Statistical Properties of the Euclidean Random Assignment Problem

PhD Thesis Defense

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École Doctorale n. 569: ITFA

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Friday, October 16, 2020, live at: link

List of papers in thesis work

⑤ **MPD**, A. Sportiello, *Euclidean Random Assignment Problems at non-integer Hausdorff dimension* $d_H \in (1, 2)$, in preparation.

④ **MPD**, A. Sportiello, *Anomalous scaling of the optimal assignment in the one dimensional Random Assignment Problem: some rigorous results*, submitted, TBC.

③ D. Benedetto, E. Caglioti, S. Caracciolo, **MPD**, G. Sicuro, A. Sportiello, *Random Assignment Problems on* 2*d manifolds*, submitted, 2008.01462.

② S. Caracciolo, **MPD**, V. Erba, A. Sportiello, *The Dyck bound in the concave 1-dimensional random assignment model*, J. Phys. A 53, 1904.10867.

① S. Caracciolo, **MPD**, G. Sicuro, *Anomalous scaling of the optimal assignment in the one dimensional Random Assignment Problem*, J. Stat. Phys. 174, 1803.04723.

Introduction

Combinatorial optimization problems are ubiquitous in everyday life and involve finding the extrema of some function of interest over a large discrete set.

The assignment problem is a major example of this and finds application in a huge variety of situations, for example:

- Particle tracking of indistinguishable objects from multiple snapshots in the diffusive regime (drops in a cloud, birds in a flock...);
- Cellular networks design, given geography + spatial density of users available from historical data.

The relevance of this problem has been discussed also in various inter-disciplinary contexts:

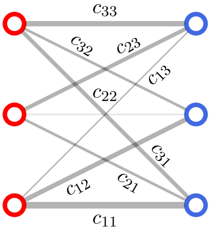
1. Economics [Koopmans–Beckmann 1957];
2. Game Theory: optimal mixed strategy in a two-player game [Von Neumann 1953, 1954].

The (linear sum) Assignment Problem (AP)

Given a $n \times n$ cost matrix c, find a bijection π s.t.

$$\mathcal{H}(\pi, c) = \sum_i c_{i\pi(i)}$$

is minimal. Call it π_{opt} and let $\mathcal{H}(\pi_{\text{opt}}; c) = \mathcal{H}_{\text{opt}}(c)$. Ex. $n = 3$:



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

The problem is P-complete: any other problem in this class can be formulated as an assignment problem. For any given cost matrix c the optimal permutation $\pi_{\text{opt}} = \pi_{\text{opt}}(c)$ can be found in time $\mathcal{O}(n^3)$ [Munkres 1957].

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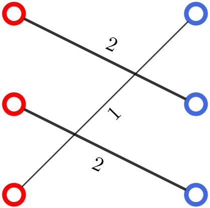

$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

$$\mathcal{H}_{\text{opt}}(c) = 5$$

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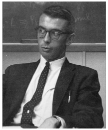
Some mathematicians involved with AP: an old problem

König
1916

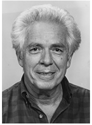
Egérvary
1931



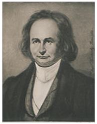
Munkres
1957

von Neumann
1953

Kuhn
1955

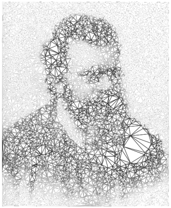


"*De investigando ordine systematis aequationum...*"
Jacobi 1860
See Ollivier 2009

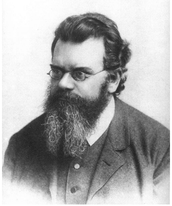


Well understood and nowadays efficiently solved even on smartphones. Why bother?

Class of CO problems $\equiv$ a disordered system



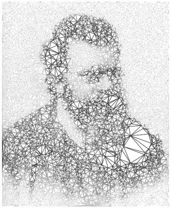
The disordered Boltzmann



Zero temperature limit (ground state) of a single and disordered physical system [Kirkpatrick *et al.* 1983].

Disordered stat-mech

Class of CO problems ≡ a disordered system



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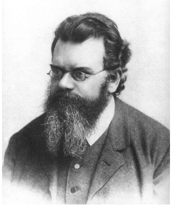

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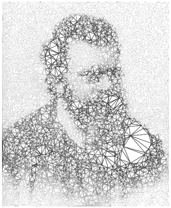
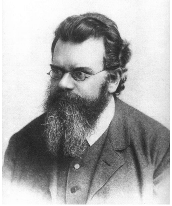
Combinatorial optimization *Disordered stat-mech*

Class of CO problems ≡ a disordered system



The disordered Boltzmann



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Combinatorial optimization

- Optimal solution

Disordered stat-mech

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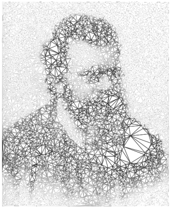

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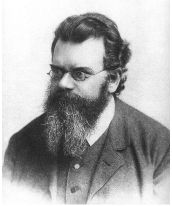

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Combinatorial optimization

- Optimal solution

Disordered stat-mech

- Ground state

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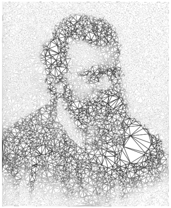

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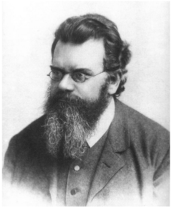


Zero temperature limit (ground state) of a single and disordered physical system [Kirkpatrick *et al.* 1983].

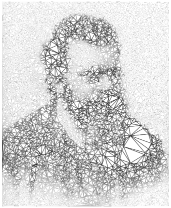
Combinatorial optimization

- Optimal solution
- Typical properties of optimal solution

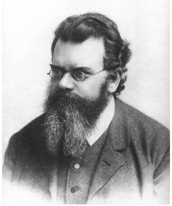
Disordered stat-mech

- Ground state

Class of CO problems ≡ a disordered system



The disordered Boltzmann



Zero temperature limit (ground state) of a single and disordered physical system [Kirkpatrick *et al.* 1983].

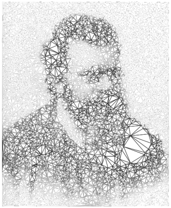
Combinatorial optimization

- Optimal solution
- Typical properties of optimal solution

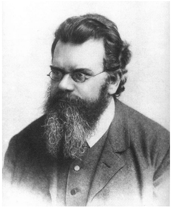
Disordered stat-mech

- Ground state
- Quenched average over the disorder

Class of CO problems $\equiv$ a disordered system



The disordered Boltzmann



Zero temperature limit (ground state) of a single and disordered physical system [Kirkpatrick *et al.* 1983].

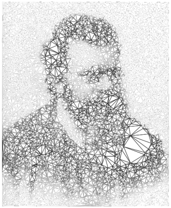
Combinatorial optimization

- Optimal solution
- Typical properties of optimal solution
- Geometrical and topological constraints

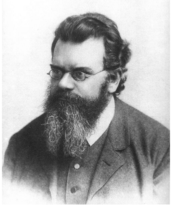
Disordered stat-mech

- Ground state
- Quenched average over the disorder

Class of CO problems ≡ a disordered system



The disordered Boltzmann



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Combinatorial optimization

- Optimal solution
- Typical properties of optimal solution
- Geometrical and topological constraints

Disordered stat-mech

- Ground state
- Quenched average over the disorder
- Frustration

Random Assignment Problems

A random assignment problem is a stochastic assignment problem in which the cost matrix c is chosen from some statistical ensemble. One is interested in the statistical properties of $\mathcal{H}_{\mathrm{opt}}$ for large n depending on the ensemble.

