ERAPs: state of art in 1d and future perspectives

Random Geometry - Géométrie Aléatoire

CIRM Luminy, Marseilles
Tuesday, 17 January 2023, 09h00-10h00 Paris time

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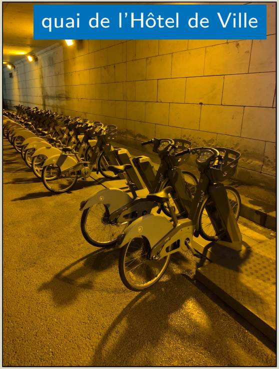
Based mainly on collaborations with:

- Dario Benedetto & Emanuele Caglioti (Rome La Sapienza)
- Sergio Caracciolo (Milan University, INFN)
- Linxiao Chen (LAGA université Sorbonne Paris Nord)
- Vittorio Erba (Lausanne EPFL)
- Yuqi Liu (former M2 student @ université Paris-Est Créteil)
- Gabriele Sicuro (London King's College)
- Andrea Sportiello (CNRS and LIPN université Sorbonne Paris Nord)

... and discussions with many others!

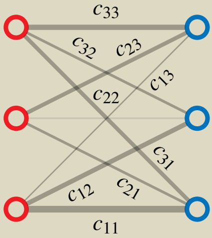
The assignment problem: A current trouble



The assignment problem: Some definitions

Def. (ASSIGNMENT PROBLEM). Consider a $n \times n$ real (cost) matrix c. For any n-permutation $\pi \in \mathscr{S}_n$, define the total cost $E(\pi) = \sum_{i=1}^{n} c_{i\pi(i)}$. Problem: Find $\pi_{\text{opt}} := \arg\min_{\pi \in \mathscr{S}_n} E(\pi)$.

Example at $n = 3$:



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

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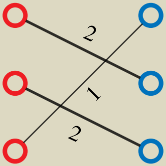
Example at $n = 3$: $\pi_{\text{opt}} = (3\,1\,2)$, $E_{\text{opt}} := E(\pi_{\text{opt}}) = 5$

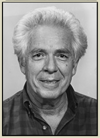


$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

- Optimization of a linear function over the **convex Birkhoff polytope**;
- P-**complete**, $\mathscr{O}(n^3)$ complexity (Munkres 1957);
- Equivalent to a two player zero-sum game (von Neumann 1953, 1954).

Assignment problem: Historical remarks

von
Neumann
1953



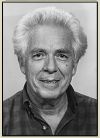
Kuhn
1955

König
1916

Egérvary
1931

Assignment problem: Historical remarks

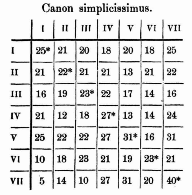
von Neumann 1953



Kuhn 1955

König 1916

Egérvary 1931



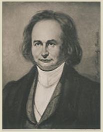
Canon simplicissimus.

	I	II	III	IV	V	VI	VII
I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

"*De investigando ordine systematis aequationum differentialium vulgarium cujuscunque*"

See also (Ollivier 2009)

Jacobi 1860



Assignment problem: Historical remarks

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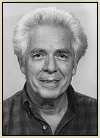

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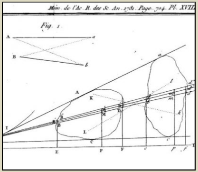
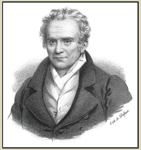


"*Le mémoire sur les déblais et les remblais*"

See this introduction at images.math.cnrs.fr (Ghys 2012)

Monge 1784



Random Assignment Problems: The case of independent weights

Now c is a **random matrix** and E_{opt} a **random variable**.

- Pioneered in Physics in the 80s by Mézard–Parisi and Orland;
- Entered Probability mostly via works of Aldous in the 90s.

Basic result: if $(c_{ij})_{i,j=1}^{n}$ are i.i.d. r.v.s of pdf $\rho(l) = l^r + o(l^r)$, then

$$\mathbb{E}[E_{\text{opt}}]_n \underset{n\to\infty}{\sim} c_r n^{1-\frac{1}{r+1}}.$$

Only "infinitely short" edges are relevant for large n: r plays the role of a "**universal exponent**".

Fact: At $r = 0$ (i.e. ρ is e.g. uniform or $\mathrm{Exp}(\lambda)$ distribution),

$$c_0 = \zeta(2) = \frac{\pi^2}{6}.$$

The Parisi conjecture (1998)

Let $c_{ij} \sim \mathrm{Exp}(1)$, $i, j = 1, \ldots, n$, iid. Then

$$\mathbb{E}[E_{\min}]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

- Extension to rectangular cost matrices (Coppersmith-Sorkin 1998);
- Proof of $\zeta(2)$ limit (among other things) (Aldous 2001);
- **Proof of Parisi conjecture** (Nair–Prabhakar–Sharma 2001);
- Extension to k-partite graphs (which is **NP-hard** for $k \geq 3$) (Martin–Mézard–Rivoire 2004,2005);
- Proof of $\exists!$ of Mézard–Parisi order parameter $\forall r \geq 0$ (Wästlund 2012, Larsson 2014, Salez 2015).

NOT discussed today...

Euclidean Random Assignment Problems (ERAPs)

Let $\mathscr{B} = (B_1, \dots, B_n)$ be blue points and $\mathscr{R} = (R_1, \dots, R_n)$ be red ones: n-samples of i.i.d. r.v. with pdf $\nu_{\mathscr{B}(\mathscr{R})} : \Omega \to \mathbb{R}$ (**disorder**). Let $(\Omega, \mathscr{D})$ be a Polish metric space (mostly an **Euclidean** space with $\mathscr{D}$ **Euclidean** distance). For $p \in \mathbb{R}$ and an assignment (or n-permutation) $\pi \in \mathscr{S}_n$, consider the *Hamiltonian*

$$\mathscr{H}(\pi) = \sum_{i=1}^{n} \mathscr{D}^p(B_i, R_{\pi(i)})$$

and the random variable **(ground state energy)**:

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = \min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi) \quad (\text{and } \pi_{\mathrm{opt}} = \arg\min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi)).$$

Def. Euclidean Random Assignment Problem (**ERAP**)
Understand the statistical properties of $\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}$ (*exact* or *asymptotic* law, moments, etc.) depending on (Ω, p, d) and $\nu_{\mathscr{B}(\mathscr{R})}$.

