Journées de Probabilités 2021, Guidel, BZH

Euclidean Random Assignment Problems: origin, state of the art and some open problems in one dimension

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LAMA UPEC & CNRS   Matteo D'Achille

Based on several papers in collaboration with

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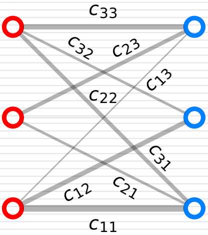
Section 1	Background and Definition
1	The Assignment Problem
2	Adding randomness to AP
3	Euclidean Random Assignment Problems

The (linear sum) Assignment Problem (AP)

For a $n \times n$ cost matrix c, find a bijection π (a permutation) s.t. $E = \sum_i c_{i\pi(i)}$ is minimal. Let $E_{\min}$ be the minimal value.

Example at $n = 3$:



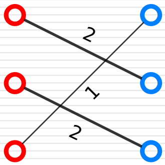
$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

- Simple formulation → good model in applications
- P-complete with $\mathcal{O}(n^3)$ complexity [Munkres 1957]

The (linear sum) Assignment Problem (AP)

For a $n \times n$ cost matrix c, find a bijection π (a permutation) s.t. $E = \sum_i c_{i\pi(i)}$ is minimal. Let $E_{\min}$ be the minimal value.

Example at $n = 3$: $E_{\min} = 5$



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

Swap columns (rows) s.t. $\mathrm{Tr}(c)$ is minimal [Koopmans–Beckmann 1957]; Optimal mixed strategy in a “hide and seek” game [Von Neumann 1953, 1954]

- Simple formulation → good model in applications
- P-complete with $\mathcal{O}(n^3)$ complexity [Munkres 1957]

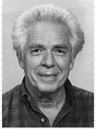
AP: an old problem

König
1916

Egérvary
1931

von Neumann
1953

Kuhn
1955



Canon simplicissimus.

	I	II	III	IV	V	VI	VII
I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

"De investigando ordine systematis aequationum ..."
[Jacobi 1860]
See also [Ollivier 2009]

Section 1 Background and Definition

The Random Assignment Problem I

c is a random matrix (c_{ij} i.i.d. r.v. $\sim \rho(l) = l^r + o(l^r)$).

$$\mathbb{E}[E_{\min}]_n \underset{n\to\infty}{\sim} c_r n^{1-\frac{1}{r+1}}.$$

- Pioneered in Physics in the 80s by Mézard–Parisi and Orland
- Entered Probability mostly through Aldous in the 90s

Result: only "short" edges are relevant for large n and r can be considered a "universal exponent".

Nice fact: at $r = 0$ (i.e. ρ is e.g. uniform or $\mathrm{Exp}(\lambda)$ distribution)

$$c_0 = \zeta(2) = \frac{\pi^2}{6}.$$

The Random Assignment Problem II

If $c_{ij} \sim \mathrm{Exp}(1)$, Parisi conjectured (1998):

$$\mathbb{E}[E_{\min}]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

1. Rectangular matrices [Coppersmith-Sorkin 1998]
2. Proof of $\zeta(2)$ limit (among other things) [Aldous 2001]
3. Proof of Parisi conjecture [Prabhakar–Sharma 2001]
4. Extension to the k-partite case (NP-hard for $k \geq 3$) [Martin–Mézard–Rivoire 2004,2005]
5. $\exists!$ solution to "cavity" equation [Wästlund 2012, Larsson 2014, Salez 2015]

<u>NOT discussed today...</u>

Section 1	Background and Definition
1	The Assignment Problem
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The Euclidean Random Assignment Problem (ERAP)

Let $\mathcal{B} = (B_1, \ldots, B_n)$ be blue points and $\mathcal{R} = (R_1, \ldots, R_n)$ be red ones: n-samples of i.i.d. r.v. of pdf $\rho_{\mathcal{B}(\mathcal{R})} : \Omega \to \mathbb{R}$ ("disorder"), $(\Omega, \mathcal{D})$ is a metric space (mostly an **Euclidean** space with $\mathcal{D}$ **Euclidean** distance). For $p \in \mathbb{R}$ and an assignment (permutation) π, consider the *Hamiltonian*

$$\mathcal{H}(\pi) = \sum_{i=1}^{n} \mathcal{D}^p(B_i, R_{\pi(i)})$$

and the random variable (ground state energy)

$$\mathcal{H}^{(p)}_{\mathrm{opt},(n,d)} = \min_{\pi \in \mathcal{S}_n} \mathcal{H}(\pi) \quad (\pi_{\mathrm{opt}} = \arg\min_{\pi \in \mathcal{S}_n} \mathcal{H}(\pi)).$$

Problem: the rate of $E_{p,d}(n) := \mathbb{E}[\mathcal{H}^{(p)}_{\mathrm{opt},(n,d)}]$ as $n \to \infty$.

