Almost Gibbsian Measures on a Cayley Tree

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Abstract:

We consider the ferromagnetic $n.n$ Ising model on Cayley trees in absence of external fields submitted to a modified majority rule transformation with overlapping cells already known to lead to non-Gibbsian measures. We describe the renormalized measures within the Generalized Gibbs framework and prove that they are Almost Gibbs at any temperature.

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1 Introduction : models and transformations

In Mathematical Statistical Mechanics, *Gibbs Measures* have been rigorously designed to represent equilibrium states and to model Phase Transitions, following the pioneering works of Dobrushin, Lanford and Ruelle who described them as an extension of Markov chains, both dynamically and spatially, in terms of the specification of their conditional probabilities w.r.t. outside of finite sets [9, 35]. This *DLR approach* has been fully rigorized in the eighties by Georgii [21], as also described in [11].

In the mean time, some pathologies of the formal definition of Gibbs measures arose within numerical studies[1] of Critical Phenomena [22, 31], identified afterwards to be manifestation of possible non-Gibbsianness of the renormalized measures. This non-Gibbsianness has been since then mainly coined by the exhibition of bad configurations which are points of essential discontinuity of the renormalized measures. This leads to a *Dobrushin program of restoration of Gibbsianness*, launched by Dobrushin himself in a talk in Renkum in 1995 [10]. Within this program, two main restoration notions have been proposed, *Weak Gibbsianness vs. Almost Gibbs*, the latter being stronger than the former, see [42] or [33] where one can learn that the Weakly Gibbsian representation is indeed too weak. On lattices, most initial efforts have been adressed to the decimation transformation, leading to an almost Gibbsian description at any temperature in a series of papers [11, 14, 37], while positive results concerning preservation of Gibbsianness near the critical point and in uniqueness regions have been proposed by *e.g.* Bertini *et al.* [2], Kennedy *et al.* [28, 32] or Martinelli *et al.* [43].

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[1] Remarkably, the presence of such pathologies is detectable through numerical studies *via* a discontinuity of the (restricted) RG map, see Wilson [52] and more recently *e.g.* Salas [49].

As it is usually difficult to evaluate the measure of the set of discontinuity points (*bad configurations*), the approach leading to almost Gibbsianness on $\mathbb{Z}^d$ heavily relies on a specification-dependent variational principle (*Zero Relative Entropy* characterization) which is known to fail on trees [17]. Due to this difficulty, while considering trees, most efforts have been addressed to the detection of non Gibbsian measures, by renormalization [26] or *via* stochastic evolution (van Enter *et al.*, see *e.g.* [12]), with the notable exception of the *failure* of almost quasilocality for the random-cluster measures proved in [25]. While decimation is known to preserve Gibbs property on trees, we focus here on a majority rule transformation already known to cause failure of Gibbsianness, and prove that it is *Almost Gibbsian at any temperature* using elementary arguments such as transfer matrices, Markov chains, dependent percolation and moment estimates, and profiting of the recursivity inherent to treatments on trees.

2 Gibbsian and non-Gibbsian Measures on Trees

In this paper we consider *Ising spins* in which the single-spin space is $E = \{-1, 1\}$. For a given lattice S, the configuration space is

$$\Omega = E^S \ , \ \mathcal{E} = \mathcal{P}(E) \ , \ \rho_0 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1} \ ,$$

where δ_i is the Dirac measure on $i \in E$. For integers $k > 0$, the lattices we consider are Cayley Trees $S = \mathcal{T}^k$, that is, $(k+1)$-regular infinite trees (see [48] for further details). We focus here on the case $k = 2$ (binary trees).

Basic notations and topology :
We denote by $\mathcal{S}$ the set of all the finite subsets of $\mathcal{T}^k$. Moreover $\forall \Lambda \subset S$, we note $\Omega_\Lambda = E^\Lambda$ and ω_Λ the canonical projection of ω on Ω_Λ (sometimes called "restrictions" [44, 45]). For all $\Lambda \subset S$, $\mathcal{F}_\Lambda$ is the σ-algebra generated by the functions ($\omega \mapsto \omega_i$ for $i \in \Lambda$). $\forall \Lambda, \Lambda' \subset S$ such that $\Lambda \cap \Lambda' \neq \emptyset, \forall \omega, \sigma \in \Omega$, we shall denote by $\omega_\Lambda \sigma_{\Lambda'}$ the element of $\Omega_{\Lambda \cup \Lambda'}$ which agrees with ω in Λ and with σ in Λ'. For any set $\Lambda \in \mathcal{S}$, $|\Lambda|$ will denote the cardinality of Λ.

Ising Potential (*n.n.*) : we consider the family $\Phi = (\Phi_A)_{A \in \mathcal{S}}$, where

$$\Phi_A(\omega) = \begin{cases} -J(i,j)\ \omega_i \omega_j & A = \{i, j\} \\ -h(i)\ \omega_i & A = \{i\} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here, $J : \mathcal{T}^k \times \mathcal{T}^k \longrightarrow \mathbb{R}$ is the coupling function and $h : \mathcal{T}^k \longrightarrow \mathbb{R}$ is often called an external magnetic field. The Ising potential (1) is an example of uniformly absolutely convergent potential (or "UAC" in short).

Gibbs specifications : For Φ a UAC potential, and for all configurations $\sigma \in \Omega$, we call $\mathcal{H}_\Lambda^{\Phi,\omega}(\sigma)$ the finite-volume Hamiltonian with boundary condition ω, defined by

$$\mathcal{H}_\Lambda^{\Phi,\omega}(\sigma) = \mathcal{H}_\Lambda^{\Phi,\omega}(\sigma \mid \omega) := \mathcal{H}_\Lambda^{\Phi,\omega}(\sigma_\Lambda \omega_{\Lambda^c}) = \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \omega_{\Lambda^c}). \tag{2}$$

Associated to (2), there are *Boltzmann-Gibbs weights* $e^{-\beta \mathcal{H}_\Lambda^{\Phi,\omega}(\sigma)}$ and *partition functions at inverse temperature* $\beta > 0$ *with boundary condition* ω,

$$Z_\Lambda^{\beta\Phi} = \int_{\Omega_\Lambda} e^{-\beta \mathcal{H}_\Lambda^{\Phi,\omega}(\sigma|\omega)} \rho_\Lambda(d\sigma_\Lambda), \tag{3}$$

where $\rho_\Lambda := \rho_0^{\otimes\Lambda}$ is the *a priori* product measure. For Φ any UAC potential, the set of probability kernels $\gamma^{\beta\Phi} = (\gamma_\Lambda^{\beta\Phi})_{\Lambda\in\mathcal{S}}$, defined for all $\Lambda \in \mathcal{S}$ and $\sigma, \omega \in \Omega$, as

$$\gamma_\Lambda^{\beta\Phi}(\sigma \mid \omega) = \frac{1}{Z_\Lambda^{\beta\Phi}} e^{-\beta\mathcal{H}_\Lambda^{\Phi,\omega}(\sigma|\omega)}, \tag{4}$$

is called a *Gibbs specification* for potential Φ at inverse temperature β. More generally, a specification is a family of probability kernels satisfying extra properties (properness and consistency) so that they can represent a regular system of conditional probabilities of *some* probability measures μ, in such a way that the *DLR equations*

$$\mu(\sigma \mid \mathcal{F}_{\Lambda^c})(\cdot) = \gamma_\Lambda(\sigma \mid \cdot),\ \mu - \text{a.s.}, \tag{5}$$

are valid for any $\Lambda \in \mathcal{S},\ \sigma \in \Omega$.

This *DLR approach* to describe probability measures on infinite product probability spaces is crucial in our framework because it allows the definition of *many* different measures specified by the same specification. In this case, we say that there is *Phase Transition*.

