Lattice Helmholtz decomposition in a two-dimensional ERAP

Optimal Transport and Uncertainty Second Workshop

Mathematics Dept. of the University of Naples "Federico II"
Tuesday, 6 September 2022, 10h20-11h10 Rome time

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Based mainly on a collaboration with:

- Sergio Caracciolo (Milan University, INFN)
- Andrea Sportiello (CNRS, université Sorbonne Paris Nord)

and on several papers in collaboration with:

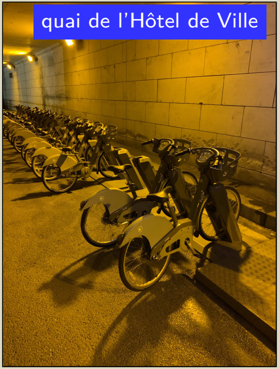
- Dario Benedetto (Rome La Sapienza)
- Emanuele Caglioti (Rome La Sapienza)
- Sergio Caracciolo (Milan University, INFN)
- Vittorio Erba (Lausanne EPFL)
- Yuqi Liu (Master 2 student, université Paris-Est Créteil)
- Gabriele Sicuro (London King's College)
- Andrea Sportiello (CNRS and université Sorbonne Paris Nord)

References

Main references: (here and afterwards all paper references are clickable) preprint at

• J. Stat. Phys. **183**, 34, 2021	`2008.01462`
• PhD Thesis, Université Paris-Saclay, 2020	`tel:03098672`
• J. Phys. A Math. Theor. **53**, 6, 2020	`1904.10867`
• J. Stat. Phys. **174**, 4, 2019	`1803.04723`
• Phys. Rev. E **96**, 4, 2017	`1707.05541`

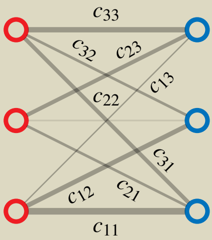
The assignment problem: A current trouble



The assignment problem: Some definitions

Definition (assignment problem). Consider a $n \times n$ real (cost) matrix c. For any n-permutation $\pi \in \mathscr{S}_n$, define the total cost $E(\pi) = \sum_{i=1}^{n} c_{i\pi(i)}$. Problem: Find $\pi_{\text{opt}} := \arg\min_{\pi \in \mathscr{S}_n} E(\pi)$.

Example at $n = 3$:



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

The assignment problem: Some definitions

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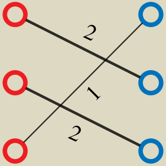
Example at $n = 3$: $\pi_{\text{opt}} = (3\,1\,2)$, $E_{\text{opt}} := E(\pi_{\text{opt}}) = 5$

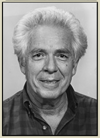


$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

- Optimization of a linear function over the **convex Birkhoff polytope**;
- P-**complete**, $\mathscr{O}(n^3)$ complexity (Munkres 1957);
- Equivalent to a two player zero-sum game (von Neumann 1953, 1954).

Assignment problem: Historical remarks

von Neumann 1953



Kuhn 1955

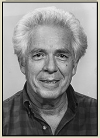
König 1916

Egérvary 1931

Assignment problem: Historical remarks

von Neumann 1953

König 1916



Kuhn 1955

Egérvary 1931

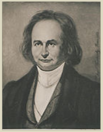
Canon simplicissimus.

	I	II	III	IV	V	VI	VII
I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

"*De investigando ordine systematis aequationum differentialium vulgarium cujuscunque*"

See also (Ollivier 2009)

Jacobi 1860



Assignment problem: Historical remarks

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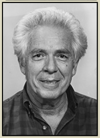

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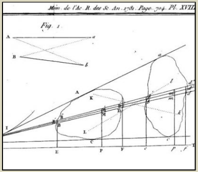
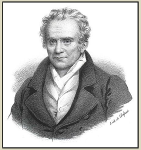


"*Le mémoire sur les déblais et les remblais*"

See this introduction at images.math.cnrs.fr (Ghys 2012)

Monge 1784



Random Assignment Problems: The case of independent weights

Here c is a random matrix, E_{opt} a **(non-trivial) random variable**.

- Pioneered in Physics in the 80s by Mézard–Parisi and Orland;
- Entered Probability Theory mostly via Aldous in the 90s.

Basic result: if $(c_{ij})_{i,j=1}^{n}$ are i.i.d. r.v.s of pdf $\rho(l) = l^r + o(l^r)$, then

$$\mathbb{E}[E_{\text{opt}}]_n \underset{n\to\infty}{\sim} c_r\, n^{1-\frac{1}{r+1}}.$$

Only "infinitely short" edges are relevant for large n. r plays the role of a "**universal exponent**".

Nice fact: At $r = 0$ (i.e. ρ is e.g. uniform or $\mathrm{Exp}(\lambda)$ distribution),

$$c_0 = \zeta(2) = \frac{\pi^2}{6}.$$

The Parisi conjecture

If $c_{ij} \sim \mathrm{Exp}(1)$, Parisi conjectured (1998):

$$\mathbb{E}[E_{\min}]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

- Extension to rectangular cost matrices (Coppersmith-Sorkin 1998);
- Proof of $\zeta(2)$ limit (among other things) (Aldous 2001);
- Proof of Parisi conjecture (Nair–Prabhakar–Sharma 2001);
- Extension to k-partite graphs (which is **NP-hard** for $k \geq 3$) (Martin–Mézard–Rivoire 2004,2005);
- Proof of $\exists!$ of Mézard–Parisi order parameter $\forall r \geq 0$ (Wästlund 2012, Larsson 2014, Salez 2015).

