EECS 127/227AT **Fall 2020**
Discussion 11

1 SDP duality

Consider the following SDP in inequality form:

$$\begin{aligned} \min_{x \in \mathbb{R}} \ & x \\ \text{s.t.} \ & \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0. \end{aligned} \tag{1}$$

(a) Draw the feasible set. Is it convex?

(b) Write the conic dual SDP. This will be in standard form.

Remark: The textbook of Calafiore and El Ghaoui only describes how write a conic dual for a primal SDP in standard form (in which case the dual will be in inequality form) and the formula there is wrong because of a sign error. The duality formulas in the textbook of Boyd and Vandenberghe can be used (but note that their sign convention for linear matrix inequalities is different from ours). You could also use the formulas in the lecture slides in Lecture 21 and Lecture 22.

(c) Is the primal SDP feasible? Is it strictly feasible?

Remark: The SDP in inequality form

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \ & c^T x \\ \text{s.t.} \ & F_0 + \sum_{i=1}^m x_i F_i \succeq 0, \end{aligned}$$

where $F_0, F_1, \dots, F_m \in \mathbb{S}^n$, $c \in \mathbb{R}^n$, is said be strictly feasible if there is some $x \in \mathbb{R}^n$ such that $F(x) \in \mathbb{S}^n_{++}$, i.e. $F(x)$ is positive definite. Here $F(x)$ denotes $F_0 + \sum_{i=1}^m x_i F_i$.

(d) Is the dual SDP feasible? Is it strictly feasible?

Remark: The SDP in standard form

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \ & \text{trace}(CX) \\ \text{s.t.} \ & \text{trace}(A_i X) = b_i, \ i = 1, \dots, m, \\ & X \succeq 0, \end{aligned}$$

where $C, A_1, \dots, A_m \in \mathbb{S}^m$, $b_1, \dots, b_m \in \mathbb{R}$, is said to be strictly feasible if there is some $X \in \mathbb{S}^n_{++}$ satisfying the equality constraints $\text{trace}(A_i X) = b_i$ for $i = 1, \dots, m$.

(e) Find the optimal primal value p^* and the optimal dual value d^*. Does strong duality hold?

2 Ellipsoid intersection

Consider the following QCQP with a single quadratic constraint

$$\begin{aligned} p^* := \min_{x \in \mathbb{R}^n} \ & x^T A_0 x + 2b_0^T x + c_0 \\ \text{s.t.} \ & x^T A_1 x + 2b_1^T x + c_1 \leq 0, \end{aligned} \tag{2}$$

where $A_0, A_1 \in \mathbb{S}_+^n, b_0, b_1 \in \mathbb{R}^n, c_0, c_1 \in \mathbb{R}$.

Note that this problem formulation allows us to determine whether a given ellipsoid in $\mathbb{R}^n$ intersects another ellipsoid. This is because, checking whether the ellipsoid

$$\{x \in \mathbb{R}^n : (x - x_1)^T A_1 (x - x_1) \leq d_1\}$$

intersects the ellipsoid

$$\{x \in \mathbb{R}^n : (x - x_0)^T A_0 (x - x_0) \leq d_0\}$$

is the same as checking whether the set of $x \in \mathbb{R}^n$ satisfying

$$x^T A_1 x - 2x_1^T A_1 x + x_1^T A_1 x_1 - d_1 \leq 0$$

intersects the set of $x \in \mathbb{R}^n$ satisfying

$$x^T A_0 x - 2x_0^T A_0 x + x_0^T A_0 x_0 - d_0 \leq 0.$$

If we could solve the optimization problem in (2) with

$$\begin{aligned} b_0 &:= -A_0^T x_0, \quad c_0 := x_0^T A_0 x_0 - d_0, \\ b_1 &:= -A_1^T x_1, \quad c_1 := x_1^T A_1 x_1 - d_1 \end{aligned}$$

then this intersection would be nonempty iff the value of this optimization problem is nonpositive.

(a) Write the Lagrangian of the primal problem in (2) and calculate the dual objective function.

(b) Using the Schur complement rule and a slack variable to represent the objective function, show that the dual problem can be written as the following SDP in inequality form:

$$\begin{aligned} d^* := \max_{\lambda \in \mathbb{R}, \gamma \in \mathbb{R}} \ & \gamma \\ \text{s.t.} \ & \begin{bmatrix} A_0 + \lambda A_1 & b_0 + \lambda b_1 \\ b_0^T + \lambda b_1^T & c_0 + \lambda c_1 - \gamma \end{bmatrix} \succeq 0, \\ & \lambda \geq 0. \end{aligned} \tag{3}$$

Remark: For this problem you will need to use to a generalized version of the Schur complement rule. A proof is provided in the the addendum of the discussion set, but you can take this for granted if you wish.

The traditional Schur complement rule, which we have proved in class, says that if we have a symmetric block matrix

$$M := \begin{bmatrix} A & X \\ X^T & B \end{bmatrix},$$

and if we assume that A is positive definite, then:

(i) M is positive semidefinite iff $B - X^T A^{-1} X$ is positive semidefinite;

(ii) M is positive definite iff $B - X^T A^{-1} X$ is positive definite.

