# Almost Gibbsian Measures on a Cayley Tree

**Seminário de Probabilidade e Mecânica Estatística, IMPA**

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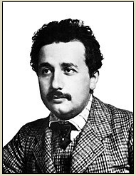
# Statistical mechanics and the ensembles of Gibbs

"to derive the laws of thermal equilibrium [...] using only the equations of mechanics and the probability calculus"

**Einstein** 1902

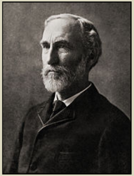
**Peliti–Rechtman** 2016



*Elementary Principles in Statistical Mechanics*

**Gibbs** 1902

**Klein** 1990



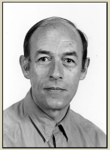
For describing the **Gibbs** (equilibrium) measure(s) of **spatially** $\infty$ lattice spin systems, **two main problems**:

- Microscopic hamiltonian is **divergent**;
- **Unicity** in phase transitions (Kolmogorov extension Thm).

# Dobrushin–Lanford–Ruelle (DLR) approach

**Dobrushin**
1968

**Lanford–Ruelle**
1969



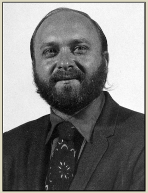
Marginal probabilities $\Longrightarrow$ **Conditional** probabilities wrt prescribed **boundary conditions**

Put on rigorous ground by **Georgii** (**Friedli–Velenik** 2017)

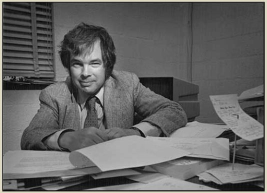
# Phase transitions and the renormalisation group

- Critical opalescence, **Cagniard de Latour** 1822
- Para–ferromagnetic transition, **Pierre Curie** 1895

**Kadanoff**
1966



**Wilson**
1983



**J. Zinn-Justin** 2005

# A few motivations

**Aim**: different **global behaviors** compatible w. given **local laws**.

**Observables** are local functions on a configuration space $(\Omega, \mathscr{F})$.

$\mathscr{F} = \mathscr{P}(\{-1, +1\})$ for Ising, a Borel $\sigma$-algebra (continuous spins)

**Physical states** are modeled by **Gibbs measures**, which are well understood if $|\Omega| < \infty$. For $\Lambda$ a **finite** subset of a lattice $\mathscr{L}$:

- **Measurability** for events supported outside $\Lambda$, which can be interpreted as **boundary conditions**;
- Concentration of the resulting measure on $\Lambda$ (**properness**);
- "Nested" conditioning (**consistency**).

The synthesis is called **local specification** (**Georgii** 1988)

# Local specifications: extended Markov chains

(**Föllmer** 1975, **Preston** 1976) A **local specification** is a family $(\gamma_\Lambda)_{\Lambda \Subset \mathscr{L}}$ of probability kernels $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ satisfying also **properness** and **consistency**.

1. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(\cdot|\omega)$ is a **probab. measure**;
2. $\forall$ event $A \in \mathscr{F}$: $\gamma_\Lambda(A|\cdot)$ is $\mathscr{F}_{\Lambda^c}$-measurable;
3. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$, $B \in \mathscr{F}_{\Lambda^c}$ (**properness**);
4. $\forall$ boxes $\Lambda \subset \Lambda'$, **finite**, $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$ (**consistency**).

See **D'A–van Enter–Le Ny** 2022a for **global** specifications for XY models

# Quasilocality

A function $f$ is **quasilocal** iff it is a limit (in the sup norm) of a sequence of **local** functions (taking a finite number of values in any finite set). Equivalently,

$$\lim_{\Lambda \uparrow \mathscr{L}} \sup_{\sigma,\omega:\sigma_\Lambda=\omega_\Lambda} |\, f(\omega) - f(\sigma) \,| = 0.$$

**Neighborhoods**: $\mathscr{N}^{\Lambda}(\sigma) = \{\omega \in \Omega$ coinciding w $\boldsymbol{\sigma}$ in $\Lambda \Subset \mathscr{L}\}$

