Decimation and the spin-flop transition in the XY model on $\mathbb{Z}^2$

Séminaire de Probabilités et Statistiques

Laboratoire Analyse, Géométrie et Applications
Institut Galilée, Université Sorbonne Paris Nord
6th December 2022, 11:00 AM (Local Time to France)

Matteo D'Achille

Talk based on works in collaboration with

- Aernout C.D. van Enter — Bernoulli Institute, Univ. of Groningen
- Arnaud Le Ny — LAMA, Université Paris-Est Créteil

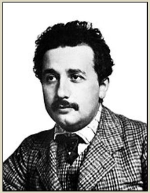
Main references:

DVLII **MD'A**, A. C. D. van Enter and A. Le Ny, "Decimations for Two Dimensional Ising and Rotator Models II: Continuous versus Discrete Symmetries", to appear in J. Math. Phys. **63**, 2022
2206.06990

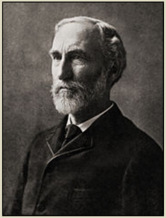
DVLI **MD'A**, A. C. D. van Enter and A. Le Ny, "Decimations for two-dimensional Ising and rotator models", J. Math. Phys. **63**, 27, 2022
2105.07950

Statistical mechanics and the ensembles of Gibbs

Einstein 1902: "... to derive the thermal equilibrium and the second law of thermodynamics from the equation of motion and probability theory"



Gibbs 1902



V volume, N particles, E energy. 3 "ensembles":

- Microcanonical - V, N and E fixed (isolated system);
- ☛ **Canonical** - V and N fixed, E fluctuates (closed system);
- Grandcanonical - V fixe, N and E fluctuate (open system).

$$\text{Micro.} \xrightarrow[\text{Laplace transform}]{} \textbf{canon.} \xrightarrow[\text{Laplace transform}]{} \text{grand.}$$

Variational principle for the Gibbs measure

If V is finite, let Ω be the configuration space and $\mathscr{H}_V : \Omega \to \mathbb{R}$ the hamiltonian. For P a prob. measure over Ω, consider

$$\begin{cases} U[P] = & \sum_{\sigma \in \Omega} P[\sigma] \mathscr{H}_V[\sigma] \quad \text{(internal energy)}, \\ S[P] = & -\sum_{\sigma \in \Omega} P[\sigma] \log P[\sigma] \quad \text{(entropy)}. \end{cases}$$

If β (inv. temperature) is fixed, take the *generalized free energy*

$$F[P;\beta] := U[P] - \frac{1}{\beta} S[P].$$

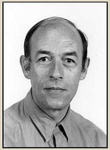
Lemma. The *Gibbs measure* $P[\sigma] := \frac{e^{-\beta \mathscr{H}_V[\sigma]}}{Z(\beta)}$ minimizes F.

OK if V is finite (albeit hard to evaluate $Z(\beta)$...). What if $V = \infty$?

The Dobrushin–Lanford–Ruelle (DLR) approach

Dobrushin 1968

Lanford-Ruelle 1969

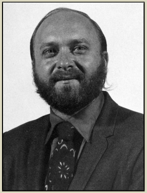


When $|V| \to \infty$ issues can arise in the hamiltonian $\mathscr{H}_V$. Gibbs measures are defined via an extension of the notion of conditional probability.

This notion is called a **specification** and will be a central ingredient of this talk.

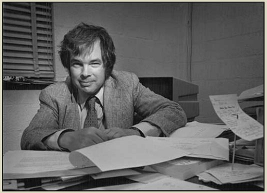
See *e.g.* Friedli-Velenik 2017

Kadanoff and Wilson: the renormalisation group



Kadanoff
1966

Wilson
1983



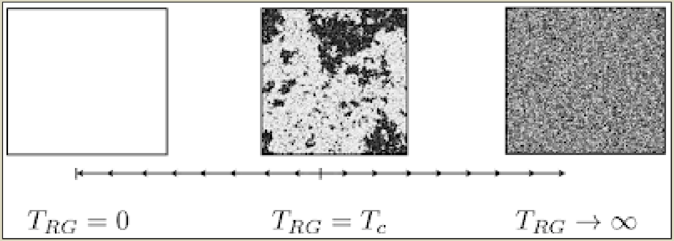
- Phase transitions
  (liquid-gas, para-ferromagnetic P. Curie 1892);
- Critical phenomena
  (critical opalescence Cagniard de Latour 1822);
- Landau Theory.

