Macroeconomics I

Solutions To Problem Set Four

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Exercise 1. Consumption of durables and non-durables.

Consider an agent who is planning his intertemporal consumption. The agent derives utility from two goods: c_t and d_t. Good c is non-durable (e.g. food), good d is durable (such as cars, refrigerators, housing, etc.). For simplicity, we assume that the durable good fully depreciates after one period. The agent can freely borrow or lend money and decides consumption in date t after observing income y_t. Income is stochastic and follows a Markov Chain process with transition matrix Π and initial condition $y_0 \in Y$. The problem of the agent is:

$$\max_{\{c_t, d_t, a_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t, d_t + d_{t-1}) \right]$$

subject to

$$a_t = (1+r)a_{t-1} + y_t - c_t - pd_t,$$

where a_t is the level of (risk-free) assets in period t; r is the real interest rate, and p is the relative price of good d. Both r and p are fixed over time. The initial conditions a_{-1} and d_{-1} are given.

1. Write down the Bellman equation. What are the state variables? What are the control variables? Write clearly the laws of motion for the states.

In order to write the Bellman equation for the problem, it is necessary to identify the state variables and the controls:

a. The **states** are a_{-1}, d_{-1} and y

b. The **controls** are c, a and d

c. The **law of motions** are:

   - $a = (1+r)a_{-1} + y - c - pd$
   - $d = (d_{-1})'$
   - $y' \sim \Pi(y, \cdot)$

Therefore the Bellman equation takes the form:

$$V(a_{-1}, d_{-1}, y) = \max_{c,d,a} u(c, d + d_{-1}) + \beta\mathbb{E}[V(a, d, y')|y] \quad \text{s.t.} \quad a = (1 + r)a_{-1} + y - c - pd \tag{1}$$

2. Write down the first order conditions of optimality involving the marginal utilities, including the optimal intertemporal condition for asset's choice. Explain the economic intuition behind them.

The FOCs are:

$$\beta\mathbb{E}[\frac{\partial V}{\partial a}] = \lambda \tag{2}$$

$$u_c = \lambda \tag{3}$$

$$u_d + \beta\mathbb{E}[\frac{\partial V}{\partial d}] = \lambda p \tag{4}$$

Solving the two derivatives in the expected values using the envelope theorem:

$$\frac{\partial V}{\partial a} = \lambda'(1 + r) \tag{5}$$

$$\frac{\partial V}{\partial d} = u'_d \tag{6}$$

Therefore we can derive the **two optimal intertemporal conditions**:

$$\beta(1 + r)\mathbb{E}[u'_c] = u_c \tag{7}$$

$$\frac{u_d + \beta\mathbb{E}[u'_d]}{u_c} = p \tag{8}$$

Equation (7) can be interpreted as the optimal intertemporal condition for asset's choice. Indeed, it equates the marginal utility of consumption of one unit of income today to its expected value tomorrow discounted by β if invested in a risk-free asset with return $1 + r$. Equation (8) is the classic Euler equation for durable goods, where the relative marginal utility of consumption of the durable today and tomorrow is equated to the relative price of the durable.

Exercise 2. Permanent income hypothesis.

A consumer seeks to solve the following consumption–saving problem:

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right)$$

subject to

$$b_{t+1} = (1+r)(b_t - c_t + y_t),$$

$$c_t \geq 0, \qquad b_0, \, y_0 \text{ given},$$

and subject to the no–Ponzi–game condition:

$$\lim_{T\to\infty} \left(\frac{1}{1+r} \right)^{T+1} b_{T+1} \geq 0 \quad \text{for all } \{b_{t+1}\}_{t=0}^{\infty},$$

which is assumed to hold almost surely for every possible history of income shocks. In the above formulation, b_t is the consumer's financial wealth at the beginning of period t, y_t is income in period t, and $r > 0$ is the (net) interest rate.

Part I

Assume that there is no uncertainty and that $y_t = y$ for all t throughout this part.

1. Use the no–Ponzi–game condition to derive the intertemporal budget constraint for this consumer. What is the economic intuition behind the expression that you have derived?

