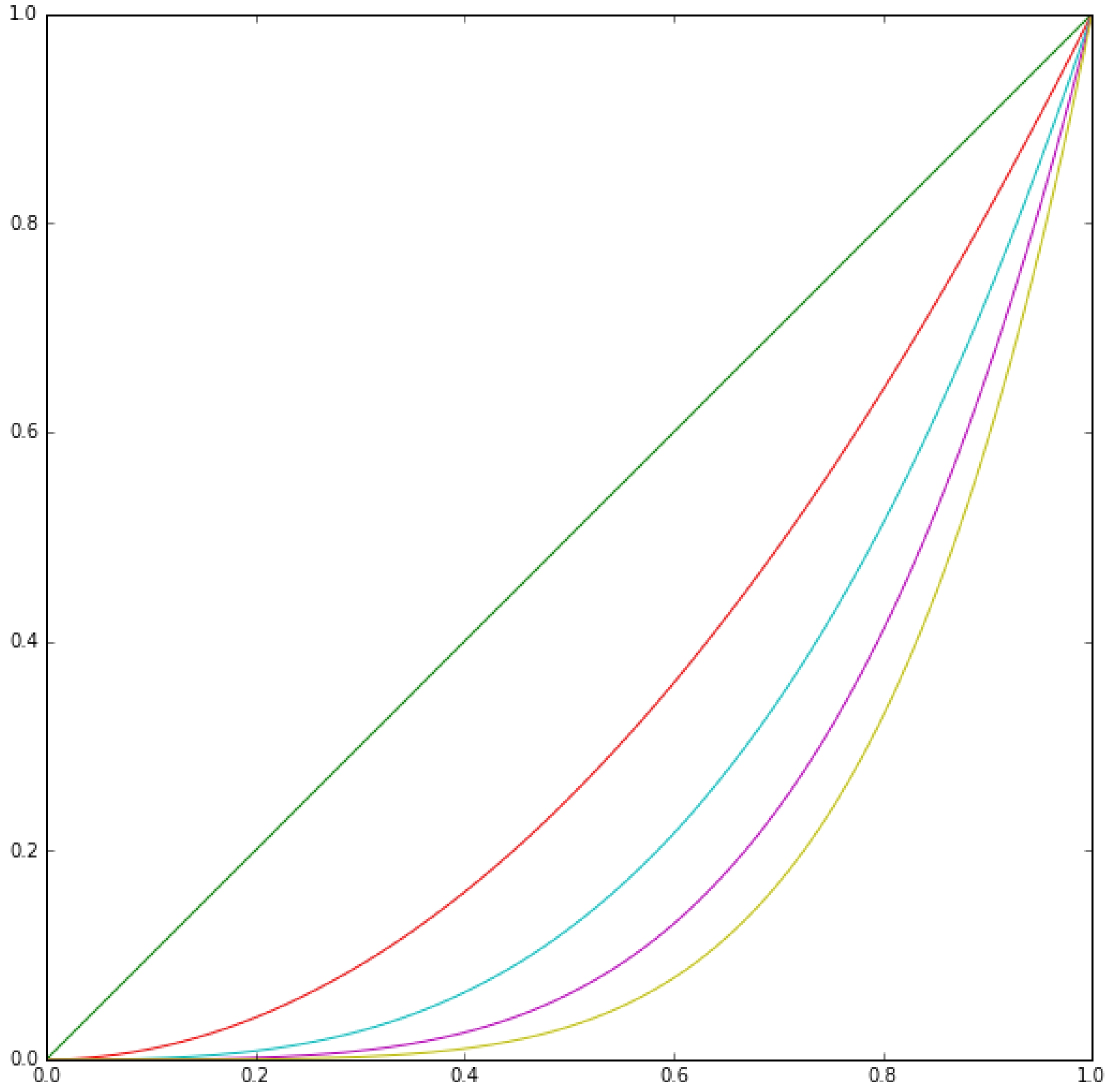
Let's start by importing the usual suspects:

[1]:
```
%matplotlib inline
from numpy import *
from pylab import *
```

In what follows, we will plot several functions in the interval $[0,1]$, so we start by defining a linear space used for plotting. As a good habit, we choose a number of points which would generate intervals that are exactly representable in terms of a binary base.