Three motivations: Physics, Mathematics and Computer Science

- **Spin Glasses** - ERAP is a toy model of spin-glass (a **disordered** and **frustrated system**) in finite dimension, which is numerically simple in comparison to e.g. Edwards–Anderson spin glass [Mézard–Parisi 1988];

- **Optimal Transport** - ERAP is a Monge-Kantorovitch problem associated to empirical measures $\rho_{\mathcal{B}}$, $\rho_{\mathcal{R}}$:

  $$\mathcal{H}_{\text{opt}} = n W_p^p(\rho_{\mathcal{B}}, \rho_{\mathcal{R}})$$

  where W_p is the p**-Wasserstein** distance [Villani 2009, Vershik 2013, Brezis 2018];

- **Computational Complexity Theory** - ERAP is a small (but crucial) modification of random TSP, however finding π_{opt} is **easy** (recall that AP is P**-complete**).

Section 2	State of the art
1	The phase diagram
2	State of the art

ERAP: the phase diagram

We shall put $\rho_{\mathcal{B}} = \rho_{\mathcal{R}} := \rho$. We wish to study

$$E_{p,d}(n) := \mathbb{E}[\mathcal{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1 + o(1))$$

as $n \to \infty$, depending on (p, d) and the choice of ρ.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be "**universal**", i.e. largely independent on the microscopic details (which may affect the constant $K_{p,d}$).

Remark: non-uniform disorder is more subtle!

Example: standard Gaussian disorder at $(p, d) = (2, 1)$

$$E_{2,1}(n) \sim 2 \ln \ln n \qquad (\text{i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

[Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019, Berthet–Fort 2020]

See [Benedetto–Caglioti 2020] for non-uniform case at $d = 2$.

Section 2	State of the art
1	The phase diagram
2	State of the art

$d \geq 3$, $p \geq 1$

“Simple”:

$$E_{p,d}(n)|_{d\geq 3} \underset{n\to\infty}{\sim} c_{p,d}\, n^{\gamma_{\mathrm{LB}}}$$

where

$$\gamma_{\mathrm{LB}} := 1 - \frac{p}{d} = \gamma_{p,d} \qquad \text{[Mézard–Parisi 1988]}$$

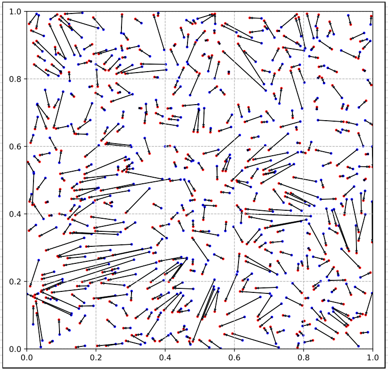
(if the disorder is uniform, otherwise unknown).

Remark: the constant $c_{p,d}$ is **unknown** (upper and lower bounds in [Talagrand 1992]).

- Almost-sure limits of Euclidean functionals of finite random point sets [Barthe–Bordenave 2013 and refs. therein];
- Recurrent interest in Optimal Transport [Goldman–Trevisan 2020].

$d = 2$: a challenge for both mathematicians and physicists

Example configuration for $\Omega = [0,1]^2$ and $\mathcal{D}$ Euclidean distance:



Optimal assignment typically involves $O(\ln n)$–nearest-neighbors:

$(\gamma_{p,d}, \gamma'_{p,d}) = (\gamma_{\mathrm{LB}}, \frac{p}{2})$ if $p \geq 1$ [Ajtai–Komlós–Tusnády 1984]