Initially, we use the intertemporal budget constraint for the financial wealth enlisted among the constraints of the problem:

$$b_{t+1} = (1+r)(b_t - c_t + y_t),$$

From which we get:

$$b_t = \frac{b_{t+1}}{(1+r)} - y + c_t \tag{9}$$

Replacing equation (9) lagged one period in place of b_{t+1} we obtain:

$$b_t = \frac{b_{t+2}}{(1+r)^2} - \frac{y}{(1+r)} - y + \frac{c_t}{(1+r)} + c_t \tag{10}$$

If we lag the equation for T+1 periods (setting the initial $t = 0$):

$$b_0 = \frac{b_{T+1}}{(1+r)^{T+1}} - \sum_{j=0}^{T} \frac{y}{(1+r)^j} + \sum_{j=0}^{T} \frac{c_j}{(1+r)^j} \tag{11}$$

Finally, we impose the no-Ponzi game condition as required:

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T+1} b_{T+1} \geq 0 \quad \text{for all } \{b_{t+1}\}_{t=0}^{\infty},$$

therefore, Equation (11) in the limit becomes:

$$b_0 + \frac{(1+r)}{r} y \geq \sum_{j=0}^{\infty} \frac{c_j}{(1+r)^j} \tag{12}$$

The economic interpretation of Equation (12) is the following: the discounted sum of life-time consumption must be always lower or equal than the discounted sum of life-time income and initial wealth, i.e. you cannot consume in your life more than your income and initial wealth.

2. Rather than the no–Ponzi–game condition we can require that the consumer's asset holdings never fall below a fixed amount B, where B is allowed to be negative. In particular, B is the maximum amount which can be borrowed by an individual in a given period t, conditionally on that agent being able to repay that debt. Find an expression for B in terms of r and y. *[Hint: Use the non–negativity constraint on consumption.]*

To find the maximum amount which can be borrowed by an individual at time t, conditional on the agent being able to repay the debt, we start from Equation 12, i.e. the intertemporal budget constraint, using the fact that $c_t \geq 0 \quad \forall t$. This implies that:

$$b_0 + \frac{(1+r)}{r} y \geq \sum_{j=0}^{\infty} \frac{c_j}{(1+r)^j} \geq 0 \implies b_0 \geq -\frac{(1+r)}{r} y \tag{13}$$

where $B = -\frac{(1+r)}{r} y$ is the borrowing limit.

Part II.

We now introduce uncertainty. Assume that labour income follows a stationary AR(1) process, i.e. $|\rho| < 1$:

$$y_{t+1} = (1-\rho)\,\bar{y} + \rho\, y_t + \varepsilon_{t+1},$$

where $\{\varepsilon_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. random shocks with bounded support $[\underline{\varepsilon}, \overline{\varepsilon}]$ satisfying $\mathbb{E}\{\varepsilon_t\} = 0$ for all t. Furthermore, assume that $\beta(1+r) = 1$ holds.

3. Formulate the consumer problem recursively. That is, indicate the states, the controls and the law of motion for the states. Write then the Bellman equation for the problem. *[Hint: use the fact that in this setting we can substitute the no–Ponzi–game restriction by imposing a time–invariant borrowing limit $\hat{B}(y)$ on asset holdings].*

In order to write the Bellman equation for the problem, it is necessary to identify the state variables and the controls:

a. The **states** are b and y

b. The **controls** are c and b'

c. The **law of motions** are:

   - $b' = (1+r)(b - c + y)$
   - $y' = (1-\rho)\bar{y} + \rho y + \epsilon'$

Therefore the Bellman equation takes the form:

$$V(b,y) = \max_{c,b'} u(c) + \beta\mathbb{E}[V(b',y')|y] \tag{14}$$

$$\begin{aligned} \textbf{s.t.} \quad b' =& (1+r)(b-c+y) \quad (15) \\ y' =& (1-\rho)\bar{y} + \rho y + \epsilon' \quad (16) \\ c \geq& 0 \quad (17) \\ b' \geq& \hat{B}(y) \quad (18) \end{aligned}$$

Constraints 15, 17 and 18 can be summarised in the unique constraint:

$$\hat{B}(y) \leq b' \leq (b+y)(1+r) \tag{19}$$

Part III

Assume that u is quadratic:

$$u(c) = -\frac{1}{2}(\bar{c}-c)^2.$$

4. Derive the consumer's Euler equation, and argue that for all t we have $\mathbb{E}_0\{c_t\} = c_0$. *[Hint: The law of iterated expectations implies that $E_{t-1}\{E_t\{c_{t+1}\}\} = E_{t-1}\{c_{t+1}\}$.]*

