Ideal Poisson–Voronoi tessellations on hyperbolic spaces

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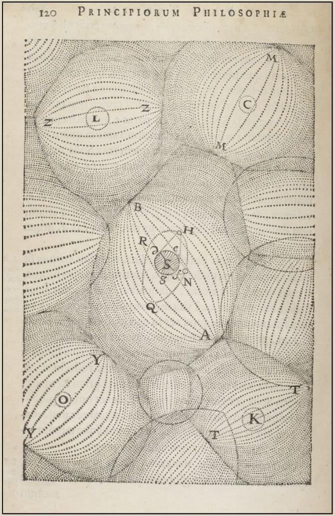
Voronoi tessellations: a classic construction

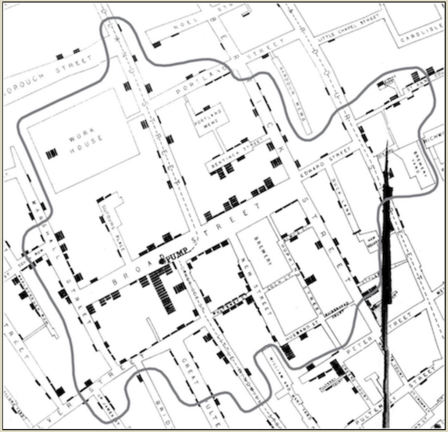
Principia
Philosophiæ
Descartes 1644



Voronoi tessellations: a classic construction

Water pump neighborhood
in London's Broad Street
cholera outbreak of 1854

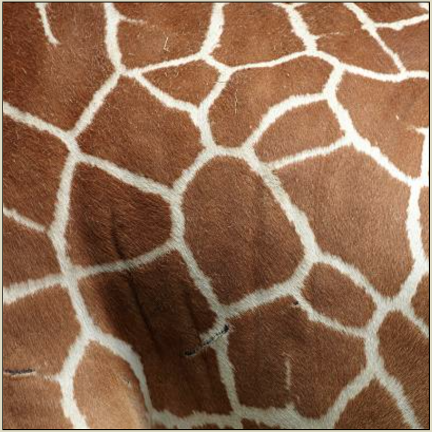
John Snow 1855



Voronoi tessellations: a classic construction

Growth models in morphogenesis

Gómez-Gálvez *et al.* 2018



Voronoi tessellations: basic definition

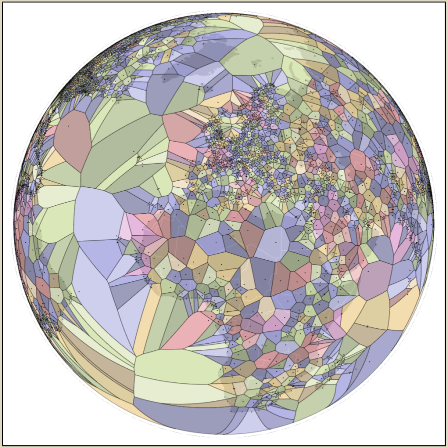
One wants to compare **distances**.

- A metric **space** (E, d)
- **Nuclei**: a list of points $(x_i\,;\, i \geq 1) \in E$
- **Tiles**:

$$C_i \coloneqq \{w \in E\,;\; d(w, x_i) \leq d(w, x_j),\; \forall j \geq 1\}\,.$$

The **Voronoi tessellation** is $\mathrm{Vor}(x_i : i \geq 1) = \{C_i\,;\; i \geq 1\}$

Voronoi airports



jasondavies.com/maps/voronoi/airports/

Poisson Point Processes

In applications (giraffe's fur, wireless networks) it is natural to assume some **complete randomness** for the nuclei. A natural candidate is a **Poisson Point Process (PPP)** of nuclei.

Def. A **PPP** on a measurable space S is a random countable subset $\Pi \subset S$ s.t. the random variable $N(A) = \#\{A \cap \Pi\}$ satisfies:

- $\forall A \subset S$, $N(A) \sim \mathrm{Pois}(\mu(A))$, for a measure μ (**intensity**);
- If $A, B \subset S$ are **disjoint** sets, $N(A) \perp N(B)$.