Several Gibbs measures at the critical point?
Symmetries of the system at the critical point?

See *e.g.* J. Zinn-Justin 2005

Renorm. group (RG) & critical phenomena: $d = 2$

- Before RG transformation: Invisible, original, non-primed spins.
- After RG transformation: Visible, transformed, primed spins.



Credits: D. Ashton

van Enter–Fernandez–Sokal 1993, pt. I

Works on hierarchical models (à-la-Dyson), non-trivial on lattices such as $\mathbb{Z}^d$ even for the most basic renorm. group transformations (decimation, majority rule).

Decimation provides an example for illustrating this non-triviality (Griffiths, Pearce and Israel), later interpreted as loss of Gibbs property by van Enter–Fernandez–Sokal in 1993 (EFS).

☛ EFS strategy: to show that **conditional expectation** of some microscopic variable after the transformation in a fixed site (e.g. magnetization at the origin) is **essentially discontinuous** as a **function of boundary conditions**.

van Enter–Fernandez–Sokal 1993, pt. II

Basic steps of the EFS strategy:

1. Build marginal measure on visible spins by integrating out invisible ones;
2. Conditioning wrt events on infinite sub-graphs of the lattice (model dependent, existence of a **global specification**);
3. Conditioning wrt "**bad configuration**" (model dependent), coexistence of several Gibbs measures $\rightarrow$ phase transition (even if original model does not have any!);
4. Show that the transformed/primed configuration in an annulus around a finite box is "good" and can act as a "**boundary condition**".
5. Unfix the origin: the choice of invisible phase, conditioned to all other primed/visible spins, influences the expectation of visible spin at the origin.

The XY (or planar rotator) model: basic definitions

A spin model with continuous symmetry $O(2)$ sitting on $\mathbb{Z}^2$.

- **Product measurable structure**:

$$\Omega = \left(\mathbb{S}^1\right)^{\mathbb{Z}^2}, \ \mathscr{F} = \mathscr{P}(\mathbb{S}^1)^{\otimes \mathbb{Z}^2}, \ \rho = \left(\frac{d\theta}{2\pi}\right)^{\otimes \mathbb{Z}^2}$$

- **Ferromagnetic potential**: for $J : \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^+$, $J(i,j) \geq 0$, the interaction is $\Phi = (\Phi_A)_{A \in \mathscr{S}}$. For $A = \{i,j\}$

$$\Phi_{\{i,j\}}(\vec{\sigma}) = -J(i,j)\, \vec{\sigma}_i \cdot \vec{\sigma}_j = -J(i,j)\, \left(\sigma_{i,1}\sigma_{j,1} + \sigma_{i,2}\sigma_{j,2}\right)$$

and 0 if $A \neq \{i,j\}$ (two-body potential).

- **Hamiltonian**: in a finite volume Λ with boundary condition $\vec{\omega}$

$$\mathscr{H}_{\Lambda}^{\Phi}(\vec{\sigma} \mid \vec{\omega}) \stackrel{\text{def}}{=} \sum_{A \in \mathscr{S}, A \cap \Lambda \neq \emptyset} \Phi_A(\vec{\sigma}_{\Lambda} \vec{\omega}_{\Lambda^c}). \tag{1}$$

A few milestones for the XY model

1928 Heisenberg: model with $O(3)$ symmetry

1964 Schultz–Mattis–Lieb gave the name XY

1970 Ginibre obtained correlation functions via Griffiths inequalities

1971 Berezinsky and Kosterlitz–Thouless (1973): transition from coupled (anti-) vortices when $\beta > \beta_c$ to free vortices at $\beta < \beta_c$ (Physics Nobel Prize 2016)

1977 McBryan–Spencer 1977, Fröhlich–Spencer 1981: proof of existence of these two phases (Dannie Heineman Prize 1991)

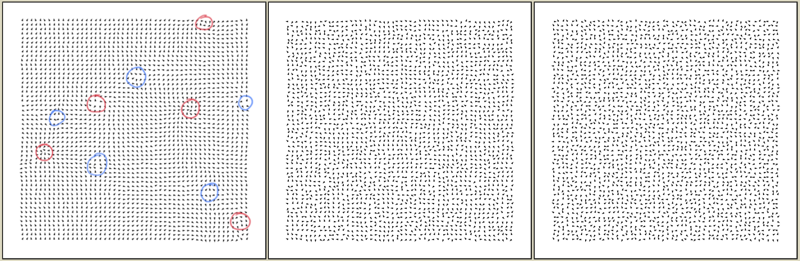
1980 Aizenmann-Simon: $\mathbb{E}^{XY}[\overrightarrow{\sigma_i} \cdot \overrightarrow{\sigma_j}]_{J,2\beta} \leq \mathbb{E}^{\text{Ising}}[\sigma_i \sigma_j]_{J,\beta}$, $\forall$ dimension d

- ...

