EECS 127/227AT Optimization Models in Engineering

Spring 2020 — Midterm

This exam has a total of 114 points. However, a score of 100 on 114 will be considered a perfect score, so 14 points on the exam are bonus.

1. Convexity (12 points)

State whether the following functions/sets are convex and **justify your answer**. Answers without justification will receive no credit.

(a) (4 points) Function $f(\vec{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution:

Not a convex function because the Hessian matrix is not positive semi-definite.

(b) (4 points) Set $S = \{(\vec{x}, y) \mid \|A\vec{x} - \vec{b}\|_2^2 \leq y\}$. *Hint: Consider the epigraph of a function. Other proofs may also work.*

Solution: Convex set because the epigraph of a convex function is convex.

(c) (4 points) Function $f(\vec{x}) = \max_{\vec{b}} \left[\vec{b}^\top A\vec{b} + \vec{x}^\top \vec{b}\right]$, where A is a fixed arbitrary matrix. *Hint: Note that the maximization is over $\vec{b}$.*

Solution: Convex function because the point-wise maximum of convex functions (in this case affine functions) in $\vec{x}$ is convex.

2. Gradient descent (10 points)

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, where

$$f(\vec{x}) = \frac{1}{4} \|\vec{x}\|_2^4 .$$

Let $\vec{x}^* \doteq \arg\min_{\vec{x}} f(\vec{x})$.

Recall that the gradient descent update equation for minimizing f is given by

$$\vec{x}_{t+1} = \vec{x}_t - \eta \nabla f(\vec{x}_t),$$

where $\eta > 0$ is the step size.

(a) (2 points) Find $\vec{x}^*$. You need not show any work for this subpart.

Solution: $f(\vec{x}) = \frac{1}{4} \|\vec{x}\|_2^4 \geq 0$.
Also, $f(\vec{0}) = 0$. Thus, $\vec{x}^* = \vec{0}$.

(b) (8 points) Suppose $\|\vec{x}_0\|_2 = c \neq 0$. **Find the range of η (in terms of c) such that gradient descent converges to $\vec{x}^*$.** Justify your answer.

Hint: If you are having trouble solving this part for general dimension n, solve it for $n = 1$ for partial credit.

Solution: Using chain rule, we can compute the gradient of f to get,

$$\begin{aligned}\nabla f(\vec{x}_t) &= \frac{1}{4}.2\,\|\vec{x}_t\|_2^2\,\nabla(\|\vec{x}_t\|_2^2)\\ &= \|\vec{x}_t\|_2^2\,\vec{x}_t.\end{aligned}$$

Using this along with part (a) and the gradient step,

$$\begin{aligned}|\vec{x}_{t+1} - \vec{x}^*| &= \left|\vec{x}_t - \eta\,\|\vec{x}_t\|_2^2\,\vec{x}_t - 0\right|\\ &= |\vec{x}_t - 0|\left|(1 - \eta\,\|\vec{x}_t\|_2^2)\right|.\end{aligned}$$

To guarantee convergence to $\vec{x}^*$ we require for all t,

$$\begin{aligned}&\left|(1 - \eta\,\|\vec{x}_t\|_2^2)\right| < 1\\ \implies\ &0 < \eta < \frac{2}{\|\vec{x}_t\|_2^2}.\end{aligned}$$

But observing that if $\left|(1 - \eta\,\|\vec{x}_t\|_2^2)\right| < 1$ then $|\vec{x}_{t+1}| < |\vec{x}_t|$ so the lowest upper bound for η will be for $t = 0$. Thus, we need,

$$0 < \eta < \frac{2}{\|x_0\|_2^2}.$$

or equivalently,

$$0 < \eta < \frac{2}{c^2}.$$

3. PCA (12 points)

In this problem, we will find the principal components of data points on a regularly spaced grid.[1] Consider a set S of $n = 15$ data points that lie at each integer node of a 5×3 grid:

$$S = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \in \{-2, -1, 0, 1, 2\},\; x_2 \in \{-1, 0, 1\} \right\}.$$

A plot of these points is shown in Fig. 1.

