EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Homework 8

This homework is due Friday, March 20, 2020 at 23:00 (11pm).
Self grades are due Friday, April 3, 2020 at 23:00 (11pm).

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Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook with solutions saved as a PDF.

Note to students: The final problem on the exam brought together many different concepts we studied through the first half of the class. We want to give you a chance to think through the problem and learn from it in a relaxed and collaborative way. We provide some solution skeletons and additional hints below to help you work through the problems.

1. **Duality**

Consider the function

$$f(\vec{x}) = \vec{x}^\top A\vec{x} - 2\vec{b}^\top \vec{x}.$$

First, we consider the unconstrained optimization problem

$$p^* = \min_{\vec{x}\in\mathbb{R}^n} f(\vec{x}) = \min_{\vec{x}\in\mathbb{R}^n} \vec{x}^\top A\vec{x} - 2\vec{b}^\top \vec{x} \tag{1}$$

for a real $n \times n$ **symmetric** matrix $A \in \mathbb{S}^n$ and $\vec{b} \in \mathbb{R}^n$. If the problem is unbounded below, then we say $p^* = -\infty$. Let $\vec{x}^*$ denote the minimizing argument of the optimization problem.

(a) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \in \mathcal{R}(A)$. Let $\operatorname{rank}(A) = n$. **Find p^*.**

Hint: What does $A \succeq 0$ tell you about the function f? How can you leverage the rank of A to compute p^*?

Solution: $p^* = -\vec{b}^\top A^{-1}\vec{b}.$

(b) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \in \mathcal{R}(A)$ as before. Let A be rank-deficient, i.e., $\operatorname{rank}(A) = r < n$. Let A have the compact/thin and full SVD as follows, with diagonal positive definite $\Lambda_r \in \mathbb{R}^{r\times r}$:

$$A = U_r\Lambda_r U_r^\top = \begin{bmatrix} U_r & U_1 \end{bmatrix} \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_r^\top \\ U_1^\top \end{bmatrix}.$$

Show that the minimizer $\vec{x}^*$ of the optimization problem (1) is not unique by **finding a general form for the family of solutions for $\vec{x}^*$** in terms of $U_r, U_1, \Lambda_r, \vec{b}$.

Hint: As before, $A \succeq 0$ gives you some information about the objective function f. Can you use this information along with the fact that $b \in \mathcal{R}(A)$ to obtain a general form for the minimizers of f? Use the fact that any vector $\vec{x} \in \mathbb{R}^n$ can be written as $\vec{x} = U_r\vec{\alpha} + U_1\vec{\beta}$ for unique $\vec{\alpha}, \vec{\beta}$.

Solution:

$$\vec{x}^* = U_r \lambda_r^{-1} U_r^\top \vec{b} + U_1 \vec{\beta},$$

for any $\beta \in \mathbb{R}^{n-r}$.

(c) If $A \nsucceq 0$ (A not positive semi-definite) **show that** $p^* = -\infty$ by finding $\vec{v}$ such that $f(\alpha \vec{v}) \to -\infty$ as $\alpha \to \infty$.

Hint: $A \nsucceq 0$ means that there exists $\vec{v}$ such that $\vec{v}^\top A \vec{v} < 0$.

Solution: $p^* = -\infty$.

(d) Suppose $A \succeq 0$ (positive semidefinite) and $\vec{b} \notin \mathcal{R}(A)$. **Find p^*.** Justify your answer mathematically.

Hint: From FTLA, we know that $\mathbb{R}^n = \mathcal{R}(A^\top) \oplus \mathcal{N}(A)$. Therefore, $\vec{b} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1 \in \mathcal{R}(A) = \mathcal{R}(A^\top)$ and $\vec{v}_2 \in \mathcal{N}(A)$.

Solution: $p^* = -\infty$.

For parts (e) and (f), consider real $n \times n$ **symmetric** matrix $A \in \mathbb{S}^n$ and $\vec{b} \in \mathbb{R}^n$. Let $\text{rank}(A) = r$, where $0 \leq r \leq n$. Now we consider the constrained optimization problem

$$\begin{aligned} p^* = \min_{\vec{x} \in \mathbb{R}^n} & \ \vec{x}^\top A \vec{x} - 2\vec{b}^\top \vec{x} \\ \text{s.t. } & \ \vec{x}^\top \vec{x} \geq 1. \end{aligned} \tag{2}$$

(e) **Write the Lagrangian** $\mathcal{L}(\vec{x}, \lambda)$, where λ is the dual variable corresponding to the inequality constraint.

(f) For any matrix $C \in \mathbb{R}^{n \times n}$ with $\text{rank}(C) = r \leq n$ and compact SVD

$$C = U_r \Lambda_r V_r^\top,$$

we define the pseudoinverse as

$$C^\dagger = V_r \Lambda_r^{-1} U_r^\top.$$

We use the "dagger" operator to represent this. If $\vec{d}$ lies in the range of C, then a solution to the equation $C\vec{x} = \vec{d}$, can be written as $\vec{x} = C^\dagger \vec{d}$, even when C is not full rank. **Show that the dual problem to the primal problem** (2) **can be written as,**

$$d^* = \max_{\substack{\lambda \geq 0 \\ A - \lambda I \succeq 0 \\ \vec{b} \in \mathcal{R}(A - \lambda I)}} -\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda.$$

Hint: To show this, first argue that when the constraints are not satisfied $\min_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) = -\infty$. Then show that when the constraints are satisfied, $\min_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) = -\vec{b}^\top (A - \lambda I)^\dagger \vec{b} + \lambda$. Compute $g(\lambda)$ and explore it's behavior under the constraints.