Poisson–Voronoi tessellations

(See e.g. **Møller** 1992)

Def. A **Poisson–Voronoi tessellation** is the Voronoi tessellation of a PPP of **nuclei** with intensity measure μ.

In this talk the **PPP** of nuclei is **homogeneous**

$\mu \propto \mathrm{Leb}_{\mathbb{R}^2}, \mathrm{Vol}_{\mathrm{M}} \ldots$ with some λ called **intensity**

Appearances of Poisson–Voronoi tessellations

☞: general case - s: λ small - **L**: λ large

- ☞ **Euclidean** space motivated by crystal growth (**Meijering** 1953, **Gilbert** 1961)
- ☞ **Hyperbolic** plane (with Bernoulli percolation on cells) (**Benjamini–Schramm** 2000)
- s **Mean characteristics** in hyperbolic plane and 3-space (**Isokawa** 2000)
- s Bernoulli–Voronoi **percolation** on regular trees, hyperbolic plane (**Bhupatiraju** 2019)
- **L** "Euclidean" limit on **surfaces**, Gauss–Bonnet Theorem (**Calka–Chapron–Enriquez** 2019)
- s, **L** Critical probabilities in **percolation** on the hyperbolic plane (**Hansen–Müller** 2021)

“Pointless” Poisson–Voronoi tessellation

(**Budzinski–Curien–Petri** 2022)

$\mathcal{S}_g$: a closed hyperbolic surface of genus g (large)

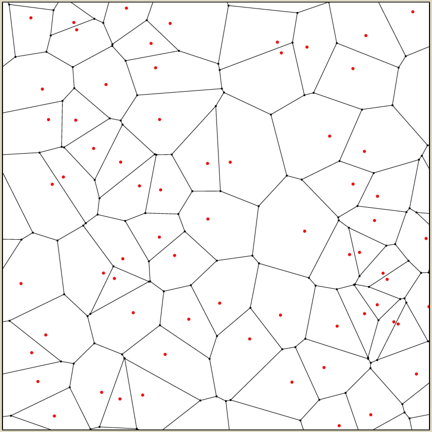
Cheeger's constant:

$$h(\mathcal{S}_g) \stackrel{\text{def}}{=} \inf_{A \subset \mathcal{S}_g} \left\{ \frac{\text{Perim}(A)}{\text{Area}(A)} \,;\, 2\,\text{Area}(A) \le \text{Area}(\mathcal{S}_g) \right\}$$

: $\lim_{\lambda \downarrow 0}$ of Poisson–Voronoi tessellation of **hyperbolic plane**

? What happens as $\lambda \downarrow 0$?

Euclidean plane $\mathbb{R}^2$



Ideal tessellations

Sketch of recipe

Tenet: one needs to compare "**differences**" of distances to nuclei. However these are ∞ quantities.

Two main deterministic **ingredients** as $\lambda \to 0$:

- **Convergence** of the nuclei to the boundary;
- **Convergence** of difference of distances (**delays**).

Probabilistic extension (convergence in distribution) if underlining space has **exponential** volume growth (tightness of delays).

Ideal tessellations

More detailed recipe

Let (E, d) be a metric space with a special point $\mathbf{o}$ (**origin**).
Let nuclei $x_i^{(\lambda)}$ be **sorted** according to $\mathrm{d}(\mathbf{o}, x_i^{(\lambda)})$.
If:

I. $x_i^{(\lambda)} \underset{\lambda\downarrow 0}{\overset{\text{Gromov}}{\longrightarrow}} \theta_i \in \partial E$ (Gromov boundary);

II. $\mathrm{d}(\mathbf{o}, x_i^{(\lambda)}) - \mathrm{d}(\mathbf{o}, x_{1}^{(\lambda)}) = \delta_i^{(\lambda)} \underset{\lambda\downarrow 0}{\longrightarrow} \delta_i$ increasing **delays**;