See *e.g.* Kardar 2007, Simon 2022

Simulation of the two phases of the XY model

Numerical protocol: fixed 50×50 grid, periodic boundary conditions, $J(i,j) = J \ \forall A = \{i,j\}$:



$\beta \gg \beta_c$ $\beta \approx \beta_c$ $\beta \ll \beta_c$

Remark: Simulations delicate at $\beta \approx \beta_c$ (critical slowing down).

Topological structure of the XY model

Let $\vec{\sigma} \in \Omega$ be a generic configuration, $\vec{\sigma}_\Lambda$ the restriction on $\Lambda \subset \mathbb{Z}^2$

Angle representation of $\vec{\sigma}$: for $\vec{e_1} = (1,0)$, the angle at site i is

$$\theta_i = \theta(\vec{\sigma}_i) = (\vec{\sigma}_i, \vec{e_1}) \in]-\pi, +\pi] \, .$$

Basis of neighborhoods of $\vec{\omega}$: set of configurations roughly collinear with $\vec{\omega}$ in a finite volume Λ and arbitrary outside Λ.

For $\varepsilon_k > 0$, the family $\left(\mathscr{N}_{\Lambda,\varepsilon_k}(\vec{\omega})\right)_{\Lambda \in \mathscr{S}}$ with, $\forall \Lambda \in \mathscr{S}$,

$$\mathscr{N}_{\Lambda,\varepsilon_k}(\vec{\omega}) = \left\{ \vec{\sigma} \in \Omega : (\vec{\sigma}_i, \vec{\omega}_i) \leq \varepsilon_k, \forall i \in \Lambda; \vec{\sigma}_{\Lambda^c} \text{ arbitrary} \right\}.$$

Quasilocality

A function f is **quasilocal** if

$$\lim_{\Lambda\uparrow\mathscr{S}} \sup_{\vec{\sigma},\vec{\omega}:\vec{\sigma}_\Lambda=\vec{\omega}_\Lambda} |\, f(\vec{\omega}) - f(\vec{\sigma}) \,| = 0.$$

In any model with finite state space (e.g. **Ising model**)

Continuity $\iff$ Quasilocality.

Remark: false for XY model.

Counterexample (Georgii, van Enter–Fernández–Sokal): the characteristic function of the spin at the origin being inside some interval is quasilocal but not continuous.

Local specification (Föllmer 1975, Preston 1976)

A **local specification** is a family of probability kernels
$\gamma_\Lambda : \Omega \times \mathscr{F} \longrightarrow [0,1];\ (\omega, A) \mapsto = \gamma_\Lambda(A \mid \omega)$ s.t. $\forall \Lambda \in \mathscr{S}$:

1. $\forall$ config. $\omega \in \Omega$, $\gamma_\Lambda(\cdot|\omega)$ is a **probab. measure**;
2. $\forall$ event $A \in \mathscr{F}$, $\gamma_\Lambda(A|\cdot)$ is $\mathscr{F}_{\Lambda^c}$-measurable;
3. $\forall \omega \in \Omega$, $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$ if the event $B \in \mathscr{F}_{\Lambda^c}$ (**property**);
4. $\forall \Lambda \subset \Lambda' \in \mathscr{S}$, $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$, where (**consistency**)

$$\forall\,\text{event}\, A \in \mathscr{F}, \forall \omega \in \Omega, (\gamma_{\Lambda'}\gamma_\Lambda)(A|\omega) = \int_\Omega \gamma_\Lambda(A|\omega')\gamma_{\Lambda'}(d\omega'|\omega).$$

Action of local specification on functions and mesures:

$$\gamma_\Lambda f(\omega) := \int_\Omega f(\sigma)\gamma_\Lambda(d\sigma|\omega) = \gamma_\Lambda[f|\omega]\,,$$

$$\mu\gamma_\Lambda[f] := \int_\Omega (\gamma_\Lambda f)(\omega)d\mu(\omega) = \int_\Omega \gamma_\Lambda[f|\omega]\mu(d\omega)\,.$$

