International Trade and Globalization

Solutions To Problem Set One

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Exercise 1 - CES demand with utility weights

Consider a representative consumer with income $\mathcal{W}$ and with preferences

$$U\left(c_1, c_2, \ldots, c_I\right)=\left(\sum_{i=1}^{I}\left(\alpha_i c_i\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{1}$$

over goods numerated from 1 to $I . \alpha_1, \alpha_2, \ldots \alpha_I$ and ε are positive constants.

1. **Derive the demand equation for each good (the equivalent of Equation (6) of the LN).**

The consumer solves the following problem:

$$V(p, \mathcal{W})=\max_{c1, \ldots, c_I}\left(\sum_{i=1}^{I}\left(\alpha_i c_i\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

subject to

$$\sum_{i=1}^{I} p_i c_i=\mathcal{W}$$

Where since the utility is strictly increasing we applied Walras' law and hence the budget constraint holds with equality. Note moreover that we should impose non negativity constraints on consumption, anyway we solve first this "relaxed" problem and then check that indeed consumption is non negative for all indices. The lagrangian reads:

$$\mathcal{L}=\left(\sum_{i=1}^{I}\left(\alpha_i c_i\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}-\lambda\left(\sum_{i=1}^{I} p_i c_i-\mathcal{W}\right)$$

First order conditions equated to 0 are $\forall i \in I$:

$$\left(\sum_{i=1}^{I}\left(\alpha_i c_i\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{1}{\varepsilon-1}}\left(\alpha_i c_i\right)^{-\frac{1}{\varepsilon}} \alpha_i=\lambda p_i$$

Taking the ratio of two generic first order conditions we get that:

$$\frac{c_i}{c_j} = (\frac{p_i}{p_j})^{-\varepsilon}(\frac{\alpha_i}{\alpha_j})^{\varepsilon-1}$$

using this expression in the budget constraint we get that:

$$c_j = \frac{\mathcal{W}a_j^{\varepsilon-1}}{\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon} p_j^{\varepsilon}} = \frac{\mathcal{W}p_j^{-\varepsilon}}{\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\alpha_j\right)^{1-\varepsilon}} \tag{2}$$

2. **Comment on the differences between your result and Equation (6). In particular, how does the demand for a good $\boldsymbol{i}$ react to an increase in $\boldsymbol{\alpha_i}$? To an increase in $\boldsymbol{\alpha_j}$(with $\boldsymbol{i \neq j}$) ?**

Equation (6) of the LN reads:

$$c_n = \frac{p_n^{-\varepsilon}\mathcal{W}}{\sum_{i=1}^{I} p_i^{1-\varepsilon}}$$

The difference between the two is that now the prices are scaled on the utility weights of the consumer. the value function of the consumer is not longer $V(p, \mathcal{W})$ where p is the vector of all prices (so it depends on the good price, and the price of other good and income) but becomes $V(p, \mathcal{W}, \alpha)$, specifically now depends also on the relative utility weight the consumer assign to that specific good. To see how the demand changes in case of a change in its own utility weight or in other utility weight we can compute the two derivatives of interest:

$$\frac{\partial c_j}{\partial a_j} = (\varepsilon-1)\frac{\mathcal{W}a_j^{\varepsilon-2}(\varepsilon-1)p_j^{-\varepsilon}}{\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}} - (\varepsilon-1)\frac{\mathcal{W}a_j^{\varepsilon-2}p_j^{-\varepsilon}(\frac{p_j}{\alpha_j})^{1-\varepsilon}}{\left[\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}\right]^2} = (\varepsilon-1)\frac{\mathcal{W}\alpha_j^{\varepsilon-2}p_j^{-\varepsilon}}{\left[\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}\right]^2}\left[\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon} - \left(\frac{p_j}{\alpha_j}\right)^{1-\varepsilon}\right] \tag{3}$$

$$\frac{\partial c_j}{\partial a_s} = (1-\varepsilon)\frac{\mathcal{W}\alpha_j^{\varepsilon-1}p_j^{-\varepsilon}\left(\frac{p_s}{\alpha_s}\right)^{1-\varepsilon}}{\alpha_s\left[\sum_{i=1}^{I}\left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}\right]^2} \tag{4}$$

Now its immediate to state that:

1. $\varepsilon > 1$ an increase in α_j will produce an increase in demand of good j and decrease consumption of good s.

