EECS 127/227AT **Fall 2020**
Discussion 7

1 LP duality in a combinatorial auction with divisible goods

An auctioneer is auctioning K divisible items. Here *divisible* means that each item can be broken up into arbitrary proportions. To be concrete, you can think of the items as being perfumes. The amount of each type of perfume the auctioneer has is 1.

The auctioneer has received B bids, where $B \geq 1$ is an integer. The bids are *combinatorial*. What this means is that each bid is for a subset of the items (i.e. the perfumes) and also states an amount which the bidder is willing to pay for that subset. Thus bid b, $1 \leq b \leq B$, will have the form (S_b, v_b), where $\emptyset \neq S_b \subseteq \{1, \ldots, K\}$, and $v_b \geq 0$.

It is assumed that each bidder will be willing to accept partial satisfaction of their bid. Namely, for any $0 \leq x_b \leq 1$, if the auctioneer gives the bidder b the amount x_b of each perfume in the set S_b, then the bidder will pay $x_b v_b$ for this. However, no bidder will accept any allocation except those that give them the items in the set S_b that they bid for in exactly equal amounts, and give them no other items.

Let A denote the $K \times B$ matrix with entries a_{ib} where

$$a_{ib} := \begin{cases} 1 & \text{if } i \in S_b, \\ 0 & \text{otherwise.} \end{cases}.$$

(a) The auctioneer wants to maximize revenue. Using the matrix A, pose the auctioneer's problem as a linear programming problem.

(b) Write down the dual LP in terms of dual variables y_i, $1 \leq i \leq K$.

(c) Give an intuitive interpretation of the dual problem, based on strong duality and complementary slackness.

2 LP relaxation of a Boolean LP

In a *Boolean linear program* the objective function and the constraint functions are affine, as in a LP, but the variables are constrained to take on the values 0 and 1. Despite the name, a Boolean linear program is *not* a linear program. In general it is not even a convex optimization problem, and is hard to solve. Here is a typical Boolean LP:

$$\begin{array}{rl} \min\limits_{x} & c^T x \\ \text{subject to:} & Ax \leq b, \\ & x_i \in \{0, 1\},\ i = 1, \ldots, n. \end{array}$$

Here $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

(a) Consider the following *LP relaxation* of the Boolean LP. The relaxation is an LP.

$$\begin{array}{rl} \min\limits_{x} & c^T x \\ \text{subject to:} & Ax \leq b, \\ & x_i \in [0, 1],\ i = 1, \ldots, n. \end{array}$$

Show that the optimal value of the LP relaxation is a lower bound for the optimal value of the original Boolean LP.

(b) Find the dual of the LP relaxation of the original Boolean LP. This should be expressed in terms of the dual variables for the constraints $Ax \leq b$ and the dual variables for the constraints $x \leq \mathbb{1}$, where $\mathbb{1}$ denotes the all-ones column vector.

(c) Simplify the dual LP found in the preceding part of this question by eliminating the dual variables corresponding to the constraints $x \leq \mathbb{1}$.

3 Lagrangian relaxation of a Boolean LP

We return to the Boolean LP considered in the preceding question, namely:

$$\begin{array}{rl} \min\limits_{x} & c^T x \\ \text{subject to:} & Ax \leq b, \\ & x_i \in \{0,1\},\ i = 1,\ldots,n, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

This can be rewritten in the equivalent form:

$$\begin{array}{rl} \min\limits_{x} & c^T x \\ \text{subject to:} & Ax \leq b, \\ & x_i(1 - x_i) = 0,\ i = 1,\ldots,n. \end{array}$$

Note that the Boolean constraint has now been expressed as a quadratic equality constraint. However this is *not* a convex optimization problem. The feasibility set has not changed, and is not convex in general. In the terminology of the textbook of Boyd and Vandenberghe this problem would not be called a QCQP, while in the terminology of the book of Calafiore and El Ghaoui this problem would be called a QCQP but one that is not a convex optimization problem.

(a) Nevertheless, we know how to subject even non-convex problems to traditional Lagrangian duality. Determine the dual of the Boolean LP expressed in the second form.

Remark: By virtue of weak duality, the value of this dual problem will give a lower bound to the value of the original Boolean LP. This method of finding a lower bound to the primal value of a given primal problem is called *Lagrangian relaxation*.

(b) Show that the dual optimization problem derived in part (b) of this question is equivalent to the dual of the LP relaxation of the Boolean LP, which was derived in part (b) of the previous question.

Hint: In part (c) of the preceding question we derived another optimization problem equivalent to the dual LP in part (b) of the preceding question.