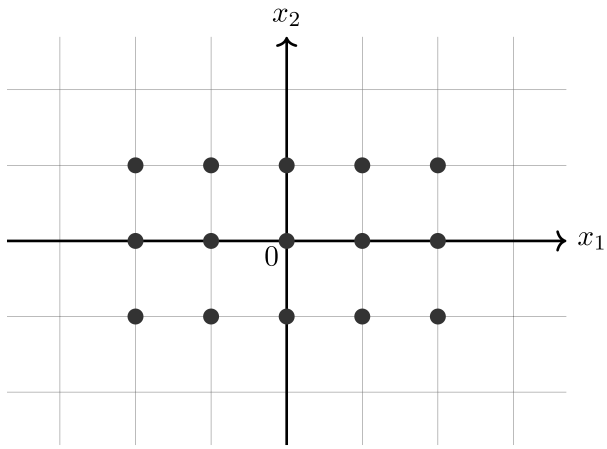

Figure 1: Point data.

Note that the empirical covariance matrix of these data points is given by

$$C = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}.$$

(a) (6 points) Recall that for data with empirical covariance matrix C, the variance $\sigma^2(\vec{u})$ along any unit vector $\vec{u}$ is given by

$$\sigma^2(\vec{u}) = \vec{u}^\top C \vec{u}.$$

The data's first principal component $\vec{u}_1$ is the unit vector direction that maximizes variance, i.e.,

$$\vec{u}_1 = \underset{\|\vec{u}\|_2 = 1}{\operatorname{argmax}}\, \sigma^2(\vec{u}).$$

Compute both $\vec{u}_1$ and $\sigma^2(\vec{u}_1)$. Show your work.

Solution: The first principal component $\vec{u}_1$ is the eigenvector corresponding to the largest eigenvalue of C, so $\vec{u}_1 = \vec{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, and $\sigma^2(\vec{u}_1) = \vec{u}_1^\top C \vec{u}_1 = 2$. The plot of $\vec{u}_1$ is shown in Fig. 2.

(b) (6 points) Let $\vec{x}_i$ for $i = 1, \cdots, 15$ represent the elements of set S. Suppose we transform every point $\vec{x} \in S$ by multiplying by an arbitrary orthonormal matrix W to generate new data points $\vec{z}_i = W\vec{x}_i$, where $i = 1, \ldots, 15$ indexes over every element of S. Let $\vec{v}_1$ denote

[1] You may find this scenario contrived, but it's actually based on a real research problem encountered by one of your GSIs when analyzing point cloud data from a robot's sensor. To figure out where the robot should place its gripper along a beam to pick it up, they used PCA!

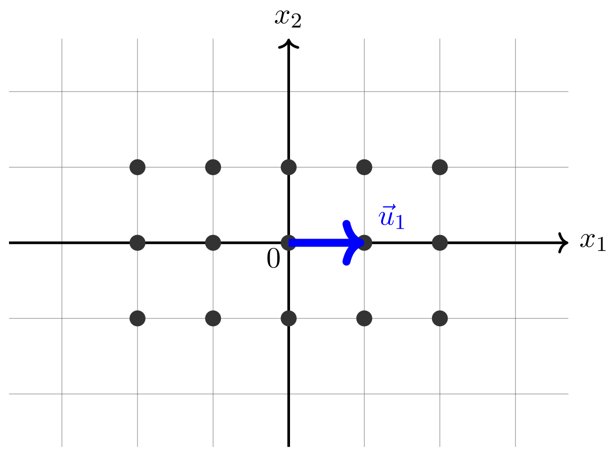

Figure 2: First principal component $\vec{u}_1$.

the first principal component of the transformed data and let $\vec{v}_2$ denote its second principal component. **Find $\vec{v}_1$ and $\vec{v}_2$ in terms of $\vec{u}_1$, $\vec{u}_2$, and W.**
Hint: It may be useful to find the new empirical covariance of this transformed data in terms of C and W.
Solution: We first calculate the transformed data's covariance matrix C_W:

$$\begin{aligned} C_W &= \frac{1}{n}\sum_{i=1}^{n} \vec{z}_i\vec{z}_i^\top \\ &= \frac{1}{n}\sum_{i=1}^{n} W\vec{x}_i\vec{x}_i^\top W^\top \\ &= W\left[\frac{1}{n}\sum_{i=1}^{n}\vec{x}_i\vec{x}_i^\top\right]W^\top \\ &= WCW^\top. \end{aligned}$$