One possible choice is to have all the c_{ij}'s to be i.i.d. random variables. This problem is usually called "the Random Assignment Problem" and was pioneered by Mézard–Parisi and Orland in 1985 to address questions arising in mean field spin-glasses using methods from the latter field.

They understood that only "short" edges are relevant in the asymptotics for $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}(c)]_n$ for large n. If c_{ij}'s have a pdf $\rho(l) = l^r + o(l^r)$ for l small, the only relevant parameter is the exponent r.

They also argued that at $r = 0$ $\lim_{n\to\infty} \mathbb{E}[\mathcal{H}_{\mathrm{opt}}(c)]_n = \zeta(2) = \pi^2/6$.

Brief history of the Random Assignment Problem

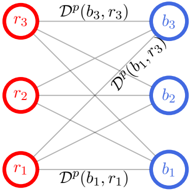
1. If $c_{ij} \sim \mathrm{Exp}(1)$, Conjecture [Parisi 1998]:

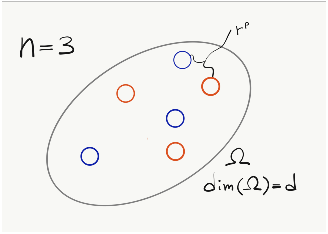
$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}(c)]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

2. Parisi conjecture can be extended to rectangular cost matrices [Coppersmith-Sorkin 1998]
3. Proof of $\zeta(2)$ limit (among other things) [Aldous 2001]
4. Proof of Parisi conjecture [Nair–Prabhakar–Sharma 2001]
5. Extension to the k-partite case (NP-hard for $k \geq 3$), [Martin–Mézard–Rivoire 2004,2005]
6. $\exists!$ of the solution to cavity equation [Wästlund 2012, Larsson 2014+Salez 2015]

Nice example of cross-fertilization. Unsatisfactory from a physicist's perspective because it is mean field.
NOT discussed today...

The Euclidean Random Assignment Problem





<u>A different statistical ensemble</u>: the c_{ij}'s are not independent. Rows and columns are naturally associated to wells and sinks (blue and red points), which often have a geographical distrib in an ambient space, and the cost is a function of the distance in this ambient space Ω. The ambient space Ω has dimension d, and a blue-red couple contributes to the energy through their Euclidean distance to the power p. This choice sharing translational, rotational and scale invariance appears meaningful as any other details of the cost function would be washed out in the large n limit.

The Euclidean Random Assignment Problem

This formulation is relevant since:

- it can be considered a toy model simplification of finite dimensional spin glass, with which it shares essential properties (frustration) while remaining more treatable both analytically and numerically;
- it is an instance of the Monge-Kantorovich problem in Optimal Transport: associating the empirical measures $\rho_{\mathcal{B}}$ and $\rho_{\mathcal{R}}$ to blue and red points, one has

$$\mathcal{H}_{\text{opt}} = n W_p^p(\rho_{\mathcal{B}}, \rho_{\mathcal{R}})$$

  where W_p is the p-Wasserstein distance.

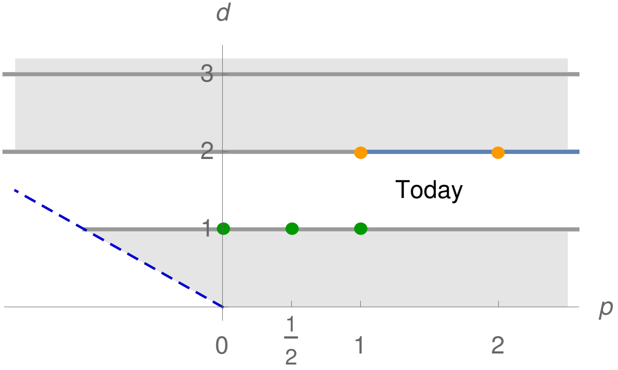
Our aim: understand the phase diagram of this problem.

$$E_{p,d}(n) = \mathbb{E}_{n,p,d}\left[\mathcal{H}_{\text{opt}}\right] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} \,(\ln n)^{\gamma'_{p,d}}\,(1 + o(1))$$

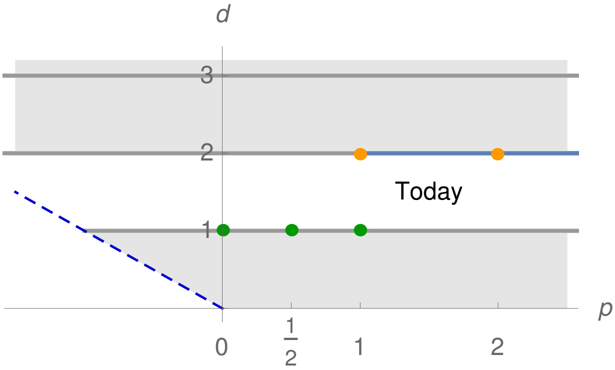
for large n, depending on d, p and choice of disorder (i.e. how we sample blue and red points).

Phase diagram of the ERAP

I describe here the known facts on the phase diagram for the uniform measure.
The non-uniform cases are more subtle, and too complicated to be summarised here (e.g., at $d = 1$ and $p = 2$, if the density is Gaussian, we have an extra factor $\sim \ln \ln n$ [Bobkov–Ledoux 2019]).

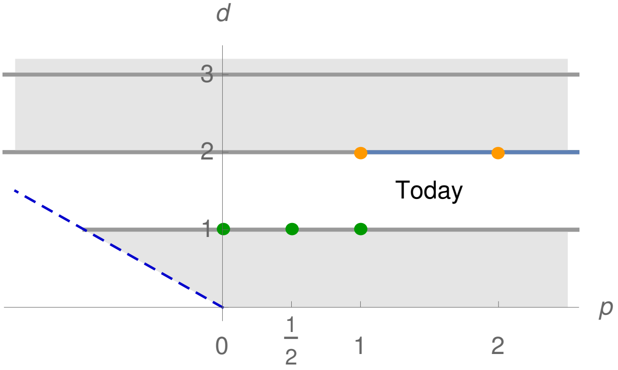


Phase diagram of the ERAP



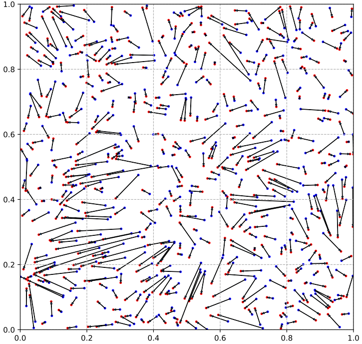
$d \geq 3$ "simple", as if there is "no geometry" (mean field at exponent $r = 1 - \frac{p}{d}$) [Mézard–Parisi 1988].
Almost sure limits Barthe–Bordenave 2013, Goldman–Trevisan 2020

Phase diagram of the ERAP



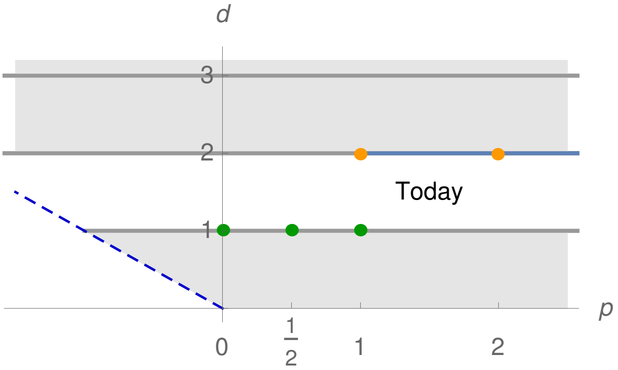
$d = 1$ "simple" (yet rich!) due to mathematical structure of the solution.

