

Ideal Poisson–Voronoi tessellations on hyperbolic spaces

Journée Cartes à l'IRIF, Université Paris-Cité

28th June 2023, 2:15 PM (Paris Time)

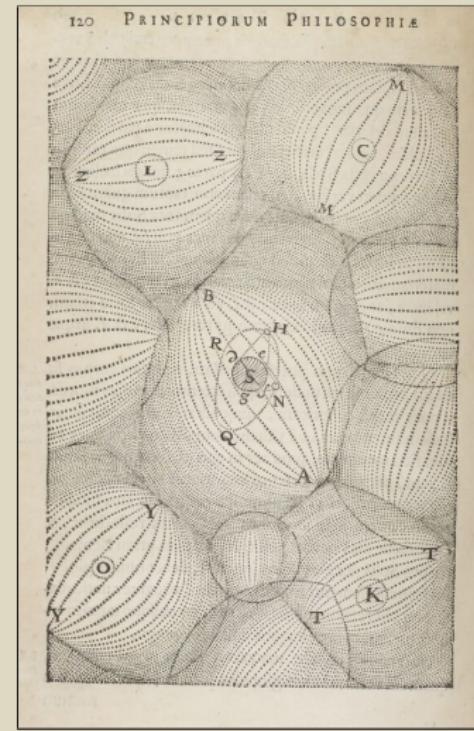
Matteo D'Achille



joint with N. Curien, N. Enriquez, R. Lyons, M. Ünel (2303.16831)

Voronoi tessellations: a classic construction

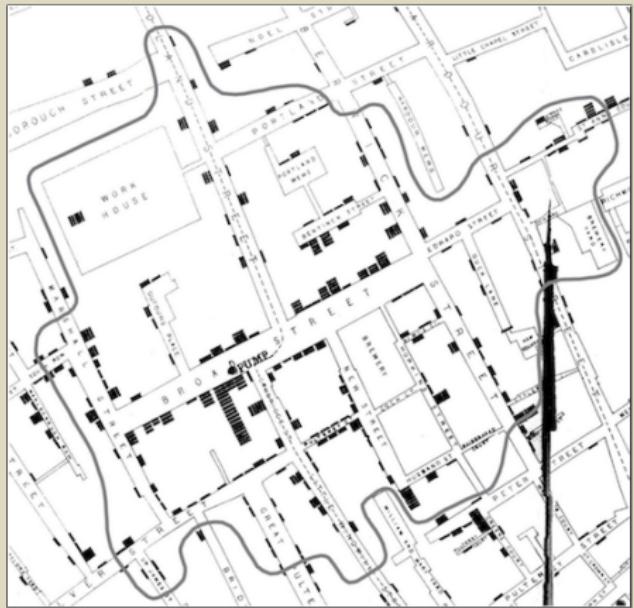
*Principia
Philosophiæ
Descartes 1644*



Voronoi tessellations: a classic construction

Water pump neighborhood
in London's Broad Street
cholera outbreak of 1854

John Snow 1855



Voronoi tessellations: a classic construction

Growth models in
morphogenesis

Gómez-Gálvez *et al.* 2018



Voronoi tessellations: basic definition

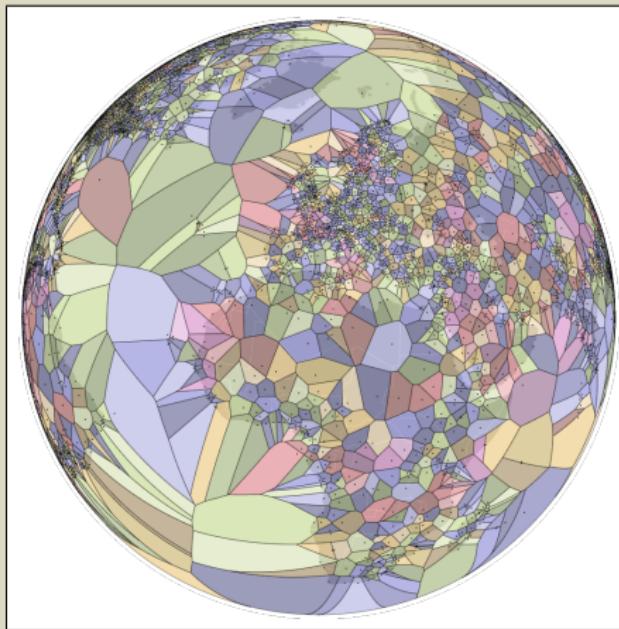
One wants to compare **distances**.