Three main motivations for ERAPs

- **Spin Glasses.** ERAP provides a toy-model of **spin-glass in finite dimension**. Besides disorder, the assignment constraint provides **frustration**. But the model is numerically simpler than e.g. Edwards–Anderson spin glass (Mézard–Parisi 1988). x
- **Optimal Transport.** ERAP = **Monge-Kantorovitch** transportation problem on Ω $(\dim(\Omega) = d)$ associated to the empirical measures $\rho_{\mathscr{B}(\mathscr{R})} = \frac{1}{n}\sum_j \delta_{B_j(R_j)}$. In particular

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = n W_p^p(\rho_{\mathscr{B}}, \rho_{\mathscr{R}}),$$

with W_p is p-**Wasserstein dist.** (Villani 2009, Brezis 2018).
- **Computational Complexity Theory.** ERAPs are small modifications of random TSPs; but TSP is NP-**complete**.

A tool for understading ERAPs: Phase diagram

Let us put $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu$. We can start studying

$$E_{p,d}(n) := \mathbb{E}_{\nu^n \otimes \nu^n}[\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1+o(1)),$$

as $n \to \infty$, depending on (p,d) and the choice of ν.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be *largely insensitive* on the choice of ν (which may alter the constant $K_{p,d}$).

Remark: Non-uniform disorder ν is more subtle!

Example: Take $\nu =$ standard Gaussian and $(p,d) = (2,1)$. Then

$$E_{2,1}(n) \underset{n\to\infty}{\sim} 2 \ln\ln n \qquad \text{(i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

(Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019, Berthet–Fort 2020)

See (Benedetto–Caglioti 2020) for non-uniform case at $d = 2$.

$d \geq 3$, $p \geq 1$, $\Omega =$ a bounded domain

"Simple": Solution is realized at the scale of nearest-neighbors

$$E_{p,d}(n)\Big|_{d\geq 3, p\geq 1} \underset{n\to\infty}{\sim} K_{p,d}\, n^{\gamma_{\mathrm{LB}}},$$

where

$$\gamma_{p,d} = \gamma_{\mathrm{LB}} := 1 - \frac{p}{d}, \quad \gamma'_{p,d} = 0 \qquad \text{(Mézard–Parisi 1988)}$$

(if the disorder ν is uniform on Ω, otherwise **unknown**).

Remark: The constants $K_{p,d}$ are **universal** (Barthe–Bordenave 2013 and refs. therein for $p < \frac{d}{2}$, and Goldman–Trevisan 2020 for an extension to $p \geq 1$) but **unknown explicitly**. Upper and lower bounds on some $K_{p,d}$ for $\Omega = [0,1]^d$ are in (Talagrand 1992), numerical estimates for $\Omega = [0,1]^d, \mathbb{T}^d$ are in (Caracciolo–Sicuro 2015, **D'A** MSc Thesis 2016).

$d = 2$: A challenge for mathematicians and physicists

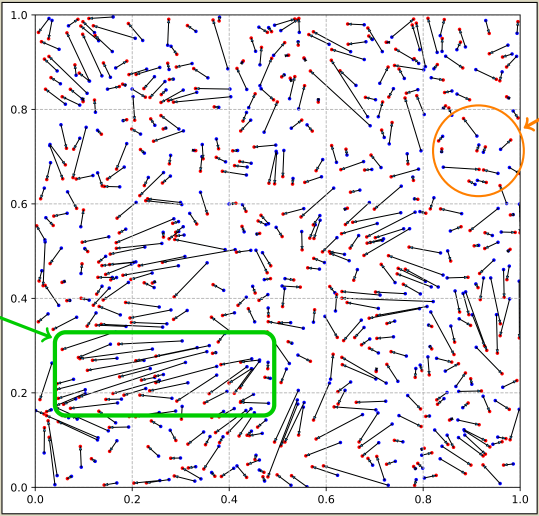
Example: $\Omega = [0,1]^2$, $\mathscr{D} = |\cdot|$. A solution at $p = 2$:

The solution $\pi_{\rm opt}$ connects typically $O(\ln n)$–**nearest-neighbors**.

$$\gamma_{p,d} = \gamma_{\rm LB} = 1 - \frac{p}{2}$$

$$\gamma'_{p,d} = \frac{p}{2}$$

(Ajtai–Komlós–Tusnády 1984).



A few recent developments at $(p,d) = (2,2)$

2014 Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E): Using a **classical field-theoretical approach**, first predicted $K_{2,2} = \frac{1}{2\pi}$.

2019 Ambrosio–Stra–Trevisan (PTRF): **Proof** of $K_{2,2} = \frac{1}{2\pi}$

2020 Ambrosio–Glaudo (JEP): Refinement on the **remainder term**

2020 Benedetto–Caglioti (J. Stat. Phys.): non-constant densities (UB)

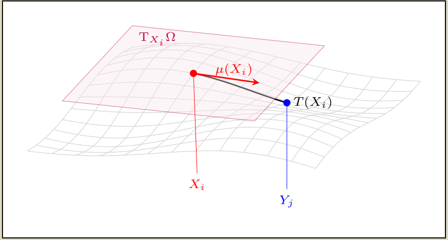
2021 Benedetto–Caglioti-Caracciolo–**D'A**–Sicuro–Sportiello (J. Stat. Phys.): **Exact formula** for $\lim_{n\to\infty} [E_\Omega(n) - E_{\Omega'}(n)]$, where Ω, Ω' are two manifolds.

2021 Caracciolo *et al.*: Link with loop-erased **SARW**.

2022 Ambrosio–Goldman–Trevisan: **proof** of universality of $K_{2,2}$.

The Caracciolo–Lucibello–Parisi–Sicuro approach

A (classical) field theory for general d and $p \geq 1$ (PRE 2014).