The set of **Gibbs measures** specified by the specification $\gamma^{\beta\Phi}$ is denoted by $\mathcal{G}(\gamma^{\beta\Phi})$, and the study of its properties is a central aim in rigorous statistical mechanics, see [21, 11, 38].

A criterion for Gibbsianness : Quasilocality.

An important feature in the theory of Gibbs measures is *Quasilocality* which, heuristically, ensures that a measurement done at a precise point in the system is not impacted too much by far away perturbations. A function $f\colon \Omega \to \mathbb{R}$ is said to be *local* if $\exists \Lambda \in \mathcal{S}$ s.t. f is $\mathcal{F}_\Lambda$ measurable (that is, f depends only on a finite number of spins). A function $f\colon \Omega \to \mathbb{R}$ is said to be *quasilocal* if it is the uniform limit of a sequence of local functions f_n, *i.e.*

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid = 0.$$

A specification is *quasilocal* if its kernels leave invariant the set of quasilocal functions $\mathcal{F}_{\text{qloc}}$. In particular, for any finite-volume Λ, $\gamma_\Lambda f$ is a *continuous* function for any $f \in \mathcal{F}_{\text{loc}}$. A measure is quasilocal if it is specified by a quasilocal specification. Remarkably, it has been shown by Kozlov [34] and Sullivan [51] that quasilocality plus a natural positivity requirement called *non-nullness*[2] fully characterize Gibbs measures. Thus, if Φ is a UAC potential, any Gibbs specification $\gamma^{\beta\Phi}$ is quasilocal (see [38], Theorem 3.33 for a proof inspired by Fernández [15], Kozlov [34] and Sullivan [51]), so that **any Gibbs measure is quasilocal**.

The relevance of these remarks in our framework comes from the fact that, for Ising spins, *continuity is equivalent to quasilocality.* Thus, the Kozlov–Sullivan characterization can be used (as we shall do) as a proxy to prove non-Gibbsianness: it suffices to exhibit essential discontinuity of some conditional probabilities as a function of the boundary conditions, such as magnetization at a given site in our Ising context.

Phase Transition and description of the set of Gibbs measures:

If $|\mathcal{G}(\gamma)| > 1$ we say that the system exhibits a *Phase Transition*, and one is interested in the study of the extreme measures (points) of this set. In the case of Cayley trees, a considerable number of extremal Gibbs measures has been constructed first by Bleher and Ganikhodgaev [5],

[2] A specification γ is *non-null* if $\rho(A) > 0 \Longrightarrow \gamma_\Lambda(A \mid \omega) > 0$ for all $\Lambda \in \mathcal{S}, A \in \mathcal{F}, \omega \in \Omega$. Nonnullness has been sometimes named "finite-energy condition".

Theorem 2.1. *[5] For $\beta > \beta_c$, the number of extreme points of $\mathcal{G}(\gamma)$ is uncountable.*

Their extreme points can be selected by uncountably many different boundary conditions for which "half" of the Cayley tree is occupied by the "plus" and the other half by the "minus". Higuchi [27] constructed other extreme points where this ratio is different. It has been an open question for a long time to know if we have then described *all* the extreme points of $\mathcal{G}(\gamma)$, and to derive the convex decomposition of any Gibbs measures w.r.t. these extremal ones. Other uncountable sets of limiting Gibbs measures different from the Bleher–Ganikhodgaev and the Higuchi ones are known [1], and it was eventually proven recently that they are the one entering into the convex decomposition of the free state into extremal points *below* the spin-glass transition temperature [18]. For a proof that the limit Gibbs measure with free boundary conditions is extremal at temperatures *above* this temperature, see [30, 7, 29, 6].

From now on let us consider the binary Cayley tree at $k = 2$ (our discussion is easily adaptable to higher $k \geq 3$). A crucial point of the strategy is to reduce the Ising model on the full Cayley tree to the Ising model on *rooted* Cayley trees. Indeed, let us consider an arbitrary edge $\langle i_0 i_1 \rangle$ and delete it from the Cayley tree $\mathcal{T}^2$. We get two (identical) rooted Cayley trees, let us call them $\mathcal{T}_0^2$ and $\mathcal{T}_1^2$. Then we have the following

Theorem 2.2. *[5] In order for μ to be an extreme Gibbs distribution on $\mathcal{T}^2$, it is necessary and sufficient that there exists extreme Gibbs distributions μ_0, μ_1 (which are determined uniquely by μ) on $\mathcal{T}_0^2$ and $\mathcal{T}_1^2$ respectively, such that the following splitting property holds*

$$\mu = \mu_0 \mu_1 Z^{-1} \exp^{\beta J \sigma_{i_0} \sigma_{i_1}},$$

where Z is a normalizing constant depending on β.

3 Modified Majority Rule on a Cayley Tree

Let μ be any Gibbs measure for the Ising model on the rooted Cayley tree $\mathcal{T}_0^2$. We choose the root as the origin and we denote it r. Define

$$\Omega = \{-1, +1\}^{\mathcal{T}_0^2} \text{ and } \Omega' = \{-1, 0, +1\}^{\mathcal{T}_0^2}.$$

Let R be any non negative integer. We define the *closed ball* of radius R, $V_R = \{i \in \mathcal{T}_0^2 \mid d(r,i) \leq R\}$ and the *sphere* of radius R (or sometimes level or generation R), $W_R = \{i \in \mathcal{T}_0^2 \mid d(r,i) = R\}$, where d is the canonical metric on $\mathcal{T}_0^2$. Vertices of $\mathcal{T}_0^2$ are represented by sequences of bits by means of the following recurrence:

- Origin r is represented by is the void sequence and its neighbours by 0 and 1.
- Let $R > 0$ and let $i \in W_R$ be represented by i^*. The representations of the neighbours of i in W_{R+1}, called k and l, are $k^* = i^*0$ and $l^* = i^*1$.

In this way we obtain a representation of all the vertices of $\mathcal{T}_0^2$ (see Figure 1).

To avoid notational overloading we shall use the same symbol i for both the vertex or its binary representation i^*. Define the root cell $C_r = \{r, 0, 1\}$ and let, $\forall j \in \mathcal{T}_0^2$, $j \neq r$ a general cell $C_j = \{j, j0, j1\}$, where $j0$ and $j1$ are the two neighbours (or children) of j from the "following" level (for example, $C_0 = \{0, 00, 01\}$, see Figure 2). Define as well $c_j = |C_j|$. In this paper we shall consider $c_j = c = 3$.

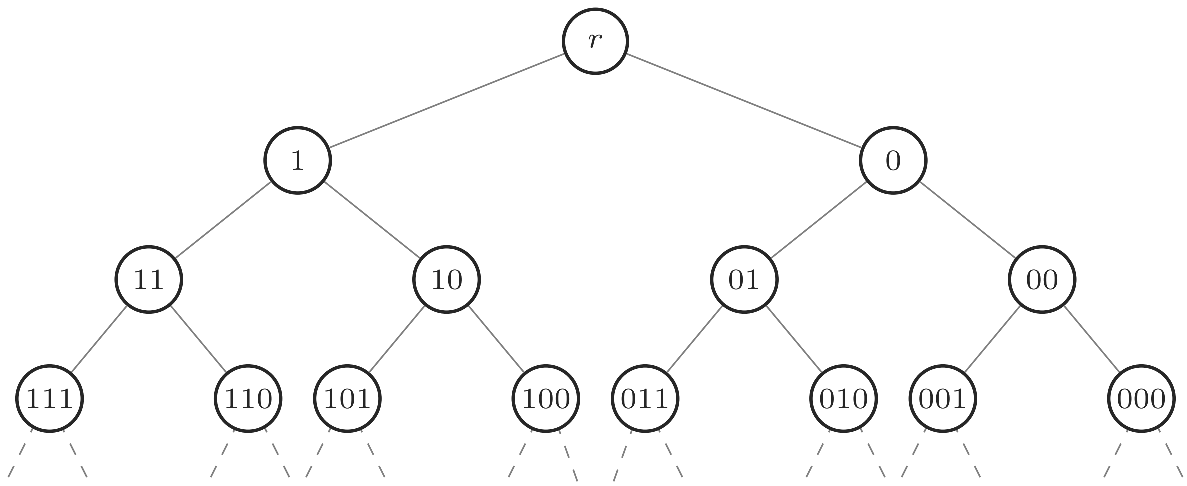

Figure 1: Binary representation of the vertices of $\mathcal{T}_0^2$, the rooted Cayley tree of degree 2. Here vertices up to level/generation $R = 3$ are represented.

