Question 1. Unit root testing.

1.1

Compute the empirical distribution of the OLS estimator in the case of a pure random walk (no drift) with $T = 250$ (you are free to choose the variance of the innovation).

Solution.
In this first part we examine how the ordinary least squares (OLS) estimator behaves when the data are generated by a pure random walk. The model is

$$y_t = y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \sigma^2 = 1$$

so the true autoregressive coefficient is $\phi = 1$. This process contains a unit root and is therefore non-stationary: its variance grows over time and shocks have permanent effects. Because of this non-stationarity, the standard asymptotic properties of the OLS estimator no longer apply, and its finite-sample distribution is not normal. Since the scale of the innovations does not affect the presence of a unit root, we are free to choose the variance of the shocks; for simplicity, we set $\sigma^2 = 1$ in all simulations. The goal of this experiment is to approximate the sampling distribution of the OLS estimator empirically using Monte Carlo simulation.

A total of $N = 1000$ independent series of length $T = 250$ are simulated from the pure random walk above, with ε_t drawn from a standard normal distribution. For each simulated series the regression

$$y_t = \rho\, y_{t-1} + u_t$$

is estimated without an intercept, and the OLS estimate $\hat{\rho}$ is stored. The collection of these 1000 estimates approximates the empirical sampling distribution of the OLS coefficient under the null hypothesis of a unit root. From these draws we compute the mean, standard deviation, and median of $\hat{\rho}$, and plot its empirical density.

The average estimate is expected to be slightly below one, indicating a small downward finite-sample bias that is typical when the true process has a unit root. The empirical distribution of $\hat{\rho}$ is typically left-skewed and clearly non-normal, confirming that standard inference methods are invalid in this context.

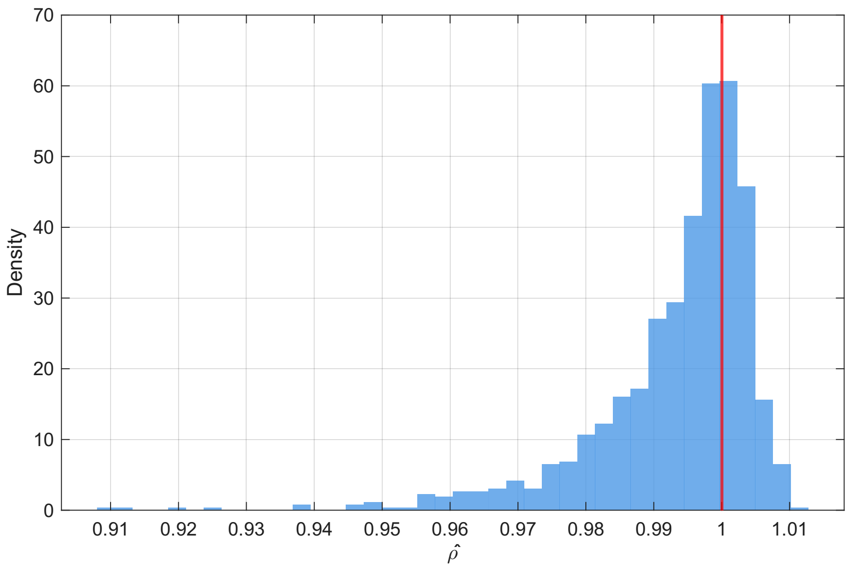


Figure 1-1: Empirical distribution of $\hat{\rho}$ from 1000 Monte Carlo simulations of a pure random walk ($T = 250$). The red vertical line marks the true value $\rho = 1$.

Table 1-1: Summary statistics of the OLS estimates $\hat{\rho}$

Statistic	Mean	Std. Dev.	Median
Monte Carlo estimates	0.9933	0.0121	0.9970

Overall, this exercise demonstrates that when the data-generating process contains a unit root, the OLS estimator of the autoregressive parameter is biased and its distribution deviates strongly from normality. This motivates the use of special testing procedures, such as the Dickey–Fuller test, which provide the correct non-standard critical values for inference in the presence of unit roots.

```
%% === Monte Carlo parameters ===
T        = 250;          % sample size
nSim     = 1000;         % number of Monte Carlo replications
sigma2   = 1;            % variance of the innovation [I set the variance ==1]
sigma    = sqrt(sigma2);

% Store all OLS estimates of rho
rho_hat = zeros(nSim,1);

%% === Monte Carlo loop ===
for s = 1:nSim

    % ----- 1. Generate a pure random walk -----
    y   = zeros(T,1);
    eps = sigma * randn(T,1);    % innovations ~ N(0, sigma2)

    for t = 2:T
        y(t) = y(t-1) + eps(t);    % random walk without drift
    end

    % ----- 2. Estimate OLS: y_t = rho * y_{t-1} + u_t -----
    y_lag = y(1:end-1);
    y_now = y(2:end);

    % OLS without intercept
    rho_hat(s) = (y_lag' * y_now) / (y_lag' * y_lag);

end

%% === Output summary ===
fprintf('Mean of OLS estimates (rho): %.4f\n', mean(rho_hat));
fprintf('Std. deviation             : %.4f\n', std(rho_hat));
fprintf('Median of OLS estimates    : %.4f\n', median(rho_hat))
```

□

1.2

Construct a t-test for the null hypothesis $H_0 : \rho = \phi - 1 = 0$, in a test regression: $\Delta y_t = \alpha + \rho y_{t-1} + \varepsilon_t$; against a one-sided alternative $H_0 : \rho < 0$. Using a standard Normal distribution, how often do you reject the null hypothesis at the 95% confidence level? Is the actual distribution of the t-test symmetric? Discuss.

