Euclidean Random Assignment Problems, old and new

Optimal Transport and Uncertainty

Mathematics Department of the University of Pisa
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LAMA LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES – UMR 8050

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Based on several works in collaboration with:

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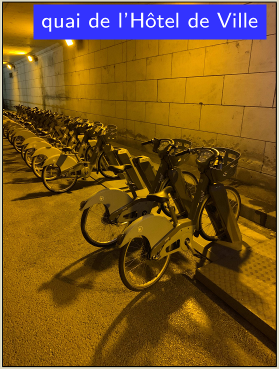
Main references: (here and afterwards all paper references are clickable) preprint at

- J. Stat. Phys. **183**, 34, 2021 `2008.01462`
- PhD Thesis, Université Paris-Saclay, 2020 `tel:03098672`
- J. Phys. A Math. Theor. **53**, 6, 2020 `1904.10867`
- J. Stat. Phys. **174**, 4, 2019 `1803.04723`
- Phys. Rev. E **96**, 4, 2017 `1707.05541`

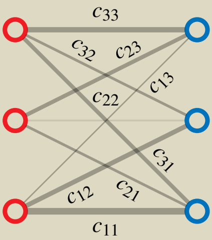
The assignment problem: A current trouble



The assignment problem: Some definitions

Definition (assignment problem). Consider a $n \times n$ real (cost) matrix c. For any n-permutation $\pi \in \mathscr{S}_n$, define the total cost $E(\pi) = \sum_{i=1}^{n} c_{i\pi(i)}$. Problem: Find $\pi_{\mathrm{opt}} := \arg\min_{\pi \in \mathscr{S}_n} E(\pi)$.

Example at $n = 3$:



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

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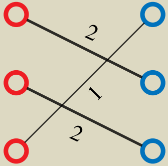
Example at $n = 3$: $\pi_{\text{opt}} = (3\,1\,2)$, $E_{\text{opt}} := E(\pi_{\text{opt}}) = 5$

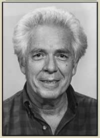


$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

- Optimization of a linear function over the **convex Birkhoff polytope**;
- P-**complete**, $\mathscr{O}(n^3)$ complexity (Munkres 1957);
- Equivalent to a two player zero-sum game (von Neumann 1953, 1954).

Assignment problem: Historical remarks

von
Neumann
1953



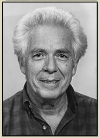
Kuhn
1955

König
1916

Egérvary
1931

Assignment problem: Historical remarks

von Neumann 1953



Kuhn 1955

König 1916

Egérvary 1931

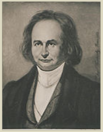
Canon simplicissimus.

	I	II	III	IV	V	VI	VII
I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

"*De investigando ordine systematis aequationum differentialium vulgarium cujuscunque*"

See also (Ollivier 2009)

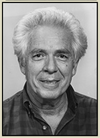
Jacobi 1860



Assignment problem: Historical remarks

von Neumann 1953

König 1916



Kuhn 1955

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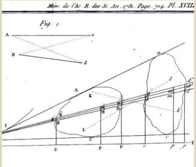


"*Le mémoire sur les déblais et les remblais*"

See this introduction at images.math.cnrs.fr (Ghys 2012)

Monge 1784

Random Assignment Problems: the case of independent weights

Here c is a random matrix, E_{opt} a **(non-trivial) random variable**.

- Pioneered in Physics in the 80s by Mézard–Parisi and Orland;
- Entered Probability Theory mostly via Aldous in the 90s.

Basic result: if $(c_{ij})_{i,j=1}^{n}$ are i.i.d. r.v.s of pdf $\rho(l) = l^r + o(l^r)$, then

$$\mathbb{E}[E_{\mathrm{opt}}]_n \underset{n\to\infty}{\sim} c_r n^{1-\frac{1}{r+1}}.$$

Only "infinitely short" edges are relevant for large n. r plays the role of a "**universal exponent**".

Nice fact: At $r = 0$ (i.e. ρ is e.g. uniform or $\mathrm{Exp}(\lambda)$ distribution),

$$c_0 = \zeta(2) = \frac{\pi^2}{6}.$$

The Parisi conjecture

If $c_{ij} \sim \mathrm{Exp}(1)$, Parisi conjectured (1998):

$$\mathbb{E}[E_{\min}]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

- Extension to rectangular cost matrices (Coppersmith-Sorkin 1998);
- Proof of $\zeta(2)$ limit (among other things) (Aldous 2001);
- Proof of Parisi conjecture (Nair–Prabhakar–Sharma 2001);
- Extension to k-partite graphs (which is **NP-hard** for $k \geq 3$) (Martin–Mézard–Rivoire 2004,2005);
- Prof of $\exists!$ of Mézard–Parisi order parameter $\forall r \geq 0$ (Wästlund 2012, Larsson 2014, Salez 2015).

NOT discussed today...

Euclidean Random Assignment Problems (ERAPs)

Let $\mathscr{B} = (B_1, \ldots, B_n)$ be blue points and $\mathscr{R} = (R_1, \ldots, R_n)$ be red ones: n-samples of i.i.d. r.v. with pdf $\nu_{\mathscr{B}(\mathscr{R})} : \Omega \to \mathbb{R}$ (**disorder**). Let $(\Omega, \mathscr{D})$ be a metric space (mostly an **Euclidean** space with $\mathscr{D}$ **Euclidean** distance). For $p \in \mathbb{R}$ and an assignment (n-permutation) $\pi \in \mathscr{S}_n$, consider the *Hamiltonian*

$$\mathscr{H}(\pi) = \sum_{i=1}^{n} \mathscr{D}^p(B_i, R_{\pi(i)})$$

and the random variable "**ground state energy**"

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = \min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi) \quad (\pi_{\mathrm{opt}} = \arg\min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi)).$$

Definition (Euclidean Random Assignment Problem).
Understand the statistical properties of $\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}$ (*exact* or *asymptotic* law, moments, etc.) depending on (Ω, p, d) and $\nu_{\mathscr{B}(\mathscr{R})}$.

Three main motivations for ERAPs

- **Spin Glasses.** ERAP provides a toy-model of **spin-glass in finite dimension**. Besides disorder, the assignment constraint provides **frustration**. But the model is numerically simpler than e.g. Edwards–Anderson spin glass (Mézard–Parisi 1988).

- **Optimal Transport.** ERAP = **Monge-Kantorovitch** transportation problem on Ω ($\dim(\Omega) = d$) associated to the empirical measures $\rho_{\mathscr{B}(\mathscr{R})} = \frac{1}{n}\sum_j \delta_{B_j(R_j)}$. In particular

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = n W_p^p(\rho_{\mathscr{B}}, \rho_{\mathscr{R}}),$$

  with W_p is p-**Wasserstein dist.** (Villani 2009, Brezis 2018).

