EECS 16B Designing Information Devices and Systems II
Fall 2019 Discussion Worksheet **Discussion 12A**

Questions

1. Linear Approximation

A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point x_0 is given by

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0),$$

where $f'(x_0) := \left.\frac{df(x)}{dx}\right|_{x=x_0}$ is the derivative of $f(x)$ at $x = x_0$.

Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x,y)$, the linear approximation of $f(x,y)$ at a point (x_0,y_0) is given by

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0) \cdot (x - x_0) + f_y(x_0,y_0) \cdot (y - y_0).$$

where $f_x(x_0,y_0)$ is the partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0):

$$f_x(x_0,y_0) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}$$

and $f_y(x_0,y_0)$ is the partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0).

(a) Now, let's see how we can derive partial derivatives. When we are given a function $f(x,y)$, we calculate the partial derivative of f with respect to x by fixing y and taking the derivative with respect to x. **Given the function $f(x,y) = x^2 y$, find the partial derivatives $f_y(x,y)$ and $f_x(x,y)$.**

(b) **Write out the linear approximation of f near (x_0,y_0).**

(c) **Compare the approximation of f at the point $(2.01, 3.01)$ using $(x_0,y_0) = (2,3)$ versus $f(2.01, 3.01)$.**

(d) When the function $f(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ takes in vectors and outputs a real number, how can we adapt our previous linearization method?

One way to linearize the function f is to do it for every single element in $\vec{x} = \begin{bmatrix} x[1] \\ x[2] \\ \dots \\ x[n] \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y[1] \\ y[2] \\ \dots \\ y[k] \end{bmatrix}$.

Then, when we are looking at $x[i]$ or $y[j]$, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_0,\vec{y}_0) + \sum_{i=1}^{n} \frac{\partial f(\vec{x},\vec{y})}{\partial x[i]}(x[i]-x_0[i]) + \sum_{j=1}^{k} \frac{\partial f(\vec{x},\vec{y})}{\partial y[j]}(y[i]-y_0[i]). \tag{1}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x[1]} & \cdots & \frac{\partial f}{\partial x[n]} \end{bmatrix}.$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y[1]} & \cdots & \frac{\partial f}{\partial y[k]} \end{bmatrix}.$$

Then, Equation (1) can be rewritten as

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_0,\vec{y}_0) + (D_{\vec{x}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{x}-\vec{x}_0) + (D_{\vec{y}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{y}-\vec{y}_0).$$

Assume that $n = k$ **and the function** $f(\vec{x},\vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x[i]y[i]$**. Find** $D_{\vec{x}} f$ **and** $D_{\vec{y}} f$**.**

(e) **Following the above part, find the linear approximation of** $f(\vec{x},\vec{y})$ **near** $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **and** $\vec{y}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$**.**

(f) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector, we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x},\vec{y}) = \begin{bmatrix} f_1(\vec{x},\vec{y}) \\ f_2(\vec{x},\vec{y}) \\ \dots \\ f_m(\vec{x},\vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_0,\vec{y}_0) + D_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}_0) + D_{\vec{y}} f_1 \cdot (\vec{y}-\vec{y}_0) \\ f_2(\vec{x}_0,\vec{y}_0) + D_{\vec{x}} f_2 \cdot (\vec{x}-\vec{x}_0) + D_{\vec{y}} f_2 \cdot (\vec{y}-\vec{y}_0) \\ \dots \\ f_m(\vec{x}_0,\vec{y}_0) + D_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}_0) + D_{\vec{y}} f_m \cdot (\vec{y}-\vec{y}_0) \end{bmatrix}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} D_{\vec{x}}f_1 \\ D_{\vec{x}}f_2 \\ \cdots \\ D_{\vec{x}}f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x[1]} & \cdots & \frac{\partial f_1}{\partial x[n]} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x[1]} & \cdots & \frac{\partial f_m}{\partial x[n]} \end{bmatrix},$$

and similarly

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y[1]} & \cdots & \frac{\partial f_1}{\partial y[k]} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial y[1]} & \cdots & \frac{\partial f_m}{\partial y[k]} \end{bmatrix}.$$

Then, the linearization becomes

$$\vec{f}(\vec{x},\vec{y}) \approx \vec{f}(\vec{x}_0,\vec{y}_0) + (D_{\vec{x}}\vec{f})\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{x}-\vec{x}_0) + (D_{\vec{y}}\vec{f})\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{y}-\vec{y}_0).$$

Let $\vec{x} = \begin{bmatrix} x[1] \\ x[2] \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x[1]^2x[2] \\ x[1]x[2]^2 \end{bmatrix}$. Find $D_{\vec{x}}\vec{f}$.

