From local to global specifications: the spin-flop transition in the XY model on $\mathbb{Z}^2$ under decimation

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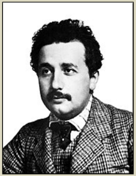
Statistical mechanics and the ensembles of Gibbs

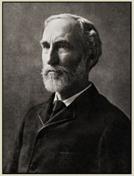
"to derive the laws of thermal equilibrium [...] using only the equations of mechanics and the probability calculus"

Einstein 1902



Elementary Principles in Statistical Mechanics

Gibbs 1902



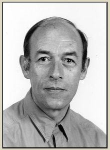
For describing the **Gibbs** (equilibrium) measure(s) of **spatially** ∞ lattice spin systems, **two main problems**:

1. Microscopic hamiltonian is **divergent**
2. **Uniqueness** in phase transitions (Kolmogorov extension Thm)

1. Dobrushin–Lanford–Ruelle (DLR) approach

Dobrushin
1968

Lanford–Ruelle
1969



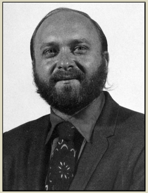
Marginal probabilities $\Longrightarrow$ **Conditional** probabilities wrt prescribed **boundary conditions**

Put on rigorous ground by **Georgii** (**Friedli–Velenik** 2017)

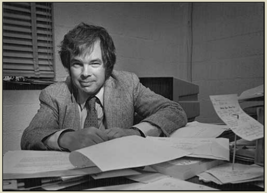
2. Phase transitions and the renormalisation group

- Critical opalescence, **Cagniard de Latour** 1822
- Para-ferromagnetic transition, **Pierre Curie** 1895

Kadanoff
1966



Wilson
1983



J. Zinn-Justin 2005

A few motivations

Aim: different **global behaviors** compatible w. given **local laws**.

Observables are local functions on a configuration space $(\Omega, \mathscr{F})$.

$\mathscr{F} = \mathscr{P}(\{-1, +1\})$ for Ising, a Borel σ-algebra (continuous spins)

Physical states are modeled by **Gibbs measures**, which are well understood if $|\Omega| < \infty$. For Λ a **finite** subset of a lattice $\mathscr{L}$:

- **Measurability** for events supported outside Λ, which can be interpreted as **boundary conditions**;
- Concentration of the resulting measure on Λ (**properness**);
- "Nested" conditioning (**consistency**).

The synthesis is called **local specification** (**Georgii** 1988)

Local specifications

Def. (**Föllmer** 1975, **Preston** 1976). A **local specification** is a family $(\gamma_\Lambda)_{\Lambda \Subset \mathscr{L}}$ of probability kernels $\gamma_\Lambda : \mathscr{F} \times \Omega \to [0,1]$ satisfying **properness** and **consistency**.

1. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(\cdot|\omega)$ is a **probab. measure**;
2. $\forall$ event $A \in \mathscr{F}$: $\gamma_\Lambda(A|\cdot)$ is $\mathscr{F}_{\Lambda^c}$-measurable;
3. $\forall$ config. $\omega \in \Omega$: $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$, $B \in \mathscr{F}_{\Lambda^c}$ (**properness**);
4. $\forall$ boxes $\Lambda \subset \Lambda'$, **finite**, $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$ (**consistency**).

Gibbs specification, measures, and the set $\mathscr{G}(\gamma)$

Gibbs specification: for $\beta > 0$, Λ finite and a priori measure ρ

$$\gamma_\Lambda(d\sigma \mid \omega) \stackrel{\text{def}}{=} \frac{1}{Z_\Lambda^{\beta\Phi}(\omega)} e^{-\beta H_\Lambda^\Phi(\sigma|\omega)} (\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}})(d\sigma).$$

A measure μ is **specified** by (or **consistent with**) γ_Λ if it satisfies the following **DLR equations**

$$\mu[A \mid \mathscr{F}_{\Lambda^c}](\sigma) = \gamma_\Lambda(A \mid \sigma),\ \mu\text{-a.e. } \sigma \in \Omega\,.$$

A **Gibbs measure** is a measure specified by a Gibbs specification.