- **Computational Complexity Theory.** ERAPs are small modifications of random TSPs; but TSP is NP-**complete**.

A tool for understading ERAPs: The phase diagram

Let us put $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu$. We can start studying

$$E_{p,d}(n) := \mathbb{E}_{\nu^n \otimes \nu^n}[\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1 + o(1)),$$

as $n \to \infty$, depending on (p,d) and the choice of ν.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be **largely insensitive** on the choice of ν (which may alter the constant $K_{p,d}$).

Remark: Non-uniform disorder ν is more subtle!

Example: Take $\nu =$ standard Gaussian and $(p,d) = (2,1)$. Then

$$E_{2,1}(n) \underset{n\to\infty}{\sim} 2 \ln\ln n \qquad (\text{i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

(Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019, Berthet–Fort 2020)

See (Benedetto–Caglioti 2020) for non-uniform case at $d = 2$.

$d \geq 3$, $p \geq 1$, $\Omega =$ a bounded domain

"Simple": Solution is realized at the scale of nearest-neighbors

$$E_{p,d}(n)\Big|_{d\geq 3, p\geq 1} \underset{n\to\infty}{\sim} K_{p,d} n^{\gamma_{\mathrm{LB}}},$$

where

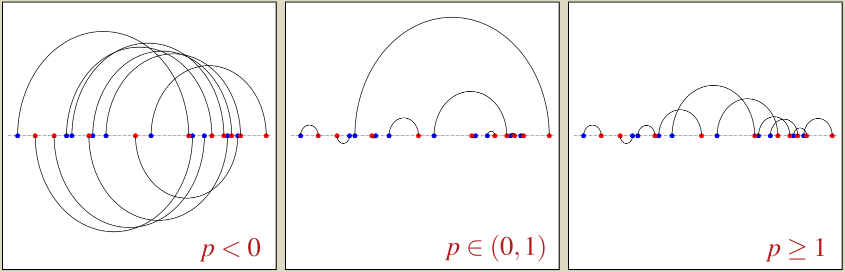
$$\gamma_{p,d} = \gamma_{\mathrm{LB}} := 1 - \frac{p}{d}, \quad \gamma'_{p,d} = 0 \qquad \text{(Mézard–Parisi 1988)}$$

(if the disorder ν is uniform on Ω, otherwise **unknown**).

Remark: The constants $K_{p,d}$ are **universal** (Barthe–Bordenave 2013 and refs. therein for $p < \frac{d}{2}$, and Goldman–Trevisan 2020 for an extension to $p \geq 1$) but **unknown explicitly**. Upper and lower bounds on some $K_{p,d}$ for $\Omega = [0,1]^d$ are in (Talagrand 1992), numerical estimates for $\Omega = [0,1]^d, \mathbb{T}^d$ are in (Caracciolo–Sicuro 2015, **D'A** MSc Thesis 2016).

$d = 1$: Properties of the solution

For any ν, $p = 0$ and $p = 1$ separate **three qualitative regimes**:



If $(b_1, \ldots, b_n)$ and $(r_1, \ldots, r_n)$ are sorted in natural order π_{opt} is:

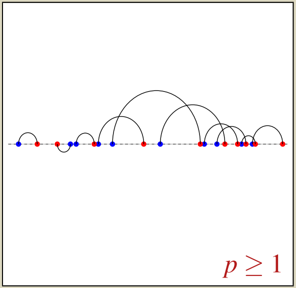
- **Cyclical** for $p < 0$: $\pi_{\text{opt}} = i + k \pmod{n}$ (Caracciolo–**D'A**–Sicuro 2017);
- **Non-crossing** for $p \in (0,1)$: Intervals "covered by edges" are either disjoint or one is included into the other (McCann 1999);
- **Ordered** for $p \geq 1$: π_{opt} is the identity permutation.

$\Omega = [0,1]$ (or $\mathbb{R}, \mathbb{R}^+, \dots$), $\mathscr{D} = |\,|$ and $p \geq 1$

Take $B_{j+1} \geq B_j$, $R_{j+1} \geq R_j$, for $j = 1, \dots, n-1$.

Optimality + **(strict) convexity** + (strict) monotonicity of $\mathscr{D}^p$

$$\Downarrow$$



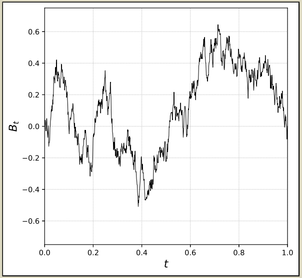
$$\mathscr{H}_{\mathrm{opt},(n,p)}\Big|_{p\geq 1} = \sum_{i=1}^{n} |B_i - R_i|^p.$$

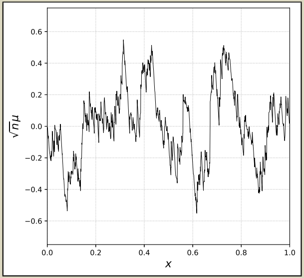
$\Longrightarrow$ Stairway to the **Brownian world!**

Brownian Bridge for $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$, $p \geq 1$

Let the **transport field** be $\mu_i := b_i - r_i$, for $i = 1, \ldots n$ and put $i = nt + \frac{1}{2}$. Then by **Donsker's Theorem**,

$$\sqrt{n}\mu_i \underset{weakly}{\longrightarrow} B_t, \text{ the Brownian Bridge.}$$





Recall $\mathscr{H}_{\text{opt},(n,p)} = \sum_{i=1}^{n} |\mu_i|^p$. Then $E_{p,1}(n)\Big|_{p\geq 1} \underset{n\to\infty}{\sim} \mathbb{E}[B_t^p]\, n^{1-\frac{p}{2}}$.

(Boniolo–Caracciolo–Sportiello 2014, Caracciolo–Sicuro 2014, Caracciolo–**D'A**–Sicuro 2017)

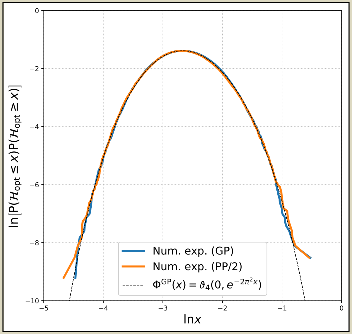
$\Omega = \mathbb{S}_1$, $\mathscr{D} =$ arc-distance, $p = 2$: The limit distribution of $\mathscr{H}_{\text{opt}}$

The cdf of $\mathscr{H}_{\text{opt},(n,2)}$ on $\mathbb{S}_1$ is **explicit**. This is because:

1. At $p = 2$ we have **Parseval identity** for μ;
2. The mgf $\mathbb{E}\left[e^{-w\sum_{s\neq 0}|\hat{\mu}|_s^2}\right] = \prod_{s\geq 1}\frac{1}{1+\frac{w}{2\pi^2 s^2}}$ can be inverse-Laplace transformed in closed form (Watson 1961).