Let us consider now the following, deterministic *Majority rule transformation*:

$$T \quad : \quad \Omega \longrightarrow \Omega' \\ \omega \longmapsto \omega',$$

where the image spin ω' at site j is defined by

$$\omega'_j = \begin{cases} +1 & \text{iff} \quad \frac{1}{c}\sum_{i \in C_j} \omega_i = +1 \\ 0 & \text{iff} \quad \frac{1}{c} \mid \sum_{i \in C_j} \omega_i \mid < 1 \\ -1 & \text{iff} \quad \frac{1}{c}\sum_{i \in C_j} \omega_i = -1. \end{cases}$$

3.1 Result : Non-Gibbsianness at any temperature

Now, let T be the modified majority rule on a rooted Cayley tree of degree 2, $\mathcal{T}_0^2$. In [36], one of us proved the following

Theorem 3.1. *[36] Let μ be any Gibbs measure for the Ising model on $\mathcal{T}_0^2$ and let ν be the image of μ by T. Then ν is non quasilocal at any temperature and cannot be a Gibbs measure.*

The situation reminds us an example on the integer lattice mentioned *e.g.* in [40]. In this example, the authors also exhibit a randomization of their transformation in order to obtain weak Gibbsianness or almost Gibbsianness, according to the level of randomness. It could be interesting to study a randomized version of our transformation (3) to see if it implies a possible restoration of Gibbsianness, as discussed in our conclusion section.

The result of [36] is due to the following

Lemma 3.1. *The "null"-configuration ω'^0 is a point of essential discontinuity for the conditional probabilities of the image measure ν under the majority rule T.*

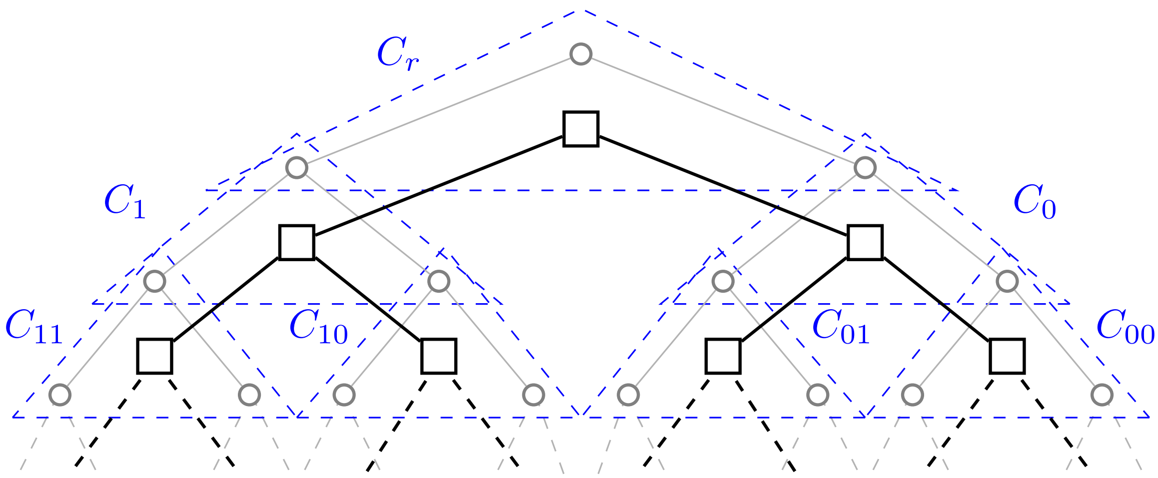

Figure 2: Pictorial representation of the transformation T acting on $\mathcal{T}_0^2$. Starting vertices are represented by small circles and image vertices by squares, conventionally put at the center of the corresponding cells C_j.

This proves that $\nu = T\mu$ is not quasilocal: it is not a Gibbs measure. We know that the "null configuration" is a bad configuration and that the image measure is not quasilocal. To incorporate the renormalized measure within the Generalized Gibbs measures of the Dobrushin program of restoration of Gibbsianness, our purpose is to evaluate the measures of the set of such "bad configurations".

Let us show first that good configurations exist. Let η be any configuration in Ω'. In the sequel, to characterize discontinuous effects, we will need the following

Definition 3.1. [Path of zeroes] :
For an integer R, a *path of zeroes from the origin to level* R is the sequence of sites $\pi = (i_k)_{k=0,\dots,R}$ s.t. $i_0 = r$, and $\forall k = 1 \dots R,\ i_k \in W_k,\ i_k, i_{k-1}$ are $n.n.$ and $\forall k \leq R,\ \eta(i_k) = 0$.

Obviously the total number of paths of zeros ending at level R is 2^R; we denote by $N_R(\eta)$ the number of paths of zeroes connecting the origin to level R in configuration η, and $N(\eta) = \lim_{R\to\infty} = N_R(\eta)$. If $N(\eta) \neq 0$, we say that there is *percolation* of zeroes.

4 Almost-Gibbsianness at any temperature

4.1 Continuity of the magnetization when $N(\eta) = 0$

Let us consider first the case of absence of percolation of zeros, that is, a configuration η such that $N(\eta) = 0$. Then there exists an integer $R_0 \geq 0$ such that $\forall R \geq R_0, N_R(\eta) = 0$. Let us prove that any such η is a good configuration for the image measure $\nu = T\mu$. In order to do so, consider the basis of neighbourhoods $\mathcal{N}^R(\eta)$ for configuration η,

$$\mathcal{N}^R(\eta) = \{\omega' \in \Omega',\ \omega'_{V_R} = \eta_{V_R},\ \omega' \text{ arbitrary elsewhere}\}.$$

We shall study the continuity, as a function of the boundary condition, of the value of the image spin at the origin r in the neighbourhood of η, namely

$$\langle \sigma'_r \rangle^{\eta,R} = \nu[\sigma'_r \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)].$$

Lemma 4.1.

$$\forall R \geq R_0, \langle \sigma'_r \rangle^{\eta,R} = \nu[\sigma'_r \mid \sigma'_{V_{R_0}\setminus\{r\}} = \eta_{V_{R_0}\setminus\{r\}}] = \nu[\sigma'_r \mid \sigma'_{\{r\}^c} = \eta_{\{r\}^c}]$$

is independent of R.

Thus, the magnetization in a neighbourhood of such an η is a function of $\eta_{V_{R_0}\setminus\{r\}}$, so implying essential continuity at this point.

