Macroeconomics I

Solutions To Problem Set Two

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Exercise 1. Recursive formulation.

Consider the following modification of the optimal growth model:

$$V^*(k_0) = \max_{\{k_{t+1},\, c_t\}_{t=0}^{\infty}} \tilde{u}(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_{t-1}, c_t)$$

subject to

$$c_t + k_{t+1} = f(k_t), \quad c_t,\, k_{t+1} \geq 0; \quad k_0 \text{ given.}$$

1. Write the problem in recursive form by detecting the states, the controls, and the transition functions for the states. [*Hint*: you might want to assume the existence of a non negative number $x \geq 0$ such that $\tilde{u}(c_0) = u(x, c_0)$ for any c_0].

Given the hint, we suppose that for each c_0 there exists a non-negative value of consumption at time -1 $x = c_{-1}$ at t=0 such that the optimal value function in the modified growth model can be written as:

$$V^*(k_0) = \max_{\{k_{t+1},\, c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{t-1}, c_t) \tag{1}$$

Defining $k_t = k$, $k_{t+1} = k'$ and $c_{t-1} = c_{-1}$:

1. The **states** are k and c_{-1}
2. The **control** is k'
3. The **transition functions** are:
    a. $c = f(k) - k'$
    b. $k' = k'$

And the recursive (or Bellman) form of the problem is:

$$V(k, c_{-1}) = \max_{0 \leq k' \leq f(k)} [(u(c_{-1}, f(k) - k') + \beta V(f(k) - k', k')] \tag{2}$$

Exercise 2. The optimal growth model with Dynamic Programming

Consider the optimal growth model with logarithmic utility, Cobb-Douglas production function, and 100% of capital depreciation:

$$V^*(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to

$$c_t + k_{t+1} = k_t^{\alpha}, \quad c_t,\, k_{t+1} \geq 0; \quad k_0 \text{ is given.}$$

1. Write the Bellman equation associated to the problem. Clearly indicate the state(s) and the control(s).

Given the full capital depreciation, the Cobb-Douglas $F(k)$ and $n_t = 1$, we have $F(k) = f(k)$. Defining $k_t = k$ and $k_{t+1} = k'$:

1. The **state** is k
2. The **control** is k'
3. The **transition function** is: $k' = k$

The Bellman equation associated with the problem is the following:

$$V(k) = \max_{0 \leq k' \leq k^{\alpha}} \ln(k^{\alpha} - k') + \beta V(k') \tag{3}$$

2. Guess that the value function takes the form $V(k) = A + B\ln(k)$ and derive the first-order conditions for the problem.

From the previous point, we know that (3) is the Bellman equation associated with the problem.

Plugging in $V(k') = A + B \ln k'$ we get:

$$V(k) = \max_{0 \leq k' \leq k^\alpha} \ln(k^\alpha - k') + \beta[A + B \ln k'] \tag{4}$$

The FOC are:

$$-\frac{1}{k^\alpha - k'} + \frac{\beta B}{k'} = 0 \tag{5}$$

3. Derive the policy function that gives k' as a function of k, A and B. Derive the policy function that gives the value of c as a function of k, A and B.

To find the policy function $g_B(k)$ as a function of k, A and B we need to rewrite the FOC with respect to k' so that:

$$g_B(k) = k' \implies g_B(k) = \frac{\beta B k^\alpha}{1 + \beta B} \tag{6}$$

Knowing $c = k^\alpha - k'$, the policy function $c_B(k)$ is given by:

$$c_B(k) = k^\alpha - k' = k^\alpha - \frac{\beta B k^\alpha}{1 + \beta B} \implies \frac{k^\alpha}{1 + \beta B} \tag{7}$$

4. Substitute your results back into the Bellman equation and write your value function as a function of k, A and B. Using the guessed form of the value function determine the values of A and B. What are the policy functions for k' and c?