We now wish to calculate first principal component v_1:

$$\begin{aligned} \vec{v}_1 &= \operatorname*{argmax}_{\|\vec{v}\|_2=1} \vec{v}^\top C_W \vec{v} \\ &= \operatorname*{argmax}_{\|\vec{v}\|_2=1} \vec{v}^\top WCW^\top \vec{v} \\ &= \operatorname*{argmax}_{\left\|W^\top\vec{v}\right\|_2=1} \vec{v}^\top WCW^\top \vec{v} \end{aligned}$$

where the third line follows because multiplication by an orthonormal matrix does not change the norm of a vector. Defining $\vec{y} \doteq W^\top\vec{v}$ (and thus $\vec{y}_1 \doteq W^\top\vec{v}_1$), we can write

$$\vec{y}_1 = \operatorname*{argmax}_{\|\vec{y}\|_2=1} \vec{y}^\top C\vec{y},$$

which is exactly equivalent to finding the first principal component of the untransformed data, so $\vec{y}_1 = \vec{u}_1$. Thus, $\vec{v}_1 = W\vec{y}_1 = W\vec{u}_1$. The second transformed principal component must be orthogonal to the first, so $\vec{v}_2 = W\vec{u}_2$.

4. All I need is Q (22 points)

Consider a partially known matrix $A \in \mathbb{R}^{3\times 2}$ given by

$$A = \begin{bmatrix} ? & 1 \\ ? & 1 \\ ? & 1 \end{bmatrix},$$

where question marks denote unknown entries of A. We can write the compact QR decomposition of A in terms of $Q_1 \in \mathbb{R}^{3\times 2}$ and $R_1 \in \mathbb{R}^{2\times 2}$ as

$$A = Q_1 R_1 = \begin{bmatrix} 1 & q_{12} \\ 0 & q_{22} \\ 0 & q_{23} \end{bmatrix} \begin{bmatrix} ? & r_{12} \\ 0 & r_{22} \end{bmatrix}. \tag{1}$$

for some unknown entry '?' and entries r_{12}, r_{22}, q_{12}, q_{22} and q_{23}, which you will calculate below. Remember that the columns of Q_1 are orthonormal. Note that the '?' entries of A and R_1 are unknown and will remain unknown; you are **NOT** required to compute them.

(a) (5 points) Suppose $r_{22} > 0$. **Compute r_{12}, r_{22}, q_{12}, q_{22} and q_{23}.** Show all your work.

Solution:

Using the Gram Schmidt procedure we have,

$$\begin{aligned} r_{12} &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= 1. \end{aligned}$$

Denoting $\vec{q} = \begin{bmatrix} q_{12} \\ q_{22} \\ q_{23} \end{bmatrix}$ and by using the fact that $\vec{q}$ must be unit-norm and that $r_{22} > 0$ we have,

$$r_{22}\vec{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - r_{12} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which gives us $r_{22} = \sqrt{2}$ and $\vec{q} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) (12 points) Suppose we can write the full QR decomposition of A as

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \tag{2}$$

where Q_1 and R_1 are as defined in Equation (1). Consider the least squares problem

$$p^* = \min_{\vec{x}} \left\| A\vec{x} - \vec{b} \right\|_2^2 \tag{3}$$

for A given in Equation (2) and some $\vec{b} \in \mathbb{R}^3$. Consider the following two possible ways of rewriting this least squares problem in terms of Q_1, Q_2, and R_1:

Strategy 1:

$$\left\|\vec{b} - A\vec{x}\right\|_2^2 \overset{(I)}{=} \left\|Q^\top\vec{b} - Q^\top A\vec{x}\right\|_2^2$$
$$= \left\|Q_1^\top\vec{b} - R_1\vec{x}\right\|_2^2 + \left\|Q_2^\top b\right\|_2^2.$$

Strategy 2:

$$\left\|\vec{b} - A\vec{x}\right\|_2^2 = \left\|\vec{b} - Q_1R_1\vec{x}\right\|_2^2$$
$$\overset{(II)}{=} \left\|Q_1^\top\vec{b} - Q_1^\top Q_1R_1\vec{x}\right\|_2^2$$
$$\overset{(III)}{=} \left\|Q_1^\top\vec{b} - R_1\vec{x}\right\|_2^2.$$

Determine whether the following labeled steps in the reformulations above are correct or incorrect and justify your answer. When evaluating the correctness of an equality, consider *only that specific equality's correctness* — i.e., ignore all earlier steps.

i. Equality (I): $\left\|\vec{b} - A\vec{x}\right\|_2^2 \overset{(I)}{=} \left\|Q^\top\vec{b} - Q^\top A\vec{x}\right\|_2^2$

ii. Equality (II): $\left\|\vec{b} - Q_1R_1\vec{x}\right\|_2^2 \overset{(II)}{=} \left\|Q_1^\top\vec{b} - Q_1^\top Q_1R_1\vec{x}\right\|_2^2$

iii. Equality (III): $\left\|Q_1^\top\vec{b} - Q_1^\top Q_1R_1\vec{x}\right\|_2^2 \overset{(III)}{=} \left\|Q_1^\top\vec{b} - R_1\vec{x}\right\|_2^2.$

