Macroeconomics I

Solutions To Problem Set Five

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Exercise 1. The price-dividend ratio in a CCAPM model.

Consider a pure endowment economy, where the representative agent makes his choices regarding consumption (c) and savings into risk-free bonds (b) and stocks (s) solving the following problem:

$$\max_{\{c_t, b_{t+1}, s_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + q_t b_{t+1} + p_t s_{t+1} \leq (d_t + p_t) s_t + b_t \quad \forall t, \qquad c_t \geq 0 \quad b_0, s_0 \text{ given}$$

together with No-Ponzi game conditions for the two assets. The term q_t is the price of the risk-free bond, p_t is the stock price, and d_t denotes the stock dividends, which are assumed to follow the exogenous stochastic process:

$$d_{t+1} = d_t \, \varepsilon_{t+1}, \qquad \log \varepsilon_t \sim \mathcal{N}\left(-\frac{\sigma_\varepsilon^2}{2}, \, \sigma_\varepsilon^2\right).$$

1. Derive the Euler equations describing the equilibrium prices in the bond and the stock markets, as well as the market clearing conditions.

To compute the Euler equations, it is necessary to compute the first order conditions of the problem:

$$\begin{aligned}
\frac{\partial V}{\partial c_t} = 0 &\implies \beta^t c_t^{-\sigma} = \lambda_t \\
\frac{\partial V}{\partial b_{t+1}} = 0 &\implies \lambda_t q_t = \mathbb{E}_t[\lambda_{t+1}] \\
\frac{\partial V}{\partial s_{t+1}} = 0 &\implies \lambda_t p_t = \mathbb{E}_t[\lambda_{t+1}(d_{t+1} + p_{t+1})]
\end{aligned}$$

Rearranging we obtain ($q_t = \frac{1}{1+r_t}$ and $\lambda_{t+1} = \beta^{t+1}\mathbb{E}[c_{t+1}^{-\sigma}]$):

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}_t[c_{t+1}^{-\sigma}] \tag{1}$$

$$p_t = \mathbb{E}_t[(d_{t+1}+p_{t+1})\frac{c_{t+1}^{-\sigma}\beta}{c_t^{-\sigma}}] \tag{2}$$

where Equation 1 represents the classic intertemporal EE for risk-free assets and Equation 2 represents the pricing condition in the stock market, where $\frac{c_{t+1}^{-\sigma}\beta}{c_t^{-\sigma}}$ is the SDF_t.

The **market clearing conditions** are:

- Goods' market: $c_t = d_t$
- Equity market: $s_t = 1$
- Bonds' market: $b_t = 0$

2. Show that in an equilibrium with no bubbles, it must be that the price dividend ratio p_t/d_t is constant. [HINT: you can use the fact that that if a random variable X follows a log-normal distribution, then $E(X) = E(e^x) = e^{E(x)+\frac{1}{2}\mathrm{Var}(x)}$.]

From the stocks pricing condition:

$$p_t = \beta\mathbb{E}_t[(p_{t+1}+d_{t+1})(\frac{d_{t+1}}{d_t})^{-\sigma}] \quad \Rightarrow \quad \frac{p_t}{d_t} = \beta\mathbb{E}_t\left[\left(1+\frac{p_{t+1}}{d_{t+1}}\right)\left(\frac{d_{t+1}}{d_t}\right)^{1-\sigma}\right]$$

We can rename the following variables:

$$x_t = \frac{p_t}{d_t}, \quad G_{t+1} = \frac{d_{t+1}}{d_t} \quad \Rightarrow$$

$$x_t = \beta\mathbb{E}_t[(1+x_{t+1})G_{t+1}^{1-\sigma}] \tag{3}$$

Using the hint to solve $\mathbb{E}[G^{1-\sigma}]$:

$$\mathbb{E}[G^{1-\sigma}] = \mathbb{E}[\varepsilon^{1-\sigma}] = \exp\left((1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\sigma_\varepsilon^2\right)$$

$$\mu = -\frac{1}{2}\sigma_\varepsilon^2 \quad \Rightarrow \quad \mathbb{E}[G^{1-\sigma}] = \exp\left(\frac{1}{2}\sigma_\varepsilon^2(\sigma^2 - \sigma)\right)$$