To write the value function as a function of k, A and B, we need to plug in the results of the previous point in (4) to get:

$$V(k) = \ln(k^\alpha - \frac{\beta B k^\alpha}{1 + \beta B}) + \beta[A + B \ln \frac{\beta B k^\alpha}{1 + \beta B}] \tag{8}$$

To derive the policy functions for k' and c, we need to substitute the guessed functional form for V on the left hand side of the identity as well:

$$A + B \ln k = \ln(k^\alpha - \frac{\beta B k^\alpha}{1 + \beta B}) + \beta[A + B \ln \frac{\beta B k^\alpha}{1 + \beta B}] \tag{9}$$

Therefore, we must solve the equation as an identity. Working on the right hand side of (9):

$$\ln(k^\alpha - \frac{\beta B k^\alpha}{1+\beta B}) + \beta[A + B \ln \frac{\beta B k^\alpha}{1+\beta B}] \tag{10}$$

Which becomes:

$$\alpha \ln k + \ln(1 - \frac{\beta B}{1+\beta B}) + \beta A + \beta B \ln \frac{\beta B}{1+\beta B} + \beta B \alpha \ln k \implies A^* + B^* \ln(k) \tag{11}$$

Restating the left hand side back into equation:

$$A + B \ln k = A^* + B^* \ln k \tag{12}$$

The coefficients of (12) must be equated, therefore:

$$\begin{cases} A^* = A \implies \ln(1 - \frac{\beta B}{1+\beta B}) + \beta A + \beta B \ln \frac{\beta B}{1+\beta B} = A \\ B^* = B \implies (\alpha + \beta B \alpha) = B \end{cases}$$

The solutions are:

$$\begin{cases} A = \frac{-\ln(1-\beta\alpha) - \frac{\beta\alpha}{1-\alpha\beta}\ \ln(\beta\alpha)}{\beta - 1} \\ B = \frac{\alpha}{1-\alpha\beta} \end{cases}$$

Plugging in the derived B into (6) and (7) we get:

$$k' = g(k) = \alpha\beta k^\alpha \tag{13}$$

$$c = c(k) = (1 - \alpha\beta)k^\alpha \tag{14}$$

5. Use the obtained policy function $k' = g(k)$ to derive an expression for the steady-state level of capital, k^*. Give an economic interpretation for the equation $\beta^{-1} = \alpha(k^*)^{\alpha-1}$ that you might have seen during your derivation. [*Hint*: It is not necessary that you have seen this equation during the derivation].

k^* is such that $k' = k = k^*$. Rewriting the previous policy function (13):

$$k^* = \alpha\beta(k^*)^\alpha \implies \frac{1}{\beta} = \alpha(k^*)^{\alpha-1} \implies k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} \tag{15}$$

The second equation in (15) can be interpreted as the Euler equation of the original problem in steady state, i.e. the condition according to which the marginal rate of substitution and the marginal rate of transformation must be equated under steady state. Indeed, considering the original problem

$$V^*(k_0) = \max_{0 \leq k_{t+1} \leq k_t^\alpha} \sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - k_{t+1}) \tag{16}$$

and taking the FOC with respect to k_{t+1} gives:

$$\beta \frac{1}{k_{t+1}^\alpha - k_{t+2}} \alpha k_{t+1}^{\alpha-1} = \frac{1}{k_t^\alpha - k_{t+1}} \tag{17}$$

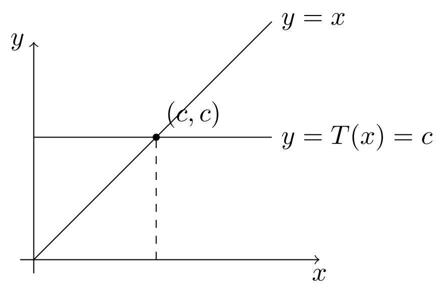
Finally, imposing the steady state condition gives the middle equation in (15).

Exercise 3. Contraction Mapping (1)

Notice that in the Contraction Mapping Theorem we allow for $\beta = 0$.

1. Draw a mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ which is a contraction of modulus $\beta = 0$ and show graphically that T must have a unique fixed point.

Let $T : \mathbb{R}_+ \to \mathbb{R}_+$ be $T(x) = c$ with $c \geq 0$ and endow $\mathbb{R}_+$ with a distance. Then for all x, y we have $d(T(x); T(y)) = 0 \leq 0 \cdot d(x; y)$, hence T is a contraction with modulus 0. A fixed point solves $x = T(x)$, i.e. $x = c$, which is unique.



2. Now, formally show the statement, that is, show that if T is a contraction of modulus $\beta = 0$ in a complete metric space then T admits a unique fixed point.