Gibbs specification, measures and the set $\mathscr{G}(\gamma)$

Let Φ be Uniformly Absolutely Convergent (UAC)

$$\forall i \in S, \qquad \sum_{i \in A, A \in \mathscr{S}} \sup_{\omega \in \Omega} |\Phi_A(\omega)| < +\infty$$

Definition. The *Gibbs specification* at inverse temperature $\beta > 0$ $\gamma^J = \gamma^{\beta\Phi}$ is defined for any finite volume Λ by

$$\gamma^J_\Lambda(d\sigma \mid \omega) = \frac{1}{Z^{\beta\Phi}_\Lambda(\omega)} e^{-\beta H^\Phi_\Lambda(\sigma|\omega)} (\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}})(d\sigma).$$

A *Gibbs measure* is any measure specified by a Gibbs specification.
$\mathscr{G}(\gamma)$: set of Gibbs measures specified by γ^J.

Graal of mathematical statistical mechanics: understand $\mathscr{G}(\gamma)$.

Gibbs $\equiv$ non-null and quasilocal

A specification γ is **quasilocal** if, $\forall \Lambda \in S$, the image of any f *via* γ is a quasilocal function :

$$f \in \mathscr{F}_{\text{loc}} \implies \gamma_\Lambda f \in \mathscr{F}_{\text{qloc}}.$$

A measure μ is quasilocal iff $\mu \in \mathscr{G}(\gamma)$, with γ a quasilocal specification.

Why should one be interested to quasilocal measures?

Theorem (Kozlov 1974, Sullivan 1973).

μ is a Gibbs measure $\iff$ μ is non-null & quasilocal.

From local to global specifications

Issue: how to perform conditioning on the outside of **infinite** sets?

➽ **Idea**: use a partial order over configs. + and monotone convergence theorem (Beppo Levi theorem).
XY model: we consider the partial order $\leq_{\sin}$:

$$\vec{\omega} \leq_{\sin} \vec{\omega}' \text{ iff } \sin\theta_i \leq \sin\theta_i',\ \forall\, i \in \mathbb{Z}^2\ ,$$

implies partial *stochastic order* between mesures

$$\mu \leq_{\sin} \mu' \quad \text{iff} \quad \forall f \text{ increasing, } \mu[f] \leq \mu'[f].$$

Remark: the configs. $\pm\frac{\pi}{2}$ $(\theta_i = \pm\frac{\pi}{2}, \forall i \in \mathbb{Z}^2)$ **are extremal** and the XY local specification is **attractive** (Griffiths inequalities).

The global specification of the XY model

Theorem (DVLI)

Let $\Gamma^+ = (\Gamma^+_S)_{S \subset \mathbb{Z}^2}$ be a family of probability kernels s.t. :

- $\Gamma^+_\Lambda(d\vec{\sigma}|\vec{\omega}) := \gamma^J_\Lambda(d\vec{\sigma}|\vec{\omega})$, $\forall \Lambda$ finite and $\forall$ config. $\vec{\omega} \in \Omega$;
- For S **infinite**, $\forall \vec{\omega} \in \Omega$, $\Gamma^+_S(d\vec{\sigma}|\vec{\omega}) := \mu_S^{+,\vec{\omega}} \otimes \delta_{\vec{\omega}_{S^c}}(d\vec{\sigma})$, where $\mu_S^{+,\vec{\omega}}$ is the weak limit obtained by fixing $\vec{+}_S \vec{\omega}_{S^c}$ on the complementary set of S, S^c :

$$\mu_S^{+,\vec{\omega}}(d\vec{\sigma}_S) := \lim_{\Delta \uparrow S} \gamma^J_\Delta(d\vec{\sigma} \mid \vec{+}_S \vec{\omega}_{S^c}).$$

Then Γ^+ is a **global specification** s.t. $\mu^+ \in \mathscr{G}(\Gamma^+)$ (*idem* for Γ^- s.t. $\mu^- \in \mathscr{G}(\Gamma^-)$).

Sketch of the proof (DVLI)

Main difficulty: in XY model the state space $E = \mathbb{S}^1$ is continuous (whereas for Ising $E = \{-1, +1\}$ is discrete).
Need to check:

- the partial (pre)-order does not break preservation of monotonicity of the kernel;
- the "candidate" kernels preserve measurability and use of monotone convergence theorem.