Recent developments in Mathematics and Physics

2014 Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E): using a (classical) field-theoretical approach, predicted

$$K_{2,2} = \frac{1}{2\pi};$$

2019 Ambrosio–Stra–Trevisan (PTRF): proof of $K_{2,2} = \frac{1}{2\pi}$ (among other things) via PDE methods;

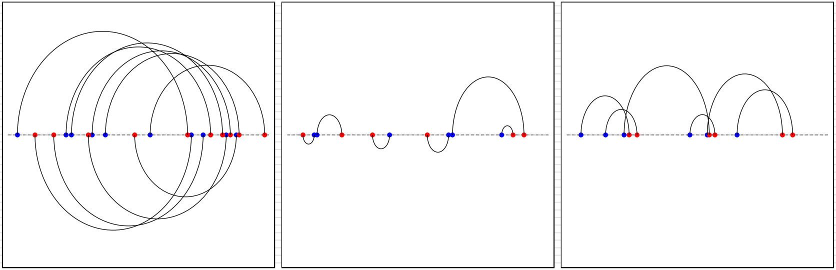
2020 Ambrosio–Glaudo (JEP): refinement on the remainder term (among other things);

2021 Benedetto–Caglioti-Caracciolo–**D'A**–Sicuro–Sportiello (JStatPhys): exact energy differences for ERAPs on two manifolds Ω, Ω'.

See https://www.youtube.com/watch?v=4RcOiW20C_E for a discussion of the latter results in the light of Weyl's law in spectral theory (and extension to ERAPs at $d = 3$).

$d = 1$: properties of the solution

$p = 0$ and $p = 1$ separate **three qualitative regimes**:



If realizations are sorted in natural order, π_{opt} is:

- $p < 0$ **cyclical**: $\pi_{\text{opt}} = i + k \pmod{n}$ [Caracciolo–**D'A**–Sicuro 2017];
- $p \in (0, 1)$ **non-crossing**: intervals "covered by edges" are either disjoint or one is included into the other [McCann 1999];
- $p \geq 1$ **ordered**: π_{opt} is the identity permutation.

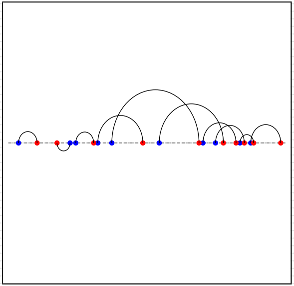
Remark: combinatorial properties hold for any disorder ρ.

Section 3	ERAPs at $d = 1$
1	Convex regime $p \geq 1$
2	Concave regime $p \in (0, 1)$
3	Anomalous scalings

$p \geq 1, \Omega = \mathbb{R}, [0,1], \mathbb{R}^+$ and $\mathcal{D} = |\,|$

Convexity and strict monotonicity of $\mathcal{D}^p \implies$ if realizations $(b_1, \ldots, b_n)$ and $(r_1, \ldots, r_n)$ are sorted in **natural order**, then

$$H^{(p)}_{\mathrm{opt},(n,1)}|_{p\geq 1} = \sum_{i=1}^{n} |b_i - r_i|^p.$$

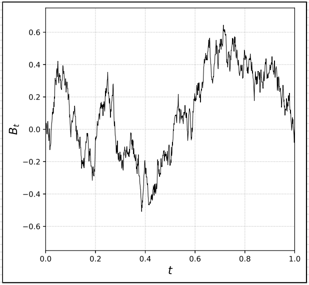


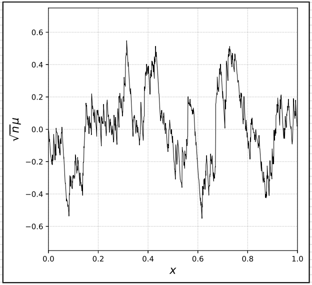
$\implies$ a path to the Brownian world.

$p \geq 1$, $\Omega = \mathbb{R}, [0,1], \mathbb{R}^+$ and $\mathcal{D} = |\,|$: Brownian Bridge

Let the **transport field** be $\mu_i := b_i - r_i$. Then

$$\sqrt{n}\,\mu_i \underset{weakly}{\longrightarrow} B_t, \text{ the Brownian Bridge}$$





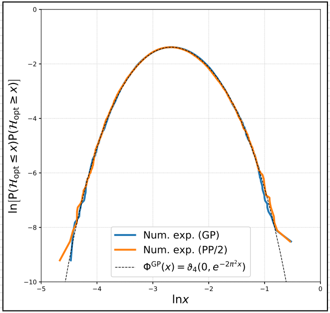
$$\text{Thus } \mathcal{H}_{\mathrm{opt},(n,p)} = \sum_i |\mu_i|^p, \text{ energy} \sim \text{moments of } B_t.$$

[Boniolo–Caracciolo–Sportiello 2014, Caracciolo–Sicuro 2014, Caracciolo–**D'A**–Sicuro 2017)]