[2]:
```
ref = 1025 # So that x_i+1 - x_i is exactly representable in base 2
x = linspace(0,1,ref)

n = 5 # Polynomials of order 5, with dimension 6

# We store the basis of Pn row-wise. This is memory efficient if we want to␣
↪access
# all the values of the basis at once
Pn = zeros((n+1, len(x)))

for i in xrange(n+1):
    Pn[i] = x**i
```

```
# The _ = in front of the plot command is only there to ignore the output of␣
↪the plot
# command
figure(figsize=[10,10])
_ = plot(x, Pn.T)
```



If we want to construct the Lagrange interpolation of a given function on $n+1$ equispaced points in $[0,1]$, then we are actively looking for an element of $\mathcal{P}^n$ that coincides with the function at these given points.

Given a basis $\{p_i\}_{i=0}^n$, any element of $\mathcal{P}^n$ can be written as a linear combination of the basis, i.e.,

$$\forall u \in \mathcal{P}^n, \quad \exists! \{u^i\}_{i=0}^n \quad | \quad u(x) = \sum_{i=0}^n u^i p_i(x)$$

in what follows, we'll use Einstein summation convention, and call u both the function of $\mathcal{P}^n$, or the R^{n+1} vector representing its coefficients.

Remark on the notation

We use upper indices to indicate both "contravariant" coefficients and the *canonical basis of the dual space*, i.e., the linear functionals in $(\mathcal{P}^n)^*$ such that

$$(\mathcal{P}^n)^* := \text{span}\{p^i\}_{i=0}^n \qquad | \qquad p^i(p_j) = \delta^i_j \qquad i, j = 0, \ldots, n$$

With this notation, we have that the coefficients of a polynomial are uniquely determined by

$$u^i = p^i(u)$$

where the u on the right hand side is an element of $\mathcal{P}^n$ (not its coefficients).

If we want to solve the interpolation problem above, then we need to find the coefficients u^i of the polynomial u that interpolates f at the points q_i:

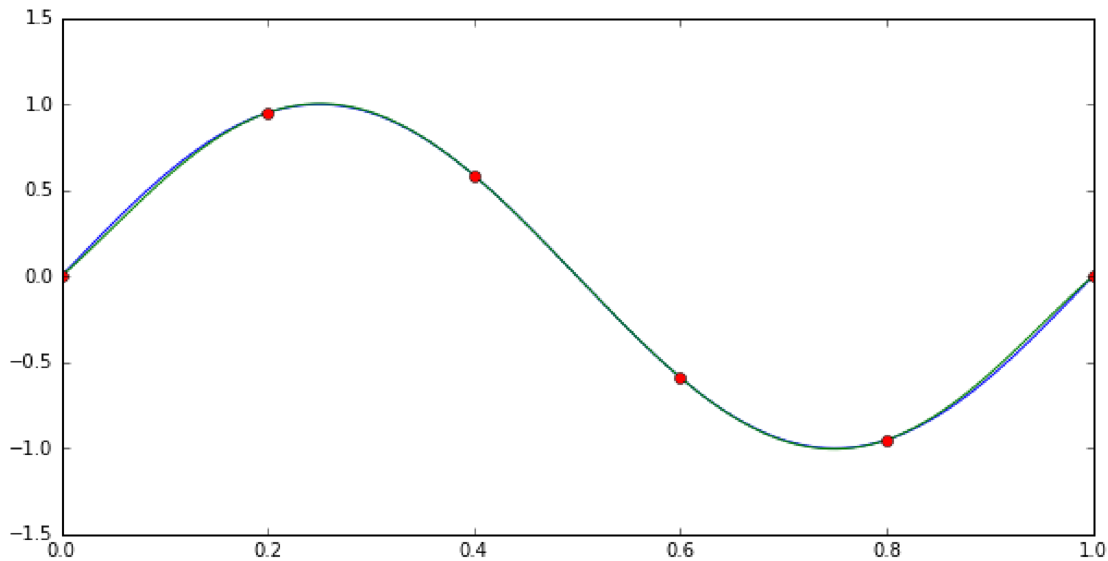
$$p_j(q_i)u^j = f(q_i)$$

(Remember Einstein summation convention)

