EECS 127/227AT Optimization Models in Engineering
Spring 2020 — Discussion 8

1. **Complementary slackness**
Consider the problem:

$$
\begin{aligned}
p^* = \min_{x \in \mathbb{R}} \; & x^2 \\
\text{s.t.} \; & x \geq 1, \; x \leq 2.
\end{aligned}
$$

(a) Does Slater's condition hold? Is the problem convex? Does strong duality hold?
Solution: We have a strictly feasible point $x = 1.5$ that lies in the relative interior of the domain of the objective function; thus, Slater's condition holds. The objective function x^2 is convex and the inequality constraints are affine and thus convex, so the problem is convex. Since Slater's condition holds and the problem is convex, strong duality holds.

(b) Find the Lagrangian $\mathcal{L}(x, \lambda_1, \lambda_2)$.
Solution: $\mathcal{L}(x, \lambda_1, \lambda_2) = x^2 + \lambda_1(-x + 1) + \lambda_2(x - 2)$.

(c) Solve for $x^*, \lambda_1^*, \lambda_2^*$ that satisfy KKT conditions.
Solution: We have:
From stationarity,

$$
\begin{aligned}
& \nabla_x \mathcal{L}(x, \lambda_1, \lambda_2) = 0 \\
\implies \; & 2x - \lambda_1 + \lambda_2 = 0. \qquad (1)
\end{aligned}
$$

From primal feasibility,

$$
\begin{aligned}
x &\geq 1 \\
x &\leq 2.
\end{aligned}
$$

From dual feasibility,

$$
\begin{aligned}
\lambda_1 &\geq 0 \\
\lambda_2 &\geq 0.
\end{aligned}
$$

Finally from complementary slackness,

$$
\begin{aligned}
\lambda_1(-x + 1) &= 0 \\
\lambda_2(x - 2) &= 0.
\end{aligned}
$$

First observe that we cannot have $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ since in this case complementary slackness would not have any feasible solutions for x. Next assume that $\lambda_1 = 0, \lambda_2 \neq 0$. Then from complementary slackness, $x = 2$. Substituting this in equation 1, we get $\lambda_2 = -4$ which violates dual feasibility. Next assume that $\lambda_1 = 0, \lambda_2 = 0$. Then from 1 we have $x = 0$ which violates primal feasibility. Finally assume that $\lambda_1 \neq 0, \lambda_2 = 0$. From complementary slackness we have $x = 1$ and from 1 we have $\lambda_1 = 2$ which satisfies dual feasibility.
Thus $x^* = 1, \lambda_1^* = 2, \lambda_2^* = 0$ satisfy KKT conditions.

(d) Can you spot a connection between the values of λ_1^*, λ_2^* in relation to whether the corresponding inequality constraints are strict or not at the optimal x^*?
Solution: We have $\lambda_1 \neq 0$ and the corresponding inequality $x \geq 1$ is satisfied with equality (and hence is not strict) at $x^* = 1$.
We have $\lambda_2 = 0$ and the corresponding inequality is strict at $x^* = 1$. The non-zero λ_1 tells us that if we relax the constraint $x \geq 1$ (for example, to $x \geq 0.9$) we can reduce the objective function further.

(e) Find the dual function $g(\lambda_1, \lambda_2)$ so that the dual problem is given by,

$$d^* = \max_{\lambda_1, \lambda_2 \in \mathbb{R}^+} g(\lambda_1, \lambda_2). \tag{2}$$

Solution:

$$g(\lambda_1, \lambda_2) = \inf_x \mathcal{L}(x, \lambda_1, \lambda_2).$$

Note that $\mathcal{L}$ is convex with respect to x, thus setting the gradient with respect to x to 0 we obtain, $x = \frac{\lambda_1 - \lambda_2}{2}$. Thus,

$$g(\lambda_1, \lambda_2) = -\frac{(\lambda_2 - \lambda_1)^2}{4} + \lambda_1 - 2\lambda_2.$$

(f) Solve the dual problem in (2) for d^*.
Solution: Let us first try setting gradient with respect to λ_1 and λ_2 to 0. This gives us,

$$\begin{aligned} \frac{\lambda_2 - \lambda_1}{2} + 1 &= 0 \\ -\frac{\lambda_2 - \lambda_1}{2} - 2 &= 0. \end{aligned}$$

