EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 7

1. A simple example of strong duality

Consider the following minimization problem, with $\epsilon \in \mathbb{R}$:

$$p^\star = \min_{x \in \mathbb{R}} \; x^2 \quad \text{s.t. } x \geq \epsilon$$

(a) Solve this optimization problem for p^*.

Solution: $x^\star = \max\{0, \epsilon\}$

so $p^\star = \begin{cases} 0 & \text{if } \epsilon \leq 0 \\ \epsilon^2 & \text{otherwise} \end{cases}$

(b) Write the Lagrangian function $\mathcal{L}(x, \lambda)$.

Solution: $\mathcal{L}(x, \lambda) = x^2 + \lambda(\epsilon - x)$

(c) Write the Lagrangian dual function $g(\lambda)$.

Solution:

$$g(\lambda) = \min_x \; \mathcal{L}(x, \lambda)$$

$\mathcal{L}$ is convex and differentiable in x, so $x^\star$ is such that $\nabla_x \mathcal{L}(x^\star, \lambda) = 0$.
This gives that $x^\star = \frac{\lambda}{2}$.
So $g(\lambda) = \frac{1}{4}\lambda^2 + \lambda(\epsilon - \frac{\lambda}{2}) = -\frac{1}{4}\lambda^2 + \lambda\epsilon$

(d) Solve the dual problem. Does strong duality hold?

Solution: $g(\lambda) = -\frac{1}{4}\lambda^2 + \lambda\epsilon = -\frac{1}{4}(\lambda - 2\epsilon)^2 + \epsilon^2$.

$d^\star = \max_{\lambda \geq 0} g(\lambda) = \begin{cases} 0 & \text{if } \epsilon \leq 0 \\ \epsilon^2 & \text{otherwise} \end{cases}$

Here $d^\star = p^\star$, strong duality holds.

(e) Give the value of the dual variable λ that maximizes the dual problem as a function of ϵ.

Solution: $\lambda^\star = \max\{0, 2\epsilon\}$

2. Dual of an LP

Consider the general form of a linear program:

$$\min_{\vec{x}} \ \vec{c}^\top \vec{x}$$
$$\text{s.t. } A\vec{x} = \vec{b}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\mu})$.

Solution:

$$\mathcal{L}(\vec{x}, \vec{\mu}) = \vec{c}^\top \vec{x} + \vec{\mu}^\top (A\vec{x} - \vec{b})$$

(b) Write the Lagrangian dual function $g(\vec{\mu})$.

Solution:

$$g(\mu) = \inf_{\vec{x}} \ \mathcal{L}(\vec{x}, \vec{\mu}) = \begin{cases} -\vec{b}^\top \vec{\mu} & \text{if } \vec{c} + A^\top \vec{\mu} = \vec{0} \\ -\infty & \text{otherwise} \end{cases}$$

(c) Write the dual problem.

Solution: Substituting, we get

$$\begin{aligned} d^\star = \max_{\vec{\mu}} \quad & -\vec{b}^\top \vec{\mu} \\ \text{s.t. } & A^\top \vec{\mu} + \vec{c} = \vec{0} \end{aligned}$$