Macroeconomics I

Solutions To Problem Set Three

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Exercise 1. Numerical methods: a cake eating problem with taste shocks.

Utility. Period utility is log-utility:

$$u(c_t, z_t) = z_t \log c_t.$$

Taste shocks. Taste shocks z_t follow a first-order Markov chain and take $N = 3$ discrete values $[1 - \gamma/2,\ 1,\ 1 + \gamma/2]$. The transition probability matrix is given by:

$$\mathcal{P} = \begin{bmatrix} \rho & 1-\rho & 0 \\ (1-\rho)/2 & \rho & (1-\rho)/2 \\ 0 & 1-\rho & \rho \end{bmatrix}.$$

Timing. At the beginning of each period, the taste shifter is realized, and then the household decides how much cake to eat and how much to save for next period. The future is discounted at rate $\beta \in (0, 1)$.

1. Write the cake-eating problem in recursive form. Clearly identify the state(s), the control(s), the laws of motion for the state(s).

The value function is:

$$V^*(k_0) = \max_{c_t, k_{t+1}} \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t (z_t \log c_t)] \quad \text{s.t.} \quad k_{t+1} = k_t - c_t \tag{1}$$

To write the stochastic dynamic problem in recursive form we need to identify the states of the system, the controls and the law of motions:

a. **States**: k, z

b. **Controls**: k', c

c. **Law of motions**:

- $k' = k' = k - c$
- $z' \sim P(z_t, \cdot)$

The Bellman form of the problem can be written as:

$$V(k,z) = \max_{k',c}\{[z \log c] + \beta \mathbb{E}[V(k',z')]\} \quad \text{s.t} \quad k' = k - c \quad z' \sim P(z,\cdot) \tag{2}$$

And therefore:

$$V(k,z_i) = \max_{k',c}\{[z_i \log c] + \beta \sum_{j=1}^{3} \pi_{i,j}[V(k',z_j)]\} \quad \text{s.t} \quad k' = k - c \quad \forall i \tag{3}$$

2. Consider the code `ConsumerChoiceSimple.m` uploaded on Blackboard. This is a code which shows you how to solve by Value Function Iteration the following household problem:

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \quad \text{s.t.} \quad \begin{aligned} & c_t + a_{t+1} = Ra_t + y_t, \\ & c_t \geq 0 \quad a_{t+1} \geq \underline{a} \quad \forall t, \\ & a_0 \text{ given.} \end{aligned}$$

and $y_t \in \{0,1\}$ follows a Markov chain process with transition matrix:

$$\Pi = \begin{bmatrix} \rho & 1-\rho \\ 1-\rho & \rho \end{bmatrix}.$$

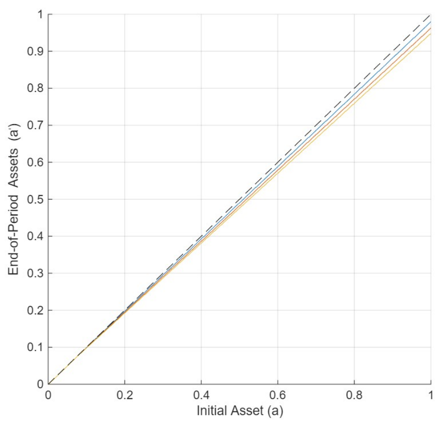
Notice that c, a and $\underline{a}$ denote, respectively, consumption, asset holdings and the exogenous borrowing limit. Furthermore, R indicates the interest rate and y income. Finally, $u(c_t)$ is a CRRA utility function.

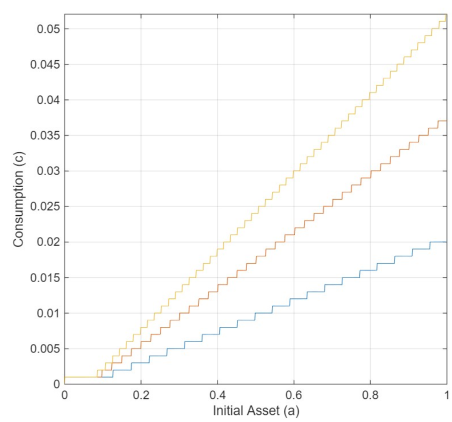
Taking as a starting point the code `ConsumerChoiceSimple.m`, write a Matlab code that solves the cake-eating problem described in this exercise by Value Function Iteration. In order to do that use the parameter values indicated in Table 1 below, and set the nodes for the endogenous state at your discretion *[Hint: think about what are plausible values for the lower bound and upper bound]*.