At $p = 2$, for d-dimensional Riemannian manif. Ω, consider

$$\mathscr{L}[\vec{\mu}, \phi] := \overbrace{\int_\Omega \frac{1}{2}\vec{\mu}^2(x)\nu_{\mathscr{B}}(\mathrm{d}x)}^{\text{Energy}} + \overbrace{\int_\Omega [\phi(x+\vec{\mu}(x))\nu_{\mathscr{B}}(x) - \phi(x)\nu_{\mathscr{R}}(\mathrm{d}x)]}^{\text{Transport constraint}}.$$

$\nu_{\mathscr{B}(\mathscr{R})}$ is the "charge" density of blues (reds) and ϕ is a Lagrange multiplier enforcing the transport constraint.

Caracciolo–Lucibello–Parisi–Sicuro (CLPS) theory

Heuristic: $|\vec{\mu}|$ "small" when $n \to \infty \implies$ Taylor expansion in $\varepsilon = |\nabla \cdot \vec{\mu}(x)|$ "small". Linearize $\mathscr{L}[\vec{\mu}, \phi]$ to

$$\mathscr{L}_{\mathrm{lin}}[\vec{\mu}, \phi] := \int_\Omega \left[\frac{1}{2} \vec{\mu}^2(x) + \vec{\mu}(x) \cdot \nabla \phi(x) \right] \mathrm{d}x + \int_\Omega \delta \nu(x) \phi(x) \, \mathrm{d}x.$$

Euler-Lagrange eqs. at leading order in ε is Poisson eq. for the potential ϕ with source $\delta \nu := \nu_{\mathscr{B}} - \nu_{\mathscr{R}}$

$$\Delta_\Omega \phi = \delta \nu, \qquad -\Delta_\Omega = \textbf{Laplace-Beltrami op. on } \Omega$$

to be solved with Neumann bc on Ω (if $\partial\Omega \neq \varnothing$).
Then $\vec{\mu} = -\nabla \phi$ and E_Ω "$=$" $\int_\Omega |\vec{\mu}|^2$. Following CLPS 2014, the magic formula is

$$E_\Omega(n) \text{ "$=$" } -2 \, \mathrm{Tr} \, \Delta_\Omega^{-1}$$

which is **very ill defined** !! $\implies$ Need for regularization(s)

Random assignment problems on 2d manifolds

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Generalized **Caracciolo–Lucibello–Parisi–Sicuro's magic formula** (conjecture): for a compact Riemannian manifold Ω

$$E_\Omega(n) = -2\,\mathrm{Tr}\,\Delta_\Omega^{-1} \simeq 2 \sum_{\lambda \in \Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n^{1/d}(\log n)^{b(d)}}\right)}{\lambda}$$

for an unknown cutoff function F **independent on** Ω (but possibly dependent on the local randomness of the ERAP), satisfying

$$\begin{cases} \lim_{z \to 0+} F(z) = 1 \\ \lim_{z \to \infty} F(z) = 0 \end{cases} .$$

Random assignment problems on 2d manifolds

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Weyl's law (Ivrii 1980, Neumann b.c. case)

Let Ω be a d-dimensional manifold and $\Lambda(\Omega)$ be the spectrum of $-\Delta_\Omega$ with Neumann b.c. if $\partial\Omega \neq \varnothing$ without $\lambda = 0$. Let $\mathscr{N}_\Omega(\lambda)$ be the eigenvalue counting function. Then, for λ large,

$$\mathscr{N}_\Omega(\lambda) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{\frac{d}{2}} + \frac{\omega_{d-1}}{4(2\pi)^{d-1}}|\partial\Omega|\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})$$

- $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ = volume of unit d-ball;
- $|\Omega|$ d-dimensional volume of Ω;
- $|\partial\Omega|$ $(d-1)$-dimensional volume of $\partial\Omega$.

On asymptotic energy differences at $d = 2$

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Argument: Precise form of F is inessential in $d=2$. For $\Omega \neq \Omega'$,

$$\lim_{n\to\infty}\left(E_\Omega(n) - E_{\Omega'}(n)\right) = 2\lim_{n\to\infty}\left(\sum_{\lambda\in\Lambda(\Omega)}\frac{F\left(\frac{\lambda}{n}\right)}{\lambda} - \sum_{\lambda\in\Lambda(\Omega')}\frac{F\left(\frac{\lambda}{n}\right)}{\lambda}\right)$$

$$= 2\lim_{n\to\infty}\int_{0^+}^{\infty} F\left(\frac{\lambda}{n}\right)\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

$$= 2\int_{0^+}^{\infty}\frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

as $\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right) = O(\sqrt{\lambda}\ln\lambda)$ at $d=2$ (and near the origin the integral is regularized by the spectral gap).

Explicit evaluation of energy differences

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

Main result: $\lim_{n\to\infty}(E_\Omega(n) - E_{\Omega'}(n))$ through **regularization**.
Done it in two ways:

- R_Ω or "**Robin mass**": Integrals of the diagonal of Green's function for Poisson eq.;
- K_Ω or "**Kronecker mass**": Residue of the spectral zeta function $Z_\Omega(s)$ associated to $-\Delta_\Omega$ at $s = 1$ (simple pole).

Remark 1: Robin and Kronecker masses satisfy (Morpurgo 2002)

$$\forall\Omega \text{ compact,} \qquad R_\Omega - K_\Omega = \frac{\ln 2}{2\pi} - \frac{\gamma_E}{2\pi} = 0.0184511\ldots$$

Remark 2: other regularizations are possible ...

