EECS 127/227AT Optimization Models in Engineering
Spring 2020

Homework 9

This homework is due Friday, April 3, 2020 at 23:00 (11pm).
Self grades are due Friday, April 10, 2020 at 23:00 (11pm).

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Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook with solutions saved as a PDF.

1. Does strong duality hold?
Consider

$$\min_{(x,y)\in\mathcal{D}} e^{-x}$$
$$\text{s.t. } x^2/y \leq 0$$

where $\mathcal{D} = \{(x, y) \mid y > 0\}$.

(a) Prove the problem is convex. Find the optimal value.
Hint: To prove the constraint function is convex, you will have to prove it is convex with respect to the vector $\begin{bmatrix} x & y \end{bmatrix}^\top$. *Consider computing the Hessian of the constraint function, its determinant and trace, and show that it is PSD by analyzing signs of its eigenvalues.*

(b) Next, we will proceed to find an optimal solution and an optimal value for the dual problem. The Lagrangian dual function $g(\lambda)$, can be written as:

$$g(\lambda) = \inf_{(x,y)\in\mathcal{D}} e^{-x} + \lambda \frac{x^2}{y}.$$

Explain why $g(\lambda)$ is lower bounded by 0 for $\lambda \geq 0$. *Note: Here we are not dualizing the constraint* $y > 0$ *that is in the definition of* $\mathcal{D}$ *— this is only dualizing the other constraint.*

(c) Show that $g(\lambda) = 0$ for $\lambda \geq 0$.
Hint 1: To show that the infimum in ((b)) is 0, we want to show there exist (x, y) *such that both* e^{-x} *and* $\lambda\frac{x^2}{y}$ *can get arbitrarily close to 0.*
Hint 2: Consider a sequence $\{x_k\}$ *going to* $+\infty$ *and a sequence* $\{y_k\}$ *also going to* $+\infty$ *such that* $\lim_{k\to\infty} \frac{x_k^2}{y_k} = 0$. *Simply put, we want to drive* x *to infinity in order to drive* e^{-x} *to* 0, *while having* y *grow faster than* x^2, *so that the second term also goes to* 0.

(d) Now, write the dual problem and find an optimal solution λ^* and an optimal value d^* for the dual problem using the results above. What is the duality gap?

(e) Does Slater's Condition hold for this problem? Does Strong Duality hold?

2. Visualizing the dual problem

Download the Jupyter notebook `visualize_dual.ipynb`; complete the code where designated and answer the questions given in the space provided. (If you prefer, for questions that do not involve writing code, you can write solutions on separate paper or LaTeX PDF, just make sure to correctly mark the relevant pages when uploading to Gradescope.)

3. Sensitivity and dual variables

In this problem, we explore the interpretation of dual variables as sensitivity parameters of the primal problem. Recall the canonical **convex** primal problem

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^n} \quad & f_0(\vec{x}) \\ \text{s.t.} \quad & f_i(\vec{x}) \leq 0, \quad i = 1, \ldots, m \\ & h_j(\vec{x}) = 0, \quad j = 1, \ldots, p \end{aligned}$$

where f_i are convex for all $i = 0, \ldots, m$ and h_j are affine for all $j = 1, \ldots, p$. **Assume strong duality holds.**

Here, we consider the *perturbed* problem

$$\begin{aligned} \min_{\vec{x}\in\mathbb{R}^n} \quad & f_0(\vec{x}) \\ \text{s.t.} \quad & f_i(\vec{x}) \leq u_i, \quad i = 1, \ldots, m \\ & h_j(\vec{x}) = v_j, \quad j = 1, \ldots, p \end{aligned}$$

and define

$$p^*(\vec{u}, \vec{v}) = \inf \{ f_0(\vec{x}) \mid f_i(\vec{x}) \leq u_i \;\; \forall i, \;\; h_j(\vec{x}) = v_j \;\; \forall j \}$$

for perturbation vectors $\vec{u} = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}^\top$ and $\vec{v} = \begin{bmatrix} v_1 & \ldots & v_p \end{bmatrix}^\top$. In other words, $p^*(\vec{u}, \vec{v})$, **is a function of** $\vec{u}$ and $\vec{v}$ that gives the optimal value for the perturbed problem (if it is feasible). If the problem is infeasible (i.e. no points exist that satisfy the constraints), we say that $p^*(\vec{u}, \vec{v}) = +\infty$ otherwise. Note that $p^*(\vec{0}, \vec{0})$ is the original problem.