Then

$$\mathrm{Vor}(x_i^{(\lambda)}) \underset{\lambda\downarrow 0}{\overset{\text{loc. Hausdorff}}{\longrightarrow}} \mathrm{Vor}((\theta_i, \delta_i); i \geq 1)$$ (**Ideal tessellation**)

Exponential volume growth: regular trees, **hyperbolic space**

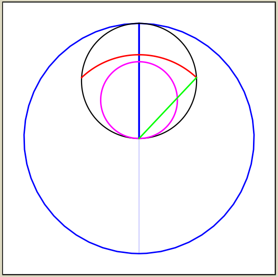
Hyperbolic space $\mathbb{H}_d$

$\mathbb{H}_d$: d-dimensional Riemannian manifold of const. sectional curv. $= -1$

Poincaré **ball** model

$$\mathbb{B}_d = \{x \in \mathbb{R}^d;\ |x| < 1\}$$

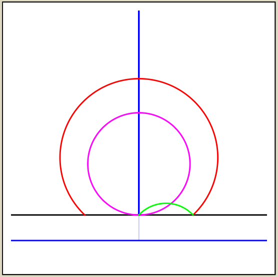
$$ds^2 = 2\,\frac{\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_d^2}{(1 - |x|^2)^2}$$



Upper-half space model

$$\mathbb{U}_d = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$$

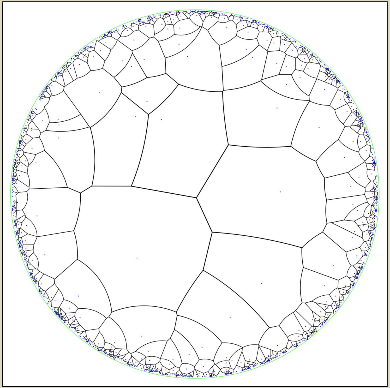
$$ds^2 = \frac{\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_{d-1}^2 + \mathrm{d}z^2}{z^2}$$



Nuclei, delays and the group Möb_d

- **Nuclei**:

$\mathbf{X}^{(\lambda)} = (X_i^{(\lambda)}\,;\, i \geq 1)$ a PPP of intensity measure $\lambda^{d-1} \cdot \mathrm{Vol}_{\mathbb{H}_d}$



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With this choice of **intensity measure**, the point closest to $\mathbf{o}$ is roughly at distance $\log 1/\lambda$ as $\lambda \to 0$ in any dimension d.

- **Delays** (a.s. strictly increasing):

  $D_i^{(\lambda)} \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda)$

Nuclei, delays and the group Möb$_d$

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- **Isometries** of $\mathbb{H}_d$: Möbius transf. form the group Möb$_d$.

Convergence of delays

Convergence of delays

$\left(D_i^{(\lambda)}\right)_{i\geq 1} \xrightarrow[\lambda\downarrow 0]{\text{law}} (D_i)_{i\geq 1}$, a **PPP** on $\mathbb{R}$ of intensity measure

$$c_d e^{(d-1)s}\mathrm{d}s\ , \qquad c_d = 4^{1-\frac{d}{2}}\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\ .$$

Equivalently, the set $(R_i)_{i\geq 1} \stackrel{\text{def}}{=} \left(\frac{c_d}{d-1}e^{(d-1)D_i}\right)_{i\geq 1}$ is a rate-1 homogeneous PPP on $\mathbb{R}_+$.

Extended ideal boundary and corona

Let us work in the ball model $\mathbb{B}_d$ in spherical coordinates:

$$X_i^{(\lambda)} = \Big(\underbrace{\Theta_i^{(\lambda)}}_{d-1\,\text{angles}}, \mathrm{d}_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda) = D_i^{(\lambda)} \Big)$$

- **Conditionally** on $D_i^{(\lambda)}$, $\left(\Theta_i^{(\lambda)}\right)_{i \geq 1}$ are i.i.d. **uniform** on $\partial \mathbb{B}_d = \mathbb{S}_{d-1}$.
- A.s. converg. $(\Theta_i^{(\lambda)}, D_i^{(\lambda)}) \to (\Theta_i, D_i)$ via Skorokhod (or dilations).