Example: Rectangles $\mathbb{R}$, 2-torus $\mathbb{T}$, Boy surface $\mathbb{B}$

Obtained from rectangle of aspect ratio ρ by appropriately gluing sides

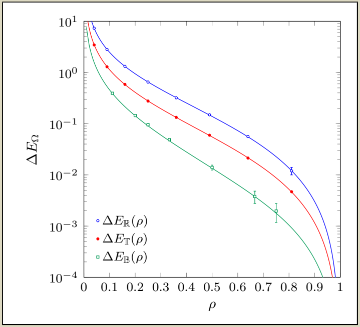
Energy shift w.r.t. manifold at aspect ratio $\rho = 1$:

$$\Delta E_\Omega(\rho) = 2(R_\Omega(\rho) - R_{\Omega'}(1)) = 2(K_\Omega(\rho) - K_{\Omega'}(1))$$

$$K_{\mathbb{R}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho|\eta(i\rho)|^4)}{4\pi} + \frac{1}{2\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$

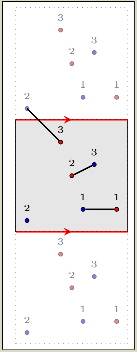
$$K_{\mathbb{T}}(i\rho) = \frac{\gamma_E - \ln(4\pi\sqrt{\rho})}{2\pi} - \frac{1}{\pi}\ln|\eta(i\rho)|$$

$$K_{\mathbb{B}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{\ln\eta(i\rho)}{\pi} - \frac{1}{4\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$

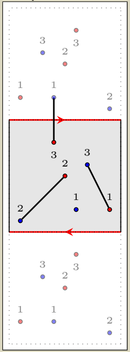


More manifolds

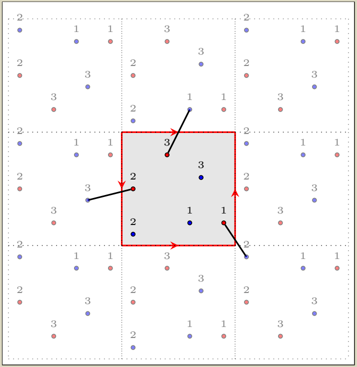
cylinder $\mathbb{C}$



Möebius strip $\mathbb{M}$



Klein bottle $\mathbb{K}$



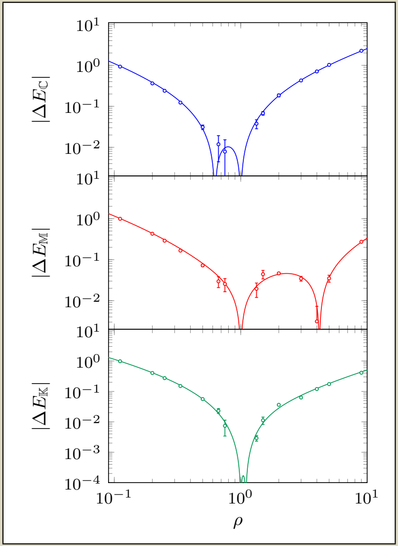
More manifolds

$$|\Delta E_\Omega| = |K_\Omega(\rho) - K_\Omega(1)|$$

$$K_{\mathbb{C}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(16\pi^2\rho)}{4\pi} - \frac{1}{\pi}\log\eta(2i\rho) + \frac{\zeta(2)}{4\pi^2\rho}$$

$$K_{\mathbb{M}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{1}{\pi}\log\frac{\eta^3(i\rho)}{\eta(i2\rho)\eta\left(i\frac{\rho}{2}\right)} + \frac{\zeta(2)}{4\pi^2\rho}$$

$$K_{\mathbb{K}}(\rho) = \frac{\gamma_E}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{1}{\pi}\ln\eta\left(i\frac{\rho}{2}\right) - \frac{\zeta(2)}{2\pi^2\rho}$$

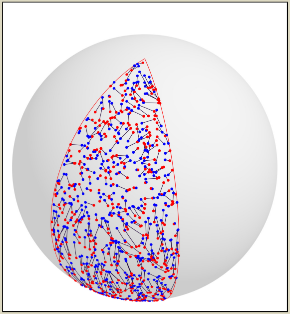


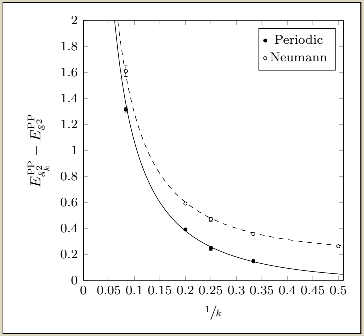
Remark: $\Delta E(\rho)$ displays minima at non-trivial values of ρ!

Manifolds with singular curvature

Benedetto–Caglioti–Caracciolo–**D'A**–Sicuro–Sportiello J.Stat.Phys. 2021

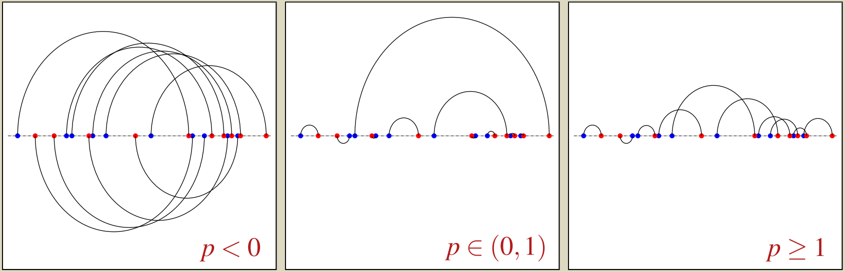
Empirical observation: approach works also for non-flat surfaces (2-sphere) and surfaces with singular curvature (conical singularities), such as the circular sector or the spherical lune:





$d = 1$: Qualitative properties of a solution π_{opt}

For any ν, $p = 0$ and $p = 1$ separate **three qualitative regimes**:



If $(b_1, \ldots, b_n)$ and $(r_1, \ldots, r_n)$ are sorted in natural order π_{opt} is:

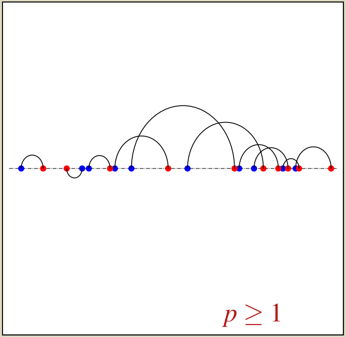
- **Cyclical** for $p < 0$: $\pi_{\mathrm{opt}} = i + k \pmod{n}$ (Caracciolo–**D'A**–Sicuro 2017);
- **Non-crossing** for $p \in (0, 1)$: Intervals "covered by edges" are either disjoint or one is included into the other (McCann 1999);
- **Ordered** for $p \geq 1$: π_{opt} is the identity permutation.