$\Omega = \mathbb{S}_1$, $\mathcal{D}$ = arc-distance, $p = 2$: distribution of $\mathcal{H}_{\text{opt}}$

The asymptotic CDF of $\mathcal{H}_{\text{opt},(n,2)}$ on the unit circle can be obtained via **Fourier analysis** [**D'A**, PhD Thesis 2020]:

$$\Pr[\mathcal{H}_{\text{opt},(n,2)} \leq x] \overset{n\to\infty}{=} \vartheta_4(0, e^{-2\pi^2 x}) := \sum_{s\in\mathbb{Z}} e^{i\pi s} e^{-2\pi^2 s^2 x}$$



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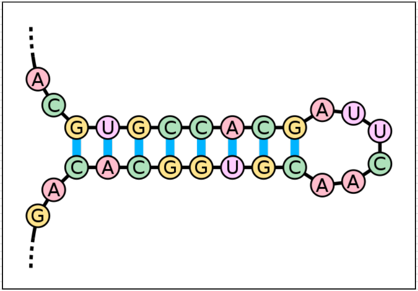
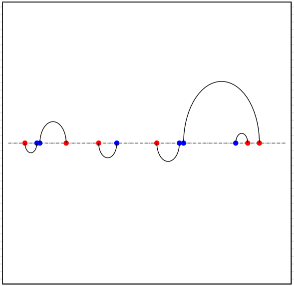
ERAPs at $p \in (0, 1)$: further motivation in biology

Besides economics, interest due to the non-crossing property of the solution: toy-model for the **secondary structure of RNA** (discarding pseudo-knots).

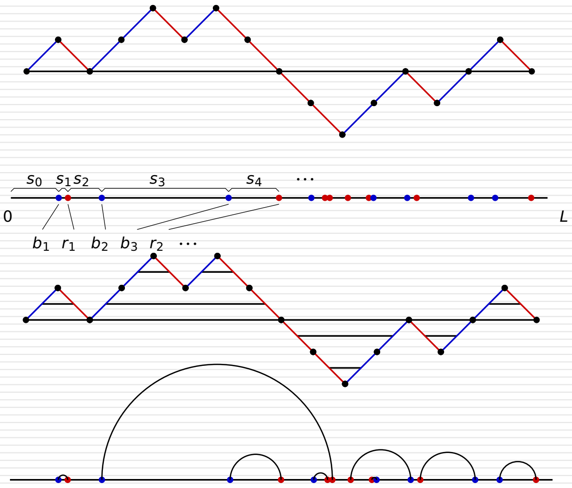




Despite this, poorly understood. [McCann 1999]

IDEA: try a canonical construction, an approximate solution sharing non-crossing property with π_{opt} → Dyck matchings!

The Dyck matching [Caracciolo–**D'A**–Erba–Sportiello 2020]



The Dyck Upper Bound at $p \in (0, 1)$

Expected energy of Dyck matchings grows asymptotically as

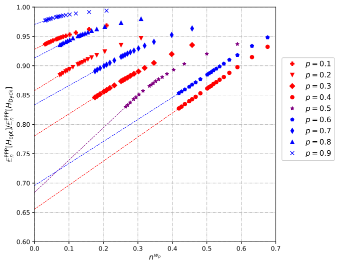
$$\mathbb{E}_n(\mathcal{H}_{\mathrm{Dyck}}) \simeq \begin{cases} n^{1-p} & \text{if} \quad 0 \le p < \frac{1}{2} \\ \sqrt{n}\ln n & \text{if} \quad p = \frac{1}{2} \\ n^{\frac{1}{2}} & \text{if} \quad \frac{1}{2} < p \le 1 \end{cases}$$

Conjecture [Caracciolo–**D'A**–Erba–Sportiello 2020]:

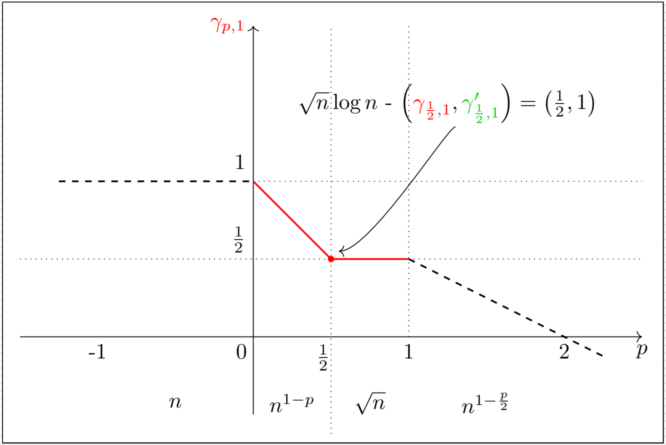
$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathcal{H}_{\mathrm{opt}})}{\mathbb{E}_n(\mathcal{H}_{\mathrm{Dyck}})} = k_p$$

for some constant $0 < k_p < 1$.

