EECS 16B Fall 2019 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 12A

Questions

1. Linear Approximation

A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point x_0 is given by

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0),$$

where $f'(x_0) := \left.\frac{df(x)}{dx}\right|_{x=x_0}$ is the derivative of $f(x)$ at $x = x_0$.

Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x,y)$, the linear approximation of $f(x,y)$ at a point (x_0,y_0) is given by

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0) \cdot (x - x_0) + f_y(x_0,y_0) \cdot (y - y_0).$$

where $f_x(x_0,y_0)$ is the partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0):

$$f_x(x_0,y_0) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}$$

and $f_y(x_0,y_0)$ is the partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0).

(a) Now, let's see how we can derive partial derivatives. When we are given a function $f(x,y)$, we calculate the partial derivative of f with respect to x by fixing y and taking the derivative with respect to x. **Given the function $f(x,y) = x^2y$, find the partial derivatives $f_y(x,y)$ and $f_x(x,y)$.**

Answer: We have

$$f_y(x,y) = x^2$$

and

$$f_x(x,y) = 2xy.$$

(b) **Write out the linear approximation of f near (x_0,y_0).**

Answer:

$$f(x,y) \approx f(x_0,y_0) + 2x_0y_0 \cdot (x - x_0) + x_0^2 \cdot (y - y_0).$$

(c) **Compare the approximation of f at the point $(2.01, 3.01)$ using $(x_0,y_0) = (2,3)$ versus $f(2.01,3.01)$.**

Answer: Let $\delta = 0.01$. Then, the true value of $f(2.01,3.01)$ is

$$f(2.01,3.01) = (2+\delta)^2(3+\delta) = (4+4\delta+\delta^2)(3+\delta) = 12+16\delta+7\delta^2+\delta^3.$$

On the other hand, our approximation is

$$f(2.01,3.01) \approx f(2,3) + 2 \cdot 2 \cdot 3 \cdot \delta + 2^2 \cdot \delta = 12 + 16\delta.$$

As we can see, our approximation removes the terms with δ^2 and δ^3. When δ is sufficiently small, these terms become very small, and hence our approximation is reasonable.

(d) When the function $f(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ takes in vectors and outputs a real number, how can we adapt our previous linearization method?

One way to linearize the function f is to do it for every single element in $\vec{x} = \begin{bmatrix} x[1] \\ x[2] \\ \cdots \\ x[n] \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y[1] \\ y[2] \\ \cdots \\ y[k] \end{bmatrix}$.

Then, when we are looking at $x[i]$ or $y[j]$, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_0,\vec{y}_0) + \sum_{i=1}^{n} \frac{\partial f(\vec{x},\vec{y})}{\partial x[i]}(x[i]-x_0[i]) + \sum_{j=1}^{k} \frac{\partial f(\vec{x},\vec{y})}{\partial y[j]}(y[i]-y_0[i]). \tag{1}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x[1]} & \cdots & \frac{\partial f}{\partial x[n]} \end{bmatrix}.$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y[1]} & \cdots & \frac{\partial f}{\partial y[k]} \end{bmatrix}.$$

Then, Equation (1) can be rewritten as

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_0,\vec{y}_0) + (D_{\vec{x}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{x}-\vec{x}_0) + (D_{\vec{y}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{y}-\vec{y}_0).$$

Assume that $n = k$ and the function $f(\vec{x},\vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x[i]y[i]$. Find $D_{\vec{x}} f$ and $D_{\vec{y}} f$.

Answer: Here, we have

$$D_{\vec{x}} f = \vec{y}^\top$$

and

$$D_{\vec{y}} f = \vec{x}^\top.$$

(e) **Following the above part, find the linear approximation of $f(\vec{x},\vec{y})$ near $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.**

Answer: From the solution in the previous part, we can write

$$\begin{aligned} f(\vec{x},\vec{y}) &\approx f(\vec{x}_0,\vec{y}_0) + (D_{\vec{x}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{x}-\vec{x}_0) + (D_{\vec{y}} f)\Big|_{(\vec{x}_0,\vec{y}_0)} \cdot (\vec{y}-\vec{y}_0) \\ &= \vec{x}_0^\top \vec{y}_0 + \vec{y}_0^\top (\vec{x}-\vec{x}_0) + \vec{x}_0^\top (\vec{y}-\vec{y}_0) \end{aligned} \tag{2}$$