$$\mathrm{Vor}(\mathbf{X}^{(\lambda)}) \underset{\lambda \to 0}{\overset{\text{dist.}}{\to}} \mathrm{Vor}(\boldsymbol{\Theta}, \mathbf{D}) \overset{\text{def}}{=} \mathcal{V}_d \quad \textbf{IPVT}$$

Equivalently, $(\Theta, \mathbf{R})$ (recall $R_i = \frac{c_d}{d-1} e^{(d-1)D_i}$ for all i) is a PPP on on $\mathbb{S}_{d-1} \times \mathbb{R}_+$ (**corona**) of intensity measure

$$\mu_d = \mathsf{Unif}_{\mathbb{S}_{d-1}} \otimes \mathsf{Leb}_{\mathbb{R}_+}.$$

Action of Möb$_d$ on the corona

Let $\phi \in$ Möb$_d$ and (θ_i, r_i) be a point in the corona $\mathbb{S}_{d-1} \times \mathbb{R}_+$.

Lemma (Möbius action extended to the corona)

$$\phi(\theta_i, r_i) \stackrel{\text{def}}{=} \left(\phi(\theta_i), \frac{r_i}{K(\phi^{-1}(\mathbf{o}), \theta_i)}\right)$$

where $K(z,\theta) = \left(\frac{1-|z|^2}{|z-\theta|^2}\right)^{d-1}$ is hyperbolic **Poisson kernel**.

μ_d is invariant under the action of any Möbius transformation.

Properties of $\mathcal{V}_d$

Theorem (Tiles have one end)

Almost surely the tiles of $\mathcal{V}_d$ have disjoint interiors,

- Each tile of $\mathcal{V}_d$ is an **unbounded**, convex hyperbolic polytope with an infinite number of bounded facets and with a **unique end**,
- at most $d+1$ tiles can share a common point,
- $\mathcal{V}_d$ is **locally finite**,
- the law of $\mathcal{V}_d$ is Möb$_d$-**invariant**.

Geometry of perpendicular bisectors

We can **canonically** assign a $z \in \mathbb{H}_d$ to a cell without a nucleus!

Given two points $(\theta_1, r_1), (\theta_2, r_2)$ in the **corona**, solve

$$\frac{r_1}{K(z,\theta_1)} \overset{?}{\lessgtr} \frac{r_2}{K(z,\theta_2)} \; .$$

Perpendicular bisectors are Euclidean semi-spheres:

$$|z-\theta_1| r_1^{\frac{1}{2(d-1)}} = |z-\theta_2| r_2^{\frac{1}{2(d-1)}} \; , \; z \in \mathbb{B}_d \, .$$

Geometry of perpendicular bisectors

Let us focus on the cell $\mathcal{C}_d \subset \mathcal{V}_d$ **containing the origin** $\mathrm{o} \in \mathbb{H}_d$.

Geometry of perpendicular bisector

In $\mathbb{U}_d$ where θ_1 is put at ∞ and θ writes $(C, 0, \ldots, 0)$, the $(d-1)$-hyperplane bisecting (θ_1, r_1) and (θ, r) is a **half-sphere** centered at $(C, 0, \ldots, 0)$ with radius

$$\rho = \sqrt{1 + C^2} \left(\frac{r_1}{r} \right)^{\frac{1}{2(d-1)}} .$$

Main result: the deposition model for $\mathcal{C}_d$

Theorem (Description of $\mathcal{C}_d$)

$\mathcal{C}_d$ is the complement of all **open half-balls** whose **centers** and **radii** are a **PPP** of intensity measure

$$\mu_d = 2 \cdot \mathcal{E}_d \cdot \ \mathrm{d}x \ \rho^{1-2d} \mathrm{d}\rho \ \mathbf{1}_{\rho \leq \sqrt{1+x^2}} \quad \text{on} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+ \ .$$