2. $\varepsilon < 1$ the reverse applies.

3. **Derive an expression for the ideal price index P (the equivalent of Equation (7) of the LN).**

For the optimal price index the following must hold:

$$V(\alpha, p, \mathcal{W}) = \frac{\mathcal{W}}{P}$$

Where we have indicated with $V(\cdot)$ the value function of the consumer, hence the utility he obtains if he consume the optimal bundle. It's now trivial to state that:

$$V(\alpha, p, \mathcal{W}) = \frac{\mathcal{W}}{\sum_i^I \left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}} \cdot \left(\sum_i^I \left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

And hence it's immediate to get that:

$$P := \left[\sum_i^I \left(\frac{p_i}{\alpha_i}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

Now, assume that the representative consumer gets her income from inelastically supplying L units of labour (so $\mathcal{W} = wL$). Firms operating under perfect competition can produce any good i with the production function

$$y_i = z_i l_i \tag{5}$$

4. **Solve for the equilibrium production and consumption of each good.**

From the perfect competition we get that for each producer optimality conditions read:

$$p_i = \frac{w}{z_i}$$

Recall that from the consumer optimization problem we obtained that:

$$\frac{c_i}{c_j} = \left(\frac{p_i}{p_j}\right)^{-\varepsilon} \left(\frac{\alpha_i}{\alpha_j}\right)^{\varepsilon-1}$$

From which we can now obtain:

$$\frac{c_i}{c_j} = \left(\frac{z_i}{z_j}\right)^{\varepsilon} \left(\frac{\alpha_i}{\alpha_j}\right)^{\varepsilon-1}$$

And using the production function we get:

$$\frac{z_i l_i}{z_j l_j} = \frac{c_i}{c_j} = \left(\frac{z_i}{z_j}\right)^{\varepsilon} \left(\frac{\alpha_i}{\alpha_j}\right)^{\varepsilon-1}$$

And now we get that:

$$l_i = l_j \left(\frac{z_i \alpha_i}{z_j \alpha_j}\right)^{\varepsilon-1}$$

Summing over i and recalling the inelastic labor supply we get that:

$$l_j = L \frac{(z_j \alpha_j)^{\varepsilon-1}}{\sum_{i=1}^{I} \left(z_i \alpha_i\right)^{\varepsilon-1}}$$

$$z_j l_j = c_j = y_j = L \frac{z_j^{\varepsilon} (\alpha_j)^{\varepsilon-1}}{\sum_{i=1}^{I} \left(z_i \alpha_i\right)^{\varepsilon-1}}$$

Where the first equality comes from the market clearing condition on the individual market.

5. **Imagine that for good 1 , productivity z_1 is multiplied by 2 . What will be the consequences for production and employment in the good-1 industry, and for the relative price of good 1 with respect to good 2? Repeat the same exercise for consumer's preference for good 1, α_1. What are the differences between a**

productivity and a preference shock?

Note that the production and employment in sector i move in the same direction given a change in productivity, apply the chain rule to get that:

$$\frac{\partial y_i}{\partial z_i} = l_i + \frac{\partial l_i}{\partial z_i}$$

And again the chain rule confirms that even in a case of taste shock production and employment in a sector move in the same direction. It's also immediate to check that:

$$\text{sgn}\frac{\partial l_i}{\partial z_i} = \text{sgn}\frac{\partial l_i}{\partial \alpha_i}$$

We can now compute one of the 4 derivative, we choose labor with respect to productivity and we get that:

$$\frac{\partial l_j}{\partial z_j} = \frac{(\varepsilon - 1)(z_j\alpha_j)^{\varepsilon-2}L\alpha_j}{\left[\sum_{i=1}^{I}(z_i\alpha_i)^{\varepsilon-1}\right]^2}\left[\sum_{i=1}^{I}(z_i\alpha_i)^{\varepsilon-1} - (z_j\alpha_j)^{\varepsilon-1}\right]$$

We need to differentiate two cases:

1. $\varepsilon > 1$ then an increase in productivity in sector i increases the production and employment in that sector. Similar conclusions for the taste shock.

