EECS 127/227AT Optimization Models in Engineering
Spring 2020 **Discussion 11**

1. **Dual norms and SOCP**

Consider the problem

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \ \|A\vec{x} - \vec{y}\|_1 + \mu \|\vec{x}\|_2,$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$, and $\mu > 0$.

(a) Express this (primal) problem in standard SOCP form.

(b) Find a dual to the problem and express it in standard SOCP form.
Hint: Recall that for every vector $\vec{z}$, the following dual norm equalities hold:

$$\|\vec{z}\|_2 = \max_{\vec{u} \,:\, \|\vec{u}\|_2 \leq 1} \vec{u}^\top \vec{z}, \quad \|\vec{z}\|_1 = \max_{\vec{u} \,:\, \|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z}.$$

(c) Assume strong duality holds[1] and that $m = 100$ and $n = 10^6$, i.e., A is 100×10^6. Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

2. Squaring SOCP constraints

When considering a second-order cone (SOC) constraint, you might be tempted to square it to obtain a classical convex quadratic constraint. This problem explores why that might not always work, and how to introduce additional constraints to maintain equivalence and convexity.

(a) For $\vec{x} \in \mathbb{R}^2$, consider the constraint

$$x_1 - 2x_2 \geq \|\vec{x}\|_2,$$

and its squared counterpart

$$(x_1 - 2x_2)^2 \geq \|\vec{x}\|_2^2.$$

Are the two sets equivalent? Are they both convex?

[1] In fact, you can show that strong duality holds using Sion's theorem, a generalization of the minimax theorem that is beyond the scope of this class.

(b) What additional constraint must be imposed alongside the squared constraint to enforce the same feasible set as the unsquared SOC constraint?

3. Casting optimization problems as SOCPs

Cast the following problem as an SOCP in its standard form:

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^n} \quad & \sum_{i=1}^{p} \frac{\|F_i\vec{x}+\vec{g}_i\|_2^2}{\vec{a}_i^\top\vec{x}+b_i} \\ \text{s.t.} \quad & \vec{a}_i^\top\vec{x}+b_i > 0, \quad i=1,\ldots,p, \end{aligned}$$

where $F_i \in \mathbb{R}^{m\times n}$, $\vec{g}_i \in \mathbb{R}^m$, $\vec{a}_i \in \mathbb{R}^n$, and $b_i \in \mathbb{R}$, for $i = 1,\ldots,p$.

4. A review of standard problem formulations

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) ***Linear programming (LP).***

i. Write the most general form of a linear program (LP) and list its defining attributes.

ii. Under what conditions is an LP convex?

(b) ***Quadratic programming (QP).***

i. Write the most general form of a quadratic program (QP) and list its defining attributes.

ii. Under what conditions is a QP convex?

(c) ***Quadratically-constrained quadratic programming (QCQP).***

i. Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

ii. Under what conditions is a QCQP convex?

(d) ***Second-order cone programming (SOCP).***

i. Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

ii. Under what conditions is an SOCP convex?

(e) ***Relationships.*** Recall that

$$LP \subset QP_{\text{convex}} \subset QCQP_{\text{convex}} \subset SOCP \subset \{\text{all convex programs}\},$$

where LP denotes the set of all linear programs, QP_{convex} denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?