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- $\rho^{1-2d}\mathrm{d}\rho$ on $\mathbb{R}_+$ for the **radii** of the half-balls;

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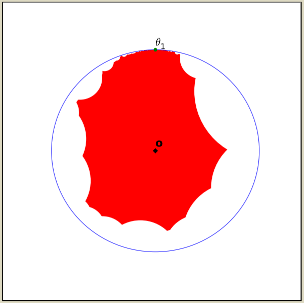
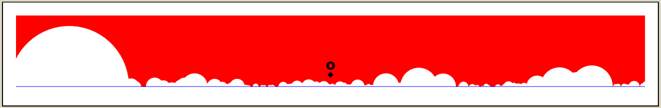
- $\rho^{1-2d}\mathrm{d}\rho$ on $\mathbb{R}_+$ for the **radii** of the half-balls;
- $\mathbf{1}_{\rho \leq \sqrt{1+x^2}}$: condition of **origin exclusion**.

Portrait of $\mathcal{C}_2$



$\mathbb{B}_2$

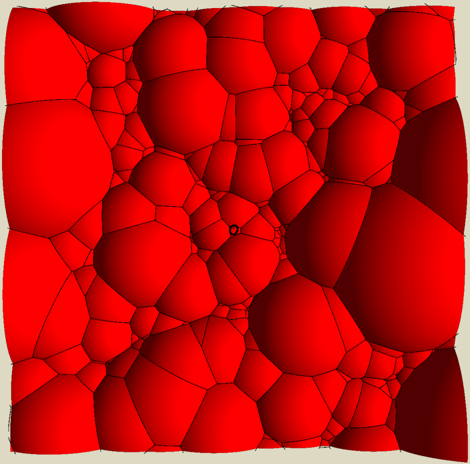
$$\varphi(z) = i\frac{i+z}{i-z} \quad \Downarrow \quad \Uparrow \quad \varphi^{-1}(w) = i\frac{w-i}{w+i}$$



$\mathbb{U}_2$

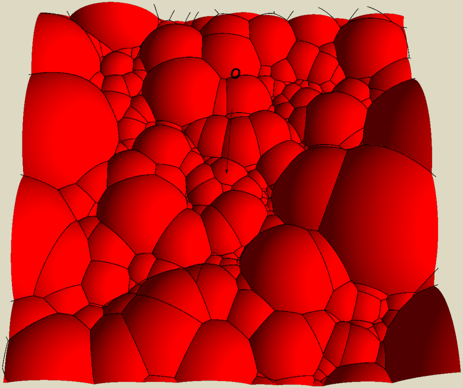
Portrait of $\mathcal{C}_3$ in $\mathbb{U}_3$ seen from above

Interactive version at sketchfab.com/foam



Portrait of $\mathcal{C}_3$ in $\mathbb{U}_3$

Interactive version at sketchfab.com/foam



Portrait of $\mathcal{C}_3$ in $\mathbb{U}_3$

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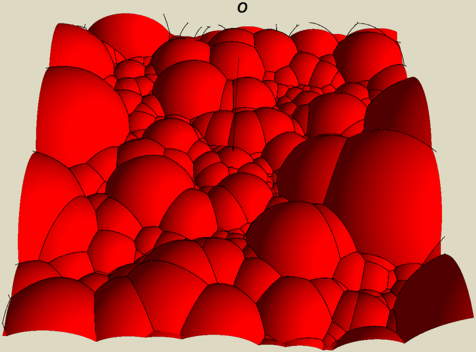

