EECS 127/227AT Optimization Models in Engineering
Spring 2020

Discussion 9

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}.$$

(a) Consider the minimization problem

$$\begin{aligned} p^* = \inf_{x \in \mathbb{R}} \ & f_0(x) \\ \text{s.t. } & -1 \leq x, \ \ x \leq 1. \end{aligned} \tag{1}$$

i. Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

ii. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda_1, \lambda_2 \geq 0} g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\},$$

with

$$\begin{aligned} g_1(\vec{\lambda}) &= \inf_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1) \\ g_2(\vec{\lambda}) &= \inf_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1). \end{aligned}$$

iii. Next, show that

$$\begin{aligned} g_1(\vec{\lambda}) &\leq -3\lambda_1 + \lambda_2 \\ g_2(\vec{\lambda}) &\leq \lambda_1 - 3\lambda_2. \end{aligned}$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

iv. Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

v. Does strong duality hold?

(b) Now, consider a problem equivalent to the minimization in (1):

$$\begin{aligned} p^* = \inf_{x \in \mathbb{R}} \ & f_0(x) \\ \text{s.t. } & x^2 \leq 1. \end{aligned} \tag{2}$$

Observe that $p^* = 2$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part (a).

i. Show that the dual problem can be represented as

$$d^* = \sup_{\lambda \geq 0} g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$\begin{aligned} g_1(\lambda) &= \inf_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ g_2(\lambda) &= \inf_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1). \end{aligned}$$

ii. Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

iii. Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

iv. Does strong duality hold?

2. Linear programming

Express the following problems as LPs.

(a)

$$\min_{\vec{x}\in\mathbb{R}^k} \left[\max_{i=1,\ldots,k} x_i - \min_{j=1,\ldots,k} x_j\right]$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$

(b)

$$\min_{\vec{x}\in\mathbb{R}^k} \sum_{i=1}^{k} |x_i|$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$