Solution.
We start from the general autoregressive process

$$y_t = \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,1),$$

and rewrite it as

$$\Delta y_t = y_t - y_{t-1} = (\phi - 1)y_{t-1} + \varepsilon_t = \rho y_{t-1} + \varepsilon_t, \quad \text{where} \quad \rho = \phi - 1$$

Testing for a unit root, i.e. testing whether $\phi = 1$, is therefore equivalent to testing

$$H_0 : \rho = 0 \quad \text{(unit root)} \qquad \text{against} \qquad H_1 : \rho < 0.$$

Under H_0, the process is a pure random walk and hence non-stationary. Under H_1, it is a stationary AR(1).

We perform a Monte Carlo experiment with $N = 1000$ replications, each of length $T = 250$, and innovations $\varepsilon_t \sim \mathcal{N}(0, 1)$. For each simulated series we estimate

$$\Delta y_t = \alpha + \rho y_{t-1} + u_t$$

by OLS, obtain $\hat{\rho}$ and its standard error $\text{se}(\hat{\rho})$, and compute

$$t_\rho = \frac{\hat{\rho}}{\text{se}(\hat{\rho})}.$$

Assuming (incorrectly) that t_ρ follows a standard Normal distribution, the 5% left-tailed critical value is -1.645, so the theoretical rejection region is

$$\mathcal{R} = \{t_\rho < -1.645\}.$$

In practice we reject H_0 whenever the simulated statistic falls inside $\mathcal{R}$.

Running the simulation gives a rejection frequency of 0.476, meaning that the null is rejected in 47.6% of the samples, even though it is true by construction. Hence, the standard-Normal t-test is severely oversized. Figure 1-2 displays the empirical distribution of the 1000 simulated t_ρ statistics together with the (wrong) critical value -1.645.

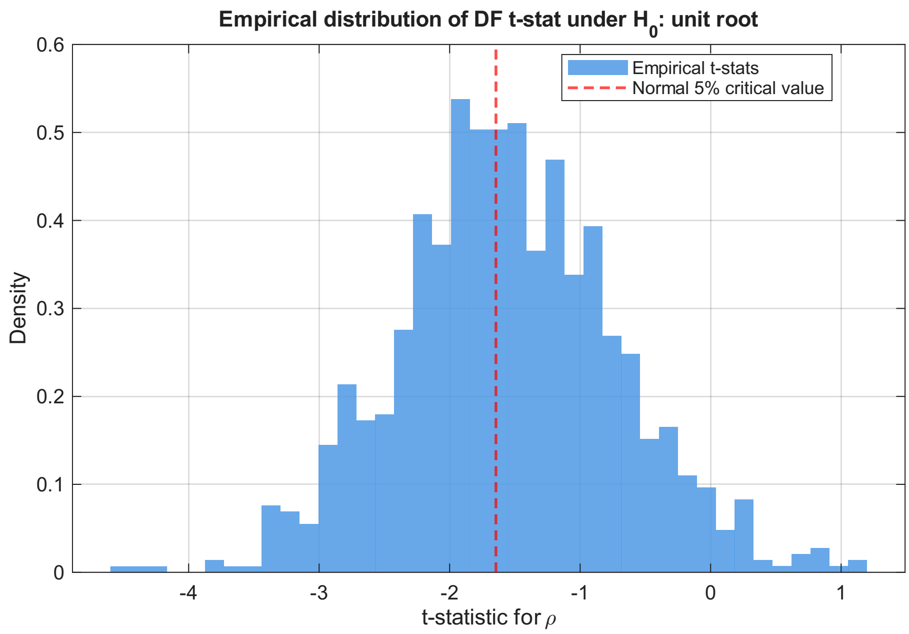


Figure 1-2: Empirical distribution of the simulated t_ρ under H_0 (unit root), $N = 1000$, $T = 250$. The red dashed line marks the 5% left-tailed critical value from the standard Normal distribution (-1.645).

The histogram shows that most simulated t_ρ values are clustered between -3 and -0.5, with the Normal critical value cutting through the centre of the empirical distribution. This explains the 47.6% rejection rate. The result reflects the fact that, when the process contains a unit root, the OLS estimator of ϕ (and thus of ρ) is biased downwards and its limiting distribution is non-standard. In particular, the numerator and denominator of the t-ratio diverge at different rates, so t_ρ does not converge to $N(0,1)$.

To further characterise the empirical distribution, additional statistics were computed. Their values are reported in Table 1-2.

Table 1-2: Empirical distribution of t_ρ under H_0 (unit root)

Rejection rate using $N(0,1)$ 5% left-tail	0.476
Mean of t-statistics	−1.560
Median of t-statistics	−1.578
Skewness	0.100
Kurtosis	3.277
1st percentile	−3.4164
5th percentile	−2.9099
10th percentile	−2.6492
50th percentile	−1.5781
90th percentile	−0.5134
95th percentile	−0.1496
99th percentile	0.5822

Both the mean and median are clearly negative, confirming that the empirical distribution is shifted to the left. The skewness (0.10) and kurtosis (3.28) show only mild asymmetry and slightly heavier tails relative to the standard Normal. The fundamental issue, however, is the systematic left-shift of the entire distribution, which leads to massive over-rejection when using Normal critical values.