Phase diagram of the ERAP



$d = 2$ challenging: logarithmic corrections to scaling [Ajtai–Komlós–Tusnády 1984].
Typically, assignment is not performed in the Euclidean neighborhood of a point, but rather to a $\sim (\ln n)$-th neighbour.

Phase diagram of the ERAP



Despite interest at $d \geq 3$ (and some work in progress!) today we will be confined at $1 \leq d \leq 2$.

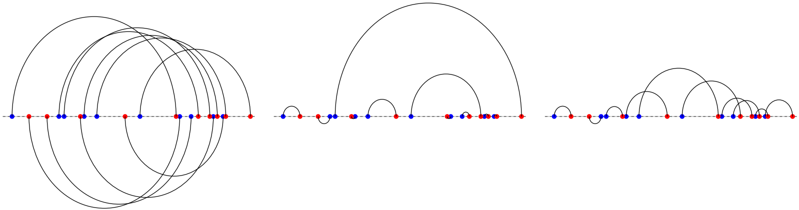
Topics that I discuss today

1. $d = 1, p = 2$: the full distribution of the energy
2. $d = 1, p \geq 1$: non-uniform distributions and anomalous scaling
3. $d = 1, p < 1$: the Dyck upper bound
4. $d = 2, p = 2$: sub-leading corrections and arbitrary manifolds
5. The problem on fractals with Hausdorff dimension $d_H \in (1, 2)$
6. Provisional conclusions and (some) perspectives

Co-authors:
Dario Benedetto[4] & Emanuele Caglioti[4] (Rome La Sapienza)
Sergio Caracciolo[1,2,3,4] & Vittorio Erba[3] (Milan Univ., INFN)
Gabriele Sicuro[2,4] (ENS Paris)
Andrea Sportiello[2,3,4,5] (CNRS, UP13)

$d = 1$: properties of the solution

$p = 0$ and $p = 1$ separate three qualitatively different regimes:



The optimal permutation must necessarily be:

- $p < 0$ cyclical: $\pi_{\text{opt}}(i) = i + k \pmod{n}$ [Caracciolo–MPD–Sicuro];
- $p \in (0, 1)$ non-crossing: if AB, CD are edges then either $AB \subset CD$ or $AB \cap CD = \emptyset$ [McCann 1999].
- $p > 1$ ordered: π_{opt} is the identity permutation;

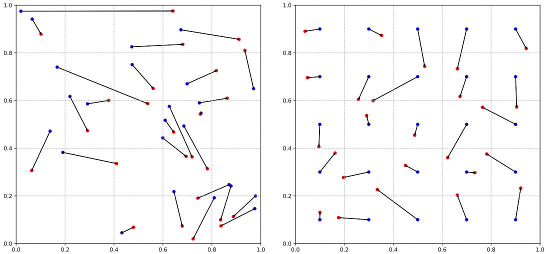
Note: these combinatorial properties are true <u>for every instance</u>.

Adding disorder: Poisson-Poisson, Grid-Poisson problems and the transport field

Two usual ways of adding randomness:

1. Poisson-Poisson: blue and red points uniformly distr. on Ω
2. Grid-Poisson: red points uniformly distr. on Ω and blue on a fixed grid

Example: $d = 2$, $n = 25$, $p = 1$:



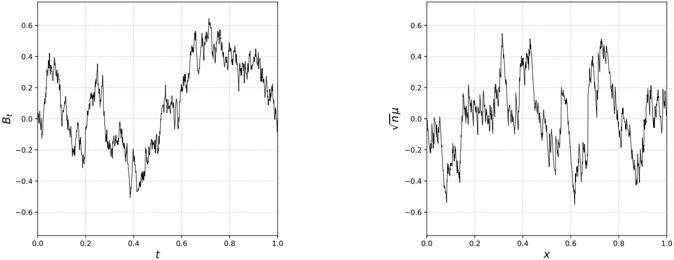
<u>Transport field</u> $\vec{\mu}$: position of optimal red companion if you sit on a blue point:

$$\vec{\mu}(b_i) := \vec{r}_{\pi_{\mathrm{opt}}(i)} - \vec{b}_i \qquad i = 1, \ldots, n.$$

$p > 1$: the Brownian Bridge

In $d = 1$ on $\Omega = [0, 1] \to \mu \sim$ "Brownian motion"

$\sqrt{n}\,\mu(b_{\lceil nt \rceil}) \underset{weakly}{\longrightarrow} B_t$ for B_t the Brownian Bridge [1]



Since $\mathcal{H}_{\mathrm{opt},(n,p)} = \sum_i |\mu_i|^p$, for $n \to \infty$, energy $\sim p-$th moment of B_t up to trivial scaling [Caracciolo–MD–Sicuro 2017]

$$E_p(n) \sim n^{1-\frac{p}{2}}\,\mathbb{E}[|B_t|^p] = \frac{\Gamma(1 + p/2)}{p+1} n^{1-\frac{p}{2}}$$

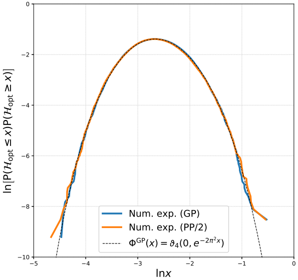
[1] Brownian motion W_t conditioned to $W_1 = 0$, or equiv. $B_t := W_t - tW_1$.

$d = 1, p = 2$: the full distribution of the energy

Fourier modes are centered and uncorrelated gaussians [Biane–Pitman–Yor 2001].
Thanks to Fourier duality, we can get an explicit expression for the Laplace transform of $\mathcal{H}_{\mathrm{opt}}$.
One can perform both finite n calculation and continuum calculation of the Laplace transform. For $\Omega = \mathbb{S}_1$, we have the nice facts:

- Zero mode: shift in order to have zero area.
- Mode contribs. GP vs PP differ by exactly a factor 2
- Explicit expression for the whole asymptotic distribution of $\mathcal{H}_{\mathrm{opt}}$!

$d = 1, p = 2$: the full distribution of the energy



$$\Pr[\mathcal{H}_{\mathrm{opt},(n,2)} \le x] \overset{n\to\infty}{=} \vartheta_4(0, e^{-2\pi^2 x}) := \sum_{s\in\mathbb{Z}} e^{i\pi s} e^{-2\pi^2 s^2 x}$$

For general $p > 1$ the distribution of $\mathcal{H}_{\mathrm{opt}}$ <u>is not known</u>. However, on the unit interval $\mathbb{Q}_1$ the expected value at fixed n <u>is known</u> [Caracciolo et al. 2019]:

$$E_{p,\mathbb{Q}_1}(n) = n\frac{\Gamma(1+\frac{p}{2})}{p+1}\frac{\Gamma(n+1)}{\Gamma(n+1+\frac{p}{2})} = \underbrace{\frac{\Gamma(1+\frac{p}{2})}{p+1} n^{1-\frac{p}{2}}}^{\text{“bulk” scaling}} (1+o(1))$$

Anomalous scaling pt I [Paper ①]

What if we have a non-uniform disorder distribution?
Analogous questions from a different perspective have been considered in the optimal transport of continuum measures [Bobkov & Ledoux 2019]
The problem can be "reduced to quadratures" in the Brownian Bridge picture: for ρ pdf and Φ its cdf,

$$E_{p,\rho}(n) = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) n^{1-p/2} \int_0^1 \left[\frac{\sqrt{s(1-s)}}{\Psi(s)}\right]^p \mathrm{d}\, s + o\left(n^{1-p/2}\right).$$

where $\Psi(s) \coloneqq \left(\rho \circ \Phi^{-1}\right)(s)$.
At the points were the density ρ vanishes, the integrand may have non-integrable singularities.