NOT discussed today...

Euclidean Random Assignment Problems (ERAPs)

Let $\mathscr{B} = (B_1, \dots, B_n)$ be blue points and $\mathscr{R} = (R_1, \dots, R_n)$ be red ones: n-samples of i.i.d. r.v. with pdf $\nu_{\mathscr{B}(\mathscr{R})} : \Omega \to \mathbb{R}$ (**disorder**). Let $(\Omega, \mathscr{D})$ be a metric space (mostly an **Euclidean** space with $\mathscr{D}$ **Euclidean** distance). For $p \in \mathbb{R}$ and an assignment (n-permutation) $\pi \in \mathscr{S}_n$, consider the *Hamiltonian*

$$\mathscr{H}(\pi) = \sum_{i=1}^{n} \mathscr{D}^p(B_i, R_{\pi(i)})$$

and the random variable "**ground state energy**"

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = \min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi) \quad (\pi_{\mathrm{opt}} = \arg\min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi)).$$

Definition (Euclidean Random Assignment Problem).
Understand the statistical properties of $\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}$ (*exact* or *asymptotic* law, moments, etc.) depending on (Ω, p, d) and $\nu_{\mathscr{B}(\mathscr{R})}$.

Three main motivations for ERAPs

- **Spin Glasses.** ERAP provides a toy-model of **spin-glass in finite dimension**. Besides disorder, the assignment constraint provides **frustration**. But the model is numerically simpler than e.g. Edwards–Anderson spin glass (Mézard–Parisi 1988).

- **Optimal Transport.** ERAP = **Monge-Kantorovitch** transportation problem on Ω $(\dim(\Omega) = d)$ associated to the empirical measures $\rho_{\mathscr{B}(\mathscr{R})} = \frac{1}{n}\sum_j \delta_{B_j(R_j)}$. In particular

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = n W_p^p(\rho_{\mathscr{B}}, \rho_{\mathscr{R}}),$$

  with W_p is p-**Wasserstein dist.** (Villani 2009, Brezis 2018).

- **Computational Complexity Theory.** ERAPs are small modifications of random TSPs; but TSP is NP-**complete**.

A tool for understading ERAPs: Phase diagram

Let us put $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu$. We can start studying

$$E_{p,d}(n) := \mathbb{E}_{\nu^n \otimes \nu^n}[\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1 + o(1)),$$

as $n \to \infty$, depending on (p,d) and the choice of ν.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be **largely insensitive** on the choice of ν (which may alter the constant $K_{p,d}$).

Remark: Non-uniform disorder ν is more subtle!

Example: Take $\nu =$ standard Gaussian and $(p,d) = (2,1)$. Then

$$E_{2,1}(n) \underset{n\to\infty}{\sim} 2 \ln \ln n \qquad (\text{i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

(Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019, Berthet–Fort 2020)

See (Benedetto–Caglioti 2020) for non-uniform case at $d = 2$.

$d \geq 3$, $p \geq 1$, $\Omega =$ a bounded domain

"Simple": Solution is realized at the scale of nearest-neighbors

$$E_{p,d}(n)\Big|_{d\geq 3, p\geq 1} \underset{n\to\infty}{\sim} K_{p,d} n^{\gamma_{\mathrm{LB}}},$$

where

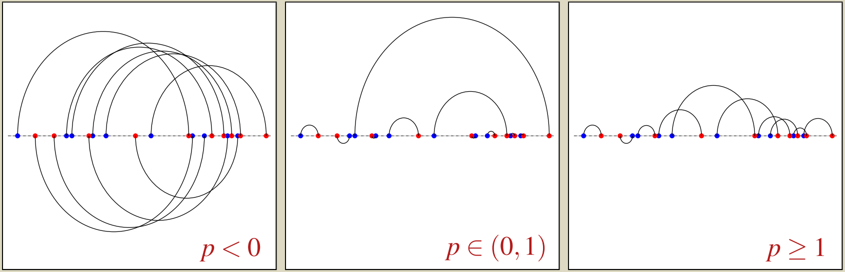
$$\gamma_{p,d} = \gamma_{\mathrm{LB}} := 1 - \frac{p}{d}, \ \ \gamma'_{p,d} = 0 \qquad \text{(Mézard–Parisi 1988)}$$

(if the disorder ν is uniform on Ω, otherwise **unknown**).

Remark: The constants $K_{p,d}$ are **universal** (Barthe–Bordenave 2013 and refs. therein for $p < \frac{d}{2}$, and Goldman–Trevisan 2020 for an extension to $p \geq 1$) but **unknown explicitly**. Upper and lower bounds on some $K_{p,d}$ for $\Omega = [0,1]^d$ are in (Talagrand 1992), numerical estimates for $\Omega = [0,1]^d, \mathbb{T}^d$ are in (Caracciolo–Sicuro 2015, **D'A** MSc Thesis 2016).