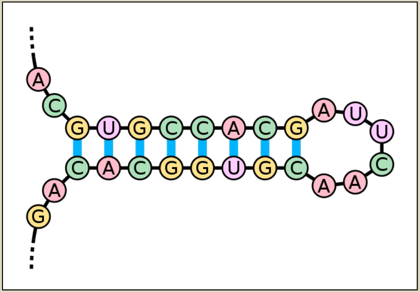
We have (**D'A**, 2020):

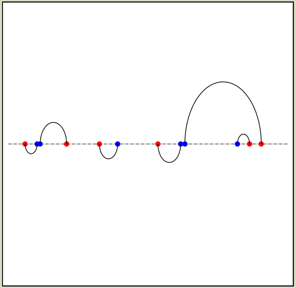
$$\mathbb{P}[\mathscr{H}_{\text{opt},(n,2)} \leq x] \underset{n\to\infty}{\to} \vartheta_4(0, e^{-2\pi^2 x}) := \sum_{s\in\mathbb{Z}} e^{i\pi s} e^{-2\pi^2 s^2 x}.$$



ERAPs at $p \in (0,1)$: Further motivation in biology

Besides economics, interest due to the non-crossing property of the solution: Toy-model for the **secondary structure of RNA** (discarding pseudo-knots).



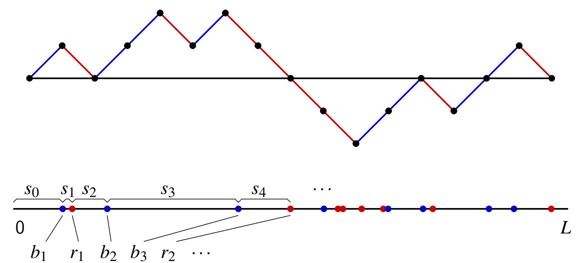


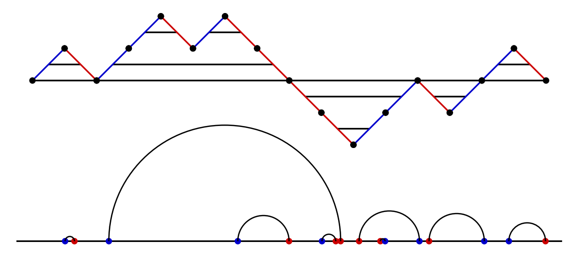
Despite this, poorly understood (McCann 1999).

IDEA: Try a canonical construction, an approximate solution sharing non-crossing property with $\pi_{\text{opt}} \to$ Dyck matchings!

The Dyck matching (Caracciolo–**D'A**–Erba–Sportiello 2020)

Construction:





The Dyck Conjecture at $p \in (0,1)$

Expected energy of Dyck matchings grows asymptotically as

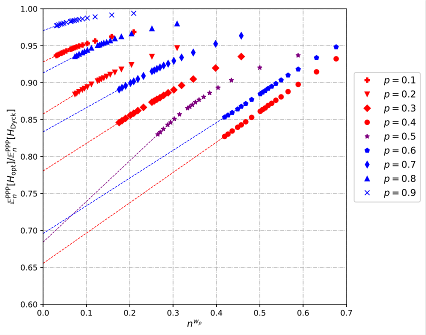
$$\mathbb{E}_n(\mathscr{H}_{\mathrm{Dyck}}) \underset{n\to\infty}{\sim} \begin{cases} n^{1-p} & \text{if} \quad 0 \le p < \frac{1}{2} \\ \sqrt{n}\ln n & \text{if} \quad p = \frac{1}{2} \\ n^{\frac{1}{2}} & \text{if} \quad \frac{1}{2} < p \le 1 \end{cases} .$$

Conjecture (Caracciolo–**D'A**–Erba–Sportiello 2020):

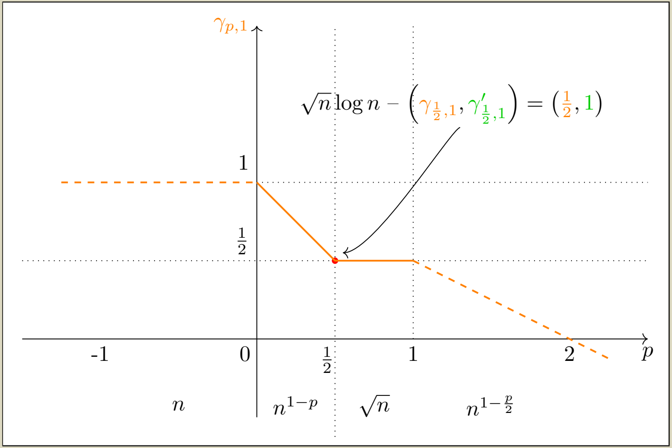
$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathscr{H}_{\mathrm{opt}})}{\mathbb{E}_n(\mathscr{H}_{\mathrm{Dyck}})} = k_p \,,$$

for some $k_p \in (0,1)$.

Upper bound implied by (Fournier–Guillin 2014).



Section of the Phase Diagram at $d = 1$



Product formula for number of solutions at $d = p = 1$ (Caracciolo–Erba–Sportiello 2021).

Beyond uniform disorder: Bulk and anomalous scaling

Recall that for $\nu = \mathbf{1}_{[0,1]}$ and $p \geq 1$ (Caracciolo *et al.* 2019):

$$E^{\mathrm{U}}_{p,1}(n) = n\frac{\Gamma\left(1+\frac{p}{2}\right)}{p+1}\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}$$
$$= c_p n^{1-\frac{p}{2}}\left(1+o(1)\right) = c_p n^{\gamma_{p,1}}\left(1+o(1)\right).$$

Thus $E^{\mathrm{U}}_{p,1}(n)$ exhibits bulk scaling.

<u>What if ν is non-uniform?</u>

Definition. If $\gamma_{p,1} \neq 1-\frac{p}{2}$ or if $\gamma'_{p,1} \neq 0$, we say that $E_{p,1}(n)$ exhibits an **anomalous scaling**.