Graal of statistical mechanics: to understand the set of all Gibbs measures $\mathscr{G}(\gamma)$ depending on parameters (temperature, coupling...)

Quasilocality

A function f is **quasilocal** iff it is a limit (in the sup norm) of a sequence of **local** functions. Equivalently,

$$\lim_{\Lambda \uparrow \mathscr{L}} \sup_{\sigma, \omega : \sigma_\Lambda = \omega_\Lambda} | f(\omega) - f(\sigma) | = 0.$$

Remark: In any model with finite state space (e.g. **Ising, Potts**)

quasilocality $\iff$ (uniform) continuity

The Kozlov–Sullivan Theorem

Action of a local specification on functions: for $\omega \in \Omega$,

$$\gamma_\Lambda f(\omega) = \int_\Omega f(\sigma)\gamma_\Lambda(d\sigma|\omega) = \gamma_\Lambda[f|\omega] \qquad \text{(sums for Ising)}$$

A specification is **quasilocal** if it preserves quasilocal functions

$$f \ \text{ is quasilocal} \implies \gamma_\Lambda f \ \text{ is quasilocal}$$

μ is a Gibbs measure $\iff$ μ is specified by a **non–null** and **quasilocal** specification.

Kozlov 1974, **Sullivan** 1973 (see also **Barbieri** *et al.* 2021)

Renormalization group and local specifications

Let $\nu = T\mu$, where T a renorm. group transformation (decimation, majority rule...)

- Existence of ν (**Griffiths**, **Pearce**, **Israel**)
- RG pathologies interpreted as loss of Gibbs property (**van Enter–Fernandez–Sokal** 1993)

"The other way round": ν not quasilocal $\implies$ ν non-Gibbsian

How: points of **ess. discontinuity** of **renormalized measures**

D'A–Le Ny 2022 for a majority rule in Ising on Cayley tree $\mathscr{T}^{(2)}$

?

Conditioning wrt some $\Lambda^c \subset \mathscr{L}$, where $|\Lambda| = \infty$?

Global specifications

Fernandez–Pfister 1997

A **global specification** is a local specification in which **consistency** holds also for ∞ subsets of the lattice $\mathscr{L}$: $\forall \Lambda_1, \Lambda_2 \subset \mathscr{L}$

$$\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2}, \quad \Lambda_1 \subset \Lambda_2 \,.$$

Fernandez–Pfister addressed the problem of existence of global specifications when quasilocality $\iff$ continuity (**Ising model**,...).

Today: extend FP to **XY models**, which have **continuous** spins.

XY model: definitions

A **ferromagnetic** spin model on $\mathbb{Z}^2$ enjoying a $O(2)$ **symmetry.**

- **Configuration space, events, a priori measure:**

$$\Omega = (\mathbb{S}_1)^{\mathbb{Z}^2}, \ \mathscr{F} = \left[\mathscr{E}(\mathbb{S}^1)\right]^{\otimes \mathbb{Z}^2} \quad \text{with} \quad \rho = \left(\frac{d\theta}{2\pi}\right)^{\otimes \mathbb{Z}^2}$$

- **Ferromagnetic**, two-body potential: for a config. $\vec{\sigma}$ and $J(i,j) > 0$

$$\Phi_{\{i,j\}}(\vec{\sigma}) = -J(i,j)\, \vec{\sigma}_i \cdot \vec{\sigma}_j = -J(i,j)\, \cos\left(\theta_j - \theta_i\right)$$