Table 1: Benchmark Parameters

Parameter	a_0	β	γ	ρ
Value	1	0.96	1	0.8

3. Plot the value function and the policy function(s) and interpret your results.





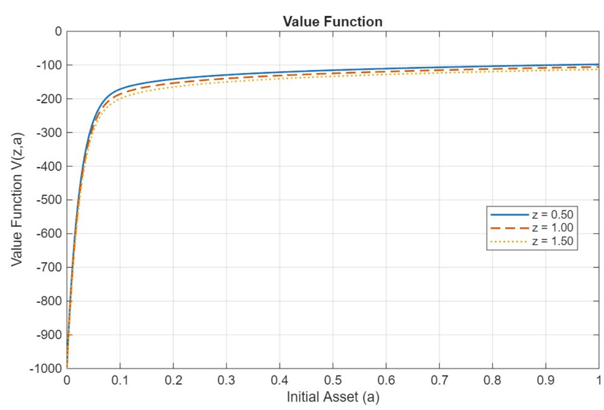


Figure one represents the three policy functions in the three states of capital accumulation, while figure two represents the evolution of the three consumption paths in the three states. When the taste shock is greater (yellow line) individuals prefer to allocate more resources towards consumption rather than savings. This explains the reasons behind the behavior of the yellow line, which tops the others in the second graph while at the same time being the lowest in the first graph.

Figure two represents the value function in the three states. It preserves the property of concavity (diminishing marginal returns) while initial cake size increases. As k_0 increases, the value function associated to a lower taste shock is higher. This can be explained by the fact that, given consumption between zero and one, its log is negative, and therefore a greater taste shock amplifies the "negativity" of such log. What matters therefore in the interpretation of the graph are the properties of the value function (such as concavity) and the relative distances of the curves, which increase when the initial size gets bigger, signaling a difference in the three states arising only with higher levels of the initial cake.

Exercise 2. Markov chains I

Consider an economy where the consumer receives an endowment which follows a stochastic process $\{y_t\}_{t=0}^{\infty}$ described by the following transition matrix

$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$$

The endowment takes value = 1 in the first state and value = 2 in the second state.

1. Compute the stationary distribution(s) associated to this Markov chain. Is it unique?

The stationary distribution p associated to the Markov chain is unique because all the entries in the matrix are strictly higher than zero. p is such that $p = p\Pi$, therefore:

$$[p_1 \;\; p_2] \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} = [p_1 \;\; p_2] \tag{4}$$

Solving the system:

$$\begin{cases} p_1 \cdot 0.3 + p_2 \cdot 0.6 = & p_1 \\ p_1 + p_2 = & 1 \end{cases} \tag{5}$$

The result is:

$$[p_1 = 0.46 \ ; \ p_2 = 0.54] \tag{6}$$

2. Assume that the initial endowment is $y_0 = 1$. What is the conditional probability of observing $y_1 = 2$?

The transition matrix tells us on the rows the origin state and on the columns the destination state. Each entry, therefore, can be interpreted as:

$$\pi_{ij} = \Pr\{z' = z_j | z = z_i\} \tag{7}$$

As a result, the second entry of the matrix tells us exactly the probability of y taking value 2 at time 1 conditional on it taking value 1 at time 0, i.e. 0.7.

3. Compute the matrix Π^2. What is the probability of observing $y_2 = 2$ and $y_1 = 2$ given that $y_0 = 1$?

$$\Pi^2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.51 & 0.49 \\ 0.42 & 0.58 \end{bmatrix}$$

Given $y_0 = 1$, the probability of observing $y_1 = 2$, as stated in exercise 2.2, is 0.7. Given that $y_1 = 2$, the probabiity of observing $y_2 = 2$ is given by the fourth entry of Π^2, i.e. 0.58. Finally, the probability of both events happening is the product between them, i.e. 0.41.

Exercise 3. Markov chains II

Consider a Markov chain with a transition matrix

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix}.$$

A stationary distribution is hence a vector $(q^*, 1-q^*)$ with $0 \leq q^* \leq 1$ such that

$$[q^* \;;\; 1-q^*]\,\Pi = [q^* \;;\; 1-q^*].$$

We know that at least one such q^* must exist.

1. Show that q^* solves $(2-\pi_{22}-\pi_{11})q^* = (1-\pi_{22})$ and discuss conditions under which q^* might take multiple values.

$$[q^* \;;\; (1-q^*)] \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix} = [q^* \;;\; (1-q^*)] \tag{8}$$

$$\begin{cases} \pi_{11}q^* + (1-q^*)(1-\pi_{22}) & = q^* \\ q^* + (1-q^*) = 1 \end{cases}$$

Solving the first equation of the system:

$$(1-\pi_{22}) = 2q^* - q^*\pi_{11} - q^*\pi_{22} \implies (1-\pi_{22}) = (2-\pi_{11}-\pi_{22})q^* \tag{9}$$

If all the entries of the transition matrix Π are strictly higher than zero, the stationary distribution is unique. However, if $2-\pi_{11}-\pi_{22} = 0 \implies \pi_{11} = \pi_{22} = 1$, then all possible values of $q^* \in [0,1]$ can solve the stationary distribution condition.

2. Now set $\pi_{11} = \pi_{22} = \pi$ and state conditions for q^* to be unique.

For q^* to be unique is enough to have $\pi \in (0,1)$. If the condition is respected, then $(1-\pi) = (2-2\pi)q^* \implies q^* = \frac{1}{2}$ and the stationary distribution $[\frac{1}{2}; \frac{1}{2}]$ is unique.