"Reduction to quadratures" in the bulk scaling

For R the cdf of ν, let $\Psi^{(\nu)} := \nu \circ R_\nu^{-1}$. Bobkov–Ledoux 2019 get

$$E_{p,1}^{(\nu)}(n) = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \int_0^1 \left[\frac{\sqrt{s(1-s)}}{\Psi^{(\nu)}(s)}\right]^p \mathrm{d}s\, n^{1-p/2} + o\left(n^{1-p/2}\right).$$

Caracciolo–**D'A**–Sicuro 2018: **Regularize the integral** inspired by cutoff regularization in Quantum Field Theory.

Example: $\nu(x) = e^{-x}$, $\Psi^{\exp}(s) = 1 - s$.

Cutoff method: Stop integration at distance $\frac{c}{n}$ away from the singularity:

$$E_{p,1}^{\exp}(n) = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) \int_0^{1-c/n} \left(\frac{s}{1-s}\right)^{\frac{p}{2}} ds.$$

At $p = 2$, this gives:

$$E_{2,1}^{\exp}(n) = 2 \ln n - 2 \log c - 2 + o(1).$$

Exact result, $p = 2$ (Beta integrals):

$$E_{2,1}^{\exp}(n) = 2 \sum_{k=1}^{n} \frac{1}{k} = 2 \ln n + 2\gamma_E + o(1).$$

Rigorous approach to anomalous scaling

(**D'A**-Sportiello 2020-)

Guiding principle: Only the local properties of ν in a neighbourhood of the zero will determine the leading anomalous behaviour. So we decided to estimate contrib. of k-th edge at **fixed** n in

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}\big|_{p\geq 1} = \sum_{k=1}^{n} |b_k - r_k|^p.$$

The general strategy:

1. **Binomial expand** $\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}\big|_{p\geq 1}$ for $p > 1$ even and fixed n;
2. Use **linearity** to compute expected contrib. of k-th edge;
3. **Asymptotic analysis** for $n \to \infty$ ($\implies$ local properties of ν);
4. **Analytic continuation** of result $\forall p \geq 1$.

A few details on the general strategy

For a disorder ν, we have for Step 1-2:

$$E_{p,1}^{(\nu)}(n) = \sum_{k=1}^{n} \mathbb{E}\left[(b_k - r_k)^p\right] = \sum_{k=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M_{n,k,q}^{(\nu)} M_{n,k,p-q}^{(\nu)} ,$$

where the ℓ-moment of the quantile function is

$$M_{n,k,\ell}^{(\nu)} = \langle R_\nu^{-1}(u)^\ell \rangle_{P_{n,k}}$$

and $\langle \dots \rangle_{P_{n,k}}$ denotes expectation w.r.t. $\mathrm{Beta}(k, n-k+1)$:

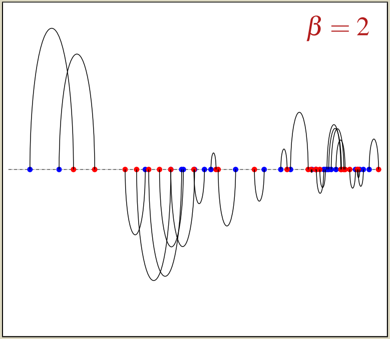
$$P_{n,k}(u)du := \frac{n!}{(k-1)!(n-k)!} u^{k-1} (1-u)^{n-k} du.$$

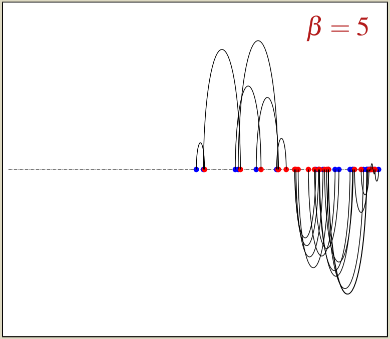
Example: Internal endpoint, algebraic zero $\nu_{\mathrm{fa},\beta}$

Consider the following pdf:

$$\nu_{\mathrm{fa},\beta}(x) = \beta x^{\beta-1}, \qquad x \in [0,1].$$

Example: Solutions at $n = 25$:





$$\text{Thus } R_{\mathrm{fa},\beta}(x) = x^{\beta} \implies R^{-1}_{\mathrm{fa},\beta}(u) = u^{\frac{1}{\beta}}.$$

Origin of bulk and anomalous regimes

Here the ℓ-moment of the quantile function is

$$M^{(\mathrm{fa}),\beta}_{k,n;\ell} = \left\langle (u^{\frac{1}{\beta}})^{\ell} \right\rangle_{P_{n,k}} = \frac{\Gamma(k+\frac{\ell}{\beta})\Gamma(n+1)}{\Gamma(k)\Gamma(n+1+\frac{\ell}{\beta})}$$

$$\underset{n\to\infty}{\sim} \begin{cases} \frac{\Gamma(k+\frac{\ell}{\beta})}{\Gamma(k)} n^{-\ell/\beta} & k \ \text{small } (\textbf{anomalous}) \\ \exp\left(-\left(\frac{\ell^2}{\beta}\right)\left(\frac{1-x^{-1}}{2n}\right)\right) x^{\ell/\beta} & k = xn' \ (\textbf{bulk}) \end{cases} .$$

Leading asymptotics in the bulk regime

Result

Lemma 1 (bulk regime, **D'A**–Sportiello 2020)

For the family of distributions $\nu_{\mathrm{fa},\beta} = \beta x^{\beta-1}$, $\beta \geq 1$,

$$E_p^{\beta}(n) \underset{n\to\infty}{\sim} b_{\beta,p}\, n^{1-p/2}, \qquad 2\beta + 2p - p\beta > 0,$$

with

$$b_{\beta,p} = \frac{1}{\beta^p} \frac{\Gamma(1+p)\,\Gamma\left(1 - \frac{p(\beta-2)}{2\beta}\right)}{\Gamma\left(2 + \frac{p}{\beta}\right)}.$$

Leading asymptotics in the anomalous regime

Result

Lemma 2 (anomalous regime, **D'A**–Sportiello 2020).