Numerical evidence $\Longrightarrow$



Section of the Phase Diagram at $d = 1$



Product formula for number of solutions at $d = p = 1$ [Caracciolo–Erba–Sportiello 2021].

Section 3	ERAPs at $d = 1$
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2	Concave regime $p \in (0, 1)$
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Beyond uniform disorder: bulk and anomalous scaling

Recall the Selberg result for points $\sim U[0,1]$ and $p \geq 1$ (see [Caracciolo *et al.* 2019]):

$$E^{\mathrm{U}}_{p,1}(n) = n\frac{\Gamma\left(1+\frac{p}{2}\right)}{p+1}\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}$$
$$= c_p n^{1-\frac{p}{2}}\left(1+o(1)\right) = c_p n^{\gamma_{p,1}}\left(1+o(1)\right) \ (\textit{Bulk scaling}).$$

What if ρ is non-uniform?

If $\gamma_{p,1} \neq 1-\frac{p}{2}$ or in presence of a logarithmic correction, we talk about an **anomalous scaling**.

"Reduction to quadratures" in the bulk scaling

Let R be the cdf of ρ. Set $\Psi^{(\rho)} := \rho \circ R^{-1}$. Then

$$E^{(\rho)}_{p,1}(n) = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \int_0^1 \left[\frac{\sqrt{s(1-s)}}{\Psi^{(\rho)}(s)}\right]^p \mathrm{d}s \, n^{1-p/2} + o\left(n^{1-p/2}\right).$$

Caracciolo–**D'A**–Sicuro (JStatPhys 2018): non-rigorous **regularization methods** inspired by cutoff regularization in QFT. Analogous problem "in the continuum" by Bobkov–Ledoux (AMS 2019).

Example: $\rho(x) = e^{-x}$, $\Psi^{\exp}(s) = 1 - s$.

- Cutoff method: $E^{\exp}_{p,1}(n) = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \int_0^{1-c/n} \left(\frac{s}{1-s}\right)^{\frac{p}{2}} ds.$
  At $p = 2$, we get $E_2(n) = 2 \ln n - 2 \log c - 2 + o(1)$;
- Exact calculation: at $p = 2$ (via Beta integrals):
  $E^{\exp}_{2,1}(n) = 2 \sum_{k=1}^{n} \frac{1}{k} = 2 \ln n + 2\gamma_{\mathrm{E}} + o(1).$

(by comparison $c = e^{-\gamma_{\mathrm{E}}-1} = 0.20655\ldots$)

On **D'A**-Sportiello 2020 (in preparation)

Emergence of logarithmic corrections to scaling $\implies$ detailed study of contribution of k-th edge at fixed n, then $n \to \infty$.

General strategy: 1) Binomial expand $\mathcal{H}^{(p)}_{\mathrm{opt},(n,1)}|_{p\geq 1}$ at fixed (n, p) (for $p > 1$ even); 2) exploit linearity to compute expectation w.r.t. a certain Beta distrib.; 3) asymptotic analysis (case by case depending on ρ).

More precisely

$$E^{(\rho)}_{p,1}(n) = \sum_{k=1}^{n} \mathbb{E}\left[(b_k - r_k)^p\right] = \sum_{k=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M^{(\rho)}_{n,k,q} M^{(\rho)}_{n,k,p-q}$$

for

$$M^{(\rho)}_{n,k,\ell} = \langle R^{-1}_{\rho}(u)^{\ell} \rangle_{P_{n,k}}, \quad R^{-1}_{\rho}(u) \quad \text{the quantile function,}$$

where $\langle \ldots \rangle_{P_{n,k}}$ is expectation w.r.t. Beta$(k, n-k+1)$:

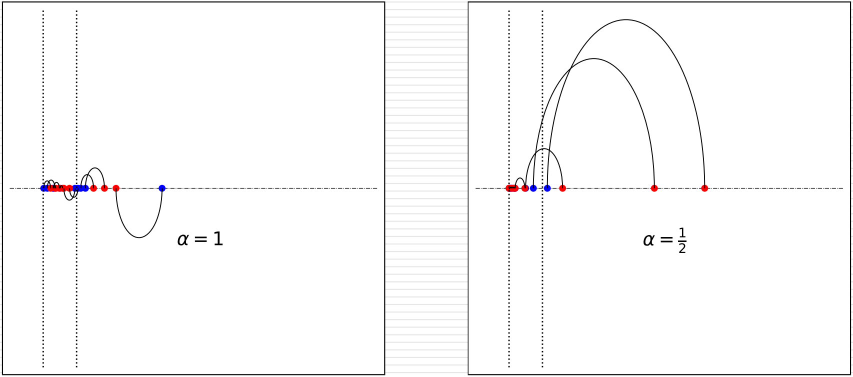
$$P_{n,k}(u)du := \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k} du.$$

Example: stretched exponential distribution

Let us consider the following pdf (θ is Heaviside's function)

$$\rho_{\mathrm{ie},\alpha}(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha})\theta(x).$$

Example: solutions at $n = 10$



$$\text{Thus } R_{\alpha}(x) = \exp(-x^{\alpha})\theta(x) \implies R_{\alpha}^{-1}(u) = (-\ln u)^{\frac{1}{\alpha}}.$$

Sketch of the computation

For integer $s = \frac{1}{\alpha}$, we wish to evaluate

$$M_{n,k;s} := \int_0^1 \mathrm{d}u \, P_{n,k}(u)(-\ln u)^s.$$

Lemma I (D'A-Sportiello 2020)

$$M_{n,k;s} = s! \, h_s(A_{k,n}),$$

for the alphabet $A_{k,n} := \left\{ \frac{1}{k}, \frac{1}{k+1}, \ldots, \frac{1}{n} \right\}$, where h_s is the *complete homogeneous symmetric function* of degree s^*.

Two hints: 1) use the representation $-\ln u = \lim_{x \to 0} \frac{u^{-x}-1}{x}$ and 2) recall that for a polynomial $A(q) = a_p q^p + a_{p-1} q^{p-1} + \ldots + a_0$,

$$\sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} A(q) = p! \, a_p.$$

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*I.e. for alphabet $(x_1, \ldots, x_m)$, $h_s(x_1, \ldots, x_m) := \sum_{1 \le j_1 \le \ldots \le j_s \le m} x_{j_1} x_{j_2} \cdots x_{j_s}$.

Sketch of the computation

The contribution $E_{(s,p),n}(k)$ of the k-th edge in the solution to the total energy $E_{(s,p)}(n) = \sum_{k=1}^{n} E_{(s,p),n}(k)$ is thus

$$E_{(s,p),n}(k) = \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M_{n,k;sq} M_{n,k;s(p-q)}$$

$$= \sum_{q=0}^{p} (-1)^{q} \binom{p}{q} (sq)!(s(p-q))!\, h_{sq}(A_{k,n}) h_{s(p-q)}(A_{k,n}).$$

In the next step, we make use of **generating functions**.

Exact result in the case $s = 1$ (exponential)

For exp. distrib. points, we have a remarkable simplification. Set

$$H(t, X) = \sum_{\ell \geq 0} h_\ell(X) t^\ell = \prod_{x \in X} (1 - tx)^{-1}.$$

Then

$$\begin{aligned} E_{(1,p),n}(k) &= p! \sum_{q=0}^{p} (-1)^q h_q(A_{k,n}) h_{p-q}(A_{k,n}) \\ &= p! [t^p] H(t, A_{k,n}) H(-t, A_{k,n}). \end{aligned}$$

Obviously for any alphabet X, $H(t, X)H(-t, X) = H(t^2, X^2)$, so that

$$E_{(1,p),n}(k) = p! [t^p] H(t^2, A_{k,n}^2) = p! h_{p/2}\left(A_{k,n}^2\right).$$

Asymptotic result at integer $s > 1$ (stretched-exponential)

For the power-sum functions $p_k(X) = \sum_{x \in X} x^k$, define

$$P(t, X) = \sum_{\ell \geq 1} \ell^{-1} p_\ell(X) t^\ell = \log H(t, X).$$

Fact: $(p_\ell(A_{k,n})^{1/\ell})_\ell$ is monotonic decreasing (due to the generalised-mean ineq.). In a regime in which $p_1 \gg \sqrt{p_2}$ we consider a **perturbative expansion**. By setting $P_+(t, X) = P(t, X) - tp_1(X)$ we then have

$$\begin{aligned} h_k(X) &= [t^k] H(t, X) = [t^k] \exp(P(t, X)) \\ &= \sum_{\ell=0}^{k} \frac{1}{(k-\ell)!} p_1(X)^{k-\ell} [t^\ell] \exp(P_+(t, X)). \end{aligned}$$

