EECS 127/227AT Discussion 5 Slides

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Q1

General form of a constrained optimization problem:

$$\begin{array}{rl} \min_x & f(x) \\ \text{subject to} & g(x) \leq 0 \\ & h(x) = 0 \end{array}$$

Definition (Feasible Set)

Set $\mathcal{F} = \{x \mid g(x) \leq 0, h(x) = 0\}$ – set of x that fulfills the constraints.

Approach to solving **simple** constrained problems:

1. Sketch feasible set.
2. If f is simple enough, guess optimal value of x.
3. Check your solution is optimal – if f is convex:
   (i) Greedy/exchange argument – perturb by small amounts and check that new value is worse (re: CS170)
   (ii) If $\mathcal{F}$ is convex, first/second order conditions (for all y, $\left(\nabla_x f(x)\right)^T (y - x) \geq 0$, and $\nabla_x^2 f(x) \succeq 0$) sufficient for optimality
   (iii) Ad-hoc methods

Q2/Q3

Definition (Supremum)

Least upper bound of a set of real numbers – a "generalized maximum." Interpretation: if $d^* = \sup_{x \in \mathcal{X}} f(x)$ then there is a sequence $\{x_n \in \mathcal{X}\} = \{x_1, x_2, \ldots\}$ such that $\lim_{n \to \infty} f(x_n) = d^*$. "Points in the set can get arbitrarily close to the supremum."

Example

Some sets have supremums but not maximums.
$S = \{0.9, 0.99, 0.999, \ldots\}$ does not have a maximum (since $1 \notin S$) but its supremum is 1. More general/complicated examples in $\mathbb{R}^n$.

Definition (Infimum)

Greatest lower bound of a set of real numbers – "generalized minimum." Interpretation: if $p^* = \inf_{x \in \mathcal{X}} f(x)$, then there is a sequence $\{x_n \in \mathcal{X}\}$ such that $\lim_{n \to \infty} f(x_n) = p^*$.

Definition (Open Set)

Colloquially, "set with an interior". Meaning: if $\mathcal{X}$ is open then every $x \in \mathcal{X}$ has some small region around it contained entirely within $\mathcal{X}$.

Example

Open ball: set $B_r(x_0) = \{x \mid \|x - x_0\| < r\}$.

Definition (Closed Set)

Colloquially, "set which contains its limits". Meaning: if $\mathcal{X}$ is closed then for any sequence $\{x_n \in \mathcal{X}\}$ which converges to x has $\lim_{n\to\infty} x_n = x \in \mathcal{X}$.

Consequence: if f is continuous and $\mathcal{X}$ is closed then $\sup_{x\in\mathcal{X}} f(x) = \max_{x\in\mathcal{X}} f(x)$, and $\inf_{x\in\mathcal{X}} f(x) = \min_{x\in\mathcal{X}} f(x)$. So no need for special treatment.