Putting in $\vec{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y}_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and let's find the approximation of $f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right)$, we have

$$\begin{aligned} f\left(\begin{bmatrix} 1+\delta_1 \\ 2+\delta_2 \end{bmatrix}, \begin{bmatrix} -1+\delta_3 \\ 2+\delta_4 \end{bmatrix}\right) &\approx \vec{x}_0^\top \vec{y}_0 + \vec{y}_0^\top (\vec{x}-\vec{x}_0) + \vec{x}_0^\top (\vec{y}-\vec{y}_0) \\ &= 3 + \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \delta_3 \\ \delta_4 \end{bmatrix} \\ &= 3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4. \end{aligned} \tag{3}$$

Let's compare this with the true value $f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix},\begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right)$. We have

$$f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix},\begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right)=(1+\delta_1)(-1+\delta_3)+(2+\delta_2)(2+\delta_4)=3-\delta_1+2\delta_2+\delta_3+2\delta_4+\delta_1\delta_3+\delta_2\delta_4.$$

As we can see, our approximation removes the second order δ terms $\delta_1\delta_3$ and $\delta_2\delta_4$.

(f) When the function $\vec{f}(\vec{x},\vec{y}):\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^m$ takes in vectors and outputs a vector, we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x},\vec{y})=\begin{bmatrix}f_1(\vec{x},\vec{y})\\f_2(\vec{x},\vec{y})\\\cdots\\f_m(\vec{x},\vec{y})\end{bmatrix}\approx\begin{bmatrix}f_1(\vec{x}_0,\vec{y}_0)+D_{\vec{x}}f_1\cdot(\vec{x}-\vec{x}_0)+D_{\vec{y}}f_1\cdot(\vec{y}-\vec{y}_0)\\f_2(\vec{x}_0,\vec{y}_0)+D_{\vec{x}}f_2\cdot(\vec{x}-\vec{x}_0)+D_{\vec{y}}f_2\cdot(\vec{y}-\vec{y}_0)\\\cdots\\f_m(\vec{x}_0,\vec{y}_0)+D_{\vec{x}}f_m\cdot(\vec{x}-\vec{x}_0)+D_{\vec{y}}f_m\cdot(\vec{y}-\vec{y}_0)\end{bmatrix}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}}\vec{f}=\begin{bmatrix}D_{\vec{x}}f_1\\D_{\vec{x}}f_2\\\cdots\\D_{\vec{x}}f_m\end{bmatrix}=\begin{bmatrix}\frac{\partial f_1}{\partial x[1]}&\cdots&\frac{\partial f_1}{\partial x[n]}\\\cdots&\cdots&\cdots\\\frac{\partial f_m}{\partial x[1]}&\cdots&\frac{\partial f_m}{\partial x[n]}\end{bmatrix},$$

and similarly

$$D_{\vec{y}}\vec{f}=\begin{bmatrix}\frac{\partial f_1}{\partial y[1]}&\cdots&\frac{\partial f_1}{\partial y[k]}\\\cdots&\cdots&\cdots\\\frac{\partial f_m}{\partial y[1]}&\cdots&\frac{\partial f_m}{\partial y[k]}\end{bmatrix}.$$

Then, the linearization becomes

$$\vec{f}(\vec{x},\vec{y})\approx\vec{f}(\vec{x}_0,\vec{y}_0)+(D_{\vec{x}}\vec{f})\Big|_{(\vec{x}_0,\vec{y}_0)}\cdot(\vec{x}-\vec{x}_0)+(D_{\vec{y}}\vec{f})\Big|_{(\vec{x}_0,\vec{y}_0)}\cdot(\vec{y}-\vec{y}_0).$$

Let $\vec{x}=\begin{bmatrix}x[1]\\x[2]\end{bmatrix}$ and $\vec{f}(\vec{x})=\begin{bmatrix}x[1]^2x[2]\\x[1]x[2]^2\end{bmatrix}$. Find $D_{\vec{x}}\vec{f}$.

Answer: Here, we have

$$D_{\vec{x}}\vec{f}=\begin{bmatrix}2x[1]x[2]&x[1]^2\\x[2]^2&2x[1]x[2]\end{bmatrix}.$$

(g) **Compare the approximation of $\vec{f}$ at the point $\begin{bmatrix}2.01\\3.01\end{bmatrix}$ using $\vec{x}_0=\begin{bmatrix}2\\3\end{bmatrix}$ versus $\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right)$.**

Answer: Let $\delta=0.01$. The true value is

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right)=\begin{bmatrix}(2+\delta)^2(3+\delta)\\(2+\delta)(3+\delta)^2\end{bmatrix}=\begin{bmatrix}12+16\delta+7\delta^2+\delta^3\\18+21\delta+8\delta^2+\delta^3\end{bmatrix}.$$