2. $\varepsilon < 1$ Now it's reversed. (This is the case in which goods are complements)

Note that the relative price as we stated before obey to:

$$\frac{p_i}{p_j} = \frac{z_j}{z_i}$$

Hence the relative price decreases if sector i get technological advancement (in the case in which technology doubles the relative price will decrease of 50%). Nothing happens if the shock is a taste shock. The intuition is simple: we have perfect competitive markets hence the price is fixed from there. Conclusions would drastically change in the case of monopolistic competition.

Exercise 2 - Introducing the world's simplest trade model

This exercise introduces you to the simplest model of international trade, the Armington model (developed by Paul Armington in 1969).

The world is made up by I countries, and each country can produce exactly one good. Good n (produced by country n) is produced under perfect competition, with the production function

$$y_n = z_n l_n$$

Each country has a representative consumer that inelastically supplies L_n units of labour, and has CES preferences over all I goods produced in the world, with

$$U_n = \left(\sum_{i=1}^{I} (c_{n,i})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $c_{n,i}$ is country n 's consumption of the good produced by country i. There are no trade costs or tariffs for shipping goods across countries.

1. **What is the equilibrium level of production for each good i ?**

In the model the agent does not have disutility from working (equivalently utility of leisure) hence the equilibrium level of production for each good is:

$$y_i = L_i z_i$$

2. **Using the CES demand functions, show that in equilibrium, we have**

$$\frac{\sum_{n=1}^{I} c_{n,i}}{\sum_{n=1}^{I} c_{n,i'}} = \left(\frac{p_i}{p_{i'}} \right)^{-\varepsilon}$$

for any pair of goods i, i'. Which property of the CES demand function is key for this result?

The previous expression comes from utility maximization of the consumer:

$$\max_{c_{n,i}} \left(\sum_{i=1}^{I} (c_{n,i})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

such that

$$\sum_{i}^{I} p_i c_{n,i} \leq \mathcal{W}$$

The utility function is strictly increasing hence locally non satiated and hence the budget constraint holds with equality. The lagrangian reads:

$$\mathcal{L} = \left(\sum_{i}^{I} (c_{n,i})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \cdot \left(\sum_{i}^{I} p_i c_{n,i} - \mathcal{W} \right)$$

The FOC equated to 0 are:

$$\left(\sum_{i}^{I} (c_{n,i})^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{1}{\varepsilon-1}} c_{n,i}^{-\frac{1}{\varepsilon}} = \lambda p_i$$

Now we get that:

$$\frac{c_{n,i}}{c_{n,j}} = \left(\frac{p_i}{p_j} \right)^{-\varepsilon}$$

From which we can write that:

$$c_{n,i} = \left(\frac{p_i}{p_j} \right)^{-\varepsilon} c_{n,j}$$

And hence summing over the countries we get the expression required:

$$\frac{\sum_{n=1}^{I} c_{n,i}}{\sum_{n=1}^{I} c_{n,j}} = \left(\frac{p_i}{p_j} \right)^{-\varepsilon} \tag{6}$$

The property fundamental to get this expression is the fact that each country is comprised of a RA and the agents are equal across countries and have preferences with constant elasticity of substitution across goods. Specifically eq.(6) relies on the **constant elasticity of substitution**. De facto eq. (6) is telling us that the relative consumption of good i (remember that all n countries can consume this good hence why we made the summation run over the countries) over good j response to a relative price change is constant and equal to $-\varepsilon$. To see it formally:

$$\frac{\partial}{\partial \ln \frac{p_i}{p_j}} \ln\left(\frac{\sum_{n=1}^{I} c_{n,i}}{\sum_{n=1}^{I} c_{n,j}}\right) = -\varepsilon$$

3. **Using your results from questions 1 and 2, derive an expression for the relative wage of country n with respect to country n', $\frac{w_n}{w_{n'}}$.**