This can be written as a linear problem $Au = F$, with system matrix $A_{ij} := p_j(q_i)$ and right hand side $F_i = f(q_i)$.

[3]:
```
# The interpolation points
q = linspace(0,1,n+1)

A = zeros((n+1, n+1))
for j in xrange(n+1):
    A[:,j] = q**j

# The interpolation function
f = lambda x: sin(2*pi*x)

# The right hand side
F = f(q)

# The coefficients
u = linalg.solve(A, F)

# Make a nice looking plot
figure(figsize=[10,5])
_ = plot(x, f(x))
```

```
_ = plot(x, Pn.T.dot(u))
_ = plot(q, f(q), 'ro')
```



Is this a good way to proceed with the interpolation? How about the condition number of A? Is it good?

Let's try with an increasing number of points (and degrees):

[4]:
```
for i in xrange(3,15):
    qtmp = linspace(0,1,i)
    Atmp = zeros((i,i))
    for j in xrange(i):
        Atmp[:,j] = qtmp**j

    print("Condition number: (n=", i, ":", linalg.cond(Atmp))
```

```
('Condition number: (n=', 3, ':', 15.099657722502098)
('Condition number: (n=', 4, ':', 98.867738507227671)
('Condition number: (n=', 5, ':', 686.43494181859796)
('Condition number: (n=', 6, ':', 4924.3710566110803)
('Condition number: (n=', 7, ':', 36061.160880212541)
('Condition number: (n=', 8, ':', 267816.70090785547)
('Condition number: (n=', 9, ':', 2009396.3800224287)
('Condition number: (n=', 10, ':', 15193229.677173976)
('Condition number: (n=', 11, ':', 115575244.51733549)
('Condition number: (n=', 12, ':', 883478685.78224337)
('Condition number: (n=', 13, ':', 6780588379.9816332)
('Condition number: (n=', 14, ':', 52214927160.937332)
```

As we see, the condition number of this matrix explodes as n increases. Since the interpolation problem reduces to solving the matrix constructed as $A_{ij} := p_j(x_i)$, one way to ensure a good condition number is to choose the basis such that A is the identity matrix, i.e., to choose the basis such that $p_j(x_i) = \delta_{ij}$. Such a basis is called the **Lagrange basis**, and it is constructed explicitly as:

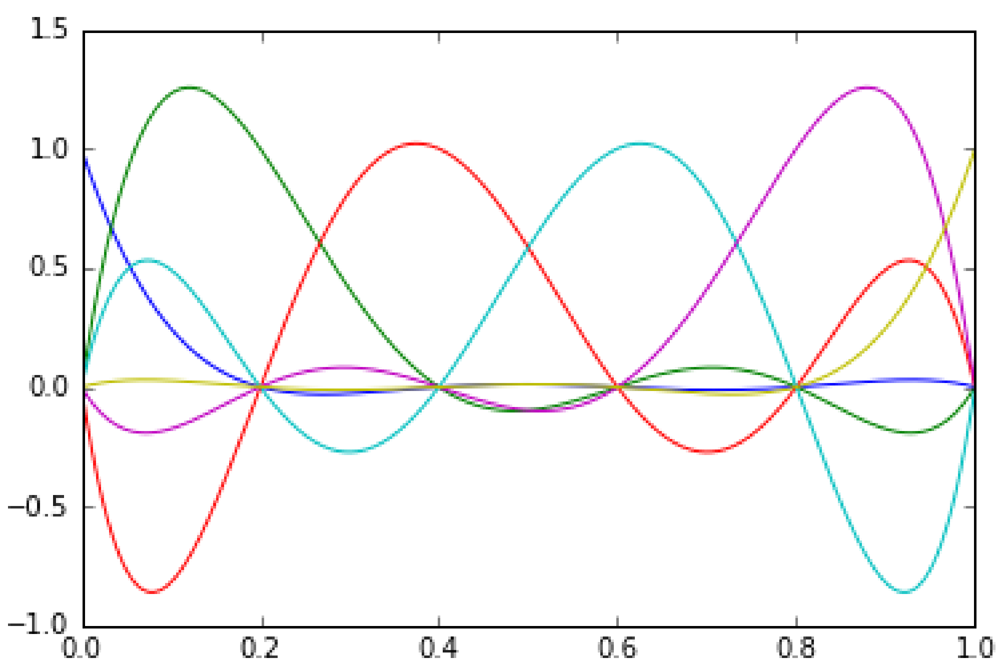
$$l_i^n(x) := \prod_{j=0, j\neq i}^{n} \frac{(x-x_j)}{(x_i - x_j)} \qquad i = 0, \ldots, n$$