$\Omega = [0, 1]$ (or $\mathbb{R}, \mathbb{R}^+, \ldots$), $\mathscr{D} = |\cdot|$ and $p \geq 1$

Take $B_{j+1} \geq B_j$, $R_{j+1} \geq R_j$, for $j = 1, \ldots, n-1$.

Optimality + **(strict) convexity** + (strict) monotonicity of $\mathscr{D}^p$

$$\Downarrow$$



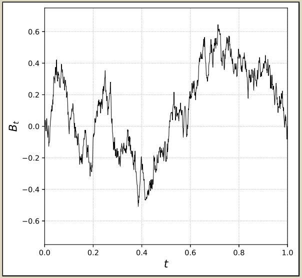
$$\mathscr{H}_{\text{opt},(n,p)}\Big|_{p\geq 1} = \sum_{i=1}^{n} |B_i - R_i|^p.$$

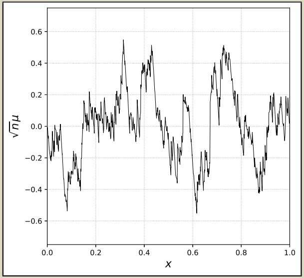
$\Longrightarrow$ Stairway to the **Brownian world!**

Brownian Bridge for $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$, $p \geq 1$

Let the **transport field** be $\mu_i := B_i - R_i$, for $i = 1, \dots n$ and put $i = nt + \frac{1}{2}$. Then by **Donsker's Theorem**,

$$\sqrt{n}\,\mu_i \underset{weakly}{\longrightarrow} B_t, \text{ the Brownian Bridge.}$$





Recall $\mathscr{H}_{\mathrm{opt},(n,p)} = \sum_{i=1}^{n} |\mu_i|^p$. Then $E_{p,1}(n)\big|_{p\geq 1} \underset{n\to\infty}{\sim} \mathbb{E}[B_t^p]\, n^{1-\frac{p}{2}}$.

(Boniolo–Caracciolo–Sportiello 2014, Caracciolo–Sicuro 2014, Caracciolo–**D'A**–Sicuro 2017)

$\Omega=\mathbb{S}_1$, $\mathscr{D}=$ arc-distance, $p=2$: The limit distribution of $\mathscr{H}_{\mathrm{opt}}$

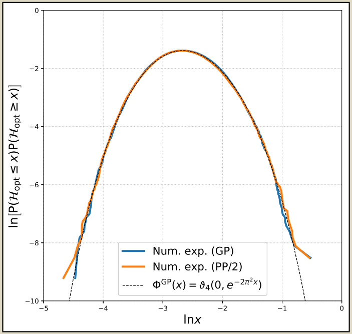
The $n\to\infty$ cdf of $\mathscr{H}_{\mathrm{opt},(n,2)}$ on $\mathbb{S}_1$ is **explicit**. Sketch of proof:

1. At $p=2$ we have **Parseval identity** for μ;
2. The mgf $\mathbb{E}\left[e^{-w\sum_{s\neq 0}|\hat{\mu}|_s^2}\right]=\prod_{s\geq 1}\frac{1}{1+\frac{w}{2\pi^2 s^2}}$ can be inverse-Laplace transformed in closed form (Watson 1961).

We have (D'A, 2020):

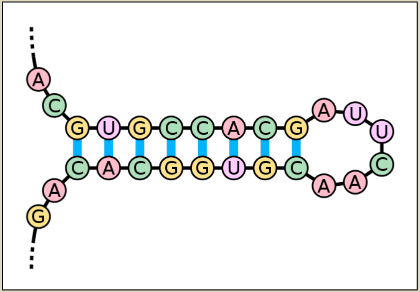
$$\mathbb{P}[\mathscr{H}_{\mathrm{opt},(n,2)}\leq x]\underset{n\to\infty}{\to}\vartheta_4(0,e^{-2\pi^2 x})$$
$$:=\sum_{s\in\mathbb{Z}}e^{i\pi s}e^{-2\pi^2 s^2 x}.$$

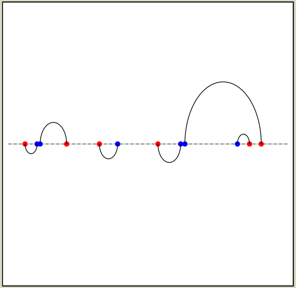
$$\implies \mathscr{H}_{\mathrm{opt},(\infty,2)}\overset{\text{law}}{=}\textstyle\sum_{k\geq 1}\frac{E_k}{k^2}$$
(Biane—Pitman–Yor 2001)



ERAPs at $p \in (0,1)$: Further motivation in biology

Besides economics, interest due to the non-crossing property of the solution: Toy-model for the **secondary structure of RNA** (discarding pseudo-knots).



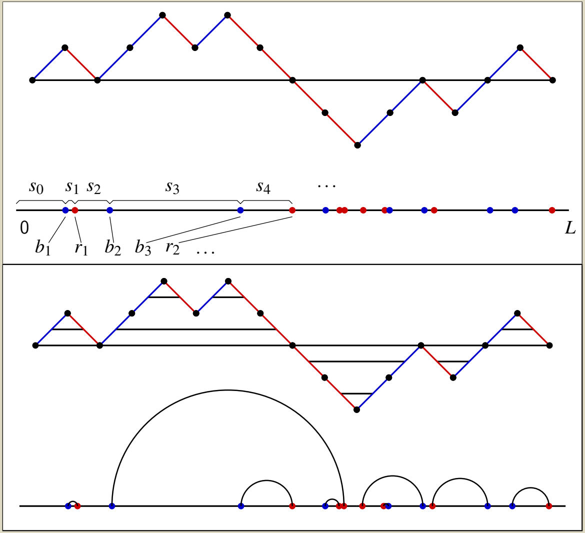


Despite this, poorly understood (McCann 1999).

IDEA: Try a canonical construction, an approximate solution sharing non-crossing property with $\pi_{\text{opt}} \to$ Dyck matchings!

