Almost Gibbsian Measures on a Cayley Tree

Seminário de Probabilidade e Mecânica Estatística, IMPA

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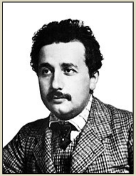
Statistical mechanics and the ensembles of Gibbs

"to derive the laws of thermal equilibrium [...] using only the equations of mechanics and the probability calculus"

Einstein 1902

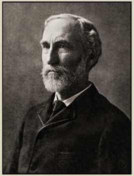
Peliti–Rechtman 2016



Elementary Principles in Statistical Mechanics

Gibbs 1902

Klein 1990



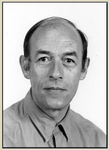
For describing the **Gibbs** (equilibrium) measure(s) of **spatially** ∞ lattice spin systems, **two main problems**:

- Microscopic hamiltonian is **divergent**;
- **Unicity** in phase transitions (Kolmogorov extension Thm).

Dobrushin–Lanford–Ruelle (DLR) approach

Dobrushin
1968

Lanford–
Ruelle
1969



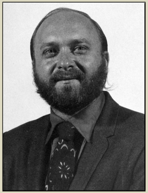
Marginal probabilities $\Longrightarrow$ **Conditional** probabilities wrt prescribed **boundary conditions**

Put on rigorous ground by **Georgii** (**Friedli–Velenik** 2017)

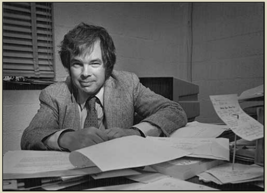
Phase transitions and the renormalisation group

- Critical opalescence, **Cagniard de Latour** 1822
- Para–ferromagnetic transition, **Pierre Curie** 1895

Kadanoff
1966



Wilson
1983



J. Zinn-Justin 2005

A few motivations

Aim: different **global behaviors** compatible w. given **local laws**.

Observables are local functions on a configuration space $(\Omega, \mathscr{F})$.

$\mathscr{F} = \mathscr{P}(\{-1, +1\})$ for Ising, a Borel σ-algebra (continuous spins)

Physical states are modeled by **Gibbs measures**, which are well understood if $|\Omega| < \infty$. For Λ a **finite** subset of a lattice $\mathscr{L}$:

- **Measurability** for events supported outside Λ, which can be interpreted as **boundary conditions**;
- Concentration of the resulting measure on Λ (**properness**);
- "Nested" conditioning (**consistency**).

The synthesis is called **local specification** (**Georgii** 1988)

Local specifications: extended Markov chains

(**Föllmer** 1975, **Preston** 1976) A **local specification** is a family $(\gamma_\Lambda)_{\Lambda \Subset \mathscr{L}}$ of probability kernels $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ satisfying also **properness** and **consistency**.

1. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(\cdot|\omega)$ is a **probab. measure**;
2. $\forall$ event $A \in \mathscr{F}$: $\gamma_\Lambda(A|\cdot)$ is $\mathscr{F}_{\Lambda^c}$-measurable;
3. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$, $B \in \mathscr{F}_{\Lambda^c}$ (**properness**);
4. $\forall$ boxes $\Lambda \subset \Lambda'$, **finite**, $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$ (**consistency**).

See **D'A–van Enter–Le Ny** 2022a for **global** specifications for XY models

Quasilocality

A function f is **quasilocal** iff it is a limit (in the sup norm) of a sequence of **local** functions (taking a finite number of values in any finite set). Equivalently,

$$\lim_{\Lambda \uparrow \mathscr{L}} \sup_{\sigma,\omega:\sigma_\Lambda=\omega_\Lambda} |\, f(\omega) - f(\sigma)\, | = 0.$$

Neighborhoods: $\mathscr{N}^{\Lambda}(\sigma) = \{\omega \in \Omega$ coinciding w $\boldsymbol{\sigma}$ in $\Lambda \Subset \mathscr{L}\}$

Remark: In any model with finite state space (e.g. **Ising, Potts**)

