EECS 127/227AT **Fall 2020**
Homework 6
Homework 6 is due on Gradescope by Friday 10/16 at 11.59 p.m.

1 Active Constraints

This problem explores basic features of convex optimization problems. The relevant sections from the textbooks are Secs. 8.3, 8.4 and 8.5 of the textbook of Calafiore and El Ghaoui and Secs. 4.1 and 4.2 of the textbook of Boyd and Vandenberghe.

Consider the following optimization problem:

$$p^* = \min_{x,y} 2x + y \quad : \quad |x| \leq 1, \quad x + y \geq 0, \quad y \geq 0.$$

(a) Show that this is a convex optimization problem and rewrite it in standard form.

(b) Solve for p^* by drawing the feasible set and determining the optimal feasible points from this plot. The optimal point will turn out to be unique. Which of the constraints are active at the optimal point?
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(c) Write a Lagrangian for this optimization problem.

(d) Use Slater's condition to verify that strong duality holds.

(e) Solve for p^* again, using strong duality.

2 Water filling

This problem explores basic features of convex optimization problems and the KKT conditions in the framework of an important application in communication theory. The relevant sections from the textbooks are Secs. 8.3, 8.4 and 8.5 of the textbook of Calafiore and El Ghaoui and Secs. 4.1 and 4.2 of the textbook of Boyd and Vandenberghe.

Consider the following problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{s.t.} \quad & x_i \geq 0, \; i = 1, \ldots, n, \\ & \mathbb{1}^\top x = 1, \end{aligned}$$

where $\mathbb{1}$ denotes the all-ones column vector and we have $\alpha_i > 0$ for $i = 1, \ldots, n$.

This problem arises in communication and information theory when we wish to allocate a fixed amount of power between a set of n communication channels. Each variable x_i represents the transmitter power allocated to the i^{th} channel, and $\log(\alpha_i + x_i) - \log \alpha_i$ gives the capacity or maximum communication rate of the channel, where α_i characterizes the noisiness of the channel (the larger α_i is, the worse the channel is). The problem is to allocate a total power of 1 between the channels, in order to maximize the total communication rate. We drop the term $\sum_{i=1}^{n} \log \alpha_i$ from the physically relevant objective function in the mathematical formulation since it is not affected by the power allocation.

(a) Verify that this is a convex optimization problem with differentiable objective and constraint functions. Find the domain $\mathcal{D}$ of the optimization problem.

(b) Let $\lambda \in \mathbb{R}^n$ and $\nu \in \mathbb{R}$ be the dual variables corresponding to the constraints $x_i \geq 0, i = 1, \ldots, n$ and $\mathbb{1}^\top x = 1$, respectively. Write a Lagrangian for the optimization problem based on these dual variables.

(c) Write the KKT conditions for the problem.

(d) Since our problem is a convex optimization problem with differential objective and constraint functions, we know that if we can find x^* and (λ^*, ν^*) that verify the KKT conditions, then x^* will be a primal optimal point, (λ^*, ν^*) will be dual optimal, and strong duality will hold. We therefore attempt to find solutions for the KKT conditions. As a first step, show how to simplify the KKT conditions so that they are expressed in terms of only x^* and ν^*, i.e. we show how λ^* can be eliminated from these conditions.

(e) Solve for x_i^*, $1 \leq i \leq n$, in terms of ν^* from the simplified KKT conditions derived in the preceding part of this question.

(f) Show that there is a unique dual optimizer ν^*, and describe an algorithm for finding it.

3 LP duality

This problem explores basic features of linear programming duality. The relevant sections from the textbooks are Secs. 8.3, 8.4, 8.5 and 9.3 of the textbook of Calafiore and El Ghaoui and Secs. 4.1, 4.2 and 4.3 of the textbook of Boyd and Vandenberghe.

(a) Consider the following linear programming problem

$$p^* := \min_{x_1, x_2, x_3} \; x_1 + x_3 \; : \; x_1 + 2x_2 \leq 5, \; x_1 + 2x_3 = 6, \; x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.$$

i. Show that the problem is feasible and that the feasible set is a polytope. Recall that a polytope is a bounded polyhedron, so what this problem is asking you to show is that the feasible set is nonempty, is a polyhedron, and is bounded.

ii. Determine the vertices of the feasible polytope, show that the optimal primal value is $p^* = 3$, and find all the optimal feasible points.

iii. Write the Lagrangian for traditional Lagrange duality for this problem. There will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

iv. Find the dual objective function and show that the dual problem can be simplified to read

$$\max_{u,v} \; -5u - 6v \; : \; u \geq 0, \; v \geq -\frac{1}{2}.$$

Note that this is also a linear program.

v. Verify that the optimal dual value is $d^* = 3$ and so strong duality holds.

vi. Is Slater's condition satisfied in this problem?

(b) Now consider the following linear programming problem

$$p^* := \min_{x_1, x_2, x_3} \; x_1 + x_3 \; : \; x_1 + 2x_2 \leq -5, \; x_1 + 2x_3 = 6, \; x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0.$$

i. Show that the problem is infeasible and therefore the optimal primal value is $p^* = \infty$.

ii. Write the Lagrangian for traditional Lagrange duality for this problem. Again, there will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

iii. Find the dual objective function and show that the dual problem can be simplified to a linear program involving two variables, as in part (a) of this question.

iv. Show that the optimal dual value is $d^* = \infty$ and so strong duality once again holds.

v. Is Slater's condition satisfied in this problem?

4 Visualizing the Dual

The jupyter notebook for this problem can be found here. Complete the code where designated and answer the questions given in the space provided. (If you prefer, for questions that do not involve writing code, you can write solutions on separate paper or LaTeX PDF, just make sure to correctly mark the relevant pages when uploading to Gradescope.)

5 Dual of a QP

This problem explores basic features of quadratic programming duality. The relevant sections from the textbooks are Secs. 8.3, 8.4, 8.5, 9.1, 9.2 and 9.4 of the textbook of Calafiore and El Ghaoui and Secs. 4.1, 4.2 and 4.4.1 of the textbook of Boyd and Vandenberghe.

Consider a quadratic program of the form

$$p^* = \min_x \frac{1}{2} x^\top Q x + c^\top x \;:\; Ax \le b,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $Q \in \mathbb{S}^n_{++}$. We assume that the problem is feasible.

(a) Form the dual of the QP. Show that strong duality holds.

(b) Show that p^*, considered as a function of c (resp. b), is concave (resp. convex).

(c) Let $\epsilon > 0$. Suppose we now scale down the quadratic penalty in the objective function in the QP considered in part (a) of this question by a factor of ϵ. Namely, we consider the QP, which we call the ϵ-QP, given by:

$$p^*_\epsilon = \min_x \frac{\epsilon}{2} x^\top Q x + c^\top x \;:\; Ax \le b,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $Q \in \mathbb{S}^n_{++}$ and the problem is assumed to be feasible.

Because the objective function of the ϵ-QP is pointwise monotonically decreasing as ϵ decreases, the limit of the ϵ-QP as $\epsilon \to 0$ becomes the LP:

$$p^*_0 = \min_x \; c^\top x \;:\; Ax \le b,$$

What is the limit as $\epsilon \to 0$ of the dual of the ϵ-QP?