Again from the perfect competition inside each country we get that $\forall n \in \{1, \dots, I\}$:

$$p_n = \frac{w_n}{z_n}$$

Now taking the ration with a generic country $n\prime$ we get that:

$$\frac{p_n}{p_{n'}} = \frac{w_n}{w_{n'}} \frac{z_{n\prime}}{z_n} \tag{7}$$

From resource constraint we have that $y_i = \sum_{n=1}^{I} c_{n,i}$ hence we get that from the utility maximization:

$$\frac{\sum_{n=1}^{I} c_{n,i}}{\sum_{n=1}^{I} c_{n,j}} = \frac{y_n}{y_n^{'}} = \frac{z_n L_n}{z_{n'} L_{n'}} = \left(\frac{p_n}{p_{n'}}\right)^{-\varepsilon} \tag{8}$$

Where the second last equality comes from point (1) Second exercise of the problem set (production technology and inelastic supply of capital). Using eq. (7) into the last equality of eq. (8) we get that:

$$\frac{w_n}{w_{n'}} = \left(\frac{z_n}{z_{n'}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \left(\frac{L_n}{L_{n'}}\right)^{-\frac{1}{\varepsilon}}$$

4. **Finally, combine these results to show that the real income of country n is given by**

$$\boldsymbol{\frac{w_n L_n}{P} = (z_n L_n)^{\frac{c-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}}}$$

where $Y \equiv \left(\sum_{i=1}^{I} (y_i)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}$ is a measure of world production.

We know that:

$$y_i = \left(\frac{p_i}{p_{i'}}\right)^{-\varepsilon} y_{i'}$$

Elevate both sides to the power of $\frac{\varepsilon-1}{\varepsilon}$ to get that:

$$y_i^{\frac{\varepsilon-1}{\varepsilon}} = \left(\frac{p_i}{p_{i'}}\right)^{1-\varepsilon} y_{i'}^{\frac{\varepsilon-1}{\varepsilon}}$$

Summing across i and noting that $i\prime$ is a constant in the summation

$$\sum_i y_i^{\frac{\varepsilon-1}{\varepsilon}} = \sum_i \left(\frac{p_i}{p_{i'}}\right)^{1-\varepsilon} y_{i'}^{\frac{\varepsilon-1}{\varepsilon}}$$

Now elevate both sides to the power of $\frac{\varepsilon}{\varepsilon-1}$ to get that:

$$Y = p_{i'}{}^{\varepsilon}\left(\frac{1}{P}\right)^{\varepsilon} y_{i'}$$

Now recall that:

$$p_{i'} = \frac{w_{n\prime}}{z_{n\prime}}$$

Hence using also the production function in country $n\prime$ we get that:

$$Y = \left(\frac{1}{P}\right)^{\varepsilon} (\frac{w_{n\prime}}{z_{n\prime}})^{\varepsilon} z_{i\prime} L_{n\prime} \longleftrightarrow Y = \left(\frac{1}{P}\right)^{\varepsilon} w_{n\prime}^{\varepsilon} z_{i\prime}^{1-\varepsilon} L_{n\prime}$$

Now multiplying the right side by$(\frac{L_{n\prime}}{L_{n\prime}})^{\varepsilon}$

$$Y = \left(\frac{1}{P}\right)^{\varepsilon} (\frac{w_{n\prime}}{z_{n\prime}})^{\varepsilon} z_{i\prime} L_{n\prime} \longleftrightarrow Y = \left(\frac{1}{P}\right)^{\varepsilon} w_{n\prime}^{\varepsilon} z_{i\prime}^{1-\varepsilon} L_{n\prime} (\frac{L_{n\prime}}{L_{n\prime}})^{\varepsilon} \rightarrow Y = \left(\frac{L_{n\prime} w_{n\prime}}{P}\right)^{\varepsilon} z_{n\prime}^{1-\varepsilon} \left(L_{n\prime}\right)^{1-\varepsilon}$$

Now as we said elevating to to the power of $\frac{1}{\varepsilon}$ and rearranging we get that:

$$\frac{w_{n\prime} L_{n\prime}}{P} = \left(L_{n\prime} z_{i\prime}\right)^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}} \tag{9}$$

5. **Show that:**

$$\boldsymbol{\sum_{n=1}^{I} \frac{w_n L_n}{P} = Y}$$

This shows that Y is equal to world (real) income

Take equation eq. (9) and sum across countries:

$$\sum_{n\prime=1}^{I} \frac{w_{n\prime} L_{n\prime}}{P} = \sum_{n\prime=1}^{I} \left(L_{n\prime} z_{i\prime} \right)^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}}$$