- A metric **space** (E, d)
- **Nuclei**: a list of points $(x_i ; i \geq 1) \in E$
- **Tiles**:

$$C_i := \{w \in E ; d(w, x_i) \leq d(w, x_j), \forall j \geq 1\}.$$

The **Voronoi tessellation** is $\text{Vor}(x_i : i \geq 1) = \{C_i ; i \geq 1\}$

Voronoi airports



jasondavies.com/maps/voronoi/airports/

Poisson Point Processes

(See e.g. Kingman 1992)

In applications (giraffe's fur, wireless networks) it is natural to assume some **complete randomness** for the nuclei. A natural candidate is a **Poisson Point Process (PPP)** of nuclei.

Def. A **PPP** on a measurable space S is a random countable subset $\Pi \subset S$ s.t. the random variable $N(A) = \# \{A \cap \Pi\}$ satisfies:

- $\forall A \subset S, N(A) \sim \text{Pois}(\mu(A))$, for a measure μ (**intensity**);
- If $A, B \subset S$ are **disjoint** sets, $N(A) \perp N(B)$.

Poisson–Voronoi tessellations

(See e.g. Møller 1992)

Def. A **Poisson–Voronoi tessellation** is the Voronoi tessellation of a PPP of **nuclei** with intensity measure μ .



In this talk the **PPP** of nuclei is **homogeneous**
 $\mu \propto \text{Leb}_{\mathbb{R}^2}, \text{Vol}_M \dots$ with some λ called **intensity**

Appearances of Poisson–Voronoi tessellations

☞: general case - s: λ small - L: λ large

☞ Euclidean space motivated by crystal growth (Meijering 1953, Gilbert 1961)

☞ Hyperbolic plane (with Bernoulli percolation on cells) (Benjamini–Schramm 2000)

s Mean characteristics in hyperbolic plane and 3-space (Isokawa 2000)

s Bernoulli–Voronoi percolation on regular trees, hyperbolic plane (Bhupatiraju 2019)

L “Euclidean” limit on surfaces, Gauss–Bonnet Theorem (Calka–Chapron–Enriquez 2019)

s, L Critical probabilities in percolation on the hyperbolic plane (Hansen–Müller 2021)

“Pointless” Poisson–Voronoi tessellation

(Budzinski–Curien–Petri 2022)

\mathcal{S}_g : a closed hyperbolic surface of genus g (large)

Cheeger's constant:

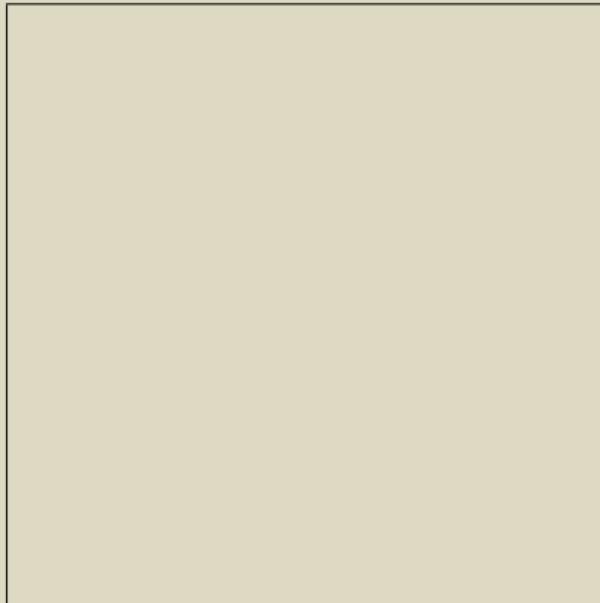
$$h(\mathcal{S}_g) \stackrel{\text{def}}{=} \inf_{A \subset \mathcal{S}_g} \left\{ \frac{\text{Perim}(A)}{\text{Area}(A)} ; \ 2 \text{Area}(A) \leq \text{Area}(\mathcal{S}_g) \right\}$$

: $\lim_{\lambda \downarrow 0}$ of Poisson–Voronoi tessellation of **hyperbolic plane**

?

