Using stochastic order to compare different Euclidean Random Assignment Problems

Matteo D'Achille[1] and Yuqi Liu[1]

[1]Univ Paris Est Créteil, Univ Gustave Eiffel, CNRS, LAMA UMR8050, F-94010 Créteil, France

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Abstract

This paper provides a theorem to compare the minimum total cost of two different Euclidean Random Assignment Problems with the same number of points, using the stochastic order of the costs of one of the pairs in these two problems. The subsequent sections provide two applications of the theorem, including studies of the problem on the one-dimensional k-star graph and comparisons between some problems in higher dimensions. More possible applications and limitations of the theorem are also discussed.

1 Introduction

Many random combinatorial optimization problems [21], like the traveling salesman problem [24] and random K-satisfiability (K-SAT) [32] have long been questions of great interest in a wide range of fields. The assignment problem [26] is also a classic type of them and has received increased attention across a number of disciplines in recent years, like in education [20], transportation [36], and healthcare [19]. Among the several extensions of the assignment problem, a prominent one is the Euclidean Random Assignment Problem (ERAP) [17], which goes as follows. Consider an n-sample $\mathcal{B} = (B_1, \dots, B_n)$ of i.i.d. random variables of law μ_B over the defined domain $\Omega_b \subset \mathbb{R}^d$, and another n-sample $\mathcal{R} = (R_1, \dots, R_n)$ of i.i.d. r.v.s of law μ_R over $\Omega_r \subset \mathbb{R}^d$. We are interested in the statistical properties of the random variable

$$\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r} := \min_{\pi \in S_n} \sum_{i=1}^{n} D^p(B_i, R_{\pi(i)}) \ , \tag{1}$$

where S_n is the symmetric group over n elements, and $D(B_1, R_1)$ is the Euclidean distance between points B_1 and R_1. In some cases, we use the notation $\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r,n}$ to stress the dependence on n. However, without causing any misunderstanding, we will also abbreviate it to $\mathcal{H}_{\text{opt}}$.

Interest in ERAPs has arisen originally in statistical physics, under the name of "Euclidean Bipartite Matching Problem", which were introduced by Mézard and Parisi as toy models of "spin-glass" [30,31]. Moreover, the ERAP is the discrete counterpart of the well-known Monge-Kantorovich problem via the identification

$$\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r} \stackrel{d}{=} nW_p^p(\rho_{\mathcal{B}}, \rho_{\mathcal{R}}) \ ,$$

where $\rho_{\mathcal{B}}, \rho_{\mathcal{R}}$ are the empirical measures of $\mathcal{B}$ and $\mathcal{R}$ and $W_p^p(\mu,\nu)$ is the p-th power of the Wasserstein distance W_p among probability measures μ, ν (for the relevant definitions, see *e.g.* [22]). It turns out that establishing the law of large numbers or the (generalized) central limit theorem for the random variable $\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}$ are already highly non-trivial questions and one can focus on the order of the expected value $\mathbb{E}\left[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}\right]$ depending on the choice of disorder, Ω_b-Ω_r and p. However, despite such an elementary formulation, already the study of $\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}$ (or equivalently, $nW_p^p(\rho_{\mathcal{B}}, \rho_{\mathcal{R}})$) turned out to be rather challenging in general, and attracted interest of several communities. Results include convergence in general dimension with uniform disorder as n grows [4,23], asymptotics for $\mathbb{E}\left[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}\right]$ for Ω a one-dimensional interval, the real line or the unit circle [8–10,14,29], also for a possibly non-uniform disorder [6,7,11,18].

Of special interest is also the $2d$ case, which has a long history since a pioneering work of Ajtai-Komlós-Tusnády [1]. Here, it is now known that, if Ω is a $2d$ compact manifold and $p \geqslant 1$,

$$\mathbb{E}\left[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}\right] = c_p n^{1-\frac{p}{2}} (\log n)^{\frac{p}{2}} (1 + o(1)) \, , \quad n \to \infty \ .$$

On the basis of a celebrated linearisation ansatz of the Monge-Ampère equation to the Poisson equation proposed by Caracciolo-Lucibello-Parisi-Sicuro (CLPS) [13] (see also [15]), which was later proven under appropriate assumptions on the domain by Ambrosio-Stra-Trevisan [3] via PDE methods, it is known that $c_2 = \frac{1}{2\pi}$. More recently, there has been an increasing amount of literature on PDE methods for studying this $2d$ problem [2,23,27]. Further consequences of the CLPS ansatz have been investigated in the case Ω a $2d$ manifold [5]. See [17] for further details.

In this paper, we prove a theorem to compare $\mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega_1,\Omega_2}]$ and $\mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega_3,\Omega_4}]$ by using stochastic order of $c(B_1^{(\Omega_1)}, R_1^{(\Omega_2)})$ and $c(B_1^{(\Omega_3)}, R_1^{(\Omega_4)})$, i.e., the cost functions of a single couple (B_1, R_1) without the assignment constraint. We then focus on the comparison of $\mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega,\Omega}]$ vs $\mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega',\Omega'}]$, with $\Omega \neq \Omega'$. More precisely, we discuss a special one-dimensional structure in Section 3 and Theorem 2 in Section 4 for comparing a particular class of problems ($|\Omega| = |\Omega'|$, in dimension $d > 1$).

A motivation of our study comes from recent numerical experiments regarding one dimensional k-star graph (see Section 3.2 for definition and Figure 3.1(β) for a visualisation).

2 Basic definition and main result

Stochastic order is useful for research in many branches of probability theory, such as stochastic processes [34, Section 9], discrete probability distributions [25], and decision theory [33]. In this paper, we only use the basic definition of the stochastic order of random variables. Further properties and applications about stochastic dominances can be found *e.g.* in [35].

Definition 1 (Stochastic order of random variables)**.** Two random variables X and Y are in *stochastic order*, written $X \leq_{\rm st} Y$, if

$$\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t), \quad \forall t \in \mathbb{R}\,.$$

Equivalently, $X \leq_{\rm st} Y \iff F_X(t) \geqslant F_Y(t)$, $\forall t \in \mathbb{R}$, where $F_X(t)$ and $F_Y(t)$ are the cumulative distribution functions.

The next theorem is the main result of this paper, and its proof is reported in Section 5.1.

