EECS 127/227AT Optimization Models in Engineering
Spring 2020 **Discussion 4**

1. **Convexity of Sets**

Definition. A set C is convex if and only if the line segment between any two points in C lies in C:

$$C \text{ is convex } \iff \forall \vec{x}_1, \vec{x}_2 \in C,\ \forall \theta \in [0,1],\ \theta \vec{x}_1 + (1-\theta)\vec{x}_2 \in C$$

(a) Show that the **intersection of convex sets is convex**:

$$C_1, C_2 \text{ are convex } \implies C = C_1 \cap C_2 \text{ is convex}$$

(b) Show that the following sets are convex:

i. [**Optional**] A vector subspace of $\mathbb{R}^n$

ii. [**Optional**] A hyperplane, $\mathcal{L} = \{\vec{x} \mid \vec{a}^\top \vec{x} = b\}$.

iii. A halfspace, $\mathcal{H} = \{\vec{x} \mid \vec{a}^\top \vec{x} \leq b\}$.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is the sum of a linear function and a constant,

$$f(\vec{x}) = A\vec{x} + \vec{b},$$

for $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$.

(c) [**Optional**] **Conservation of convexity through affine transformation**. Prove that if $S \subseteq \mathbb{R}^n$ is convex, then the image of S under an affine function f,

$$f(S) = \{f(\vec{x}) \mid \vec{x} \in S\},$$

is convex.

2. **Convexity of Functions**

Definition. A function $f :: \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is a convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and $\theta \in [0,1]$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \leq \theta f(\vec{x}) + (1-\theta) f(\vec{y}). \tag{1}$$

The function f is strictly convex if the inequality is strict.

Definition. A function $f :: \mathbb{R}^n \to \mathbb{R}$ is concave if $\mathrm{dom}(f)$ is a convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and θ with $0 \leq \theta \leq 1$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \geq \theta f(\vec{x}) + (1-\theta) f(\vec{y}).$$

The function f is strictly concave if the inequality is strict.

Property. A function f is concave if and only if $-f$ is convex. An affine function is both convex and concave.

Property: Jensen's inequality. The inequality in Equation (1) is known as **Jensen's Inequality**. This can be extended to convex combinations of more than one point. If f is convex, and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \text{dom}(f)$, and $\theta_1, \theta_2, \ldots, \theta_k \geq 0$ with $\sum_{i=1}^{k} \theta_i = 1$ then,

$$f(\theta_1 \vec{x}_1 + \theta_2 \vec{x}_2 + \cdots + \theta_k \vec{x}_k) \leq \theta_1 f(\vec{x}_1) + \theta_2 f(\vec{x}_2) + \cdots + \theta_k f(\vec{x}_k).$$

Property: First order condition. Suppose f is differentiable. Then f is convex if and only if $\text{dom}(f)$ is convex and

$$f(\vec{y}) \geq f(\vec{x}) + \nabla f(\vec{x})^\top (\vec{y} - \vec{x}),$$

for all $\vec{x}, \vec{y} \in \text{dom}(f)$.

Property: Second order condition. Suppose f is twice differentiable. Then f is convex if and only if, $\text{dom}(f)$ is convex and the Hessian of f, $\nabla^2 f(\vec{x})$, is positive semi-definite for all $\vec{x} \in \text{dom}(f)$.

(a) Under what condition on $A \in \mathbb{R}^{n \times n}$, where A is symmetric, is the function $f : \vec{x} \to \vec{x}^\top A \vec{x}$ convex?

(b) **[Optional] Restriction to a line**. Show that a function f is convex if and only if for all $\vec{x} \in \text{dom}(f)$ and all $\vec{v}$, the function $g : \text{dom}(g) \to \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ is convex for $\text{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \text{dom}(f)\}$.

(c) **[Optional] Non-negative weighted sum**. Show that the non-negative weighted sum of convex functions is convex: i.e. if $f_1, \ldots, f_n$ are n convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $w_1, \ldots, w_n \in \mathbb{R}_+$ are n positive scalars, then the function:

$$f = \sum_{i=1}^{n} w_i f_i$$

is convex. To make the question easier, you can assume that the functions $f_1, \ldots, f_n$ are twice-differentiable.

(d) **[Optional] Point-wise maximum** Show that if f_1 and f_2 are convex functions then their pointwise maximum f, defined by

$$f(\vec{x}) = \max(f_1(\vec{x}), f_2(\vec{x})),$$

with $\text{dom}(f) = \text{dom}(f_1) \cap \text{dom}(f_2)$, is also convex.

(e) Show that a piece-wise linear function that can be written as,

$$f(\vec{x}) = \max(\vec{a}_1^\top \vec{x} + \vec{b}_1, \vec{a}_2^\top \vec{x} + \vec{b}_2, ..., \vec{a}_m^\top \vec{x} + \vec{b}_m),$$

is convex.

3. Disproving convexity: Finding counter-examples

Though we spend a lot of time in this course learning how to prove convexity of sets and functions, in practical scenarios we may not have a mathematical representation of a set/function and so it is not possible to prove convexity. Instead, we may be able to represent this set/function in terms of a query $Q(\vec{x})$ that returns some information about the element $\vec{x}$ in relation to the set/function.

For example, instead representing the set $S = \{\vec{x} \mid \text{some condition on } \vec{x}\}$ we only have $Q(\vec{x})$ which returns whether or not $\vec{x} \in S$.

In these cases we can **disprove** convexity by showing that one or more of the properties of convex sets/functions are violated by finding counterexamples. In this problem we will see how we can disprove convexity for sets/functions given limited information that can be accessed via certain types of queries.

(a) **Disproving convexity of set S (Proving non-convexity of set S)**
Assume that we know that the set lies within some $\mathcal{D}$.
Query: $Q(\vec{x})$: For $\vec{x} \in \mathcal{D}$ that returns True if $\vec{x} \in S$ and False if $\vec{x} \notin S$. How can you use Q to check/disprove convexity of S?

(b) **Disproving convexity of function f (Proving non-convexity of function f).**
Assume that we know $\text{dom}(f)$, denoted as $\mathcal{D}$ and that $\mathcal{D}$ is convex.

i. Query: $G(\vec{x})$:For $\vec{x} \in \mathcal{D}$, returns function value $f(\vec{x})$.
How can you use G to check/disprove convexity of f?

ii. Query: $H(\vec{x})$: For $\vec{x} \in \mathcal{D}$, returns $f(\vec{x})$ and $\nabla f(\vec{x})$. (Here we assume that f is differentiable). How can you use H to check/disprove convexity of f?