Suppose T is a contraction of modulus $\beta = 0$ in a metric space (S, d) and x,y are two fixed points of the contraction, i.e. $T(x) = x$ and $T(y) = y$. By definition of contraction of modulus $\beta = 0$, $d(T(x), T(y)) = 0$. By definition of distance, $d(T(x), T(y)) = 0 \implies T(x) = T(y) \implies x = y$ therefore the contraction admits a unique fixed point.

3. Is the completeness assumption on $(\mathbb{R}^+, d)$ (where d denotes the usual Euclidean distance) crucial in this case with $\beta = 0$? Explain.

No, as shown in the proof, the completeness assumption is not necessary to prove the uniqueness of the fixed point when the modulus is equal to zero. In a generic case where $\beta < 1$, instead, the completeness is necessary as shown by the Banach or contraction mapping theorem.

Exercise 6. Non-additive utility.

Consider the following modification of the optimal growth model. The representative consumer has the same production and investment technologies as in the most standard model of optimal growth. The representative consumer does not enjoy utility from leisure and has one unit of time endowment per period. Using the constant return to scale assumption for the production function F transforming capital and labor into consumption goods, and a depreciation rate of δ, the feasibility set in each period is standard:

$$0 \le k_{t+1} \le f(k_t)$$

where

$$f(k_t) = F(k_t, 1) + (1-\delta)k_t.$$

The preferences of the consumer over consumption streams $c := \{c_n\}_{n=0}^{\infty}$ are represented by a time-stationary (but non-additive) utility U solving the recursion:

$$U(c) = G(u(c_0), U(\underline{c'})),$$

where $c' = \{c_n\}_{n=1}^{\infty}$ is the one-period continuation of c and u and G are real-valued, concave, bounded, continuous, differentiable, and strictly increasing functions (with G having these properties in both arguments u and U).

1. Derive the Bellman equation associated to this dynamic problem.

We start by defining the value function as:

$$V(k) = \max_{0 \leq k' \leq f(k)} U(\underline{c}) = \max_{0 \leq k' \leq f(k)} G(u(c_0), U(\underline{c'})) \tag{18}$$

We know that:

$$\max_{0 \leq k' \leq f(k)} u(c_0) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') \quad \text{by the feasibility} \tag{19}$$

$$\max_{0 \leq k' \leq f(k)} U(\underline{c'}) = \max_{0 \leq k' \leq f(k)} V(k') \quad \text{by the definition of V} \tag{20}$$

Therefore:

$$V(k) = \max_{0 \leq k' \leq f(k)} G(u(f(k) - k'), V(k')) \tag{21}$$

3. Derive the Euler equation associated to this dynamic problem and specify the conditions for the steady state levels of consumption and capital. [*Hint*: It might be useful to use the following envelope condition associated to the Bellman equation you derived in (a): $V'(k) = u'(c)f'(k)G_1(u, U)$, where V is the value function of the problem.]

We might want to rewrite the Bellman equation in (21) as:

$$V(k_t) = \max_{0 \le k_{t+1} \le f(k_t)} G(u(f(k_t) - k_{t+1}), V(k_{t+1})) \tag{22}$$

Taking the FOC with respect to k_{t+1}:

$$-u'(c_t)G_1(u(c_t), V(k_{t+1})) + V'(k_{t+1})G_2(u(c_t), V(k_{t+1})) = 0$$

Substituting the envelope in the hint we obtain the Euler equation of the dynamic problem:

$$u'(c_t)\, G_1(u(c_t), V(k_{t+1})) = G_2(u(c_t), V(k_{t+1}))u'(c_{t+1})\, f'(k_{t+1})\, G_1(u(c_{t+1}), V(k_{t+2}))$$

Or, writing it as MRS=MRT,

$$\underbrace{\frac{u'(c_t)}{u'(c_{t+1})} \frac{G_1(u(c_t), V(k_{t+1}))}{G_1(u(c_{t+1}), V(k_{t+2}))} \frac{1}{G_2(u(c_t), V(k_{t+1}))}}_{\text{MRS}_{t,t+1}} = \underbrace{f'(k_{t+1})}_{\text{MRT}_{t,t+1}} . \tag{23}$$

The conditions for the steady state can be found by imposing $k_{t+2} = k_{t+1} = k_t = k^*$. Therefore equation (23) can be rewritten as:

$$\frac{1}{G_2(u(c_t), V(k^*))} = f'(k^*) \tag{24}$$

The correspondent condition for steady state consumption is determined by the feasibility:

$$c^* = f(k^*) - k^* \tag{25}$$