Theorem 1. Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ be four domains. $\mathcal{B}^{(\Omega_1)}$ are n i.i.d. r.v.s over Ω_1. Analogously, $\mathcal{R}^{(\Omega_2)}$, $\mathcal{B}^{(\Omega_3)}$ and $\mathcal{R}^{(\Omega_4)}$ are respectively n i.i.d. r.v.s over Ω_2, Ω_3, and Ω_4. Consider the cost function defined by

$$\begin{aligned} c\ :\ & \Omega_1 \times \Omega_2 \cup \Omega_3 \times \Omega_4 \to \mathbb{R}^+ \\ & (x, y) \qquad\quad\ \ \mapsto c(x, y), \end{aligned}$$

and let $F_{c(x,y)}(t)$ be its cumulative distribution function. Assume that, if we take arbitrary points $B^{(\Omega_1)}$, $R^{(\Omega_2)}$, $B^{(\Omega_3)}$ and $R^{(\Omega_4)}$, one in each n-sample $\mathcal{B}^{(\Omega_1)}$, $\mathcal{R}^{(\Omega_2)}$, $\mathcal{B}^{(\Omega_3)}$ and $\mathcal{R}^{(\Omega_4)}$, then the following inequality holds

$$c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) \leq_{\rm st} c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)\,. \tag{2}$$

Then

$$\mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega_3,\Omega_4,n}] \leqslant \mathbb{E}[\mathcal{H}_{\rm opt}^{\Omega_1,\Omega_2,n}],\ \forall n \geqslant 1. \tag{3}$$

Remark 1.

(i) The domains $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in the hypothesis are generalised, that means, for instance, the boundaries could be less than regular or the domains could be not bounded, as eq.(2) is a very strong assumption. This theorem holds even if the cost function $c(x, y)$ can be equal to $+\infty$.

(ii) If neither $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)$ nor $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ stochastically dominates the other, then the direction of the inequality in eq.(3) generally depends on n. We illustrate in section 3.2 a one-dimensional example.

(iii) Even if $\Omega_1 = \Omega_3$ and $\Omega_2 = \Omega_4$, it is possible that $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)$ is not equal to $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ by definition of the cost function $c(x, y)$. For instance, one can set $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right) = D^2\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ and $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) = D^4\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$.

(iv) Theorem 1 holds for *any* choice of disorder (as long as $B_i(R_i) \perp\!\!\!\perp B_j(R_j)$, $i \neq j$, $i, j = 1, \ldots, n$).

(v) As in the case of Remark 3 (iii) as an example, if we have equality

$$F_{c\left(B^{(\Omega_3)},R^{(\Omega_4)}\right)}(t) = F_{c\left(B^{(\Omega_1)},R^{(\Omega_2)}\right)}(t),\ \forall t \geqslant 0,$$

then we get equality among *all moments*

$$\mathbb{E}\left[\left(\mathcal{H}_{\rm opt}^{\Omega_1,\Omega_2,n}\right)^m\right] = \mathbb{E}\left[\left(\mathcal{H}_{\rm opt}^{\Omega_3,\Omega_4,n}\right)^m\right],\ \forall n \geqslant 1,\ m \in \mathbb{N}\,.$$

3 Applications of Theorem 1 in dimension $d = 1$

We show in section 3.2 how Theorem 1 can be used to study some structures (see Figure 3.1 as examples) in one dimension. Before the applications, our discussion starts with possibly the simplest case, that is, both domains Ω_b and Ω_r are line segments. In this section, we assume that $\mathcal{B}$ ($\mathcal{R}$) are distributed with uniform disorder unless otherwise stated.

3.1 Ω_b and Ω_r are line segments: Results valid in general

In dimension $d = 1$, one can get a lot more information due to the particularly simple combinatorial properties of the optimal permutation π_{opt}. For example, it is well-known that for an ERAP over an interval, if $p \geq 1$, (strict) convexity of the cost function and optimality imply $\pi_{\text{opt}}(i) = i$, $\forall i = 1, \ldots, n$, if the points are sorted in natural order (see *e.g.* [10,17]). More precisely, in the case $\Omega_r = \Omega_b = \Omega = [0, 1]$, the reformulation of ERAP in terms of generalized Selberg integrals [12] allows to write, for $p \geq 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega,n}] = \frac{\Gamma\left(1+\frac{p}{2}\right)}{(p+1)}\, n\, \frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}, \qquad \forall n \in \mathbb{N}. \tag{4}$$

We can thus address (here in the case $p = 2$ but with minor modifications for generic $p \geq 1$, p even) the evaluation of $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}]$, which we resume in the following.

Proposition 1. Let $\Omega_b = [0, l_b]$ and $\Omega_r = [0, l_r]$, for $l_b,\ l_r > 0$, and let $\Omega = [0, 1]$. Then, at $p = 2$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] = \big((n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r\big)\, \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}],\ \forall n \in \mathbb{N}. \tag{5}$$

Proof of Proposition 1. Let $\rho_b,\ \rho_r$ be the probability density functions for red and blue points respectively, $R_{\rho_b},\ R_{\rho_r}$ the cumulative distribution functions, and $R_{\rho_b}^{-1},\ R_{\rho_r}^{-1}$ their inverse functions (usually called "quantile functions" in Statistics)[1]. More precisely,

$$\rho_b(x) = \frac{1}{l_b}\mathbf{1}_{(0,l_b)}(x) \ \Rightarrow\ R_{\rho_b}(x) = \frac{x}{l_b}\cdot \mathbf{1}_{(0,l_b)}(x) \ \Rightarrow\ R_{\rho_b}^{-1}(u) = l_b u \cdot \mathbf{1}_{(0,1)}(u).$$
$$\rho_r(y) = \frac{1}{l_r}\mathbf{1}_{(0,l_r)}(y) \ \Rightarrow\ R_{\rho_r}(y) = \frac{y}{l_r}\cdot \mathbf{1}_{(0,l_r)}(y) \ \Rightarrow\ R_{\rho_r}^{-1}(v) = l_r v \cdot \mathbf{1}_{(0,1)}(v).$$

After re-labeling the n points in order of distance from an endpoint, the probability of the k-th point being in interval $[u, u + \mathrm{d}u]$, denoted by $P_{n,k}(u)$, is

$$P_{n,k}(u)\mathrm{d}u = \frac{n!}{(k-1)!(n-k)!}u^{k-1}(1-u)^{n-k}\mathrm{d}u.$$