Proof. By definition,

$$\langle \sigma'_r \rangle^{\eta,R} = \nu\big[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)\big] - \nu\big[\sigma'_r = - \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)\big].$$

We first deal with the event $\{\sigma'_0 = +\}$. We have

$$\nu[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)] = \frac{\nu[\sigma'_r = +,\ \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)]}{\nu[\sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)]}.$$

Notice that this conditioning is possible because the denominator is not zero when $\eta \in T(\Omega)$. Now take $R \geq R_0$ and define $A'_{V_R} = \{\sigma' \in \Omega', \sigma'_{V_R\setminus\{r\}} = \eta_{V_R\setminus\{r\}}\}$ and $A_{V_R} = T^{-1}(A'_{V_R})$. Now, $A'_{V_R} = A'_{V_{R_0-1}} \cap A'_{V^c_{R_0-1}}$ and $A_{V_R} = A_{V_{R_0-1}} \cap A_{V^c_{R_0-1}}$ and hence

$$\nu[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)] = \frac{\nu[\{\sigma'_r = +\} \cap A'_{V_R}]}{\nu[A'_{V_R}]}.$$

We also have $\nu[A'_{V_R}] = \mu[A_{V_R}] = \mu[A_{V_{R_0-1}} \cap A_{V^c_{R_0-1}}] = \mu[A_{V^c_{R_0-1}}]\mu[A_{V_{R_0-1}} \mid A_{V^c_{R_0-1}}]$. Proceeding similarly for $\nu[\{\sigma'_r = +\} \cap A'_{V_R}]$, due to the Markov property satisfied by μ, we get

$$\begin{aligned} \nu\big[\sigma'_r = + \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)\big] &= \frac{\mu[A_{V^c_{R_0-1}}]\mu[\{\sigma'_r = +\} \cap A_{V_{R_0-1}} \mid A_{W_{R_0}}]}{\mu[A_{V^c_{R_0-1}}]\mu[A_{V_{R_0-1}} \mid A_{W_{R_0}}]} \\ &= \frac{\mu[\{\sigma'_r = +\} \cap A_{V_{R_0-1}} \mid A_{W_{R_0}}]}{\mu[A_{V_{R_0-1}} \mid A_{W_{R_0}}]}. \end{aligned}$$

The configuration in V_{R_0} is fixed in η, by defintion of R_0, and we get

$$\begin{aligned} \nu\big[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c},\ \omega' \in \mathcal{N}^R(\eta)\big] &= \nu\big[\sigma'_r = + \mid \sigma'_{V_{R_0}\setminus\{r\}} = \eta_{V_{R_0}\setminus\{r\}}\big] \\ &= \nu\big[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \eta_{\{r\}^c}\big]. \end{aligned}$$

The statement of Lemma 4.1 easily follows. $\diamond$

A direct consequence of Lemma 4.1 is the following

Theorem 4.1. *The magnetization is essentially continuous as a function of the boundary condition for all $\eta \in T(\Omega)$ without percolation of zeroes.*

4.2 Continuity of the magnetization when $N(\eta) = 1$

Let us consider now, for $\eta \in T(\Omega)$ such that $N(\eta) = 1$, the set $\pi(\eta)$ to be the only infinite cluster from the origin when the system lies in configuration η.
First, we shall restrict ourselves to configurations where the spin in the cells in the neighborhood of the path is never zero and denote the path by π. We can assume without loss of generality that π is the set of sites having binary representation "1", except the first one.

Since the length of all the other paths of zeroes is necessarily finite, we can always consider an integer R_1 which is the maximal length of these other paths (similarly to the definition of R_0 adopted in Section 4.1). We shall now consider a region which is sufficiently far away level from the origin r, with $R \geq R_1$, where no such terminating path of zero can penetrate. Define also the projection Y' of η onto the only infinite path π, by $Y'_R = \eta_{W_R \cap \pi}$, and it will be useful to extend Y'_R on the other neighbours of the origin by $Y'_{-1} = \eta_0$. Write as usual $Y = T^{-1}(Y')$ and define the shortcut $X_n = Y_{R-n+1}$ for all integers $n \leq R$. We study the asymptotic behaviour of the ν-magnetization when we have around the origin a b.c. in a neighbourhood of η :

$$\langle \sigma'_r \rangle^{\eta,R} = \nu[\sigma'_r = + \mid A'_R] - \nu[\sigma'_r = - \mid A'_R],$$

where A'_R has been defined in the proof of Lemma 4.1 and again $A_R = T^{-1}(A'_R)$. Up to now, $\eta_0 \neq 0$ and we shall first assume that $\eta_0 = +$. This implies $\sigma_0 = +$ and

$$\begin{aligned} \langle \sigma'_r \rangle^{\eta,R} &= \mu[\sigma_r = \sigma_0 = \sigma_1 = + \mid A_R] \\ &= \mu[\sigma_r = \sigma_1 = + \mid A_R \cap \{\sigma_0 = +\}]\mu[\sigma_0 = + \mid A_R] \\ &= \mu[\sigma_r = \sigma_1 = + \mid A_R \cap \{\sigma_0 = +\}]. \end{aligned}$$

When $\eta_0 = -$, we obtain $\langle \sigma'_r \rangle^{\eta,R} = \mu[\sigma_r = \sigma_1 = - \mid A_R \cap \{\sigma_0 = -\}]$. Thus we only have to study the behaviour of

$$\mu[\sigma_r = \sigma_1 = + \mid A_R \cap \{\sigma_0 = +\}] \text{ and } \mu[\sigma_r = \sigma_1 = - \mid A_R \cap \{\sigma_0 = -\}]$$

knowing that the configuration in η along the path is zero with the previous assumptions on η: we have to study the law of X with this environment η.

Now, if X_n is the spin at a site $n \in \pi$, let h_n be the spin value at the neighbour of n which is not on the path and define $h = (h_n)_{n\in\mathbb{N}}$. All the possible values of h, which can be seen as an external field for the process X, are determined by the configuration η but are independent of R because of the uniqueness of the path of zeroes. Let us denote by $\mathbb{P}$ the probability measure $\mu[. \mid A_R]$. We shall study first the law of X under $\mathbb{P}$.

Forget first the constraint due to the path of zeroes and study the law of X without it. With the fixed spin at the origin and a fixed external configuration (field) $h = h(\eta) = (h_n)_{n\in\mathbb{N}}$, the law of X is exactly the law of a one dimensional Ising model at inverse temperature β with coupling $J > 0$ and external magnetic field g defined by

$$g_{R+1} = 0 \text{ and } \forall n \in \mathbb{N},\ g_n = \beta J h_n.$$

We shall now incorporate the coupling in the temperature, assume $J = 1$ and denote γ the inhomogeneous specification of this Ising chain on $\mathbb{Z}$ (see *e.g.* [21] for details). From now on we can proceed *via* a classical transfer matrix approach: define $\forall n \in \mathbb{N}$ a 2×2 matrix Q'_n by

$$\forall n \in \mathbb{N},\ \forall x, y \in \{-,+\},\ Q'_n(x,y) = \exp(\beta xy + \frac{\beta}{2}(h_{n-1}x + h_n y));$$

one could rewrite the specification in terms of this transfer matrix: Define $V = \{i+1, \dots, k-1\} \subset \mathbb{N}$ and let $\sigma, \omega \in \{-,+\}^{\mathbb{N}}$. Then

$$\gamma_V(\sigma_V \mid \omega) = \frac{Q'_{i+1}(\omega_i, \sigma_{i+1})(\prod_{j=i+2}^{k-1} Q'_j(\sigma_{j-1}, \sigma_j))Q'_k(\sigma_{k-1}, \omega_k)}{(\prod_{j=i+1}^{k} Q'_j)(\omega_i, \omega_k)}.$$

Before dealing with a general environment h (leading to inhomogeneous Markov chains), we shall first deal with the homogeneous case $h_n = + \; \forall n \in \mathbb{N}$, assuming[3] for example $\eta_0 = +$.