Solution:

Equality (I) is CORRECT. Since Q is an orthogonal matrix we have $QQ^\top = I$. Thus,

$$\begin{aligned}\left\|\vec{b} - A\vec{x}\right\|_2^2 &= (\vec{b} - A\vec{x})^\top(\vec{b} - A\vec{x}) \\ &= (\vec{b} - A\vec{x})^\top QQ^\top(\vec{b} - A\vec{x}) \\ &= \left\|Q^\top(\vec{b} - A\vec{x})\right\|_2^2\end{aligned}$$

Equality (II) is INCORRECT. We can try to apply same approach as before but now a crucial difference is that Q_1 is **NOT** an orthogonal matrix. In fact $Q_1Q_1^\top$ cannot be full rank since $\text{rank}(Q_1) = 2 < 3$, thus it cannot be I.

Equality (III) is CORRECT. Even though Q_1 is not an orthogonal matrix, its columns are orthonormal thus $Q_1^\top Q_1 = I$.

(c) (5 points) Now consider a **different matrix** $A = QR$**, unrelated to the matrix** A **in previous parts**. Here, let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where $R \in \mathbb{R}^{3\times 2}$ and $R_1 \in \mathbb{R}^{2\times 2}$ is a completely unknown **invertible** upper triangular matrix. Let

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Again consider the least squares optimization problem:

$$p^* = \min_{\vec{x}} \left\| A\vec{x} - \vec{b} \right\|_2^2.$$

Find the optimal value p^*. Your answer should be a real number; it should **NOT** be an expression involving A, Q, R, R_1, or $\vec{b}$. **Solution:** Using Equation (**??**) from the correct solution in the previous part, we have

$$\begin{aligned} p^* &= \left\| Q_2^\top \vec{b} \right\|_2^2 \\ &= \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)^2 \\ &= 4. \end{aligned}$$

Further, $\vec{x}^* = R_1^{-1} Q_1^\top \vec{b}$. Thus,

$$\vec{y}^* = Q_1 R_1 \vec{x}^* = Q_1 Q_1^\top \vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

.

5. Subspace projection (18 points)

Consider a set of points $\vec{z}_1, \dots, \vec{z}_n \in \mathbb{R}^d$. The first principal component of the data, $\vec{w}^*$, is the direction of the line that minimizes the sum of the squared distances between the points and their projections on $\vec{w}^*$, i.e.,

$$\vec{w}^* = \underset{\|\vec{w}\|_2=1}{\operatorname{argmin}} \sum_{i=1}^{n} \|\vec{z}_i - \langle \vec{w}, \vec{z}_i \rangle \vec{w}\|^2 .$$

In this problem, we generalize to finding the r-dimensional subspace (instead of a 1-dimensional line) that minimizes the sum of the squared distances between the points $\vec{z}_i$ and their projections on the subspace. We assume that $1 \le r \le \min(n, d)$. We can represent an r-dimensional subspace by its orthonormal basis $(\vec{w}_1, \dots, \vec{w}_r)$, and we want to solve:

$$(\vec{w}_1^*, \dots, \vec{w}_r^*) = \underset{\substack{\|\vec{w}_i\|_2=1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \le i,j \le r}}{\operatorname{argmin}} \sum_{i=1}^{n} \min_{\alpha_1,\dots,\alpha_r} \left\| \vec{z}_i - \sum_{k=1}^{r} \alpha_k \vec{w}_k \right\|^2 . \tag{4}$$

Note that the inner minimization projects the point $\vec{z}_i$ onto the subspace defined by $(\vec{w}_1, \dots, \vec{w}_r)$. The variables $\alpha_k \in \mathbb{R}$. This means that for an arbitrary point $\vec{z}$, this inner minimization

$$(\alpha_1^*, \dots, \alpha_r^*) = \underset{\alpha_1,\dots,\alpha_r}{\operatorname{argmin}} \left\| \vec{z} - \sum_{k=1}^{r} \alpha_k \vec{w}_k \right\|^2$$

has minimizers $\alpha_k^* = \langle \vec{w}_k, \vec{z} \rangle$.

