EECS 127/227AT Optimization Models in Engineering
Spring 2020 Discussion 6

1. Simple constrained optimization problem with duality

Consider the optimization problem

$$\begin{aligned}\min_{x_1,x_2\in\mathbb{R}} & f(x_1,x_2)\\ \text{subject to } & 2x_1+x_2\geq 1\\ & x_1+3x_2\geq 1\\ & x_1\geq 0,\ x_2\geq 0\end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$

Solution:

$$\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)=f(x_1,x_2)+\lambda_1(-2x_1-x_2+1)+\lambda_2(1-x_1-3x_2)-\lambda_3x_1-\lambda_4x_2$$

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.

Solution: $-\mathcal{L}$ is convex in $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as a affine function of $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. So $\mathcal{L}$ is concave in $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$

(c) Express the dual function of the problem, and show that it is concave.

Solution: $g(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\min_{x_1,x_2}\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$.
We can show that by showing that $-g$ is convex.

$$\begin{aligned}-g(\vec{\lambda}) &= -\min_{x_1,x_2}\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\ &= \max_{x_1,x_2}-\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\end{aligned}$$

When (x_1,x_2) is fixed, the function $-\mathcal{L}$ is linear in $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, therefore convex.
Because the max of convex functions is convex, $-g$ is convex. Therefore g is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1,x_2).

Solution: $\mathcal{L}$ is convex in (x_1,x_2) because it is the sum of convex functions.

(e) Denoting $\mathcal{X}=\{(x_1,x_2)\mid 2x_1+x_2\geq 1,\ x_1+3x_2\geq 1,\ x_1\geq 0,\ x_2\geq 0\}$, show that

$$\max_{\lambda_1\geq 0,\lambda_2\geq 0,\lambda_3\geq 0,\lambda_4\geq 0}\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\begin{cases}f(x_1,x_2) & \text{if } (x_1,x_2)\in\mathcal{X}\\ +\infty & \text{otherwise}\end{cases}$$

Solution: Let's just do it for λ_4:

$$\begin{aligned}
\max_{\lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \max_{\lambda_4 \geq 0} \left(f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2\right) \\
&= f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 + \max_{\lambda_4 \geq 0} -\lambda_4 x_2
\end{aligned}$$

$$\max_{\lambda_4 \geq 0} -\lambda_4 x_2 = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

One can show the same results for λ_1, λ_2 and λ_3, resulting in:

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases}$$

(f) Conclude that $\min_{(x_1,x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2) \in \mathcal{X}} f(x_1, x_2)$

Solution:

$$\begin{aligned}
\min_{x_1, x_2} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \min_{x_1, x_2} \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \\
&= \min_{(x_1,x_2) \in \mathcal{X}} f(x_1, x_2)
\end{aligned}$$

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

Solution:

$$\begin{aligned}
\nabla_{x_1, x_2} \mathcal{L}(x_1^\star, x_2^\star, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0 & \\
&= \nabla_{x_1, x_2} f(x_1^\star, x_2^\star) + \begin{pmatrix} -2\lambda_1 - \lambda_2 - \lambda_3 \\ -\lambda_1 - 3\lambda_2 - \lambda_4 \end{pmatrix}
\end{aligned}$$

2. Lagrangian Dual of a QP

Consider the general form of a convex quadratic program, with $Q \succ 0$:

$$\begin{aligned}
\min_{\vec{x}} \quad & \frac{1}{2} \vec{x}^\top Q \vec{x} \\
\text{s.t. } & A\vec{x} \leq \vec{b}
\end{aligned}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

Solution:

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = \frac{1}{2} \vec{x}^\top Q \vec{x} + \vec{\lambda}^\top (A\vec{x} - \vec{b})$$

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

Solution:

$$g(\vec{\lambda}) = \inf_{\vec{x}} \ \mathcal{L}(\vec{x}, \vec{\lambda})$$

We can find this infimum by setting $\nabla_{\vec{x}} \mathcal{L}(\vec{x}^*, \vec{\lambda}) = 0$:

$$Q\vec{x}^* + A^\top \vec{\lambda} = 0 \implies \vec{x}^* = -Q^{-1} A^\top \vec{\lambda}$$

Substituting, we get

$$\begin{aligned} g(\vec{\lambda}) &= \mathcal{L}(\vec{x}^*, \vec{\lambda}) \\ &= \frac{1}{2} \vec{\lambda}^\top A Q^{-\top} A^\top \vec{\lambda} - \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} \\ &= -\frac{1}{2} \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} \end{aligned}$$

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?

Solution: The Lagrangian dual problem writes

$$\max_{\vec{\lambda} \geq 0} \ g(\vec{\lambda}) = \max_{\vec{\lambda} \geq 0} \ -\frac{1}{2} \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b},$$

the maximization of a concave function of $\vec{\lambda}$ over the convex region given by the non-negative orthant $\vec{\lambda} \geq 0$. The dual problem is therefore convex.

While in this problem, the primal problem was convex, it turns out that the Lagrangian dual problem is a convex problem even when the primal is not. To see this, examine its general form:

$$\max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) = \max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \left[f_0(\vec{x}) + \sum_{i=1}^{n} \lambda_i f_i(\vec{x}) \right]$$

This represents the pointwise minimum of affine functions of $\vec{\lambda}$, which we know to be concave. The resulting maximization problem of a concave objective in $\vec{\lambda}$ over the convex region $\vec{\lambda} \geq 0$ is then a convex optimization problem!