- **Hamiltonian** in finite volume $\Lambda \Subset \mathbb{Z}^2$ and **boundary condition** $\vec{\omega}$:

$$H_\Lambda^\Phi(\vec{\sigma} \mid \vec{\omega}) \overset{\text{def}}{=} \sum_{\{i,j\},\, \Lambda \cap \{i,j\} \neq \emptyset} \Phi_{\{i,j\}}(\vec{\sigma}_\Lambda \vec{\omega}_{\Lambda^c}).$$

XY model: golden age

'64 **Schultz–Mattis–Lieb**: coined the name XY (quantum case)

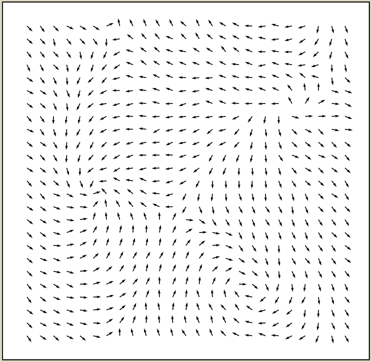
'70 **Ginibre**: Griffiths inequalities, including XY model (model 3)

'71 **Berezinsky**, **Kosterlitz–Thouless** ('73): free vortices to vortices–antivortices transition (*Nobel Prize in Physics 2016*)

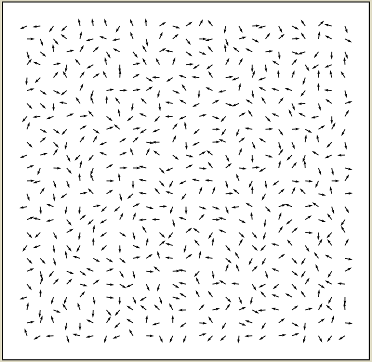
'77 **Fröhlich–Spencer** ('81): proof of existence of these two phases (*Dannie Heineman Prize 1991*)

'80 **Aizenmann–Simon**: lower bound on the BKT transition temperature in any lattice dimension (**Simon** 2022)

Portrait of the two phases of the XY model



$\beta \gg \beta_{\mathrm{BKT}}$ (low T)



$\beta \ll \beta_{\mathrm{BKT}}$ (high T)

Numerical protocol: Metropolis–Hastings, free bc, $\Lambda = [0,25]^2 \cap \mathbb{Z}^2$, $J(i,j) = \text{const} > 0$

Angle representation and partial preorder

- **Angle representation** of $\vec{\sigma}$:

  for $\vec{e_1} = (1,0)^T$, define the angle at $i \in \mathbb{Z}^2$ as

  $$\theta_i = \theta(\vec{\sigma}_i) = \underbrace{(\vec{\sigma}_i, \vec{e_1})}_{\text{angle bw } \vec{\sigma_i} \text{ and } \vec{e_1}} \in \,]-\pi, +\pi]\,.$$

- **Partial preorder** on configurations:

  for $\vec{\sigma}, \vec{\sigma}' \in \Omega$,

  $$\vec{\sigma} \leq_{\sin} \vec{\sigma}' \iff \sin\theta_i \leq \sin\theta_i', \, \forall\, i \in \mathbb{Z}^2\,.$$

Gibbs specification and extremal measures

Gibbs (local) **specification** for the XY model: for $\Lambda \Subset \mathbb{Z}^2$ finite

$$\gamma^J_\Lambda(d\vec{\sigma} \mid \vec{\omega}) \overset{\text{def}}{=} \frac{1}{Z_\Lambda(\vec{\omega})} \exp\left[-\beta H^\Phi_\Lambda(\vec{\sigma} \mid \vec{\omega})\right] \left(\rho_\Lambda \times \delta_{\vec{\omega}^c_\Lambda}\right)(d\vec{\sigma})$$

Fact: take the configs. $\vec{\pm} = \left(\pm\frac{\pi}{2}\right)_{i\in\mathbb{Z}^2}$. The Gibbs measures obtained as weak limits are **extremal** wrt $\leq_{\sin}$: $\forall f : \Omega \to \mathbb{R}$

$$\mu^-[f] \leq \mu[f] \leq \mu^+[f]$$

See talks by Külske and Coquille.