Anomalous scaling part I [Paper ①]

Main idea: in analogy with cutoff regularization in QFTs, regularize the integral using a cutoff on the scale of $\frac{1}{n}$ with an unknown constant c. This allows to obtain the leading scaling since the numerical value of c enters only at sub-leading order.

Example: $\rho(x) = e^{-x}\theta(x)$ (i.e. $\Omega = \mathbb{R}^+$).

- *Exact* fixed n calculation at $p = 2$ (via Beta integrals):

$$E_{2,\rho}(n) = 2\sum_{k=1}^{n} \frac{1}{k} = 2\ln n + 2\gamma_{\mathrm{E}} + o(1).$$

- Cutoff method:

$$E_{p,\rho}(n) = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right) n^{1-p/2} \int_0^{1-c/n} \left(\frac{s}{1-s}\right)^{\frac{p}{2}} ds,$$

which at $p = 2$ gives $E_{2,\rho}(n) = 2\ln n - 2\log c - 2 + o(1)$

(and a posteriori $c = e^{-\gamma_{\mathrm{E}}-1} = 0.20655\ldots$)

Anomalous scaling part I [Paper ①]

An asymptotics $\neq n^{1-\frac{p}{2}}$ is called an *anomalous scaling*.
We have performed an extensive study of anomalous scalings, including :

- Gaussian tail (Rayleigh distribution $\rho(x) = xe^{-x^2/2}\theta(x)$): $E_{2,\rho}(n) = \ln\ln n + \gamma_{\mathrm{E}} + o(1)$
- Power law tails and zeros: three different scaling regimes depending on parameter, logarithmic corrections.

The method applies also to distributions with algebraic tails or with zeros of a given order.
The method is both simple and elegant. <u>But</u>:

- There are cases in which even the cut-offed integral diverges.
- Mathematical motivations for choice of cutoff is not clear.
- Sub-leading constants <u>are unaccessible</u> to the theory.

<u>IDEA</u>: study moments of transport field <u>at fixed n</u>, and only afterwards take $n \to \infty$.

Anomalous scaling pt II [Paper ④]

HOW: write exact formulas for the k-th edge (expectation wrt Beta distrib.), and then perform asymptotic analysis.

$$\langle \mathcal{H}^{(p)}_{\mathrm{opt,n}} \rangle_{\rho,n} = \sum_{k=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} M^{(\rho)}_{n,k,q} M^{(\rho)}_{n,k,p-q}$$

(valid only if $p \in 2\mathbb{N}$), where

$$M^{(\rho)}_{n,k,\ell} = \langle \Phi^{-1}_{\rho}(u)^{\ell} \rangle_{P_{n,k}}$$

for $\Phi^{-1}_{\rho}(u)$ the inverse cdf (quantile function) and

$$P_{n,k}(u)du := \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k} du.$$

By universality we considered only certain "classes of zeros":

- A finite or internal endpoint;
- An endpoint at infinity (with algebraic or stretched exponential decay).

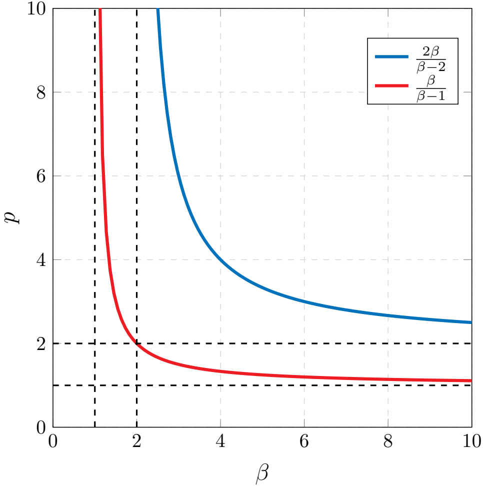
Anomalous scaling pt II [Paper ④]

Families of prob. dens.	leading scaling of $E_{p,\rho}(n)$
Stretch exp $\rho(x) \approx \exp(-x^\alpha)$ $(s = {}^1/_\alpha)$	$\begin{cases} 2s^2\,[\ln(n)]^{2s-1} & p=2 \\ 2\zeta(p-1)s^p p!\,[\ln(n)]^{(s-1)p} & p \geq 4 \text{ even} \end{cases}$
Finite endpoint, algebraic zero	
$\rho_{\mathrm{fa},\beta}(x) = \beta x^{\beta-1}$ $(x \in [0,1])$	$\begin{cases} b_{\beta,p} n^{1-p/2} & \text{Bulk regime} \\ a_{\beta,p} n^{-p/\beta} & \text{Anomalous regime} \\ Q(p) n^{2/(2-\beta)} \ln n & \text{Critical line } \beta = {}^{2p}/_{(p-2)} \end{cases}$
Internal endpoint, algebraic zero	
$\rho_{\mathrm{sa},\beta}(x) = \frac{\beta}{2}\lvert x\rvert^{\beta-1}$ $(x \in [-1,1])$	$\begin{cases} 2B_{\beta,p} n^{1-p/2} & \text{Bulk regime} \\ (2A_{\beta,p} + K_{\beta,p})\, n^{\frac{1}{2}(1-p/\beta)} & \text{Anomalous regime} \\ R(p) n^{(2-\beta)/(2(1-\beta))} \ln n & \text{Critical line } \beta = {}^{p}/_{(p-1)} \end{cases}$

The bottom line:

- stretched exponential: $E_{p,s}(n) \sim (\ln n)^c$, thus always anomalous for $p \geq 2$;
- algebraic zeros: logarithmic corrections to scaling appear along critical hyperbolae, algebraic corrections beyond it;
- sub-leading corrections are also accessible.

Anomalous scaling pt II [Paper ④]



Blue:
zero endpoint
example:
$\rho_\beta(x) = \beta x^{\beta-1}, x \in [0,1]$

Red:
internal endpoint
example:
$\rho_\beta(x) = \frac{\beta}{2}|x|^{\beta-1}, x \in [-1,1]$

An internal zero is stronger than an endpoint zero.
The diagram allows to deal easily for a general pdf, e.g.
$\rho(x) = c\,x\sin(x^3)|\mathbb{1}_I(x)$, for which $p_{\text{crit}} = \frac{10}{3}$ if $I = [0,1]$ ($\beta = 5$).