On the other hand, our approximation is

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) \approx \vec{f}\left(\begin{bmatrix}2\\3\end{bmatrix}\right) + \begin{bmatrix}12 & 4\\9 & 12\end{bmatrix}\cdot\begin{bmatrix}\delta\\\delta\end{bmatrix} = \begin{bmatrix}12+16\delta\\18+21\delta\end{bmatrix}.$$

Again, our approximation essentially removes the higher order terms of δ.
When we plug in $\delta = 0.01$, we have

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) = \begin{bmatrix}12.160701\\18.210801\end{bmatrix}$$

and our approximation is

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) = \begin{bmatrix}12.16\\18.21\end{bmatrix}.$$

(h) (Do at home) **Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^\top\vec{w}$. Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.**

Answer: Here, recall that

$$\vec{f} = \begin{bmatrix}x[1]\\x[2]\end{bmatrix}\cdot\begin{bmatrix}y[1] & y[2]\end{bmatrix}\cdot\begin{bmatrix}w[1]\\w[2]\end{bmatrix} = \begin{bmatrix}x[1]y[1] & x[1]y[2]\\x[2]y[1] & x[2]y[2]\end{bmatrix}\cdot\begin{bmatrix}w[1]\\w[2]\end{bmatrix} = \begin{bmatrix}x[1]y[1]w[1]+x[1]y[2]w[2]\\x[2]y[1]w[1]+x[2]y[2]w[2]\end{bmatrix}.$$

Then,

$$D_{\vec{x}}\vec{f} = \begin{bmatrix}\frac{\partial f_1}{x[1]} & \frac{\partial f_1}{x[2]}\\\frac{\partial f_2}{x[1]} & \frac{\partial f_2}{x[2]}\end{bmatrix} = \begin{bmatrix}y[1]w[1]+y[2]w[2] & 0\\0 & y[1]w[1]+y[2]w[2]\end{bmatrix}$$

and

$$D_{\vec{y}}\vec{f} = \begin{bmatrix}\frac{\partial f_1}{y[1]} & \frac{\partial f_1}{y[2]}\\\frac{\partial f_2}{y[1]} & \frac{\partial f_2}{y[2]}\end{bmatrix} = \begin{bmatrix}x[1]w[1] & x[1]w[2]\\x[2]w[1] & x[2]w[2]\end{bmatrix}.$$

We can also write

$$D_{\vec{x}}\vec{f} = \vec{y}^\top\vec{w}\cdot I$$

and

$$D_{\vec{y}}\vec{f} = \vec{x}\vec{w}^\top,$$

which can be derived by noticing that $\vec{y}^\top\vec{w} = \vec{w}^\top\vec{y}$.

(i) (Do at home) **Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix}1\\1\end{bmatrix}$ and with $\vec{w} = \begin{bmatrix}2\\1\end{bmatrix}$.**

Answer: We have

$$\begin{aligned}\vec{f}(\vec{x},\vec{y}) &\approx \vec{f}(\vec{x}_0,\vec{y}_0) + D_{\vec{x}}\vec{f}\cdot(\vec{x}-\vec{x}_0) + D_{\vec{y}}\vec{f}\cdot(\vec{y}-\vec{y}_0)\\ &= \begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}x[1]-1\\x[2]-1\end{bmatrix} + \begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\cdot\begin{bmatrix}y[1]-1\\y[2]-1\end{bmatrix}\end{aligned} \tag{4}$$

Let's do an approximation of $\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right)$, then,

$$\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right)\approx\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}+\begin{bmatrix}2&1\\2&1\end{bmatrix}\cdot\begin{bmatrix}\delta_3\\\delta_4\end{bmatrix}=\begin{bmatrix}3+3\delta_1+2\delta_3+\delta_4\\3+3\delta_2+2\delta_3+\delta_4\end{bmatrix}.$$

We can compare with the true value

$$\begin{aligned}\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right)&=\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix}\begin{bmatrix}1+\delta_3&1+\delta_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\\&=\begin{bmatrix}1+\delta_1+\delta_3+\delta_1\delta_3&1+\delta_1+\delta_4+\delta_1\delta_4\\1+\delta_2+\delta_3+\delta_2\delta_3&1+\delta_2+\delta_4+\delta_2\delta_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\\&=\begin{bmatrix}3+3\delta_1+2\delta_3+\delta_4+2\delta_1\delta_3+\delta_1\delta_4\\3+3\delta_2+2\delta_3+\delta_4+2\delta_2\delta_3+\delta_2\delta_4\end{bmatrix},\end{aligned}\tag{5}$$

and we see that our approximation removes the second order δ terms $\delta_1\delta_3$, $\delta_1\delta_4$, $\delta_2\delta_3$ and $\delta_2\delta_4$.