Quasilocality $\iff$ (uniform) continuity

Gibbs specification, measures, and the set $\mathscr{G}(\gamma)$

Gibbs specification: for $\beta > 0$, Λ finite and a priori measure ρ

$$\gamma_\Lambda(d\sigma \mid \omega) \stackrel{\text{def}}{=} \frac{1}{Z_\Lambda^{\beta\Phi}(\omega)} e^{-\beta H_\Lambda^\Phi(\sigma|\omega)} (\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}})(d\sigma).$$

A measure μ is **specified** by (or **consistent with**) γ_Λ if it satisfies the **DLR equations:**

$$\mu[A \mid \mathscr{F}_{\Lambda^c}](\sigma) = \gamma_\Lambda(A \mid \sigma),\ \mu\text{-a.e. } \sigma \in \Omega\,.$$

A **Gibbs measure** is a measure specified by a Gibbs specification.

The set of all Gibbs measures $\mathscr{G}(\gamma)$ is a **Choquet symplex** and is thus uniquely represented by a proba. on **extremal measures**

The Kozlov–Sullivan Theorem

Action of a local specification on functions: for $\omega \in \Omega$,

$$\gamma_\Lambda f(\omega) = \int_\Omega f(\sigma)\gamma_\Lambda(d\sigma|\omega) = \gamma_\Lambda[f|\omega] \qquad \text{(sums for Ising)}$$

A specification is **quasilocal** if it preserves quasilocal functions:

$$f \text{ is quasilocal} \implies \gamma_\Lambda f \text{ is quasilocal}$$

μ is a Gibbs measure $\iff$ μ is specified by a **non–null** and **quasilocal** specification.

Kozlov 1974, **Sullivan** 1973 (see also **Barbieri** *et al.* 2021)

Some motivations: Renormalization Group (RG)

In RG one wants to transform your Gibbs measure (decimation, majority rule...). Main **mathematical challenges**:

- Existence (**Griffiths**, **Pearce**, **Israel**);
- RG pathologies, later interpreted as loss of Gibbs property (**van Enter–Fernandez–Sokal** 1993)

☛ Kozlov–Sullivan as proxy: ν not quasilocal $\Longrightarrow$ ν non-Gibbsian

Show/measure the set of points of (ess.) **discontinuity** of **renormalized measures** ("bad configurations")

Extensions of Gibbsianness: Almost and Weakly Gibbs

Dobrushin famously advocated for a restoration program.

A measure μ specified by a Gibbs specific. γ with potential Φ is:

- **Almost** Gibbsian if $\mu(\Omega_\gamma) = 1$, where Ω_γ is the set of good configurations of γ;
- **Weakly** Gibbsian if $\mu(\Omega_\Phi) = 1$, where Ω_Φ is the set on which Φ is convergent.

$$\text{Almost} \implies \text{Weakly}$$

(**Maes–Redig–van Moffaert** 1999)

Ising model on $\mathscr{T}^k$: definition

Let $\mathscr{T}^k$ be the $(k+1)$-regular infinite tree (a.k.a. Bethe lattice)

- **Configuration space, events, a priori measure:**

$$\Omega = \{-1,1\}^{\mathscr{T}^k}, \mathscr{F} = [\mathscr{P}(\{-1,+1\})]^{\otimes \mathscr{T}^k}, \quad \rho = \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}\right)^{\otimes \mathscr{T}^k}$$

- **Ferromagnetic** potential $(\Phi_A)_{A \Subset \mathscr{T}^k}$: for all $\sigma \in \Omega$ and $J(i,j) > 0$

$$\Phi_{\{i,j\}}(\sigma) = -J(i,j)\,\sigma_i\sigma_j, \qquad \Phi_{\{k\}} = -h_k\sigma_k$$

- **Hamiltonian** in finite volume $V \Subset \mathscr{T}^k$ and **boundary condition** ω:

$$H_V^{\Phi}(\sigma \mid \omega) \stackrel{\text{def}}{=} \sum_{A \Subset \mathscr{T}^k, V \cap A \neq \emptyset} \Phi_A(\sigma_V \omega_{V^c}).$$