(g) **Compare the approximation of $\vec{f}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$ using $\vec{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ versus $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$.**

(h) (Do at home) **Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^\top\vec{w}$. Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.**

(i) (Do at home) **Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

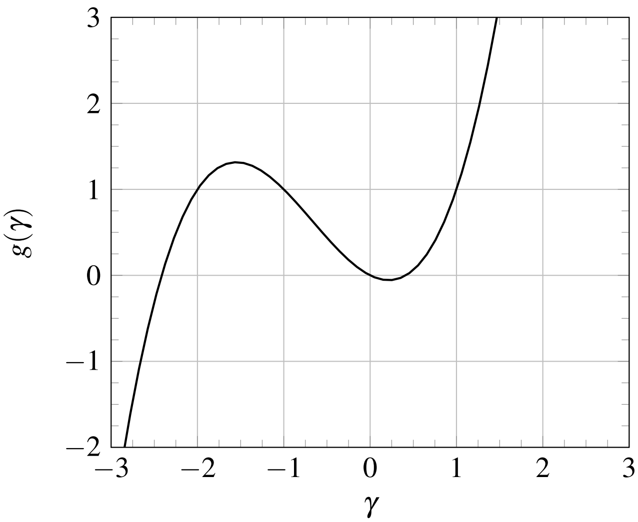
These linearizations are important for us because we can do many easy computations using linear functions.

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{d}{dt}\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix} = \frac{d}{dt}\vec{x}(t) = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t))+u(t)\end{bmatrix} = \vec{f}(\vec{x}(t),u(t)) \tag{2}$$

where $\vec{x}(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a DC operating point.

(a) **If we have fixed $u^\star(t) = -1$, what values of γ and β will ensure $\frac{d}{dt}\vec{x}(t) = \vec{0}$?**

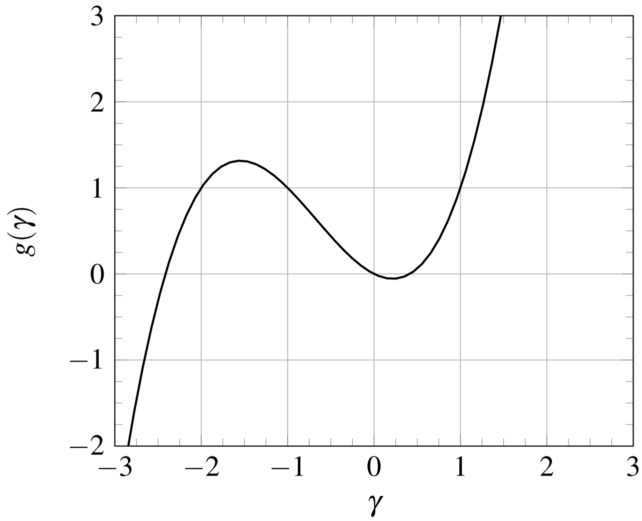
(b) Now that you have the three DC operating points, **linearize the system abou t the DC operating point $(\vec{x}_3^\star, u^\star)$ that has the largest value for γ**. Specifically, what we want is as follows. Let $\vec{\delta x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1,2,3$, and $\delta u(t) = u(t) - u^\star$. We can in principle write the *linearized system* for each DC operating point in the following form:

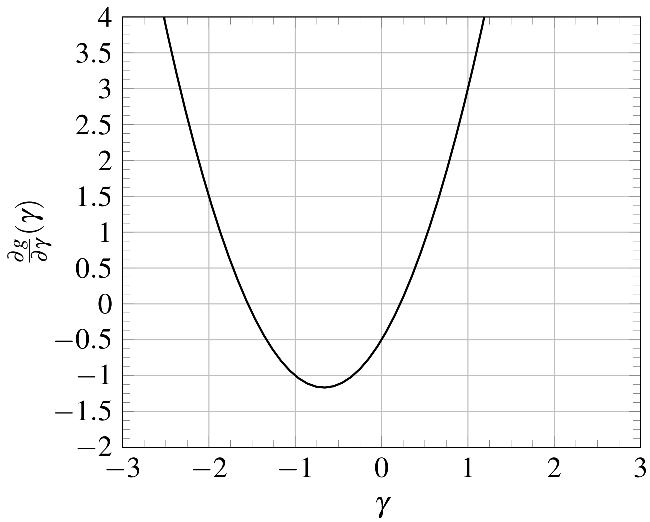
$$\text{(linearization about } (\vec{x}_i^\star, u^\star)) \qquad \frac{d}{dt}\vec{\delta x}_i(t) = A_i\vec{\delta x}_i(t) + B_i\delta u(t) + \vec{w}_i(t) \tag{3}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_3 **and** B_3.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





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