What happens as $\lambda \downarrow 0$?

Euclidean plane \mathbb{R}^2



Ideal tessellations

Sketch of recipe

Tenet: one needs to compare **differences** of distances to nuclei.

Two main deterministic **ingredients** as $\lambda \rightarrow 0$:

-  **Convergence** of the nuclei to the boundary;
-  **Convergence** of difference of distances (**delays**).

Probabilistic extension (convergence in distribution) if underlining space has **exponential** volume growth (tightness of delays).

Ideal tessellations

More detailed recipe

Let (E, d) be a metric space with a special point \mathbf{o} (**origin**).

Let nuclei $x_i^{(\lambda)}$ be **sorted** according to $d(\mathbf{o}, x_i^{(\lambda)})$.

If:

- I. $x_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{} \theta_i \in \partial E$ (Gromov boundary);
- II. $d(\mathbf{o}, x_i^{(\lambda)}) - d(\mathbf{o}, x_{\textcolor{red}{1}}^{(\lambda)}) = \delta_i^{(\lambda)} \xrightarrow[\lambda \downarrow 0]{} \delta_i$ increasing **delays**;

Then

$$\text{Vor}(x_i^{(\lambda)}) \xrightarrow[\lambda \downarrow 0]{\text{loc. Hausdorff}} \text{Vor}((\theta_i, \delta_i); i \geq 1) \quad (\textbf{Ideal tessellation})$$

Exponential volume growth: regular trees, hyperbolic space

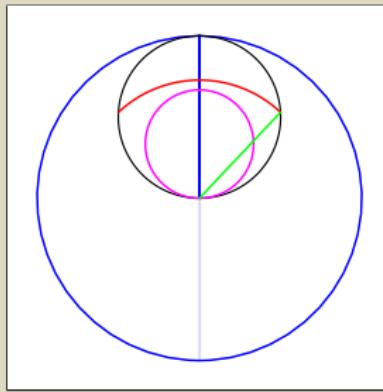
Hyperbolic space \mathbb{H}_d

\mathbb{H}_d : d -dimensional Riemannian manifold of const. sectional curv. $= -1$

Poincaré **ball** model

$$\mathbb{B}_d = \{x \in \mathbb{R}^d; |x| < 1\}$$

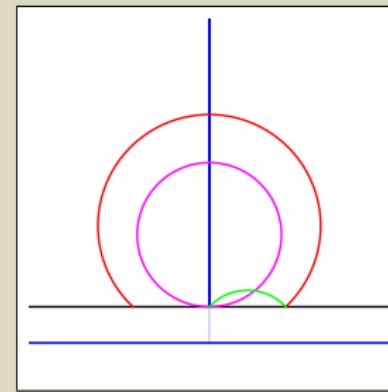
$$ds^2 = 4 \frac{dx_1^2 + \dots + dx_d^2}{(1 - |x|^2)^2}$$



Upper-half space model

$$\mathbb{U}_d = \mathbb{R}^{d-1} \times \mathbb{R}_{>0}$$

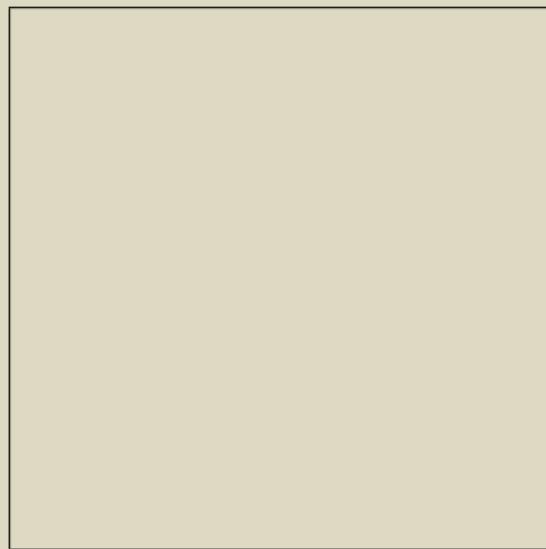
$$ds^2 = \frac{dx_1^2 + \dots + dx_{d-1}^2 + dz^2}{z^2}$$



Setup and delays

- **Nuclei:**

$\mathbf{X}^{(\lambda)} = (X_i^{(\lambda)} ; i \geq 1)$ a PPP of intensity measure $\lambda^{d-1} \cdot \text{Vol}_{\mathbb{H}_d}$



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With this choice of **intensity measure**, the point closest to \mathbf{o} is roughly at distance $\log 1/\lambda$ as $\lambda \rightarrow 0$ in any dimension d .