Portrait of $\mathcal{C}_3$ in $\mathbb{U}_3$

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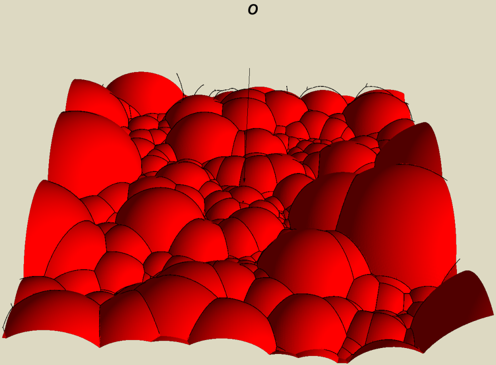
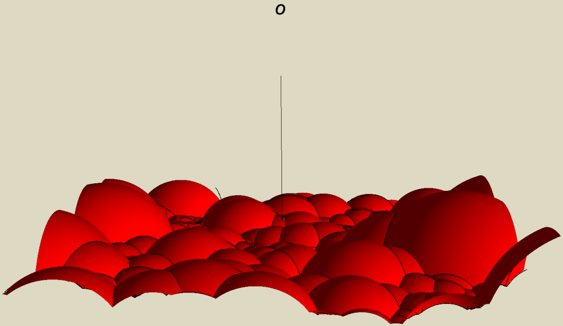


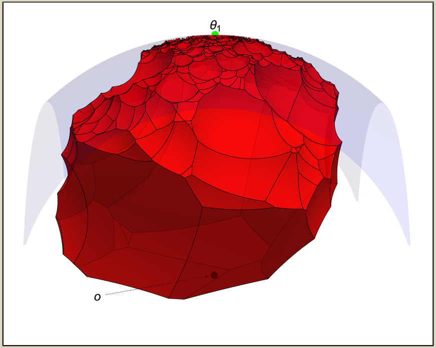
Portrait of $\mathcal{C}_3$ in $\mathbb{U}_3$ seen from the side

Interactive version at sketchfab.com/foam



Portrait of $\mathcal{C}_3$ in $\mathbb{B}_3$

Interactive version at sketchfab.com/jewel (or in your hands)



Local properties of $\mathcal{C}_d$: hole probability

The probability that the hyperbolic ball $B_r(\mathbf{o})$ centered at $\mathbf{o}$ with hyperbolic radius r is contained in $\mathcal{C}_d$ is given, $\forall d \geq 2$ by

(i) **Conditional** on $R_1 = s$,

$$\mathbb{P}\left(B_r(\mathbf{o}) \subset \mathcal{C}_d \;\middle|\; R_1 = s\right) = \exp\left(-sI_d(r)\right) \; ;$$

(ii) **Averaging** on the value of R_1,

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_d) = \frac{1}{1 + I_d(r)} \, ,$$

where

$$1 + I_d(r) = \frac{1}{c_d} \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}x}{\left(\sqrt{\cosh^2 r + |x|^2} - \sinh r\right)^{2d-2}} \, .$$

Local properties of $\mathcal{C}_d$: hole probability

- $d = 2$

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_2) = \frac{\pi}{4\left(\arctan\left(e^r\right)\cosh^2 r + 2\sinh r\right)}$$

- $d > 2$ odd: a rational function of e^r, *e.g.*

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_3) = \frac{3e^{-2r}}{2 + e^{2r}}$$

Generalize results of Isokawa in $d = 2, 3$ to arbitrary dimension d

Asymptotic properties: height and azimuth

Let $\mathcal{M}_d \subset \mathbb{U}_d$ be the random closed obtained by removing balls whose centers and radii follow a **PPP** of intensity measure

$$2 \cdot \mathcal{E}_d \cdot \; \mathrm{d}x \; \rho^{1-2d} \mathrm{d}\rho \; \cancel{\mathbf{1}_{\rho \leq \sqrt{1+x^2}}} \quad \text{on} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+ \, .$$

The law of $\partial \mathcal{M}_d$ is **explicit** in terms of **height** $\mathcal{H}(x_0)$ **above** $x_0 \in \mathbb{R}^{d-1}$ and **angle** $\Theta(x_0)$ of the tangent hyperplane to $\partial \mathcal{M}_d$ above x_0 wrt vertical direction (**azimuth**).
For $d \geq 2$, the law of $(\mathcal{H}, \Theta)$ is, **conditional** on $R_1 = s$,

$$\left(\frac{1}{\mathcal{H}^{d-1}}, \sin^2(\Theta) \right) \sim \mathrm{Exp}\,(s) \otimes \mathrm{Beta}\left(\frac{d+1}{2}, \frac{d-1}{2} \right).$$

