

# Macroeconomics I

## Solutions To Problem Set One

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## Exercise 2 - The Solow Model with population growth

Consider the Solow model seen in class with the following Cobb-Douglas production and saving functions:

$$Y_t = K_t^\alpha (AN_t)^{1-\alpha}, \quad S_t = sY_t,$$

where  $A$  is constant and population grows at a constant rate, that is:

$$N_{t+1} = (1 + g_N)N_t.$$

1. Derive the law of motion for capital per worker,  $k_t = K_t/N_t$ . How does  $g_N$  affect that relationship?

We start from capital accumulation and population growth:

$$K_{t+1} = (1 - \delta)K_t + sY_t, \quad N_{t+1} = (1 + g_N)N_t.$$

Define  $k_t \equiv K_t/N_t$  and note that, with  $Y_t = K_t^\alpha (AN_t)^{1-\alpha}$  (with constant  $A$ ),

$$y_t \equiv \frac{Y_t}{N_t} = \frac{K_t^\alpha (AN_t)^{1-\alpha}}{N_t} = A^{1-\alpha} \left( \frac{K_t}{N_t} \right)^\alpha = A^{1-\alpha} k_t^\alpha.$$

Divide the capital accumulation equation by  $N_{t+1} = (1 + g_N)N_t$ :

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{(1 - \delta)K_t + sY_t}{(1 + g_N)N_t} = \frac{1 - \delta}{1 + g_N} k_t + \frac{s}{1 + g_N} y_t.$$

Substituting  $y_t = A^{1-\alpha} k_t^\alpha$  yields

$$k_{t+1} = \frac{1 - \delta}{1 + g_N} k_t + \frac{s A^{1-\alpha}}{1 + g_N} k_t^\alpha$$

so a higher  $g_N$  dilutes capital per worker via the factor  $(1 + g_N)$  in the denominator.

2. Derive the steady-state level of capital per worker  $k^*$  in terms of the model's parameters only.

What happens to the aggregate value of capital in the steady state?

From Exercise 2 we have  $Y_t = K_t^\alpha (AN_t)^{1-\alpha}$  with constant  $A$ ,  $S_t = sY_t$ ,  $K_{t+1} = (1-\delta)K_t + sY_t$ , and  $N_{t+1} = (1+g_N)N_t$ .

Define  $k_t = K_t/N_t$  and  $y_t = Y_t/N_t$ . Then

$$y_t = \frac{K_t^\alpha (AN_t)^{1-\alpha}}{N_t} = A^{1-\alpha} \left( \frac{K_t}{N_t} \right)^\alpha = A^{1-\alpha} k_t^\alpha.$$

Divide the accumulation equation by  $N_{t+1} = (1+g_N)N_t$ :

$$k_{t+1} = \frac{1-\delta}{1+g_N} k_t + \frac{s}{1+g_N} y_t = \frac{1-\delta}{1+g_N} k_t + \frac{s}{1+g_N} A^{1-\alpha} k_t^\alpha.$$

In steady state  $k_{t+1} = k_t = k^*$ , hence

$$k^* = \frac{(1-\delta)}{(1+g_N)} k^* + \frac{s}{1+g_N} A^{1-\alpha} (k^*)^\alpha$$

$$(1+g_N)k^* = (1-\delta)k^* + sA^{1-\alpha}(k^*)^\alpha \implies (\delta+g_N)k^* = sA^{1-\alpha}(k^*)^\alpha.$$

Therefore

$$k^* = \left( \frac{sA^{1-\alpha}}{\delta+g_N} \right)^{\frac{1}{1-\alpha}}$$

At the steady state, with  $k^*$  constant and

$$K_{t+1} = k^* N_{t+1}, \quad K_t = k^* N_t, \quad N_{t+1} = (1+g_N)N_t$$

$$\frac{K_{t+1}}{K_t} = \frac{N_{t+1}}{N_t} = (1+g_N)$$

hence aggregate capital grows at population rate  $g_N$ .