**Remark:** In any model with finite state space (e.g. **Ising, Potts**)

Quasilocality $\iff$ (uniform) continuity

# Gibbs specification, measures, and the set $\mathscr{G}(\gamma)$

**Gibbs specification**: for $\beta > 0$, $\Lambda$ finite and a priori measure $\rho$

$$\gamma_\Lambda(d\sigma \mid \omega) \overset{\text{def}}{=} \frac{1}{Z_\Lambda^{\beta\Phi}(\omega)} e^{-\beta H_\Lambda^\Phi(\sigma|\omega)} (\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}})(d\sigma).$$

A measure $\mu$ is **specified** by (or **consistent with**) $\gamma_\Lambda$ if it satisfies the **DLR equations:**

$$\mu[A \mid \mathscr{F}_{\Lambda^c}](\sigma) = \gamma_\Lambda(A \mid \sigma), \ \mu\text{-a.e. } \sigma \in \Omega\,.$$

A **Gibbs measure** is a measure specified by a Gibbs specification.

The set of all Gibbs measures $\mathscr{G}(\gamma)$ is a **Choquet symplex** and is thus uniquely represented by a proba. on **extremal measures**

# The Kozlov–Sullivan Theorem

**Action** of a local specification on functions: for $\omega \in \Omega$,

$$\gamma_\Lambda f(\omega) = \int_\Omega f(\sigma)\gamma_\Lambda(d\sigma|\omega) = \gamma_\Lambda[f|\omega] \qquad \text{(sums for Ising)}$$

A specification is **quasilocal** if it preserves quasilocal functions:

$$f \text{ is quasilocal} \implies \gamma_\Lambda f \text{ is quasilocal}$$

$\mu$ is a Gibbs measure $\iff$ $\mu$ is specified by a **non–null** and **quasilocal** specification.

**Kozlov** 1974, **Sullivan** 1973 (see also **Barbieri** *et al.* 2021)

# Some motivations: Renormalization Group (RG)

In RG one wants to transform your Gibbs measure (decimation, majority rule...). Main **mathematical challenges**:

- Existence (**Griffiths**, **Pearce**, **Israel**);
- RG pathologies, later interpreted as loss of Gibbs property (**van Enter–Fernandez–Sokal** 1993)

☛ Kozlov–Sullivan as proxy: $\nu$ not quasilocal $\Longrightarrow$ $\nu$ non-Gibbsian

Show/measure the set of points of (ess.) **discontinuity** of **renormalized measures** ("bad configurations")

# Extensions of Gibbsianness: Almost and Weakly Gibbs

**Dobrushin** famously advocated for a restoration program.

A measure $\mu$ specified by a Gibbs specific. $\gamma$ with potential $\Phi$ is:

- **Almost** Gibbsian if $\mu(\Omega_\gamma) = 1$, where $\Omega_\gamma$ is the set of good configurations of $\gamma$;
- **Weakly** Gibbsian if $\mu(\Omega_\Phi) = 1$, where $\Omega_\Phi$ is the set on which $\Phi$ is convergent.

$$\text{Almost} \implies \text{Weakly}$$

(**Maes–Redig–van Moffaert** 1999)

# Ising model on $\mathscr{T}^k$: definition

Let $\mathscr{T}^k$ be the $(k+1)$-regular infinite tree (a.k.a. Bethe lattice)

- **Configuration space, events, a priori measure:**

$$\Omega = \{-1,1\}^{\mathscr{T}^k}, \ \mathscr{F} = [\mathscr{P}(\{-1,+1\})]^{\otimes \mathscr{T}^k}, \quad \rho = \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\otimes \mathscr{T}^k}$$

- **Ferromagnetic** potential $(\Phi_A)_{A \Subset \mathscr{T}^k}$: for all $\sigma \in \Omega$ and $J(i,j) > 0$

$$\Phi_{\{i,j\}}(\sigma) = -J(i,j)\,\sigma_i\sigma_j, \qquad \Phi_{\{k\}} = -h_k\sigma_k$$