This has no solution. We can see that a quadratic objective function could be unbounded even if it was convex. To get meaningful solutions we must check for optimal values at the boundaries. Checking at boundary $\lambda_1 = 0$.

$$g(0, \lambda_2) = -\frac{\lambda_2^2}{4} - 2\lambda_2.$$

This is a concave function so taking gradient with respect to λ_2 and setting it to zero we obtain,

$$\begin{aligned} -\frac{\lambda_2}{2} - 2 &= 0 \\ \implies \lambda_2 &= -4. \end{aligned}$$

This is not feasible so we must check value at $\lambda_2 = 0$. We have $g(0, 0) = 0$. Finally let us check at the other boundary $\lambda_2 = 0$.

$$g(\lambda_1, 0) = -\frac{\lambda_1^2}{4} + \lambda_1.$$

Again this is a concave function so taking gradient with respect to λ_1 and setting it to zero we obtain,

$$\begin{aligned} -\frac{\lambda_1}{2} + 1 = 0 \\ \implies \lambda_1 = 2. \end{aligned}$$

We have $g(2, 0) = -1 + 2 = 1$. Thus $d^* = 1$.

2. [Optional] Simple constrained optimization problem with duality

Consider the optimization problem

$$\begin{aligned}
&\min_{x_1,x_2\in\mathbb{R}} f(x_1,x_2)\\
&\text{subject to } 2x_1+x_2\geq 1\\
&\qquad\qquad x_1+3x_2\geq 1\\
&\qquad\qquad x_1\geq 0,\ x_2\geq 0
\end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)$

Solution:

$$\mathcal{L}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3,\lambda_4)=f(x_1,x_2)+\lambda_1(-2x_1-x_2+1)+\lambda_2(1-x_1-3x_2)-\lambda_3x_1-\lambda_4x_2$$

Solve the following problems analytically and give the minimizing x_1^*,x_2^*: *Hint:* Use duality if the problem is hard to solve. Use the graphs in Figure 1 to "dualize" only some constraints:

(b) $f(x_1,x_2)=x_1+x_2$

Solution: From the drawing, it seems that at the optimal point $x_1\geq 0$ and $x_2\geq 0$ are not active. So let's assume that $x_1^\star>0$ and $x_2^\star>0$ so $\lambda_3=0$ and $\lambda_4=0$ by Complementary Slackness.

So we can solve the following problem

$$\begin{aligned}
&\min_{\vec{x},\vec{x}\geq 0}\max_{\lambda_1,\lambda_2\geq 0} f(x_1,x_2)+\lambda_1(-2x_1-x_2+1)+\lambda_2(1-x_1-3x_2)\\
=&\max_{\lambda_1,\lambda_2\geq 0}\min_{\vec{x},\vec{x}\geq 0} f(x_1,x_2)+\lambda_1(-2x_1-x_2+1)+\lambda_2(1-x_1-3x_2) &&\text{(Strong duality)}\\
&\implies \nabla_{x_1,x_2}\mathcal{L}(x_1^\star,x_2^\star,\lambda_1,\lambda_2)=0 &&\text{(Stationarity)}\\
&\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\lambda_1+\lambda_2\\\lambda_1+3\lambda_2\end{pmatrix}
\end{aligned}$$

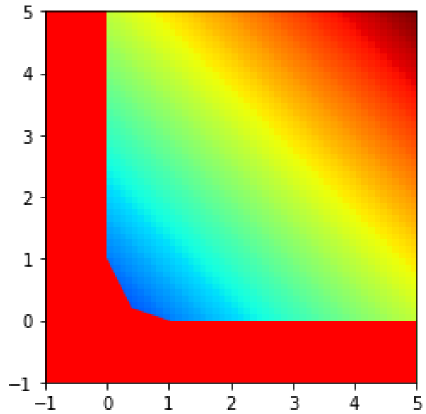
This gives $\lambda_1=\frac{2}{5}>0$ and $\lambda_2=\frac{1}{5}>0$.
And $(-2x_1^\star-x_2^\star+1)=0$ and $(1-x_1^\star-3x_2^\star)=0$.
So $x_1^\star=\frac{2}{5}$ and $x_2^\star=\frac{1}{5}$, which satisfies the assumption condition $x_1^\star>0$ and $x_2^\star>0$.
The optimal point is $x_1^\star=\frac{2}{5}$ and $x_2^\star=\frac{1}{5}$ and $f(x_1^\star,x_2^\star)=\frac{3}{5}$.