- **Delays** (a.s. strictly increasing):

$$D_i^{(\lambda)} \stackrel{\text{def}}{=} d_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda)$$

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- **Isometries** of \mathbb{H}_d : Möbius transf. form the group M\"ob_d .

Convergence of delays



Proposition (convergence of delays)

$(D_i^{(\lambda)})_{i \geq 1} \xrightarrow[\lambda \downarrow 0]{\text{law}} (D_i)_{i \geq 1}$, a PPP on \mathbb{R} of intensity measure

$$c_d e^{(d-1)s} ds , \quad c_d = 4^{1-\frac{d}{2}} \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} .$$

Equivalently, the set $(R_i)_{i \geq 1} \stackrel{\text{def}}{=} \left(\frac{c_d}{d-1} e^{(d-1)D_i} \right)_{i \geq 1}$ is a rate-1 homogeneous PPP on \mathbb{R}_+ .

Extended ideal boundary and corona

Let us work in the ball model \mathbb{B}_d in spherical coordinates:

$$X_i^{(\lambda)} = \left(\underbrace{\Theta_i^{(\lambda)}}_{d-1 \text{ angles}}, d_{\mathbb{H}_d}(\mathbf{o}, X_i^{(\lambda)}) - \log(1/\lambda) = D_i^{(\lambda)} \right)$$

- **Conditionally** on $D_i^{(\lambda)}$, $(\Theta_i^{(\lambda)})_{i \geq 1}$ are i.i.d. **uniform** on $\partial \mathbb{B}_d = \mathbb{S}_{d-1}$.
- A.s. convergence $(\Theta_i^{(\lambda)}, D_i^{(\lambda)}) \rightarrow (\Theta_i, D_i)$ via Skorokhod.

$$\text{Vor}(\mathbf{X}^{(\lambda)}) \xrightarrow[\lambda \rightarrow 0]{\text{dist.}} \text{Vor}(\boldsymbol{\Theta}, \mathbf{R}) \stackrel{\text{def}}{=} \mathcal{V}_d \quad \mathbf{IPVT}$$

$(\boldsymbol{\Theta}, \mathbf{R})$ is a PPP on the **corona** $\mathbb{S}_{d-1} \times \mathbb{R}_+$ of intensity measure

$$\mu_d = \text{Unif}_{\mathbb{S}_{d-1}} \otimes \text{Leb}_{\mathbb{R}_+}.$$

Action of Möb_d on the corona

Let $\phi \in \text{Möb}_d$ and (θ_i, r_i) be a point in the corona $\mathbb{S}_{d-1} \times \mathbb{R}_+$.



Lemma (Möbius action extended to the corona)

$$\phi(\theta_i, r_i) \stackrel{\text{def}}{=} \left(\phi(\theta_i), \frac{r_i}{K(\phi^{-1}(\mathbf{o}), \theta_i)} \right)$$

where $K(z, \theta) = \left(\frac{1 - |z|^2}{|z - \theta|^2} \right)^{d-1}$ is hyperbolic **Poisson kernel**.

μ_d is invariant under the action of any Möbius transformation.

Geometry of perpendicular bisectors

We can now canonically say to which cell $z \in \mathbb{B}_d$ belongs to!

Given two points $(\theta_1, r_1), (\theta_2, r_2)$ in the **corona**, solve

$$\frac{r_1}{K(z, \theta_1)} \stackrel{?}{\leqslant} \frac{r_2}{K(z, \theta_2)} .$$

Perpendicular bisectors are Euclidean semi-spheres:

$$|z - \theta_1| r_1^{\frac{1}{2(d-1)}} = |z - \theta_2| r_2^{\frac{1}{2(d-1)}} , \quad z \in \mathbb{B}_d .$$

Geometry of perpendicular bisectors

Let us focus on the cell $\mathcal{C}_d \subset V_d$ containing the origin $\mathbf{o} \in \mathbb{H}_d$.



