

# EXERCISE SESSION 1\*

for the lecture “The Assignment Problem from von Neumann to the Hungarian algorithm”

Nesin Mathematics Village, Turkey, 28/07/2024–4/08/2024

Matteo D’ACHILLE<sup>†</sup>

*Nota bene:* The exercises are ranked in order of increasing difficulty, from direct application of classroom material (\*) to research-level problems (\*\*\*\*\*).

**Exercise 1 (\*)** Using the Hungarian algorithm, find an optimal assignment and the optimal cost for the following cost matrices:

(a)

$$c_1 = \begin{pmatrix} 3 & 5 & 2 & 3 \\ 7 & 1 & 2 & 9 \\ 2 & 4 & 4 & 1 \\ 3 & 5 & 2 & 7 \end{pmatrix}$$

(b)

$$c_2 = \begin{pmatrix} -2 & 1 & 4 & 2 \\ 1 & 4 & 1 & -2 \\ 3 & -2 & 0 & 1 \\ 5 & 2 & 1 & 4 \end{pmatrix}$$

(c)

$$c_3 = \begin{pmatrix} 6 & 1 & 4 & 1 & 8 \\ 4 & 5 & 9 & 4 & 3 \\ 6 & 1 & 5 & 6 & 4 \\ 6 & 2 & 3 & 1 & 4 \\ 1 & 6 & 5 & 8 & 0 \end{pmatrix}$$

(d)

$$c_4 = \begin{pmatrix} 7 & 5 & 8 & 10 & 2 \\ 11 & 6 & 13 & 11 & 6 \\ 15 & 8 & 7 & 8 & 7 \\ 9 & 6 & 10 & 17 & 6 \\ 7 & 5 & 7 & 8 & 2 \end{pmatrix}$$

---

\*Available electronically at: <https://matteodachille.github.io/teaching>

<sup>†</sup>[matteo.dachille@universite-paris-saclay.fr](mailto:matteo.dachille@universite-paris-saclay.fr), solutions welcome!

**Exercise 2 (\*\*)** Let  $(x_i)_{i=1}^n$  be a strictly increasing real sequence and  $(y_i)_{i=1}^n$  a strictly decreasing real sequence. Consider the assignment problem for the following cost matrix:

$$c_{ij} \stackrel{\text{def}}{=} x_i y_j, \quad i, j = 1, \dots, n.$$

📖 Prove that  $\pi_{\text{opt}} = (1 \dots n)$ , and hence  $c_{\text{opt}} = \text{Tr } c$ .

– *Hint:* reason either by induction or by contradiction.

**Exercise 3 (\*\*\*\*)** (VON NEUMANN'S GAME) The exercise consists of two independent parts.

**3.1** Let  $(\alpha_i)_{i=1}^n$  be a sequence of positive reals and let  $p = (p_i)_{i=1}^n$  be a probability distr. over  $[n]$ .

📖 The goal is to prove that a probability distr. over  $[n]$  such that

$$p^* = \arg \min_{p: p \text{ is a proba over } [n]} \max_{i \in [n]} (\alpha_i p_i)$$

is given by

$$p_i^* = \frac{\alpha_i}{\text{HM}(\alpha_1, \dots, \alpha_n)}, \quad i = 1, \dots, n,$$

where  $\text{HM}(\alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \sum_{i=1}^n \frac{1}{\alpha_i}$  is the *harmonic mean* of  $\alpha_1, \dots, \alpha_n$ .

We will do it in two steps:

a) Prove that a  $p^*$  must satisfy  $\alpha_1 p_1 = \dots = \alpha_n p_n = \Lambda$ , where  $\Lambda > 0$  is strictly positive.

– *Hint:* reason by contradiction.

b) Solve for  $p$  and  $\Lambda$ , and then comment.

**3.2** Let  $(\alpha_{i,j})_{i,j=1}^n$  be  $n^2$  positive reals and let  $\mathbf{B}_n$  be the Birkhoff polytope of  $n \times n$  doubly stochastic matrices. The goal is to find a matrix  $X^* \in \mathbf{B}_n$  which satisfies the following:

$$X^* = \arg \min_{X \in \mathbf{B}_n} \max_{\substack{r \in [n] \\ s \in [n]}} \left( \sum_{j=1}^n \alpha_{rj} X_{rj}, \sum_{i=1}^n \alpha_{is} X_{is} \right).$$

We will do it in four steps:

a) Let  $\Lambda > 0$  be the maximum value attained in the previous display (why can we do this?).

Prove that

$$\zeta_{ij} \stackrel{\text{def}}{=} \frac{\alpha_{ij} X_{ij}^*}{\Lambda} \in \mathbf{B}_n.$$

b) Prove that  $\zeta$  is a permutation matrix, for some permutation  $\pi_{\text{opt}}$ .

– *Hint:* use the Birkhoff–von Neumann theorem and the optimality requirement.

c) Deduce that  $\sum_{j=1}^n \alpha_{ij} X_{ij}^* = \Lambda = \sum_{i=1}^n \alpha_{ij} X_{ij}^*$ , and hence

$$\Lambda = \frac{1}{\text{HM}(\alpha_{1\pi_{\text{opt}}(1)}, \dots, \alpha_{n\pi_{\text{opt}}(n)})}.$$

d) Provide a cost matrix  $c$  depending on  $(\alpha_{i,j})_{i,j=1}^n$  whose optimal assignment is  $\pi_{\text{opt}}$ .