EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 6

1. **Simple constrained optimization problem with duality**

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \; & f(x_1, x_2) \\ \text{subject to } & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(c) Express the dual function of the problem, and show that it is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2).

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1, \; x_1 + 3x_2 \geq 1, \; x_1 \geq 0, \; x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases}$$

(f) Conclude that $\min_{(x_1, x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1, x_2) \in \mathcal{X}} f(x_1, x_2)$

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

2. **Lagrangian Dual of a QP**

Consider the general form of a convex quadratic program, with $Q \succ 0$:

$$\begin{aligned} \min_x \; & \frac{1}{2} x^\top Q x \\ \text{s.t. } & Ax \leq b \end{aligned}$$

(a) Write the Lagrangian function $\mathcal{L}(x, \lambda)$.

(b) Write the Lagrangian dual function, $g(\lambda)$.

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?