Now noting that $L_{n\prime} z_{i\prime} = y_{n\prime}$ we get that:

$$\sum_{n\prime=1}^{I} \frac{w_{n\prime} L_{n\prime}}{P} = \sum_{n\prime=1}^{I} \left(y_{n\prime} \right)^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}} = Y^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}} = Y$$

6. **Assume first that $\epsilon > 1$. What is the consequence of a rise in productivity z_n on the relative price of country n goods? On the (real) national income of country n? And on the share of country n in world income?**

The relative price is given by:

$$\frac{z_n L_n}{z_{n'} L_{n'}} = \left(\frac{p_n}{p_{n'}} \right)^{-\varepsilon} \longrightarrow \frac{p_n}{p_{n\prime}} = \left(\frac{z_n L_n}{z_{n'} L_{n'}} \right)^{-\frac{1}{\varepsilon}}$$

It's easy to see that since $\varepsilon > 1 \rightarrow -\frac{1}{\varepsilon} < 0$ or more formally we can compute the derivative of this expression, hence:

$$\frac{\partial}{\partial z_n} \frac{p_n}{p_{n\prime}} = -\frac{1}{\varepsilon} \left(\frac{z_n L_n}{z_{n'} L_{n'}} \right)^{-\frac{\varepsilon+1}{\varepsilon}} \left(\frac{L_n}{z_{n'} L_{n'}} \right)$$

And hence an increase in the productivity z_n will lead to a **decrease** in the relative price. The expression for the real national income is:

$$\frac{w_n L_n}{P} = (z_n L_n)^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}} = y_n^{\frac{\varepsilon-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}}$$

Now it's immediate to check that $\frac{\partial}{\partial z_n} Y > 0$ as well as $\frac{\partial}{\partial z_n} y_n > 0$ hence the real national income **increases** if $z_n \uparrow$ since $\frac{\varepsilon - 1}{\varepsilon} > 0$ and $\frac{1}{\varepsilon} > 0$. The country share of world income is given by:

$$\frac{\frac{w_n L_n}{P}}{\sum_{n\prime=1}^{I} \frac{w_{n\prime} L_{n\prime}}{P}}$$

But recall that fm the previous section we proved that $\sum_{n\prime=1}^{I} \frac{w_{n\prime} L_{n\prime}}{P} = Y$ hence the expression simplifies to:

$$\frac{\frac{w_n L_n}{P}}{\sum_{n\prime=1}^{I} \frac{w_{n\prime} L_{n\prime}}{P}} = \left(\frac{y_n}{Y} \right)^{\frac{\varepsilon - 1}{\varepsilon}}$$

Note again that since $\varepsilon > 1 \rightarrow \frac{\varepsilon - 1}{\varepsilon} > 0$ hence the sign of the derivative will depend on:

$$\frac{\partial \frac{y_n}{Y}}{\partial z_n} = \frac{L_n}{\left[\sum_{i=1}^{I} (z_i L_i)^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon - 1}{\varepsilon}}} \left(1 - \frac{(z_n L_n)^{\frac{\varepsilon - 1}{\varepsilon}}}{\sum_{i=1}^{I} (z_i L_i)^{\frac{\varepsilon - 1}{\varepsilon}}} \right) > 0$$

Hence the share of country n if z_n increases **increases** as a share of world income.

7. **Now, assume that $\epsilon < 1$, and answer the same questions as in 6..**

It's immediate from previous computations that in this case the relative price **decreases**. Use computations before but and remember that even if $\varepsilon < 1$ we bound the lower value to be such that $\varepsilon \in (0, 1)$.

The real income change is **unclear**. Note indeed that:

$$\frac{\partial}{\partial z_n} \frac{w_n L_n}{P} = \frac{y_n^{-\frac{1}{\varepsilon}}}{\varepsilon} L_n Y^{\frac{1}{\varepsilon}} \left(\varepsilon - 1 + \frac{y_n^{\frac{\varepsilon - 1}{\varepsilon}}}{\sum_{i=1}^{I} (y_i)^{\frac{\varepsilon - 1}{\varepsilon}}} \right)$$

and that $\frac{\partial}{\partial L_n} Y > 0$.