Then, for $p \geqslant 2$ even, we can write

$$\begin{aligned}
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] &= \sum_{k=1}^{n} \int_0^1\!\!\int_0^1 P_{n,k}(u)P_{n,k}(v)\left|R_{\rho_b}^{-1}(u) - R_{\rho_r}^{-1}(v)\right|^p \mathrm{d}v\mathrm{d}u \\
&= \sum_{k=1}^{n}\sum_{q=0}^{p}\binom{p}{q}\int_0^1\!\!\int_0^1 \frac{n!u^{k-1}(1-u)^{n-k}}{(k-1)!(n-k)!}\frac{n!v^{k-1}(1-v)^{n-k}}{(k-1)!(n-k)!}\left(R_{\rho_b}^{-1}(u)\right)^q\left(-R_{\rho_r}^{-1}(v)\right)^{p-q}\mathrm{d}v\mathrm{d}u \\
&= \sum_{k=1}^{n}\left(\frac{n!}{(k-1)!}\right)^2\sum_{q=0}^{p}\binom{p}{q}(l_b)^q(-l_r)^{p-q}\frac{(k-1+q)!}{(n+q)!}\frac{(k-1+p-q)!}{(n+p-q)!}\,.
\end{aligned} \tag{6}$$

At $p = 2$, eq.(6) simplifies to

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] = \frac{n}{3}l_r^2 + \frac{n}{3}l_b^2 - \frac{n(2n+1)}{3(n+1)}l_b l_r\,,$$

which is a homogeneous quadratic polynomial of (l_b, l_r), invariant under $l_b \leftrightarrow l_r$. Upon recalling the standard result at $p = 2$, namely $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] = \frac{1}{3}\frac{n}{n+1}$, eq.(5) follows by simple algebra. $\square$

[1] This proof follows the assumptions and ideas of [17, Lemma 2.6.1]'s proof.

3.2 On the k-star graph

As an application of Theorem 1, we study now the upper and lower bounds of $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ for the ERAP on the k-star graph presented in [28], that is, $\mathcal{B}$ and $\mathcal{R}$ are uniformly distributed over a tree with k edges of length $\frac{1}{k}$ that share a common vertex O', $k \in \mathbb{N}^*$. In this structure, distances are defined by cases, i.e.,

$$D(B_1, R_1) = \begin{cases} |(D(B_1, O') - D(R_1, O')|, & \text{if } B_1 \text{ and } R_1 \text{ are on the same edge,} \\ D(B_1, O') + D(R_1, O'), & \text{otherwise.} \end{cases}$$

Proposition 2. Let Ω^* be 3-star graph and let the cost function be $D^2(B^{(\Omega^*)}, R^{(\Omega^*)})$. Then we have

$$\frac{4n}{27(n+1)} \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*, \Omega^*, n}] \leqslant \frac{n}{3(n+1)}, \ \forall n \geqslant 1.$$

Proof. We use Theorem 1 to prove this. We bound the ERAP on Ω^* above and below using two ERAPs with the same cost function but defined, respectively, on the line segments:

$$\Omega := \Omega_b = \Omega_r = [0, 1], \quad \Omega' := \Omega_b = \Omega_r = \left[0, \frac{2}{3}\right].$$

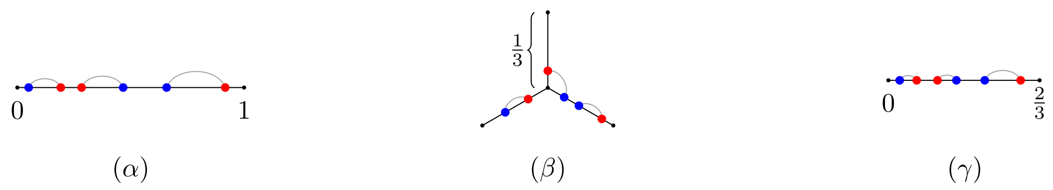

Figure 3.1: (α): $\Omega_b = \Omega_r = [0, 1]$, ($\beta$): Ω_b, Ω_r are 3-star graphs, (γ): $\Omega_b = \Omega_r = [0, \frac{2}{3}]$.

Now we use the results of Section 3.1 for domains of the form $\Omega_b = \Omega_r = [0, l]$ in the case of the 3-star graph. There are 3 choices of 2 arms among 3 available (see Figure 3.2 (a)-(c)). Once this choice is made, the blue $B^{(\Omega)}$ and the red $R^{(\Omega)}$ will fall on some interval of length $\frac{2}{3}$. Notice that case (a), (b) and (c) are not independent: each one carries a probability $\frac{4}{9}$ that that two points are distributed on exactly the bold edges (either on the same edge or not). And with reference to Figure 3.2 (d)-(f), it is clear that each case has a probability of occurring with $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. That is, an edge is specified, and both points are distributed on this edge.

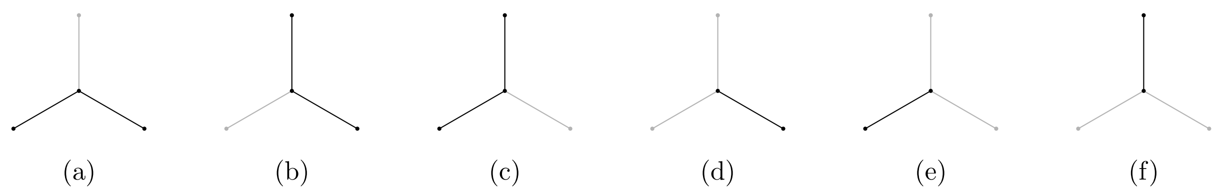

Figure 3.2: 3-star graphs.

Therefore, with reference to Figure 3.1, we can write the cumulative distribution functions as

$$(\alpha): \ \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant t\right) = \begin{cases} 0, \ \forall t < 0 \\ -t^2 + 2t, \ \forall t \in [0, 1] \\ 1, \ \forall t \geqslant 1 \end{cases},$$

$$\begin{aligned} (\beta): \ \mathbb{P}\left(D\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leqslant t\right) &= 3 \times \frac{4}{9}\mathbb{P}\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leqslant t\right) - 3 \times \frac{1}{9}\mathbb{P}\left(D\left(B^{(\overline{\Omega})}, R^{(\overline{\Omega})}\right) \leqslant t\right) \\ &= \frac{4}{3}\mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2}t\right) - \frac{1}{3}\mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant 3t\right), \text{ where } \overline{\Omega} := \left[0, \frac{1}{3}\right], \end{aligned}$$

and

$$(\gamma): \ \mathbb{P}\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2}t\right).$$

Therefore,

$$D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} D\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} D\left(B^{(\Omega)}, R^{(\Omega)}\right),$$

this means

$$c\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} c\left(B^{(\Omega)}, R^{(\Omega)}\right).$$

By applying Theorem 1 and Proposition 1,

$$\frac{4n}{27(n+1)} = \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega',n}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*,n}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega,n}] = \frac{n}{3(n+1)}, \ \forall n \geqslant 1.$$

□

Remark 2. An analogous argument (with minor technical modifications) will give upper and lower bounds for the ERAP on the k-star graph for general exponent $p \geq 1$. The k-star graph also provides an easy situation for illustrating Remark 1-(ii).