Proposition (Geometry of perpendicular bisector)

In \mathbb{U}_d where θ_1 is put at ∞ and θ writes $(C, 0, \dots, 0)$, the $(d-1)$ -hyperplane bisecting (θ_1, r_1) and (θ, r) is a **half-sphere** centered at $(C, 0, \dots, 0)$ with radius

$$\rho = \sqrt{1 + C^2} \left(\frac{r_1}{r} \right)^{\frac{1}{2(d-1)}}.$$

From which a very appealing description of \mathcal{C}_d follows:

Main result: the deposition model for \mathcal{C}_d

↙ Theorem (Description of \mathcal{C}_d)

\mathcal{C}_d is the complement of all **open half-balls** whose **centers** and **radii** are a **PPP** of intensity measure

$$\mu_d = 2 \cdot \mathcal{E}_d \cdot dx \rho^{1-2d} d\rho \mathbf{1}_{\rho \leq \sqrt{1+x^2}} \quad \text{on} \quad \mathbb{R}^{d-1} \times \mathbb{R}_+ .$$

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- \mathcal{E}_d : **random variable** of law $\text{Exp}(c'_d)$, $c'_d = \frac{4^{1-d/2}\pi^{d/2}}{\Gamma(d/2)}$;

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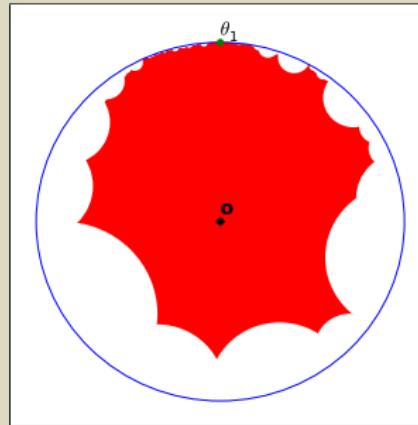
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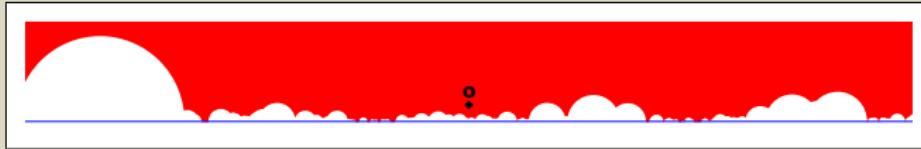
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- dx : Lebesgue on \mathbb{R}^{d-1} for the **centers** of the half-balls;
- $\rho^{1-2d} d\rho$ on \mathbb{R}_+ for the **radii** of the half-balls;
- $\mathbf{1}_{\rho \leq \sqrt{1+x^2}}$: condition of **origin exclusion**.

Portrait of \mathcal{C}_2

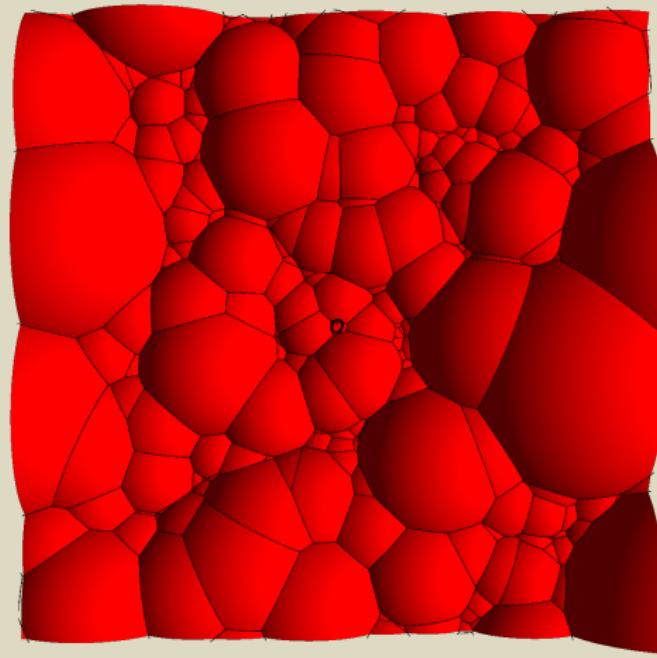
 \mathbb{B}_2

$$\varphi(z) = i \frac{z+i}{z-i} \quad \Downarrow \quad \Updownarrow \quad \varphi^{-1}(w) = i \frac{w-i}{w+i}$$