(a) Prove that $p^*(\vec{u}, \vec{v})$ is jointly convex [1] in $(\vec{u} \in \mathbb{R}^m, \;\; \vec{v} \in \mathbb{R}^p)$.
Hint: Let $\mathcal{D} = \{\vec{x} \in \mathbb{R}^n \mid f_i(\vec{x}) \leq u_i \;\; \forall i, \;\; h_j(\vec{x}) = v_j \;\; \forall j\}$ denote the feasible set. Now define $F(\vec{x}, \vec{u}, \vec{v})$ to be a function that is equal to $f_0(\vec{x})$ on $\mathcal{D}$ and $+\infty$ otherwise. Show $F(\vec{x}, \vec{u}, \vec{v})$ is convex and then observe that

$$p^*(\vec{u}, \vec{v}) = \min_{\vec{x}} F(\vec{x}, \vec{u}, \vec{v}).$$

(b) Show that

$$p^*(\vec{u}, \vec{v}) \geq p^*(\vec{0}, \vec{0}) - \vec{\lambda}^{*\top} \vec{u} - \vec{\nu}^{*\top} \vec{v}$$

where $\vec{\lambda}^* \in \mathbb{R}^m$ and $\vec{\nu}^* \in \mathbb{R}^p$ are the optimal dual variables for the dual of the unperturbed primal problem (corresponding to inequality and equality constraints, respectively). *Hint: Consider the dual function $g(\vec{\lambda}, \vec{\nu})$ for the perturbed problem at $\lambda = \lambda^*, \nu = \nu^*$ and upper-bound it by the Lagrangian at a feasible point $\vec{x}$. Here λ^* and ν^* are the optimal dual variables for the original unperturbed problem.*

[1] Recall that a function $f : A \times B \to \mathbb{R}$ is jointly convex in $(\vec{a} \in A, \;\; \vec{b} \in B)$ if for all $\theta \in [0, 1]$, and for all $\vec{a}_1, \vec{a}_2 \in A, \;\; \vec{b}_1, \vec{b}_2 \in B$, we have that $f(\theta\vec{a}_1 + (1-\theta)\vec{a}_2, \;\; \theta\vec{b}_1 + (1-\theta)\vec{b}_2) \leq \theta f(\vec{a}_1, \vec{b}_1) + (1-\theta) f(\vec{a}_2, \vec{b}_2)$.

(c) Suppose we only have one equality and one inequality constraint (i.e., $\vec{u} = u$ and $\vec{v} = v$ are scalars). For each of the following situations, argue whether:

(A) the value of $p^*(u, v)$ increases as compared with $p^*(0, 0)$,

(B) the value of $p^*(u, v)$ decreases as compared with $p^*(0, 0)$, or

(C) we can make no claims on the relationship between $p^*(u, v)$ and $p^*(0, 0)$.

Hint: Use the bound you computed in (b).

i. λ^* is large (as compared with ν^*) and $u < 0$.
ii. λ^* is large (as compared with ν^*) and $u > 0$.
iii. ν^* is large (as compared with λ^*) and positive and $v < 0$.
iv. ν^* is large (as compared with λ^*) and negative and $v > 0$.

Note that we can think of u and v as variables we choose — by examining how the solution to our original primal problem changes, we can describe how "sensitive" our problem is to its different constraints!

4. KKT with circles

Consider the problem

$$\begin{aligned} \min_{\vec{x} \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \end{aligned}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$.

(a) Sketch the feasible region and the level sets of the objective function. Find the optimal point $\vec{x}^*$ and the optimal value p^*.

(b) Does strong duality hold?

(c) Write the KKT conditions for this optimization problem. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove the optimality of $\vec{x}^*$?

5. Water filling

Consider the following problem:

$$\begin{aligned} \min_{\vec{x}} \; & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{s.t.} \quad & x_i \geq 0, \; i = 1, \ldots, n, \\ & \vec{1}^\top \vec{x} = 1, \end{aligned}$$

where each scalar $\alpha_i > 0$ for $i = 1, \ldots, n$.

This problem arises in information theory when we wish to allocate power to a set of n communication channels. Each variable x_i represents the transmitter power allocated to the i^{th} channel, and $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the channel, so the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate.
Note: This is Example 5.2 on Page 245 of Boyd's book.

(a) Write the KKT conditions.

(b) Find the primal solution $\vec{x}^*$ using the KKT conditions.

6. Homework process

Whom did you work with on this homework? List the names and SIDs of your group members.