The Dyck matching (Caracciolo–**D'A**–Erba–Sportiello 2020)

Construction:



The Dyck Conjecture at $p \in (0,1)$

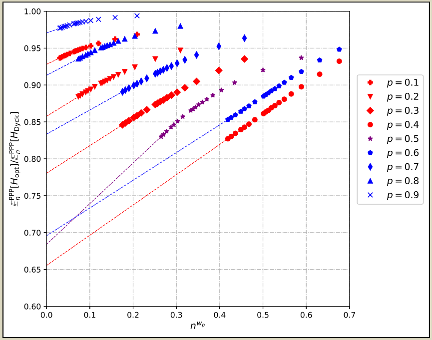
Expected energy of Dyck matchings grows asymptotically as

$$\mathbb{E}_n(\mathscr{H}_{\text{Dyck}}) \underset{n\to\infty}{\sim} \begin{cases} n^{1-p} & \text{if} \quad 0 \le p < \frac{1}{2} \\ \sqrt{n}\ln n & \text{if} \quad p = \frac{1}{2} \\ n^{\frac{1}{2}} & \text{if} \quad \frac{1}{2} < p \le 1 \end{cases} .$$

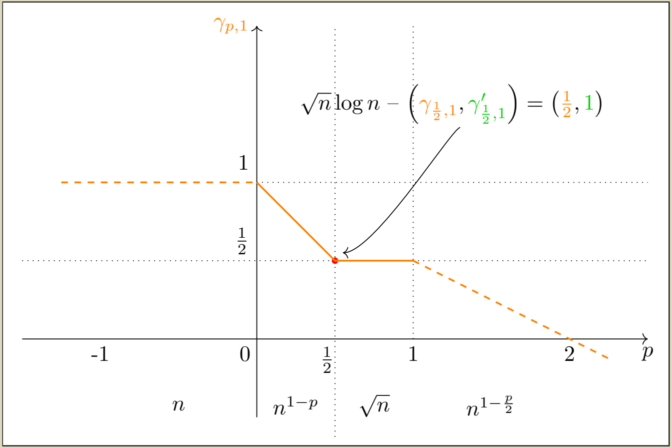
Conjecture (Caracciolo–**D'A**–Erba–Sportiello 2020):

$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathscr{H}_{\text{opt}})}{\mathbb{E}_n(\mathscr{H}_{\text{Dyck}})} = k_p \,,$$

for some $k_p \in (0,1)$.
Upper bound implied by (Fournier–Guillin 2014).



Section of the phase diagram at $d = 1$



Product formula for number of solutions at $d = p = 1$ (Caracciolo–Erba–Sportiello 2021).

Beyond uniform disorder: Bulk and anomalous scaling

Recall that for $\nu = \mathbf{1}_{[0,1]}$ and $p \geq 1$ (Caracciolo *et al.* 2019):

$$E^{\mathrm{U}}_{p,1}(n) = n\frac{\Gamma\left(1+\frac{p}{2}\right)}{p+1}\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}$$
$$= c_p n^{1-\frac{p}{2}}\left(1+o(1)\right) = c_p n^{\gamma_{p,1}}\left(1+o(1)\right).$$

Thus $E^{\mathrm{U}}_{p,1}(n)$ exhibits bulk scaling.

What if ν is non-uniform?

Def.(ANOMALOUS SCALING). If $\gamma_{p,1} \neq 1-\frac{p}{2}$ or if $\gamma'_{p,1} \neq 0$, we say that $E_{p,1}(n)$ exhibits an **anomalous scaling**.

"Reduction to quadratures" in the bulk scaling

For R the cdf of ν, let $\Psi^{(\nu)} := \nu \circ R_\nu^{-1}$. Bobkov–Ledoux 2019 get

$$E_{p,1}^{(\nu)}(n) = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\int_0^1 \left[\frac{\sqrt{s(1-s)}}{\Psi^{(\nu)}(s)}\right]^p \mathrm{d}s\, n^{1-p/2} + o\left(n^{1-p/2}\right).$$

Caracciolo–**D'A**–Sicuro 2018: **Regularize the integral** inspired by cutoff regularization in Quantum Field Theory.

Example: $\nu(x) = e^{-x}$, $\Psi^{\exp}(s) = 1 - s$.

Cutoff method: Stop integration at distance $\frac{c}{n}$ away from the singularity:

$$E_{p,1}^{\exp}(n) = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\int_0^{1-c/n}\left(\frac{s}{1-s}\right)^{\frac{p}{2}} ds.$$

At $p = 2$, this gives:

$$E_{2,1}^{\exp}(n) = 2\ln n - 2\log c - 2 + o(1).$$

Exact result, $p = 2$ (Beta integrals):

$$E_{2,1}^{\exp}(n) = 2\sum_{k=1}^{n}\frac{1}{k} = 2\ln n + 2\gamma_{\mathsf{E}} + o(1).$$

A rigorous approach to anomalous scaling

(**D'A**-Sportiello 2020-)

Guiding principle: Only the local properties of ν in a neighbourhood of the zero should matter for determining the leading anomalous behaviour.

Idea: estimate contrib. of k-th edge at **fixed** n in

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}\big|_{p\geq 1} = \sum_{k=1}^{n} |b_k - r_k|^p.$$

General strategy:

1. Fix n and $p > 1$ even, then **binomial expand** $\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}\big|_{p\geq 1}$;
2. Use **linearity** to compute expected contrib. of k-th edge;
3. **Asymp. analysis** $n \to \infty$ (local props. of ν enter the game);
4. **Analytic continuation** of result $\forall p \geq 1$.

