EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Homework 2

This homework is due Friday, February 7, 2020 at 23:00 (11pm).
Self grades are due Friday, February 14, 2020 at 23:00 (11pm).

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Questions marked practice will not be graded.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook with solutions saved as a PDF.

1. **Proof of the Fundamental Theorem of Linear Algebra**

In this question, we will prove the fundamental theorem of linear algebra. For any $A \in \mathbb{R}^{m \times n}$, let $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $\text{rank}(A)$ denote the null space, range and rank of A respectively.

For any subspace, $\mathcal{S}$ with dimension, $\dim(\mathcal{S})$, let $\mathcal{S}^\perp$ denote its the subspace orthogonal to $\mathcal{S}$.

The fundamental theorem of linear algebra states that,

$$\mathcal{N}(A) \oplus \mathcal{R}(A^\top) = \mathbb{R}^n.$$

The proof technique we employ will first show that,

$$\mathcal{N}(A) = (\mathcal{R}(A^\top))^\perp.$$

Then we will prove that we can find orthonormal vectors $\vec{e}_1, \vec{e}_2, \dots \vec{e}_n$ such that $\mathcal{N}(A) = \text{span}(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_l)$ and $\mathcal{R}(A^\top) = \text{span}(\vec{e}_{l+1}, \vec{e}_{l+2}, \dots, \vec{e}_n)$. As a corollary we get the rank-nullity theorem:

$$\dim(\mathcal{N}(A)) + \text{rank}(A) = n.$$

(a) First, show that $\mathcal{N}(A) \subseteq (\mathcal{R}(A^\top))^\perp$. *Hint: Consider $\vec{u}$ in $\mathcal{N}(A), \vec{v} \in \mathcal{R}(A^\top)$ and show that $\vec{u}^\top \vec{v} = 0$.*

(b) Now show that: $(\mathcal{R}(A^\top))^\perp \subseteq \mathcal{N}(A)$

Hint 1: Sometimes moving from symbols to words makes things clearer. Another way of stating what you want to prove is that any vector $\vec{v}$ that is orthogonal to all vectors in the range of $A^\top$, must satisfy $A\vec{v} = 0$.

Hint 2: Consider $\vec{v} \in (\mathcal{R}(A^\top))^\perp$. What can you say about $\vec{v}^\top A^\top$?

(c) Let $\dim(\mathcal{N}(A)) = l$ and let $\vec{e}_1, \dots, \vec{e}_l$ be an orthonormal basis for $\mathcal{N}(A)$. Consider an extension of the basis to an orthonormal basis, $\vec{e}_1, \dots, \vec{e}_n$ for $\mathbb{R}^n$. We will prove that $\vec{e}_{l+1}, \dots, \vec{e}_n$ form a basis for $\mathcal{R}(A^\top)$ and as a consequence, the dimension of $\mathcal{R}(A^\top)$ is $n - l$.

   i. Show that $\mathcal{R}(A^\top)$ lies in the span of $\vec{e}_{l+1}, \dots, \vec{e}_n$. To do this, first express any vector $\vec{u} \in \mathcal{R}(A^\top)$ in terms of the basis vectors e_i and use $\mathcal{N}(A) = (\mathcal{R}(A^\top))^\perp$, which you proved in parts (a) and (b).

   Hint: If a vector $\vec{u}$ in a vector space is orthogonal to one of the basis vectors $\vec{e}_i$, what is the value of the coefficient α_i when writing $\vec{u} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \dots$?

ii. From part (i) we know that $\mathcal{R}(A^\top) \subseteq \text{span}(\vec{e}_{l+1}, \ldots, \vec{e}_n)$, but we want something stronger. Show that in fact $\mathcal{R}(A^\top) = \text{span}(\vec{e}_{l+1}, \ldots, \vec{e}_n)$.
Hint 1: First show that the dimension of $\mathcal{R}(A^\top)$ is the same as the dimension of the space spanned by the basis vectors $\vec{e}_{l+1}, \ldots, \vec{e}_n$, i.e., show $\dim(\mathcal{R}(A^\top)) = n - l$. You can show this via a contradiction: assume that $\dim(\mathcal{R}(A^\top)) = k < n - l$, and show that a vector $\vec{u} \notin \mathcal{R}(A^\top)$ and $\vec{u} \in Span\{\vec{e}_{l+1}, \ldots, \vec{e}_n\}$ cannot exist. For the proof by contradiction, one approach is to consider an orthonormal basis $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k$ for $\mathcal{R}(A^\top)$, so we can find non-zero $\vec{u}' = \vec{u} - \sum_{i=1}^{k}(\vec{f}_i^\top \vec{u})\vec{f}_i$ that is orthogonal to $\mathcal{R}(A^\top)$. Does $\vec{u}'$ lie in $\mathcal{N}(A)$? Does $\vec{u}'$ also lie in $\text{span}(\vec{e}_{l+1}, \ldots, \vec{e}_n)$? Does this lead to a contradiction?
Hint 2: Think of this in easily visualizable dimensions. Take $n - l = 3$ and $k = 2$.
Hint 3: You may use a fact that for two subspaces, $\mathcal{S}_1$ and $\mathcal{S}_2$, if $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\dim(\mathcal{S}_1) = \dim(\mathcal{S}_2)$ then $\mathcal{S}_1 = \mathcal{S}_2$.