Iterating forward Equation 3:

$$\implies x_t = \sum_{j=1}^{T} \beta^j \, \mathbb{E}_t\Big[\prod_{k=1}^{j} G_{t+k}^{1-\sigma}\Big] + \beta^T \, \mathbb{E}_t\Big[\prod_{k=1}^{T} G_{t+k}^{1-\sigma} \, x_{t+T}\Big]$$

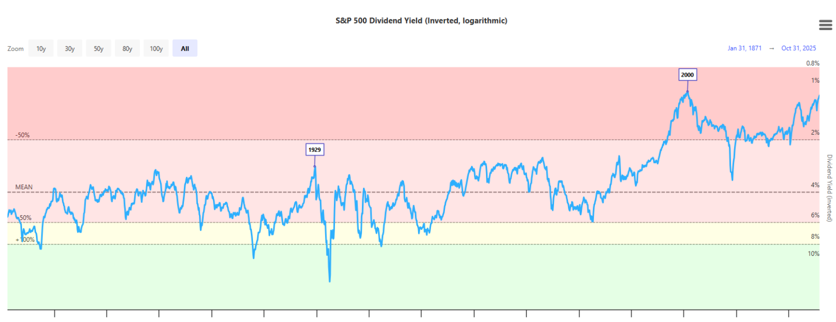
Imposing the no-bubble condition means that the last term is zero in the limit. Therefore:

$$A = \beta \exp\left(\frac{1}{2}\sigma_\varepsilon^2(\sigma^2 - \sigma)\right)$$

$$x_t = \sum_{j=1}^{\infty} A^j = \frac{A}{1-A} \quad \Rightarrow \quad \frac{p_t}{d_t} = \frac{\beta \exp\big(\frac{1}{2}\sigma_\varepsilon^2(\sigma^2 - \sigma)\big)}{1 - \beta \exp\big(\frac{1}{2}\sigma_\varepsilon^2(\sigma^2 - \sigma)\big)}$$

which does not depend on time.

3. Report a plot representing the historical evolution of the S&P500 price-dividend ratio. Do you think that this model can rationalize the observed volatility of prices and dividends? Why (not)?



The dividend yield is the reciprocal of the price-dividend ratio. As shown, the theory seems not to predict correctly the volatility in prices. An explanation could be time-varying discount rates and expected returns. If the SDF varies over time, for example due to **Epstein–Zin preferences** (separating risk aversion and EIS), prices can swing a lot even when dividends don't.

Exercise 2. Investor's optimality conditions

Consider an infinitely lived investor whose preferences for consumption at time t are represented by the utility function $u(c_t)$ and whose discount factor is $\beta \in (0,1)$. Assume that she can buy an asset whose price is p_t which provides the dividend stream $\{d_t\}_{t=0}^{\infty}$. The first order conditions of the investor's problem can be written as:

$$p_t = E_t\left[\sum_{j=1}^{\infty} \beta^j \, \frac{u'(c_{t+j})}{u'(c_t)} \, d_{t+j}\right] \quad \forall t \tag{4}$$

Instead, the first-order conditions for buying a security with price p_t and payoff $x_{t+1} = d_{t+1} + p_{t+1}$ are:

$$p_t = E_t\left[\beta \, \frac{u'(c_{t+1})}{u'(c_t)} \, (p_{t+1} + d_{t+1})\right] \quad \forall t \tag{5}$$

1. Derive (5) from (4).

We start by lagging Equation 4 one period and taking expectations at time t:

$$\begin{aligned}
\mathbb{E}_t[p_{t+1}] &= \mathbb{E}_t\left[\mathbb{E}_{t+1}\left(\sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+1+j})}{u'(c_{t+1})}\, d_{t+1+j}\right)\right] \\
&= \mathbb{E}_t\left[\beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\, d_{t+2} + \beta^2 \frac{u'(c_{t+3})}{u'(c_{t+1})}\, d_{t+3} + \cdots\right]
\end{aligned}$$