With this basis, no matrix inversion is required, and we can simply write the Lagrange interpolation as

$$\mathcal{L}^n f := \sum_{i=0}^{n} f(x_i) l_i^n(x),$$

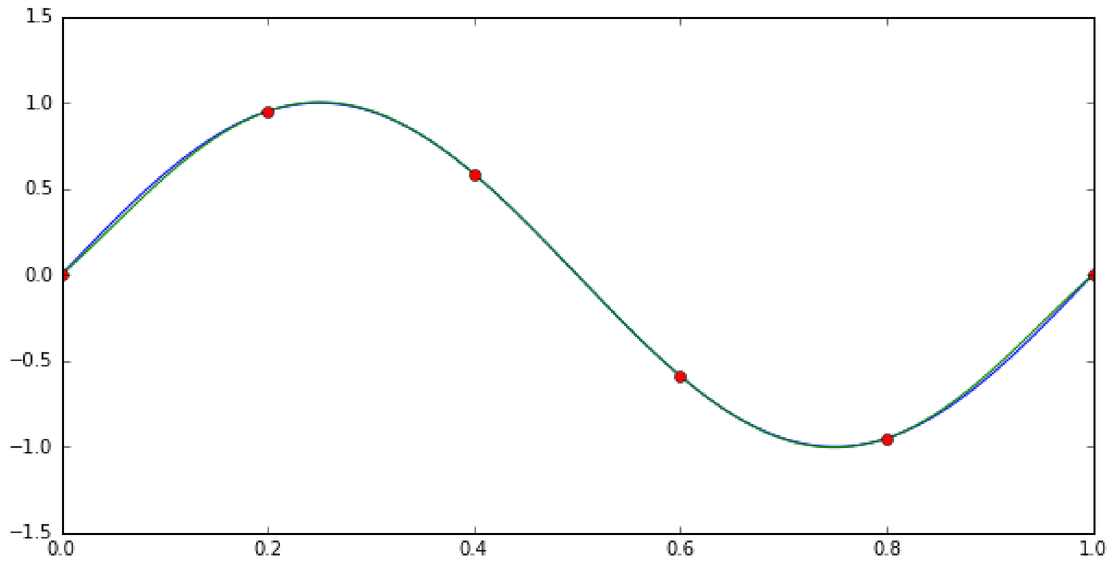
> Given a set of $(n+1)$ distinct points $\{x_i\}_{i=0}^n$, there exist a unique Lagrange interpolation of order n.