(c) $f(x_1,x_2)=-x_1-x_2$

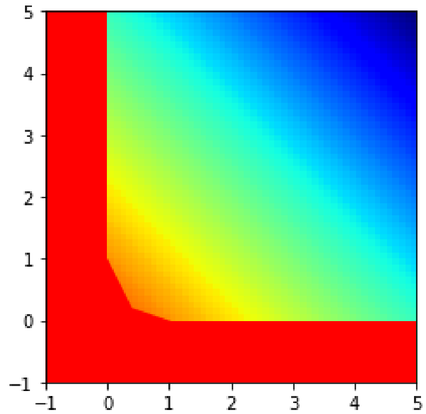
Solution: Here, $t\cdot(1,1)$ is a feasible point when $t\geq 1$. And $f(t\cdot(1,1))=-2t$ which is unbounded below. So the problem is unbounded below.

(d) $f(x_1,x_2)=x_1$

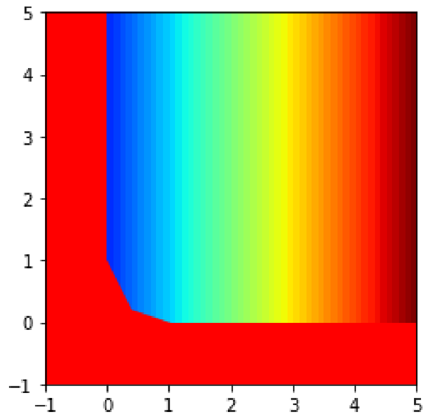
Solution: Under the constraints $x_1\geq 0$, we know that $f(x_1,x_2)\geq 0$.
Also $f(x_1,x_2)=0$ implies that $x_1=0$ and the constraints becomes equivalent to $x_2\geq 1$.
Therefore the set of solutions is $S=\{\vec{x},x_1=0 \text{ and } x_2\geq 1\}$ and for all $\vec{x}\in S$, $f(\vec{x})=0$.



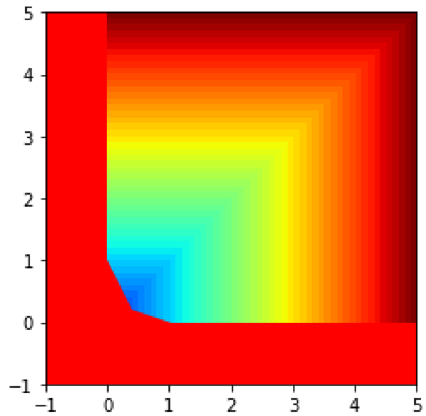
(b)



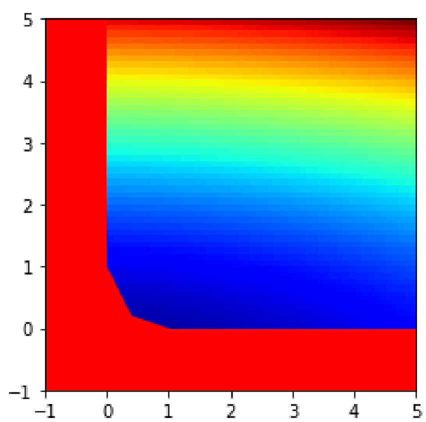
(c)



(d)



(e)



(f)

Figure 1: Heatmap of 2(b), 2(c), 2(d), 2(e) and 2(f): $\vec{x}^{\star} = (\frac{2}{5}, \frac{1}{5})$. In red is the unfeasible points, then the level sets are shown with colors; blue points are points (x_1, x_2) with the lowest value $f(x_1, x_2)$, red points are the ones with highest value.