1) Local γ^J and candidate global Γ^+ specific. coincide on finite Λ

$$\Gamma^+_\Lambda \equiv \gamma^J_\Lambda, \qquad \forall \Lambda \subset \mathbb{Z}^2 \text{ **finite**.}$$

2) $\forall$ infinite set $S \subset \mathbb{Z}^2$ and $\forall$ finite set Λ

$$\Gamma^+_S = \Gamma^+_S \Gamma^+_\Lambda = \Gamma^+_S \gamma^J_\Lambda.$$

Sketch of the proof (DVLI) - bis

For all local function f and all finite Λ, $\forall$ boundary condition $\vec{\omega}$,

$$\Gamma_S^+(f \mid \vec{\omega}) = \int_\Omega \gamma_\Lambda^J(f \mid \vec{\sigma})\Gamma_S^+(d\vec{\sigma} \mid \vec{\omega})$$
$$\underset{\text{property}}{=} \int_\Omega \gamma_\Lambda^J(f \mid \vec{\sigma}_S \vec{\omega}_{S^c})\Gamma_S^+(d\vec{\sigma} \mid \vec{\omega}) .$$

Now, the measure $\Gamma_S^+(\cdot \mid \vec{\omega})$ is specified by the constrained specification $\gamma^{S,\vec{\omega}}$, defined $\forall$ configuration $\vec{\eta}$ by

$$\gamma_\Lambda^{S,\vec{\omega}}(\cdot \mid \vec{\eta}) := \gamma_\Lambda^J(\cdot \mid \vec{\eta}_S \vec{\omega}_{S^c}).$$

The weak limits (<u>after</u> the freezing of $\vec{\omega}$ on S^c) exists:

$$\mu_S^{+,\vec{\omega}}(\cdot) := \lim_{\Delta \uparrow S} \gamma_\Delta^J(\cdot \mid \vec{+}_S \, \vec{\omega}_{S^c}).$$

Sketch of the proof (DVLI) - tris

Thus $\forall S \subset \mathbb{Z}^2$ **infinite**, the kernels

$$\Gamma_S^+(d\vec{\sigma} \mid \vec{\omega}) := \mu_S^{+,\vec{\omega}}(d\vec{\sigma}_S) \otimes \delta_{\vec{\omega}_{S^c}}(d\vec{\sigma}_{S^c})$$

are consistent.

What is left to do: prove that

$$\mathbb{E}_{\mu^+}(f_1 f_2) = \mathbb{E}_{\mu^+}(\Gamma_{D_1}^+(f_1 \mid \cdot) f_2).$$

$\forall f_1, f_2$ * Λ_1-local and Λ_2-local, with $\Lambda_1 \subset D_1$, $\Lambda_2 \subset D_2 \setminus D_1$.

Proof: $\leq$ and $\geq$.

* f_1 and f_2 positive and increasing in the sense of $\leq_{\sin}$.

From specification to decimation over $\mathbb{Z}^2$

➽ **Idea**: transform Gibbs measure μ as $\mu \mapsto \nu = T\mu$ (only one RG step!), where T is the decimation of spacing 2, and show that ν is not quasilocal and hence non-Gibbs by Kozlov–Sullivan theorem.

Definition. For an "invisible" configuration $\vec{\omega} \in \Omega$, the **decimation transformation** is the map $T\colon (\Omega, \mathscr{F}) \longrightarrow (\Omega', \mathscr{F}') = (\Omega, \mathscr{F})$ acting on an invisible configuration $\vec{\omega}$ and returning a "visible" configuration $\vec{\omega}' \in \Omega'$ as follows:

$$\vec{\omega} \longmapsto \vec{\omega}' = (\omega_i')_{i \in \mathbb{Z}^2}, \text{ with } \omega_i' = \omega_{2i} \,.$$

Essential discontinuity

Definition. A configuration $\vec{\omega} \in \Omega$ is an **essential discontinuity** for a conditional proba μ, if the conditional expectations of a local function wrt. to two configurations coinciding in a finite box cannot be made arbitrary close.