(a) (6 points) With the following definition of matrices Z and W:

$$Z = \begin{bmatrix} \uparrow & \dots & \uparrow \\ \vec{z}_1 & \dots & \vec{z}_n \\ \downarrow & \dots & \downarrow \end{bmatrix}, \qquad W = \begin{bmatrix} \uparrow & \dots & \uparrow \\ \vec{w}_1 & \dots & \vec{w}_r \\ \downarrow & \dots & \downarrow \end{bmatrix},$$

show that we can rewrite the optimization problem in Equation (4) **as:**

$$(\vec{w}_1^*, \dots, \vec{w}_r^*) = \underset{\substack{\|\vec{w}_i\|_2=1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \le i,j \le r}}{\operatorname{argmin}} \left\| Z - W W^\top Z \right\|_F^2 . \tag{5}$$

Solution:

First, consider a single vector $\vec{z} \in \mathbf{R}^d$. For this vector, consider the optimization problem:

$$\min_{\alpha_1,\dots,\alpha_r} \left\| \vec{z} - \sum_{i=1}^{r} \alpha_i \vec{w}_i \right\|_2^2 .$$

We first expand the term inside the minimization problem as follows:

$$\begin{aligned} \left\| \vec{z} - \sum_{i=1}^{r} \alpha_i \vec{w}_i \right\|_2^2 &= \|\vec{z}\|_2^2 + \left\| \sum_{i=1}^{r} \alpha_i \vec{w}_i \right\|_2^2 - 2\langle \sum_{i=1}^{r} \alpha_i \vec{w}_i, \vec{z} \rangle = \|\vec{z}\|_2^2 + \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \langle \vec{w}_i, \vec{w}_j \rangle - 2 \sum_{i=1}^{r} \alpha_i \langle \vec{w}_i, \vec{z} \rangle \\ &= \|\vec{z}\|_2^2 + \sum_{i=1}^{r} (\alpha_i^2 - 2\alpha_i \langle \vec{w}_i, \vec{z} \rangle) \end{aligned}$$

where for the final equality, we have used the fact that $\langle \vec{w}_i, \vec{w}_j \rangle = 0$ for $i \neq j$ and $\|\vec{w}_i\| = 1$ for all i. By taking derivatives, we see that the optimal value for α_i is $\langle \vec{w}_i, \vec{z} \rangle$. From this, we can conclude that for a fixed vector, $\vec{z}$, we get:

$$\min_{\alpha_1,\dots,\alpha_r} \left\| \vec{z} - \sum_{i=1}^{r} \alpha_i \vec{w}_i \right\|^2 = \left\| \vec{z} - \sum_{i=1}^{r} \langle \vec{w}_i, \vec{z} \rangle \vec{w}_i \right\|^2. \tag{6}$$

In this question you the optimizers α_j^* were given and it was sufficient to plug those in to arrive at this step. Now, observe that for a single vector, $\vec{z}$, we have:

$$WW^\top \vec{z} = W \begin{bmatrix} \langle \vec{w}_1, \vec{z} \rangle \\ \vdots \\ \langle \vec{w}_r, \vec{z} \rangle \end{bmatrix} = \sum_{i=1}^{r} \langle \vec{w}_i, \vec{z} \rangle \vec{w}_i.$$

Therefore, we get using the fact that the squared Frobenius norm of a matrix is the sum of the squared lengths of its columns:

$$\left\| Z - WW^\top Z \right\|_F^2 = \sum_{i=1}^{n} \left\| \vec{z}_i - WW^\top \vec{z}_i \right\|^2 = \sum_{i=1}^{n} \left\| \vec{z}_i - \sum_{j=1}^{r} \langle \vec{z}_i, \vec{w}_j \rangle \vec{w}_j \right\|^2.$$

From Equation 6, we conclude that the above expression is equivalent to 4.

Next, we will solve the optimization problem in Equation (5) using the SVD of Z.

(b) (6 points) Let σ_i refer to the i^{th} largest singular value of Z, and $l = \min(n, d)$. First **show** that,

$$\min_{\substack{\|\vec{w}_i\|_2 = 1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \le i,j \le r}} \left\| Z - WW^\top Z \right\|_F^2 \ge \sum_{i=r+1}^{l} \sigma_i^2.$$

Solution:

