EECS 127/227AT Optimization Models in Engineering
Spring 2020

Discussion 5

1. **Simple constrained optimization problem**

   Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \; & f(x_1, x_2) \\ \text{subject to } & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

   (a) Make a sketch of the feasible set.

   For each of the following objective functions, give the optimal set or the optimal value.

   (b) $f(x_1, x_2) = x_1 + x_2$
   (c) $f(x_1, x_2) = -x_1 - x_2$
   (d) $f(x_1, x_2) = x_1$
   (e) $f(x_1, x_2) = \max\{x_1, x_2\}$
   (f) $f(x_1, x_2) = x_1^2 + 9x_2^2$

2. **About general optimization**

   In this exercise, we test your understanding of the general framework of optimization and its language. We consider an optimization problem in standard form:

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \; f_0(\vec{x}) \; : \; f_i(\vec{x}) \leq 0, \;\; i = 1, \ldots, m.$$

   In the following we denote by $\mathcal{X}$ the feasible set. Note that the feasible set is a subset of R^n that satisfies the inequalities $f_i(\vec{x}) \leq 0$, i.e $\mathcal{X} = \{\vec{x} \in \mathbb{R}^n \mid f_i(\vec{x}) \leq 0, i = 1, \ldots, m\}$. We make no assumption about the convexity of $f_0(\vec{x})$ and $f_i(\vec{x})$, $i = 1, \ldots, m$. For the following statements, provide a proof or counter-example.

   (a) A general optimization problem can be expressed as one with a linear objective.
   (b) A general optimization problem can be expressed as one without any constraints.
   (c) If at the optimal point $\vec{x}^\star$, one constraint is not active (i.e. $f_i(\vec{x}^\star) < 0$), then we can remove the constraint from the original problem and obtain the same optimum value.
   (d) If the problem is convex, and at the optimal point $\vec{x}^\star$, one constraint is not active ($f_i(\vec{x}^\star) < 0$), then we can remove the constraint from the original problem and obtain the same optimum value.

   Assume that the minimum is attained for some $\vec{x}^* \in \mathbb{R}^n$.

3. Convexity and composition of functions

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. Define the composition of f with g as $h = f \circ g : \mathbb{R}^n \to \mathbb{R}$ such that $h(\vec{x}) = f(g(\vec{x}))$.

(a) Show that if f is convex and non decreasing and g is convex, then h is convex.

(b) Show that there exists f non decreasing and g convex, such that $h = f \circ g$ is not convex.

(c) Show that there exists f convex and g convex such that $h = f \circ g$ is not convex.