4.2.1 Homogeneous external field case

Let us assume $\eta_0 = +$ and $h_n(\eta) = +, \; \forall n \in \mathbb{N}$, and let us use the description of the one dimensional homogeneous Ising model as a Markov chain. In this homogeneous case, we have

$$\forall n \in \mathbb{N}, \; Q'_n = Q' = \begin{pmatrix} 1 & e^{-\beta} \\ e^{-\beta} & e^{2\beta} \end{pmatrix}.$$

It is well known that *there is a one to one correspondence between the set of all positive homogeneous Markov specifications and the set of all stochastic matrices on E with no vanishing entries* (see *e.g.* Theorem 3.5 of [21] for a detailed proof). Hence, under the environment h (without the constraint on the path of zeroes), the law of X is that of an homogeneous Markov Chain with a transition matrix P and a priori law α_P[4] such that

$$P = \begin{pmatrix} a & 1-a \\ 1-b & b \end{pmatrix},$$

with $a = \frac{e^{-\beta}}{\cosh\beta + \sqrt{e^{-\beta} + (\sinh\beta)^2}} \in]0,1[$ and $b = e^{2\beta}$.

Let us now introduce the constraint of being on the path of zeroes. The law of X is still Markovian, but now some transitions are forbidden. In this case with $h_n = +, \; \forall n \in \mathbb{N}$, the transition matrix becomes

$$P_+ = \begin{pmatrix} a & 1-a \\ 1 & 0 \end{pmatrix}$$

because, with this constraint, necessarily

$$\{h_{n+1} = +, X_n = +\} \Longrightarrow \{X_{n+1} = -\}$$

otherwise the infinite path of zeros would end at cell C_n.

Now, recall that in order to study the continuity of the magnetization under the environment η, we have to study the asymptotic behaviour, on the neighbourhoods of η and when R goes to infinity, of $\langle \sigma'_r \rangle^{\eta,R}$. Under the assumption of $\eta_0 = +$, we get

$$\begin{aligned}
\langle \sigma'_r \rangle^{\eta^+,R} =& \mu\big[\{\sigma_1 = \sigma_r = +\} \mid A_R \cap \{\sigma_0 = +\}\big] \\
=& \gamma_R\big[X_R = X_{R+1} = + \mid X_0 = x_0\big] \\
=& \mu_P\big[X_R = X_{R+1} = + \mid X_0 = x_0\big] \\
=& \mu_P\big[X_{R+1} = + \mid X_R = +\big]\mu_P\big[X_R = + \mid X_0 = x_0\big] \\
=& P(+,+)(P_+^{R+1})_{x_0,+},
\end{aligned} \tag{6}$$

[3]The cases $\eta_0 = -$ or $h_n = -, \; \forall n \in \mathbb{N}$, are treated similarly.

[4]α_P is the only probability (row) vector such that $\alpha_P = \alpha_P P$.

in accordance with the expression of the Ising random field as a Markov chain (here, $(.)_{x_0,+}$ denotes the column $+$ and line x_0 of the matrix and $x_0 = +$ or $-$ depending on which neighbourhood of η we are studying). On the first step, we keep the matrix P because there is no constraint. Thus, studying continuity of the magnetization is reduced to the study of the dependence on x_0 of the expression in (6). By the usual diagonalization trick, we get $\forall n \in \mathbb{N}$:

$$P_+^n = \begin{pmatrix} \frac{1}{2-a} + \frac{(a-1)^{n+1}}{2-a} & \frac{1-a}{2-a} - \frac{(a-1)^{n+1}}{2-a} \\ \\ \frac{1}{2-a} + \frac{(a-1)^{n}}{2-a} & \frac{1-a}{2-a} - \frac{(a-1)^{n}}{2-a} \end{pmatrix},$$

and hence $M := \lim_{n\to\infty} P_+^n$ is just

$$M = \begin{pmatrix} \frac{1}{2-a} & \frac{1-a}{2-a} \\ \\ \frac{1}{2-a} & \frac{1-a}{2-a} \end{pmatrix}. \tag{7}$$

Notice that the elements of each column in M are equal, implying that $\lim_{R\to\infty}(P_+^{R+1})_{x_0,+}$ is independent on x_0: *the magnetization is a continuous function of the boundary condition.* It is given by

$$\langle \sigma'_r \rangle^{\eta^+} = e^{2\beta} a \frac{1-a}{2-a}$$

for a configuration η^+ with one infinite path of zeroes and $+$ everywhere else. For a configuration η^- with only one path of zeroes and $-$ everywhere else, the same continuity result holds, and we simply have

$$\langle \sigma'_r \rangle^{\eta^-} = -e^{2\beta} a \frac{1-a}{2-a} = -\langle \sigma'_r \rangle^{\eta^+},$$

where we recall that $a = \frac{e^{-\beta}}{\cosh\beta + \sqrt{e^{-\beta} + (\sinh\beta)^2}} \in]0,1[$.

4.2.2 Inhomogeneous external field case

Here, we deal with an inhomogeneous Markov Chain without knowing exactly the transition matrix (we only know a transfer matrix and the notion of boundary laws, see [21], def 12.10). With the path of zeros constraint, the law of X is still Gibbsian but the potential now becomes

$$\Phi_A(\omega) = \begin{cases} -\beta\omega_{n-1}\omega_n - \frac{\beta}{2}(h_n\omega_n + h_{n-1}\omega_{n-1}) \\ \quad \text{iff } A = \{n-1,n\} \text{ and } h_n\omega_{n-1} = h_n\omega_n = + \\ \\ +\infty \ \text{ iff } A = \{n-1,n\} \text{ and } \omega_n = \omega_{n-1} = h_n \\ \\ 0 \ \text{ otherwise,} \end{cases}$$

because some transitions are forbidden under the constraint. The potential is infinite and it provides a kind of *hard core exclusion potential*, but the formalism is the same as in the finite potential case (provided the objects are defined). We shall have to deal with various products of the matrices $Q_{h_n h_{n-1}}$ depending on h, so that it is useful to introduce:

$$Q_{++} = \begin{pmatrix} 1 & e^{-\beta} \\ \\ e^{-\beta} & 0 \end{pmatrix}, \; Q_{--} = \begin{pmatrix} 0 & e^{-\beta} \\ \\ e^{-\beta} & 1 \end{pmatrix},$$

$$Q_{-+} = \begin{pmatrix} 0 & e^{-2\beta} \\ & \\ 1 & e^{\beta} \end{pmatrix} \text{ and } Q_{+-} = \begin{pmatrix} e^{\beta} & 1 \\ & \\ e^{-2\beta} & 0 \end{pmatrix}.$$

Keeping the same notations as in Section 4.2.1, we first deal with the case $\eta_0 = +$. Let $x_0 \in \{-1, +1\}$ and define, for $R \geq 0$,

$$\langle \sigma'_r \rangle_{x_0}^{\eta,R} = \nu\big[\sigma'_r \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c}, \omega' \in \mathcal{N}_R(\eta), X_0 = x_0\big].$$