$d = 1$: Qualitative properties of a solution π_{opt}

For any ν, $p = 0$ and $p = 1$ separate **three qualitative regimes**:



If $(b_1, \ldots, b_n)$ and $(r_1, \ldots, r_n)$ are sorted in natural order π_{opt} is:

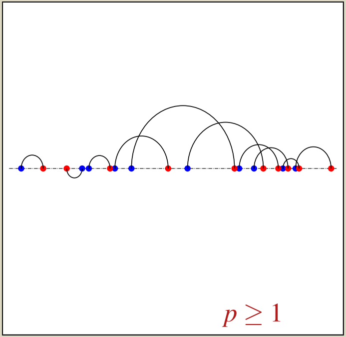
- **Cyclical** for $p < 0$: $\pi_{\text{opt}} = i + k \pmod n$ (Caracciolo–**D'A**–Sicuro 2017);
- **Non-crossing** for $p \in (0,1)$: Intervals "covered by edges" are either disjoint or one is included into the other (McCann 1999);
- **Ordered** for $p \geq 1$: π_{opt} is the identity permutation.

$\Omega = [0,1]$ (or $\mathbb{R}, \mathbb{R}^+, \ldots$), $\mathscr{D} = |\cdot|$ and $p \geq 1$

Take $B_{j+1} \geq B_j$, $R_{j+1} \geq R_j$, for $j = 1, \ldots, n-1$.

Optimality + **(strict) convexity** + (strict) monotonicity of $\mathscr{D}^p$

$$\Downarrow$$



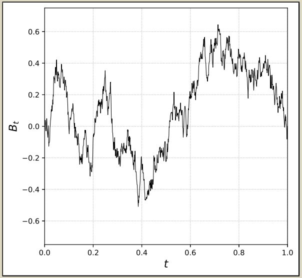
$$\mathscr{H}_{\mathrm{opt},(n,p)}\Big|_{p\geq 1} = \sum_{i=1}^{n} |B_i - R_i|^p.$$

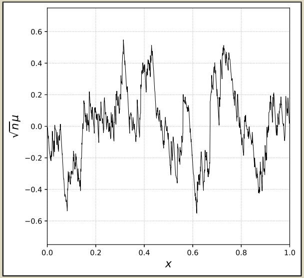
$\Longrightarrow$ Stairway to the **Brownian world!**

Brownian Bridge for $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$, $p \geq 1$

Let the **transport field** be $\mu_i := b_i - r_i$, for $i = 1, \dots n$ and put $i = nt + \frac{1}{2}$. Then by **Donsker's Theorem**,

$$\sqrt{n}\,\mu_i \underset{weakly}{\longrightarrow} B_t, \text{ the Brownian Bridge.}$$





Recall $\mathscr{H}_{\mathrm{opt},(n,p)} = \sum_{i=1}^{n} |\mu_i|^p$. Then $E_{p,1}(n)\Big|_{p \geq 1} \underset{n\to\infty}{\sim} \mathbb{E}[B_t^p]\, n^{1-\frac{p}{2}}$.

(Boniolo–Caracciolo–Sportiello 2014, Caracciolo–Sicuro 2014, Caracciolo–**D'A**–Sicuro 2017)

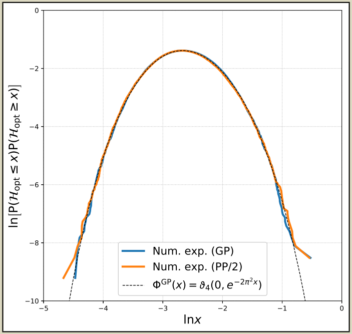
$\Omega=\mathbb{S}_1$, $\mathscr{D}=$ arc-distance, $p=2$: The limit distribution of $\mathscr{H}_{\mathrm{opt}}$

The $n\to\infty$ cdf of $\mathscr{H}_{\mathrm{opt},(n,2)}$ on $\mathbb{S}_1$ is **explicit**. Sketch of proof:

1. At $p=2$ we have **Parseval identity** for μ;
2. The mgf $\mathbb{E}\left[e^{-w\sum_{s\neq 0}|\hat{\mu}|_s^2}\right]=\prod_{s\geq 1}\frac{1}{1+\frac{w}{2\pi^2 s^2}}$ can be inverse-Laplace transformed in closed form (Watson 1961).

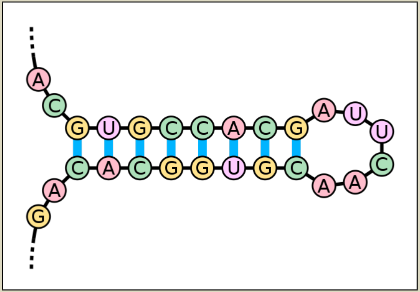
We have (**D'A**, 2020):

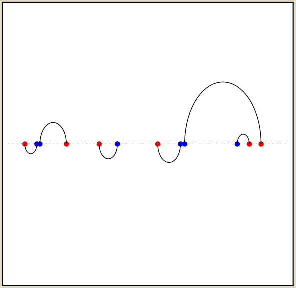
$$\mathbb{P}[\mathscr{H}_{\mathrm{opt},(n,2)}\leq x]\underset{n\to\infty}{\to}\vartheta_4(0,e^{-2\pi^2 x}) := \sum_{s\in\mathbb{Z}} e^{i\pi s}e^{-2\pi^2 s^2 x}.$$



ERAPs at $p \in (0,1)$: Further motivation in biology

Besides economics, interest due to the non-crossing property of the solution: Toy-model for the **secondary structure of RNA** (discarding pseudo-knots).