For the family of distributions $\nu_{\mathrm{fa},\beta} = \beta x^{\beta-1}$, for $\beta \geq 1$,

$$E_p^{\beta,\mathrm{anom}}(n) \underset{n\to\infty}{\sim} a_{\beta,p}\, n^{-p/\beta} ,$$

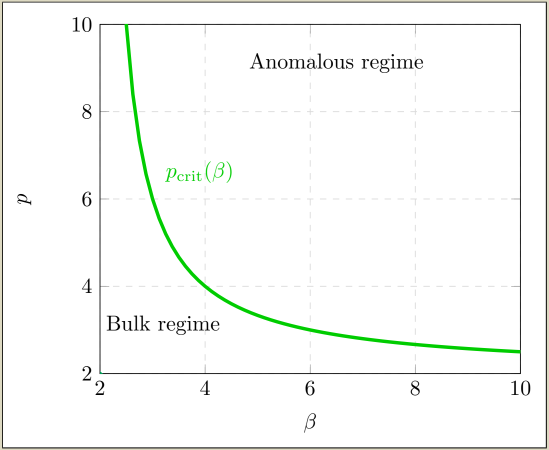
with

$$a_{\beta,p} = \Gamma(-1-p/\beta) \sum_{q=0}^{p} \binom{p}{q} (-1)^q \frac{\Gamma(1+\frac{q}{\beta})\Gamma(1+\frac{p-q}{\beta})}{\Gamma(-\frac{q}{\beta})\Gamma(-\frac{p-q}{\beta})}.$$

Bulk meets anomalous: The critical hyperbola

Equating leading exponents gives **the critical hyperbola**:

$$2\beta + 2p = p\beta, \qquad p \geq 1, \beta \geq 1.$$



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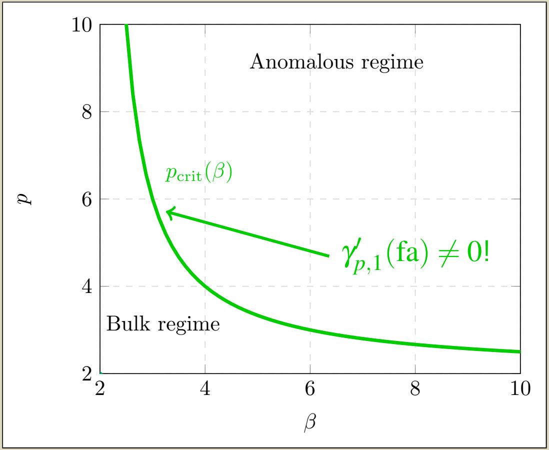


$d = 2$: A challenge for both mathematicians and physicists

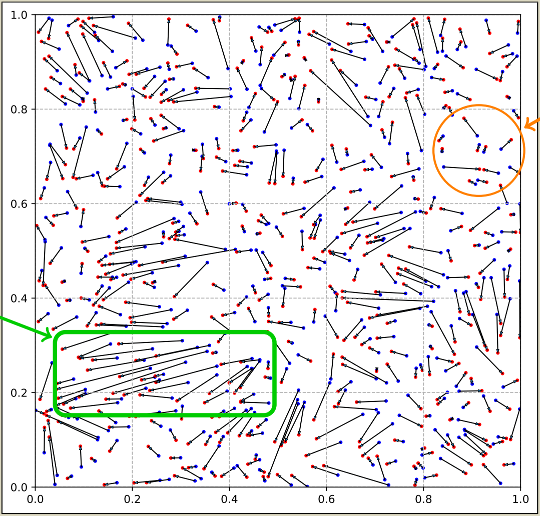
Example: $\Omega = [0,1]^2$, $\mathscr{D} = |\cdot|$. A solution at $p = 2$:

The solution π_{opt} connects typically $O(\ln n)$–**nearest-neighbors**.

$$\gamma_{p,d} = \gamma_{\text{LB}} = 1 - \frac{p}{2}$$

$$\gamma'_{p,d} = \frac{p}{2}$$

(Ajtai–Komlós–Tusnády 1984).



A few recent developments at $(p, d) = (2, 2)$

2014 Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E): Using a **classical field-theoretical approach**, first predicted $K_{2,2} = \frac{1}{2\pi}$.

2019 Ambrosio–Stra–Trevisan (PTRF): **Proof** of $K_{2,2} = \frac{1}{2\pi}$ (among other things) via PDE methods.

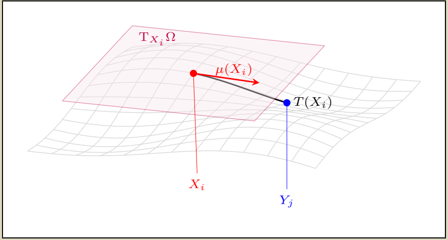
2020 Ambrosio–Glaudo (JEP): Refinement on the remainder term (among other things).

2021 Benedetto–Caglioti-Caracciolo–**D'A**–Sicuro–Sportiello (J. Stat. Phys.): Among other things, **exact formula** for $\lim_{n\to\infty} [E_\Omega(n) - E_{\Omega'}(n)]$, where Ω, Ω' are two manifolds.

2021 Caracciolo *et al.*: Excitations are compatible with a loop-erased self-avoiding random walk process.

The Caracciolo–Lucibello–Parisi–Sicuro approach

A (classical) field theory for general d and $p \geq 1$ (PRE 2014).



At $p = 2$, for a d-dimensional manifold Ω, the Lagrangian is

$$\mathscr{L}[\vec{\mu}, \phi] := \overbrace{\int_\Omega \frac{1}{2}\vec{\mu}^2(x)\nu_{\mathscr{B}}(\mathrm{d}x)}^{\text{Energy}} + \overbrace{\int_\Omega [\phi(x+\vec{\mu}(x))\nu_{\mathscr{B}}(x) - \phi(x)\nu_{\mathscr{R}}(\mathrm{d}x)]}^{\text{Transport constraint}}.$$

$\nu_{\mathscr{B}(\mathscr{R})}$ is the "charge" density of blues (reds) and ϕ is a Lagrange multiplier enforcing the transport constraint.

The Caracciolo–Lucibello–Parisi–Sicuro approach

The argument: If $|\vec{\mu}|$ is "small" when $n \to \infty \implies$ Taylor expansion $\varepsilon = |\nabla \cdot \vec{\mu}(x)|$ "small". The linearized Lagrangian is

$$\mathscr{L}_{\text{lin}}[\vec{\mu}, \phi] := \int_\Omega \left[\frac{1}{2} \vec{\mu}^2(x) + \vec{\mu}(x) \cdot \nabla \phi(x) \right] \mathrm{d}x + \int_\Omega \delta \nu(x) \phi(x) \, \mathrm{d}x.$$

The Euler-Lagrange eqs. at leading order in ε give Poisson eq. for ϕ with source $\delta \nu := \nu_{\mathscr{B}} - \nu_{\mathscr{R}}$

$$\Delta_\Omega \phi(x) = \delta \nu(x), \qquad -\Delta_\Omega = \text{Laplace-Beltrami op. on } \Omega$$

to be solved with Neumann bc on Ω (if $\partial \Omega \neq \varnothing$). Then $\vec{\mu} = -\nabla \phi$ and $E_\Omega = \int_\Omega |\vec{\mu}|^2$. Following Caracciolo–Lucibello–Parisi–Sicuro 2014, the energy writes

$$E_\Omega(n) = -2 \operatorname{Tr} \Delta_\Omega^{-1}$$

which is **bad defined** !! $\implies$ Regularizations are needed.