3. Derive the golden rule saving rate,  $s_{GR}$ , in terms of the model's parameters only.

By definition, the golden rule saving rate  $s^{GR}$  maximizes consumption per worker in steady

state. We know:

$$c = y - i = (1 - s)y,$$

Since  $y = A^{1-\alpha}k^\alpha$  and

$$k^* = \left( \frac{sA^{1-\alpha}}{\delta + g_N} \right)^{\frac{1}{1-\alpha}},$$

steady-state consumption per worker can be written as

$$c^*(s) = (1 - s)y^*(s) = (1 - s)A^{1-\alpha}(k^*(s))^\alpha.$$

Substituting for  $k(s)$ :

$$c^*(s) = (1 - s)A^{1-\alpha} \left( \frac{sA^{1-\alpha}}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}}.$$

To find  $s^{GR}$ ,

$$\max_s c^*(s).$$

:

$$FOC = \frac{\partial c^*(s)}{\partial s} = 0 \implies s^{GR} = \alpha.$$

$s^{GR} = \alpha$

## Exercise 3 - The Solow Model with technological progress and different saving rates

Consider the Solow model with population growth analyzed in Exercise 2. Let's now introduce also technological progress. The Cobb-Douglas production function becomes:

$$Y_t = K_t^\alpha (A_t N_t)^{1-\alpha},$$

now besides population growing at constant rate  $g_N$  also technology grows at constant rate  $g_A$ :

$$N_{t+1} = (1 + g_N)N_t, \quad A_{t+1} = (1 + g_A)A_t.$$

1. Derive the law of motion for capital per unit of effective worker

$$\hat{k}_t = \frac{K_t}{A_t N_t}.$$

How does  $g_A$  affect this relationship?

[Hint: to simplify the algebra you can use the following approximation:  $g_N g_A \approx 0$ .]

We define capital per unit of effective labor as

$$\hat{k}_t = \frac{K_t}{A_t N_t}, \quad \hat{y}_t = \frac{Y_t}{A_t N_t} = \hat{k}_t^\alpha.$$

Starting from capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + sY_t,$$

and dividing through by  $A_{t+1}N_{t+1} = (1 + g_A)(1 + g_N)A_t N_t$ , we obtain

$$\hat{k}_{t+1} = \frac{(1 - \delta)K_t + sY_t}{(1 + g_A)(1 + g_N)A_t N_t}.$$

Substituting  $K_t = A_t N_t \hat{k}_t$  and  $Y_t = A_t N_t \hat{y}_t$  gives

$$\hat{k}_{t+1} = \frac{1 - \delta}{(1 + g_A + g_N)} \hat{k}_t + \frac{s}{(1 + g_A + g_N)} \hat{y}_t.$$

Since  $\hat{y}_t = \hat{k}_t^\alpha$ , we finally get

$$\boxed{\hat{k}_{t+1} = \frac{1 - \delta}{(1 + g_A + g_N)} \hat{k}_t + \frac{s}{(1 + g_A + g_N)} \hat{k}_t^\alpha}$$

This shows that  $g_A$  acts as an additional “dilution” term in the denominator.

2. Suppose that at  $t_0$  the economy is on its balanced growth path and the saving rate is  $s_0$ .

Derive the steady state level of capital per unit of effective labor  $\hat{k}$ . Now suppose that at  $t_1$  the saving rate increases permanently to  $s_1 > s_0$ . Show in a diagram with  $\hat{k}_t$  on the horizontal axis and investment per unit of effective worker on the vertical axis what happens to the steady state capital per effective worker  $\hat{k}$  after  $t_1$ .

From the law of motion:

$$\hat{k}_{t+1} = \frac{1 - \delta}{(1 + g_A + g_N)} \hat{k}_t + \frac{s}{(1 + g_A + g_N)} \hat{k}_t^\alpha,$$

in steady state we have  $\hat{k}_{t+1} = \hat{k}_t = \hat{k}$ , so

$$\hat{k}^* = \frac{1 - \delta}{(1 + g_A + g_N)} \hat{k}^* + \frac{s}{(1 + g_A + g_N)} (\hat{k}^*)^\alpha.$$

Rearranging:

$$\hat{k}^* \left[ 1 - \frac{1 - \delta}{(1 + g_A + g_N)} \right] = \frac{s}{(1 + g_A + g_N)} (\hat{k}^*)^\alpha.$$