(e) $f(x_1, x_2) = \max\{x_1, x_2\}$

Solution: In order to make the Lagrangian of the problem differentiable with respect to $\vec{x}$, we can use a slack variable. The problem is equivalent to:

$$\begin{aligned}
&\min_{x_1,x_2,t} t \\
\text{subject to } & t \geq x_1, t \geq x_2 \\
& 2x_1 + x_2 \geq 1 \\
& x_1 + 3x_2 \geq 1 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

We have the augmented Lagrangian:

$$\begin{aligned}
\mathcal{L}(t, x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = t &+ \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2 \\
&+ \lambda_5(x_1 - t) + \lambda_6(x_2 - t)
\end{aligned}$$

Using the drawing (see figure 1), we see that several constraints are not active at the optimum and we can make the assumption that $\lambda_2 = 0$, $\lambda_3 = 0$ and $\lambda_4 = 0$ by complementary slackness. So we can solve the following problem (with $\mathcal{X} = \{\vec{x} \mid x_1 + 3x_2 \geq 1,\ x_1 \geq 0 \text{ and } x_2 \geq 0\}$)

$$\begin{aligned}
&\min_{(\vec{x},t),\vec{x}\in\mathcal{X}} \max_{\lambda_1,\lambda_5,\lambda_6} t + \lambda_1(-2x_1 - x_2 + 1) + \lambda_5(x_1 - t) + \lambda_6(x_2 - t) \\
&= \max_{\lambda_1,\lambda_5,\lambda_6} \min_{(\vec{x},t),\vec{x}\in\mathcal{X}} t + \lambda_1(-2x_1 - x_2 + 1) + \lambda_5(x_1 - t) + \lambda_6(x_2 - t) && \text{(Strong duality)} \\
&\implies \nabla_{x_1,x_2,t}\mathcal{L}(x_1^\star, x_2^\star, t^\star, \lambda_1, \lambda_5, \lambda_6) = 0 && \text{(Stationarity)} \\
&\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 - \lambda_5 \\ \lambda_1 - \lambda_6 \\ \lambda_5 + \lambda_6 \end{pmatrix}
\end{aligned}$$

This gives that $\lambda_1 = \frac{1}{3}$, $\lambda_5 = \frac{1}{3}$ and $\lambda_6 = \frac{1}{3}$.
Complementary slackness gives that

$$t^\star = x_1^\star = x_2^\star = \frac{1}{3}$$

Here, $x_1^\star + 3x_2^\star > 1$, $x_1^\star > 0$ and $x_2^\star > 0$, so the assumptions are valid.

(f) $f(x_1, x_2) = x_1^2 + 9x_2^2$

Solution: Using the drawing (see figure 1), we can make the assumptions that $\lambda_1 = 0$, $\lambda_3 = 0$ and $\lambda_4 = 0$.
So we can solve the following problem (with $\mathcal{X} = \{\vec{x} \mid 2x_1 + x_2 \geq 1,\ x_1 \geq 0 \text{ and } x_2 \geq 0\}$)

$$\begin{aligned}
&\min_{\vec{x},\vec{x}\geq 0} \max_{\lambda_2} f(x_1, x_2) + \lambda_2(1 - x_1 - 3x_2) \\
&= \max_{\lambda_2} \min_{\vec{x},\vec{x}\in\mathcal{X}} f(x_1, x_2) + \lambda_2(1 - x_1 - 3x_2) && \text{(Strong duality)} \\
&\implies \nabla_{x_1,x_2}\mathcal{L}(x_1^\star, x_2^\star, \lambda_2) = 0 && \text{(Stationarity)} \\
&\begin{pmatrix} 2x_1^\star \\ 18x_2^\star \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 3\lambda_2 \end{pmatrix}
\end{aligned}$$

So $\lambda_2 = 2x_1^\star = 6x_2^\star$. $(1 - x_1 - 3x_2) = 0$ gives that $x_1^\star = \frac{1}{2}$, $x_2^\star = \frac{1}{6}$ and $\lambda_2 = 1$.
We have that $2x_1^\star + x_2^\star > 1$, $x_1^\star > 0$ and $x_2^\star > 0$, so the assumptions are valid.