The regularized spectral expansion and Weyl's law

A way of rewriting Caracciolo–Lucibello–Parisi–Sicuro's regularization is

$$E_\Omega(n) = -2\,\mathrm{Tr}\,\Delta_\Omega^{-1} \simeq 2 \sum_{\lambda\in\Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n^{1/d}(\log n)^{b(d)}}\right)}{\lambda}$$

for an unknown cutoff function F **independent on** Ω (but possibly dependent on the local randomness of the ERAP), satisfying

$$\begin{cases} \lim\limits_{z\to 0+} F(z) = 1 \\ \lim\limits_{z\to\infty} F(z) = 0 \end{cases} .$$

The regularized spectral expansion and Weyl's law

Weyl's law (Ivrii 1980, Neumann b.c. case)

Let Ω be a d-dimensional manifold and $\Lambda(\Omega)$ be the spectrum of $-\Delta_\Omega$ with Neumann b.c. if $\partial\Omega \neq \varnothing$ without $\lambda = 0$. Let $\mathcal{N}_\Omega(\lambda)$ be the eigenvalue counting function. Then, for λ large,

$$\mathcal{N}_\Omega(\lambda) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{\frac{d}{2}} + \frac{\omega_{d-1}}{4(2\pi)^{d-1}}|\partial\Omega|\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})$$

- $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ = volume of standard d-ball;
- $|\Omega|$ d-dimensional volume of Ω;
- $|\partial\Omega|$ surface area of the boundary of Ω.

Example: Asymptotic energy differences at $d = 2$

Benedetto-Caglioti-Caracciolo-**D'A**-Sicuro-Sportiello, JStatPhys 2021

The precise form of F is inessential as, for two manifolds Ω, Ω',

$$\lim_{n\to\infty}\left(E_\Omega(n) - E_{\Omega'}(n)\right) = 2\lim_{n\to\infty}\left(\sum_{\lambda\in\Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda} - \sum_{\lambda\in\Lambda(\Omega')} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda}\right)$$

$$= 2\lim_{n\to\infty}\int_{0^+}^{\infty} F\left(\frac{\lambda}{n}\right)\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

$$= 2\int_{0^+}^{\infty}\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

as $\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right) = O(\sqrt{\lambda}\ln\lambda)$ at $d = 2$ (and near the origin the integral is regularized by the spectral gap).

Explicit evaluation of energy differences

Benedetto-Caglioti-Caracciolo-**D'A**-Sicuro-Sportiello, JStatPhys 2021

Main result: Even if the field theory is ill-posed (room for rigorous work!), we can give a precise experimental (and predictive!) meaning to energy differences $E_{\Omega}(n) - E_{\Omega'}(n)$ through **regularization**. We did it in two ways :

- R_{Ω} or "**Robin mass**": Integrals of the diagonal of Green's function for Poisson eq.;
- K_{Ω} or "**Kronecker mass**": Expand spectral function $Z_{\Omega}(s)$ associated to $-\Delta_{\Omega}$ around the simple pole $s = 1$.

Remark 1: Robin and Kronecker masses satisfy (Morpurgo 2002)

$$\forall \Omega, \qquad R_{\Omega} - K_{\Omega} = \frac{\ln 2}{2\pi} - \frac{\gamma_E}{2\pi} = 0.0184511\ldots$$

Remark 2: other regularizations are possible.

Example: square $\mathbb{R}$, 2-torus $\mathbb{T}$, Boy surface $\mathbb{B}$

Obtained from rectangle of aspect ratio ρ by appropriately gluing sides

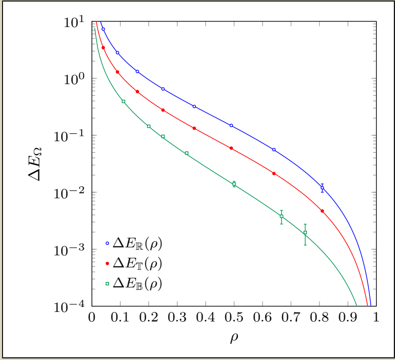
Energy shift w.r.t. manifold at aspect ratio $\rho = 1$:

$$\Delta E_\Omega(\rho) = 2(R_\Omega(\rho) - R_{\Omega'}(1)) = 2(K_\Omega(\rho) - K_{\Omega'}(1))$$

$$K_{\mathbb{R}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho|\eta(i\rho)|^4)}{4\pi} + \frac{1}{2\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$

$$K_{\mathbb{T}}(i\rho) = \frac{\gamma_E - \ln(4\pi\sqrt{\rho})}{2\pi} - \frac{1}{\pi}\ln|\eta(i\rho)|$$

$$K_{\mathbb{B}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{\ln\eta(i\rho)}{\pi} - \frac{1}{4\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$



(See JStatPhys 2021 for more manifolds).

1. Consequences of the field-theoretical approach in ERAPs at $d \geq 3$

(W.i.p. with Sergio Caracciolo, Gabriele Sicuro and Andrea Sportiello, 2020-)

Let us recall Weyl's law at $d = 3$

$$\mathcal{N}_\Omega(\lambda) = \frac{\omega_3}{(2\pi)^3}|\Omega|\lambda^{\frac{3}{2}} + \frac{\omega_{3-1}}{4(2\pi)^{3-1}}|\partial\Omega|\lambda^{\frac{3-1}{2}} + o(\lambda^{\frac{3-1}{2}}) \quad \text{as} \quad \lambda \to \infty.$$

For an ERAP at $(p,d) = (2,3)$ with unif. disorder, assuming a n.n. cutoff scale at $n^{-1/3}$, we get (omitting terms $o(\log n)$)

$$\begin{aligned} E_\Omega(n) &\simeq Kn^{\frac{1}{3}} + \frac{\pi}{4(2\pi)^2}|\partial\Omega|\, 2\int_{0^+}^{\infty} \mathrm{d}\lambda\, \frac{F\left(\frac{\lambda}{n^{2/3}}\right)}{\lambda} \\ &= Kn^{\frac{1}{3}} + \frac{1}{8\pi}|\partial\Omega|\frac{2}{3}\log n = Kn^{\frac{1}{3}} + \frac{|\partial\Omega|}{12\pi}\log n \end{aligned}$$

where $|\partial\Omega|$ is the surface area of the boundary of Ω !

