EECS 127/227AT **Fall 2020**
Discussion 8

1 Least squares with equality constraints

Consider the least squares problem with equality constraints

$$\min_x \|Ax - b\|_2^2 \;:\; Gx = h, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $G \in \mathbb{R}^{p \times n}$ and $h \in \mathbb{R}^p$. For simplicity, we will assume that $\text{rank}(A) = n$ and $\text{rank}(G) = p$.

Using the KKT conditions, determine the optimal solution of this optimization problem.

2 Distance between polytopes

Let $p^{(1)}, \dots, p^{(r)}$ and $q^{(1)}, \dots, q^{(s)}$ be points in $\mathbb{R}^d$, where $r, s \geq 1$. Let $\mathcal{P}$ denote the polytope defined as the convex hull of $\{p^{(1)}, \dots, p^{(r)}\}$, and $\mathcal{Q}$ the polytope defined as the convex hull of $\{q^{(1)}, \dots, q^{(s)}\}$. Thus every point in $\mathcal{P}$ can be written as $\sum_{i=1}^{r} x_i p^{(i)}$ for som $x_i \geq 0$, $1 \leq i \leq r$ such that $\sum_{i=1}^{r} x_i = 1$, and every point in $\mathcal{Q}$ can be written as $\sum_{j=1}^{s} x_{r+i} q^{(j)}$ for some $x_i \geq 0$, $r+1 \leq i \leq n$ such that $\sum_{i=r+1}^{n} x_i = 1$, where $n := r + s$.

Define the matrix $C \in \mathbb{R}^{d \times n}$ whose i-th column is $p^{(i)}$, $1 \leq i \leq r$ and whose $r+j$-th column is $-q^{(j)}$, $1 \leq j \leq s$.

(a) Pose the problem of finding the minimum squared ℓ_2 distance between points in $\mathcal{P}$ and points in $\mathcal{Q}$ as a quadratic program with objective function $\|Cx\|_2^2$, viewed as a function on $\mathbb{R}^n$.

(b) Define $y := Cx$. Show that QP found in the preceding part of this question can be expressed as a QP with the objective function $\|y\|_2^2$, viewed as a function of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$.

(c) Show that the dual to the QP in the preceding part of this question takes the form of the unconstrained QP

$$\max_z \left(-\frac{1}{4} z^T z + \min_{1 \leq i \leq r} z^T p^{(i)} - \max_{1 \leq j \leq s} z^T q^{(j)} \right),$$

(d) Provide a geometric interpretation of the dual problem formulated in the preceding part of this question.