(d) Using part (c) argue why $\mathcal{N}(A) \oplus \mathcal{R}(A^\top) = \mathbb{R}^n$ and why the rank nullity theorem holds.

2. Eigenvectors of a symmetric matrix

Let $\vec{p}, \vec{q} \in \mathbb{R}^n$ be two linearly independent vectors, with unit norm ($\|\vec{p}\|_2 = \|\vec{q}\|_2 = 1$). Define the symmetric matrix $A \doteq \vec{p}\vec{q}^\top + \vec{q}\vec{p}^\top$. In your derivations, it may be useful to use the notation $c \doteq \vec{p}^\top \vec{q}$.

(a) Show that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are eigenvectors of A, and determine the corresponding eigenvalues.

(b) Determine the nullspace and rank of A.

(c) Find an eigenvalue decomposition of A, in terms of $\vec{p}, \vec{q}$. *Hint:* use the previous two parts.

(d) (Practice) Now consider general $\vec{p}, \vec{q}$ that are not necessarily norm 1. Write A as a function $\vec{p}, \vec{q}$ and their norms and the new eigenvalues as a function of $\vec{p}, \vec{q}$ and their norms.

3. Norms

(a) Show that the following inequalities hold for any vector $\vec{x} \in \mathbb{R}^n$:

$$\frac{1}{\sqrt{n}} \|\vec{x}\|_2 \le \|\vec{x}\|_\infty \le \|\vec{x}\|_2 \le \|\vec{x}\|_1 \le \sqrt{n} \|\vec{x}\|_2 \le n \|\vec{x}\|_\infty .$$

As an aside: note that we can interpret different norms as different ways of computing distance between two points $\vec{x}, \vec{y} \in \mathbb{R}^2$. The ℓ_2 norm is the distance as the crow flies (i.e. point-to-point distance), the ℓ_1 norm, also known as the Manhattan distance is the distance you would have to cover if you were to navigate from $\vec{x}$ to $\vec{y}$ via a rectangular street grid, and the ℓ_∞ norm is the maximum distance that you have to travel in either the north-south or the east-west direction.

(b) Show that for any non-zero vector x,

$$\text{card}(\vec{x}) \ge \frac{\|\vec{x}\|_1^2}{\|\vec{x}\|_2^2},$$

where $\text{card}(\vec{x})$ is the *cardinality* of the vector $\vec{x}$, defined as the number of non-zero elements in $\vec{x}$. Find all vectors $\vec{x}$ for which the lower bound is attained.

4. Distinct Eigenvalues, Orthogonal Eigenspaces

Let $A \in \mathbb{S}^n$ (i.e. the set of $n \times n$ symmetric matrices) and $(\lambda_1, \vec{u}_1), (\lambda_2, \vec{u}_2), \lambda_1 \neq \lambda_2$ be distinct eigen-pairs of A. Show that $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$, i.e eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.

5. PSD Matrices In this problem, we will analyze properties of PSD matrices. Assume $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix.

(a) Show that $\forall \vec{x} \;\; \in \mathbb{R}^n, \vec{x}^\top A\vec{x} \geq 0 \iff$ all eigenvalues of A are non-negative.

(b) Show that A having non-negative eigenvalues allows us to decompose $A = P^\top P$ where $P \in \mathbb{S}^n_+$ (i.e. the set of $n \times n$ positive semidefinite matrices).

(c) (Practice) Show that any matrix of the form $B = C^\top C \succeq 0$.

(d) If $A \succeq 0$, all diagonal entries of A are non-negative, $A_{ii} \geq 0$.

6. SVD Transformation

In this problem we will interpret the linear map corresponding to a matrix $A \in \mathbb{R}^{n\times n}$ by looking at its singular value decomposition, $A = UDV^\top$. Recall that here $U, D, V \in \mathbb{R}^{n\times n}$ and U, V are orthonormal(orthogonal) matrices while D is a diagonal matrix. We will first look at how $V^\top$, D and U each separately transform the unit circle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and then look at their effect as a whole. This problem has an associated jupyter notebook, "svd_transformation.ipynb" that contains several parts (b,c,d,e) of the problem. These sub-parts can be answered in the notebook itself in the space provided and can be submitted as a pdf using the 'Download as pdf' feature that jupyter notebook supports.

(a) Show that $V^\top\vec{x}$ represents $\vec{x}$ in the basis defined by the columns of V. Recall: $V^\top V = I$.

For rest of the problem we restrict ourselves to the case where $A \in \mathbb{R}^{2\times 2}$ and move to the Jupyter notebook.

7. Homework process

Whom did you work with on this homework? List the names and SIDs of your group members.