More formally, $\exists \Lambda_0 \in \mathscr{S}$, a local function f, $\delta > 0$, s.t. $\forall \Lambda$ containing Λ_0, 2 neighborhoods of $\vec{\omega}$ $\mathscr{N}_\Lambda^1(\vec{\omega})$ and $\mathscr{N}_\Lambda^2(\vec{\omega})$ exists s.t.

$$\forall \vec{\omega}^1 \in \mathscr{N}_\Lambda^1(\vec{\omega}),\ \forall \vec{\omega}^2 \in \mathscr{N}_\Lambda^2(\vec{\omega}),$$
$$\left| \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\vec{\omega}^1) - \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\vec{\omega}^2) \right| > \delta\ ,$$

or equivalently

$$\lim_{\Delta \uparrow \mathbb{Z}^2} \sup_{\vec{\omega}^1, \vec{\omega}^2 \in \Omega} \left| \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\vec{\omega}_\Delta \omega^1_{\Delta^c}) - \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\vec{\omega}_\Delta \vec{\omega}^2_{\Delta^c}) \right| > \delta.$$

The bad configuration, and conditioning

For a special configuration $\overrightarrow{\omega}'_{\mathrm{spe}}$, for ν^+-a.e. $\overrightarrow{\omega}' \in \mathscr{N}_{\Lambda',\varepsilon}(\overrightarrow{\omega}'_{\mathrm{spe}})$

$$\nu^+[f(\overrightarrow{\sigma}')|\mathscr{F}_{\{(0,0)\}^c}](\overrightarrow{\omega}') = \Gamma_S^+[f(\overrightarrow{\sigma}')|\overrightarrow{\omega}] \;\; \mu^+\text{−a.e.}(\overrightarrow{\omega}),$$

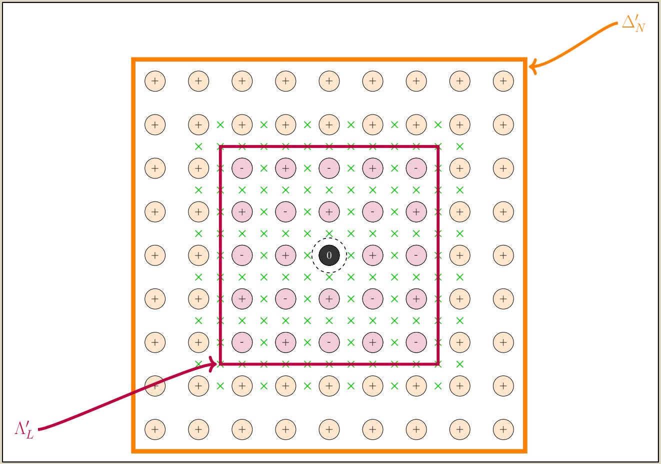
with $S = (2\mathbb{Z}^2)^c \cup \{(0,0)\}$ and $\overrightarrow{\omega} \in T^{-1}\{\overrightarrow{\omega}'\}$ which coincide with $\overrightarrow{\omega}'_{\mathrm{spe}}$ over $2\mathbb{Z}^2$. $\forall \overrightarrow{\omega}' \in \mathscr{N}_{\Lambda'}(\overrightarrow{\omega}'_{\mathrm{spe}})$,

$$\nu^+[f(\overrightarrow{\sigma}')|\mathscr{F}_{\{(0,0)\}^c}](\overrightarrow{\omega}') = \mu^{+,\overrightarrow{\omega}}_{(2\mathbb{Z}^2)^c\cup\{(0,0)\}} \otimes \delta_{\overrightarrow{\omega}_{2\mathbb{Z}^2\cap\{(0,0)\}^c}}[f(\overrightarrow{\sigma}')].$$

It is obtained as monotone weak limit with b.c. $+\frac{\pi}{2}$ fixed *after* freezing $\overrightarrow{\omega}$ on even sites: $\forall \overrightarrow{\omega}' \in \mathscr{N}_{\Lambda'}(\overrightarrow{\omega}'_{\mathrm{alt}}), \forall \overrightarrow{\omega} \in T^{-1}\{\overrightarrow{\omega}'\}$,

$$\mu^{+,\overrightarrow{\omega}}_{(2\mathbb{Z}^2)^c\cup\{(0,0)\}}(\cdot) = \lim_{\Delta\in\mathscr{S},\Delta\uparrow(2\mathbb{Z}^2)^c\cup\{0,0)\}} \gamma^J_\Delta(\cdot \mid \overrightarrow{+}_{(2\mathbb{Z}^2)^c\cup\{0,0)\}}) \,\overrightarrow{\omega}_{2\mathbb{Z}^2\cap\{0,0)\}^c}).$$

The alternating configuration $\overrightarrow{\omega}'_{\text{alt}}$



$$\overrightarrow{\omega}'_{\text{alt}} = (-1)^{i'_1 + i'_2} \overrightarrow{e_1}, \ \forall (i'_1, i'_2) \in \mathbb{Z}^2.$$

Non-Gibbsianness of ν^+ in the XY model

Theorem (DVLI)

For any XY model with ferromagnetic interaction

$$J^{iso,\alpha}(i,j) := J \cdot |i-j|^{-\alpha} \cdot \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle,$$

for $\beta > \beta_c^{J(\alpha)}$, the renormalized measure $\nu^+ = T\mu^+$ is non-quasilocal and hence non-Gibbsian $\forall\ 2 < \alpha \leq 4$.