Overall, the simulation demonstrates that the usual t-test for $\rho = 0$ is not valid in the presence of a unit root, since the sampling distribution of t_ρ is non-standard and the Normal approximation fails. This motivates the use of special critical values—those tabulated by Dickey and Fuller—which will be analysed in point 1(c).

```
%% === simulation settings ===
T       = 250;       % sample size
nSim    = 1000;      % number of Monte Carlo replications
sigma   = 1;         % std of innovations

% preallocate
t_stats          = zeros(nSim,1);    % DF t-statistics
rejected_normal = false(nSim,1);    % indicator: reject using N(0,1)?

% 5% *left-tailed* critical value from standard Normal
crit_normal_5 = -1.645;

%% === Monte Carlo loop ===
for s = 1:nSim

    % ----- 1. generate a pure random walk (unit root) -----
    y   = zeros(T,1);
    eps = sigma * randn(T,1);
    for t = 2:T
        y(t) = y(t-1) + eps(t);
    end

    % ----- 2. build DF regression variables -----
    dy    = diff(y);         %  y_t   , length T-1
```

```
    ylag = y(1:end-1);      % y_{t-1}, length T-1

    % ----- 3. OLS:  y_t  =    +    y_{t-1} + u_t -----
    X = [ones(T-1,1), ylag];
    b = X \ dy;             % OLS estimates: [alpha_hat; rho_hat]

    % residuals
    uhat = dy - X*b;

    % estimate error variance: (T-1) obs, 2 parameters
    dof  = (T-1) - size(X,2);     % degrees of freedom = 248
    s2   = (uhat' * uhat) / dof;

    % variance-covariance matrix of OLS
    varb = s2 * inv(X' * X);

    rho_hat = b(2);
    se_rho  = sqrt(varb(2,2));

    % ----- 4. DF t-stat -----
    t_val = rho_hat / se_rho;
    t_stats(s) = t_val;

    % ----- 5. rejection using WRONG N(0,1) critical value -----
    rejected_normal(s) = (t_val < crit_normal_5);
end

%% === summary results ===
rej_rate   = mean(rejected_normal);     % how often do we reject?
t_mean     = mean(t_stats);
t_median   = median(t_stats);
t_skew     = skewness(t_stats);
t_kurt     = kurtosis(t_stats);
t_prc      = prctile(t_stats, [1 5 10 50 90 95 99]);
```

□

1.3

Compute some percentiles of the empirical distribution of the t-test you generated at point b. and check that they are close to those tabulated by Dickey and Fuller. (hint: you can find tables with the DF percentiles in Enders or Hamilton or on the internet).

Solution.
From the 1000 simulated t_ρ statistics obtained in point (b), we compute the empirical quantiles and compare them with the Dickey–Fuller critical values tabulated for the case with an intercept and no trend (Enders, 4th ed., Table 4.A.2).

Table 1-3: Empirical percentiles of t_ρ vs. Dickey–Fuller critical values

Significance level	Empirical quantile	DF tabulated value
1%	−3.42	−3.43
5%	−2.91	−2.86
10%	−2.65	−2.57

The empirical quantiles are very close to the tabulated DF critical values. This confirms that the simulated t_ρ statistics follow the non-standard Dickey–Fuller distribution derived theoretically, rather than a standard Normal one. Hence, the Monte Carlo simulation in point (b) successfully reproduces the correct finite-sample behaviour of the test statistic.

□

Question 2. Spurious regression.

Design a Monte Carlo to show that the regression coefficient, the t-test and the R^2 are meaningless in the case of a spurious regression.

In particular, show what happens in each of the 4 cases discussed in Enders, Edition 3 or 4, pp. 195-199.

Solution.

The problem of spurious regression arises when ordinary least squares (OLS) is applied to time series that are nonstationary in levels. As originally demonstrated by Granger and Newbold (1974) and further discussed by Enders (2014, pp. 195–199), regressions between unrelated trending variables may produce apparently strong results—large t-statistics and high R^2—even though the variables share no true relationship. The regression output "looks good," but the estimated coefficients are inconsistent and standard inference procedures are invalid because the residuals are nonstationary.

Following Enders, we distinguish four representative cases that clarify when a regression in levels is meaningful and when it is not.

- **Case 1 — Both variables stationary.** When both $\{y_t\}$ and $\{x_t\}$ are stationary, the classical regression model is appropriate. The OLS estimator is consistent, the usual asymptotic distributions apply, and R^2 measures genuine explanatory power. The residuals fluctuate around zero without any systematic trend, and standard hypothesis testing remains valid.

- **Case 2 — Variables integrated of different orders.** Suppose that y_t follows a unit root process while x_t is stationary, for example

  $$y_t = y_{t-1} + \varepsilon_{y,t}, \qquad x_t = \phi x_{t-1} + \varepsilon_{x,t}, \quad |\phi| < 1.$$

  In this situation the regression is meaningless. The stationary regressor cannot explain the stochastic trend in y_t, so the residuals inherit the nonstationarity of the dependent variable. Although the R^2 may remain small, the t-statistics become unreliable because the error term violates the assumptions required for valid OLS inference.

- **Case 3 — Both variables nonstationary and independent.** Consider two independent random walks,

  $$y_t = y_{t-1} + \varepsilon_{y,t}, \qquad x_t = x_{t-1} + \varepsilon_{x,t},$$

  with $\varepsilon_{y,t}$ and $\varepsilon_{x,t}$ independent white-noise innovations. Although the two series are unrelated by construction, their trending behaviour often appears correlated in finite samples. OLS interprets this co-movement as a genuine relationship, producing large t-statistics and spuriously high R^2 values. The residuals remain nonstationary, their variance grows with time, and deviations from the fitted line never revert to zero. This is the classic spurious regression identified by Granger and Newbold: the estimated relationship has no economic meaning, yet it appears statistically significant.

- **Case 4 — Both variables nonstationary but cointegrated.** When y_t and x_t share a common stochastic trend, some linear combination of them is stationary:

  $$y_t = \beta_0 + \beta_1 x_t + u_t, \qquad u_t \sim I(0).$$

  In this case the regression in levels is again meaningful. The residuals are stationary, R^2 is high for the right reason, and the estimated coefficient β_1 represents a valid long-run equilibrium relation between the two variables.

These four cases summarize the results discussed by Enders (2014, pp. 195–199). OLS inference is valid only when the variables, or at least the regression residuals, are stationary. Regressing unrelated nonstationary variables, as in Case 3, leads to spuriously significant results and misleading conclusions.