Let $Z = U\Sigma V^\top = \sum_{i=1}^{l} \sigma_i \vec{u}_i \vec{v}_i^\top$ denote the SVD of Z and let $Z_r = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top$. Note that for any $W \in \mathbb{R}^{d \times r}$, $WW^\top Z$ is a matrix of rank at most r. Therefore, we get from the Eckart-Young theorem that:

$$\min_{\substack{\|\vec{w}_i\| = 1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \le i,j \le r}} \left\| Z - WW^\top Z \right\|_F^2 \ge \|Z - Z_r\|_F^2 = \sum_{i=r+1}^{l} \sigma_i^2.$$

(c) (6 points) Again σ_i refers to the i^{th} largest singular value of Z, and $l = \min(n, d)$. **Show** that,

$$\min_{\substack{\|\vec{w}_i\|_2 = 1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \le i,j \le r}} \left\| Z - WW^\top Z \right\|_F^2 \le \sum_{i=r+1}^{l} \sigma_i^2.$$

Hint: Find a W that achieves this upper bound.

Solution:

As before, let $Z = U\Sigma V^\top = \sum_{i=1}^{l} \sigma_i \vec{u}_i \vec{v}_i^\top$ denote the SVD of Z and $Z_r = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^\top$. By picking $\vec{w}_i = \vec{u}_i$ for $i \in [r]$ in (5), we get that:

$$\min_{\substack{\|\vec{w}_i\|=1 \\ \langle \vec{w}_i, \vec{w}_j \rangle = 0 \ \forall i \neq j \\ 1 \leq i,j \leq r}} \left\| Z - WW^\top Z \right\|_F^2 \leq \|Z - Z_r\|_F^2 = \sum_{i=r+1}^{l} \sigma_i^2.$$

From the previous part and this result, we conclude that an optimal solution to 4 are the top-r left singular vectors of Z which can be computed via the SVD of Z.

6. Duality (36 points)

Consider the function

$$f(\vec{x}) = \vec{x}^\top A\vec{x} - 2\vec{b}^\top \vec{x}.$$

First, we consider the unconstrained optimization problem

$$p^* = \min_{\vec{x}\in\mathbb{R}^n} f(\vec{x}) = \min_{\vec{x}\in\mathbb{R}^n} \vec{x}^\top A\vec{x} - 2\vec{b}^\top \vec{x} \tag{7}$$

for a real $n \times n$ **symmetric** matrix $A \in \mathbb{S}^n$ and $\vec{b} \in \mathbb{R}^n$. If the problem is unbounded below, then we say $p^* = -\infty$. Let $\vec{x}^*$ denote the minimizing argument of the optimization problem.

(a) (6 points) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \in \mathcal{R}(A)$. Let $\text{rank}(A) = n$. **Find p^*.**

Solution: If $\text{rank}(A) = n$, then $A \succ 0$, and therefore the objective is strictly convex. Setting the gradient to 0 we obtain,

$$\begin{aligned}
\nabla_{\vec{x}} f(\vec{x}) &= 2A\vec{x} - 2\vec{b} = 0 \\
&\implies A\vec{x} = \vec{b} \\
&\implies \vec{x}^* = A^{-1}\vec{b}
\end{aligned}$$

Where in the last step, we used that fact that a full rank square matrix is invertible. Plugging this back into our objective function we get,

$$\begin{aligned}
f(\vec{x}^*) &= (\vec{b}^\top (A^{-1})^\top) A (A^{-1}\vec{b}) - 2\vec{b}^\top (A^{-1}\vec{b}) \\
&= \vec{b}^\top (A^\top)^{-1} \cancel{AA^{-1}} \vec{b} - 2\vec{b}^\top A^{-1}\vec{b} \\
&= \vec{b}^\top A^{-1}\vec{b} - 2\vec{b}^\top A^{-1}\vec{b} \\
p^* &= -\vec{b}^\top A^{-1}\vec{b}
\end{aligned}$$

(b) (8 points) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \in \mathcal{R}(A)$ as before. Let A be rank-deficient, i.e., $\text{rank}(A) = r < n$. Let A have the compact/thin and full SVD as follows, with diagonal positive definite $\Lambda_r \in \mathbb{R}^{r\times r}$:

$$A = U_r \Lambda_r U_r^\top = \begin{bmatrix} U_r & U_1 \end{bmatrix} \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_r^\top \\ U_1^\top \end{bmatrix}.$$

Show that the minimizer $\vec{x}^*$ of the optimization problem (7) is not unique by **finding a general form for the family of solutions for $\vec{x}^*$** in terms of $U_r, U_1, \Lambda_r, \vec{b}$.

Solution: Since $A \succeq 0$, $f(\vec{x})$ is convex and we can attempt to find the minimizer by setting the gradient to zero. Doing this we obtain,

$$A\vec{x} = b, \tag{8}$$

as in the part (a) of this problem.