Global specification for XY model

Let $S \subset \mathbb{Z}^2$ **infinite**, and let us *freeze* into $\vec{\omega}$ on S^c

$$\mu_S^{+,\vec{\omega}}(\cdot) = \lim_{\Lambda \uparrow S} \gamma_\Lambda^J(\cdot \mid \vec{+}_S \vec{\omega}_{S^c}).$$

For all boundary condition $\vec{\omega} \in \Omega$, $\Gamma_S^+(d\vec{\sigma}|\vec{\omega}) \overset{\text{def}}{=} \mu_S^{+,\vec{\omega}} \otimes \delta_{\vec{\omega}_{S^c}}(d\vec{\sigma})$.

Theorem (D'A–van Enter–Le Ny 2022a)

$\Gamma^+ = \left(\Gamma_S^+\right)_{S \subset \mathbb{Z}^2}$ is a **global** specification.

Sketch of proof

- $\leq_{\sin}$ preserves **attractivity/monotonicity** (FKG inequalities)
- Continuous spin maintain **measurability** of kernels (basis...)

Choose $\Gamma_\Lambda^+ \equiv \gamma_\Lambda^J$ for Λ finite.

A very useful fact: For all configuration $\vec{\sigma}$

$$\vec{\sigma} \leq_{\sin} \vec{\sigma}_\Lambda \vec{+}_{\Lambda^c}$$

Consistency: $\forall f_1, f_2$* Λ_1-local & Λ_2-local, $\Lambda_1 \subset D_1$, $\Lambda_2 \subset D_2 \setminus D_1$:

$$\mathbb{E}_{\mu^+}(f_1 f_2) = \mathbb{E}_{\mu^+}(\Gamma_{D_1}^+(f_1 \mid \cdot) f_2)$$

Proof: $\leq$ and $\geq$.

■

* f_1 and f_2 positive and increasing in the sense of $\leq_{\sin}$.

Essential discontinuity/bad configuration

Essential discontinuities or **bad configurations** model the effect that changes at ∞ have deep into the bulk.

A configuration $\vec{\sigma}_{\text{bad}} \in \Omega$ is an **essential discontinuity** for a cond. proba μ, if the cond. expectations of a local function wrt. two configurations coinciding in a finite box **cannot be made arbitrarily close**.

If we are able to exhibit such configuration(s), we can easily prove **non-Gibbsianness** via Kozlov–Sullivan Theorem!

Non–Gibbsianness of decimated measure

Decimation is the map in which we read spins at a sub-lattice.

$T : (\Omega, \mathscr{F}) \longrightarrow (\Omega', \mathscr{F}') = (\Omega, \mathscr{F})$ acting as follows

$$\vec{\omega} \longmapsto \vec{\omega}' = (\vec{\omega}'_i)_{i \in \mathbb{Z}^2}, \text{ with } \vec{\omega}'_i = \vec{\omega}_{2i} \,.$$

The **decimated measure** is defined as, $\forall A' \in \mathscr{F}'$

$$\nu^+(A') = \mu^+(T^{-1}A')$$

Fact: We can drop the $+$ (or any θ..) and write just ν (or μ)!

Mermin-Wagner “ban” and the spin-flop transition

The n.n. XY model does not spontaneously break $O(2)$ symmetry when $\beta \to \infty$ due to the **Mermin-Wagner Theorem**.

If spontaneous magnetization $= 0$ then unique (extremal) phase in the equilibrium states (**Bricmont–Fontaine–Landau** 1977)

By choosing a particular configuration to condition for the primed spins, the $O(2)$ symmetry for the non-primed spins **gets reduced to a discrete one**.