A few details on the general strategy

For a disorder ν, and $p > 1$ even, we have for Step 1-2:

$$E_{p,1}^{(\nu)}(n) = \sum_{k=1}^{n} \mathbb{E}\left[(b_k - r_k)^p\right] = \sum_{k=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M_{n,k,q}^{(\nu)} M_{n,k,p-q}^{(\nu)} ,$$

where the ℓ-moment of the quantile function is

$$M_{n,k,\ell}^{(\nu)} = \langle R_\nu^{-1}(u)^\ell \rangle_{P_{n,k}}$$

and $\langle \dots \rangle_{P_{n,k}}$ denotes expectation w.r.t. $\mathrm{Beta}(k, n-k+1)$:

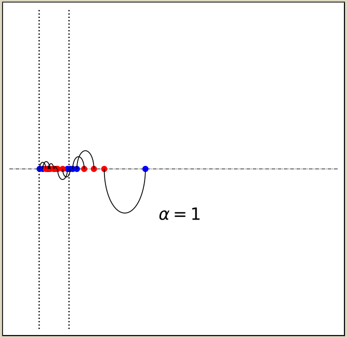
$$P_{n,k}(u)du := \frac{n!}{(k-1)!(n-k)!} u^{k-1} (1-u)^{n-k} du.$$

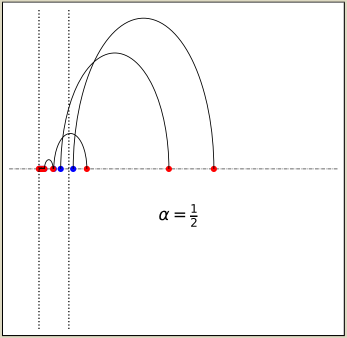
Application: stretched exponential distribution

Consider for $\alpha > 0$ the following pdf:

$$\rho_{\mathrm{ie},\alpha}(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha}), \qquad x \geq 0.$$

Example: solutions at $n = 10$





So $R_\alpha(x) = \exp(-x^\alpha) \implies R_\alpha^{-1}(u) = (-\ln u)^{\frac{1}{\alpha}}$.

Step I. Reduction to complete homogeneous symmetric functions

For $s = \frac{1}{\alpha} \in \mathbb{Z}^+$ we need the s-th moment of the quantile function

$$M_{n,k;s} \stackrel{\text{def}}{=} \int_0^1 \mathrm{d}u \, P_{n,k}(u)(-\ln u)^s.$$

Theorem I (D'A-Sportiello).

$$M_{n,k;s} = s! \, h_s(A_{k,n}),$$

where $A_{k,n} := \left\{\frac{1}{k}, \frac{1}{k+1}, \dots, \frac{1}{n}\right\}$ and h_s is the *complete homogeneous symmetric function* of degree s*.

*I.e. for alphabet $(x_1, \dots, x_m)$, $h_s(x_1, \dots, x_m) := \sum_{1 \le j_1 \le \dots \le j_s \le m} x_{j_1} x_{j_2} \cdots x_{j_s}$.

Step I. A simple preliminary Lemma

Lemma (D'A–Sportiello). For $p \in \mathbb{N}$, let $A(q) = a_p q^p + a_{p-1} q^{p-1} + \ldots + a_0$ be a polynomial of degree at most p. Then

$$\sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} A(q) = p!\, a_p.$$

Proof. 1. $A(q) = \sum_{k=0}^{p} b_k q(q-1)\cdots(q-k+1)$ ($\implies b_p = a_p$).

2. $$\sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} A(q) = \sum_{k=0}^{p} b_k \sum_{q=k}^{p} \frac{p!}{q!(p-q)!} (-1)^{p-q} q(q-1)\cdots(q-k+1)$$

$$= \sum_{k=0}^{p} \frac{b_k\, p!}{(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} (-1)^{(p-k)-r} = \sum_{k=0}^{p} \frac{b_k\, p!}{(p-k)!} \delta_{p,k} = p!\, b_p. \qquad \blacksquare$$

Step I. Proof of Theorem I

Main idea: a rigorous replica method-like trick.

First, use $\ln \frac{1}{t} = \lim_{u \downarrow 0} \frac{t^{-u}-1}{u}$, write $M_{n,k;p} = \lim_{u \downarrow 0} M_{n,k;p}(u)$, where ($p \geq 2$ even)

$$M_{n,k;p}(u) := \int_0^1 dt\, t^{k-1}(1-t)^{n-k} \left(\frac{t^{-u}-1}{u} \right)^p \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} .$$

Second, expand binomial and integrate term by term (Beta integrals)

$$\begin{aligned} M_{n,k;p}(u) &= u^{-p} \langle (t^{-u}-1)^p \rangle_{P_{n,k}} = u^{-p} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} \prod_{h=k}^{n} \left(1 - \frac{qu}{h}\right)^{-1} \\ &= u^{-p} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} \sum_{\ell \geq 0} q^{\ell} u^{\ell} h_{\ell}(A_{k,n}) . \end{aligned}$$

Finally use Lemma and take the limit $u \downarrow 0$. $\blacksquare$

Step II. Generating functions

Let $E_{(s,p),n}(k)$ be the contribution of the k-th edge to the total energy $E_{(s,p)}(n) = \sum_{k=1}^{n} E_{(s,p),n}(k)$. Then

$$E_{(s,p),n}(k) = \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M_{n,k;sq} M_{n,k;s(p-q)}$$

$$\overset{\text{Thm I}}{=} \sum_{q=0}^{p} (-1)^{q} \binom{p}{q} (sq)!(s(p-q))!\, h_{sq}(A_{k,n}) h_{s(p-q)}(A_{k,n}).$$

In the next step, we make use of **generating functions**.

Step II. Generating functions

$$H(t,X) = \sum_{\ell \geq 0} h_\ell(X) t^\ell = \prod_{x \in X} (1 - tx)^{-1}.$$

Remark. For $\alpha = s = 1$ (i.e. disorder $\sim$ Exp(1)), we have a first miracle: since for $p \geq 2$ even†

$$\begin{aligned} E_{(1,p),n}(k) &= p! \sum_{q=0}^{p} (-1)^q h_q(A_{k,n}) h_{p-q}(A_{k,n}) \\ &= p! [t^p] H(t, A_{k,n}) H(-t, A_{k,n}) \,, \end{aligned}$$

and $\forall$ alphabet X $H(t,X)H(-t,X) = H(t^2, X^2)$,

$$E_{(1,p),n}(k) = p! [t^p] H(t^2, A_{k,n}^2) = p! \, h_{p/2} \left(A_{k,n}^2 \right).$$

† $[t^p] f(t)$ is the coeff. of t^p in the Taylor expansion of $f(t)$ around $t = 0$.