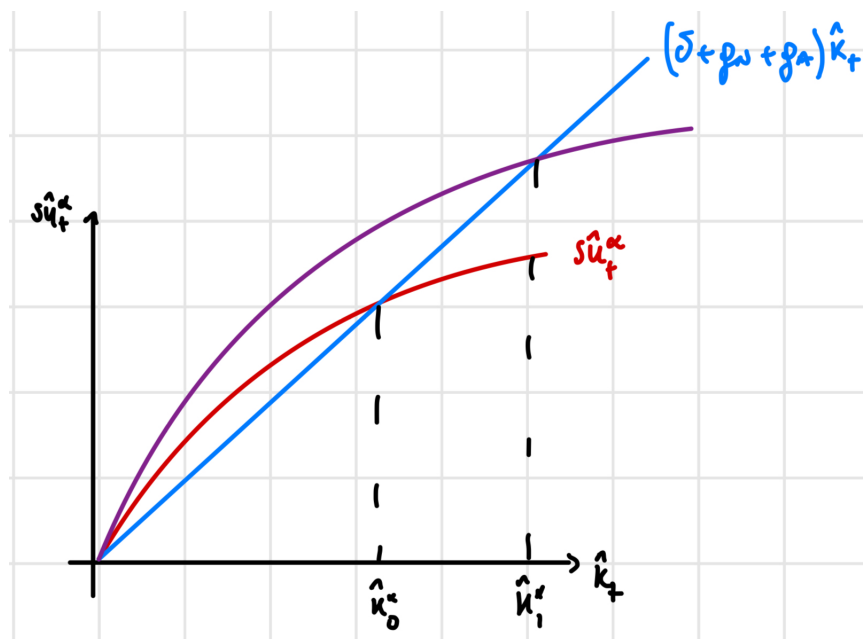
$$\hat{k}^* \frac{\delta + g_A + g_N}{(1 + g_A + g_N)} = \frac{s}{(1 + g_A + g_N)} (\hat{k}^*)^\alpha.$$

$$(\delta + g_A + g_N) \hat{k} = s (\hat{k})^\alpha.$$

Therefore,

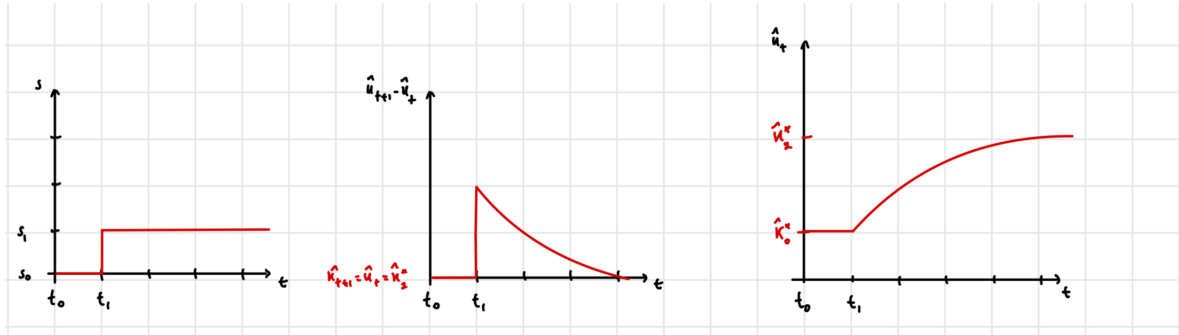
$$\hat{k}^* = \left( \frac{s_0}{\delta + g_A + g_N} \right)^{\frac{1}{1-\alpha}}$$

which is the steady-state level of capital per effective worker.



An increase in the saving rate from  $s_0$  to  $s_1 > s_0$  shifts the investment curve  $s\hat{k}_t^\alpha$  upward. The break-even line  $(\delta + g_N + g_A)\hat{k}_t$  is unchanged. As a result, the steady-state capital per effective worker rises from  $\hat{k}_0$  to  $\hat{k}_1$ . Immediately after the increase, the economy is still at  $\hat{k}_0$ , where investment exceeds break-even investment, so  $\hat{k}_t$  starts to grow over time until it converges to the new steady state  $\hat{k}_1$ .

3. Suppose again (as in point 2.) that before  $t_1$  the economy was on its balanced growth path with saving rate  $s_0$ , while after  $t_1$  the saving rate remains permanently at  $s_1$ , with  $s_1 > s_0$ . Draw 3 plots representing time on the x-axis and on the y-axis the evolution of  $s$ ,  $\hat{k}_{t+1} - \hat{k}_t$ , and  $\hat{k}_t$ , before and after  $t_1$ . Comment on them.