The Bellman equation is:

$$V(b,y) = \max_{c,b'} -\frac{1}{2}(\bar{c}-c)^2 + \beta\mathbb{E}[V(b',y')|y]$$

using (15) as constraint, the FOCs are:

$$(\bar{c}-c) = \lambda(1+r) \tag{20}$$

$$\beta\mathbb{E}[\frac{\partial V}{\partial b'}] = \lambda \tag{21}$$

Using the envelope, the derivative in the expected value in equation 21 becomes:

$$\frac{\partial V}{\partial b'} = \lambda'(1+r)$$

Using equation 20, $\lambda' = \frac{(\bar{c}-c')}{(1+r)}$ and the Euler equation is:

$$\beta(1+r)\mathbb{E}[(\bar{c}-c')] = (\bar{c}-c) \tag{22}$$

Using the condition that $\beta(1+r) = 1$ and that $\mathbb{E}[\bar{c}] = \bar{c}$, Equation 22 can be simplified to:

$$\mathbb{E}[c'] = c \tag{23}$$

This Euler equation is valid for any time period. For instance:

$$\begin{aligned}
\mathbb{E}_t[c_{t+1}] &= c_t \quad \text{at time } t \\
\mathbb{E}_{t+1}[c_{t+2}] &= c_{t+1} \quad \text{at time } t+1 \\
\text{therefore} \quad \mathbb{E}_t[E_{t+1}[c_{t+2}]] &= c_t \quad \text{which by LIE} \\
\implies \mathbb{E}_t[c_{t+2}] &= c_t
\end{aligned}$$

5. Find the consumer's optimal consumption at time t, c_t^*, using the results of point 4 and the intertemporal budget constraint you have derived. *[Hint: notice that* $E_t(y_{t+k}) = (1-\rho^k)\bar{y} + \rho^k y_t]$

We can rewrite Equation 12, i.e. the IBC, for a non-stationary y as:

$$b_0 + \sum_{j=0}^{\infty} \frac{y_j}{(1+r)^j} \geq \sum_{j=0}^{\infty} \frac{c_j}{(1+r)^j} \tag{24}$$

Taking expected values on both sides:

$$b_t + \mathbb{E}_t[\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j}] \geq \mathbb{E}_t[\sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}] \tag{25}$$

We can work on both expected values separately. Using the martingale process of consumption derived before for the first expected value on the right hand side:

$$\mathbb{E}_t[\sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}] = \sum_{j=0}^{\infty} \frac{c_t}{(1+r)^j} = \frac{(1+r)}{r} c_t$$

Using the AR(1) process of income for the second expected value on the left hand side:

$$\mathbb{E}_t[\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j}] = \mathbb{E}_t[y_t + \frac{(1-\rho)\bar{y} + \rho y_t}{(1+r)} + ... + \frac{(1-\rho^j)\bar{y} + \rho^j y_t}{(1+r)^j}] = \sum_{j=0}^{\infty} \frac{(1-\rho^j)\bar{y} + \rho^j y_t}{(1+r)^j} \tag{26}$$

We can finally rewrite Equation 25 as:

$$\frac{(1+r)}{r} c_t^* = \sum_{j=0}^{\infty} \frac{(1-\rho^j)\bar{y} + \rho^j y_t}{(1+r)^j} + b_t \tag{27}$$

Which can be re-arranged as:

$$c_t^* = \frac{r}{(1+r)} [\sum_{j=0}^{\infty} \frac{(1-\rho^j)\bar{y} + \rho^j y_t}{(1+r)^j} + b_t] \tag{28}$$

where $\sum_{j=0}^{\infty} \frac{(1-\rho^j)\bar{y}+\rho^j y_t}{(1+r)^j} = H_t$

8. How much does current consumption increase if there is a *permanent* increase in income of one euro, i.e. if income increases by one euro in the current period and in all future periods? How much does current consumption increase if there is a *temporary* increase in income of one euro, i.e. if labour income in the current period increases by one euro but it remains unchanged in all future periods?

We can compare the two cases combining equations 25 and 28:

$$\frac{r}{(1+r)}\Bigg\{b_t + \mathbb{E}_t[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}]\Bigg\} = c_t^* \tag{29}$$

- A **permanent** increase in income raises by one unit the numerator of the expression inside H_t, therefore increasing H_t by $\frac{(1+r)}{r}$ $(\sum_{j=0}^{\infty}\frac{y_{t+j}+1}{(1+r)^j} = \sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j} + \frac{1+r}{r})$, thus optimal consumption today will eventually increase by one unit.

- A **temporary** increase in income affects the first element of the sum inside H_t, therefore resulting in the following:

$$\frac{r}{(1+r)}\Bigg\{b_t + (y_t + 1) + \mathbb{E}_t[\sum_{j=1}^{\infty}\frac{y_{t+j}}{(1+r)^j}]\Bigg\} = c_t^* \tag{30}$$

  Since the expression in curly brackets differs from the standard case by one, the increase in consumption will be of $\frac{r}{(1+r)}$ units.