XY model: recent works using spin–flop transition

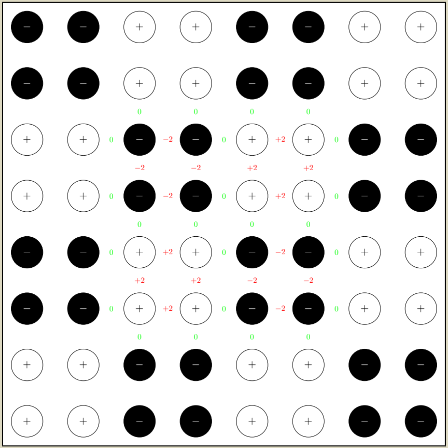
'08 **van Enter–Ruszel**: "transient" Gibbsianness in diffusive dynamics

'10 **van Enter–Külske–Opoku–Ruszel**: review ($O(n)$ models)

'11 **Crawford**: Random centered ext. field in one direction $\forall d \geq 2$

'16 **Collet–Ruszel**: mean–field XY w dichotomic external random field

The doubly alternating configuration $\vec{\sigma}'_{\mathrm{dblyalt}}$



$$\vec{\sigma}'_{\mathrm{dblyalt}} = (-1)^{\lfloor \frac{i'}{2} \rfloor + \lfloor \frac{j'}{2} \rfloor}, \quad i' j' \in \mathbb{Z}$$

Theorem (D'A–van Enter–Le Ny 2022b)

$\vec{\sigma}'_{\mathrm{dblyalt}}$ is a **bad configuration** for $\nu = T\mu$ at low temperature.

Proof ideas

First step: zero temperature ($\beta = \infty$)

Local function: **magnetization** along the x axis $f = \vec{\sigma}'^{(1)}_{\{0,0\}}$.
Technical estimate and use of XY global specification show that $\vec{\sigma}'_{\text{dblyalt}}$ is an essential discontinuity for $\nu = T\mu$
($\implies$ ν non-quasilocal $\implies$ ν non-Gibbsian by Kozlov–Sullivan).

Second step: Extension to low temp. $\beta < \infty$ **large** by **convexity**

Generalized contour method of **Malyshev *et al.*** 1983
(see also **van Enter–Külske–Opoku** 2011);

$\blacksquare$

Conclusions

If $|\Lambda| = \infty$, cond. wrt configs supported on Λ^c requires extra care!

Fortunately, **global specifications** are very useful ...

For **XY** models, **global specifications** are central ingredient for showing non-Gibbsian of decimated measure even in absence of LRO of the original model.

Using a well-known **spin-flop mechanism** we have shown an explicit exemple

Some perspectives

☛ Extension of Malyshev Thm to long-range models

☛ **Global specifications** and **essential enhancements** in percolation with obstacles (Aizenman–Grimmett 1991)

- Gibbs specifications for Ising on **Penrose lattices**; and **regular tiling** of $\mathbb{H}_2$ (Series–Sinai, Gandolfo–Ruiz–Shlosman)
- Spin–flop mechanism for XY on **regular trees**?

Thanks!

The bad configuration, and conditioning

For a special configuration $\vec{\omega}'_{\text{spe}}$, for ν^+-a.e. $\vec{\omega}' \in \mathcal{N}_{\Lambda',\varepsilon}(\vec{\omega}'_{\text{spe}})$

$$\nu^+[f(\vec{\sigma}')|\mathcal{F}_{\{(0,0)\}^c}](\vec{\omega}') = \Gamma_S^+[f(\vec{\sigma}')|\vec{\omega}] \;\; \mu^+\text{−a.e.}(\vec{\omega}),$$

with $S = (2\mathbb{Z}^2)^c \cup \{(0,0)\}$ and $\vec{\omega} \in T^{-1}\{\vec{\omega}'\}$ which coincide with $\vec{\omega}'_{\text{spe}}$ over $2\mathbb{Z}^2$. $\forall \vec{\omega}' \in \mathcal{N}_{\Lambda'}(\vec{\omega}'_{\text{spe}})$,

$$\nu^+[f(\vec{\sigma}')|\mathcal{F}_{\{(0,0)\}^c}](\vec{\omega}') = \mu^{+,\overrightarrow{\omega}}_{(2\mathbb{Z}^2)^c \cup \{(0,0)\}} \otimes \delta_{\overrightarrow{\omega}_{2\mathbb{Z}^2 \cap \{(0,0)\}^c}}[f(\vec{\sigma}')].$$