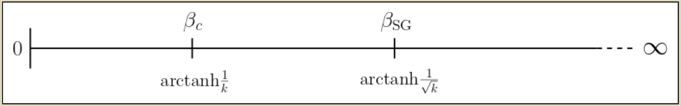
Ising model on $\mathscr{T}^k$: a few milestones

'74 **Preston**: proof of existence of phase transition;

'77 **Higuchi**: extremal and non-translation invariant measures;

'89 **Lyons**: proof of critical inverse temperature on an arbitrary infinite tree; For $\mathscr{T}^k$ with $J = 1$, $\beta_c = \operatorname{arctanh} \frac{1}{k}$;

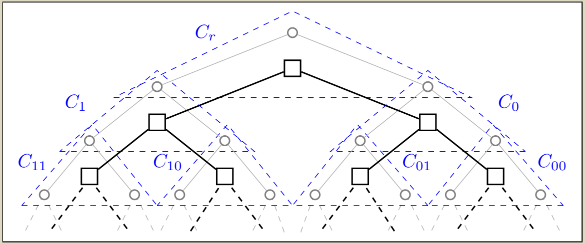
'90s $\to$ onwards - **Bleher–Ganikhodjaev** 90, **Akin–Rozikov–Temir** '11, **Gandolfo–Ruiz–Shlosman** '20, **Coquille–Külske–Le Ny** '23: zoology of extremal non-automorphism invariant Gibbs measures.



The modified majority rule T

Here and afterwards $k = 2$ ($\mathscr{T}^2$ = infinite 3-regular tree)

The majority rule $T : \Omega = \{-1,+1\}^{\mathscr{T}^2} \to \{-1,0,+1\}^{\mathscr{T}^2} = \Omega'$



$\nu = T\mu$ defined by $\nu(A') = \mu(T^{-1}A')$ $\forall A'$ measurable

Main result: almost Gibbs at all temperatures

Theorem (4.1 in D'A–Le Ny 2022)

The measures $\nu = T\mu$ are **almost Gibbsian** at any β.

Plan of the proof

: **Coupling** with β-**dependent percolation** of **zeros**.

Four major steps:

1. Magnetization at r (ess.) continuous if 0s do not percolate;
2. Detailed analysis for a **single path** of 0s;
3. **Growth estimate** for the # of percolating paths of 0s;
4. **Upper bound** (by zero) on the measure of **bad configs**.

A few definitions

Consider $\mathscr{T}_0^2 = \infty$ **binary tree** rooted at r (**Bleher–Ganikhodjaev** 90) in binary representation, for which $\Omega_0' = \{-1, 0, +1\}^{\mathscr{T}_0^2}$.

A **path of** 0**s** in η' is a seq. of *n.n.* 0 (primed) spins starting at r.

- $N_R(\eta')$= #{paths of 0s in $\eta' \in \Omega_0'$ reaching depth R};
- $N(\eta') = \lim_{R\to\infty} N_R(\eta') =$ #{∞ paths of 0s in $\eta' \in \Omega_0'$}.

If $N(\eta') \neq 0$ we say that there is **percolation of** 0**s**.

Quasilocal function: $\langle \sigma_r' \rangle^{\eta', R} = \nu[\sigma_r' \,|\, \sigma_{\{r\}^c}' = \omega_{\{r\}^c}',\, \omega' \in \mathscr{N}^R(\eta')]$

1. Magnetization ess. cont. if 0s do not percolate

☛ Consider first absence of percolation. Then

$$N(\eta') = 0 \implies \langle \sigma_r' \rangle^{\eta',R} \text{ is (ess.) cont. as a function of } \eta'.$$

Idea of proof: $\langle \sigma_r' \rangle^{\eta',R}$ is actually **independent** of R for R large enough (i.e. larger than $R_0 = \max_{\sigma_i' \in \eta' \text{s.t.} \sigma_i' = 0} \mathrm{dist}_{\mathscr{T}_0^2}(i,r)$)

Le Ny 2000 proved that $\eta' = 0_{\mathscr{T}^2}$ is a **bad configuration**.