Remark: these arguments apply also to XY on $\mathbb{Z}$ with long-range coupling with $1 < \alpha < 2$ (the case $\alpha = 2$ being more subtle).

The Mermin-Wagner theorem

The original n.n. XY model does not have breaking of $O(2)$ symmetry at low temperatures due to the **Mermin-Wagner Theorem**.

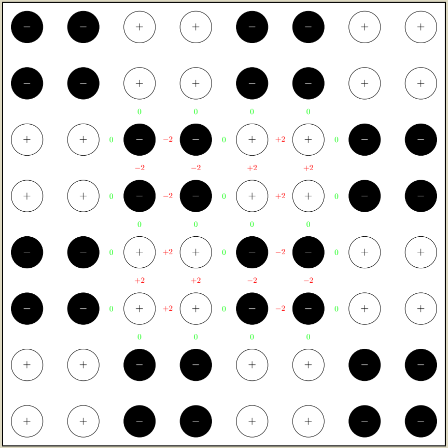
Heuristic argument (following Friedli–Velenik 2017): fix a box Λ, and homogenous b.c. (say all North). You can flip the spin at the origin at **zero price** as $|\Lambda| \to \infty$.

$\exists!$ translation invariant, extremal Gibbs measure. For all homogeneous b.c. θ, the weak limits μ^θ coincide at any temperature

$$\forall \beta > 0, \mathscr{G}(\gamma^J) = \{\mu\} .$$

Nevertheless, a special configuration exists such that the symmetry breaks for the renormalized model from $O(2) \to \mathbb{Z}_2$ at low temperatures. Such a transition is called "**spin-flop transition**".

The doubly alternating configuration $\vec{\sigma}'_{\text{dblyalt}}$



$$\vec{\sigma}'_{\text{dblyalt}} = (-1)^{\lfloor \frac{i'}{2} \rfloor + \lfloor \frac{j'}{2} \rfloor}, \quad i' j' \text{ integers}$$

Non-Gibbs at low temperatures

Theorem (DVLII)

$\vec{\sigma}'_{\mathrm{dblyalt}}$ is a bad configuration for the decimated measures $\nu = T\mu$ of the classical XY model at low enough temperatures.

Sketch of proof:
1) Zero temperature argument, $\beta = \infty$;

As in DVLI, using XY global specification provides essential discontinuity $\implies$ non-quasilocality $\implies$ non-Gibbsianness by Kozlov–Sullivan Theorem.

Sketch of proof - bis

2) Extension to $\beta < \infty$, either via *percolation of spin patterns* (Georgii 1981) or *soft contour methods* (Malyshev et al. 1983);

Along the lines of van Enter–Kulske–Opoku 2011, **for n.n. interactions.**

Heuristic argument: The renormalized Hamiltonian admits 2 different global minima (spin-flop) and satisfies a local convexity property around those 2 minima (the 2 distinct zero-temperature Gibbs measures) $\implies$ *persistence* in a neighborhood (i.e. at finite sufficiently small temperature $\beta < \infty$).

$\blacksquare$

A few provisional conclusions and perspectives

- Conditioning wrt infinite subsets requires extra care (**global specification**) if the state space is continuous!
- New source of **non-Gibbsian** examples: Mermin–Wagner thm prohibits long-range order at any β in hidden/non-primed model; **but** model becomes non-Gibbsian after one decimation step. Conditioning wrt $\vec{\sigma}'_{\mathrm{dblyalt}}$ reduces $O(2)$ symmetry to $\mathbb{Z}_2$ symmetry.

Two research directions:

☛ Extension of Malyshev theorem to long-range models by softening contours (Aernout van Enter and Arnaud Le Ny);

☛ XY model on regular d-trees and its bad configurations (after discussions with Christof Külske and Wioletta Ruszel in Berlin)

Thank you for your attention!