The share of country n will **decrease** since now the change (only in term of sign) is $-\frac{\partial \frac{y_n}{Y}}{\partial z_n}$ since $\frac{\varepsilon - 1}{\varepsilon} < 0$.

8. **Suppose you are the prime minister of country n. Country n is a small country, and therefore you can assume that its impact on world production Y is close to**

zero. To increase the free time of your citizens, you plan a reform which would reduce their labor supply Ln. Would such a reform increase or decrease real national income? Why

1. if $\varepsilon > 1$ the effect will be a decrease in real national income.

2. if $\varepsilon < 1$ the effect will be an increase in the real national income.

The idea behind the reform is to transform the perfect competitive representative firm in the country to be a representative monopolist. De facto by controlling L_n in this simple setting the prime minister is able to control $y_n = L_n z_n$. and indirectly the price. Now it's not limpid the effect in general since:

$$\frac{\partial}{\partial L_n} \frac{w_n L_n}{P} = (z_n L_n)^{\frac{\varepsilon - 1}{\varepsilon}} Y^{\frac{1}{\varepsilon}}$$

De facto we have two opposite forces: (1) an "income" effect, the world becomes poorer, hence ceteris our share of world income we loose national income. (2) a substitution effect since the amount of the good produced by our economy decreases and the price rises. Now the substitution effect is good for our economy if the elasticity of price to quantity is larger than 1, which is the case in which goods are complement with elasticity of quantity to price $\varepsilon < 1$. Now the assumption of a small open economy allow us to say that *we are negligible in* Y hence "income effect" of the fact that the world becomes poorer is negligible. The only effect will be driven by how much the price will change if we reduce our production, the inverse of the classical elasticity of quantity to price. Hence if $\varepsilon < 1$ we gain in term of real national income.

Exercise 3 - Model Equivalence

In this exercise, you will show that it is exactly equivalent to have a representative consumer with CES preferences (as in the LN), or firms producing a unique final consumption good with a CES production function.

Consider a representative consumer that consumes a unique final good and has preferences

$$U = C \tag{10}$$

where C is the amount of the final good consumed. Firms produce this final good under perfect competition, by assembling it from a continuum $[0, M]$ of intermediate goods with the production function

$$Y = \left(\int_0^M (m(i))^{\frac{\varepsilon - 1}{\tau}} di \right)^{\frac{\varepsilon}{\varepsilon - 1}} \tag{11}$$

where m_i stands for the amount of intermediate input i used in production. Intermediate inputs are produced by another set of firms operating under perfect competition using only labour, with the production function

$$m_i = z_i l_i \tag{12}$$

As in the LN, labour L is supplied inelastically by the representative consumer.

1. **Derive the intermediate input demand functions of the final good producer.**

Note that we can solve this problem both by maximizing the amount produced by the final producer (11) subject to a budget constraint hence maximizing the production given a fixed amount to be spent in varieties or we can use the dual problem and hence minimize expenditure given a fixed production. The dual problem is identical and hence we minimize expenditure given a fixed final production. The final producer problem is hence:

$$\min_{m(i)_{i \in [0,m]}} \int_0^m p(i) m(i) di$$

subject to

$$\left(\int_0^M (m(i))^{\frac{\varepsilon - 1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon - 1}} \geq \bar{y}$$

Note that we elude the use of KKT and hence we don't place any *complementary slackness* since the constraints holds with equality and hence we can use Lagrange. To see why it holds with equality let's claim:

Claim. *If the sequence of $m(i)$ is optimal then:*

$$\left(\int_0^M (m^*(i))^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = \bar{y}$$

Proof. Let's prove it by contraddiction[1]. Suppose $\hat{m}(i)_{i\in[0,m]}$ is such that $\left(\int_0^M (\hat{m}(i))^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} >$ $\bar{y}$. Consider a new allocation $\bar{m}(i)_{i\in[0,m]}$ such that $\bar{m}(i) = \hat{m}(i) - \varepsilon$ where $\varepsilon > 0$. By continuity of the production function for ε small enough the constraint is still satisfied and

$$\int_0^M p(i)(\hat{m}(i) - \varepsilon)di < \int_0^M p(i)\hat{m}(i)di$$

hence $\hat{m}(i)_{i\in[0,m]}$ is not optimal and so we have proved our claim. $\square$