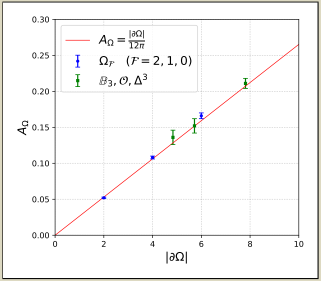
1. Consequences of Weyl's law in ERAPs at $d \geq 3$

(W.i.p. with Sergio Caracciolo, Gabriele Sicuro and Andrea Sportiello, 2020-)

This suggests the intriguing expansion

$$E_\Omega(n) = K n^{\frac{1}{3}} + \mathrm{OUT}(n) + A_\Omega \log n + o(\log n),$$

where K is a constant independent on Ω (see Barthe–Bordenave 2013), $\log n \lesssim \mathrm{OUT}(n) \lesssim n^{\frac{1}{3}}$ are Other (unknown!) Universal Terms independent on Ω, and **area term** $A_\Omega = \frac{|\partial\Omega|}{12\pi}$.

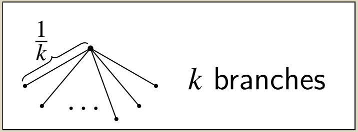


2. ERAPs on convex domains with wormholes at $d \leq 2$

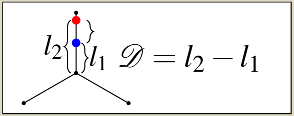
(W.i.p. with Sergio Caracciolo, Yuqi Liu and Andrea Sportiello, 2021-)

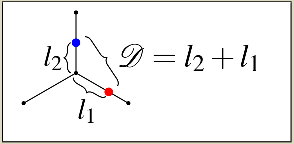
Example 1: The k-star graph. Choice of $((\Omega, \mathscr{D}), (\nu_{\mathscr{B}}, \nu_{\mathscr{R}}), p)$:

1. $\Omega = \mathbb{K}_{1,k}$ (i.e. the star graph with k branches)



with distance $\mathscr{D}$:



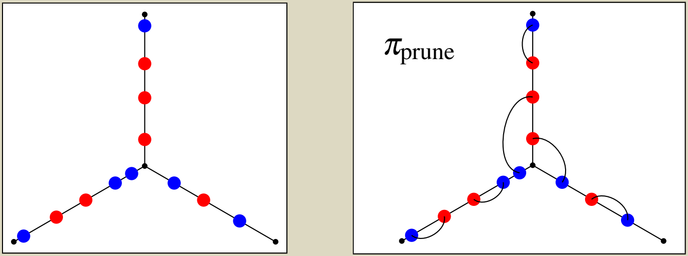


2. $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} = \nu$, $\nu \sim U[0, \frac{1}{k}] \otimes \mathscr{U}_{\{1,\dots,k\}}$;
3. $p \geqslant 1$.

On the configuration π_{prune}

IDEA: Try a canonical construction, an approximate solution sharing local ordering property with $\pi_{\mathrm{opt}} \to \underline{\pi_{\mathrm{prune}}}$!

Example: $k=3$ and $n=6$



On the configuration π_{prune}

Main result : $\mathbb{E}_\nu\left[\mathscr{H}_{\mathsf{opt}}^{(p)}\right] \underset{n\to\infty}{\sim} \mathbb{E}_\nu\left[\mathscr{H}_{\text{prune}}^{(p)}\right], \qquad p \geq 1 .$

The phase diagram: $\gamma_{p,1}^{(k)} = 1 - \frac{p}{2}$, $\gamma'^{(k)}_{p,1} = 0$.

And we get explicit (and simple) formulas for limit constants (partial moments of Brownian Bridge).

Example: At $p = 2$,

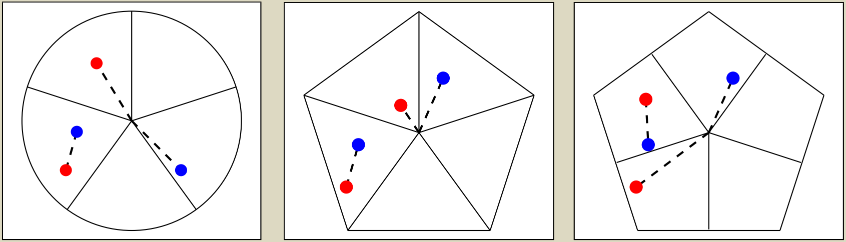
$$\lim_{n\to+\infty} \mathbb{E}\left[\mathscr{H}_{\mathsf{opt}}^{(2)}\right] = \frac{3k-2}{3k^2}.$$

2. ERAPs on convex domains with wormholes at $d \leq 2$

(W.i.p. with Sergio Caracciolo, Yuqi Liu and Andrea Sportiello, 2021-)

Example 2: k-identical $2d$ domains $(\Omega_i)_{i=1}^{k}$ "glued" by a point (wormhole) on the border.

At $k = 5$:



Model 1 Model 2 Model 3

$\nu_{\mathscr{B}} = \nu_{\mathscr{R}} \sim U(\Omega) \otimes \mathscr{U}_{\{1,\dots,k\}}$; $p = 2$. **Remark**: $\sum_{j=1}^{k} |\Omega_j| = 1$.

2. ERAPs on convex domains with wormholes at $d \leq 2$

(W.i.p. with Sergio Caracciolo, Yuqi Liu and Andrea Sportiello, 2021-)

Is the exp. contribution due to "swapping points" negligible as $n \to \infty$?

From Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E 2014), Ambrosio–Stra–Trevisan (PTRF 2019) and (Benedetto–Caglioti J. Stat. Phys. 2020), one expects for a uniform disorder:

$$E(n) \underset{n\to\infty}{\sim} \frac{\sum_{j=1}^{k} |\Omega_j|}{2\pi} \log n + o(\log n) = \frac{1}{2\pi} \log n + o(\log n).$$

However, our experimental findings show:

- The limit constant f appears to be independent on model $\ddot\smile$;
- $f > \frac{1}{2\pi}$!

2. Nice formulas in one dimensional ERAPs

(following discussions with Nathanaël Enriquez, Sophie Laruelle and Andrea Sportiello, 2021-)

Take $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu = \mathbf{1}_{[0,1]}(x)$.
Using generalised Selberg integrals, Caracciolo *et al.* 2019 have shown, for integer ℓ,

$$\mathbb{E}[|b_k - r_k|^\ell] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}.$$

This implies the exact formula for $E_{p,1}\big|_{p\geq 1}(n)$ for $f = |\cdot|^p$.

Question*: Which $f = f(|\cdot|)$ give a "nice formula" for $E(n)$ after resummation?

*Question raised by N. Enriquez (Paris-Saclay (Orsay)) at this talk for ALEA Days 2021, online at CIRM Marseilles - Luminy, 18 March 2021.