 \mathbb{U}_2

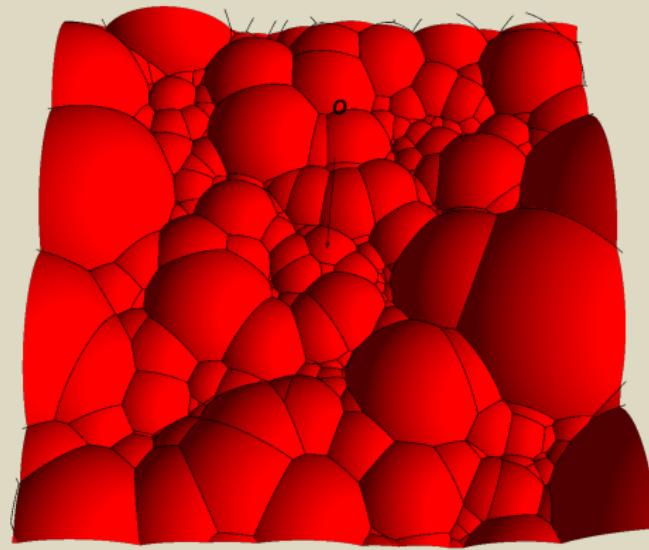
Portrait of \mathcal{C}_3 in \mathbb{U}_3 seen from above

Interactive version at sketchfab.com/foam



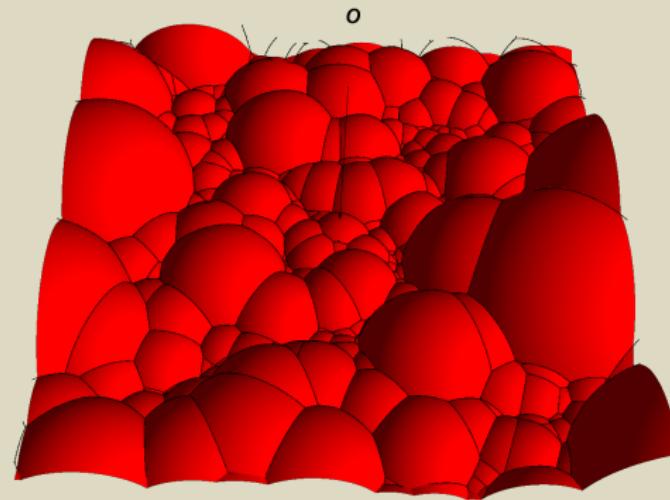
Portrait of \mathcal{C}_3 in \mathbb{U}_3

Interactive version at sketchfab.com/foam



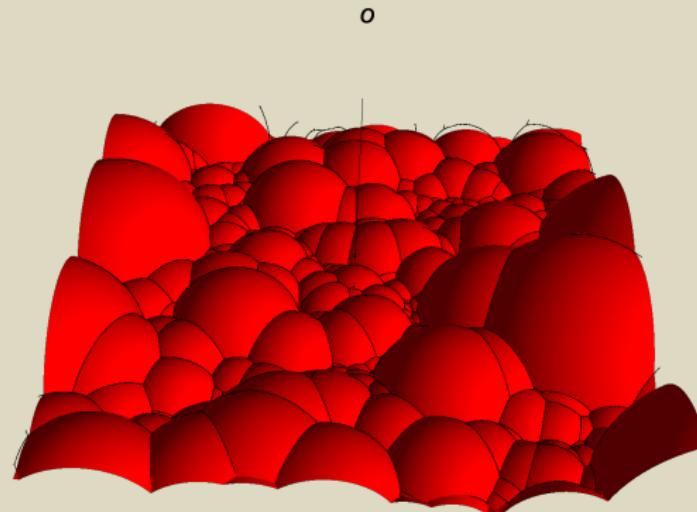
Portrait of \mathcal{C}_3 in \mathbb{U}_3