Our goal is to replicate these four cases numerically and verify the results discussed by Enders (2014). We perform a Monte Carlo experiment with $N = 1000$ independent replications, each of length $T = 250$. In each replication, we generate the data according to the appropriate data-generating process (DGP) and estimate the following simple regression model:

$$y_t = \alpha + \beta x_t + u_t, \tag{1}$$

where u_t denotes the regression residual.

For each replication we record the t-statistic associated with $\hat{\beta}$, the coefficient of determination R^2, and an indicator for whether $|t_\beta| > 1.96$ (corresponding to a 5% two-sided significance level).

Table 2-4: Data-generating processes (DGPs) for the four cases in Enders (2014, pp. 195–199).

Case	Processes for y_t and x_t	Description
1	$y_t = 0.5y_{t-1} + e_t, \quad x_t = 0.4x_{t-1} + v_t$	Both variables stationary ($I(0)$).
2	$y_t = y_{t-1} + \varepsilon_{y,t}, \quad x_t = 0.5x_{t-1} + \varepsilon_{x,t}$	y_t nonstationary, x_t stationary.
3	$y_t = y_{t-1} + \varepsilon_{y,t}, \quad x_t = x_{t-1} + \varepsilon_{x,t}$	Both variables $I(1)$ and independent (spurious regression).
4	$x_t = x_{t-1} + u_t, \quad y_t = x_t + 0.5v_t$	Both variables $I(1)$ but cointegrated (valid long-run relation).

Cases 1–2 generate stationary or mixed-order processes, Case 3 two independent random walks, and Case 4 two cointegrated random walks sharing the same stochastic trend.

Figure 2-3 displays one simulated realisation for each case. Stationary series fluctuate around a constant mean; random walks show permanent shocks and long swings; the cointegrated pair move together over time.

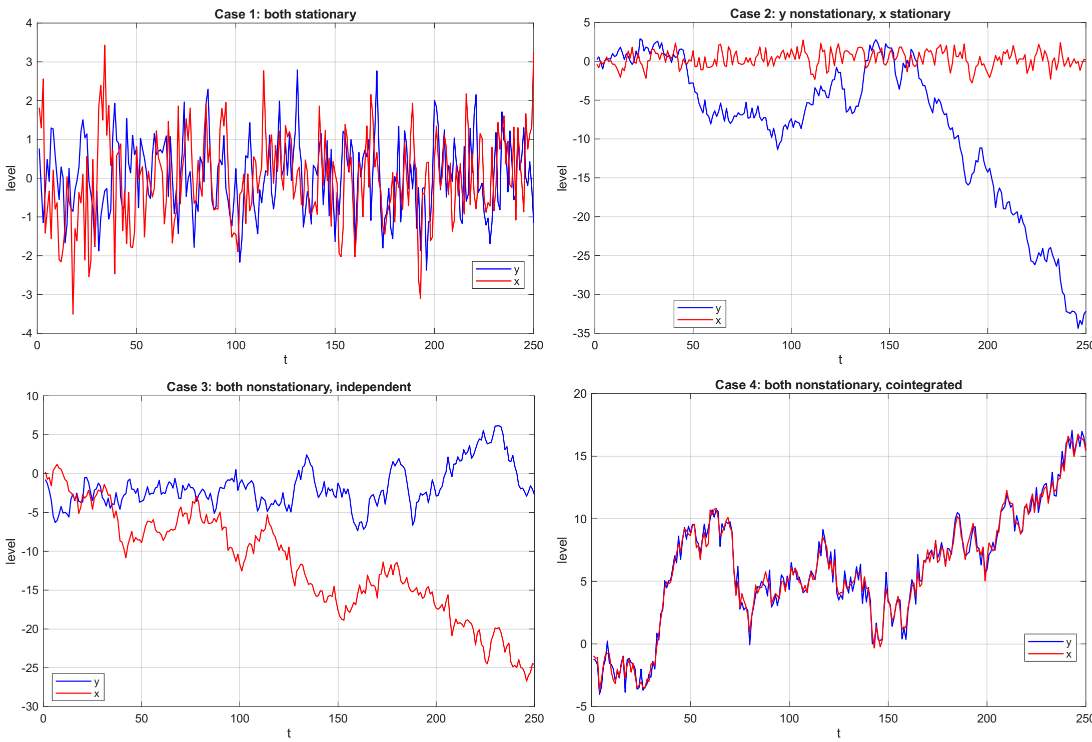


Figure 2-3: Example time series for the four DGPs.