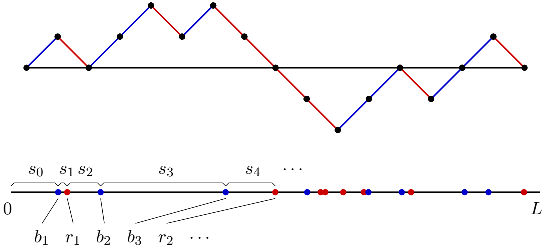
The Dyck UB at $p \in (0, 1)$ [Paper ②]

The concave regime $p \in (0, 1)$ is interesting because the ground state is not determined a priori and poorly understood [McCann 1999, Delon–Salomon–Sobolevski 2001].

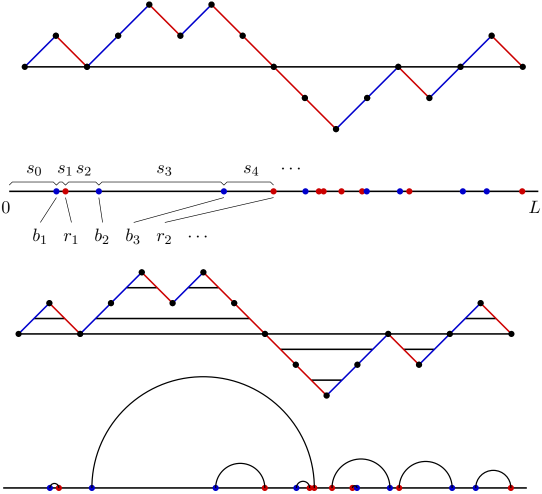
In this case, the non-crossing property of the solution has raised interest in biology, where this problem has been suggested as a toy-model for the secondary structure or RNA [Orland–Zee 2002, Bundschuh–Hwa 2002].

IDEA: try a canonical construction, an approximate solution sharing the basic combinatorial properties of π_{opt} with the hope that it shares the same asymptotics of the ground state energy $\Longrightarrow$ Dyck matchings!

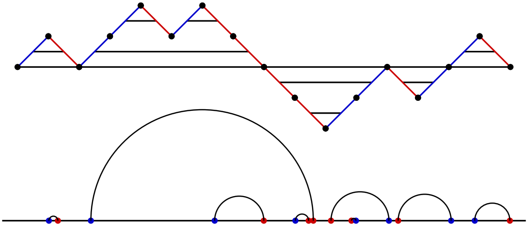
How to build a Dyck matching [Paper ②]



How to build a Dyck matching [Paper ②]



How to build a Dyck matching [Paper ②]



Remark: the Dyck matching is determined by the order of the colors of the points and is thus independent on their spacings and on the exponent p.

Asymptotic expected energy of Dyck matchings and the Dyck conjecture [Paper ②]

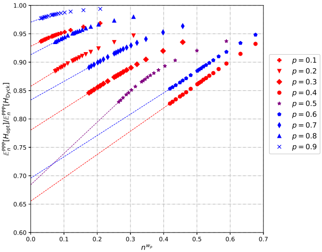
$$\mathbb{E}_n(H_{\text{Dyck}}) \simeq \begin{cases} n^{1-p} & 0 \leq p < \frac{1}{2} \\ \sqrt{n}\ln n & p = \frac{1}{2} \\ n^{\frac{1}{2}} & \frac{1}{2} < p \leq 1 \end{cases}$$

Conjecture: for all $p \in (0,1)$

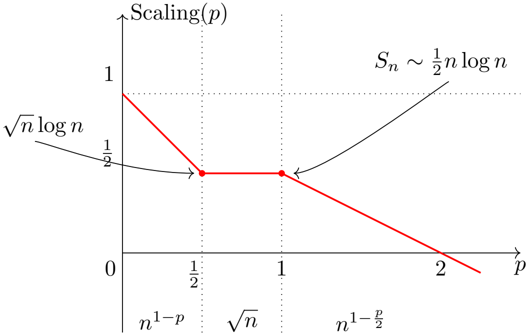
$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathcal{H}_{\text{opt}})}{\mathbb{E}_n(H_{\text{Dyck}})} = k_p$$

with $0 < k_p \leq 1$.

Numerical evidence:



Slice of the phase diagram at $d = 1$ [Paper ②]



Remarkable that for such a simple problem, at varying p, in the phase diagram there are three regimes and two singular points. Nobody suspected this.

Random assignment on $2d$ manifolds [Paper ③]

Recall: fundamental result by [Ajtai–Komlós–Tusnády 1984] that, for the Poisson-Poisson problem on $\Omega = [0,1]^2$

$$E_\Omega(n) \sim c_p\, n^{1-\frac{p}{2}} \,(\ln n)^{\frac{p}{2}} \,.$$

The value of the constant c_p was unknown for any value p and any two dimensional Ω. Caracciolo–Lucibello–Parisi–Sicuro showed through a QFT calculation that at $p = 2$ in a domain of unit area $|\Omega| = 1$:

$$E_\Omega(n) = \begin{cases} \frac{1}{2\pi} \ln n + o(\ln n) & \text{Poisson} - \text{Poisson} \\ \frac{1}{4\pi} \ln n + o(\ln n) & \text{Grid} - \text{Poisson} \end{cases}$$

Rigorous proof for leading constant provided by [Ambrosio–Stra–Trevisan 2019] and the remainder is $o(\ln n) = O(\sqrt{\ln n \ln\ln n})$ [Ambrosio–Glaudo 2019].

<u>State of the art!</u>

On energy shifts among two $2d$ manifolds [Paper ③]

Despite an ill-defined theory, which predicts infinite average energy in absence of a mysterious cutoff regularisation, we were able to give a precise operational sense to energy differences among different domains.

Main result: For Ω, Ω' two regular two-dimensional manifolds

$$\lim_{n\to\infty} [E_\Omega(n) - E_{\Omega'}(n)] = 2(R_\Omega - R_{\Omega'}) = 2(K_\Omega - K_{\Omega'})$$

where K_Ω, R_Ω are constants which can be directly computed from the spectrum of the Laplace–Beltrami operator on Ω.

Energy shift for the flat torus [Paper ③]

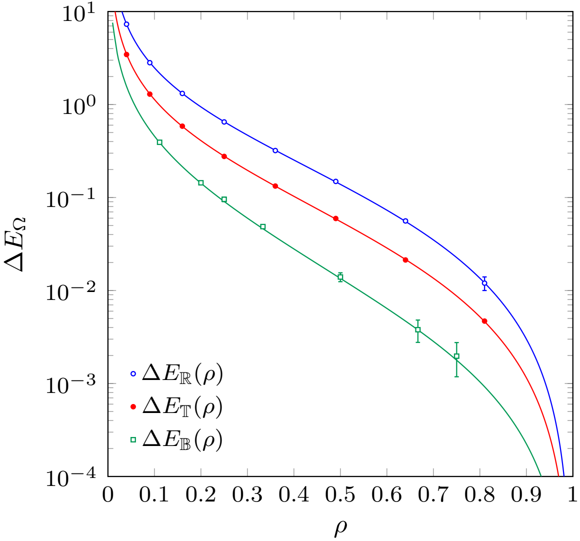
Example: let Ω be built from a rectangular fundamental polygon of aspect ratio ρ by appropriate gluing of the boundaries. For all such manifolds we can get analytic expressions for energy differences at aspect ratios ρ_1, ρ_2. In the case of the 2-torus of aspect ratio ρ we get

$$\frac{1}{2}\left(E(\rho) - E(1)\right) = K_{\mathbb{T}}(i\rho) - K_{\mathbb{T}}(i) = -\frac{1}{2\pi} \ln \frac{\eta(i\rho)\eta\left(i\rho^{-1}\right)}{\eta^2(i)}$$

where η is the Dedekind function.
Analogous expressions hold with different choices of boundary conditions on the fundamental polygon: open-open or square and antiperiodic-antiperiodic or "Boy surface".