Interactive version at sketchfab.com/foam



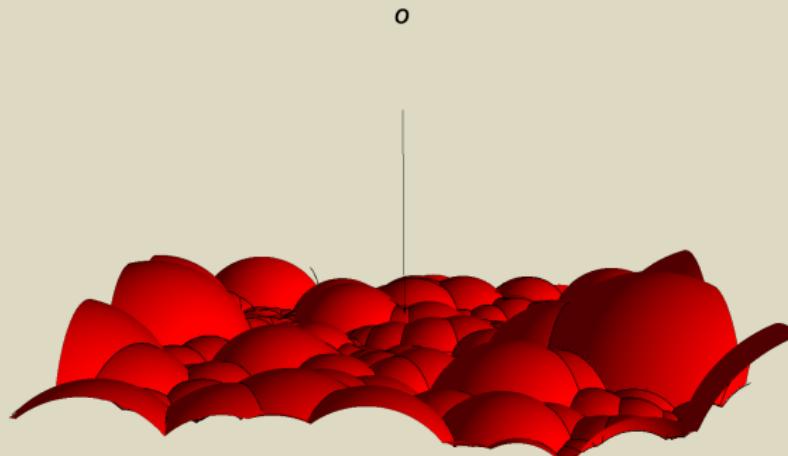
Portrait of \mathcal{C}_3 in \mathbb{U}_3

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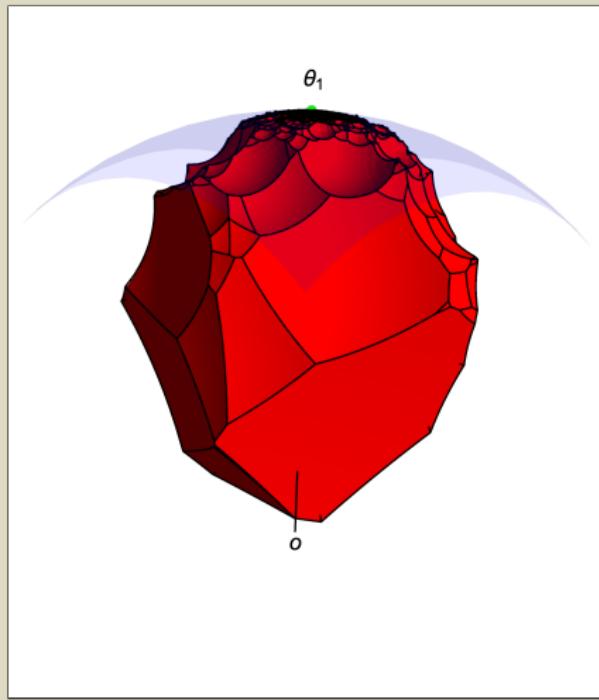
Portrait of \mathcal{C}_3 in \mathbb{U}_3 seen from the side

Interactive version at sketchfab.com/foam



Portrait of \mathcal{C}_3 in \mathbb{B}_3

Interactive version at sketchfab.com/jewel (or in your hands)



Properties of \mathcal{V}_d



Theorem (Tiles have one end)

Almost surely the tiles of \mathcal{V}_d have disjoint interiors,

- Each tile of \mathcal{V}_d is an **unbounded**, convex hyperbolic polytope with an infinite number of bounded facets and with a **unique end**,
- at most $d + 1$ tiles can share a common point,
- \mathcal{V}_d is **locally finite**,
- the law of \mathcal{V}_d is Möb _{d} -**invariant**.

Local properties of \mathcal{C}_d : hole probability

The probability that the hyperbolic ball $B_r(\mathbf{o})$ centered at \mathbf{o} with hyperbolic radius r is contained in \mathcal{C}_d is given, $\forall d \geq 2$ by

(i) **Conditional** on $R_1 = s$,

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_d \mid R_1 = s) = \exp(-sI_d(r)) ;$$

(ii) **Averaging** on the value of R_1 ,

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_d) = \frac{1}{1 + I_d(r)} ,$$

where

$$1 + I_d(r) = \frac{1}{c_d} \int_{\mathbb{R}^{d-1}} \frac{dx}{\left(\sqrt{\cosh^2 r + |x|^2} - \sinh r \right)^{2d-2}} .$$

Local properties of \mathcal{C}_d : hole probability

- $d = 2$

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_2) = \frac{\pi}{4 \left(\arctan(e^r) \cosh^2 r + 2 \sinh r \right)}$$

- $d > 2$ odd: a rational function of e^r , e.g.

$$\mathbb{P}(B_r(\mathbf{o}) \subset \mathcal{C}_3) = \frac{3e^{-2r}}{2 + e^{2r}}$$