It is obtained as monotone weak limit with b.c. $+\frac{\pi}{2}$ fixed *after* freezing $\vec{\omega}$ on even sites: $\forall \vec{\omega}' \in \mathcal{N}_{\Lambda'}(\vec{\omega}'_{\text{alt}}), \forall \vec{\omega} \in T^{-1}\{\vec{\omega}'\}$,

$$\mu^{+,\overrightarrow{\omega}}_{(2\mathbb{Z}^2)^c \cup \{(0,0)\}}(\cdot) = \lim_{\Delta \in \mathcal{S}, \Delta \uparrow (2\mathbb{Z}^2)^c \cup \{0,0)\}} \gamma^J_\Delta(\cdot \mid \overrightarrow{+}_{(2\mathbb{Z}^2)^c \cup \{0,0)\}}) \, \vec{\omega}_{2\mathbb{Z}^2 \cap \{0,0)\}^c}).$$

Essential discontinuity

Detailed definition

Def. A configuration $\vec{\omega} \in \Omega$ is an **essential discontinuity** for a conditional proba μ, if $\exists \Lambda_0 \Subset \mathscr{L}$, a local function f, and a real $\delta > 0$, s.t. $\forall \Lambda$ containing Λ_0, 2 neighborhoods of $\vec{\omega}$ $\mathscr{N}_\Lambda^1(\vec{\omega})$ and $\mathscr{N}_\Lambda^2(\vec{\omega})$ exists s.t.

$$\forall \vec{\omega}^1 \in \mathscr{N}_\Lambda^1(\vec{\omega}),\ \forall \vec{\omega}^2 \in \mathscr{N}_\Lambda^2(\vec{\omega}),$$
$$\left| \mu\left[f \middle| \mathscr{F}_{\Lambda^c}\right](\vec{\omega}^1) - \mu\left[f \middle| \mathscr{F}_{\Lambda^c}\right](\vec{\omega}^2) \right| > \delta\ .$$

Equivalently:

$$\lim_{\Delta \uparrow \mathbb{Z}^2} \sup_{\vec{\omega}^1, \vec{\omega}^2 \in \Omega} \left| \mu\left[f \middle| \mathscr{F}_{\Lambda^c}\right](\vec{\omega}_\Delta \omega^1_{\Delta^c}) - \mu\left[f \middle| \mathscr{F}_{\Lambda^c}\right](\vec{\omega}_\Delta \vec{\omega}^2_{\Delta^c}) \right| > \delta.$$

End of proof of Thm 1 (DVLI)

$\mathbb{E}_{\mu^+}(f_1 f_2) \geq \mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2)$

Use FP 1997 for Ising model & $\vec{\sigma} \leq_{\sin} \vec{\eta}_\Lambda \vec{+}_{\Lambda^c}$.

$\forall f_1, f_2$, Λ_1-local and Λ_2-local, with $\Lambda_1 \subset D_1$, $\Lambda_2 \subset D_2 \setminus D_1$:

1. $\Gamma^+_{D_1}(f_1 \mid \vec{\eta}) \underset{\text{Kernel monotonicity}}{\leq} \gamma^J_\Lambda\left(f_1(\vec{\sigma}_\Lambda) \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c}\right)$;
2. $\mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2) \leq \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_\Lambda\left(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c}\right) f_2(\vec{\eta})$;
3. Take a set Λ_2 s.t. $\Lambda_2 \cap D_1 = \Lambda$. Then

$$\int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_\Lambda\left(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c}\right) f_2(\vec{\eta}) = \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) f_1(\vec{\eta}) f_2(\vec{\eta}),$$

and hence $\mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2) \leq \mathbb{E}_{\mu^+}(f_1 f_2)$.