Here we use the fact that A is nonsingular to be able to write A^{-1}.

The more general version of the Schur complement rule works even if A is only positive semidefinite but not necessarily positive definite. It says

(iii) M is positive semidefinite iff $B - X^T A^{\dagger} X$ is positive semidefinite and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$.

Here $A^{\dagger}$ denotes the Moore-Penrose inverse of A.

(c) Since the dual is an SDP in inequality form, we can now take the dual of the dual (called the bidual), which will be an SDP in standard form. Show that, after some simplification, the bidual problem can be written as

$$\begin{aligned} \min_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \quad & \text{trace}(A_0 X) + 2b_0^T x + c_0 \\ \text{s.t.} \quad & \text{trace}(A_1 X) + 2b_1^T x + c_1 \leq 0, \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0. \end{aligned}$$

(d) What is the relation of the initial primal problem with the bidual derived in the previous part of the question?

Addendum: (**Optional**)

We will prove the more general version of the Schur complement rule. Consider the symmetric block matrix

$$M := \begin{bmatrix} A & X \\ X^T & B \end{bmatrix},$$

and assume that $A \in \mathbb{S}^n$ is positive semidefinite. Then we can write

$$A = U_r \Sigma_r U_r^T,$$

where $U_r \in \mathbb{R}^{n \times r}$ has columns forming an orthonormal basis for $\mathcal{R}(A)$ and Σ_r is an invertible $r \times r$ diagonal matrix. Here $r = \text{rank}(A)$. For concreteness, let $B \in \mathbb{S}^m$.

Suppose first that $B - X^T A^\dagger X$ is not positive semidefinite. Then there is $v \in \mathbb{R}^m$ such that

$$v^T (B - X^T A^\dagger X) v < 0.$$

Given $w \in \mathbb{R}^n$, consider

$$\begin{aligned} \begin{bmatrix} w^T & v^T \end{bmatrix} M \begin{bmatrix} w \\ v \end{bmatrix} &= \begin{bmatrix} w^T & v^T \end{bmatrix} \begin{bmatrix} U_r \Sigma_r U_r^T & X \\ X^T & B \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\ &= w^T U_r \Sigma_r U_r^T w + 2 w^T X v + v^T B v \end{aligned}$$

Now choose $w = -A^\dagger X v = -U_r \Sigma_r^{-1} U_r^T X v$. This gives

$$\begin{bmatrix} w^T & v^T \end{bmatrix} M \begin{bmatrix} w \\ v \end{bmatrix} = v^T (B - X^T A^\dagger X) v < 0,$$

which shows that M is not positive semidefinite.

Next suppose $\mathcal{R}(X) \subsetneq \mathcal{R}(A)$. Then there is $w \in \mathcal{N}(A)$ and $v \in \mathbb{R}^m$ such that $w^T X v < 0$. For $\alpha \in \mathbb{R}$ we write

$$\begin{aligned} \begin{bmatrix} w^T & \alpha v^T \end{bmatrix} M \begin{bmatrix} w \\ \alpha v \end{bmatrix} &= \begin{bmatrix} w^T & \alpha v^T \end{bmatrix} \begin{bmatrix} A & X \\ X^T & B \end{bmatrix} \begin{bmatrix} w \\ \alpha v \end{bmatrix} \\ &= w^T A w + 2\alpha w^T X v + \alpha^2 v^T B v \\ &= 2\alpha w^T X v + \alpha^2 v^T B v, \end{aligned}$$

where the last step is because $w^T A w = 0$, which comes from $w \in \mathcal{N}(A)$. For $\alpha > 0$ sufficiently small, this is negative, because $w^T X v < 0$. Hence M is not positive semidefinite.

We have shown that if M is positive semidefinite then we must have both $B - X^T A^\dagger X$ is positive semidefinite and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. We will now assume that we have both $B - X^T A^\dagger X$ is positive semidefinite and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ and show that this implies that M is positive semidefinite, which will complete the proof.

Note that $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ implies that $X = U_r U_r^T X$. Hence we have

$$M = \begin{bmatrix} U_r \Sigma_r U_r^T & X \\ X^T & B \end{bmatrix} = \begin{bmatrix} U_r \Sigma_r U_r^T & U_r U_r^T X \\ X^T U_r U_r^T & B \end{bmatrix} = \begin{bmatrix} U_r & 0_{n,m} \\ 0_{m,r} & I_m \end{bmatrix} \begin{bmatrix} \Sigma_r & U_r^T X \\ X^T U_r & B \end{bmatrix} \begin{bmatrix} U_r^T & 0_{r,m} \\ 0_{m,n} & I_m \end{bmatrix}.$$

The traditional Schur complement rule tells us that since $B - X^T A^\dagger X$ is positive semidefinite we have that $\begin{bmatrix} \Sigma_r & U_r^T X \\ X^T U_r & B \end{bmatrix}$ is positive semidefinite (to see this, recall that $A^\dagger = U_r \Sigma_r^{-1} U_r^T$). But M is written above as a congruence transformation of this matrix. Hence M is positive semidefinite. This completes the proof.