Example 1. By reusing the previous notations and by setting $\Omega'' = [0, \frac{2.4}{3}]$, we have

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2.4}t\right).$$

One instance can show that neither $c\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leq_{\text{st}} c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right)$ nor $c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} c\left(B^{(\Omega'')}, R^{(\Omega'')}\right)$ holds:

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant \frac{2}{3}\right) = 1 \leqslant \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{2}{3}\right),$$

while,

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant \frac{1}{4}\right) = \frac{5.4}{10.24} \leqslant \frac{5.12}{10.24} = \frac{1}{2} = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{1}{4}\right).$$

Using Proposition 1 and Proof of Proposition 2, we get for $n = 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] = \frac{5.76}{54}\ ,\ \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}] = 3 \times \frac{4}{9}\mathbb{E}(\mathcal{H}_{\text{opt}}^{[0,2/3],[0,2/3]}) - 3 \times \frac{1}{9}\mathbb{E}(\mathcal{H}_{\text{opt}}^{[0,1/3],[0,1/3]}) = \frac{5}{54}\ ,$$

and

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] = \frac{5.76}{27}\ .$$

The numerical results in [28] have suggested that

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}] = \frac{7}{27}.$$

Therefore,

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] < \lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}], \text{ and for } n=1,\ \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] > \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}].$$

4 Applications of Theorem 1 in general dimension $d \geqslant 2$

Theorem 2. Let Ω be a domain with dimension d, and consider one random blue point and one random red point $B_1 = (B_{1,1}, \ldots, B_{1,d})^T, R_1 = (R_{1,1}, \ldots, R_{1,d})^T \in \Omega$. We suppose that $\forall 1 \leqslant i \leqslant d$, $\forall t \geqslant 0$, $\mathbb{P}(D(B_{1,i}, R_{1,i}) \leqslant t) = \mathbb{P}(D(B_{1,1}, R_{1,1}) \leqslant t)$. Then the 3 following statements hold:

1. For $(\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d \backslash \{0\}$, consider a diagonal matrix $\Lambda \in \mathcal{M}_d(\mathbb{R})$ with determinant ± 1,

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)\ ,$$

which acts as a linear map on B_1 and R_1. Set $B_1' = \Lambda B_1 \in \Omega'$ and $R_1' = \Lambda R_1 \in \Omega'$ for some domain Ω'. Then $\forall p \in \mathbb{R}_+$[2], we prove in Section 5.2 that $D(B_1, R_1) \leq_{\text{st}} D(B_1', R_1')$ (and hence by strict monotonicity $D^p(B_1, R_1) \leq_{\text{st}} D^p(B_1', R_1')$, $\forall p \in \mathbb{R}_+$). Therefore, by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1. \tag{7}$$

2. Let Λ' and Λ'' be two diagonal $d \times d$-matrices with determinant ± 1,

$$\Lambda' = \text{diag}(\lambda_1', \lambda_2', \ldots, \lambda_d')\ , \qquad \Lambda'' = \text{diag}(\lambda_1'', \lambda_2'', \ldots, \lambda_d'')\ ,$$

which transform the points in Ω into the domain Ω' and Ω'', as follows,

$$R_1' = \Lambda' R_1 \in \Omega',\ B_1' = \Lambda' B_1 \in \Omega',\ R_1'' = \Lambda'' R_1 \in \Omega'',\ B_1'' = \Lambda'' B_1 \in \Omega''.$$

If $\text{Tr}(\text{abs}(\Lambda'')) \leqslant \text{Tr}(\text{abs}(\Lambda'))$, where $\text{abs}(M) := (|M_{ij}|)_{1\leq i,j\leq d}$, then $\forall p \in \mathbb{R}_+$, $D^p(B_1'', R_1'') \leq_{\text{st}} D^p(B_1', R_1')$ and by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1. \tag{8}$$

[2] For $p \in \mathbb{R}_-$, $D(B_1', R_1') \leq_{\text{st}} D(B_1, R_1)$, $\forall t \geqslant 0$.

3. Let $M \in \mathrm{GL}_d(\mathbb{R})$. We assume that M transforms the points in Ω into the domain $\overline{\Omega}$, thus

$$MR_1 \in \overline{\Omega},\ MB_1 \in \overline{\Omega}.$$

Then $\forall p > 0$, $D^p\left(B_1, R_1\right) \leq_{\mathrm{st}} D^p\left(MB_1, MR_1\right)$ and by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}],\ \forall n \geqslant 1.$$

We now attempt to make Theorem 2 more visible through an example, and to add some comments on this result. The example is comparing the $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}]$ in the ellipses with different eccentricities and equal areas. Unless otherwise specified, we assume in the rest of Section 4 that $\mathcal{B}$ ($\mathcal{R}$) are uniformly distributed over Ω_b (Ω_r), and that the cost function is $D^2(x,y)$.

Consider the one-parameter family of ellipses $\mathcal{E}_\lambda = \{(x,y) \in \mathbb{R}^2,\ \text{s.t. } (\lambda x)^2 + (\frac{y}{\lambda})^2 \leqslant \frac{1}{\pi},\ \lambda \in (0,1]\}$. Notice that $|\mathcal{E}_\lambda| = 1$, $\forall\lambda$, and that $\mathcal{E}_1 = \mathcal{B}\left((0,0), \frac{1}{\sqrt{\pi}}\right)$ is the unit-area 2-ball. The eccentricity of $\mathcal{E}_\lambda$ is $\sqrt{1-\lambda^4}$. Let $\mathcal{H}_{\mathrm{opt}}^{\lambda}$ be the ground state energy of the ERAP at $p = 2$ on $\mathcal{E}_\lambda$, that is

$$\mathcal{H}_{\mathrm{opt}}^{\lambda} \coloneqq \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^{n} D^2(B_i, R_{\pi(i)}),\ \text{with } B_i, R_j \text{ i.i.d.} \sim \mathcal{U}\left(\mathcal{E}_\lambda\right) \text{rvs},\ 1 \leqslant i,j \leqslant n.$$

Remark 3.

(i) In $\mathcal{E}_1$, we get

$$\forall 1 \leqslant i \leqslant d,\ \forall t \geqslant 0,\ \mathbb{P}\left(D\left(B_{1,i}, R_{1,i}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B_{1,1}, R_{1,1}\right) \leqslant t\right).$$

Then, by Theorem 2-1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\lambda}] \geqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{1}],\ \forall n \geqslant 1, \forall \lambda \in (0,1]. \tag{9}$$

(ii) Theorem 2-2 tells us that, the closer the eccentricity of the boundary is to 0 (when the boundary is a circle), in other words, the closer λ is to 1, the lower $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\lambda}]$.