Generalize results of Isokawa in $d = 2$ to arbitrary dimension d

Asymptotic properties: height and azimuth

Let $\mathcal{M}_d \subset \mathbb{U}_d$ be the random closed obtained by removing balls whose centers and radii follow a **PPP** of intensity measure

$$2 \cdot \mathcal{E}_d \cdot dx \rho^{1-2d} d\rho \mathbf{1}_{\rho \leq \sqrt{1+x^2}} \quad \text{on } \mathbb{R}^{d-1} \times \mathbb{R}_+.$$

The law of $\partial\mathcal{M}_d$ is **explicit** in terms of **height** $\mathcal{H}(x_0)$ **above** $x_0 \in \mathbb{R}^{d-1}$ and **angle** $\Theta(x_0)$ of the tangent hyperplane to $\partial\mathcal{M}_d$ above x_0 wrt vertical direction (**azimuth**).

For $d \geq 2$, the law of (\mathcal{H}, Θ) is, **conditional** on $R_1 = s$,

$$\left(\frac{1}{\mathcal{H}^{d-1}}, \sin^2(\Theta) \right) \sim \text{Exp}(s) \otimes \text{Beta}\left(\frac{d+1}{2}, \frac{d-1}{2}\right).$$

Asymptotic properties: vertex intensity

In the **stationary model** \mathcal{M}_d , the process of vertices has the following intensity at a point $(x, z) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$:

(i) **Conditional** on $R_1 = s$

$$\frac{(\mathbf{c}s)^d}{d!} \nu_d \frac{e^{-\frac{s}{z^{d-1}}}}{z^{d^2}} dx dz,$$

$$\nu_d = \frac{1}{2^d} \int_{[0,1]^d \times (\mathbb{S}_{d-2})^d} \prod_{i=1}^d [v_i(1-v_i)]^{\frac{d}{2}-1} \text{Vol}_{d-1} \left(\sqrt{\frac{v_i}{1-v_i}} u_i \right)_{1 \leq i \leq d} dv_i du_i$$

where Vol_{d-1} = Euclidean volume of a $d-1$ **dimensional simplex**.

In particular $\nu_2 = \frac{3\pi}{4}$ and $\nu_3 = \frac{\pi^4}{35}$.

(ii) **Averaging** on the value of R_1 :

$$\mathbf{c}^d \nu_d \frac{1}{z(1+z^{d-1})^{d+1}} dx dz.$$

Mean area in $d = 3$

Sketch of proof

Consider a **cylinder** $C_L = [0, L]^2 \times \mathbb{R}_+$. Then:

- Conditionally on $R_1 = s$, **area** of $\partial\mathcal{M}_d \cap C_L \sim L^2 \mathbb{E} \left(\frac{1}{\mathcal{H}^2 \sin \Theta} \right)$ a.s.;
- Conditionally on $R_1 = s$, # of vertices inside C_L is easy;
- Since $2E_L = 3V_L + o(L^2)$, (approximate) **Euler's formula** $F_L - E_L + V_L = o(L^2)$ allows to get $F_L = \frac{1}{2}V_L + o(L^2)$;
- Involved quantities all behave as $L \underset{\rightarrow \infty}{\sim} \frac{c}{s}L^2 \implies$ take the **ratio!**

The **mean area** of a face is equal to $\frac{\pi^4}{105} = 0.927705\dots$ ■

Generalization to arbitrary $d \geq 4$ via **Gusakova–Kabluchko–Thäle 2022**

Provisional conclusions

- **Poisson–Voronoi tessellations** with small intensity of nuclei have been considered or used in the literature.
- We have provided a characterization of the limit object, called **ideal Poisson–Voronoi tessellation**, focusing mostly on hyperbolic spaces *.
- Here a remarkably simple description in terms of a **deposition model** allows to study in detail the geometry of the typical cell C_d , both near and far away from a prescribed point of \mathbb{H}_d .

*Plus some results on regular trees.

Some perspectives



Many, interesting research directions!

- Observables defined on more than one cell
- A.s. **injectivity** of the map from IPVT to Voronoi vertices
- **Poisson–Delaunay** triangulations
- Degree Constrained **Minimum Spanning Tree**
- **Infinite-dimensional** hyperbolic spaces
- **Complex** hyperbolic spaces

Thanks!