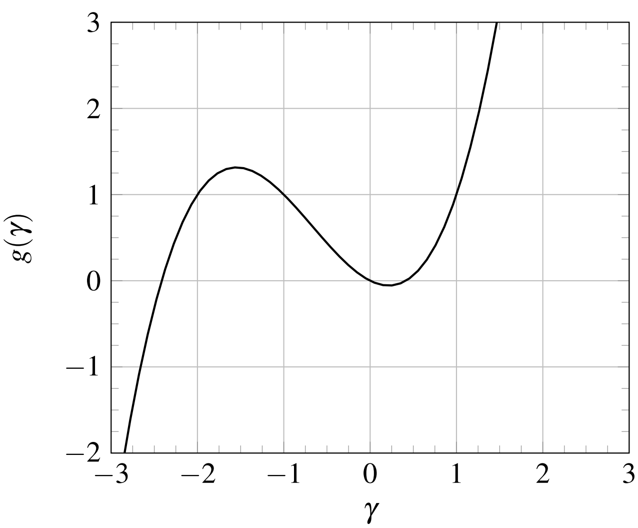
These linearizations are important for us because we can do many easy computations using linear functions.

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{d}{dt}\begin{bmatrix}\beta(t)\\\gamma(t)\end{bmatrix}=\frac{d}{dt}\vec{x}(t)=\begin{bmatrix}-2\beta(t)+\gamma(t)\\g(\gamma(t))+u(t)\end{bmatrix}=\vec{f}(\vec{x}(t),u(t))\tag{6}$$

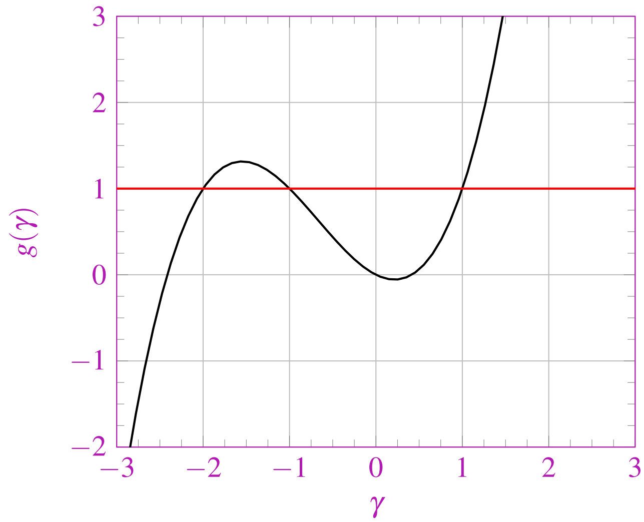
where $\vec{x}(t)=\begin{bmatrix}\beta(t)\\\gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a DC operating point.

(a) **If we have fixed $u^\star(t) = -1$, what values of γ and β will ensure $\frac{d}{dt}\vec{x}(t) = \vec{0}$?**

Answer: To find the equilibrium point, we'll start by finding the values for which $g(\gamma) + u^\star = g(\gamma) - 1 = 0$. In other words, we need to find values of γ such that $g(\gamma) = 1$. Although we don't have an equation for $g(\gamma)$, we can still find these points *graphically*, by using our graph. If we add a horizonal line at $g(\gamma) = 1$, we get the following:



Having done this, it looks like we'll have $f_2(\vec{x}, u^\star) = g(\gamma) - u^\star = 0$ for $\gamma = -2, \gamma = -1$, and $\gamma = 1$. Now we just need to find an β that sets $f_1(\vec{x}, u^\star) = -2\beta + \gamma = 0$ for each of these. Setting $\beta = \frac{1}{2} \cdot \gamma$ will do this.