(iii) In the proof of Theorem 2-3, if $M = UIU^T$, I is an identity matrix and $U \in \mathrm{SO}_d(\mathbb{R})$, then $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] = \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}]$, $\forall n \geqslant 1$. It means $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}]$ is the identical in the two congruent regions.

(iv) Further, we observe that for the second part of Theorem 2, if the absolute value of determinant is $a \in \mathbb{R}_+$ instead of 1, the stochastic order holds, so does eq.(8). Therefore, we can compare any two ellipses that are equal in area, but the area is not necessarily equal to 1.

Let us now consider the case when Ω and Ω' are triangles, $\mathcal{R}^{(\Omega)}$, $\mathcal{B}^{(\Omega)}$ are uniformly distributed on Ω, and $\mathcal{R}^{(\Omega')}$, $\mathcal{B}^{(\Omega')}$ are uniformly distributed on Ω'. We define the Cartesian coordinate system so that two vertices of the triangle are located at the same position on the positive half-axis of the horizontal and vertical axes, respectively. This means, their coordinates are $(\alpha, 0)$ and $(0, \alpha)$, with $\alpha \in \mathbb{R}_+$. When the third vertex of the triangle is on the line $y = x$, we get

$$\forall t \geqslant 0, \mathbb{P}\left(D\left(B_{i,x}, R_{i,x}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B_{i,y}, R_{i,y}\right) \leqslant t\right). \tag{10}$$

In this case, Ω is an isosceles triangle (see Figure 4.1 for examples). We do the transformation $\Lambda = \mathrm{diag}(\lambda, 1/\lambda)$, $\lambda \in \mathbb{R}_+$ on $\mathcal{R}^{(\Omega)}$, $\mathcal{B}^{(\Omega)}$ to get points $\mathcal{R}^{(\Omega')}$, $\mathcal{B}^{(\Omega')}$ (examples of Ω' with $\lambda = \frac{3}{4}, \frac{4}{3}$ are shown in Figure 4.1). Therefore, by using Theorem 2,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1.$$

We then discuss this according to the shape of the triangle:

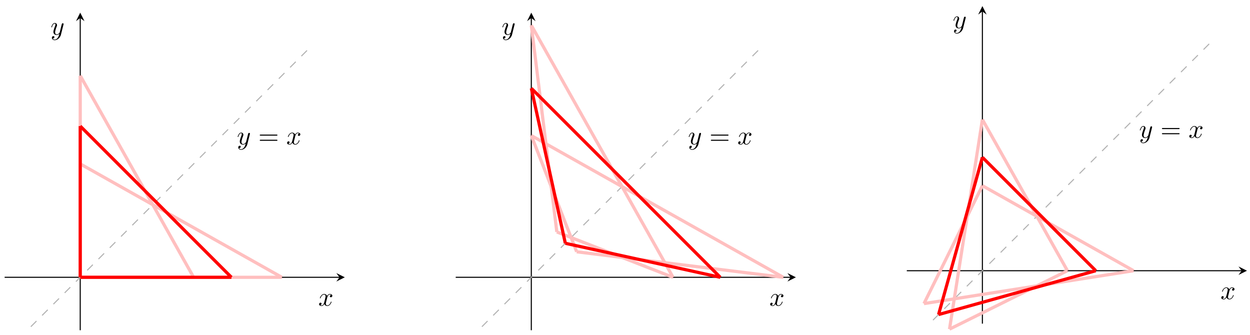

Figure 4.1: Examples of Ω (with red boundary) as right obtuse, or acute triangle. And examples of Ω' (with pink boundary)

1. If Ω is a right triangle, so is Ω'. Ω' is an isosceles triangle only if $\lambda = 1$;
2. if Ω is a obtuse triangle, so is Ω'. Ω' is an isosceles triangle only if $\lambda = 1$;
3. if Ω is an acute triangle, Ω' might be acute, right or obtuse. And except for the case when Ω is an equilateral triangle, there is only one $\lambda > 0$ such that Ω' is an isosceles triangle. The isosceles triangle Ω', which might be acute, right or obtuse (see Figure 4.2 for examples), and Ω are not congruent, since the base of Ω gets longer and becomes one leg of Ω'.

In summary, every triangle Ω_1 (including the isosceles triangle) can be obtained by picking a suitable Cartesian coordinate system then transforming from an isosceles triangle Ω_2 with $\lambda > 1$, except for the equilateral triangle. Theorem 2-1 tells us that this transformation leads to a reduction in energy. That is, for every triangle Ω_1 which is not an equilateral triangle, we can find an acute isosceles triangle Ω_2 such that

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_2,\Omega_2}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_1}],\ \forall n \geqslant 1.$$

We can therefore conclude the following.

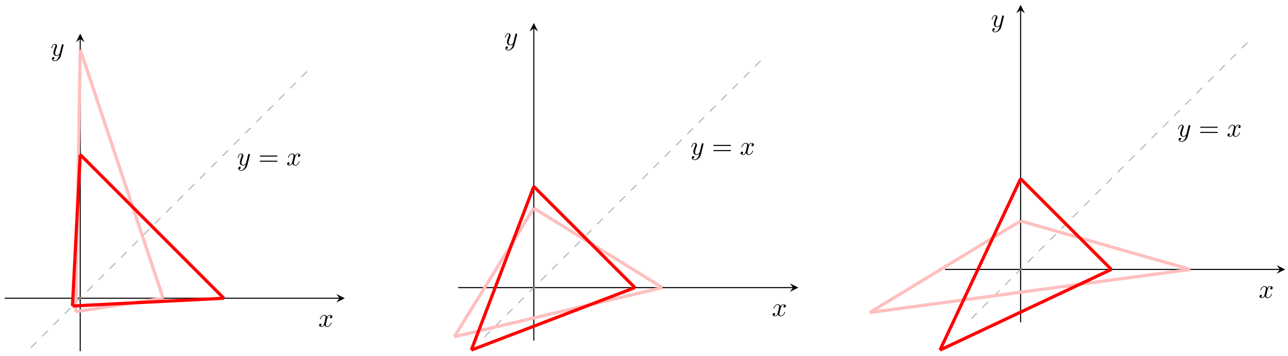

Figure 4.2: Examples of Ω (with red boundary) as acute triangles and Ω' as acute, right or obtuse triangles (with pink boundary)

Proposition 3. If Ω is an equilateral triangle and Ω' is a triangle with the same area, we always have

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1. \tag{11}$$

To verify Proposition 3, we only need to compare $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ over isosceles triangles for $n = 1$ (see Lemma 1).

