EECS 127/227AT Optimization Models in Engineering
Spring 2020 Discussion 2

1. Gradients and Hessians

(a) The *Gradient* of a scalar-valued function $g : \mathbb{R}^n \to \mathbb{R}$, is the column vector of length n, denoted ∇g, containing the derivatives of components of g with respect to the variables:

$$(\nabla g(\vec{x}))_i = \frac{\partial g}{\partial x_i}(\vec{x}),\ i = 1, \ldots n.$$

Compute the gradient, $\nabla g(\vec{x})$, of:

i. $g(\vec{x}) = \vec{c}^\top \vec{x}$

ii. $g(\vec{x}) = \vec{x}^\top \vec{x}$

iii. $g(\vec{x}) = \ln\left(\sum_{i=1}^{n} e^{x_i}\right)$

(b) The *Hessian* of a scalar-valued function $g : \mathbb{R}^n \to \mathbb{R}$, is the $n \times n$ matrix, denoted as $\nabla^2 g$, containing the second derivatives of components of g with respect to the variables:

$$(\nabla^2 g(\vec{x}))_{ij} = \frac{\partial^2 g}{\partial x_i \partial x_j}(\vec{x}),\ \ i = 1, \ldots, n,\ \ j = 1, \ldots, n.$$

Compute the Hessian, $\nabla^2 g(\vec{x})$, of:

i. g($\vec{x}$)= $\vec{c}^\top \vec{x}$

ii. g($\vec{x}$) = $\vec{x}^\top \vec{x}$.

iii. g($\vec{x}$) = $\vec{x}^\top \mathrm{A} \vec{x}$.

2. Gradients with respect to matrices (OPTIONAL)

Assume that $A \in \mathbb{R}^{p\times m}, C, X \in \mathbb{R}^{m\times n}$, $\Sigma \in \mathbb{R}^{m\times m}$ and $\vec{a} \in \mathbb{R}^m, \vec{b} \in \mathbb{R}^n$. Find the following gradients and specify the dimensions of the gradients.

(a) $\nabla_X \mathrm{tr}(X^\top C)$

(b) $\nabla_X (\vec{a}^\top X \vec{b})$

(c) $\nabla_{\Sigma^{-1}} \mathrm{tr}(X^\top \Sigma^{-1} X)$

(d) $\nabla_X \|AX\|_F^2$

3. Jacobians (OPTIONAL)

The *Jacobian* of a vector-valued function $g : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix, denoted as Dg, containing the derivatives of the components of g with respect to the variables:

$$(Dg)_{ij} = \frac{\partial g_i}{\partial x_j},\ \ i = 1, \ldots, m,\ \ j = 1, \ldots, n.$$

(a) Compute the Jacobian of $g(\vec{x}) = A\vec{x}$