EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Homework 13

This homework is NEVER DUE. All problems are intended as practice for the final exam, and problems and solutions have been released simultaneously.

1. Multiple Choice

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Consider the following optimization problems:

$$\begin{aligned} p_1^* = \min_{t\in\mathbb{R},\vec{x}\in\mathbb{R}^n} \quad & t \\ \text{s.t. } & \|\vec{x}\|_2 = t, \\ & f(\vec{x}) \le 0, \end{aligned} \tag{1}$$

$$\begin{aligned} p_2^* = \min_{t\in\mathbb{R},\vec{x}\in\mathbb{R}^n} \quad & t \\ \text{s.t. } & \|\vec{x}\|_2 \le t, \\ & f(\vec{x}) \le 0. \end{aligned} \tag{2}$$

Write the statement labels (A, B, C) corresponding to statements that are true in the box given below. More than one statement might be true; and you will get credit for this problem only if you write the labels corresponding to all statements that are true and do not write a label corresponding to any statement that is false. No justification is required.

(A) Problem (1) as written is a convex problem.
(B) Problem (2) as written is a convex problem.
(C) We necessarily have $p_1^* = p_2^*$.

2. Linear algebra meets optimization Let wide matrix $A \in \mathbb{R}^{m\times n}$ $(m < n)$ be full row rank.

(a) Consider the ridge regression problem, where $\vec{b} \in \mathbb{R}^m, x \in \mathbb{R}^n$ and the constant $\lambda > 0$ is given:

$$\min_{\vec{x}} \quad ||A\vec{x} - \vec{b}||_2^2 + \lambda||\vec{x}||_2^2 \tag{3}$$

Since this is a convex problem and the objective function is differentiable, the optimum can be found by setting the gradient to zero. Use this to find the optimal solution $\vec{x}^*$.

(b) Now we rewrite the problem in (3) by adding a constraint

$$\min_{\vec{z}=A\vec{x}-\vec{b}} ||\vec{z}||_2^2 + \lambda||\vec{x}||_2^2. \tag{4}$$

Let the Lagrangian corresponding to this problem be $\mathcal{L}(\vec{x}, \vec{z}, \vec{\nu})$, where $\vec{\nu}$ is the dual variable corresponding to the equality constraint. Write out the dual function $g(\vec{\nu}) = \inf_{\vec{x},\vec{z}} \mathcal{L}(\vec{x}, \vec{z}, \vec{\nu})$ explicitly. Solve the dual problem to get $\vec{\nu}^*$. Find the corresponding values of $\tilde{\vec{x}}, \tilde{\vec{z}}$ such that $g(\vec{\nu}^*) = \mathcal{L}(\tilde{\vec{x}}, \tilde{\vec{z}}, \vec{\nu}^*)$.

(c) Show that for every $\lambda > 0$,

$$(A^\top A + \lambda I)^{-1} A^\top \vec{b} = A^\top (AA^\top + \lambda I)^{-1} \vec{b}.$$

Hint: One approach is to start by considering $\lambda A^\top + A^\top A A^\top$. Another approach is to use the SVD of A.

3. Best Approximation in the Uniform norm

Let $(x_1, y_1), \cdots, (x_n, y_n) \in \mathbb{R}^2$ be the given data points, and define vectors $\vec{x} = [x_1, \cdots, x_n]^\top$ and $\vec{y} = [y_1, \cdots, y_n]^\top$.

(a) We want to find $a, b \in \mathbb{R}$ that minimizes $\|a\vec{x} + b\vec{1} - \vec{y}\|_\infty$, where $\vec{1}$ is an n-dimensional vector of ones. Formulate this problem as an LP.

(b) Now we want to find $a, b \in \mathbb{R}$ that minimizes $\|a\vec{x} + b\vec{1} - \vec{y}\|_1$, where $\vec{1}$ is an n-dimensional vector of ones. Formulate this problem as an LP.

4. Newton's method

Given a symmetric positive definite matrix $Q \in \mathbb{S}^n_{++}$ and $\vec{b} \in \mathbb{R}^n$, consider the minimization of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(\vec{x}) = \frac{1}{2}\vec{x}^\top Q \vec{x} - \vec{b}^\top \vec{x}.$$

Let $\vec{x}^*$ denote the point at which $f(\vec{x})$ is minimized, and define $\mathcal{B}(\vec{x}^*)$ as the ball centered at $\vec{x}^*$ with unit ℓ_2-norm:

$$\mathcal{B}(\vec{x}^*) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}^*\|_2 \leq 1\}.$$

Assume we use Newton's method to minimize f:

$$\vec{x}_{k+1} = \vec{x}_k - (\nabla^2 f(\vec{x}_k))^{-1} \nabla f(\vec{x}_k),$$

where the initial point is $\vec{x}_0 \in \mathcal{B}(\vec{x}^*)$. For any $k \in \mathbb{N}$, find

$$\max_{\vec{x}_0 \in \mathcal{B}(\vec{x}^*)} \|\vec{x}_k - \vec{x}^*\|_2.$$