Lemma 1. Let the family of isosceles triangles be

$$\Lambda\Omega = \Omega^{(\lambda)} = \left\{(x, y) \in \mathbb{R}^2 \text{ s.t. } \frac{1}{\lambda^2}(x - \lambda) \le y \le \frac{1}{\lambda^2}(\lambda - x),\ x \in [0, \lambda],\ \lambda > 0\right\}.$$

Then, we have, for $n = 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda}] \ge \mathbb{E}[\mathcal{H}_{\text{opt}}^{3^{1/4}}]\ . \tag{12}$$

Proof. Remark that the uniform distribution over $\Omega^{(1)}$ has the following probability density function

$$\rho_{X,Y}(x, y) = \mathbf{1}_{\Omega}(x, y)$$

and hence the following marginals

$$\rho_X(x) = 2(1 - x)\mathbf{1}_{[0,1]}(x), \qquad \rho_Y(y) = (1 + y)\mathbf{1}_{[-1,0]}(y) + (1 - y)\mathbf{1}_{[0,1]}(y). \tag{13}$$

For general λ, the marginals read

$$\rho_X^{(\lambda)}(x) = \frac{2}{\lambda^2}(\lambda - x)\mathbf{1}_{[0,\lambda]}(x), \qquad \rho_Y^{(\lambda)}(y) = \lambda(1 + \lambda y)\mathbf{1}_{[-\frac{1}{\lambda},0]}(y) + \lambda(1 - \lambda y)\mathbf{1}_{[0,\frac{1}{\lambda}]}(y)\ , \tag{14}$$

which recover eq.(13) at $\lambda = 1$. Now, for $s \in \mathbb{N}$, the moments write

$$\begin{aligned} \mathbb{E}\left[X^s\right] &= 2\frac{\lambda^s}{(s+2)(s+1)},\ s \in \mathbb{N}, \\ \mathbb{E}\left[Y^s\right] &= \begin{cases} 2\frac{\left(\frac{1}{\lambda}\right)^s}{(s+2)(s+1)} & s \text{ even} \\ 0 & s \text{ odd}\ . \end{cases} \end{aligned} \tag{15}$$

As $B_x, R_x \sim X$; $B_y, R_y \sim Y$ and $B \perp\!\!\!\perp R$,

$$\begin{aligned}\mathbb{E}[\mathcal{H}^{\lambda}_{\text{opt}}] =&\ \mathbb{E}\left[(B_x-R_x)^2+(B_y-R_y)^2\right]=\mathbb{E}\left[(B_x-R_x)^2\right]+\mathbb{E}\left[(B_y-R_y)^2\right]\\ =&\ 2\mathbb{E}\left[X^2\right]-2\mathbb{E}\left[X\right]^2+2\mathbb{E}\left[Y^2\right]-2\mathbb{E}\left[Y\right]^2=2\left(\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2+\mathbb{E}\left[Y^2\right]\right)\\ =&\ 2\left(\frac{\lambda^2}{6}-\left(\frac{\lambda}{3}\right)^2+\frac{\frac{1}{\lambda^2}}{6}\right)=\frac{\lambda^2}{9}+\frac{1}{3\lambda^2}\,.\end{aligned} \tag{16}$$

Then, when $\lambda = 3^{1/4}$, eq.(16) reaches the minimum. Thus, eq.(12) is proven, and $\Omega^{(3^{1/4})}$ is an equilateral triangle. $\square$

Remark 4. Similarly, we can extend Proposition 3 to the n-sided convex polygons, i.e., if Ω is an n-sided regular convex polygon, and Ω' is an n-sided convex polygon with the same area, then the eq.(11) holds.

5 Proofs

5.1 Proof of Theorem 1

Let $\{X_i, 1 \leqslant i \leqslant m\}$ be m r.v.s, and $\{Y_i, 1 \leqslant i \leqslant m\}$ be m r.v.s such that

$$F_{X_i}(t) \geqslant F_{Y_i}(t),\ \forall t \geqslant 0,\ \forall 1 \leqslant i \leqslant m. \tag{17}$$

We will discuss some findings about $\{X_i\}$ and $\{Y_i\}$ in **Steps 1** and **2**. Then applying these findings, by substitution in **Steps 3** and **4**, we will get the stochastic order of $\min\limits_{\pi\in S_n}\left\{\sum\limits_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right\}$ and $\min\limits_{\pi\in S_n}\left\{\sum\limits_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right\}$. In **Step 5** we will finish the proof.

Step 1. We now prove that if $\{X_i\}$ are i.i.d. r.v.s, $\{Y_i\}$ as well, then $\forall 1 \leqslant k \leqslant m$,

$$F_{\sum_{j=1}^k X_j}(t) \geqslant F_{\sum_{j=1}^k Y_j}(t),\ \forall t \geqslant 0. \tag{18}$$

In fact, $\forall t \geqslant 0$,

$$F_{X_i}(t) \geqslant F_{Y_i}(t)\ \Rightarrow\ \mathbb{P}(X_i \leqslant t) \geqslant \mathbb{P}(Y_i \leqslant t). \tag{19}$$

By independence, we have

$$F_{\sum_{j=1}^k X_j}(t) = \mathbb{P}\left(\sum_{j=1}^{k} X_j \leqslant t\right) = \prod_{j=1}^{k}\mathbb{P}\left(X_j \leqslant t - \sum_{i=1}^{j-1} X_i\right),\ \forall 1 \leqslant k \leqslant m. \tag{20}$$

An analogous expression holds for the cdf of $\sum_{j=1}^k Y_j$'s. Therefore, we apply eq.(19) to the right-hand side of eq.(20) to get eq.(18).

Step 2. Now we prove the following inequality by contradiction,

$$F_{\min_{1\leqslant j\leqslant m}\{X_j\}}(t) \geqslant F_{\min_{1\leqslant j\leqslant m}\{Y_j\}}(t),\ \forall t \geqslant 0. \tag{21}$$

If there exist $t \geqslant 0$ and $1 \leqslant i_1, i_2 \leqslant m$ such that

$$F_{\min_{1\leqslant j\leqslant m}\{X_j\}}(t) = F_{X_{i_1}}(t),\ F_{\min_{1\leqslant j\leqslant m}\{Y_j\}}(t) = F_{Y_{i_2}}(t),\ \text{and}\ F_{X_{i_1}}(t) < F_{Y_{i_2}}(t),$$

then

$$F_{X_{i_1}}(t) < F_{Y_{i_2}}(t) \leqslant F_{Y_{i_1}}(t),$$

which reaches a contradiction.