Multiply both sides by the SDF_t:

$$\begin{aligned}
\mathbb{E}_t\left[\beta \, \frac{u'(c_{t+1})}{u'(c_t)}\, p_{t+1}\right] &= \mathbb{E}_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \left(\beta \frac{u'(c_{t+2})}{u'(c_{t+1})}\, d_{t+2} + \beta^2 \frac{u'(c_{t+3})}{u'(c_{t+1})}\, d_{t+3} + \cdots\right)\right] \\
&= \mathbb{E}_t\left[\beta^2 \frac{u'(c_{t+2})}{u'(c_t)}\, d_{t+2} + \beta^3 \frac{u'(c_{t+3})}{u'(c_t)}\, d_{t+3} + \cdots\right]
\end{aligned}$$

Finally, add $\mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}\right]$ on both sides:

$$\begin{aligned}
\mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}\right]+\mathbb{E}_t\left[\beta\;\frac{u'(c_{t+1})}{u'(c_t)}\,p_{t+1}\right] &= \mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}+\beta^2\frac{u'(c_{t+2})}{u'(c_t)}\,d_{t+2}+\cdots\right] \\
\mathbb{E}_t\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}\,(p_{t+1}+d_{t+1})\right] &= \mathbb{E}_t\left[\sum_{j=1}^{\infty}\beta^j\,\frac{u'(c_{t+j})}{u'(c_t)}\,d_{t+j}\right] \\
\mathbb{E}_t\left[\beta\;\frac{u'(c_{t+1})}{u'(c_t)}\,(p_{t+1}+d_{t+1})\right] &= p_t\,.
\end{aligned}$$

2. Derive (4) from (5). In order to do that you need to state an extra condition. Show that this extra condition is a first-order condition for maximization. To do this, think about what strategy the consumer could follow to improve her utility if the condition did not hold.

Start from Equation 5 and plug its lagged expression in place of p_{t+1}:

$$\begin{aligned}
p_t &= \mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,(p_{t+1}+d_{t+1})\right] \\
&= \mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\left(\mathbb{E}_{t+1}\left[\beta\,\frac{u'(c_{t+2})}{u'(c_{t+1})}\,(p_{t+2}+d_{t+2})\right]+d_{t+1}\right)\right] \\
&= \mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}\right]+\mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,\mathbb{E}_{t+1}\left[\beta\,\frac{u'(c_{t+2})}{u'(c_{t+1})}\,(p_{t+2}+d_{t+2})\right]\right] \\
&= \mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}\right]+\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,\beta\,\frac{u'(c_{t+2})}{u'(c_{t+1})}\,(p_{t+2}+d_{t+2})\right]\right].
\end{aligned}$$

By LIE the second expected value becomes:

$$\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,\beta\,\frac{u'(c_{t+2})}{u'(c_{t+1})}\,(p_{t+2}+d_{t+2})\right]\right]=\mathbb{E}_t\left[\beta^2\frac{u'(c_{t+2})}{u'(c_t)}\,p_{t+2}\right]+\mathbb{E}_t\left[\beta^2\frac{u'(c_{t+2})}{u'(c_t)}\,d_{t+2}\right].$$

Iterating forward T times:

$$\mathbb{E}_t\left[\beta\,\frac{u'(c_{t+1})}{u'(c_t)}\,d_{t+1}+\beta^2\frac{u'(c_{t+2})}{u'(c_t)}\,d_{t+2}+...+\beta^T\frac{u'(c_{t+T})}{u'(c_t)}\,d_{t+T}\right]+\mathbb{E}_t\left[\beta^{T+1}\frac{u'(c_{t+T+1})}{u'(c_t)}\,p_{t+T+1}\right]$$