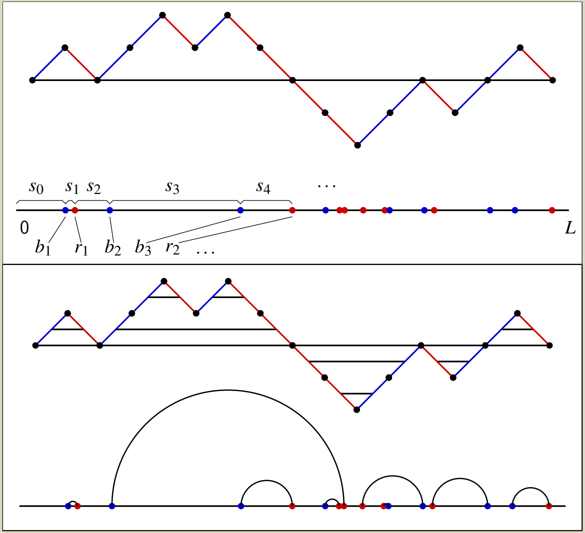


Despite this, poorly understood (McCann 1999).

IDEA: Try a canonical construction, an approximate solution sharing non-crossing property with $\pi_{\text{opt}} \to$ Dyck matchings!

The Dyck matching (Caracciolo–**D'A**–Erba–Sportiello 2020)

Construction:



The Dyck Conjecture at $p \in (0,1)$

Expected energy of Dyck matchings grows asymptotically as

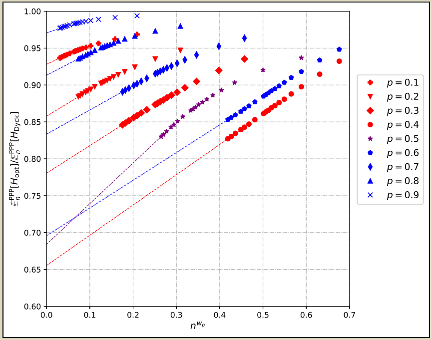
$$\mathbb{E}_n(\mathscr{H}_{\mathrm{Dyck}}) \underset{n\to\infty}{\sim} \begin{cases} n^{1-p} & \text{if} \quad 0 \le p < \frac{1}{2} \\ \sqrt{n}\ln n & \text{if} \quad p = \frac{1}{2} \\ n^{\frac{1}{2}} & \text{if} \quad \frac{1}{2} < p \le 1 \end{cases} .$$

Conjecture (Caracciolo–**D'A**–Erba–Sportiello 2020):

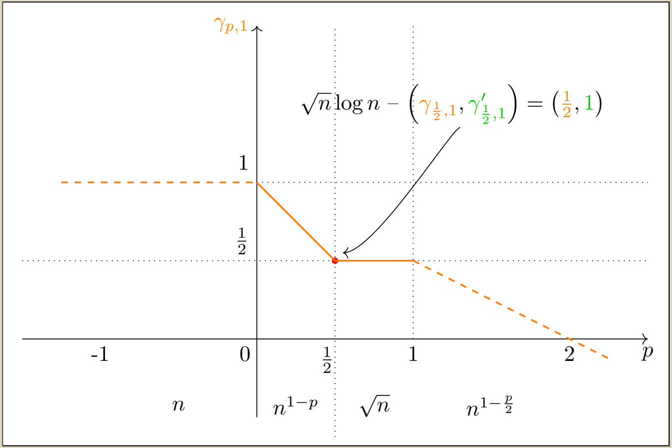
$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathscr{H}_{\mathrm{opt}})}{\mathbb{E}_n(\mathscr{H}_{\mathrm{Dyck}})} = k_p\,,$$

for some $k_p \in (0,1)$.

Upper bound implied by (Fournier–Guillin 2014).



Section of the phase diagram at $d = 1$



Product formula for number of solutions at $d = p = 1$ (Caracciolo–Erba–Sportiello 2021).

$d = 2$: A challenge for both mathematicians and physicists

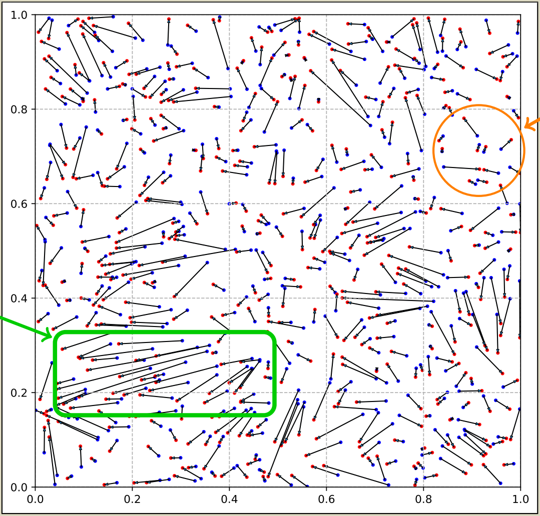
Example: $\Omega = [0,1]^2$, $\mathscr{D} = |\cdot|$. A solution at $p = 2$:

The solution π_{opt} connects typically $O(\ln n)$–**nearest-neighbors**.

$$\gamma_{p,d} = \gamma_{\text{LB}} = 1 - \frac{p}{2}$$

$$\gamma'_{p,d} = \frac{p}{2}$$

(Ajtai–Komlós–Tusnády 1984).