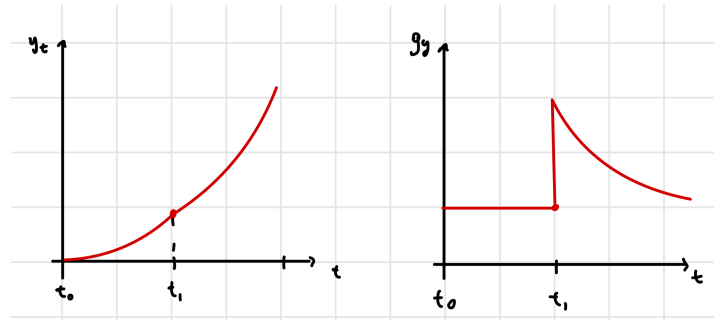
The three plots illustrate the adjustment dynamics after a permanent increase of the saving rate from  $s_0$  to  $s_1 > s_0$  at time  $t_1$ :

- **Left panel:** the saving rate  $s$  is constant at  $s_0$  up to  $t_1$ , then it jumps permanently to the higher value  $s_1$ .
- **Middle panel:** before  $t_1$ , the economy is in steady state so  $\hat{k}_{t+1} - \hat{k}_t = 0$ . Right after  $t_1$ , with the new higher saving rate, investment exceeds break-even investment and  $\hat{k}_{t+1} - \hat{k}_t > 0$ . Over time the gap declines as  $\hat{k}_t$  approaches the new steady state, and the difference goes asymptotically to zero when

$$\hat{k}_{t+1} = \hat{k}_t = \hat{k}_1^*.$$

- **Right panel:** capital per effective worker  $\hat{k}_t$  is constant at  $\hat{k}_0^*$  up to  $t_1$ . After the increase in  $s$ ,  $\hat{k}_t$  starts growing and gradually converges to the higher steady-state level  $\hat{k}_1^*$ .

4. Now, consider the behaviour of output per worker  $Y_t/N_t$ . Draw 2 plots representing time on the x-axis and on the y-axis the evolution before and after  $t_1$  of output per worker and its growth rate.



The output per worker:

$$y_t = Y_t/N_t = A_t \hat{y}_t = A_t \hat{k}_t^\alpha$$

Baseline steady state before time 1 has constant  $\hat{k}_t^\alpha$ , thus the growth rate is given only by  $g_A$ . After  $s_0 \rightarrow s_1$ , also  $\hat{k}_t^\alpha$  grows making the curve steeper but at a decreasing rate since  $\hat{k}_t^\alpha$  grows increasingly slowly towards asymptotically the new steady state  $\hat{k}_1^*$ . This pattern is captured also by graph 2 (growth rate of output per worker).

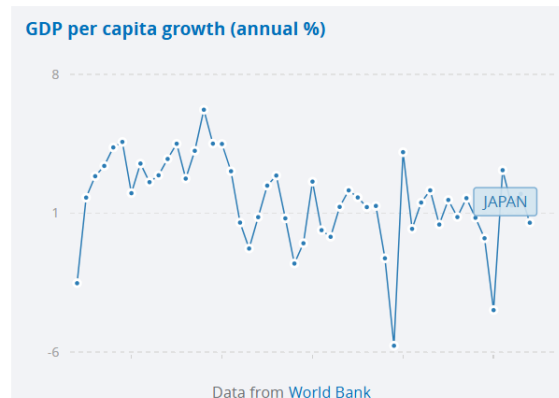
## Exercise 4 - High growth vs low growth countries

In examining a cross-country data set with information on the growth rates of real GDP- per-capita over the last fifty years, one can identify one set of countries (call them “low growth”) which experience an average growth rate of 2% per annum, this set includes the US and most Western European Countries. The data reveal another set of countries (call them “high growth”) which experience an average growth rate of 7% per annum: this set includes, for example, China and India.

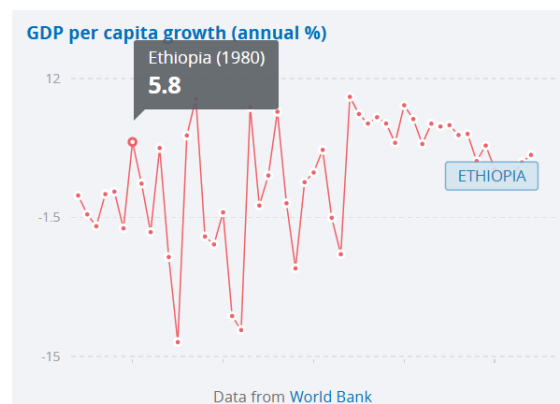
1. Add (at least) one country to both “low growth” and “high growth” groups.