However, now this equation has infinite solutions since $\vec{b}$ lies in the range of A and A is rank-deficient. Indeed we can add any vector from the (non-trivial) nullspace of A to any particular

solution $\vec{x}_0$ of Equation (8) and get another solution.
By the Fundamental Theorem of Linear Algebra we have,

$$\begin{aligned}\vec{x} &= U_r\vec{\alpha} + U_1\vec{\beta} \\ \vec{b} &= U_r\vec{\gamma},\end{aligned}$$

where we used the fact that $\vec{b} \in \mathcal{R}(A)$. Using this we obtain,

$$U_r\Lambda_r U_r^\top (U_r\vec{\alpha} + U_1\vec{\beta}) = U_r\vec{\gamma}$$

Since the columns of U_1 and U_r are orthogonal to each other and because $U_r^\top U_r = I, \Lambda_r$ is invertible we have,

$$\begin{aligned}U_r\Lambda_r U_r^\top U_r\vec{\alpha} &= U_r\vec{\gamma} \\ \implies \vec{\alpha} &= \Lambda_r^{-1}\vec{\gamma} \\ &= \Lambda_r^{-1}U_r^\top \vec{b}.\end{aligned}$$

Thus any solution to Equation (8) and hence a minimizer to the optimization problem (7) can be written as,

$$\vec{x}^* = U_r\Lambda_r^{-1}U_r^\top \vec{b} + U_1\vec{\beta}.$$

(c) (6 points) If $A \not\succeq 0$ (A not positive semi-definite) **show that** $p^* = -\infty$ by finding $\vec{v}$ such that $f(\alpha\vec{v}) \to -\infty$ as $\alpha \to \infty$.
Solution: Since $A \not\succeq 0$ there exists an eigenvalue, eigenvector pair $(\mu, \vec{v})$ such that

$$\vec{v}^\top A\vec{v} = \mu < 0.$$

Assuming without loss of generality that $-2\vec{b}^\top\vec{v} \le 0$ (If it is positive then multiply $\vec{v}$ by -1) we can take $\vec{x} = \alpha\vec{v}$ to obtain,

$$f(\vec{x}) = f(\alpha\vec{v}) = \alpha^2\vec{v}^\top A\vec{v} + \alpha(-2\vec{b}^\top\vec{v}),$$

which goes to $-\infty$ as α goes to ∞ since $\vec{v}^\top A\vec{v} < 0$ and $-2\vec{b}^\top\vec{v} \le 0$.

(d) (6 points) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \notin \mathcal{R}(A)$. **Find** p^***.** Justify your answer mathematically.
Solution: First, note that since A is symmetric, we have $\mathcal{R}(A) = \mathcal{R}(A^\top)$. We have $\vec{b} = \vec{v}_1 + \vec{v}_2$ with $\vec{v}_1 \in \mathcal{R}(A) = \mathcal{R}(A^\top)$ and $\vec{v}_2 \in \mathcal{N}(A)$ as $\mathbb{R}^n = \mathcal{R}(A) \oplus \mathcal{N}(A)$ from the Fundamental Theorem of Linear Algebra. We cannot have $\vec{v}_2 = 0$ as otherwise we'd get $\vec{b} = \vec{v}_1 \in \mathcal{R}(A)$ which is a contradiction. Now, let $\vec{v} = \vec{v}_2$. We get from this:

$$f(\alpha\vec{v}) = \alpha^2\vec{v}^\top A\vec{v} - 2\alpha(\vec{v}_1 + \vec{v}_2)^\top\vec{v}_2 = 0 - 2\alpha\left\|\vec{v}_2\right\|^2$$

where we used the fact that $\vec{v}_2 \in \mathcal{N}(A)$ and $\vec{v}_1 \in \mathcal{R}(A)$. As $\alpha \to \infty$, we get that $f(\alpha\vec{v}) \to -\infty$ from which we conclude that $p^* = -\infty$.

For parts (e) and (f), consider real $n \times n$ **symmetric** matrix $A \in \mathbb{S}^n$ and $\vec{b} \in \mathbb{R}^n$. Let $\text{rank}(A) = r$, where $0 \le r \le n$. Now we consider the constrained optimization problem

$$\begin{aligned}p^* = \min_{\vec{x}\in\mathbb{R}^n} &\ \vec{x}^\top A\vec{x} - 2\vec{b}^\top\vec{x} \\ \text{s.t. } &\ \vec{x}^\top\vec{x} \ge 1.\end{aligned} \tag{9}$$

(e) (4 points) **Write the Lagrangian** $\mathcal{L}(\vec{x}, \lambda)$, where λ is the dual variable corresponding to the inequality constraint.