We can now use lagrange and the lagrangian read:

$$\mathcal{L} = -\int_0^M p(i)m(i)di + \lambda\left[\left(\int_0^M (m(i))^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} - \bar{y}\right]$$

equating the FOCs to 0 yields:

$$\lambda p(i) = \left(\int_0^M (m(i))^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{1}{\varepsilon-1}} m(i)^{-\frac{1}{\varepsilon}} \quad \forall i \in I$$

From which is immediate to obtain:

$$m(i) = m(j)\left(\frac{p(i)}{p(j)}\right)^{-\varepsilon} \quad \forall i \in [0, M]$$

Please note that the previous equation hold for each i in the index set. We can hence plug this in

[1] The following logical statements are equivalent: $A \rightarrow B \longleftrightarrow \bar{B} \rightarrow \bar{A}$

to the budget constraint to obtain:

$$Y = m(j)p(j)^{\varepsilon}\left[\int_0^M p(i)^{1-\varepsilon}di\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

For simplicity define the aggregate price level as:

$$P = \left[\int_0^M p(i)^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}$$

It's now immediate to state that:

$$m(j) = \left(\frac{p(j)}{P}\right)^{-\varepsilon}Y$$

2. **Express the prices of all intermediate inputs and of the final good as a function of the wage w and productivities z_i in equilibrium.**

intermediate producers solve:

$$\max_{l(i)} p(i)z(i)l(i) - w(i)l(i)$$

Since factor markets are competitive by arbitrage $w(i) = w \ \forall \ i \in I$ Solving the maximization problem yields:

$$p(i) = \frac{w}{z(i)}$$

Now its immediate to see that since final producer are operating in a competitive market the free entry condition gives us:

$$\Pi = \bar{P}\left(\int_0^M (m(i))^{\frac{\varepsilon-1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^M p(i)m(i)di = 0$$

And hence immediately:

$$\bar{P} = \frac{\int_0^M p(i)m(i)di}{Y}$$

Now recalling the intermediate good producer optimization problem we get that:

$$p(i) = \left(\frac{m(i)}{Y}\right)^{-\frac{1}{\varepsilon}} P$$

Hence from the previous equation we can write:

$$\bar{P} = \frac{\int_0^M p(i)m(i)di}{Y}$$

From which we get that:

$$\bar{P} = \frac{P\int_0^M m(i)^{\frac{\varepsilon-1}{\varepsilon}}}{Y^{\frac{\varepsilon-1}{\varepsilon}}} \longleftrightarrow P = \bar{P}$$

From which we can express the aggregate price level $\bar{P} = P$ as a function of the wage and productivity:

$$\bar{P} = P = \left[\int_0^M \left(\frac{w}{z(i)}\right)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$$

3. **Solve for the equilibrium production and consumption of each good, and compare your results to the LN.**

From the production function of the intermediate producer it's immediate to obtain that:

$$\frac{p(i)}{p(j)} = \frac{z(j)}{z(i)}$$

As well as from the optimality condition of the final producer problem we get that:

$$\left(\frac{p(i)}{p(j)}\right)^{-\varepsilon} = \frac{m(i)}{m(j)} = \frac{l(i)z(i)}{l(j)z(j)}$$

Combining this two results we get:

$$\left(\frac{z(i)}{z(j)}\right)^{\varepsilon-1} = \frac{l(i)}{l(j)} \to l(i) = l(j)\left(\frac{z(i)}{z(j)}\right)^{\varepsilon-1}$$

From which is immediate to get that:

$$\int_0^M l(i)di = \int_0^M l(j)\left(\frac{z(i)}{z(j)}\right)^{\varepsilon-1} di$$

And the following is trivial:

$$\begin{aligned} l(j) &= L\frac{z(j)^{\varepsilon-1}}{\int_0^M z(i)^{\varepsilon-1}di} \\ m(j) = z(j)l(j) &= L\frac{z(j)^{\varepsilon}}{\int_0^M z(i)^{\varepsilon-1}di} \end{aligned}$$