[5]:
```
Ln = zeros((n+1, len(x)))

for i in xrange(n+1):
    Ln[i] = product([ (x-q[j])/(q[i]-q[j]) for j in xrange(n+1) if j != i],␣
 ↪axis=0)

_ = plot(x,Ln.T)
```



Now the interpolation in the sampling points is simply:

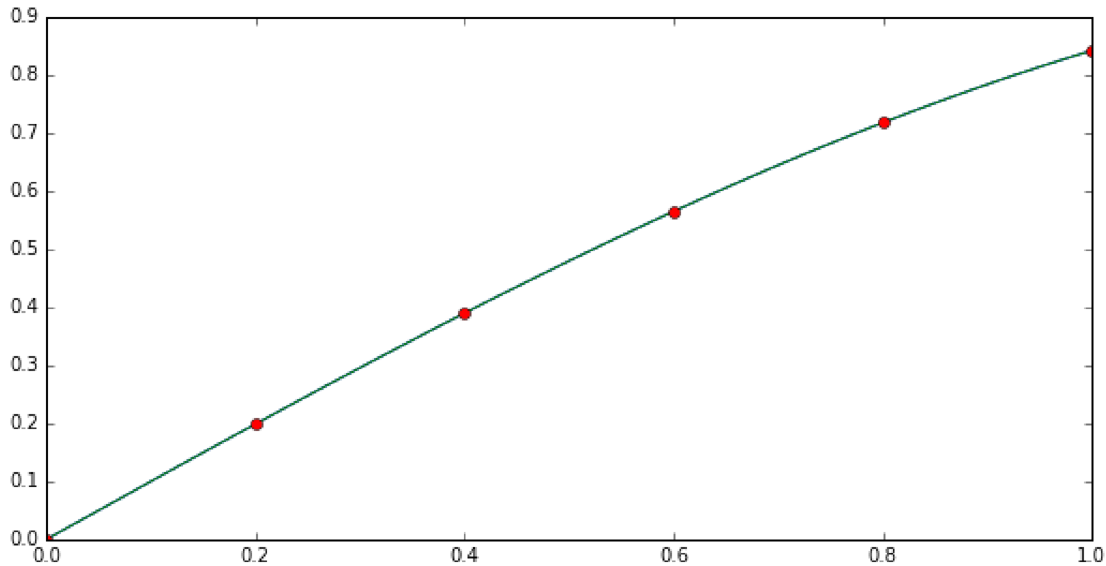
[6]:
```python
y = Ln.T.dot(f(q))

figure(figsize=[10,5])
_ = plot(x, f(x))
_ = plot(x, y)
_ = plot(q, f(q), 'ro')
```