Step 3. Due to the i.i.d. assumption,

$$F_{c\left(B_i^{(\Omega_1)}, R_j^{(\Omega_2)}\right)}(t) = F_{c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)}(t),\ \forall 1 \leqslant i, j \leqslant n,\ \forall t \geqslant 0.$$

It means, $\forall 0 \leqslant i, j \leqslant n$, $c\left(B_i^{(\Omega_1)}, R_j^{(\Omega_2)}\right)$ follow the identical distributions. We remark that, for every $\pi, \pi' \in S_n$,

$$\left\{c\left(B_i^{(\Omega_1)}, R_{\pi(j)}^{(\Omega_2)}\right),\ \forall 1 \leqslant i \leqslant n\right\}\ \text{and}\ \left\{c\left(B_i^{(\Omega_3)}, R_{\pi'(j)}^{(\Omega_4)}\right),\ \forall 1 \leqslant i \leqslant n\right\}$$

are sequences of n i.i.d. r.v.s. By the assumption (eq.(2)), $\forall 1 \leqslant i, j \leqslant n$,

$$F_{c\left(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}\right)}(t) \geqslant F_{c\left(B_j^{(\Omega_1)}, R_{\pi'(j)}^{(\Omega_2)}\right)}(t),\ \forall t \geqslant 0\ ,$$

and we bring the following substitutions into eq.(17) and eq.(18) of **Step 1**

$$m \longrightarrow n, \quad k \longrightarrow n, \quad X_j \longrightarrow c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \quad Y_j \longrightarrow c(B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)}),$$

then,

$$F_{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t) \geqslant F_{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)})}(t),\ \forall t \geqslant 0. \tag{22}$$

Step 4. We remark that

$$\left\{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}),\ \forall \pi \in S_n\right\}$$

are $n!$ identically distributed r.v.s because $\forall 0 \leqslant i, j \leqslant n,\ c\left(B_i^{(\Omega_1)}, R_j^{(\Omega_2)}\right)$ follow the identical distributions. By the same reason,

$$\left\{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}),\ \forall \pi \in S_n\right\}$$

are $n!$ identically distributed r.v.s. Using eq.(22), and we bring the following substitutions into eq.(17) and eq.(21) of **Step 2**

$$m \longrightarrow n!, \quad \{X_j\} \longrightarrow \left\{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \pi \in S_n\right\}, \quad \{Y_j\} \longrightarrow \left\{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \pi \in S_n\right\},$$

then,

$$F_{\min\left\{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \pi \in S_n\right\}}(t) \geqslant F_{\min\left\{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \forall \pi \in S_n\right\}}(t),\ \forall t \geqslant 0. \tag{23}$$

Step 5. Since $\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(x,y)$ is non-negative, $\mathbb{E}\left[\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(x,y)\right] = \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(x,y)}(t)\right) \mathrm{d}t$. Thus,

$$\begin{aligned}
&\mathbb{E}\left[\min_{\pi \in S_n} \sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right] - \mathbb{E}\left[\min_{\pi \in S_n} \sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right] \\
&= \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right) \mathrm{d}t - \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t)\right) \mathrm{d}t \\
&= \int_0^{+\infty}\left(F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t) - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right) \mathrm{d}t \geqslant 0\ .
\end{aligned}$$

We conclude that

$$\mathbb{E}\left[\min_{\pi \in S_n} \sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right] \geqslant \mathbb{E}\left[\min_{\pi \in S_n} \sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right],\ \forall n \geqslant 1. \tag{24}$$

We notice that if

$$\int_0^{+\infty}\left(1 - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right) \mathrm{d}t = +\infty,\ \text{or},\quad \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi \in S_n} \sum\limits_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t)\right) \mathrm{d}t = +\infty,$$

eq.(24) holds naturally.

5.2 Proof of Theorem 2

1. By assumption, $\forall 1 \leqslant i \leqslant d,\ \forall t \geqslant 0,\ \mathbb{P}\left(D^p\left(B_{1,i}, R_{1,i}\right) = t\right) = \mathbb{P}\left(D^p\left(B_{1,1}, R_{1,1}\right) = t\right)$. Thus,

$$\begin{aligned}
&\mathbb{P}\left(|\lambda_1|^p|B_{1,1} - R_{1,1}|^p + |\lambda_2|^p|B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p|B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= &\ \mathbb{P}\left(|\lambda_1|^p|B_{1,2} - R_{1,2}|^p + |\lambda_2|^p|B_{1,3} - R_{1,3}|^p + \cdots + |\lambda_d|^p|B_{1,1} - R_{1,1}|^p \leqslant t\right) \\
= &\ \mathbb{P}\left(|\lambda_1|^p|B_{1,d} - R_{1,d}|^p + |\lambda_2|^p|B_{1,1} - R_{1,1}|^p + \cdots + |\lambda_d|^p|B_{1,d-1} - R_{1,d-1}|^p \leqslant t\right)\ .
\end{aligned}$$

We remark that, if for every $t \leqslant 0$, $\mathbb{P}(X \leqslant t) = \mathbb{P}(Y \leqslant t)$ then $\mathbb{P}(X \leqslant t) = \mathbb{P}(X + Y \leqslant 2t)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(|\lambda_1|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(\sum_{i=1}^{d} |\lambda_i|^p |B_{1,1} - R_{1,1}|^p + \sum_{i=1}^{d} |\lambda_i|^p |B_{1,2} - R_{1,2}|^p + \cdots + \sum_{i=1}^{d} |\lambda_i|^p |B_{1,d} - R_{1,d}|^p \leqslant d \times t\right) \\
= \ &\mathbb{P}\left(\left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \times |B_{1,1} - R_{1,1}|^p + \cdots + \left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \times |B_{1,d} - R_{1,d}|^p \leqslant t\right).
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(\left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \times |B_{1,1} - R_{1,1}|^p + \cdots + \left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \times |B_{1,d} - R_{1,d}|^p \leqslant \left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \times t\right).
\end{aligned}$$

As $\left|\prod_{i=1}^{d} \lambda_i\right| = 1$, by the inequality of arithmetic and geometric means, we get

$$\left(\sum_{i=1}^{d} |\lambda_i|^p\right) / d \geqslant \left(\prod_{i=1}^{d} |\lambda_i|^p\right)^{1/d} = \left(\prod_{i=1}^{d} |\lambda_i|\right)^{p/d} = 1.$$