Therefore, the condition to impose to retrieve (4) from (5) is:

$$\lim_{T} \mathbb{E}_t \left[\beta^{T+1} \frac{u'(c_{t+T+1})}{u'(c_t)} \, p_{t+T+1} \right] \to 0 \tag{6}$$

The condition is a **FOC for maximisation**. To show this, let us suppose by contradiction that the condition does not hold. The agent is therefore allowed to:

- Sell a small quantity of stocks $p_t \varepsilon$ today
- With $p_t \varepsilon$ you buy a state-contingent portfolio that pays εd_{t+1}, εd_{t+2}, ... in every state, so your future consumption remains identical to what it would be if you held the stock.
- You are left with $\varepsilon B_t = \varepsilon(p_t - p_t^F) > 0$ to consume today; this increases utility, yielding a contradiction with optimality if $B_t > 0$.

Exercise 3. The Neoclassical Theory of Investment

Consider the neoclassical model of investment:

$$V^*(k_0) = \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (z_t f(k_t) - p_t i_t)$$

s.t.

$$k_{t+1} = (1-\delta) k_t + i_t, \qquad k_{t+1} \geq 0, \qquad k_0 \text{ given}$$

with $r > 0$. Notice that $f(k_t)$ indicates the production technology of the firm ($f(\cdot)$ is assumed to be increasing and concave), z_t denotes the (non-stochastic) technology level and p_t the price of a unit of investment good.

1. Derive the Euler equation and show that it can be re-written as:

$$k^*_{t+1} = (f')^{-1} \left(\frac{p_t(1+r) - p_{t+1}(1-\delta)}{z_{t+1}} \right).$$

Plugging in the law of motion of capital in the value function:

$$\max_{\{k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t (z_t f(k_t) - p_t[k_{t+1} - (1-\delta)k_t])$$

Taking the first order conditions:

$$\begin{aligned}
&\frac{\partial V^*}{\partial k_{t+1}} = \left(\frac{1}{1+r}\right)^{t+1} z_{t+1} f'(k_{t+1}) + (1-\delta)\left(\frac{1}{1+r}\right)^{t+1} p_{t+1} = \left(\frac{1}{1+r}\right)^t p_t \\
&\implies p_t = \frac{1}{1+r}\Big(z_{t+1} f'(k_{t+1}) + (1-\delta) p_{t+1}\Big) \\
&\implies z_{t+1} f'(k_{t+1}) = p_t(1+r) - p_{t+1}(1-\delta) \\
&\implies k^*_{t+1} = (f')^{-1}\left(\frac{p_t(1+r) - p_{t+1}(1-\delta)}{z_{t+1}}\right).
\end{aligned}$$

2. Explain intuitively, in economic terms, why investment is increasing in z_{t+1} and p_{t+1}, and decreasing in p_t, r, and δ.

Investment increases in:

- **Future productivity.** An increase in z_{t+1} increases expected profits since employing the same means of production as before, future total production will go up. This induces the firm to invest in more capital today expecting an expansion of its production and profits tomorrow.

- **Future prices.** An increase in p_{t+1} signals future higher value of capital. Therefore, the firm adjusts inter-temporally increasing its stock of capital today.

Investment decreases in:

- **Present prices.** An opposite effect with respect to the latter described. The value of capital increases today, therefore it becomes more expensive to employ capital in production today with respect to tomorrow.

- **Interest rate.** An increase in the interest rate reduces the value of future discounter profits, therefore the firm has a higher opportunity cost investing today with respect to tomorrow.

- **Depreciation rate.** A similar mechanism with respect to the interest rate influences the choice of the firm, except that now the shock comes from a higher depreciation of capital. A higher δ makes today's capital more desirable than tomorrow, thus decreasing the investment left by the firm.