Solution:

$$\begin{aligned}
\mathcal{L}(\vec{x}, \lambda) &= \vec{x}^\top A\vec{x} - 2\vec{b}^\top \vec{x} + \lambda(1 - \vec{x}^\top \vec{x}) \\
&= \vec{x}^\top A\vec{x} - \vec{x}^\top \lambda\vec{x} - 2\vec{b}^\top \vec{x} + \lambda \\
&= \vec{x}^\top (A - \lambda I)\vec{x} - 2\vec{b}^\top \vec{x} + \lambda
\end{aligned}$$

(f) (6 points) For any matrix $C \in \mathbb{R}^{n\times n}$ with $\text{rank}(C) = r \leq n$ and compact SVD

$$C = U_r \Lambda_r V_r^\top,$$

we define the pseudoinverse as

$$C^\dagger = V_r \Lambda_r^{-1} U_r^\top.$$

We use the "dagger" operator to represent this. If $\vec{d}$ lies in the range of C, then a solution to the equation $C\vec{x} = \vec{d}$, can be written as $\vec{x} = C^\dagger \vec{d}$, even when C is not full rank. **Show that the dual problem to the primal problem** (9) **can be written as,**

$$d^* = \max_{\substack{\lambda \geq 0 \\ A - \lambda I \succeq 0 \\ \vec{b} \in \mathcal{R}(A - \lambda I)}} \quad -\vec{b}^{\,\top} (A - \lambda I)^\dagger \,\vec{b} + \lambda.$$

Hint: To show this, first argue that when the constraints are not satisfied $\min_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) = -\infty$*. Then show that when the constraints are satisfied,* $\min_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) = -\vec{b}^{\,\top} (A - \lambda I)^\dagger \,\vec{b} + \lambda$*.*

Solution:

$$g(\lambda) = \min_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) = \min_{\vec{x}} \vec{x}^\top (A - \lambda I)\vec{x} - 2\vec{b}^\top \vec{x} + \lambda$$

Drawing from parts (c) and (d), we can see that if $A - \lambda I \not\succeq 0$ or if $A - \lambda I \succeq 0, \vec{b} \notin \mathcal{R}(A - \lambda I)$, then we can choose $\vec{x}$ to drive the Lagrangian to $-\infty$.

If the constraints are satisfied, however, then we can proceed like in part (b) by taking the gradient:

$$\begin{aligned}
\nabla_{\vec{x}} \mathcal{L} = 2(A - \lambda I)\vec{x} - 2\vec{b} &= 0 \\
(A - \lambda I)\vec{x} &= \vec{b} \\
\vec{x}^* &= (A - \lambda I)^\dagger \vec{b}
\end{aligned}$$

where in the last step, we used the fact that the PSD contraint on $A - \lambda I$ is satisfied and $\vec{b}$ lies in the range of $A - \lambda I$, so we can use the pseudoinverse and the gradient-zero point is indeed the minimum.

Plugging this back into the Lagrangian, we get:

$$\begin{aligned}
\mathcal{L}(\vec{x}^*, \lambda) &= \vec{b}^\top ((A - \lambda I)^\dagger)^\top (A - \lambda I)(A - \lambda I)^\dagger \vec{b} - 2\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda \\
&= \vec{b}^\top (A - \lambda I)^\dagger (A - \lambda I)(A - \lambda I)^\dagger \vec{b} - 2\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda \\
&= \vec{b}^\top (A - \lambda I)^\dagger \vec{b} - 2\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda \\
&= -\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda
\end{aligned}$$

where we used the fact that $(A - \lambda I)^\dagger$ is symmetric and by properties of pseudo inverse,

$$(A - \lambda I)^\dagger (A - \lambda I)(A - \lambda I)^\dagger = (A - \lambda I)^\dagger.$$

Now, we have a full expression for our dual function:

$$g(\lambda) = \begin{cases} -b^\top (A - \lambda I)^\dagger b + \lambda & \text{if } A - \lambda I \succeq 0, b \in \mathcal{R}\left(A - \lambda I\right) \\ -\infty & \text{else} \end{cases}$$

The dual problem follows, as it is just a maximization of the dual function:

$$d^* = \max_{\lambda \geq 0} g(\lambda)$$