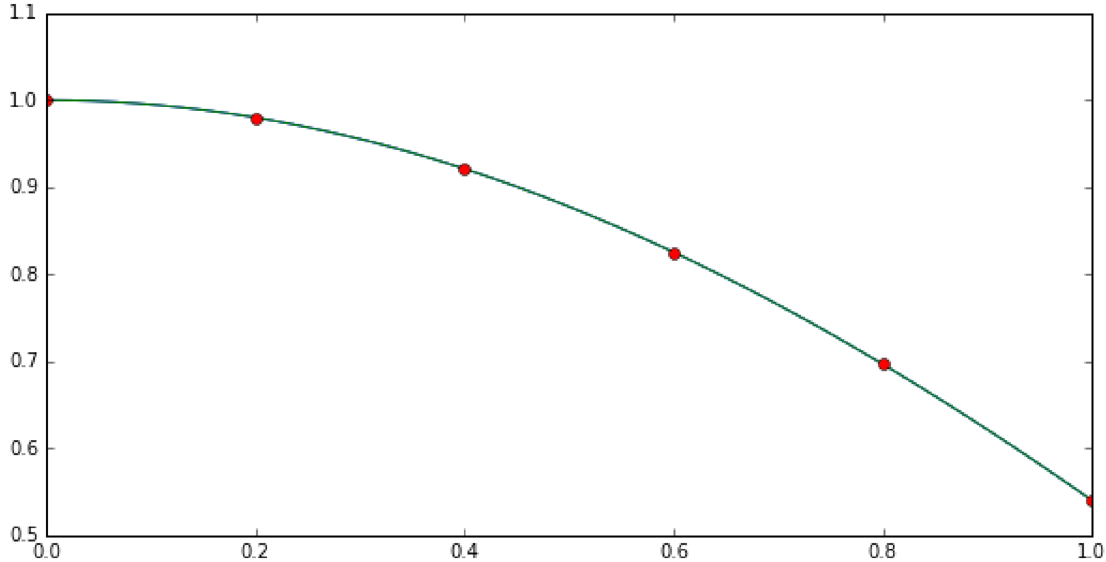


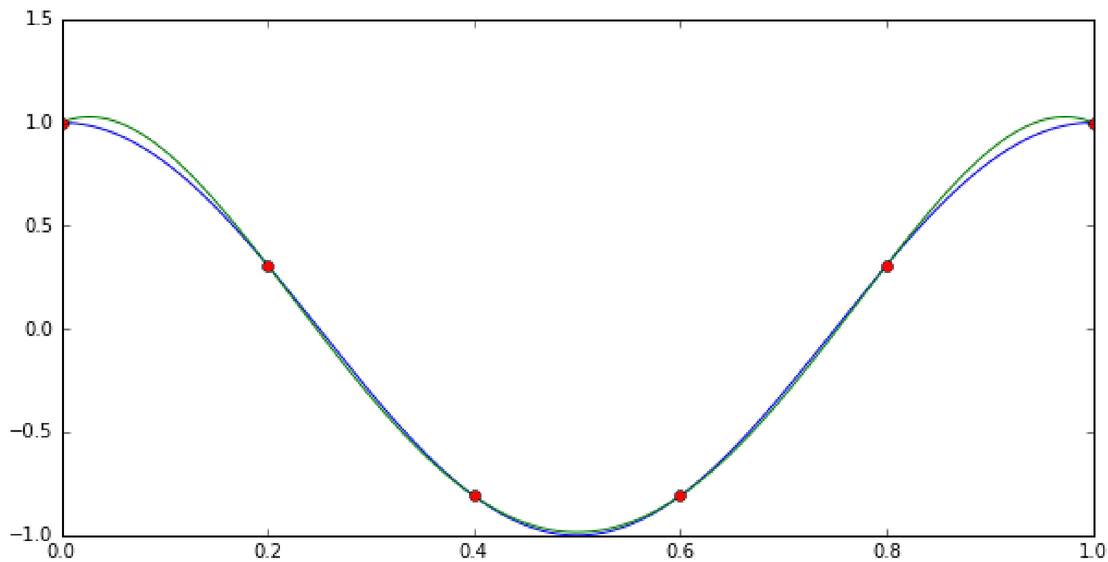
Let’s try different functions:

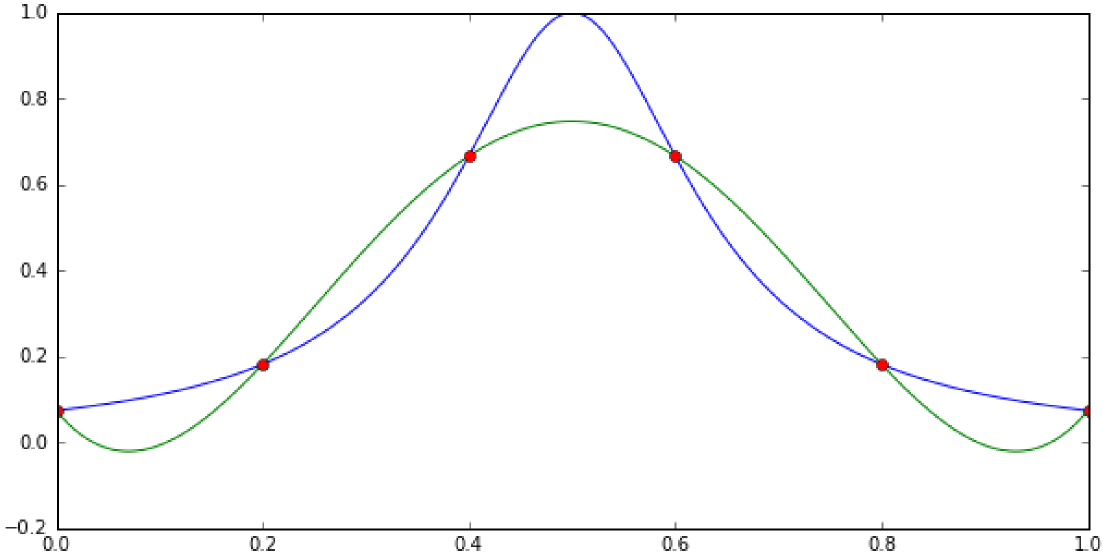
[7]:
```python
# A little "macro". This assumes Ln, q, and x are all defined
# Notice: technically this is a python function. However, it
# expects symbols and variables to be defined in the global scope
# and this is **not** good programming style. It may be very
# useful and fast at times, but try not to overdo it.
#
# I'd call it a function if internally it did not use any globally
# defined variable.

def my_plot(f):
    figure(figsize=[10,5])
    y = Ln.T.dot(f(q))
    _ = plot(x, f(x))
    _ = plot(x, y)
    _ = plot(q, f(q), 'ro')
    show()

my_plot(sin)
my_plot(cos)
```