Proceeding as usual, we get, in this case where $\eta_0 = +$,

$$\begin{aligned} \langle \sigma'_r \rangle_{x_0}^{\eta,R} &= \nu\big[\sigma'_r = + \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c}, \omega' \in \mathcal{N}_R(\eta), X_0 = x_0\big] \\ &= \mu\big[X_R = X_{R+1} = + \mid X_{R+2} = +, h, X_0 = x_0\big], \end{aligned}$$

where the notation "h" means a conditioning with the event of being under the environment $h = (h_n)$ besides the path of zeroes. Using again the expression with transfer matrices,

$$\begin{aligned} \langle \sigma'_r \rangle_{x_0}^{\eta,R} &= \frac{Q'_{R+2}(+,+)Q'_{R+1}(+,+)[Q_R \dots Q_n \dots Q_1]_{x_0,+}}{Q'_{R+2}(+,+)Q'_{R+1}(+,+)([Q_R \dots Q_n \dots Q_1]_{x_0,-} + Q'_{R+2}[Q_R \dots Q_n \dots Q_1]_{x_0,+})} \\ &= \frac{[Q_R \dots Q_n \dots Q_1]_{x_0,+}}{[Q_R \dots Q_n \dots Q_1]_{x_0,-} + [Q_R \dots Q_n \dots Q_1]_{x_0,+}} \end{aligned}$$

where, due to the constraints and $g_{R+1} = 0, g_{R+2} = 2, g_n = h_n \; \forall n \leq R$,

$$Q'_{R+2} = \begin{pmatrix} 1 & 1 \\ & \\ e^{-2\beta} & e^{2\beta} \end{pmatrix}, \; Q'_{R+1} = \begin{pmatrix} e^{\beta - \frac{\beta h_R}{2}} & e^{-\beta - \frac{\beta h_R}{2}} \\ & \\ e^{-\beta + \frac{\beta h_R}{2}} & e^{\beta + \frac{\beta h_R}{2}} \end{pmatrix}$$

and

$$\forall n \leq R, \quad Q_n = Q_{h_n h_{n-1}}.$$

Thus, in order to study the continuity of this magnetization as a function of the boundary conditions h_0, h_1 and x_0, we have to study the asymptotic behaviour, depending on h_0, h_1, x_0, of the matrix products

$$P_R = Q_R \dots Q_1 = \prod_{n=R}^{1} Q_n = \prod_{n=R}^{1} Q_{h_n h_{n-1}}$$

taking account the constraints[5]. Denote

$$P_n = \begin{pmatrix} a_n & b_n \\ & \\ c_n & d_n \end{pmatrix}$$

and assume first that, at least for $n \geq n_0 \in \mathbb{N}$, $a_n.b_n.c_n.d_n > 0$. This condition holds for all configurations η's except two special ones, called *alternating configurations*, for which we shall prove analytically a discontinuity later. Take now $n \geq n_0$ and denote $x_n = \frac{a_n}{b_n}$ and $y_n = \frac{c_n}{d_n}$. We want to study the asymptotic behaviour of the sequences $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$. We have the general pattern $P_{n+1} = P_n \times A_n$, with

$$A_n \in \{Q_{++}, Q_{+-}, Q_{--}, Q_{-+}\}$$

and we consider now those four cases separately.

[5]The constraints imply that some products such as $Q_{++}Q_{-+}$ are forbidden.

Case $A_n = Q_{++}$: We obtain

$$P_{n+1} = \begin{pmatrix} a_n + b_n e^{-\beta} & a_n e^{-\beta} \\ c_n + d_n e^{-\beta} & c_n e^{-\beta} \end{pmatrix}$$

and this yields the same evolution for $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$:

$$x_{n+1} = f_1(x_n) \text{ and } y_{n+1} = f_1(y_n),$$

where f_1 is defined for all $x > 0$ by

$$f_1(x) = 1 + \frac{e^{-\beta}}{x} \geq 1.$$

We also have

$$\forall x \geq 1 \ \mid f_1'(x) \mid = \frac{e^{-\beta}}{x^2} \leq e^{-\beta} < 1.$$

This application is then contracting as soon as $x \geq 1$, which will be true after one step because $f_1(x) \geq 1 \ \forall x > 0$. We denote the Lipschitz constant $k_1 = e^{-\beta}$, and we have

$$\forall n \geq n_0, \ \mid x_{n+1} - y_{n+1} \mid \leq k_1 \cdot \mid x_n - y_n \mid .$$

Case $A_n = Q_{+-}$: We obtain

$$P_{n+1} = \begin{pmatrix} a_n e^{\beta} + b_n e^{-2\beta} & a_n \\ c_n e^{\beta} + d_n e^{-2\beta} & c_n \end{pmatrix}$$

and the same contractive result holds with the function f_2 :

$$f_2(x) = e^{\beta} + \frac{e^{-2\beta}}{x} > e^{\beta} > 1, \text{for all } x > 0.$$

Now

$$\forall x \geq 1, \ \mid f_2'(x) \mid = \frac{e^{-2\beta}}{x^2} \leq e^{-2\beta} < 1$$

and we will denote by $k_2 = e^{-2\beta}$ this second Lipschitz constant.

Case $A_n = Q_{--}$: We obtain

$$P_{n+1} = \begin{pmatrix} b_n e^{-\beta} & a_n e^{-\beta} + b_n \\ d_n e^{-\beta} & c_n e^{-\beta} + d_n \end{pmatrix}$$

and $f_3(x) = \frac{1}{x+e^{\beta}}$, $\forall x > 0$, which gives a Lipschitz constant $k_3 = k_2 = e^{-2\beta}$.

Case $A_n = Q_{-+}$: The contraction holds with $k_4 = k_2 = e^{-2\beta}$.

Thus, whatever the configuration η, we always have

$$\forall n \geq n_0, \mid x_n - y_n \mid \leq e^{n_o \beta} \left(\frac{1}{e^\beta} \right)^n \cdot \mid x_{n_0} - y_{n_0} \mid . \tag{8}$$

Let us come back to the magnetization. Whatever h_0 and h_1 are, we have

$$\langle \sigma'_r \rangle_{x_0}^{\eta,R} = \frac{[P_R]_{x_0,+}}{[P_R]_{x_0,} + [P_R]_{x_0,+}},$$

which yields $\langle \sigma'_r \rangle_+^{\eta,R} = \frac{d_R}{c_R + d_R}$ and $\langle \sigma'_r \rangle_-^{\eta,R} = \frac{a_R}{a_R + b_R}$. If $n \geq n_0$, we get

$$\langle \sigma'_r \rangle_+^{\eta,R} = \frac{1}{1 + y_R} \text{ and } \langle \sigma'_r \rangle_-^{\eta,R} = \frac{1}{1 + x_R}$$

and

$$\mid \langle \sigma'_r \rangle_+^{\eta,R} - \langle \sigma'_r \rangle_-^{\eta,R} \mid \leq \mid x_R - y_R \mid \leq e^{n_o \beta} \cdot \left(\frac{1}{e^\beta} \right)^R \mid x_{n_0} - y_{n_0} \mid$$

which converges to zero as $R \to \infty$. Proceeding similarly in the case $\eta_0 = -$, and conditioning on the events $\{X_{R+2} = +\}$ and $\{X_{R+2} = -\}$, we get continuity of the magnetization for all the configurations having only one path of zeroes, except the so-called two *alternating configurations*. Assume again $\eta_0 = +$ and define η^1_{alt} and η^2_{alt} to be such that $N(\eta) = 1$ and

$$\forall i = 1, 2, \ \forall R \in \mathbb{N}, \forall n = 0 \dots R-1, \quad h^i_R = (-1)^i, \ h^i_n = -h^i_{n+1}$$

where h^1 (resp. h^2) is the environment under η^1_{alt} (resp. η^2_{alt}).