- **Hamiltonian** in finite volume $V \Subset \mathscr{T}^k$ and **boundary condition** $\omega$:

$$H_V^{\Phi}(\sigma \mid \omega) \stackrel{\text{def}}{=} \sum_{A \Subset \mathscr{T}^k, V \cap A \neq \emptyset} \Phi_A(\sigma_V \omega_{V^c}).$$

# Ising model on $\mathscr{T}^k$: a few milestones

'74 **Preston**: proof of existence of phase transition;

'77 **Higuchi**: extremal and non-translation invariant measures;

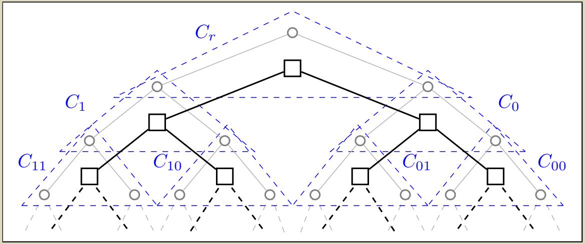
'89 **Lyons**: proof of critical inverse temperature on an arbitrary infinite tree; For $\mathscr{T}^k$ with $J = 1$, $\beta_c = \operatorname{arctanh} \frac{1}{k}$;

'90s $\rightarrow$ onwards - **Bleher–Ganikhodjaev** 90, **Akin–Rozikov–Temir** '11, **Gandolfo–Ruiz–Shlosman** '20, **Coquille–Külske–Le Ny** '23: zoology of extremal non-automorphism invariant Gibbs measures.

# The modified majority rule $T$

Here and afterwards $k=2$ ($\mathscr{T}^2$ = infinite 3-regular tree)

The majority rule $T:\Omega=\{-1,+1\}^{\mathscr{T}^2}\to\{-1,0,+1\}^{\mathscr{T}^2}=\Omega'$




$\nu = T\mu$ defined by $\nu(A')=\mu(T^{-1}A')$ $\quad\forall A'$ measurable

# Main result: almost Gibbs at all temperatures

**Theorem (4.1 in D'A–Le Ny 2022)**

The measures $\nu = T\mu$ are **almost Gibbsian** at any $\beta$.

# Plan of the proof

: **Coupling** with $\beta$-**dependent percolation** of **zeros**.

Four major steps:

1. Magnetization at $r$ (ess.) continuous if $0$s do not percolate;
2. Detailed analysis for a **single path** of $0$s;
3. **Growth estimate** for the # of percolating paths of $0$s;
4. **Upper bound** (by zero) on the measure of **bad configs**.

# A few definitions

Consider $\mathscr{T}_0^2 = \infty$ **binary tree** rooted at $r$ (**Bleher–Ganikhodjaev** 90) in binary representation, for which $\Omega_0' = \{-1, 0, +1\}^{\mathscr{T}_0^2}$.

A **path of** $0$**s** in $\eta'$ is a seq. of *n.n.* $0$ (primed) spins starting at $r$.

- $N_R(\eta')$= $\#\{$paths of $0$s in $\eta' \in \Omega_0'$ reaching depth $R\}$;
- $N(\eta') = \lim_{R\to\infty} N_R(\eta') = \#\{\infty$ paths of $0$s in $\eta' \in \Omega_0'\}$.

If $N(\eta') \neq 0$ we say that there is **percolation of** $0$**s**.

**Quasilocal function**: $\langle \sigma_r' \rangle^{\eta', R} = \nu[\sigma_r' \,|\, \sigma_{\{r\}^c}' = \omega_{\{r\}^c}', \; \omega' \in \mathscr{N}^R(\eta')]$

# 1. Magnetization ess. cont. if 0s do not percolate

**Le Ny** 2000 proved that $\eta' = 0_{\mathscr{T}^2}$ is a **bad configuration**.

By Kozlov–Sullivan $\nu$ are **non-Gibbsian** at any temperature $\beta$.

$\eta' = 0_{\mathscr{T}^2}$ (and similar configs) are **quite unlikely** under $\nu$.

: start from very “few” 0s and control the growth in $R$.

