EECS 127/227AT Optimization Models in Engineering
Spring 2020 **Discussion 0**

1. **Least squares and Gram-Schmidt**

Consider the least squares problem,

$$\vec{x}^* = \arg\min_{\vec{x}\in\mathbb{R}^n} \left\| A\vec{x} - \vec{b} \right\|_2^2,$$

where $A \in \mathbb{R}^{m\times n}, \vec{b} \in \mathbb{R}^m$ and assume A is full column rank. One way to solve this least-squares problems is to use Gram-Schmidt Orthonormalization (GSO). Using GSO, the matrix A can be written as,

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where Q is an orthonormal (orthogonal) matrix and R is an upper-triangular matrix.

The columns of Q_1 form an an orthonormal basis for the range $\mathcal{R}(A)$ and columns of Q_2 form an orthonormal basis for the range $\mathcal{R}(A)^{\perp}$. Moreover, R_1 is upper triangular and invertible.

1. Show that the squared norm of the residual is given by

$$\|\vec{r}\|_2^2 \triangleq \|\vec{b} - A\vec{x}\|_2^2 = \left\| \left(Q_1^\top \vec{b} - R_1\vec{x} \right) \right\|_2^2 + \left\| Q_2^\top \vec{b} \right\|_2^2. \quad (1)$$

2. Find $\vec{x}^*$ such that the squared norm of the residual in Equation (1) is minimized. Your expression for $\vec{x}^*$ should only use some or all of the following terms: Q_1,Q_2,R_1,$\vec{b}$.
3. Check if the expression for $\vec{x}^*$ obtained in the previous part is equivalent to the one obtained by the formula, $\vec{x}^* = (A^\top A)^{-1}A^\top\vec{b}$.

2. **Eigenvalues**

Let $A \in \mathbb{R}^{n\times n}$ have the eigendecomposition $P\Lambda P^{-1}$ where Λ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Note that this is equivalent to stating that A is diagonalizable via the transformation,

$$P^{-1}AP = \Lambda.$$

1. Show that $A^m = P\Lambda^m P^{-1}$, for integer $m \geq 1$.
2. Show that determinant of A is the product of its eigenvalues, i.e.

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Hint: We have the identity $\det(XY) = \det(X)\det(Y)$.