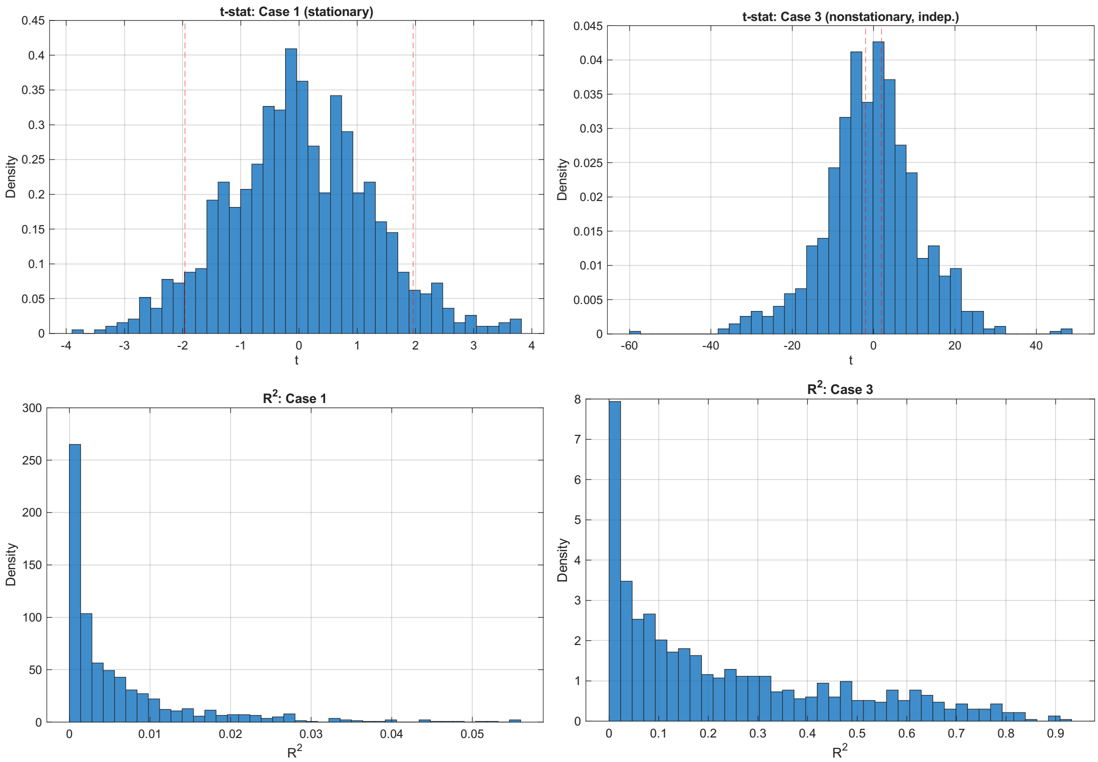


Figure 2-4: Empirical distributions of t-statistics and R^2 for Case 1 (stationary variables) and Case 3 (independent random walks). Dashed red lines mark the 5 % critical values (± 1.96).

The main outcomes of the simulation are summarised below.

Table 2-5: Rejection frequencies and mean R^2 (1000 replications, $T = 250$)

Case	Rejection rate ($|t| > 1.96$)	Mean R^2
(1) Stationary y_t, stationary x_t	0.116	0.006
(2) Nonstationary y_t, stationary x_t	0.270	0.013
(3) Both nonstationary, independent	0.852	0.233
(4) Both nonstationary, cointegrated	1.000	0.989

The results reproduce Enders's theoretical discussion. When both variables are stationary (Case 1), the regression behaves correctly: R^2 is nearly zero and the rejection rate is close to the nominal size. When only the dependent variable is nonstationary (Case 2), the regression is meaningless and inference slightly distorted. When both variables are nonstationary but independent (Case 3), the regression becomes spuriously significant: the null is rejected in 85 % of the replications and the average R^2 rises to 0.23, even though no relationship exists. Finally, when the variables are cointegrated (Case 4), the regression again becomes valid—the residuals are stationary and R^2 approaches one, reflecting a true long-run relation.

In Case 3, both y_t and x_t contain random-walk trends. Because these trends drift in the same general direction within the finite sample, the regression captures their co-movement and attributes it to $\hat{\beta}$, producing inflated R^2 and exaggerated t-statistics. The residuals remain nonstationary, so deviations never revert to zero. This confirms that OLS applied to unrelated nonstationary variables yields *spurious* significance. By contrast, the cointegrated case (Case 4) demonstrates that high R^2 is not necessarily spurious when the residuals are stationary.

Overall, the Monte Carlo exercise validates the analysis in Enders (2014): OLS inference is reliable only when the variables, or at least the regression residuals, are stationary. Regressing nonstationary variables without accounting for their stochastic trends can produce entirely misleading results.

```
%% === simulation settings ===
T    = 250;                      % sample size
nSim = 1000;                     % number of Monte Carlo replications
crit = 1.96;                     % ~5% two-sided critical value

% container for results
results = struct();

% helper OLS function
ols_stats = @(y,x) deal_stats(y,x);

%% ===============================================================
%  CASE 1: both STATIONARY (benchmark, "good regression")
%  y_t = 0.5 y_{t-1} + e_t
%  x_t = 0.4 x_{t-1} + v_t
%  -> both I(0): OLS should be fine, small R^2, rejection ~near nominal
%% ===============================================================
phi_y = 0.5;
phi_x = 0.4;

beta1 = zeros(nSim,1);
t1    = zeros(nSim,1);
R21   = zeros(nSim,1);

for s = 1:nSim
    e1 = randn(T,1);
    e2 = randn(T,1);
    y  = filter(1,[1 -phi_y], e1);     % stationary AR(1)
    x  = filter(1,[1 -phi_x], e2);     % stationary AR(1)
    [b, tval, R2] = ols_stats(y,x);
    beta1(s) = b;
    t1(s)    = tval;
    R21(s)   = R2;
    % store first replication to plot later
    if s == 1
        y_case1 = y;
        x_case1 = x;
    end
end
results.case1 = struct('t',t1,'R2',R21);

%% ===============================================================
%  CASE 2: y NONSTATIONARY (random walk), x STATIONARY
%  -> different orders of integration: regression has no meaning
%% ===============================================================
phi_x = 0.5;

t2  = zeros(nSim,1);
R22 = zeros(nSim,1);

for s = 1:nSim
    y = cumsum(randn(T,1));            % random walk -> nonstationary
    x = filter(1,[1 -phi_x], randn(T,1));  % stationary AR(1)
    [~, tval, R2] = ols_stats(y,x);
    t2(s)  = tval;
    R22(s) = R2;
    if s == 1
        y_case2 = y;
        x_case2 = x;
    end
end
```

```
results.case2 = struct('t',t2,'R2',R22);

%% ================================================================
%  CASE 3: both NONSTATIONARY, independent random walks
%  -> classic spurious regression (Enders, pp. 195  199  )
%  -> high R^2 and many large t-stats even with no true relation
%% ================================================================
t3  = zeros(nSim,1);
R23 = zeros(nSim,1);

for s = 1:nSim
    y = cumsum(randn(T,1));            % random walk
    x = cumsum(randn(T,1));            % independent random walk
    [~, tval, R2] = ols_stats(y,x);
    t3(s)  = tval;
    R23(s) = R2;
    if s == 1
        y_case3 = y;
        x_case3 = x;
    end
end
results.case3 = struct('t',t3,'R2',R23);

%% ================================================================
%  CASE 4: both NONSTATIONARY, but COINTEGRATED
%  y_t = x_t + noise, with x_t random walk
%  -> they share the same stochastic trend, residuals are I(0)
%  -> regression becomes meaningful again, high R^2 is genuine
%% ================================================================
t4  = zeros(nSim,1);
R24 = zeros(nSim,1);

for s = 1:nSim
    u = randn(T,1);
    v = randn(T,1);
    x = cumsum(u);                % common stochastic trend
    y = x + 0.5*v;                % y shares the trend -> cointegrated
    [~, tval, R2] = ols_stats(y,x);
    t4(s)  = tval;
    R24(s) = R2;
    if s == 1
        y_case4 = y;
        x_case4 = x;
    end
end
results.case4 = struct('t',t4,'R2',R24);
```