Let us first consider η^1. We have the same expression for the magnetization: $\forall x_0 \in \{-1, +1\}$, $\forall R \geq 0$,

$$\langle \sigma'_r \rangle_{x_0}^{\eta,R} = \frac{[Q_R \dots Q_n \dots Q_1]_{x_0,+}}{[Q_R \dots Q_n \dots Q_1]_{x_0,-} + [Q_R \dots Q_n \dots Q_1]_{x_0,+}}.$$

Denote $A := Q_R Q_{R+1} = Q_{-+} Q_{+-}$ so that, depending on h_1 and h_0,

$$\prod_{n=1}^{R} Q_n = A^{\frac{R-2}{2}} \cdot Q_2 Q_1$$

when R is even, and

$$\prod_{n=1}^{R} Q_n = A^{\frac{R-1}{2}+1} \cdot Q_{-+} Q_2 Q_1$$

when R is odd. We have then to investigate the asymptotic behaviour of the powers of the lower triangular matrix

$$A = Q_{-+} Q_{+-} = \begin{pmatrix} e^{-4\beta} & 0 \\ e^\beta + e^{-\beta} & 1 \end{pmatrix}.$$

Such powers of A are easily computable by recurrence:

$$A^n = \begin{pmatrix} e^{-4n\beta} & 0 \\ (e^\beta + e^{-\beta})(1 + \dots + e^{-4(n-1)\beta}) & 1 \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$

Let us check for R even that this leads to a continuity of the magnetization. We have

$$\prod_{n=1}^{R} Q_n = A^{\frac{R-2}{2}} \cdot Q_2 Q_1$$

with $Q_2 = Q_{-h_1}$. Computing all the products of matrices for the various values of h_0 and h_1, we obtain a non-quasilocality of the magnetization. The same statement holds for $\eta_0 = -$ and this remains true for η^2_{alt}.

5 (Almost) quasilocality of magnetizations

We have the following:

Theorem 5.1 (Continuity of magnetization). *Let $\eta \in \Omega'$ such that $N(\eta) = 1$, $\eta \neq \eta^1_{\mathrm{alt}}, \eta^2_{\mathrm{alt}}$, and define, $\forall \omega' \in \Omega'$, the magnetization with boundary condition ω' to be*

$$\langle \sigma'_r \rangle^{\omega'} = \nu[\sigma'_r \mid \sigma'_{\{r\}^c} = \omega'_{\{r\}^c}].$$

This magnetization is continuous at η and moreover, there exists a strictly positive constant $C > 0$ such that

$$\forall R \geq 0, \forall \omega'_1, \omega'_2 \in \mathcal{N}_R(\eta), \qquad \mid \langle \sigma'_r \rangle^{\omega'_1} - \langle \sigma'_r \rangle^{\omega'_2} \mid \leq C \cdot (e^{-\beta})^R.$$

5.1 Sufficient condition for essential continuity on $N(\eta)$

Recall that, for a configuration η, $N_R(\eta)$ denotes the number of paths of zeroes up to level R in η and $N = N(\eta) = \lim_{R \to \infty} N_{R(\eta)}$. We can now use Theorem 5.1 to show

Lemma 5.1. *For ν-a.e. $\omega'_1, \omega'_2 \in \mathcal{N}_R(\eta)$, there exists a constant C such that*

$$\forall R \geq 0, \sup_{\omega'_1, \omega'_2} \mid \langle \sigma'_r \rangle^{\omega'_1} - \langle \sigma'_r \rangle^{\omega'_2} \mid \leq C \cdot N_R(\eta) \cdot (e^{-\beta})^R. \tag{9}$$

Coupled to the particular structure of the Cayley tree (and the bound $N_R(\eta) \leq 2^R$), this lemma enables us to give a first result of almost quasilocality. Indeed, for all $\eta \in \Omega'$, we have

$$N_R(\eta) e^{-\beta R} \leq e^{R(\ln 2 - \beta)} \tag{10}$$

so that (9) is fulfilled at sufficiently low temperatures $\beta > \ln 2 = 0.69314\ldots$:

Lemma 5.2. *The transformation of any Gibbs measure for the Ising model on the Cayley tree $\mathcal{T}^2_0$ via the modified majority rule leads to an almost quasilocal measure at low temperatures such that $\beta \geq \beta_0 = \ln 2$.*

5.2 Percolation model and almost Gibbsianness

In the previous sections we have seen that the number of infinite cluster of zeros is a good discriminant for detecting failure of quasilocality. Let us consider now the probability of being in the configuration zero. Due to the overlap among neighboring cells involved in the majority rule transformation T, clearly the random variables $(\eta_j)_{j \in \mathcal{T}^2_0}$ are not independent. Nevertheless, we can obtain a few informations about η_j due to the Markovianness of the original measure. For a site $j \in \mathcal{T}_0^2$, let us call $j1$ its father, that is, if j has the binary representation $\sigma_{c_0 \cdots c_n}$, then $j1$ has the binary representation $\sigma_{c_0 \cdots c_{n-1}}$. When the conditioning is possible, for $\eta \in \Omega'$, from Bayes' formula we get

$$\begin{aligned} \nu\big[\eta_{j+1} = 0\big] =& \nu\big[\eta_{j+1} = 0 \mid \eta_j = +\big]\nu\big[\eta_j = +\big] + \nu\big[\eta_{j+1} = 0 \mid \eta_j = -\big]\nu\big[\eta_j = -\big] + \\ &+ \nu\big[\eta_{j+1} = 0 \mid \eta_j = 0\big]\nu\big[\eta_j = 0\big] \end{aligned} \tag{11}$$

(if a conditioning is not possible, an analogous formula holds without their contributions). Let us now compute separately the three different conditional probabilities appearing in (11). In order to do so, let us introduce the shortcut

$$\mu_{\pm\pm|\pm} \coloneqq \mu\big[\sigma_{j0} = \pm, \sigma_{j1} = \pm \mid \sigma_j = \pm\big].$$

First, for the event $\{\eta_j = +\}$,

$$\begin{aligned} \nu\big[\eta_{j+1} = 0 \mid \eta_j = +\big] =& \ (\mu_{+-|+} + \mu_{-+|+}) + \mu_{--|+} = \frac{2}{\left(e^{-\beta} + e^{\beta}\right)^2} + \frac{e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2} \\ =& \ \frac{2 + e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2}. \end{aligned} \tag{12}$$

Analogously for the event $\{\eta_j = -\}$ we get

$$\nu\big[\eta_{j+1} = 0 \mid \eta_j = -\big] = 2\mu_{+-|-} + \mu_{++|-} = \frac{2 + e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2}. \tag{13}$$

Lastly, the contribution due to conditioning on $\{\eta_j = 0\}$ becomes

$$\nu\big[\eta_{j+1} = 0 \mid \eta_j = 0\big] \quad = \quad \frac{2 + e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2} \left(\mu[\sigma_j = +] + \mu[\sigma_j = -]\right) = \frac{2 + e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2},$$

where the factors 2 come from $+/-$ symmetry in a progeny at fixed father. Thus, despite the spins being dependent, the three considered events are indeed uncorrelated and we just have[6]

$$\nu[\eta_j = 0] \coloneqq p(\beta) = \frac{2 + e^{-2\beta}}{\left(e^{-\beta} + e^{\beta}\right)^2} \quad \forall j \in \mathcal{T}_0^2.$$

Therefore, the problem becomes one of bond percolation on a Cayley tree *with a β-dependent probability of open bond* (see *e.g.* [23], Chapter 10). For $\mathcal{T}^2$, if $N(\eta)$ is the number of infinitely long paths of zeros (or infinite clusters of zeroes) in configuration η, we just have

$$\nu[N(\eta) > 0] = \begin{cases} 0 & \text{iff} \quad p(\beta) \leq p_{\mathrm{c}} = \frac{1}{2} \\ > 0 & \text{iff} \quad p(\beta) > p_{\mathrm{c}} = \frac{1}{2}. \end{cases}$$

Thus, there exists a critical inverse temperature β_1 such that $p(\beta_1) = \frac{1}{2}$, that is $\beta_1 = \sqrt{\ln\left(1 + \sqrt{2}\right)} = 0.93881\ldots$, and the following theorem holds:

Theorem 5.2. *The transformation of any Gibbs measure for the Ising model on $\mathcal{T}_0^2$ by the modified majority rule leads to an almost quasilocal measure at temperatures $\beta > \beta_1$.*

[6] We remark that $p(\beta) = P_2(\tanh\beta)$, where $P_2(y) = \frac{1}{4}(1 - y)(y + 3)$, and we believe this should be a polynomial in $\tanh\beta$ for general Cayley trees of order $k \geq 2$.