With that, we have our three equilibrium points, namely

$$\vec{x}_1^\star = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \qquad \vec{x}_2^\star = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} \qquad \vec{x}_3^\star = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \tag{7}$$

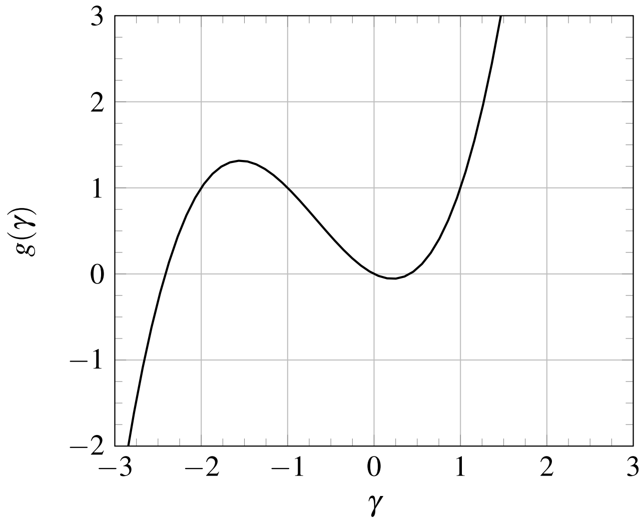
(b) Now that you have the three DC operating points, **linearize the system abou t the DC operating point $(\vec{x}_3^\star, u^\star)$ that has the largest value for γ**. Specifically, what we want is as follows. Let $\vec{\delta}x_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u^\star$. We can in principle write the *linearized system* for each DC operating point in the following form:

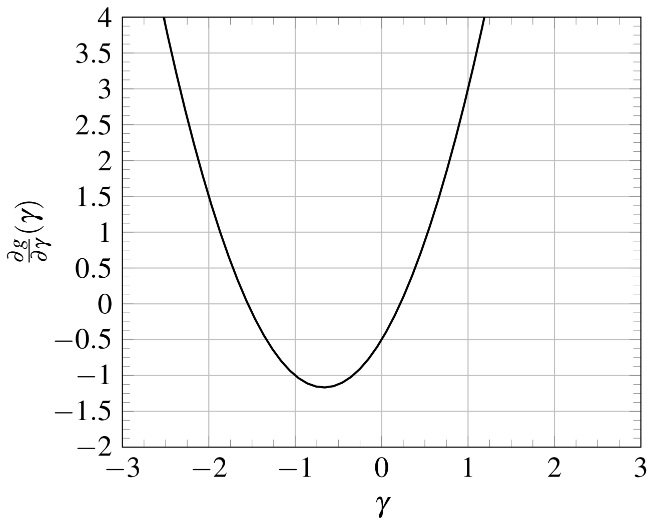
$$\text{(linearization about } (\vec{x}_i^\star, u^\star)) \qquad \frac{d}{dt}\vec{\delta}x_i(t) = A_i \vec{\delta}x_i(t) + B_i \delta u(t) + \vec{w}_i(t) \tag{8}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_3 **and** B_3.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





Answer: to linearize the system, we need to compute the two Jacobians

$$D_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial \beta} & \frac{\partial f_1}{\partial \gamma} \\ \frac{\partial f_2}{\partial \beta} & \frac{\partial f_2}{\partial \gamma} \end{bmatrix} \tag{9}$$

$$D_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u}, \end{bmatrix} \tag{10}$$

and evaluate them at the DC operating points that we found in the previous part. The Jacobian matrices evaluated at the DC operating points will be the A_i and B_i matrices.

If we work out the partial derivatives, we get

$$\frac{\partial f_1}{\partial \beta} = \frac{\partial}{\partial \beta}(-2\beta + \gamma) = -2 \tag{11}$$
$$\frac{\partial f_1}{\partial \gamma} = \frac{\partial}{\partial \gamma}(-2\beta + \gamma) = 1 \tag{12}$$
$$\frac{\partial f_2}{\partial \beta} = \frac{\partial}{\partial \beta}(g(\gamma) + u) = 0 \tag{13}$$
$$\frac{\partial f_2}{\partial \gamma} = \frac{\partial}{\partial \gamma}(g(\gamma) + u) = \frac{\partial g}{\partial \gamma} \tag{14}$$
$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(-2\beta + \gamma) = 0 \tag{15}$$
$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}(g(\gamma) + u) = 1, \tag{16}$$
$$\tag{17}$$

which gives

$$D_{\vec{x}} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix} \tag{18}$$
$$D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{19}$$

It turns out that the only part of $D_{\vec{x}}$ and D_u that depends on the DC operating point is $\partial g/\partial \gamma$, and we can read these off of the given graph. The relevant values are

$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=-2} = 1.5 \tag{20}$$
$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=-1} = -1 \tag{21}$$
$$\left.\frac{\partial g}{\partial \gamma}\right|_{\gamma=2} = 3, \tag{22}$$

which correspond to $\vec{x}_1^\star, \vec{x}_2^\star$, and $\vec{x}_3^\star$, respectively. Finally, this gives

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1.5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$
$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{24}$$
$$A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{25}$$
$$\tag{26}$$

Contributors:

- Kuan-Yun Lee.
- Alex Devonport.