□

Question 3. Invertibility.

Suppose that the DGP is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{bmatrix} 1 & L^2 \\ \frac{\beta}{1-\beta} & \frac{\beta^2}{1-\beta} + \beta L \end{bmatrix} \begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix}$$

where the shocks are uncorrelated, $u_t = [\eta_t \quad \varepsilon_t]'$, $\text{VCov}(u_t) = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$ and $0 < \beta < 1$ (assume $\beta = 0.6$).

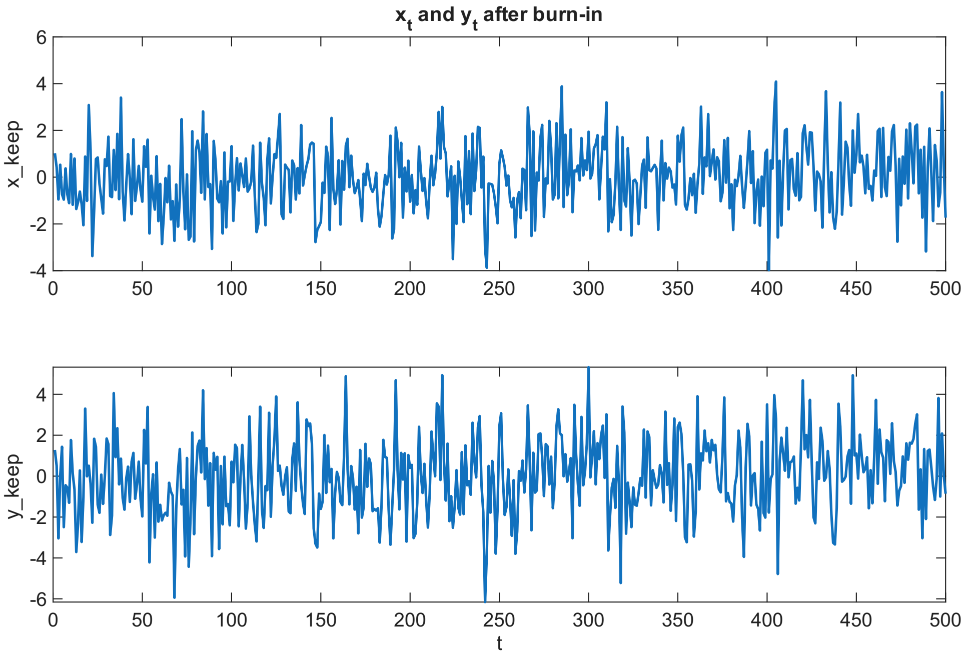
3.1

Generate 500 observations for $[x_t \quad y_t]'$, using the DGP, estimate a VAR with 4 lags and compute the impulse responses using a Cholesky identification (after all, it is compatible with the DGP, right?). Do the simulations N times and store the impulse responses for the N simulations.

Solution. Using the DGP and trivial matrix multiplication we can identify the vector $[x_t \quad y_t] = [\eta_t + \epsilon_{t-2} \quad \frac{\beta}{1-\beta}\eta_t + \frac{\beta^2}{1-\beta}\epsilon_t + \beta\epsilon_{t-1}]$. To generate a vector of 500 observation for x_t and y_t we implement a `for` loop strategy with a burn-in period $B = 200$. Lastly, we stack the two vectors in a 500 x 2 matrix X and we plot the two vectors (columns) of the matrix.

```
%%%%%% EXERCISE THREE %%%%%%

%% parameters
T_keep = 500;
B = 200;
T = B + T_keep;
beta = 0.6;
sigma_2_eta = 1;
sigma_2_eps = 0.8;

eta = sqrt(sigma_2_eta)*randn(T,1);
eps_today = sqrt(sigma_2_eps)*randn(T,1);
% eps_yesterday
eps_yesterday=zeros(T,1);
eps_yesterday(1)=0;
for t = 2:T
    eps_yesterday(t) = eps_today(t-1);
end
% eps_twodays
eps_twodays=zeros(T,1);
eps_twodays(1)=0;
for t = 2:T
    eps_twodays(t) = eps_yesterday(t-1);
end

% x and y
x = zeros(T, 1);
y = zeros(T, 1);
for t = 1:T
    x(t) = eta(t) + eps_twodays(t);
    y(t) = (beta/(1-beta)) * eta(t) + ((beta^2)/(1-beta)) * eps_today(t) + beta
   *eps_yesterday(t);
end

% burn-in
s = B+1;
x_keep = x(s:end);
y_keep = y(s:end);
X = [x_keep , y_keep];
```

□



To estimate a VAR with four lags, we used the Econometrics Toolbox of MATLAB. Through the command `estimate` and the function `varm` we set the VAR lag length p and estimate a two-variable VAR(p) on the matrix X. This fitted reduced-form VAR provides the coefficients needed to build $C(L)$ and, thus, to compute the structural IRFs via $C(L)A$.