Essential continuity of the magnetization follows directly from Theorem 4.1, and this is enough in this case to get almost quasilocality (see *e.g.* [42]). Since $\beta_1 > \beta_0$ (see Lemma 5.2), almost quasilocality holds at lower temperatures than in previous subsection.

Let us now study almost quasilocality in the low temperature regime $\beta > \beta_1$. By means of the sufficient condition of Lemma 5.1, our aim is to investigate the ν-measure of the set Ω_g of configurations for which $N_R(\eta)$ (the number of zero paths between the root and generation R in configuration η) grows slower than $e^{\beta R}$:

$$\Omega_g := \{\eta \in \Omega' \ : \ \lim_{R\to\infty} \frac{N_R}{e^{\beta R}} = 0\}. \tag{14}$$

Thus we seek $l > e^{\beta}$ such that $\lim_{R\to\infty} \nu[N(\eta) > l^R] = 0$. In this way, $\nu[\Omega_g^c] = 1$ and this implies almost quasilocality. By this estimates, we indeed recover it at any temperature in Theorem 5.3, see below. To this aim, let us consider the sequence $(N_R)_R$ of random variables on $(\Omega', \mathcal{F}')$ (we shall drop the dependence on η to avoid notational cluttering). Using percolation language, there is a probability $p(\beta)^2$ of increasing N_R by 1 (*i.e.* opening two new bonds), a probability $(1-p(\beta))^2$ of closing two bonds, etc. Therefore, for ν-a.e. η,

$$\begin{aligned}\mathbb{E}_\nu\left[N_R \mid \mathcal{F}_{V_{R-1}}\right](\eta) &= p^2\left(N_{R-1}+1\right) + 2p(1-p)N_{R-1} + (1-p)^2\left(N_{R-1}-1\right)\\ &= N_{R-1}(\eta) + 2p - 1.\end{aligned} \tag{15}$$

Thus, in the high temperature regime $\beta < \beta_1$, $p(\beta) > \frac{1}{2}$ (as $p(\beta)$ is a monotone decreasing function of β). By induction we simply get

$$\mathbb{E}_\nu\left[N_R\right] = p + (R-1)\left(2p-1\right), \tag{16}$$

and thus $\lim_{R\to\infty} \frac{\mathbb{E}_\nu[N_R]}{e^{\beta R}} = 0$. We want now to go beyond expectation and bound the probability of a deviation larger than $e^{\beta R}$.

In the context of percolation on trees, at this point one usually exploits the special tree topology to obtain recursive relations. An example is the determination of the critical temperature of the ferromagnetic Ising model with constant interaction strength (see [41], Theorem 2.1). Instead, here we obtain a (tight) bound on the probability of an exponential deviation following a slightly different (and possibly simpler) route based on a combination of an exponential Chebyshev's inequality and a uniform bound on the cumulant generating function of N_R, resumed in the following

Lemma 5.3 (Exponential deviation bound for N_R)**.** *For any $\theta \geq 0$*

$$\nu[N_R(\eta) > e^{\beta R}] \leq e^{\theta\left(R - e^{R\beta}\right)}$$

so that only sub-exponential deviations are allowed at any finite temperature $\frac{1}{\beta}$ as $R \to \infty$.

Proof. For $R > 0$, exponential Chebyshev's inequality implies, $\forall \theta \in \mathbb{R}$,

$$\nu[N_R(\eta) > e^{\beta R}] \leq \mathbb{E}_\nu\left[e^{\theta N_R(\eta)}\right] e^{-\theta e^{R\beta}}. \tag{17}$$

In analogy with (15) we just have

$$\begin{aligned}\mathbb{E}_\nu\left[e^{\theta N_R(\eta)} \mid \mathcal{F}_{R-1}\right] &= p^2 e^{\theta(N_{R-1}+1)} + 2p(1-p)e^{\theta N_{R-1}} + (1-p)^2 e^{\theta(N_{R-1}-1)}\\ &= \left(p^2 e^{\theta} + 2p(1-p) + (1-p)^2 e^{-\theta}\right) e^{\theta N_{R-1}},\end{aligned} \tag{18}$$

and hence by recurrence the full *Moment generating function* (MGF) is given by

$$\mathbb{E}_\nu\left[e^{\theta N_R(\eta)}\right] := M_R(\theta,p) = \left(p^2e^\theta + 2p(1-p) + (1-p)^2e^{-\theta}\right)^R.$$

A straightforward calculation shows that, for $\theta \geq 0$, the *Cumulant* $K_R(\theta,p) := \ln M_R(\theta,p)$ satisfies, uniformly in θ,

$$K_R(\theta,0) = -R\theta \leq K_R(\theta,p) \leq R\theta = K_R(\theta,1),$$

from which the assertion follows directly (recall that $p = p(\beta) \in [0,1]$). $\diamond$

As a direct consequence, we eventually get

Theorem 5.3. *The renormalized measures $\nu = T\mu$, obtained by the modified majority rule from any Gibbs measure for the Ising model on binary Cayley trees, is* almost quasilocal *at any temperature.*

6 Some research perspectives

It would be interesting to understand if our almost Gibbsianness result holds for Cayley trees of general order $k \geq 2$. Concerning the crucial step in Theorem 5.3, here we only observe that the MGF of the number of zeros reaching level R in primed configuration η, $N_R^{(k)}(\eta)$, is

$$M_R^{(k)}(\theta,p^{(k)}) = \left[e^{-\theta}\left(p^{(k)}\left(e^\theta - 1\right) + 1\right)^k\right]^R,$$

where now $p^{(k)} \equiv p^{(k)}(\beta)$ is the probability that a primed 0 percolates at inverse temperature β for the Cayley tree $\mathcal{T}^k$ (i.e. generalizing (5.2)), and a k *vs* β tradeoff becomes possible. Recently, ferromagnetic Ising models on Cayley trees subjects to inhomogeneous external fields have attracted some interest that could be useful for our purposes (see *e.g.* [3] for a recent work concerning spatially dependent external fields that are "small perturbations" of the critical external field value).

Moreover, the model considered in this paper can be naturally generalized to non-rooted Cayley trees and to non-uniform c_j possibly different from two, or with other sizes of cells. The majority rule could also be generalized as a stochastic transformation. For example, let $\epsilon \in [0,1]$ and ξ be a Bernoulli random variable with parameter ϵ and values 0 or 1. Define the deterministic map $t_\epsilon \colon \Omega \longrightarrow \Omega'; \omega \longmapsto \omega'$ where ω' is defined by

$$\omega'_j = \begin{cases} +1 & \text{iff} \quad \frac{1}{c}\sum_{i\in C_j}\omega_i = +1 \text{ and } \xi = 0 \\ -1 & \text{iff} \quad \frac{1}{c}\sum_{i\in C_j}\omega_i = -1 \text{ and } \xi = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Its action is described by a probabilistic kernel T_ϵ defined by:

$$\forall A' \in \mathcal{F}', \forall \omega \in \Omega, \quad T_\epsilon(\omega, A') = (1-\xi)\delta_{t_\epsilon(\omega)}(A') + \xi\delta_0(A').$$

It could be interesting to study the difference between the deterministic transformation and the stochastic one, as this could play a role on the degree of non-Gibbsianness of the image measure, similar to the van den Berg example on the integers (see [42]).

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