After some computation, we get

$$E_{(s,p),n}(k) \simeq p!\, s^p p_1(A_{k,n})^{(s-1)p} [t^p] \left(\exp(P_+(t, A_{k,n})) \exp(P_+(-t, A_{k,n}))\right).$$

Asymptotic result at $s > 1$ integer (stretched-exponential)

Since still obviously $\exp(P_+(t,X))\exp(P_+(-t,X)) = H(t^2,X^2)$, we get

$$E_{(s,p),n}(k) \simeq p!\, s^p p_1(A_{k,n})^{(s-1)p}\, h_{p/2}(A_{k,n}^2)$$

which recovers the exact result at $s = 1$. As $p_1(A_{k,n}) \sim \ln(n) - \ln(k) + \mathcal{O}(1)$ whenever $n \gg k$, while $p_h(A_{k,n}) \sim \frac{1}{h-1}(k^{1-h} - n^{1-h})$ if $n \gg k \gg 1$, in any case $p_1/\sqrt{p_2}$ is at least of order $\ln(n)$ whenever $n \gg k$. We can thus replace $p_1(A_{k,n})$ with $\ln(n)$ to get

$$E_{(s,p),n}(k) = p!\, s^p \ln(n)^{(s-1)p}\, h_{p/2}(A_{k,n}^2)\big(1 + \mathcal{O}((1+\ln k)/\ln n)\big).$$

In order to estimate $E_{(s,p)}(n) = \sum_k E_{(s,p),n}(k)$, we are left with the task of estimating ($q = \frac{p}{2}$)

$$F_{n,q} := \sum_{k=1}^{n} h_q(A_{k,n}^2).$$

Emergence of **multiple ζ^* values** as $n \to \infty$

Observe that each monomial contributing to $F_{n,q}$ is of the form $\frac{1}{i_1^2 i_2^2 \cdots i_q^2}$, and enters exactly i_1 times in the sum. Hence

$$F_{n,q} = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_q \le n} \frac{1}{i_1\, i_2^2 \cdots i_q^2} = \sum_{k=1}^{n} \frac{1}{k} h_{q-1}(A_{k,n}^2).$$

In this series† we recognize a special kind of "**multiple ζ^* values**"(MZSVs), defined as

$$\zeta^*(a_1, \ldots, a_r) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r} \frac{1}{i_1^{a_1} \cdots i_r^{a_r}}.$$

MZSVs are related to **multiple ζ values** arising e.g. in **Vassiliev–Kontsevich theory** [Zagier 1992]. Not new in **Physics**: scattering amplitudes in perturbative quantum field theory [Broadhurst–Kreimer 1997].

†Convergence as $n \to \infty$ holds since $h_{q-1}(A_{k,n}^2) \sim k^{-(q-1)}$ for $n \gg k$.

Result for $E^{(s)}_{(p,1)}(n)$

For our special MZSVs a **further remarkable simplification holds** [Ohno–Wakabayashi 2006,Ohno–Zudilin 2008, Borwein–Zudilin 2011]

$$F_{+\infty,q} = \zeta^*(1, \underbrace{2, \ldots, 2}_{q-1}) = 2\zeta(2q-1)$$

where ζ is Riemann's function. In conclusion, we have

Lemma II (D'A-Sportiello 2020)

In the ERAP with stretched-exponentially distributed points with pdf $\rho_{\mathrm{ie},\alpha}(x) = \alpha x^{\frac{1}{s}-1}\exp(-x^{\frac{1}{s}})$, for $p > 1$ even and s integer

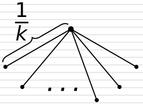
$$E^{(s)}_{p,1}(n) \simeq \begin{cases} 2s^2\ln(n)^{2s-1} & p = 2 \\ 2\zeta(p-1)s^p p!\ln(n)^{(s-1)p} & p \geq 4 \end{cases}$$

Remark: the same asymptotics holds for $\rho^+_{\mathrm{ie},\alpha}(x) = \alpha x^{\alpha-1}\exp(1-x^\alpha)$ (the argument goes with minor modifications).