We then follow the handout on invertibility and recursively compute the reduced-form VMA coefficients. The array `C(:,:,h)` contains the reduced-form impulse responses at horizon $h-1$, which will be post-multiplied by A (provided in the handout) to obtain structural IRFs.

$$A = \begin{bmatrix} 1 & 0 \\ \dfrac{\beta}{1-\beta} & \dfrac{\beta^2}{1-\beta} \end{bmatrix}.$$

```
%% VAR(4) and structural IRFs via C(L)A (using Econometrics Toolbox)
p = 4; H = 20; N = 500;
EstMdl = estimate(varm(2,p), X, 'Display','off');

% VMA coefficients C_h recursively
C = zeros(2,2,H);
C(:,:,1) = eye(2);
for h = 2:H
    tmp = zeros(2);
    for j = 1:min(p,h-1)
        tmp = tmp + EstMdl.AR{j} * C(:,:,h-j);
    end
    C(:,:,h) = tmp;
end

% matrix A from the handout
A = [1, 0; beta/(1-beta), (beta^2)/(1-beta)];
```

3.2

Compare the true impulse response functions and the estimated ones (for the estimated ones, you can compute the mean of the empirical distribution of the estimated impulse responses and, say, the 2.5th and the 97.5th percentile). Are they similar? What's wrong with the DGP? Discuss.

Solution. We then proceeded to build the true IRFs of the DGP and the VAR-implied structural IRFs via post-multiplication of the matrix A as explicitly stated in the handout. More specifically, `IRF_hat` stores the one-sample estimate for our "observed" dataset X, i.e. our original simulation. It will be plotted as a dashed dot line.

We then proceeded to repeat the whole process of simulation generation, VAR estimation, and IRFs computation for $N = 500$ times in a Monte Carlo loop to approximate the finite-sample distribution of identified IRFs implied by the DGP.

```
IRF_hat = zeros(H,2,2);
for h = 1:H
    IRF_hat(h,:,:) = C(:,:,h) * A;   % unit structural shocks
end

% "real" IRF of the DGP
IRF_true = zeros(H,2,2);
% shock 1 = eta_t
IRF_true(1,1,1) = 1;
IRF_true(1,2,1) = (beta/(1-beta));
% shock 2 = eps_t
IRF_true(1,2,2) = (beta^2/(1-beta));
IRF_true(2,2,2) = beta;
IRF_true(3,1,2) = 1;

% Monte Carlo
IRF_stack = zeros(H,2,2,N);

for n = 1:N
    % DGP + burn-in
    eta_n        = sqrt(sigma_2_eta) * randn(T,1);
    eps_today_n = sqrt(sigma_2_eps) * randn(T,1);
    eps_yest_n   = [0; eps_today_n(1:end-1)];
    eps_2day_n   = [0; eps_yest_n(1:end-1)];

    x_n = eta_n + eps_2day_n;
    y_n = (beta/(1-beta))*eta_n + ((beta^2)/(1-beta))*eps_today_n + beta*
   eps_yest_n;

    Xn = [x_n(B+1:end), y_n(B+1:end)];

    % VAR(4) and IRF
    EstMdl_n = estimate(varm(2,p), Xn, 'Display','off');
    % VMA of the estimated VAR (reduced form)
    C_n = zeros(2,2,H);
    C_n(:,:,1) = eye(2);
        for h = 2:H
            tmp = zeros(2);
            for j = 1:min(p,h-1)
                tmp = tmp + EstMdl_n.AR{j} * C_n(:,:,h-j);
            end
            C_n(:,:,h) = tmp;
        end
        for h = 1:H
            IRF_stack(h,:,:,n) = C_n(:,:,h) * A;
        end
end
```

The Monte Carlo loop shows that the true and estimated IRFs are systematically mismatched. We then computed the mean and percentiles and we plot them as shadows in the graphs to provide visual representation.

```
% percentiles
IRF_mean = mean(IRF_stack, 4);
```

```
IRF_lo   = prctile(IRF_stack, 2.5, 4);
IRF_hi   = prctile(IRF_stack, 97.5, 4);
```

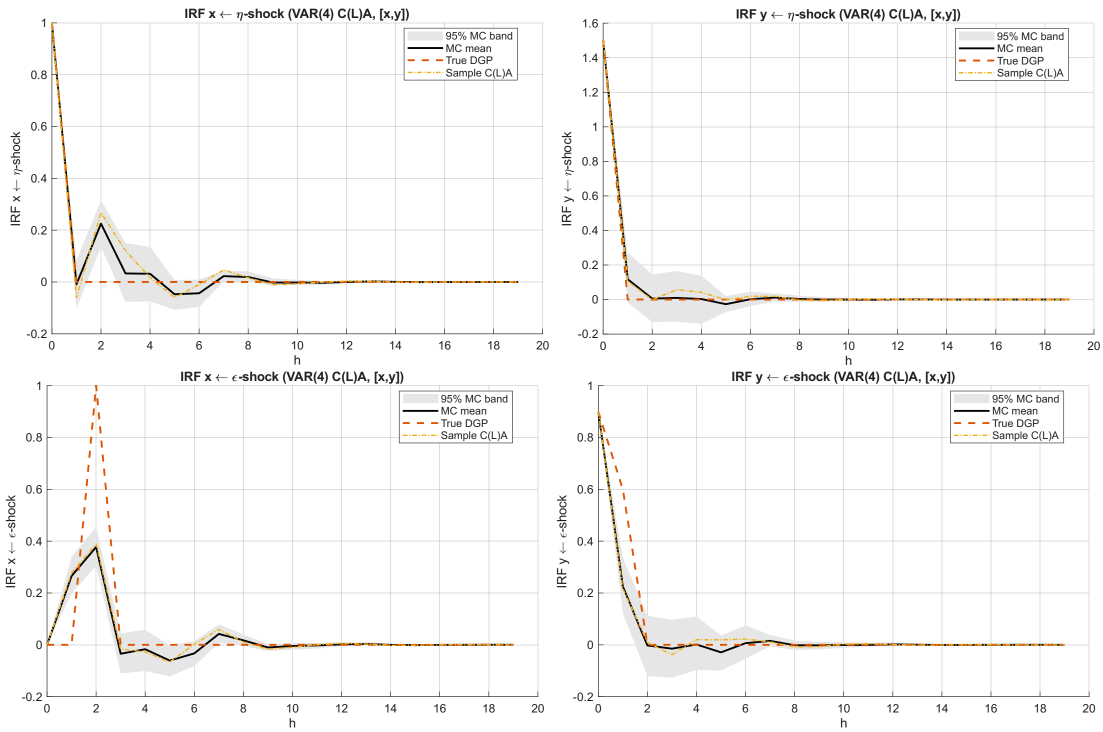
The mismatch occurs systematically and not just in one-sample IRFs, as it can be seen in the persistence of the mismatch not only in `IRF_hat` but also in Monte Carlo estimates.