A few recent developments at $(p,d)=(2,2)$

2014 Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E): Using a **classical field-theoretical approach**, first predicted $K_{2,2}=\frac{1}{2\pi}$.

2019 Ambrosio–Stra–Trevisan (PTRF): **Proof** of $K_{2,2}=\frac{1}{2\pi}$

2020 Ambrosio–Glaudo (JEP): Refinement on the **remainder term**

2020 Benedetto–Caglioti (J. Stat. Phys.): non-constant densities (UB)

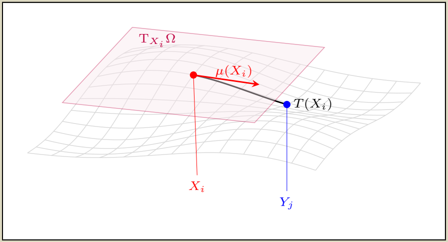
2021 Benedetto–Caglioti-Caracciolo–**D'A**–Sicuro–Sportiello (J. Stat. Phys.): **Exact formula** for $\lim_{n\to\infty}[E_\Omega(n)-E_{\Omega'}(n)]$, where Ω,Ω' are two manifolds.

2021 Caracciolo *et al.*: Link with loop-erased **SARW**.

2022 Ambrosio–Goldman–Trevisan: **proof** of universality of $K_{2,2}$.

The Caracciolo–Lucibello–Parisi–Sicuro approach

A (classical) field theory for general d and $p \geq 1$ (PRE 2014).



At $p = 2$, for a d-dimensional manifold Ω, the Lagrangian is

$$\mathscr{L}[\vec{\mu}, \phi] := \overbrace{\int_\Omega \frac{1}{2}\vec{\mu}^2(x)\nu_{\mathscr{B}}(\mathrm{d}x)}^{\text{Energy}} + \overbrace{\int_\Omega [\phi(x + \vec{\mu}(x))\nu_{\mathscr{B}}(x) - \phi(x)\nu_{\mathscr{R}}(\mathrm{d}x)]}^{\text{Transport constraint}}.$$

$\nu_{\mathscr{B}(\mathscr{R})}$ is the "charge" density of blues (reds) and ϕ is a Lagrange multiplier enforcing the transport constraint.

Caracciolo–Lucibello–Parisi–Sicuro (CLPS) theory

Heuristic: $|\vec{\mu}|$ "small" when $n \to \infty \implies$ Taylor expansion in $\varepsilon = |\nabla \cdot \vec{\mu}(x)|$ "small". Linearize $\mathscr{L}[\vec{\mu}, \phi]$ to

$$\mathscr{L}_{\rm lin}[\vec{\mu}, \phi] := \int_\Omega \left[\frac{1}{2}\vec{\mu}^2(x) + \vec{\mu}(x) \cdot \nabla\phi(x)\right] \mathrm{d}x + \int_\Omega \delta\nu(x)\phi(x)\,\mathrm{d}x.$$

Euler-Lagrange eqs. at leading order in ε is Poisson eq. for the potential ϕ with source $\delta\nu := \nu_{\mathscr{B}} - \nu_{\mathscr{R}}$

$$\Delta_\Omega \phi = \delta\nu, \qquad -\Delta_\Omega = \textbf{Laplace-Beltrami op. on } \Omega$$

to be solved with Neumann bc on Ω (if $\partial\Omega \neq \varnothing$).
Then $\vec{\mu} = -\nabla\phi$ and E_Ω " $=$ " $\int_\Omega |\vec{\mu}|^2$. Following CLPS 2014, the magic formula is

$$E_\Omega(n) \text{ "}=\text{" } -2\,\mathrm{Tr}\,\Delta_\Omega^{-1}$$

which is **very ill defined** !! $\implies$ Need for regularization(s)

Random assignment problems on 2d manifolds

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

A way of rewriting Caracciolo–Lucibello–Parisi–Sicuro's regularization is

$$E_\Omega(n) = -2\,\mathrm{Tr}\,\Delta_\Omega^{-1} \simeq 2 \sum_{\lambda\in\Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n^{1/d}(\log n)^{b(d)}}\right)}{\lambda}$$

for an unknown cutoff function F **independent on** Ω (but possibly dependent on the local randomness of the ERAP), satisfying

$$\begin{cases} \lim\limits_{z\to 0+} F(z) = 1 \\ \lim\limits_{z\to\infty} F(z) = 0 \end{cases}.$$

Random assignment problems on 2d manifolds

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Weyl's law (Ivrii 1980, Neumann b.c. case)

Let Ω be a d-dimensional manifold and $\Lambda(\Omega)$ be the spectrum of $-\Delta_\Omega$ with Neumann b.c. if $\partial\Omega \neq \varnothing$ without $\lambda = 0$. Let $\mathscr{N}_\Omega(\lambda)$ be the eigenvalue counting function. Then, for λ large,

$$\mathscr{N}_\Omega(\lambda) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{\frac{d}{2}} + \frac{\omega_{d-1}}{4(2\pi)^{d-1}}|\partial\Omega|\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})$$

- $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} =$ volume of standard d-ball;
- $|\Omega|$ d-dimensional volume of Ω;
- $|\partial\Omega|$ surface area of the boundary of Ω.