In conclusion,

$$\begin{aligned}
F_{D^p\left(B_1', R_1'\right)}(t) = \ &\mathbb{P}\left(D^p\left(B_1', R_1'\right) \leqslant t\right) = \mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(\Lambda B_1 - \Lambda R_1))^p\right) \leqslant t\right) \\
= \ &\mathbb{P}\left(|\lambda_1|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
\leqslant \ &\mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(B_1 - R_1))^p\right) \leqslant t\right) = \mathbb{P}\left(D^p\left(B_1, R_1\right) \leqslant t\right) = F_{D^p(B_1, R_1)}(t), \ \forall t \geqslant 0,
\end{aligned}$$

i.e.,

$$D^p\left(B_1, R_1\right) \leq_{\mathrm{st}} D^p\left(B_1', R_1'\right), \ \forall p \in \mathbb{R}_+,$$

and hence, by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega',\Omega'}], \ \forall n \geqslant 1.$$

2. We use monotonicity of trace functions (see *e.g.* [16, Section 2.2]) w.r.t. the function $f(t) = t^p$ (which is monotone increasing $\forall p > 0$): For any Hermitian matrix A, the function $A \to \operatorname{Tr}[A^p]$ is monotone increasing on $\mathbb{R}_+$. In our case,

$$\operatorname{Tr}\left(\operatorname{abs}(\Lambda'')\right) \leqslant \operatorname{Tr}\left(\operatorname{abs}(\Lambda')\right) \implies \operatorname{Tr}\left((\operatorname{abs}(\Lambda''))^p\right) \leqslant \operatorname{Tr}\left((\operatorname{abs}(\Lambda'))^p\right) \iff \sum_{i=1}^{d} |\lambda_i''|^p \leqslant \sum_{i=1}^{d} |\lambda_i'|^p.$$

Then, as above, by the assumption, $\forall 1 \leqslant i \leqslant d$, $\forall t \geqslant 0$, $\mathbb{P}\left(D^p\left(B_{1,i}, R_{1,i}\right) = t\right) = \mathbb{P}\left(D^p\left(B_{1,1}, R_{1,1}\right) = t\right)$, we have

$$\begin{aligned}
F_{D^p\left(B_1'', R_1''\right)}(t) = \ &\mathbb{P}\left(D^p\left(B_1'', R_1''\right) \leqslant t\right) = \mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(\Lambda'' B_1 - \Lambda'' R_1))^p\right) \leqslant t\right) \\
= \ &\mathbb{P}\left(|\lambda_1''|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2''|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d''|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(\left(\sum_{i=1}^{d} |\lambda_i''|^p\right) / d \times |B_{1,1} - R_{1,1}|^p + \cdots + \left(\sum_{i=1}^{d} |\lambda_i''|^p\right) / d \times |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant d / \left(\sum_{i=1}^{d} |\lambda_i''|^p\right) \times t\right) \\
\geqslant \ &\mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant d / \left(\sum_{i=1}^{d} |\lambda_i'|^p\right) \times t\right), \text{ as } \sum_{i=1}^{d} |\lambda_i''|^p \leqslant \sum_{i=1}^{d} |\lambda_i'|^p \\
= \ &\mathbb{P}\left(\left(\sum_{i=1}^{d} |\lambda_i'|^p\right) / d \times |B_{1,1} - R_{1,1}|^p + \cdots + \left(\sum_{i=1}^{d} |\lambda_i'|^p\right) / d \times |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(|\lambda_1'|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2'|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d'|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ &\mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(\Lambda' B_1 - \Lambda' R_1))^p\right) \leqslant t\right) = \mathbb{P}\left(D^p\left(B_1', R_1'\right) \leqslant t\right) = F_{D^p\left(B_1', R_1'\right)}(t), \ \forall t \geqslant 0.
\end{aligned}$$

Therefore, $D^p\left(B_1'', R_1''\right) \leq_{\mathrm{st}} D^p\left(B_1', R_1'\right)$ and using again Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega'',\Omega''}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega',\Omega'}], \ \forall n \geqslant 1.$$

3. Since M is real symmetric, it is diagonalizable by rotation matrices: $M = U\Lambda U^T$, where
   (a) Λ is a diagonal $d \times d$ matrix, and $|\det(\Lambda)| = 1$, since $|\det(M)| = 1$;
   (b) $U \in \mathrm{SO}_d(\mathbb{R})$.

   Clearly
$$D\left(U\Lambda U^T B_1, U\Lambda U^T R_1\right) = D\left(\Lambda U^T B_1, \Lambda U^T R_1\right), \qquad D\left(U^T B_1, U^T R_1\right) = D\left(B_1, R_1\right),$$
   so that, $\forall p$, we have
$$F_{D^p(U\Lambda U^T B_1, U\Lambda U^T R_1)}(t) = F_{D^p(\Lambda U^T B_1, \Lambda U^T R_1)}(t), \qquad F_{D^p(U^T B_1, U^T R_1)}(t) = F_{D^p(B_1,R_1)}(t),\ \forall t \geqslant 0.$$
   According to Theorem 2-1, $\forall p > 0$,
$$F_{D^p(\Lambda U^T B_1, \Lambda U^T R_1)}(t) \leqslant F_{D^p(U^T B_1, U^T R_1)}(t),\ \forall t \geqslant 0.$$
   In summary, $\forall p > 0$,
$$F_{D^p(MB_1, MR_1)}(t) = F_{D^p(U\Lambda U^T B_1, U\Lambda U^T R_1)}(t) \leqslant F_{D^p(B_1,R_1)}(t),\ \forall t \geqslant 0,$$
   i.e. $D^p(B_1, R_1) \leq_{\mathrm{st}} D^p(MB_1, MR_1)$. Therefore, after using Theorem 1,
$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}],\ \forall n \geqslant 1.$$

6 Some research perspectives

A number of limitations needs to be noted regarding Theorem 1. First, stochastic order does not appear to be a necessary condition for getting eq.(3). We believe that further study of this condition may help elucidate comparison of ERAPs in a general setting. Second, it would be interesting to study simpler sufficient conditions, as calculating the cumulative distribution function of $c(B_1, R_1)$ might be challenging (analytically), particularly for the higher dimensional problems.

In spite of its limitations, Theorem 1 offers valuable insights into studying some difficult cases. In terms of future work, we are also considering several applications. First of all, it should be possible to find upper and lower bounds for $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}]$ on the k-star graphs that are closer to the exact solution using a similar method (with some refinement) to Proposition 2. Furthermore, setting $\Omega \neq \Omega'$, the comparison of $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}]$ and $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega'}]$ could be made using Theorem 1. Finally, Theorem 1 may be a useful tool for studying ERAPs with glued domains in higher dimensions by performing projections.

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