We can enrich the two groups of countries as follows:

- **Low growth group (around 2% per annum):** United States, most Western European Countries, **Japan**.



- **High growth group (around 7% per annum):** China, India, **Ethiopia**.



2. Use the lessons learned from the standard Solow model to provide two reasons that could explain the observed growth rates of these two groups of countries, and their differences.

According to the Solow model, two key factors can explain the differences:

- *Savings and investment rates:* High-growth countries (e.g. China, India, Ethiopia) typically have higher savings and investment rates, accelerating capital accumulation and raising the growth rate during transition to a higher steady state.
- *Conditional convergence:* Countries starting from lower capital grow faster since they are further from their steady state, thus experiencing rapid productivity growth, while low-growth

countries (e.g. US, Japan, Germany) are already close to the technological frontier and converge to their steady states with lower growth.

3. Do you think that the average growth rates of the two sets of countries will look more similar or dissimilar in another fifty years? Why?

The Solow model predicts that in the long run, growth rates are expected to become more similar, converging towards the rate of technological progress  $g_A$ . High-growth countries will gradually slow down as they approach their new steady states, while low-growth countries will continue to grow at the frontier rate of technological progress.

## Exercise 5 - Endogenous growth

Assume the production function of the economy takes the following form:

$$Y = F(K, N) = AK^\beta N^{1-\beta} + BK^\alpha N^{1-\alpha},$$

with  $A > 0$ ,  $B > 0$ ,  $\beta \in (0, 1)$ ,  $\alpha \in (0, 1)$ .

1. Write down the production function in per capita terms,  $f(k)$ , where  $k$  denotes as usual  $k = K/N$ .

Given:

$$Y = AK^\beta N^{1-\beta} + BK^\alpha N^{1-\alpha},$$

with  $A, B > 0$  and  $\beta, \alpha \in (0, 1)$ , define  $k \equiv K/N$  and  $y \equiv Y/N$ . Then

$$f(k) \equiv \frac{Y}{N} = A \frac{K^\beta N^{1-\beta}}{N} + B \frac{K^\alpha N^{1-\alpha}}{N} = AK^\beta N^{-\beta} + BK^\alpha N^{-\alpha}.$$

Substituting  $k = K/N$  yields

$$\boxed{f(k) = Ak^\beta + Bk^\alpha}.$$

2. Compute the first derivative of  $f(k)$  with respect to  $k$ . Is it positive or negative?

Given  $f(k) = Ak^\beta + Bk^\alpha$  with  $A, B > 0$  and  $\beta, \alpha \in (0, 1)$ ,

$$f'(k) = \frac{d}{dk}(Ak^\beta + Bk^\alpha) = A\beta k^{\beta-1} + B\alpha k^{\alpha-1}.$$

$$f'(k) = A\beta k^{\beta-1} + B\alpha k^{\alpha-1}$$

Since  $k > 0$ ,  $A, B > 0$ , and  $\beta, \alpha \in (0, 1)$ , each term is positive, hence

$$f'(k) > 0 \quad \text{for all } k > 0, \text{ respecting the properties of standard production functions.}$$

3. Compute the second derivative of  $f(k)$  with respect to  $k$ . Is it positive or negative?

Given  $f(k) = Ak^\beta + Bk^\alpha$  with  $A, B > 0$  and  $\beta, \alpha \in (0, 1)$ ,

$$f''(k) = \frac{d^2}{dk^2}(Ak^\beta + Bk^\alpha) = A\beta(\beta-1)k^{\beta-2} + B\alpha(\alpha-1)k^{\alpha-2}.$$

$$f''(k) = A\beta(\beta-1)k^{\beta-2} + B\alpha(\alpha-1)k^{\alpha-2}$$

Since  $\beta-1 < 0$  and  $\alpha-1 < 0$ , for  $k > 0$  we have

$$f''(k) < 0$$

the per-capita production function is concave (in line with the diminishing marginal returns to  $k$  property of standard production functions).