On asymptotic energy differences at $d = 2$

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Argument: Precise form of F is inessential. For 2 manifolds Ω, Ω',

$$\lim_{n\to\infty}\left(E_\Omega(n) - E_{\Omega'}(n)\right) = 2\lim_{n\to\infty}\left(\sum_{\lambda\in\Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda} - \sum_{\lambda\in\Lambda(\Omega')} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda}\right)$$

$$= 2\lim_{n\to\infty}\int_{0^+}^{\infty} F\left(\frac{\lambda}{n}\right)\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

$$= 2\int_{0^+}^{\infty}\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

as $\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right) = O(\sqrt{\lambda}\ln\lambda)$ at $d = 2$ (and near the origin the integral is regularized by the spectral gap).

Explicit evaluation of energy differences

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Main result: Even if the field theory is ill-posed (room for rigorous work!), we can give a precise experimental (and predictive!) meaning to energy differences $E_\Omega(n) - E_{\Omega'}(n)$ through **regularization**. We did it in two ways :

- R_Ω or "**Robin mass**": Integrals of the diagonal of Green's function for Poisson eq.;
- K_Ω or "**Kronecker mass**": Expand spectral function $Z_\Omega(s)$ associated to $-\Delta_\Omega$ around the simple pole $s = 1$.

Remark 1: Robin and Kronecker masses satisfy (Morpurgo 2002)

$$\forall \Omega, \qquad R_\Omega - K_\Omega = \frac{\ln 2}{2\pi} - \frac{\gamma_E}{2\pi} = 0.0184511\ldots$$

Remark 2: other regularizations are possible.

Example: Rectangles $\mathbb{R}$, 2-torus $\mathbb{T}$, Boy surface $\mathbb{B}$

Obtained from rectangle of aspect ratio ρ by appropriately gluing sides

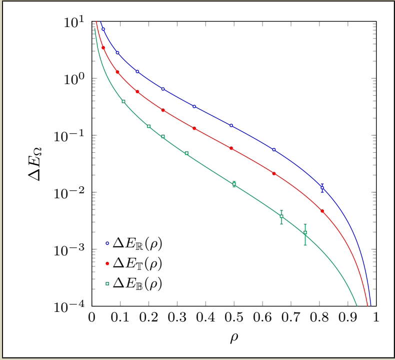
Energy shift w.r.t. manifold at aspect ratio $\rho = 1$:

$$\Delta E_\Omega(\rho) = 2(R_\Omega(\rho) - R_{\Omega'}(1)) = 2(K_\Omega(\rho) - K_{\Omega'}(1))$$

$$K_\mathbb{R}(\rho) = \frac{\gamma_\mathrm{E}}{2\pi} - \frac{\ln(4\pi^2\rho|\eta(i\rho)|^4)}{4\pi} + \frac{1}{2\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$

$$K_\mathbb{T}(i\rho) = \frac{\gamma_\mathrm{E} - \ln(4\pi\sqrt{\rho})}{2\pi} - \frac{1}{\pi}\ln|\eta(i\rho)|$$

$$K_\mathbb{B}(\rho) = \frac{\gamma_\mathrm{E}}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{\ln\eta(i\rho)}{\pi} - \frac{1}{4\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$



(See JStatPhys 2021 for more manifolds).

The model: Grid-Poisson ERAP on the 2-torus

(See **D'A** 2020, p.171 + Python code available at github.com/ERAPs2d)

$\mathscr{B} \equiv \Lambda_n = \left\{0, \frac{1}{L}, \ldots, 1 - \frac{1}{L}\right\}^2$

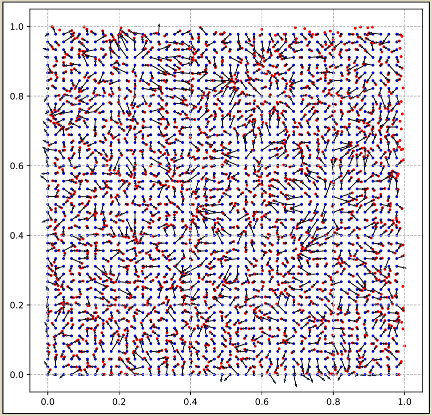
$\mathscr{R}$: n i.i.d. $\mathscr{U}(\mathbb{T}^2)$ random points

Geodesic distance, $p = 2$

Optimal transport field:

$$\vec{\mu}(b_k) := r_{\pi_{\mathrm{opt}}(k)} - b_k \pmod 1$$
$$= \begin{pmatrix} \mu_x(b_k) \\ \mu_y(b_k) \end{pmatrix}, k = 1, \ldots, n$$

Example: $n = 45 \cdot 45 = 2025$



Setup and lattice calculus

Idea: CLPS field theory is analogue to Electrostatics "at $n=\infty$"
$\implies$ let's do Electrodynamics at finite n!