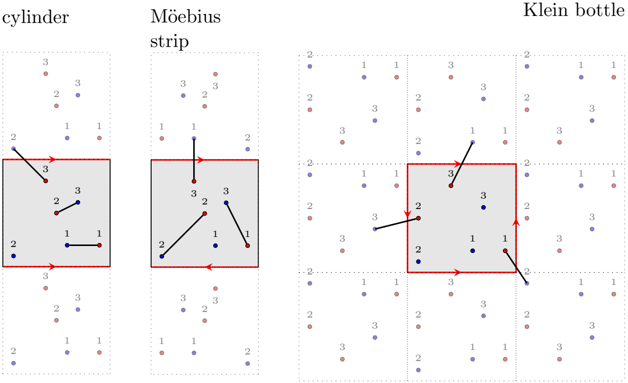
Theory vs experiments for torus, square and Boy surface

Comparison of theoretical predictions (lines) with numerical experiments (dots with error bars):



Manifolds with mixed boundary conditions [Paper ③]

We could easily obtain the homologous results for mixed boundary conditions: cylinder, Möebius strip, Klein bottle:



Remark: $\Delta E(\rho)$ displays minima at non-trivial values of ρ.

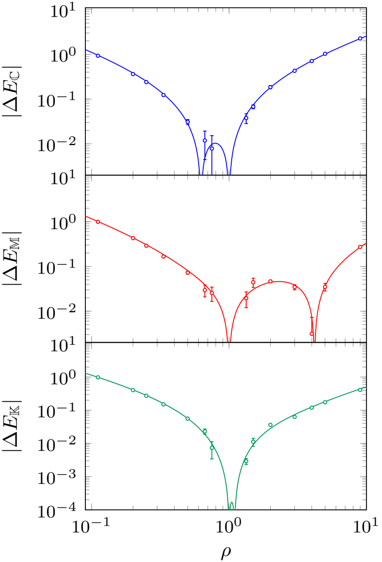
Manifolds with mixed boundary conditions [Paper ③]

$$|\Delta E_\Omega| = |K_\Omega(\rho) - K_\Omega(1)|$$

$$K_{\mathbb{C}}(\rho) = \frac{\gamma_{\mathrm{E}}}{2\pi} - \frac{\ln(16\pi^2\rho)}{4\pi} - \frac{1}{\pi}\log\eta(2i\rho) + \frac{\zeta(2)}{4\pi^2\rho}$$

$$K_{\mathbb{M}}(\rho) = \frac{\gamma_{\mathrm{E}}}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{1}{\pi}\log\frac{\eta^3(i\rho)}{\eta(i2\rho)\eta\left(i\frac{\rho}{2}\right)} + \frac{\zeta(2)}{4\pi^2\rho}$$

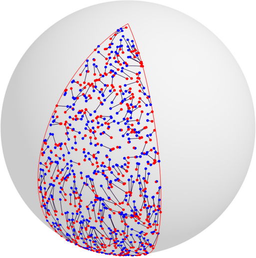
$$K_{\mathbb{K}}(\rho) = \frac{\gamma_{\mathrm{E}}}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{1}{\pi}\ln\eta\left(i\frac{\rho}{2}\right) - \frac{\zeta(2)}{2\pi^2\rho}$$

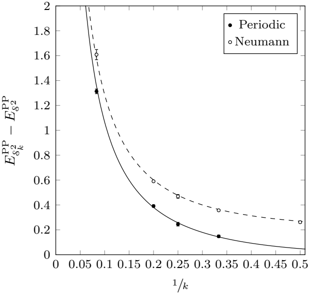


<u>Remark</u>: $\Delta E(\rho)$ displays minima at non-trivial values of ρ.

Manifolds with singular curvature [Paper ③]

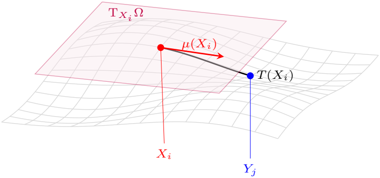
We have obtained analogous formulas for energy shifts depending on some natural parameters also for non-flat surfaces (2-sphere) and surfaces with singular curvature (conical singularities), such as the circular sector or the spherical lune:





Random assignment on $2d$ manifolds [Paper ③]

Sketch of the idea: the field theoretical approach.



On a d-dimensional Riemannian manifold Ω consider the lagrangian ($p = 2$)

$$\mathcal{L}[\mu, \phi] := \underbrace{\frac{1}{2}\int_\Omega \mu^2(x)\nu_{\mathcal{B}}(\mathrm{d}\,x)}^{\text{Energy}} + \overbrace{\int_\Omega [\phi(x+\mu(x))\nu_{\mathcal{B}}(\mathrm{d}\,x) - \phi(x)\nu_{\mathcal{R}}(\mathrm{d}\,x)]}^{\text{Transportation constraint}}$$

where ϕ is the Lagrange multiplier for transportation constraint.

Random assignment on $2d$ manifolds [Paper ③]

Since we expect μ to be small as $n \to \infty$, we linearize $\mathcal{L}[\mu, \phi]$ to

$$\hat{L}[\mu, \phi] := \int_\Omega \left[\frac{1}{2}\mu^2(x) + \mu(x) \cdot \nabla\phi(x)\right] \mathrm{d}^2\, x + \int_\Omega \delta\nu(x)\phi(x)\, \mathrm{d}^2\, x$$

Non-trivially, the Euler-Lagrange equations for $\hat{L}[\mu, \phi]$, under a suitable linear approximation, imply the Poisson equation

$$\Delta\phi(x) = \delta\nu(x)$$

to be solved with Neumann boundary conditions on Ω. Then $\mu = -\nabla\phi$. On this basis, Caracciolo–Sicuro 2015 proposed that for the Poisson-Poisson the formal result

$$E_\Omega(n) = -2\,\mathrm{Tr}\,\Delta_\Omega^{-1}$$

holds for n large. The trace of the inverse Laplace operator is ill-defined and gives rise to infinite energies. It must be regularized by introducing a cutoff (in analogy with anomalous scaling in the one-dimensional case).

Random assignment on $2d$ manifolds [Paper ③]

Following Caracciolo–Lucibello–Parisi–Sicuro, we assume the existence of a cutoff function F dumping the trace for large λ (i.e. vanishing for large argument) so that

$$E_\Omega(n) \simeq 2 \sum_{\lambda \in \Lambda(\Omega)} \frac{F\left(\lambda/n\right)}{\lambda} = 2 \int_{0^+}^{\infty} \frac{F\left(\lambda/n\right)}{\lambda} \, \mathrm{d}\, \mathcal{N}_\Omega(\lambda),$$

for $\mathcal{N}_\Omega(\lambda)$ the number of eigenvalues $\leq \lambda$ on Ω. Thus, the independence of energy shifts on the precise form of the (unknown) cutoff function F is a consequence of universality in Weyl's law: $\mathcal{N}_\Omega(\lambda) - \mathcal{N}_{\Omega'}(\lambda)$, at $d = 2$, grows no faster than $\sqrt{\lambda} \ln \lambda$, and the integral is convergent at both infinity and zero (due to the spectral gap), giving

$$\lim_{n \to \infty} \big(E_\Omega(n) - E_{\Omega'}(n)\big) = 2 \int_{0^+}^{\infty} \frac{\mathrm{d}\left(\mathcal{N}_\Omega(\lambda) - \mathcal{N}_{\Omega'}(\lambda)\right)}{\lambda}.$$

Random assignment on $2d$ manifolds [Paper ③]

We considered two regularizations:

- R_Ω or "Robin mass": integrals of diagonal part of the regular part of an appropriate Green function;
- K_Ω or "Kronecker mass": expand the spectral $\zeta_\Omega(s) = \sum_{\lambda \in \Lambda(\Omega)} \frac{1}{\lambda^s}$ function around the (simple) pole at $s = 1$.