4. Imagine that we are in a situation in which the saving rate (as a fraction of output) is exogenous and equal to  $s$ . Depreciation rate equals  $\delta$  and population growth rate equals  $n$ . Time is discrete. Write down the law of motion of capital per capita.

$$K_{t+1} = (1 - \delta)K_t + sY_t, \quad N_{t+1} = (1 + n)N_t,$$

with  $y_t \equiv Y_t/N_t = Ak_t^\beta + Bk_t^\alpha$  and  $k_t \equiv K_t/N_t$ .

Dividing  $K_{t+1}$  by  $N_{t+1} = (1+n)N_t$ :

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{(1-\delta)K_t + sY_t}{(1+n)N_t} = \frac{1-\delta}{1+n}k_t + \frac{s}{1+n}y_t.$$

Substituting  $y_t = Ak_t^\beta + Bk_t^\alpha$ :

$$\boxed{k_{t+1} = \frac{1-\delta}{1+n}k_t + \frac{s}{1+n}\left(Ak_t^\beta + Bk_t^\alpha\right)}.$$

5. Use this law of motion to get an expression of the growth rate of capital per capita at a given point in time  $t$ :

$$\gamma_{kt} = \frac{k_{t+1} - k_t}{k_t}.$$

Draw a diagram to describe  $\gamma_{kt}$  as a function of  $k_t$ .

From the law of motion:

$$k_{t+1} = \frac{1-\delta}{1+n}k_t + \frac{s}{1+n}\left(Ak_t^\beta + Bk_t^\alpha\right),$$

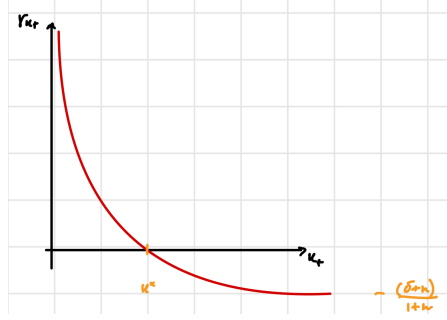
the growth rate is

$$\gamma_{kt} \equiv \frac{k_{t+1} - k_t}{k_t} = \left(\frac{1-\delta}{1+n} - 1\right) + \frac{s}{1+n}\left(Ak_t^{\beta-1} + Bk_t^{\alpha-1}\right).$$

Since  $\frac{1-\delta}{1+n} - 1 = \frac{-(\delta+n)}{1+n}$ , we get

$$\boxed{\gamma_{kt} = \frac{1}{1+n}\left[s(Ak_t^{\beta-1} + Bk_t^{\alpha-1}) - (\delta+n)\right]}.$$

- $\gamma_{kt} = 0 \iff k_t = k^*$  (steady state).
- As  $k_t \rightarrow 0^+$ :  $k_t^{\beta-1}, k_t^{\alpha-1} \rightarrow +\infty$  (since  $\beta, \alpha \in (0, 1)$ )  $\Rightarrow \gamma_{kt} \rightarrow +\infty$ .
- As  $k_t \rightarrow +\infty$ :  $k_t^{\beta-1}, k_t^{\alpha-1} \rightarrow 0 \Rightarrow \gamma_{kt} \rightarrow -\frac{\delta+n}{1+n} < 0$ .



6. Does this model generate endogenous growth? Why? If yes, explain how you could modify the parameters so that the model does not generate endogenous growth. If not, explain how you could modify the parameters so that the model does generate endogenous growth.

From

$$\gamma_{kt} = \frac{1}{1+n} \left[ s(Ak_t^{\beta-1} + Bk_t^{\alpha-1}) - (\delta + n) \right],$$

at baseline  $\alpha, \beta \in (0, 1)$ .

$$\lim_{k \rightarrow \infty} \gamma_{kt} = -\frac{\delta + n}{1+n} < 0$$

hence, no endogenous growth of capital per capita.

If  $\beta = 1$  (or  $\alpha = 1$ ), then

$$\lim_{k \rightarrow \infty} \gamma_{kt} = \frac{sA - (\delta + n)}{1+n} \quad \left( \text{or } \frac{sB - (\delta + n)}{1+n} \right).$$

$$\text{Endogenous growth if } sA > \delta + n \text{ (or } sB > \delta + n \text{)}.$$