The underlying reason for this is that the **DGP is non-invertible.** Indeed, ϵ_t hits x_t with a two-period delay, therefore it is impossible to recover ϵ_t from current or past x_t or y_t without future data. This violates the definition of invertibility.

Mathematically, to estimate the structural IRFs we require that the VMA polynomial $\Theta(L) = C(L)A$ is invertible and therefore $\det \Theta(z) \neq 0 \quad \forall |z| \leq 1$

$$\det \Theta(z) = -\frac{\beta}{1-\beta}(z-1)(z+\beta) \tag{2}$$

with zeros $z = -\beta$ and $z = 1$; since $0 < \beta < 1$, the two zeros lie respectively inside and on the unit circle.



Lastly, we provide the representation of the one-sample IRFs, the IRFs of the true DGP, the 95% MC bands and the MC mean of the IRFs for the two processes and the two shocks. As expected from the non-invertibility discussion, structural IRFs on x_t and y_t due to η shocks appear more consistent with the true IRFs, while structural IRFs on x_t and y_t due to ϵ shocks seem to diverge more visibly from the true IRFs. However, being the whole structural MA non invertible, VAR+$C(L)A$ is not guaranteed to provide the true IRF for either shock. □

Question 4. Granger causality.

Read the paper "A New Measure of Monetary Shocks: Derivation and Implications", by Christina D. Romer and David H. Romer, *The American Economic Review*, Vol. 94, No. 4 (Sep., 2004), pp. 1055-1084, available on the blackboard.

Download the file `Romer_Romer.xlsx`. There, you will find 4 time series: US inflation, US unemployment, US federal funds rate, the Romer and Romer monetary policy shocks from 1969Q1 to 1996Q4.

4.1

Run a VAR with 4 lags and test for Granger causality of the Romer and Romer shocks. Are the Romer and Romer shocks Granger-causing other variables or not? Discuss. Are the other variables Granger-causing the Romer and Romer dates? Discuss.

Solution.

```
%%%%%% EXERCISE FOUR %%%%%%

%% Import Romer and Romer Excel Shocks
fname = 'Romer_Romer.xlsx';
T = readtable(fname, 'Sheet','Foglio1', 'VariableNamingRule','preserve');
T.Properties.VariableNames = {'Date','inflation','unemployment','ffr','rr_shock'};

% Y matrix
Y = [T.inflation, T.unemployment, T.ffr, T.rr_shock];

%% VAR(4) and conditional Granger causality
p = 4;
Mdl    = varm(4,p);
EstMdl = estimate(Mdl, Y, 'Display','off');
names = T.Properties.VariableNames(2:4);   % {'inflation','unemployment','ffr'}

% RR -> others
for k = 1:3
    Z = Y(:, setdiff(1:4, [4 k]));
    [~,p] = gctest(Y(:,4), Y(:,k), Z, 'NumLags',4,'Constant',true);
    fprintf('RR shocks -> %-13s : p=%.4g\n', names{k}, p);
end

% others -> RR
for k = 1:3
    Z = Y(:, setdiff(1:4, [k 4]));
    [~,p] = gctest(Y(:,k), Y(:,4), Z, 'NumLags',4,'Constant',true);
    fprintf('%-13s -> RR shocks : p=%.4g\n', names{k}, p);
end
```

Let $y_t \in \{\pi_t, u_t, i_t^{ff}\}$ be a vector of macroeconomic fundamentals and define the RR shocks as $x_t = \epsilon_t^{RR}$ The RR shocks are residuals from a regression of *intended* changes in the Federal Funds Rates on information available to the Fed at time t. By construction, they are orthogonal to the Fed's internal forecasts and as a consequence they should be orthogonal also to every predictable macroeconomic condition at time t.

We should therefore expect that **they predict the Federal Funds rate but not the rest of the information set at time** t, and that the macroeconomic fundamentals at time t are not able to predict them.

We test for such results employing the Econometrics Toolbox command `varm` to estimate a VAR(4) model first and the command `gctest` later, which runs a Granger causality test, specifying the conditioning set. We run two different tests, on whether y_t Granger-causes the RR-shocks and viceversa.

The interpretation of the p-values for each Granger-causing relation confirms out expectations:

- **RR shocks Granger-cause FFR** in a strongly significant manner.
- RR shocks Granger-cause weakly other fundamentals (not at 5%).
- No fundamental Granger-causes the RR shocks.

□

Granger-causing relation	**p-value**
RR shocks → inflation	0.07435
RR shocks → unemployment	0.06763
RR shocks → ffr	2.578×10^{-7}
inflation → RR shocks	0.6730
unemployment → RR shocks	0.4813
ffr → RR shocks	0.2621

Table 4-6: Conditional Granger causality p-values from a VAR(4).