Step II. Asymptotic result at $s > 1$ integer

For $s > 1$ it is more cumbersome: one needs to carefully estimate the first power-sum function associated to $H(t,X)$. After some computation, we get

$$E_{(s,p),n}(k) = p!\, s^p \ln(n)^{(s-1)p}\, h_{p/2}(A_{k,n}^2)\big(1 + O((1+\ln k)/\ln n)\big).$$

In order to estimate $E_{(s,p)}(n) = \sum_{k\geq 1} E_{(s,p),n}(k)$ and conclude, we are left with the task of estimating ($q = \frac{p}{2}$)

$$F_{n,q} \stackrel{\text{def}}{=} \sum_{k=1}^{n} h_q(A_{k,n}^2), \quad n \to \infty\,.$$

Step III. Multiple ζ^* values

Obser. Each term contributing to $F_{n,q}$ is of the form $\frac{1}{i_1^2 i_2^2 \cdots i_q^2}$ and enters (say) exactly i_1 times in the sum. Hence

$$F_{n,q} = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_q \le n} \frac{1}{i_1\, i_2^2 \cdots i_q^2} = \sum_{k=1}^{n} \frac{1}{k} h_{q-1}(A_{k,n}^2).$$

Def. Let $(a_1, \ldots, a_r)$ an r-uple of positive integers, with $r \in \mathbb{N}$. Then

$$\zeta^*(a_1, \ldots, a_r) \stackrel{\text{def}}{=} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r} \frac{1}{i_1^{a_1} \cdots i_r^{a_r}}.$$

are called **multiple zeta star values** (MZSVs) of indices $a_1, \ldots a_r$. MZSVs are cousins of **multiple ζ values** (wo $=$) arising in several fields, from knot-invariants in Vassiliev–Kontsevich theory (Zagier 1992) to scattering amplitudes in perturbative QFT (Broadhurst–Kreimer 1997).

Step III. Multiple ζ^* values

For our special MZSVs [‡] a second miracle occurs (Ohno–Wakabayashi 2006,Ohno–Zudilin 2008, Borwein–Zudilin 2011)

$$F_{+\infty,q} \stackrel{\text{def}}{=} \lim_{n\to\infty} F_{n,q} = \zeta^*(1,\underbrace{2,\ldots,2}_{q-1}) = 2\zeta(2q-1)\,,$$

where ζ is the usual Riemann's function. In conclusion, we have **Theorem (D'A-Sportiello 2020)**. For a stretch.-exp. disorder $\rho_{\text{ie},\alpha}(x) = \alpha x^{\frac{1}{s}-1}\exp(-x^{\frac{1}{s}})$, for $p>1$ even and $s \in \mathbb{Z}_+$,

$$E^{(s)}_{p,1}(n) \simeq \begin{cases} 2s^2 \ln(n)^{2s-1} & p=2 \\ 2\zeta(p-1)s^p p! \ln(n)^{(s-1)p} & p \geq 4 \end{cases},$$

and therefore the scaling is always anomalous.

[‡] Absolute convergence holds since $h_{q-1}(A^2_{k,n}) \sim k^{-(q-1)}$ for $k \ll n$.

One conclusion and two main take home messages

"*There ain't no such thing as a free lunch*".

- Convexity is a strong constraint. The Brownian scaling limit of the transport field at $p \geq 1$ persists even on a general (finite) metric graph G. Particularly simple on trees (w.i.p. with Linxiao Chen and Yuqi Liu);
- There is much rigorous work to be done in $d = 2$ (some is wip with S. Caracciolo, G.Sicuro and A. Sportiello).

Thank you for your attention!

Open Exercise 1. **On a functional of the Brownian Bridge**

✪✪

Take w.l.o.g. $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} = \mathbf{1}_{[0,1]}(x)$, $p \geq 1$.

Consider the standard Brownian Bridge on $[0,1]$, $B_\tau \stackrel{\text{law}}{=} W_\tau - \tau W_1$, $\tau \in [0,1]$, with W_τ the standard Brownian motion.

Def. Let $\Phi(B_\tau; \Lambda_p) \stackrel{\text{def}}{=} \int_0^1 |B_\tau - \Lambda_p|^p d\tau$, where Λ_p is a real-valued random shift independent on time.

Q1 Discuss $\exists$ and ! of extremizers Λ_p^* depending on p.

Q2 Prove that any Λ_p^* is centered $\forall p \geq 1$. Discuss the geometrical interpretation of Λ_p^*. What is Λ_2^*? And Λ_1^*?

Q3 Prove that $\frac{d}{dp}\mathbb{E}\left[(\Lambda_p^*)^2\right] > 0$ for $p \in [1, \infty]$.

Open Problem 2. **Nice formulas in $1d$ ERAPs w. uniform disorder**

✪✪✪

Take $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} = \mathbf{1}_{[0,1]}(x)$, $p \geq 1$.
Using generalised Selberg integrals, Caracciolo *et al.* 2019 have shown, for integer $\ell \geq 1$,

$$\mathbb{E}[|b_k - r_k|^{\ell}] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}.$$

This implies the exact formula for $E_{p,1}\big|_{p\geq 1}(n)$ for $f = |\cdot|^p$.

Question[§]: Which $f = f(|\cdot|)$ give a "nice formula" for $E(n)$ after resummation?

[§] Question raised by Nathanaël Enriquez (LMO Orsay) at online ALEA Days 2021.

Open Research Question 3. **Scaling limit for $p \in (0,1)$**

✪✪✪✪✪

Take $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} = \mathbf{1}_{[0,1]}(x)$, $p \in (0,1)$.

Q1 Prove that π_{opt} depends on p and it is *non-crossing*.

Q2 Prove the lower bound on $E_{p,1}$ for n large.

Q3 The non-crossing property of π_{opt} induces a natural tree structure $T_n^{(p)}$. What is the scaling limit of $T_n^{(p)}$?

References

Main references: (here and above paper references might be clickable) preprint

	preprint
J. Stat. Phys. **183**, 34, 2021	2008.01462
PhD Thesis, Université Paris-Saclay, 2020	tel:03098672
J. Phys. A Math. Theor. **53**, 6, 2020	1904.10867
J. Stat. Phys. **174**, 4, 2019	1803.04723
Phys. Rev. E **96**, 4, 2017	1707.05541