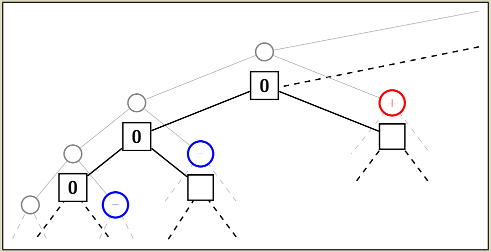
By Kozlov–Sullivan ν are **non-Gibbsian** at any temperature β.

$\eta' = 0_{\mathscr{T}^2}$ (and similar configs) are **quite unlikely** under ν.

💡: start from very “few” 0s and control the growth in R.

2. Detailed analysis of $N(\eta') = 1$

Let $\eta' \in \Omega_0'$ be such that $N(\eta') = 1$.*



Let Y' be the **projection** of η' onto the ∞ path and $Y = T^{-1}(Y')$.

Define $X_n := Y_{R-n+1}$ for $n \leq R$. Then X is an explicit **inhomogeneous Markov chain** (possibly with some forbidden transition)

*Except a few (ν-negligible) peculiar configurations

2. Detailed analysis of $N(\eta') = 1$ - bis

The law of X within η' is the one of a **1-d Ising model** in an inhomogeneous external field $h(\eta') = (h(\eta')_n)_{n\in\mathbb{N}}$.

This can be dealt via **transfer matrices** ... 

... after some work, it turns out that:

- $\langle \sigma_r' \rangle^{\eta',R}$ is (ess.) cont. at $\eta' \in \Omega_0'$ if $N(\eta') = 1$;
- For every $R \geq 0$, $\exists \mathrm{C} > 0$ (indep. on such η') s.t.

$$|\langle \sigma_r' \rangle^{\omega_1,R} - \langle \sigma_r' \rangle^{\omega_2,R}| \leq \mathrm{C} \cdot \left(e^{-\beta}\right)^R \qquad \forall \omega_1, \omega_2 \in \mathscr{N}^R(\eta').$$

3. From 1 to a finite number of paths of 0s

Let now $N_R(\eta') = 2$. We can put r at the unique common ancestor of the paths of 0s and use the **Markov property** of μ to get

$$\forall R > 0, \left|\langle\sigma'_r\rangle^{\omega'_1,R} - \langle\sigma'_r\rangle^{\omega'_2,R}\right| \leq p(\beta)\left|\langle\sigma'_{r1}\rangle^{\omega'_1,R}\langle\sigma'_{r0}\rangle^{\omega'_1,R} - \langle\sigma'_{r1}\rangle^{\omega'_2,R}\langle\sigma'_{r0}\rangle^{\omega'_2,R}\right|$$

for some $p(\beta) \in [0,1]$ depending only on β.

Now apply the following elementary inequality at RHS:

$$|xy - wz| \leq |x - w| + |y - z|, \qquad \forall x, y, w, z \in [0,1].$$

For $N_R > 2$ (finite) we can proceed by iteration.

3. Growth estimate for the # of percolating paths of 0s

Bottom line: for a finite # of percolating paths of 0s

$$\forall R > 0, \sup_{\omega_1', \omega_2' \in \mathscr{N}^R(\eta')} \left| \langle \sigma_r' \rangle^{\omega_1', R} - \langle \sigma_r' \rangle^{\omega_2', R} \right| \leq C_2 \cdot N_R(\eta') \cdot \left(e^{-\beta} \right)^R$$

This result suggests that everything is fine for configs. whose number of 0s grows at most as $e^{\beta R}$ in the depth R.

$$\Omega_g = \left\{ \eta' \in \Omega_0 \; : \; \lim_{R \to \infty} \frac{N_R}{e^{\beta R}} = 0 \right\}$$

4. Upper bound on the measure of bad configurations

Lemma (4.5 in D'A–Le Ny 2022)

$$\nu(\Omega_g) = 1.$$

Proof. First we prove i) $\lim_{R\to\infty} \frac{\mathbb{E}_\nu[N_R]}{e^{\beta R}} = 0$.

$$\begin{aligned}\mathbb{E}_\nu[N_R \mid \mathscr{F}_{R-1}] &= p^2(N_{R-1}+1) + 2p(1-p)N_{R-1} + (1-p)^2(N_{R-1}-1) \\ &= N_{R-1} + (2p-1)\end{aligned}$$

where $p = p(\beta)$ is an (explicit) bond percolation probability.
i) follows by induction.