End of proof of Thm 1 (DVLI)

$\mathbb{E}_{\mu^+}(f_1 f_2) \geq \mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2)$

Fernandez-Pfister 1997 for Ising model + $\vec{\sigma} \leq_{\sin} \vec{\eta}_\Lambda \vec{+}_{\Lambda^c}$.

$\forall f_1, f_2$, Λ_1-local and Λ_2-local, with $\Lambda_1 \subset D_1$, $\Lambda_2 \subset D_2 \setminus D_1$:

1. $\Gamma^+_{D_1}(f_1 \mid \vec{\eta}) \underset{\text{Kernel monotonicity}}{\leq} \gamma^J_\Lambda \left(f_1(\vec{\sigma}_\Lambda) \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c} \right)$;
2. $\mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2) \leq \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_\Lambda \left(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c} \right) f_2(\vec{\eta})$;
3. Take a set Λ_2 s.t. $\Lambda_2 \cap D_1 = \Lambda$. Then

$$\int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_\Lambda \left(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c} \right) f_2(\vec{\eta}) = \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) f_1(\vec{\eta}) f_2(\vec{\eta}),$$

and hence $\mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2) \leq \mathbb{E}_{\mu^+}(f_1 f_2)$.

End of proof of Thm 1 (DVLI)

$\mathbb{E}_{\mu^+}(f_1 f_2) \leq \mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2)$

For $M \subset \Lambda_2 \cap D_1$,

$$\begin{aligned}\mathbb{E}_{\mu^+}(f_1 f_2) &\leq \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) f_1(\vec{\eta}) f_2(\vec{\eta}) = \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_M(f_1 | \vec{\eta}) f_2(\vec{\eta}) \\ &\leq \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_M(f_1 | \vec{+}_{D_1} \vec{\eta}_{D_1^c}) f_2(\vec{\eta}) \\ &= \int \mu^+(d\vec{\eta}) \gamma^J_M(f_1 | \vec{+}_{D_1} \vec{\eta}_{D_1^c}) f_2(\vec{\eta}) \,.\end{aligned}$$

The conclusion follows from Beppo Levi theorem:

$$\begin{aligned}\mathbb{E}_{\mu^+}(f_1 f_2) &\leq \lim_{M \uparrow D_1} \int \mu^+(d\vec{\eta}) f_2(\vec{\eta}) \gamma_M(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c}) \\ &= \mathbb{E}_{\mu^+}\left(f_2 \Gamma^+_{D_1}(f_1 \mid \cdot)\right)\end{aligned}$$

hence consistency is extended to **infinite** sets, $\Gamma^+_{D_1} = \Gamma^+_{D_1} \Gamma^+_{D_2}$. ■

Proof of essential discontinuity (DVLI) - detail

Let $\Lambda'_L = ([-L,+L]\cap\mathbb{Z})^2$, $\Delta'_N = ([-N,+N]\cap\mathbb{Z})^2$, with $N > L$. Then a <u>bound uniform in L</u> holds for energy differences with b.c. ω_1^+ and ω_2^+: it is enough to choose $N = N(L) = O(L^{\frac{2}{\alpha-1}})$. More precisely:

$$\delta H_L^{+,\omega'_{1/2}} := \left| H_{\Lambda,\omega_1^+}(\sigma_\Lambda) - H_{\Lambda,\omega_2^+}(\sigma_\Lambda) \right| \leq \sum_{x\in\Lambda_{2L}} 2 \sum_{k>2N} \frac{1}{k^\alpha} < C < \infty.$$

Lemma (DVLI). Let $\Lambda' \subset \Delta' \in \mathscr{S}$ and let $\omega'^+ \in \mathscr{N}^+_{\Lambda',\Delta'}(\omega'_{\text{alt}})$ and $\omega'^- \in \mathscr{N}^-_{\Lambda',\Delta'}(\omega'_{\text{alt}})$. Then $\exists\delta > 0$ and $\exists\Lambda'_0$ large enough s.t. $\Delta' \supset \Lambda' \supset \Lambda'_0$ with $\Delta'\setminus\Lambda'$ much larger than Λ', s.t. $\forall\omega^+ \in T^{-1}\{\omega'^+\}$ and $\forall\omega^- \in T^{-1}\{\omega'^-\}$,

$$\left| \mu^{+,\omega^+}_{(2\mathbb{Z}^2)^c\cup\{0\}}[\sigma_0] - \mu^{+,\omega^-}_{(2\mathbb{Z}^2)^c\cup\{0\}}[\sigma_0] \right| > \delta \qquad \text{(essential discontinuity).}$$