The two constants differ by a universal constant

$$R_\Omega - K_\Omega = \frac{\ln 2}{2\pi} - \frac{\gamma_E}{2\pi} = 0.0184511\ldots$$

due to a result by Morpurgo [Morpurgo 2002, Steiner 2005, Okikiolu 2008a, Okikiolu2008b].

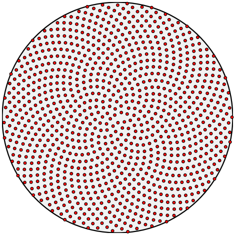
We could thus extend the previous results to

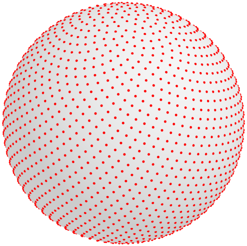
$$E_\Omega(n) = \frac{1}{2\pi} \ln n + 2c_*(n) + 2K_\Omega + o(1)$$

where $2c_*(n)$ is an unknown quantity depending on the cutoff function F <u>but not on the geometry Ω</u>, thus falling outside the scope of our current approach.

Random assignment on $2d$ manifolds [Paper ③]

As of Grid-Poisson, we have considered square, triangular as well as "Fibonacci" grids [Saff–Kuijlaars 1997]





Bottom line for Grid–Poisson:

$$E_\Omega^{\mathrm{GP}}(n) = \frac{1}{4\pi} \ln n + c_*^{\mathrm{GP}}(n) + K_\Omega + o(1)$$

for $G \in \{\mathrm{Square}, \mathrm{Tri}, \mathrm{Fibo}\}$. To be compared to Poisson-Poisson

$$E_\Omega^{\mathrm{PP}}(n) = \frac{1}{2\pi} \ln n + 2c_*^{\mathrm{PP}}(n) + 2K_\Omega + o(1)$$

Is the missing term constant in n? [Paper ③]

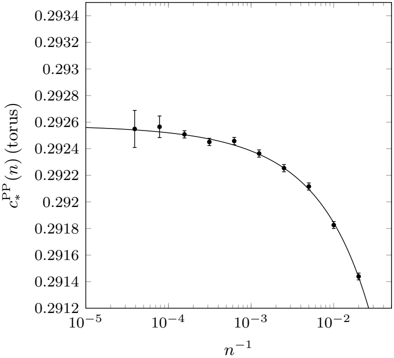
Regarding the unknown remainders $c_*(n)$, extensive numerical experiments suggest that they may be indeed constants in n, while they depend on the choice of randomness protocol.

$$c_*^{\mathrm{PP}}(n) = 0.29258(2)$$

$$c_*^{\mathrm{SP}}(n) = 0.4156(5)$$

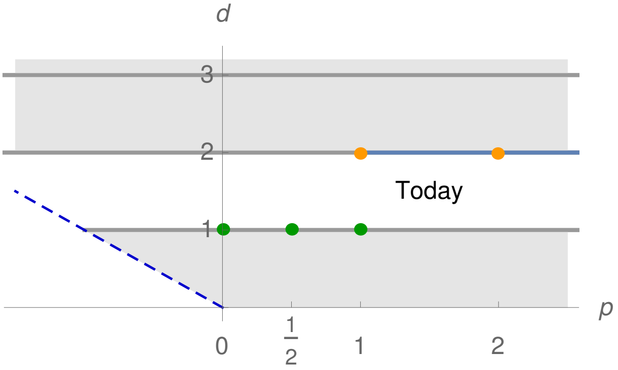
$$c_*^{\mathrm{TP}}(n) = 0.413(2)$$

Remark: for grids, 10^{-3} may also be the scale of "true" differences (regularizations).



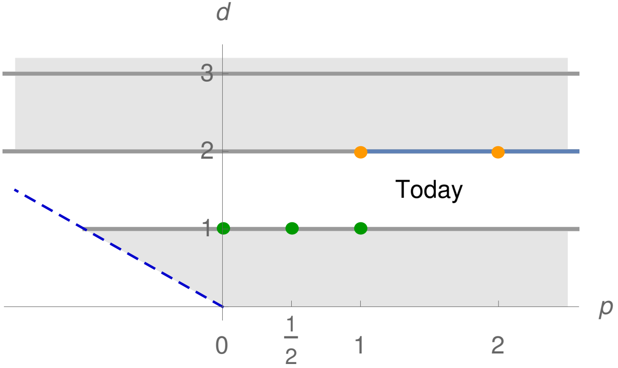
ERAP at $d_H \in (1,2)$ [Paper ⑤]

Motivation: several indications of "critical points" in the phase diagram (leading scaling exponent in the (p,d) plane):



How do the critical points at $d = 1$ "enter" the phase diagram?

ERAP at $d_H \in (1,2)$ [Paper ⑤]



IDEA: blue and red points uniformly distributed on a fractal set of controllable Hausdorff dimensions $d_H \in (1,2)$, compute energy using two-dimensional Euclidean distance. We considered two fractal sets in order to test universality.

Peano and Cesàro fractals [Paper ⑤]

<u>How</u>: for $P = (\xi, z) \in \mathbb{C}^2$ iterate a random function

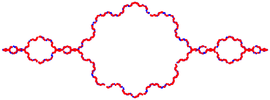
$$f(P) := (\lambda_i \phi_i \xi, z + v_i \xi)$$

where λ_i, ϕ_i and v_i are RVs depending on protocol for interpolating $d_H \in (1, 2)$.

$$\boxed{d_H = 1.1}$$

Peano

Cesàro



Peano and Cesàro fractals [Paper ⑤]

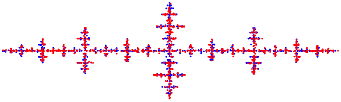
<u>How</u>: for $P = (\xi, z) \in \mathbb{C}^2$ iterate a random function

$$f(P) := (\lambda_i \phi_i \xi, z + v_i \xi)$$

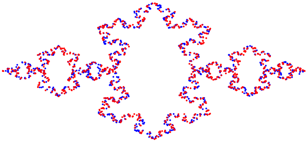
where λ_i, ϕ_i and v_i are RVs depending on protocol for interpolating $d_H \in (1, 2)$.

$$\boxed{d_H = 1.4}$$

Peano



Cesàro



Peano and Cesàro fractals [Paper ⑤]

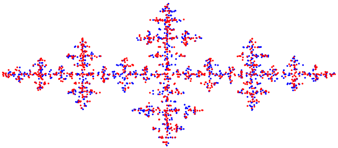
How: for $P = (\xi, z) \in \mathbb{C}^2$ iterate a random function

$$f(P) := (\lambda_i \phi_i \xi, z + v_i \xi)$$

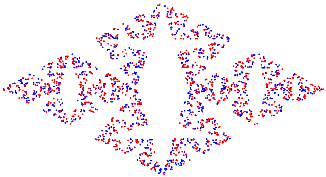
where λ_i, ϕ_i and v_i are RVs depending on protocol for interpolating $d_H \in (1, 2)$.