3. Dyck matchings and the clustering function in the KPKVB model

(W.i.p. with Pim van der Hoorn, 2021-)

- Let $\gamma_\alpha(k)$ be the local clustering coefficient avg. over all vertices of degree k in the KPKVB model of random graphs in hyperbolic geometry of parameter α (see here, Prop. 1.4);
- Let $E^{(p)}_{\mathrm{Dyck}}(n)$ be exp. energy of Dyck matchings in ERAP with $p \in (0,1)$ and (*e.g.*) uniform disorder (see Slide 17).

Then

$$k(n)^2\gamma_{\alpha(p)}(k(n)) \underset{k,n\to\infty}{\sim} Cn^{-p}E^{(p)}_{\mathrm{Dyck}}(n),$$

under

$$\begin{cases} k(n) = \sqrt{n} \\ \alpha(p) = \frac{1+p}{2} \end{cases} \quad \text{or} \quad \begin{cases} n(k) = k^2 \\ p(\alpha) = 2\alpha - 1 \end{cases}$$

where C depends only on α. **Question**†: Why?

†After talks by the author and by P. van der Hoorn (TU Eindhoven) at Bézout-Eurandom meeting, Institut Henri Poincaré, Paris, 7 July 2021.

4. Schrödinger problem approach for asymptotic constants

(W.i.p. with Alberto Chiarini and Oliver Tse, 2021-)

At $d \geq 3$ and $p = 2$, and uniform disorder, the PDE technique by Ambrosio–Stra–Trevisan (PTRF 2019) recovers the exponent $\gamma_{2,d} = \frac{d-2}{d}$, but the leading constants $K_{2,d}$ remain inaccessibles: $\mathscr{B}$ and $\mathscr{R}$ "remain gaussians" of arbitrary variance as $n \to \infty$ if $d \geq 3$.

A promising path to bound (or possibly get!) these $K_{2,d}$'s appears if one considers ERAP as a limit of the **Schrödinger problem** in the sense of Γ-convergence (Léonard 2012). We are currently investigating this route using **Evolutionary Variational Inequality** and the **dual Schrödinger problem** for the lower bound, and **path measure** techniques for the upper bound.

Thank you for your attention!

A useful lemma in asymptotic analysis

Lemma (see e.g. Whittaker–Watson, Tricomi)

As $n \to \infty$, for β finite, the function $\psi_n(\beta) := \ln\left[\frac{n^\beta \Gamma(n)}{\Gamma(n+\beta)}\right]$ has the series expansion

$$\psi_n(\beta) = \sum_{\substack{k\geq 1 \\ 0\leq \ell \leq k}} A_{k,l}\, n^{-k} \beta^{l+1} = \frac{\beta - \beta^2}{2n} + \frac{\beta - 3\beta^2 + 2\beta^3}{12n^2} + \cdots$$

where

$$A_{k,\ell} = (-1)^k B_{k-\ell} \frac{\Gamma(k)}{\Gamma(\ell)\Gamma(k-\ell+1)}, \qquad k \geq 1, \quad 0 \leq \ell \leq k,$$

and B_s the **s-th Bernoulli number**.

Leading asymptotics in the bulk regime

Sketch of proof

1. Lemma implies

$$E_{(\beta,p),n}(k) \underset{n\to\infty}{\sim} \frac{1}{\Gamma(p/2+1)} x^{p/\beta} \left(\frac{-1+x^{-1}}{n\beta^2} \right)^{p/2}, \qquad k = xn' \ (\mathbf{bulk}).$$

2. Put $\Lambda = \mathscr{O}(\sqrt{n})$. Then $E_p^{\beta,\mathrm{bulk}}(n) = \sum_{k=\Lambda}^{n} E_{(\beta,p),n}(k)$ can be transformed into an integral **without affecting the leading asymptotics**. This gives

$$E_p^{\beta,\mathrm{bulk}}(n) \underset{n\to\infty}{\sim} \int_{\Lambda/n}^{1} \mathrm{d}x\, x^{\frac{p}{\beta}-\frac{p}{2}} P(x)\, n^{1-p/2},$$

for a polynomial $P(x)$. **Remark**: Convergence if $2\beta + 2p - p\beta > 0$.

3. Change of variables + standard Beta-Γ identity.

Leading asymptotics in the anomalous regime

Sketch of proof

1. Recall that for $k = o(\sqrt{n})$ (**anomalous regime**)

$$\sum_{k=1}^{\Lambda} E_{(\beta,p),n}(k) \sim n^{-p/\beta} \sum_{k=1}^{\Lambda} \frac{1}{\Gamma^2(k)} \sum_{q=0}^{p} \binom{p}{q} (-1)^q \Gamma\left(k + \frac{q}{\beta}\right) \Gamma\left(k + \frac{p-q}{\beta}\right).$$

2. <u>**Remark**</u>: $\sum_k^{\Lambda}$ converges if $2\beta + 2p - p\beta < 0$. In this case we can take $\Lambda \to \infty$ (we "remove the infrared cutoff").

3. Recall the hypergeometric identity

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma^2(k)} = \Gamma(-1-a-b) \frac{\Gamma(a+1)}{\Gamma(-a)} \frac{\Gamma(b+1)}{\Gamma(-b)}.$$

$\gamma'_{p,1} \neq 0$ along the critical hyperbola

Main result

Theorem 1 (D'A–Sportiello 2020)

For the family of distributions $\nu_{\mathrm{fa},\beta}(x) = \beta x^{\beta-1}$, for $\beta \geq 1$,

$$E_p^\beta(n) \underset{n\to\infty}{\sim} \frac{p!}{(p/2)!} \left(\frac{p-2}{2p}\right)^p n^{1-p/2} \log n, \qquad 2\beta + 2p = p\beta.$$

Thus along the critical hyperbola:

$$\gamma_{p,1} = 1 - \frac{p}{2}, \qquad \gamma'_{p,1} = 1$$

independently on β!

$\gamma'_{p,1} \neq 0$ along the critical hyperbola

Sketch of proof

1. Here $E_{(n,p)} \sim E^{\text{bulk}}_{(n,p)} + E^{\text{anom}}_{(n,p)}$ at leading order in n. We need to keep a **finite cutoff** $1 \ll \Lambda(n) \ll n$ throughout the calculation.

2. $$E^{\text{bulk}}_{(n,p)} = \int_{\Lambda/n}^{1} \mathrm{d}x \, \frac{1 + xR(x)}{x} \underset{n\to\infty}{\sim} \int_{\Lambda/n}^{1} \mathrm{d}x \, \frac{1}{x} + \int_0^1 \mathrm{d}x \, R(x) \,,$$

where $R(x)$ is a polynomial.

3. One proves (after some work involving 1) hypergeometric identities and 2) the Gauss digamma theorem) that

$$E^{\text{anom}}_{(n,p)} = o(n^{1-p/2} \log n).$$