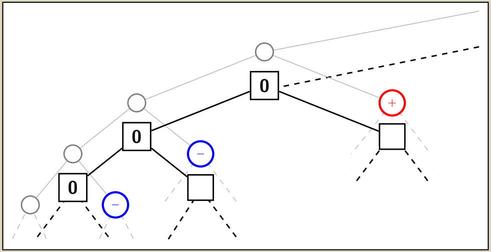
Consider absence of percolation. Then

$$N(\eta') = 0 \implies \langle \sigma_r' \rangle^{\eta',R} \text{ is (ess.) cont. as a function of } \eta'.$$

**Idea of proof**: $\langle \sigma_r' \rangle^{\eta',R}$ is actually **independent** of $R$ for $R$ large enough (i.e. larger than $R_0 = \max_{\sigma_i' \in \eta' \text{s.t.} \sigma_i' = 0} \operatorname{dist}_{\mathscr{T}_0^2}(i, r)$ )

# 2. Detailed analysis of $N(\eta') = 1$

Let $\eta' \in \Omega'_0$ be such that $N(\eta') = 1$.*




Let $Y'$ be the **projection** of $\eta'$ onto the $\infty$ path and $Y = T^{-1}(Y')$.

Define $X_n := Y_{R-n+1}$ for $n \leq R$. Then $X$ is an explicit **inhomogeneous Markov chain** (possibly with some forbidden transition)

*Except a few ($\nu$-negligible) peculiar configurations

# 2. Detailed analysis of $N(\eta')=1$ - bis

The law of $X$ within $\eta'$ is the one of a **1-d Ising model** in an inhomogeneous external field $h(\eta')=(h(\eta')_n)_{n\in\mathbb{N}}$.

This can be dealt via **transfer matrices** ... 

... after some work, it turns out that:

- $\langle \sigma_r' \rangle^{\eta',R}$ is (ess.) cont. at $\eta'\in\Omega_0'$ if $N(\eta')=1$;
- For every $R\geq 0$, $\exists \mathrm{C}>0$ (indep. on such $\eta'$) s.t.

$$|\langle \sigma_r' \rangle^{\omega_1,R} - \langle \sigma_r' \rangle^{\omega_2,R}| \leq \mathrm{C}\cdot \left(e^{-\beta}\right)^R \qquad \forall \omega_1,\omega_2\in\mathcal{N}^R(\eta').$$

# 3. From 1 to a finite number of paths of 0s

Let now $N_R(\eta') = 2$. We can put $r$ at the unique common ancestor of the paths of 0s and use the **Markov property** of $\mu$ to get

$$\forall R > 0, \left| \langle \sigma'_r \rangle^{\omega'_1,R} - \langle \sigma'_r \rangle^{\omega'_2,R} \right| \leq p(\beta) \left| \langle \sigma'_{r1} \rangle^{\omega'_1,R} \langle \sigma'_{r0} \rangle^{\omega'_1,R} - \langle \sigma'_{r1} \rangle^{\omega'_2,R} \langle \sigma'_{r0} \rangle^{\omega'_2,R} \right|$$

for some $p(\beta) \in [0,1]$ depending only on $\beta$.

Now apply the following elementary inequality at RHS:

$$|xy - wz| \leq |x - w| + |y - z|, \qquad \forall x,y,w,z \in [0,1].$$

For $N_R > 2$ (finite) we can proceed by iteration.

# 3. Growth estimate for the # of percolating paths of 0s

**Bottom line**: for a finite # of percolating paths of 0s

$$\forall R > 0, \sup_{\omega_1', \omega_2' \in \mathcal{N}^R(\eta')} \left| \langle \sigma_r' \rangle^{\omega_1', R} - \langle \sigma_r' \rangle^{\omega_2', R} \right| \leq C_2 \cdot N_R(\eta') \cdot \left( e^{-\beta} \right)^R$$

This result suggests that everything is fine for configs. whose number of 0s grows at most as $e^{\beta R}$ in the depth $R$.

$$\Omega_g = \left\{ \eta' \in \Omega_0 \,:\, \lim_{R \to \infty} \frac{N_R}{e^{\beta R}} = 0 \right\}$$

# 4. Upper bound on the measure of bad configurations

**Lemma (4.5 in D'A–Le Ny 2022)**

$$\nu(\Omega_g) = 1.$$

*Proof.* First we prove i) $\lim_{R\to\infty} \frac{\mathbb{E}_\nu[N_R]}{e^{\beta R}} = 0.$

$$\begin{aligned}\mathbb{E}_\nu[N_R \mid \mathscr{F}_{R-1}] &= p^2(N_{R-1}+1) + 2p(1-p)N_{R-1} + (1-p)^2(N_{R-1}-1) \\ &= N_{R-1} + (2p-1)\end{aligned}$$

where $p = p(\beta)$ is an (explicit) bond percolation probability.
i) follows by induction.

