EECS 127/227AT Optimization Models in Engineering
Spring 2020 **Homework 10**

This homework is due Friday, April 10, 2020 at 23:00 (11pm).
Self grades are due Friday, April 17, 2020 at 23:00 (11pm).

This version was compiled on 2020-04-04 18:07.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook with solutions saved as a PDF.

1. **(Optional) Strong Duality but no KKT**
   In this question, we will see an example of a problem where strong duality holds but the KKT conditions don't hold. Consider the following problem:

$$\begin{aligned} p^* = \min_{x_1, x_2 \in \mathbb{R}} \ & x_1^2 + x_2^2 \\ \text{s.t.} & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1. \end{aligned}$$

   (a) Sketch the feasible set. Find the optimal solution x^*.
   (b) Write the KKT conditions and solve them.
   (c) Formulate and solve the Lagrange dual problem. Does strong duality hold?

2. **The Duality of the ℓ_1 and ℓ_∞ norms**

   For this problem, we will prove the duality of the ℓ_1 and ℓ_∞ norms. Recall that the ℓ_1 and ℓ_∞ norms, denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively, are defined as follows for $\vec{x} \in \mathbb{R}^n$:

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|\vec{x}\|_\infty = \max_{i \in [n]} |x_i|.$$

   We will show that the ℓ_1 and ℓ_∞ norms are duals of each other; that is, we will show that:

$$\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty = 1} \vec{y}^\top \vec{x} \text{ and } \|\vec{x}\|_\infty = \max_{\|\vec{y}\|_1 = 1} \vec{y}^\top \vec{x}.$$

   (a) To start, we will first prove the following inequality for all $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$\vec{x}^\top \vec{y} \leq \|\vec{x}\|_1 \|\vec{y}\|_\infty.$$

   (b) Now, show that:

$$\max_{\|\vec{y}\|_\infty = 1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_1$$

   and using (a), conclude that $\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty = 1} \vec{y}^\top \vec{x}$.

(c) Finally, show the following inequality:

$$\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_\infty$$

and prove the second equality.

3. A Linear Program

Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ and $\mu > 0$. First consider the following problem:

$$p^* = \min_x \|Ax - y\|_1.$$

For $j \in \{1, \ldots, n\}$, we denote by $\vec{a}_j$ the j-th column of A, so that $A = [\vec{a}_1, \ldots, \vec{a}_n]$.

(a) Express the problem as an LP.

(b) Show that a dual to the problem can be written as

$$d^* = \max_{\vec{u}} -\vec{u}^\top \vec{y} \ : \ A^\top \vec{u} = 0, \ \ \|\vec{u}\|_\infty \leq 1.$$

Hint: use the fact that, for any vector z:

$$\max_{u\,:\,\|u\|_1\leq 1} u^T z = \|z\|_\infty, \quad \max_{u\,:\,\|u\|_\infty\leq 1} u^T z = \|z\|_1.$$

Now, consider the following more complicated problem involving both the ℓ_1 and ℓ_∞ norms:

$$p^* = \min_x \|Ax - y\|_1 + \mu\|x\|_\infty.$$

(c) Express the problem as an LP.

(d) Show that a dual to the problem can be written as

$$d^* = \max_u -u^T y \ : \ \|u\|_\infty \leq 1, \ \ \|A^T u\|_1 \leq \mu.$$

Hint: use the fact that, for any vector z:

$$\max_{u\,:\,\|u\|_1\leq 1} u^T z = \|z\|_\infty, \quad \max_{u\,:\,\|u\|_\infty\leq 1} u^T z = \|z\|_1.$$

4. Fenchel conjugate

Given a function $f : \mathbb{R} \to \mathbb{R}$, its Fenchel conjugate $f^* : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$f^*(x) = \sup_{y\in\mathbb{R}}\{\langle y, x\rangle - f(y)\}.$$

Note that this transform is always well-defined when $f(x)$ is convex.

(a) Show that for every function f, its Fenchel conjugate f^* is convex.

(b) Given $f(x) = (x-2)^2$, find its Fenchel conjugate f^*.

(c) Given $f(x) = (x-2)^2$, find the Fenchel conjugate of its Fenchel conjugate, $(f^*)^*(x)$.

The next two parts of this question are optional and you will not be required to submit a solution. However, understanding them and their solutions may be useful for the final project.

(d) **(Optional)** Consider the following piecewise linear function:

$$f(x) = \begin{cases} -x & x < 0, \\ x & 0 \leq x \leq 0.5, \\ 1-x & 0.5 \leq x \leq 1, \\ x-1 & x \geq 1. \end{cases}$$

Verify that f^* and $(f^*)^*$ are given by the following:

$$f^*(x) = \begin{cases} \infty & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & -1 \leq x \leq 0 \\ \infty & x < -1 \end{cases} \quad \text{and } (f^*)^*(x) = \begin{cases} x-1 & x \geq 1 \\ 0 & 0 \leq x \leq 1 \\ -x & x < 0 \end{cases}.$$

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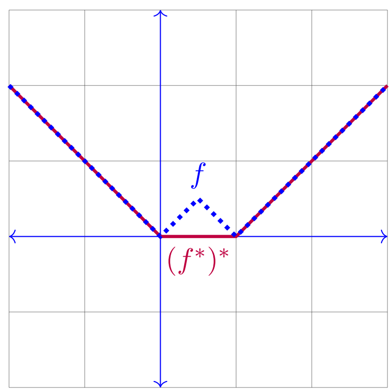
Hint: Using the plot of f might be helpful. $f^*(y)$ can be interpreted as the maximum distance between the function f and the line passing through the origin with slope y.

(e) **(Optional)** Compare the epigraph of $(f^*)^*$ with the convex hull of the epigraph of f.

Solution:



The epigraph of $(f^*)^*$ is the same as the *convex hull* of the epigraph of f.

5. Jupyter Notebook Coming soon!

6. Homework process

Whom did you work with on this homework? List the names and SIDs of your group members.