Exercise 4. The Neoclassical vs q-Theory of Investment

Consider the neoclassical model of investment:

$$V^*(k_0) \;=\; \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \Pi_t \right]$$

s.t.

$$\Pi_t \;=\; z_t f(k_t) - p_t i_t, \qquad k_{t+1} = (1-\delta)k_t + i_t, \qquad k_{t+1} \geq 0, \qquad k_0 \text{ given.}$$

1. Assume that $p_t = p$ is constant for all t, but productivity $z \in \{z_h, z_l\}$ follows a Markov chain process with transition matrix:

$$\Pi \;=\; \begin{pmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{pmatrix}.$$

   State the problem in recursive form, carefully specifying what are the control(s), the state(s) of the dynamic problem and the law(s) of motion for the state(s).

The elements to construct the Bellman are:

- **States**: k, z
- **Controls**: i, k'
- **Laws of motion**:
  a. $k' = (1-\delta)k + i$
  b. $z' \sim \Pi(z, \cdot)$

Therefore, the Bellman equation is:

$$V(k,z) = \Pi_t + \left(\frac{1}{1+r}\right)\mathbb{E}[V(k',z')\,|\,z] \tag{7}$$

$$\text{s.t.} \begin{cases} \Pi_t = zf(k) - pi, \\ k' = (1-\delta)k + i \end{cases}$$

Which is equivalent to:

$$V(z_i,k) = [z_i f(k) - pi] + \left(\frac{1}{1+r}\right)\sum_j \pi_{ij}\, V(z_j',k') \quad \forall i \qquad \text{s.t. } k' = (1-\delta)k + i \tag{8}$$

2. Compute the first-order condition with respect to $k^{'}$ and the envelope condition. By rearranging terms, you should get the Euler equation for this problem. Provide the economic intuition behind the Euler equation. What is the optimal policy function $k^{'} = g(k,z)$ as a function of the parameters and $\mathbb{E}[z^{'}|z]$?

We can derive the first-order conditions from Equation 8 plugging in the law of motion for capital. The FOCs are:

$$\frac{\partial V}{\partial k'} = \frac{1}{1+r}\mathbb{E}[\frac{\partial V}{\partial k'}|z] = p$$
$$\text{Envelope} = \mathbb{E}[\frac{\partial V}{\partial k'}|z] = \mathbb{E}[z'\partial f(k')|z] + p(1-\delta)$$

Therefore the Euler equation is:

$$\mathbb{E}[z'\partial f(k')|z] = p(r+\delta) \tag{9}$$

The economic interpretation of Equation 9 is immediate: the left hand side, i.e. the **expected marginal revenues** of the firm, must equate **expected marginal costs**, which comprise a direct component of the price of investment and an indirect component (the interest rate and the depreciation rate).

Once obtained the FOC, the optimal policy function can be retrieved through the inverse function of the derivative of f (which is increasing, concave and has the Inada conditions as a standard neoclassical production function):

$$\begin{aligned}\mathbb{E}[z'f'(k') \mid z] &= p(r+\delta)\\ f'(k')\,\mathbb{E}[z' \mid z] &= p(r+\delta)\\ f'(k') &= \frac{p(r+\delta)}{\mathbb{E}[z' \mid z]}\end{aligned}$$

$$k' \;=\; g(k,z) = (f')^{-1}\left(\frac{p(r+\delta)}{\mathbb{E}[z' \mid z]}\right) \tag{10}$$

3. Using the specified Markov process derive an expression for k'_h, the optimal choice for next period's capital when the current realization of the shock is $z = z_h$. Derive analogously k'_l.

To solve this, it is enough to render explicit the expected value in the denominator of Equation 10:

$$\mathbb{E}[z' \mid z] = \begin{pmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{pmatrix} \begin{pmatrix} z_L \\ z_H \end{pmatrix} = \begin{pmatrix} \pi z_L + (1-\pi) z_H \\ (1-\pi) z_L + \pi z_H \end{pmatrix}$$

Therefore:

$$\text{if } z_t = z_H \;\Rightarrow\; k'_H = g(k, z_H) = (f')^{-1}\left(\frac{p(r+\delta)}{(1-\pi)z_L + \pi z_H}\right) \tag{11}$$

$$\text{if } z_t = z_L \;\Rightarrow\; k'_L = g(k, z_L) = (f')^{-1}\left(\frac{p(r+\delta)}{(1-\pi)z_H + \pi z_L}\right) \tag{12}$$