Let $\hat{\Lambda}_n$ be the dual lattice. For a function $h : \Lambda_n \cup \hat{\Lambda}_n \to \mathbb{C}$ consider

$$\nabla_1 h(i,j) = \frac{1}{\sqrt{2}}\left[h\left(i+\frac{1}{2}, j+\frac{1}{2}\right) - h\left(i-\frac{1}{2}, j-\frac{1}{2}\right)\right],$$

$$\nabla_2 h(i,j) = \frac{1}{\sqrt{2}}\left[h\left(i-\frac{1}{2}, j+\frac{1}{2}\right) - h\left(i+\frac{1}{2}, j-\frac{1}{2}\right)\right].$$

The lattice Helmholtz decomposition

Recall the **divergence** and **curl** of a vector field E:

$$\nabla \cdot \overrightarrow{E} = \nabla_\alpha E_\alpha$$
$$\nabla \wedge \overrightarrow{E} = \varepsilon_{\alpha\beta} \nabla_\alpha E_\beta, \quad (\alpha, \beta = 1, 2, \text{ Einstein's notation})$$

After a rotation $\overrightarrow{\mu}_{-\pi/4} = R(-\frac{\pi}{4})\overrightarrow{\mu}$ we can write

$\overrightarrow{\mu}_{-\pi/4} = \nabla \phi - \nabla \wedge \psi$ **(Lattice Helmholtz decomposition)**,

where ϕ is the CLPS potential and ψ is a new "vector" potential.
<u>**Remark**</u>: ϕ and ψ are defined on the dual lattice $\hat{\Lambda}_n$.
The "1st and 3rd Maxwell equations" are

$$\nabla \cdot \overrightarrow{\mu}_{-\pi/4} = \nabla_\alpha \mu_\alpha = \Delta \phi,$$
$$\nabla \wedge \overrightarrow{\mu}_{-\pi/4} = \varepsilon_{\alpha\beta} \nabla_\alpha \mu_\beta = \Delta \psi.$$

Energy decomp. in longitudinal and transverse part

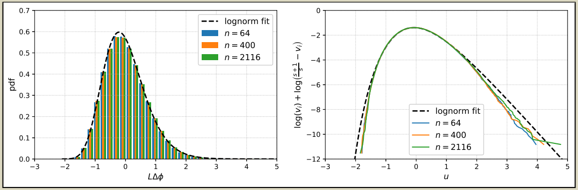
With in mind the phase diagram, we can try to split $\mathscr{H}_{\mathrm{opt},(n,2)}$ as

$$\begin{aligned}\mathscr{H}_{\mathrm{opt},(n,2)} &= (\vec{\mu},\vec{\mu}) \overset{rot.\ inv.}{=} (\vec{\mu}_{-\pi/4},\vec{\mu}_{-\pi/4}) \\ &\overset{def}{=} (\nabla_\alpha \phi - \varepsilon_{\alpha\beta}\nabla_\beta \psi, \mu_\alpha) \\ &= -(\phi,\nabla_\alpha \mu_\alpha) - (\psi, \varepsilon_{\alpha\beta}\nabla_\alpha \mu_\beta) \\ &= -(\phi,\Delta\phi) - (\psi,\Delta\psi) \\ &:= \mathscr{H}^{(\phi)} + \mathscr{H}^{(\psi)} .\end{aligned}$$

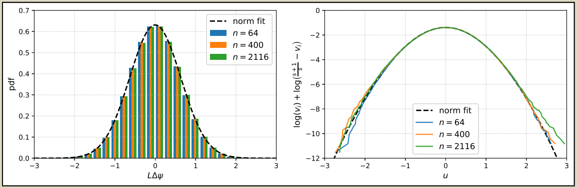
Using github.com/ERAPs2d we can study the statistical properties of $\Delta\phi$, $\Delta\psi$, their (random) Fourier modes, correlation functions ...

Numerical results in coordinate space

Histograms of $L\Delta\phi$:



Histograms of $L\Delta\psi$:



<u>Numerical protocol</u>: $n = 64, 400, 2116$, $> 10^4$ reps for each n.

Numerical results in momentum space

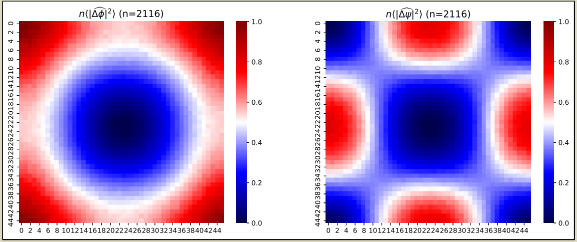
The Fourier components of $\Delta\phi$ and $\Delta\psi$ are

$$\widehat{\Delta\phi}(p) = \frac{1}{L} \sum_{z \in \hat{\Lambda}_n} e^{-iz\cdot p} \Delta\phi(z) \,,$$

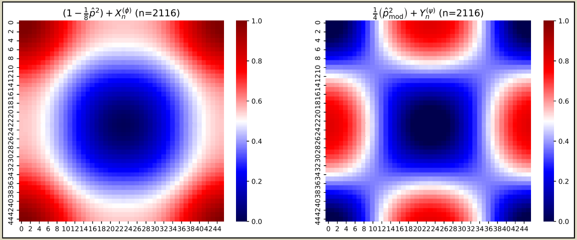
$$\widehat{\Delta\psi}(p) = \frac{1}{L} \sum_{z \in \hat{\Lambda}_n} e^{-iz\cdot p} \Delta\psi(z) \,.$$

Numerical results in momentum space

Numerical data:



$2d$ **fit**:



Numerical protocol: $n = 2116$, avg. of 10^4 reps.