# 4. Upper bound on the measure of bad configurations

Second we show that ii) $\forall\theta \geq 0,\ \nu[N_R(\eta') > e^{\beta R}] \leq e^{\theta(R - e^{R\beta})}$.

ii) follows using the same recurrence, then bounding the MGF $\mathbb{E}_\nu[e^{\theta N_r(\eta')}]$ **uniformly** in $\theta$ and exponential Chebyshev inequality. This proves the statement. ∎

In order to conclude the proof of the main Theorem, we show that those $\eta' \in \Omega_g$ **having no infinite alternating external fields** $h$ around their paths of $0$s **are also of full** $\nu$ **measure**. The growth estimate applied to such configurations concludes the proof.

∎

# Conclusions

The renormalized measure $\nu$ obtained by acting with the majority rule $T$ on the Gibbs measure $\mu$ of the Ising model on $\mathscr{T}^2$ was known to be non-Gibbsian.

By studying the problem with a $\beta$ **dependent percolation model**, we have **proved** that the set of **bad configurations** is $\nu$-negligble, rendering $\nu$ **almost Gibbsian** (hence weakly Gibbsian) at **all temperatures.**

Our result provides a neat example in which the **Dobrushin restoration program** turned out to be a **rich source** of mathematical work (already for one single RG step!)

# Three perspectives

- ☛ $k$ vs $\beta$ tradeoff in the percolation model;
- ☛ Other choices for the majority rule (size of cell, inhomogeneity);
- ☛ Study a **stochastic version** of the majority rule.

# Obrigado!

# Percolation probability $p(\beta)$

Look at cell $j$ and use the law of total expectation:

$$\nu[\eta'_{j1}=0]=\sum_{x\in\{-,0,+\}}\nu[\eta'_{j1}=0\mid\eta'_j=x]\cdot\nu[\eta'_j=x]\,.$$

Then **evaluate explicitly** each cond. prob.in terms of the measure $\mu$ (**Ising model on the complete graph** $K_3$). It turns out that those cond. prob. all **are equal**. Thus, despite the primed spin being dependent (cells overlap!), the three considered events are actually **uncorrelated**.

We get the marginal probability

$$\nu[\eta'_j=0]=\frac{2+e^{-\beta}}{(e^{\beta}+e^{-\beta})^2}:=p(\beta),\qquad\forall j\in\mathscr{T}_0^2\,.$$

# Essential discontinuity

## Detailed definition

A configuration $\omega \in \Omega$ is an **essential discontinuity** for a conditional proba $\mu$, if $\exists \Lambda_0 \Subset \mathscr{L}$, a local function $f$, and a real $\delta > 0$, s.t. $\forall \Lambda$ containing $\Lambda_0$, 2 neighborhoods of $\omega$ $\mathscr{N}_\Lambda^1(\omega)$ and $\mathscr{N}_\Lambda^2(\omega)$ exists s.t.

$$\forall \omega^1 \in \mathscr{N}_\Lambda^1(\omega),\ \forall \omega^2 \in \mathscr{N}_\Lambda^2(\omega),$$
$$\left| \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\omega^1) - \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\omega^2) \right| > \delta\ .$$

Equivalently:

$$\lim_{\Delta \uparrow \mathscr{L}} \sup_{\omega^1, \omega^2 \in \Omega} \left| \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\omega_\Delta \omega^1_{\Delta^c}) - \mu\left[f|\mathscr{F}_{\Lambda^c}\right](\omega_\Delta \omega^2_{\Delta^c}) \right| > \delta.$$