EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 1

1. Understanding ellipses

Consider the Euclidean space $\mathbb{R}^2$ with the orthogonal basis $\{\vec{e_1}, \vec{e_2}\}$. In this exercise, we study the ellipse

$$\mathcal{E} = \left\{ x_1\vec{e_1} + x_2\vec{e_2} \;\middle|\; x_1, x_2 \in \mathbb{R},\; \left(\sqrt{5}x_1 - \frac{3\sqrt{5}}{5}x_2\right)^2 + \left(\frac{4\sqrt{5}}{5}x_2\right)^2 \leq 8 \right\}.$$

1. Show that we can express the ellipse as $\mathcal{E} = \{\vec{x} \in \mathbb{R}^2 \mid \vec{x}^\top A\vec{x} \leq 1\}$ for symmetric positive definite A, where

$$A = \frac{1}{8}\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

2. Show that the ellipse $\mathcal{E}$ can be viewed as a linear transformation of the unit disk by finding B such that $\mathcal{E} = \{B\vec{v} \mid \|\vec{v}\|_2 \leq 1\}$. Is this B unique?
3. Relate the length and direction of the semi-major and semi-minor axes of $\mathcal{E}$ to the singular values of B (or eigenvalues of A).
4. Compute the area of $\mathcal{E}$.

2. SVD

Suppose we have a matrix $A \in \mathbb{R}^{m\times n}$ with rank r.

We define the *compact (or "thin") SVD* of A as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U_r}_{m\times r}\underbrace{\Sigma_r}_{r\times r}\underbrace{V_r^\top}_{r\times n}.$$

Here, $\Sigma_r \in \mathbb{R}^{r\times r}$ is a diagonal matrix containing non-zero singular values of A:

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$.

Furthermore, $U_r \in \mathbb{R}^{m\times r}$ is given by

$$U_r = \left[\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\right],$$

where u_i is a left singular vector corresponding to non-zero singular value σ_i, for $i = 1, 2, \ldots, r$. The columns of U_r are orthonormal and together they span the columnspace of A. Why? (**Exercise**).

Finally, $V_r^\top \in \mathbb{R}^{r\times n}$ is given by

$$V_r^\top = \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vdots \\ \vec{v}_r^\top \end{bmatrix},$$

where v_j is a right singular vector corresponding to non-zero singular value σ_j for $j = 1, 2, \ldots, r$. The rows of $V_r^\top$ are orthonormal and span the rowspace of A. Equivalently, the columns of V_r span the column space of $A^\top$. Why? (**Exercise**).

Note that the matrix A can be expressed as

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top + \ldots + \sigma_r \vec{u}_r \vec{v}_r^\top.$$

Assume now that $m \geq n$. Another type of SVD — which might be more familiar — is the *full SVD* of A, which is defined as follows:

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m}\underbrace{\Sigma}_{m\times n}\underbrace{V^\top}_{n\times n}.$$

Here, all non-diagonal entries of $\Sigma \in \mathbb{R}^{m\times n}$ are zero. The diagonal entries of Σ contain the singular values of A, and we can write Σ in terms of Σ_r as

$$\Sigma = \left[\begin{array}{c|c} \Sigma_r & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right].$$

Continuing as above, $U \in \mathbb{R}^{m\times m}$ is an orthogonal matrix. U can be expressed in terms of U_r as

$$U = [\underbrace{U_r}_{m\times r}\ \underbrace{\vec{u}_{r+1}\ldots\vec{u}_m}_{m\times(m-r)}].$$

The columns $\vec{u}_{r+1}, \vec{u}_{r+2}, \ldots, \vec{u}_n$ are left singular values corresponding to singular value 0 and together span the nullspace of $A^\top$. Why? (**Exercise**).
Finally, $V^\top$ is an orthogonal matrix and can be expressed in terms of $V_r^\top$ as,

$$V^\top = \begin{bmatrix} V_r \\ \vec{v}_{r+1}^\top \\ \vdots \\ \vec{v}_n^\top \end{bmatrix} \begin{matrix} \} \quad r\times n \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad (n-r)\times n \end{matrix}$$

The rows $\vec{v}_{r+1}^\top, \vec{v}_{r+2}^\top, \ldots, \vec{v}_n^\top$ when transposed are the right singular vectors corresponding to singular value of 0 and together span the nullspace of A. Why? (**Exercise**).

1. *Properties of the decomposition matrices.* Assume that $m > n > r$. Which of the following are true?
   (a) $UU^\top = I$
   (b) $U^\top U = I$

(c) $V^\top V = I$
(d) $VV^\top = I$
(e) $U_r^\top U_r = I$
(f) $U_r U_r^\top = I$
(g) $V_r V_r^\top = I$
(h) $V_r^\top V_r = I$

When dealing with equations for matrices expressed in SVD form, it is crucial that you are clear on which of the previous matrix products simplify to the identity matrix and which do not.

2. *Building the compact SVD.* Let matrix A have full SVD

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the compact SVD of A.

3. *Building the full SVD.* Let matrix A have compact SVD

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Find the full SVD of A.