ERAPs on the k-star graph (W.i.p. with Caracciolo, Liu and Sportiello 2021-)

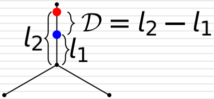
Choice of the triple $((\Omega, \mathcal{D}), (\rho_{\mathcal{B}}, \rho_{\mathcal{R}}), p)$:

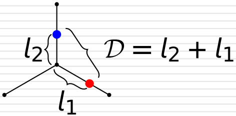
1. $\Omega = \mathbb{K}_{1,k}$ (i.e. “star” with k branches)



k branches

2. Distance $\mathcal{D}$:



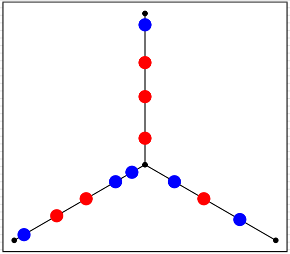


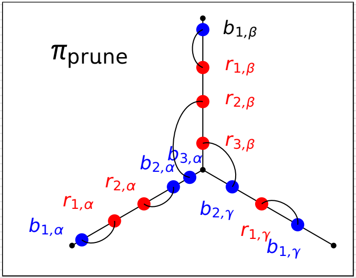
3. $\mathcal{B}$ and $\mathcal{R}$ are $\sim U[0, \frac{1}{k}] \bigotimes \mathcal{U}\{1, k\}$ random variables;
4. $p \geqslant 1$.

On the configuration π_{prune}

IDEA: try a canonical construction, an approximate solution sharing local ordering property with $\pi_{\text{opt}} \rightarrow \underline{\pi_{\text{prune}}}$!

Example: $k = 3$ and $n = 6$





On the configuration π_{prune}

Conjecture :

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}^{(p)}_{\text{opt}}\right] = \lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}^{(p)}_{\text{prune}}\right].$$

Consequence: $\gamma^{(k)}_{p,1} = 1 - \frac{p}{2}$, $\gamma'^{,(k)}_{p,1} = 0$, explicit (and simple!) formulas for limit constants.

Example: at $p = 2$

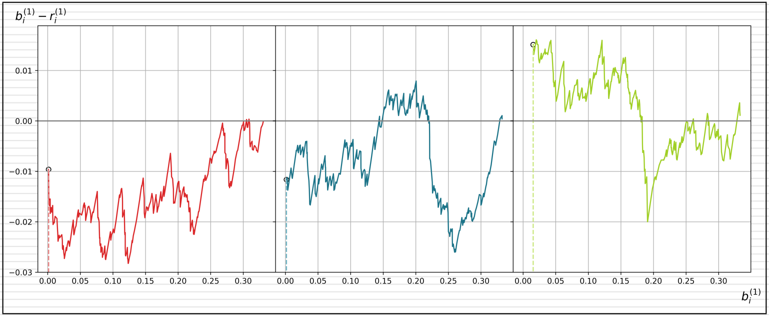
$$\lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}^{(2)}_{\text{opt}}\right] = \frac{3k-2}{3k^2}.$$

On the configuration π_{prune}

Claim:

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}_{\text{opt}}^{(2)}\right] = \frac{3k-2}{3k^2}.$$

Sketch of proof: moments of k independent Brownian Bridges stopped at time $\frac{1}{k}$.



Open problem 1: the Dyck conjecture

Proof of the Dyck conjecture for $p \in (0, 1)$:

$$\lim_{n \to \infty} \frac{\mathbb{E}_n(\mathcal{H}_{\mathrm{opt}})}{\mathbb{E}_n(\mathcal{H}_{\mathrm{Dyck}})} = k_p.$$

Open problem 2: implications of Generalised Selberg Integrals

Take only $\Omega = [0,1]$, $\mathcal{D} = |.|$ and $\rho_{\mathcal{B}} = \rho_{\mathcal{R}} := \rho = \mathbb{1}_{[0,1]}(x)$.
From Generalised Selberg Integrals, we know [Caracciolo *et al.* 2019], for integer ℓ,

$$\mathbb{E}[|b_k - r_k|^\ell] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}.$$

For the usual cost function $f = |.|^p$, we have a nice formula for $E(n)$.

Problem: are there choices of a more general cost function $f = f(|.|)$ so that a "nice expression" for $E(n)$ (i.e. not necessarily involving hypergeometric functions) can be obtained upon resummation?‡

‡This question was raised by N.Enriquez during a talk given by the author at CIRM Marseilles - Luminy in March 2021.

Thank you for your attention!