4. Upper bound on the measure of bad configurations

Second we show that ii) $\forall \theta \geq 0,\ \nu[N_R(\eta') > e^{\beta R}] \leq e^{\theta(R - e^{R\beta})}$.

ii) follows using the same recurrence, then bounding the MGF $\mathbb{E}_\nu[e^{\theta N_r(\eta')}]$ **uniformly** in θ and exponential Chebyshev inequality. This proves the statement. ■

In order to conclude the proof of the main Theorem, we show that those $\eta' \in \Omega_g$ **having no infinite alternating external fields** h around their paths of 0s **are also of full** ν **measure**. The growth estimate applied to such configurations concludes the proof.

■

Conclusions

The renormalized measure ν obtained by acting with the majority rule T on the Gibbs measure μ of the Ising model on $\mathscr{T}^2$ was known to be non-Gibbsian.

By studying the problem with a β **dependent percolation model**, we have **proved** that the set of **bad configurations** is ν-negligible, rendering ν **almost Gibbsian** (hence weakly Gibbsian) at **all temperatures.**

Our result provides a neat example in which the **Dobrushin restoration program** turned out to be a **rich source** of mathematical work (already for one single RG step!)

Three perspectives

- ☛ k vs β tradeoff in the percolation model;
- ☛ Other choices for the majority rule (size of cell, inhomogeneity);
- ☛ Study a **stochastic version** of the majority rule.

Obrigado!

Percolation probability $p(\beta)$

Look at cell j and use the law of total expectation:

$$\nu[\eta'_{j1} = 0] = \sum_{x \in \{-,0,+\}} \nu[\eta'_{j1} = 0 \mid \eta'_j = x] \cdot \nu[\eta'_j = x] \,.$$

Then **evaluate explicitly** each cond. prob.in terms of the measure μ (**Ising model on the complete graph** K_3). It turns out that those cond. prob. all **are equal**. Thus, despite the primed spin being dependent (cells overlap!), the three considered events are actually **uncorrelated**.

We get the marginal probability

$$\nu[\eta'_j = 0] = \frac{2 + e^{-\beta}}{(e^{\beta} + e^{-\beta})^2} := p(\beta), \qquad \forall j \in \mathscr{T}_0^2 \,.$$

Essential discontinuity

Detailed definition

A configuration $\omega \in \Omega$ is an **essential discontinuity** for a conditional proba μ, if $\exists \Lambda_0 \Subset \mathscr{L}$, a local function f, and a real $\delta > 0$, s.t. $\forall \Lambda$ containing Λ_0, 2 neighborhoods of ω $\mathscr{N}_\Lambda^1(\omega)$ and $\mathscr{N}_\Lambda^2(\omega)$ exists s.t.

$$\forall \omega^1 \in \mathscr{N}_\Lambda^1(\omega),\ \forall \omega^2 \in \mathscr{N}_\Lambda^2(\omega),$$
$$\left| \mu\left[f | \mathscr{F}_{\Lambda^c}\right](\omega^1) - \mu\left[f | \mathscr{F}_{\Lambda^c}\right](\omega^2) \right| > \delta\ .$$

Equivalently:

$$\lim_{\Delta \uparrow \mathscr{L}} \sup_{\omega^1, \omega^2 \in \Omega} \left| \mu\left[f | \mathscr{F}_{\Lambda^c}\right](\omega_\Delta \omega^1_{\Delta^c}) - \mu\left[f | \mathscr{F}_{\Lambda^c}\right](\omega_\Delta \omega^2_{\Delta^c}) \right| > \delta.$$