End of proof of Thm 1 (DVLI)

$\mathbb{E}_{\mu^+}(f_1 f_2) \le \mathbb{E}_{\mu^+}(\Gamma^+_{D_1}(f_1 \mid \cdot) f_2)$

For $M \subset \Lambda_2 \cap D_1$,

$$\begin{aligned}
\mathbb{E}_{\mu^+}(f_1 f_2) &\le \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) f_1(\vec{\eta}) f_2(\vec{\eta}) = \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_M(f_1 | \vec{\eta}) f_2(\vec{\eta}) \\
&\le \lim_{\Lambda_2} \int \gamma^J_{\Lambda_2}(d\vec{\eta} \mid \vec{+}) \gamma^J_M(f_1 | \vec{+}_{D_1} \vec{\eta}_{D_1^c}) f_2(\vec{\eta}) \\
&= \int \mu^+(d\vec{\eta}) \gamma^J_M(f_1 | \vec{+}_{D_1} \vec{\eta}_{D_1^c}) f_2(\vec{\eta}) \,.
\end{aligned}$$

The conclusion follows from Beppo Levi theorem:

$$\begin{aligned}
\mathbb{E}_{\mu^+}(f_1 f_2) &\le \lim_{M \uparrow D_1} \int \mu^+(d\vec{\eta}) f_2(\vec{\eta}) \gamma_M(f_1 \mid \vec{+}_{D_1} \vec{\eta}_{D_1^c}) \\
&= \mathbb{E}_{\mu^+}\left(f_2 \Gamma^+_{D_1}(f_1 \mid \cdot) \right)
\end{aligned}$$

hence consistency is extended to **infinite** sets, $\Gamma^+_{D_1} = \Gamma^+_{D_1} \Gamma^+_{D_2}$. ■

Proof of essential discontinuity (DVL2022a) - detail

Let $\Lambda'_L = ([-L,+L]\cap\mathbb{Z})^2$, $\Delta'_N = ([-N,+N]\cap\mathbb{Z})^2$, with $N > L$. Then a <u>bound uniform in L</u> holds for energy differences with b.c. ω_1^+ and ω_2^+: it is enough to choose $N = N(L) = O(L^{\frac{2}{\alpha-1}})$. More precisely:

$$\delta H_L^{+,\omega'_{1/2}} := \left| H_{\Lambda,\omega_1^+}(\sigma_\Lambda) - H_{\Lambda,\omega_2^+}(\sigma_\Lambda) \right| \le \sum_{x\in\Lambda_{2L}} 2 \sum_{k>2N} \frac{1}{k^\alpha} < C < \infty.$$

Lemma (DVL2022a). Let $\Lambda' \subset \Delta' \in \mathscr{S}$ and let $\omega'^+ \in \mathscr{N}^+_{\Lambda',\Delta'}(\omega'_{\mathrm{alt}})$ and $\omega'^- \in \mathscr{N}^-_{\Lambda',\Delta'}(\omega'_{\mathrm{alt}})$. Then $\exists\delta > 0$ and $\exists\Lambda'_0$ large enough s.t. $\Delta' \supset \Lambda' \supset \Lambda'_0$ with $\Delta'\setminus\Lambda'$ much larger than Λ', s.t. $\forall\omega^+ \in T^{-1}\{\omega'^+\}$ and $\forall\omega^- \in T^{-1}\{\omega'^-\}$,

$$\left| \mu^{+,\omega^+}_{(2\mathbb{Z}^2)^c\cup\{0\}}[\sigma_0] - \mu^{+,\omega^-}_{(2\mathbb{Z}^2)^c\cup\{0\}}[\sigma_0] \right| > \delta \qquad \text{(essential discontinuity).}$$