$$\boxed{d_H = 1.6}$$

Peano



Cesàro



Peano and Cesàro fractals [Paper ⑤]

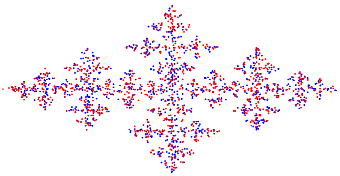
<u>How</u>: for $P = (\xi, z) \in \mathbb{C}^2$ iterate a random function

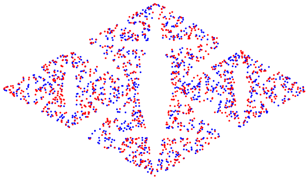
$$f(P) := (\lambda_i \phi_i \xi, z + v_i \xi)$$

where λ_i, ϕ_i and v_i are RVs depending on protocol for interpolating $d_H \in (1, 2)$.

$$\boxed{d_H = 1.7}$$

Peano *Cesàro*





Peano and Cesàro fractals [Paper ⑤]

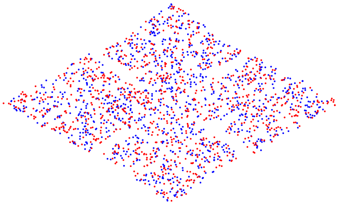
How: for $P = (\xi, z) \in \mathbb{C}^2$ iterate a random function

$$f(P) := (\lambda_i \phi_i \xi, z + v_i \xi)$$

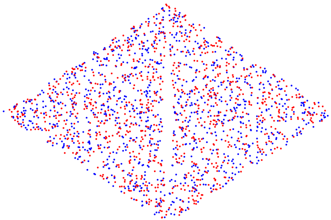
where λ_i, ϕ_i and v_i are RVs depending on protocol for interpolating $d_H \in (1, 2)$.

$$\boxed{d_H = 1.9}$$

Peano



Cesàro



The observable [Paper ⑤]

Assumption:

$$E^{\mathrm{P/C}}_{n,(p,d_H)} \sim c^{\mathrm{P/C}}_{(p,d_H)} n^{\gamma^{\mathrm{P/C}}_{(p,d_H)}}$$

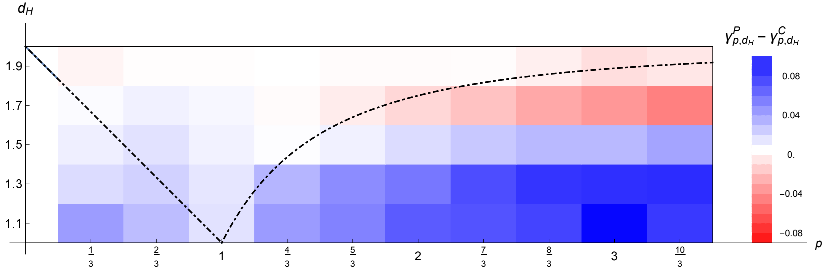
Question: do the exponents γ depend on the fractal set considered?

We have numerically estimated $\gamma^{\mathrm{P/C}}_{(p,d_H)}$ in a large region of the (p, d_H) plane: $p \in (.33, 1.33)$ and $d_H \in (1.1, 1.9)$, for a total of 50 (p, d_H) points. Results have been obtained through two different statistical methods which gave consistent results.
The exponents for possible logarithmic corrections γ' remains out of reach of our numerical protocol.

Results: scaling exponents are non-monotone and satisfy $\gamma^{\mathrm{P/C}}_{(p,d_H)} > 1 - \frac{p}{d_H}$.

Numerical findings [Paper ⑤]

Signature of universality: in a whole sub-region of the considered domain "engulfing" the line $d_H = 2$, $\gamma^{\mathrm{P}}_{(p,d_H)} - \gamma^{\mathrm{C}}_{(p,d_H)}$ is on the scale of statistical error.



Open question: is there a critical line ?

Some provisional conclusions

At $d = 1$ we have:

- Obtained the full distribution of $\mathcal{H}_{\mathrm{opt},(n,2)}$ in the $n \to \infty$ limit on the circle.
- Introduced and studied anomalous scalings at $p \geq 1$: if the Brownian Bridge qualitative picture works, the problem is "reduced to quadratures"; otherwise either regularize such integral (faster but less rigorous and no sub-leading constant), or perform a combinatorial-analytical calculation of the contributions of each edge, at finite n.
- Logarithmic corrections appear along critical hyperbolae.
- In the concave regime $p \in (0, 1)$, we have introduced Dyck matchings, obtained their average energy, and conjectured that it has the same scaling of the true ground state.

Some provisional conclusions

At $d = 2$ we have:

- extended the Caracciolo–Lucibello–Parisi–Sicuro field theoretic approach at $p = 2$ on a generic 2-manifold Ω. Conjecture that the first Ω-dependent terms in the expansion of $E_\Omega(n)$ can be obtained from the spectrum of Laplace-Beltrami on Ω (e.g. as "Kronecker masses", or equivalently "Robin masses").
- Conjectured that all (possibly) divergent terms are independent on Ω. Empirical evidences suggest that they depend on the randomness protocol (Grid-Poisson vs Poisson-Poisson, choice of grid...).
- Proposed an Helmholtz decomposition of the transport field in the Grid-Poisson case on the 2-torus (Not discussed today).
- Performed a study of the problem for configurations of points connected by a symmetry group (Not discussed today).

Some provisional conclusions

At "$d_H \in (1,2)$" we have:

- Introduced the Peano and Cesàro fractals which provide two different interpolations of $(1,2)$ in Hausdorff dimension.
- Estimated numerically the leading scaling exponent of ground state energy in a large part of the (p, d_H) plane.
- Found evidence of persistence of universality in a v-shaped region based at $(p,d) = (1,1)$ and "engulfing" the line $d_H = 2$.

Main open points

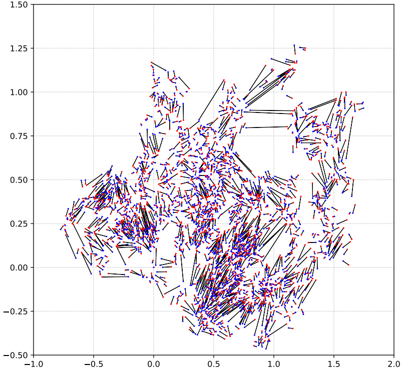
The phase diagram remains a bit obscure. Most notably:

- $p \neq 2$?
- Can we understand "anomalous scaling" at $d > 1$?
- $p \leq 1$ and $d = 2$: what is the leading scaling?
- 2-manifolds at $p = 2$: can we really understand the scaling limit?
- Can we understand the field theory beyond linearization?
- Analogous of Kronecker/Robin masses at $d > 2$?

The gap with rigorous results is still quite wide.

Some broader perspectives

1. Excited states. How to warm the system up? Elementary excitations are closed cycles of even length. What are their statistical properties?
2. We can study the ERAP on the d-dimensional Brownian loop (especially at $d = 2$). Is the energy asymptotics for such an ERAP dominated by the bulk or by the anomalous behavior around cut-times?



Thank you for your attention !

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