4. Comment on the relationship between k'_h and k'_l depending on the values that the parameter π may assume.

The parameter $\pi \in (0,1)$ determines the relationship between k'_h and k'_l. Assuming $z_H > z_L$, if $\pi > \frac{1}{2}$, $\mathbb{E}[z' \mid z_H] > \mathbb{E}[z' \mid z_L]$, thus $k'_h > k'_l$. Conversely, if $\pi < \frac{1}{2}$, $\mathbb{E}[z' \mid z_H] < \mathbb{E}[z' \mid z_L]$, thus $k'_l > k'_h$. Lastly, if $\pi = \frac{1}{2}$, $k'_h = k'_l$

5. Now drop the assumption that $p_t = p$ for all t and assume that $\Pi_t = z_t f(k_t) - p_t \left(i_t + \frac{\gamma}{2}\left(\frac{i_t}{k_t}\right)^2\right)$.

Write down the Lagrangian and derive the FOC with respect to i_t and k_{t+1}. Comment on the difference between these FOC and the ones derived in point 2.

The Bellman equation of the stochastic problem is:

$$V(k,z) = \max_{i,k'} \left\{ zf(k) - p\Big(i + \frac{\gamma}{2}\Big(\frac{i}{k}\Big)^2\Big) + \frac{1}{1+r}\,\mathbb{E}[V(k',z')] \right\} \quad \text{s.t.} \quad k' = (1-\delta)k + i \tag{13}$$

Therefore, the Lagrangian is:

$$\mathcal{L} = zf(k) - p\Big(i + \frac{\gamma}{2}\Big(\frac{i}{k}\Big)^2\Big) + \frac{1}{1+r}\,\mathbb{E}[V(k',z')] + \lambda((1-\delta)k + i - k'). \tag{14}$$

The FOCs are (with shadow prices):

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i} = 0: &\quad -p\Big(1+\gamma\frac{i}{k^2}\Big) + \lambda = 0 \implies \lambda = p\Big(1+\gamma\frac{i}{k^2}\Big)\,, \\
\frac{\partial \mathcal{L}}{\partial k'} = 0: &\quad \frac{1}{1+r}\,\mathbb{E}[V_k(k',z')] - \lambda = 0 \implies \lambda = \frac{1}{1+r}\,\mathbb{E}[V_k(k',z')]\,.
\end{aligned}$$

Solving the Envelope in the expected value of the second FOC:

$$V_k(k,z) = zf'(k) + p\,\gamma\frac{i^2}{k^3} + (1-\delta)\lambda$$

Conclusively, the FOCs are:

$$\frac{\partial \mathcal{L}}{\partial i} = 0: \quad -p\Big(1+\gamma\frac{i}{k^2}\Big) + \lambda = 0 \implies \boxed{\lambda = p\Big(1+\gamma\frac{i}{k^2}\Big)}, \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial k'} = 0: \quad \frac{1}{1+r}\,\mathbb{E}[V_k(k',z')] - \lambda = 0 \implies \boxed{\lambda = \frac{1}{1+r}\,\mathbb{E}[z'\partial f(k') + p'\,\gamma\frac{i'^2}{k'^3} + (1-\delta)\lambda'\,]}. \tag{16}$$

There are many differences between Equations 15 and 16 and the FOCs derived in Exercise 2:

- The problem is **stochastic**, therefore we preserve the possibility for z to be z_l or z_h
- The **investment FOC** equates the value of marginal contribution of capital to profits (the

shadow price) to the marginal costs of investment, which now comprise also adjustment costs $\gamma \frac{i}{k^2}$

- The **future capital FOC** equates today's marginal cost $\lambda = p(1 + \gamma i/k^2)$ to the expected marginal benefits of one extra unit of capital (next period's marginal product $z'f'(k')$, the direct effect on adjustment costs $p'\gamma i'^2/k'^3$, and the saving of future expenditure $(1-\delta)\lambda'$).