Asymptotic properties: vertex intensity

In the **stationary model** $\mathcal{M}_d$, the process of vertices has the following intensity at a point $(x, z) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$:

(i) **Conditional** on $R_1 = s$

$$\frac{(\mathbf{c}s)^d}{d!} \nu_d \frac{\mathrm{e}^{-\frac{s}{z^{d-1}}}}{z^{d^2}} \mathrm{d}x \, \mathrm{d}z \,,$$

$$\nu_d = \frac{1}{2^d} \int_{[0,1]^d \times (\mathbb{S}_{d-2})^d} \prod_{i=1}^{d} \left[v_i \left(1 - v_i\right)\right]^{\frac{d}{2}-1} \mathrm{Vol}_{d-1} \left(\sqrt{\frac{v_i}{1 - v_i}} u_i \right)_{1 \le i \le d} \mathrm{d}v_i \mathrm{d}u_i$$

where Vol_{d-1} = Euclidean volume of a $d-1$ **dimensional simplex**.
In particular $\nu_2 = \frac{3\pi}{4}$ and $\nu_3 = \frac{\pi^4}{35}$.

(ii) **Averaging** on the value of R_1:

$$\mathbf{c}^d \nu_d \, \frac{1}{z(1 + z^{d-1})^{d+1}} \mathrm{d}x \, \mathrm{d}z.$$

Explicit formulas for face intensities

Godland–Kabluchko–Thale 2022 provide mean face vector of the typical cell of $\mathcal{V}_d^{(\lambda)}$. Knowing that the volume of C_d is $\frac{1}{\lambda}$, we got **explicit formulas** for face-counting intensities $\forall d$ and $\forall k$.

The orthogonal projection of $\partial\mathcal{M}_d$ over $\mathbb{R}^{d-1}$ provides a special case of a more general point process recently studied by **Gusakova–Kabluchko–Thäle 2022**, call it $\mathcal{B}_d$.

This result, which uses Poisson–Delaunay duality, combined with our result for height and angle of $\partial\mathcal{M}_d$ allows us to get explicit formulas for mean volumes of boundary cells in any d.

Mean area of a face in $d = 3$

Sketch of proof

Consider a **cylinder** $C_L = [0, L]^2 \times \mathbb{R}_+$. Then:

- Conditionally on $R_1 = s$, **area** of $\partial \mathcal{M}_d \cap C_L \sim L^2 \mathbb{E}\left(\frac{1}{\mathcal{H}^2 \sin \Theta}\right)$ a.s.;
- Conditionally on $R_1 = s$, $\#$ of vertices inside C_L is easy;
- Since $2E_L = 3V_L + o(L^2)$, (approximate) **Euler's formula** $F_L - E_L + V_L = o(L^2)$ allows to get $F_L = \frac{1}{2} V_L + o(L^2)$;
- Involved quantities all behave as $\underset{L \to \infty}{\sim} \frac{c}{s} L^2 \implies$ take the **ratio**!

The **mean area** of a face is equal to $\frac{35}{12\pi} = 0.928404\ldots$ ∎

Provisional conclusions

- **Poisson–Voronoi tessellations** with small intensity of nuclei have been considered or used in the literature.

- We have provided a characterization of the limit object, called **ideal Poisson–Voronoi tessellation**, focusing mostly on hyperbolic spaces *.

- Here a remarkably simple description in terms of a **deposition model** allows to study in detail the geometry of the zero cell C_d, both near and far away from a prescribed point of $\mathbb{H}_d$.

*Plus some results on regular trees.

Some perspectives

Many, interesting research directions!

- Observables defined on more than one cell
- A.s. **injectivity** of the map from IPVT to Voronoi vertices
- **Poisson–Delaunay** triangulations
- Degree Constrained **Minimum Spanning Tree**
- **Infinite-dimensional** hyperbolic spaces
- **Complex** hyperbolic spaces

Merci!