A few (provisional) conclusions

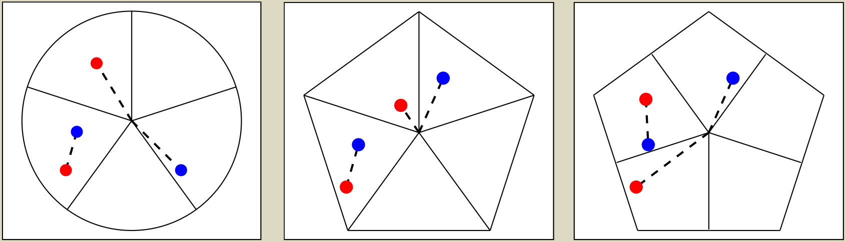
- The CLPS potential ϕ (Gauss law) is responsible accounts for the leading $\log n$ behavior;
- The contribution from "vector" potential ψ (Maxwell–Faraday law) $\mathbb{E}\left[\mathscr{H}^{(\psi)}\right]$ appears to converge to some constant in n;
- $\Delta\psi$ appears to be a gaussian field;
- Non-trivial correlation functions.

1. ERAPs on convex domains with wormholes at $d \leq 2$

(W.i.p. with Sergio Caracciolo, Yuqi Liu and Andrea Sportiello, 2021-)

Example 2: k-identical $2d$ domains $(\Omega_i)_{i=1}^{k}$ "glued" by a point (wormhole) on the border.

At $k = 5$:



Model 1 Model 2 Model 3

$\nu_{\mathscr{B}} = \nu_{\mathscr{R}} \sim U(\Omega) \otimes \mathscr{U}_{\{1,\dots,k\}}$; $p = 2$. **Remark**: $\sum_{j=1}^{k} |\Omega_j| = 1$.

1. ERAPs on convex domains with wormholes at $d \leq 2$

(W.i.p. with Sergio Caracciolo, Yuqi Liu and Andrea Sportiello, 2021-)

Is the exp. contribution due to "swapping points" negligible as $n \to \infty$?

From Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E 2014), Ambrosio–Stra–Trevisan (PTRF 2019) and Benedetto–Caglioti (J. Stat. Phys. 2020)+ Ambrosio–Goldman–Trevisan (EJP 2022), one expects for a uniform disorder:

$$\mathbb{E}\left[\mathscr{H}_{\mathrm{opt}}\right] \underset{n\to\infty}{\sim} \frac{\sum_{j=1}^{k} |\Omega_j|}{2\pi} \log n + o(\log n) = \frac{1}{2\pi} \log n + o(\log n).$$

<u>However</u>, numerical simulations show:

- The limit constant f appears to be independent on model $\ddot{\smile}$;
- $f > \frac{1}{2\pi}$!

Theorem 1 in **D'A**-Liu 2022 for providing upper and lower bounds to difficult ERAPs via simpler ones (go to poster session for more)

2. On the phase transition at $(p,d) = (\frac{1}{2}, 1)$

(after discussions with Sergio Caracciolo and Andrea Sportiello, 2021-)

The appearance of an exponent $\gamma'_{\frac{1}{2},1} \neq 0$ is somehow surprising.

- On the discrete side, the Dyck conjecture states that $\gamma'_{\frac{1}{2},1} \leq 1$;
- On the continuum side, Fournier–Guillin 2014 implies $\gamma'_{\frac{1}{2},1} \leq 1$;

Recent work by Bobkov–Ledoux 2020 using Zolotarev distances implies the refinement $\gamma'_{\frac{1}{2},1} \leq \frac{1}{2}$

Can one provide a lower bound

$$\gamma'_{\frac{1}{2},1} \geq ??$$

A proof of the lower bound (supported by numerical experiments) exploiting Dyck matchings seems in reach (Andrea Sportiello, private communication).

3. Beyond the CLPS linearization ansatz

(W.i.p. with Dario Trevisan and Francesco Mattesin, 2021-)

Let $\xi : \mathbb{T}^d \to [-1, \infty)$ be integrable and "small" such that $\int_{\mathbb{T}^d} \xi = 0$ and $\rho = (1 + \xi)$ a probability density. Starting from the Monge-Ampère equation

$$\det(\nabla^2 u) = 1 + \xi \ .$$

From the expansion $u = \sum_{\ell=0}^{\infty} u_\ell$ the optimal transport map reads

$$T(x) = \nabla u(x) = \sum_{\ell=0}^{\infty} \nabla u_\ell(x).$$

We wish to study the iterative scheme

$$\begin{cases} \Delta u_1 = \xi \\ \Delta u_\ell = -p_\ell(\nabla^2 u_1, \dots, \nabla^2 u_{\ell-1}) \quad \text{if } \ell > 1 \end{cases}$$

for polynomials p_l.

4. Nice formulas in one dimensional ERAPs

(after discussions with Nathanaël Enriquez, Sophie Laruelle and Andrea Sportiello, 2021-)

Take $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu = \mathbf{1}_{[0,1]}(x)$.
Using generalised Selberg integrals, Caracciolo *et al.* 2019 have shown, for integer ℓ,

$$\mathbb{E}[|b_k - r_k|^\ell] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}.$$

This implies the exact formula for $E_{p,1}\big|_{p\geq 1}(n)$ for $f = |\cdot|^p$.

Question*: Which $f = f(|\cdot|)$ give a "nice formula" for $E(n)$ after resummation?

*Question raised by N. Enriquez (Paris-Saclay (Orsay)) at this talk for ALEA Days 2021, online at CIRM Marseilles - Luminy, 18 March 2021.

Thank you for your attention!