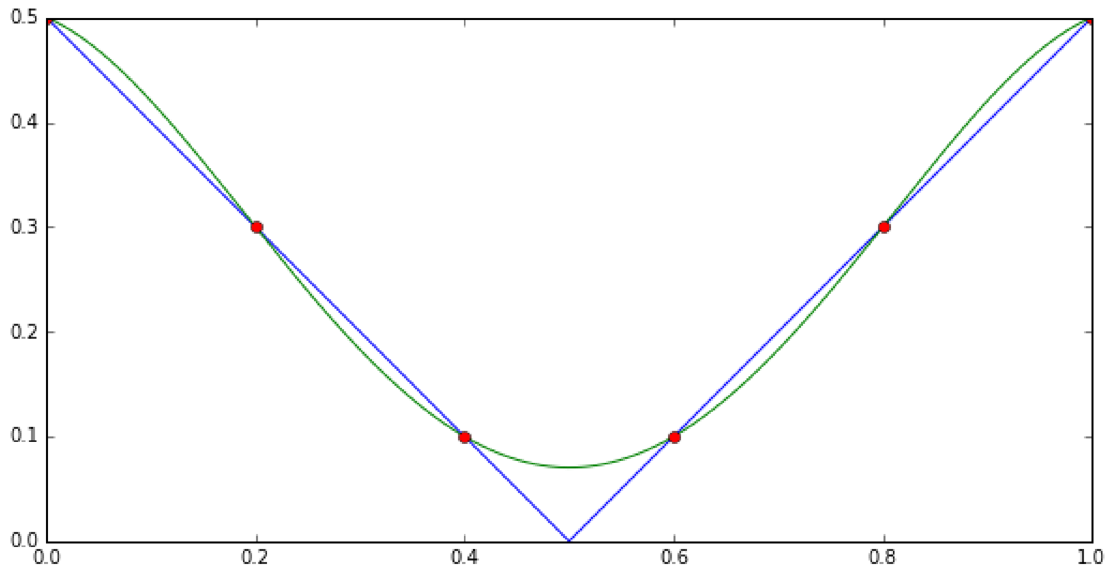
```
# When we need something more complicated than simply cos, or sin,
# we can use "on the fly" function definitions, or lambda functions:
my_plot(lambda x: cos(2*pi*x))

# Lambda functions can be assigned too, for convenience and later
# reuse...
runge = lambda x: 1.0/(1+50*(.5-x)**2)
my_plot(runge)

# Alternatively, you can define the function in the classical pythonic
# way:
def shifted_abs(x):
    return abs(x-.5)

my_plot(shifted_abs)
```




1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.0
0.2
0.4
0.6
0.8
1.0


1.5
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0


1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0


0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0

[ ]: