On the phase diagram of Euclidean Random Assignment Problems at low dimensions

Franco-Dutch meeting “Bézout-Eurandom”

CNRS International Research Project « Random Graphs, Statistical Mechanics and Networks »

Amphithéâtre Hermite, Institut Henri Poincaré, Paris
Wednesday, 7 July 2021, 10h20-11h00 CET

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Based on several works in collaboration with:

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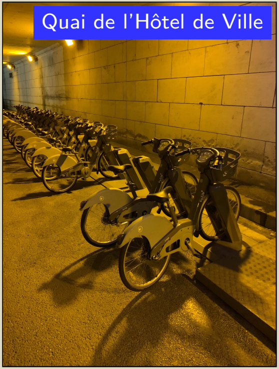
Main references: (here and afterwards all paper references are clickable) preprint at

- J. Stat. Phys. **183**, 34, 2021 `2008.01462`
- **D'A**, PhD Thesis, Université Paris-Saclay, 2020 `tel:03098672`
- J. Phys. A Math. Theor. **53**, 6, 2020 `1904.10867`
- J. Stat. Phys. **174**, 4, 2019 `1803.04723`
- Phys. Rev. E **96**, 4, 2017 `1707.05541`

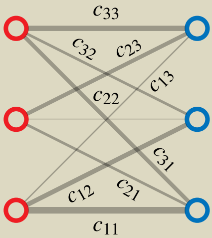
The assignment problem



The assignment problem

Definition (assignment problem). Consider a $n \times n$ real matrix c (cost matrix). For any n-permutation $\pi \in \mathscr{S}_n$, define the energy $E(\pi) = \sum_{i=1}^{n} c_{i\pi(i)}$. Find $\pi_{\text{opt}} := \arg\min_{\pi \in \mathscr{S}_n} E(\pi)$.

Example at $n = 3$:



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

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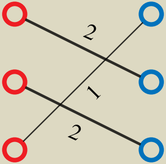
<u>Example</u> at $n = 3$: $\pi_{\text{opt}} = (3\,1\,2)$, $E_{\text{opt}} := E(\pi_{\text{opt}}) = 5$



$$c = \begin{pmatrix} 5 & 3.5 & 1 \\ 2 & 1.2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

- Optimization of a linear function over the **convex Birkhoff polytope**;
- P-**complete** with $\mathscr{O}(n^3)$ complexity (Munkres 1957);
- Equivalent to a two player zero-sum game (von Neumann 1953, 1954).

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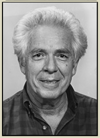
Assignment problem: Some historical remarks

von
Neumann
1953



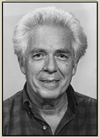
Kuhn
1955

König
1916

Egérvary
1931

Assignment problem: Some historical remarks

von Neumann 1953



Kuhn 1955

König 1916

Egérvary 1931

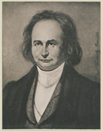
Canon simplicissimus.

	I	II	III	IV	V	VI	VII
I	25*	21	20	18	20	18	25
II	21	22*	21	21	13	21	22
III	16	19	23*	22	17	14	16
IV	21	12	18	27*	13	14	24
V	25	22	22	27	31*	16	31
VI	10	18	23	21	19	23*	21
VII	5	14	10	27	31	20	40*

"*De investigando ordine systematis aequationum differentialium vulgarium cujuscunque*"

See also (Ollivier 2009).

Jacobi 1860



The Random Assignment Problem I

Here c is a random matrix, E_{opt} a **(non-trivial) random variable**.

- Pioneered in Physics in the 80s by Mézard–Parisi and Orland;
- Entered Probability mostly through Aldous in the 90s.

For $(c_{ij})_{i,j=1}^{n}$ i.i.d. r.v. of probability density $\rho(l) = l^r + o(l^r)$, then

$$\mathbb{E}[E_{\text{opt}}]_n \underset{n\to\infty}{\sim} c_r n^{1-\frac{1}{r+1}}.$$

Result: in the graph theory interpretation, only "short" edges are relevant for large n: r is a "**universal exponent**".

Nice fact: at $r = 0$ (i.e. ρ is e.g. uniform or $\mathrm{Exp}(\lambda)$ distribution),

$$c_0 = \zeta(2) = \frac{\pi^2}{6}.$$

The Random Assignment Problem II

If $c_{ij} \sim \mathrm{Exp}(1)$, Parisi conjectured (1998):

$$\mathbb{E}[E_{\min}]_n = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

- Extension to rectangular cost matrices (Coppersmith-Sorkin 1998)
- Proof of $\zeta(2)$ limit (among other things) (Aldous 2001)
- Proof of Parisi conjecture (Nair–Prabhakar–Sharma 2001)
- Extension to k-colors (which is **NP-hard** for $k \geq 3$) (Martin–Mézard–Rivoire 2004,2005)
- Prof of $\exists!$ of Mézard–Parisi order parameter $\forall r \geq 0$ (Wästlund 2012, Larsson 2014, Salez 2015)

NOT discussed today...

Euclidean Random Assignment Problems (ERAPs)

Let $\mathscr{B} = (B_1, \ldots, B_n)$ be blue points and $\mathscr{R} = (R_1, \ldots, R_n)$ be red ones: n-samples of i.i.d. r.v. with pdf $\nu_{\mathscr{B}(\mathscr{R})} : \Omega \to \mathbb{R}$ (**disorder**). Let $(\Omega, \mathscr{D})$ be a metric space (mostly an **Euclidean** space with $\mathscr{D}$ **Euclidean** distance). For $p \in \mathbb{R}$ and an assignment (n-permutation) $\pi \in \mathscr{S}_n$, consider the *Hamiltonian*

$$\mathscr{H}(\pi) = \sum_{i=1}^{n} \mathscr{D}^p(B_i, R_{\pi(i)})$$

and the random variable "**ground state energy**"

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = \min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi) \quad \left(\pi_{\mathrm{opt}} = \arg\min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi)\right).$$

Definition (Euclidean Random Assignment Problem).
Understand the statistical properties of $\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}$ (*exact* or *asymptotic* law, moments, etc.) depending on (Ω, p, d) and $\nu_{\mathscr{B}(\mathscr{R})}$.

Three motivations for ERAPs

- **Spin Glasses.** ERAP provides a toy-model of **spin-glass in finite dimension**. Besides disorder, the assignment constraint provides **frustration**. But the model is numerically simpler than e.g. Edwards–Anderson spin glass (Mézard–Parisi 1988).
- **Optimal Transport.** ERAP $=$ **Monge-Kantorovitch** transportation problem on Ω $(\dim(\Omega) = d)$ associated to the empirical measures $\rho_{\mathscr{B}(\mathscr{R})} = \frac{1}{n}\sum_j \delta_{B_j(R_j)}$. In particular

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)} = n W_p^p(\rho_{\mathscr{B}}, \rho_{\mathscr{R}}),$$

  where W_p is the p-**Wasserstein distance** (Villani 2009, Brezis 2018).
- **Computational Complexity Theory.** ERAPs are small modifications of random TSPs. But TSP is NP-**complete**.

ERAPs: the phase diagram

We shall put $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu$. We can start studying

$$E_{p,d}(n) := \mathbb{E}_{\nu^n \otimes \nu^n}[\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1 + o(1)),$$

as $n \to \infty$, depending on (p,d) and the choice of ν.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be **largely insensitive** on the choice of ν (which may alter the constant $K_{p,d}$).

Remark: Non-uniform disorder ν is more subtle!

Example: Take $\nu =$ standard Gaussian and $(p,d) = (2,1)$. Then

$$E_{2,1}(n) \underset{n\to\infty}{\sim} 2 \ln\ln n \qquad \text{(i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

(Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019, Berthet–Fort 2020)

See (Benedetto–Caglioti 2020) for non-uniform case at $d = 2$.

$d \geq 3,\ p \geq 1,\ \Omega =$ a bounded domain

"Simple": Solution is realized at the scale of nearest-neighbors

$$E_{p,d}(n)\Big|_{d\geq 3, p\geq 1} \underset{n\to\infty}{\sim} K_{p,d} n^{\gamma_{\mathrm{LB}}},$$

where

$$\gamma_{p,d} = \gamma_{\mathrm{LB}} := 1 - \frac{p}{d}, \quad \gamma'_{p,d} = 0 \qquad \text{(Mézard–Parisi 1988)}$$

(if the disorder ν is uniform on Ω, otherwise **unknown**).

Remark: The constants $K_{p,d}$ are **universal** (Barthe–Bordenave 2013 and refs. therein for $p < \frac{d}{2}$, and Goldman–Trevisan 2020 for an extension to $p \geq 1$) but **unknown explicitly**. Upper and lower bounds on some $K_{p,d}$ for $\Omega = [0,1]^d$ are in (Talagrand 1992), numerical estimates for $\Omega = [0,1]^d, \mathbb{T}^d$ are in (Caracciolo–Sicuro 2015, **D'A** MSc Thesis 2016).

$d = 2$: A challenge for both mathematicians and physicists

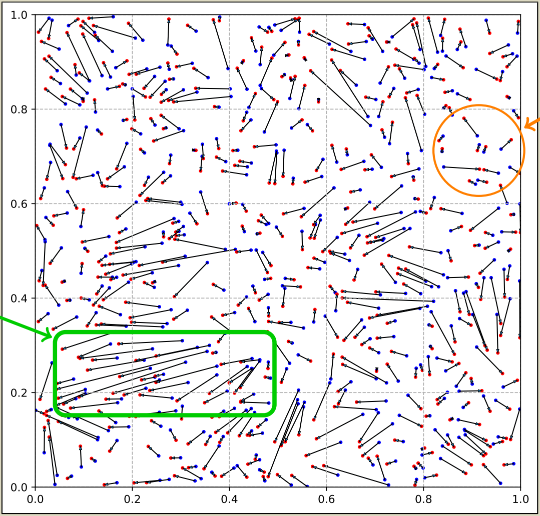
Example: $\Omega = [0,1]^2$, $\mathscr{D} = |\cdot|$. A solution at $p = 2$:

The solution π_{opt} connects typically $O(\ln n)$–**nearest-neighbors**.

$$\gamma_{p,d} = \gamma_{\text{LB}} = 1 - \frac{p}{2}$$

$$\gamma'_{p,d} = \frac{p}{2}$$

(Ajtai–Komlós–Tusnády 1984)



A few recent developments at $(p,d) = (2,2)$

2014 Caracciolo–Lucibello–Parisi–Sicuro (Phys. Rev. E): Using a **classical field-theoretical approach**, first predicted $K_{2,2} = \frac{1}{2\pi}$.

2019 Ambrosio–Stra–Trevisan (PTRF): **Proof** of $K_{2,2} = \frac{1}{2\pi}$ (among other things) via PDE methods.

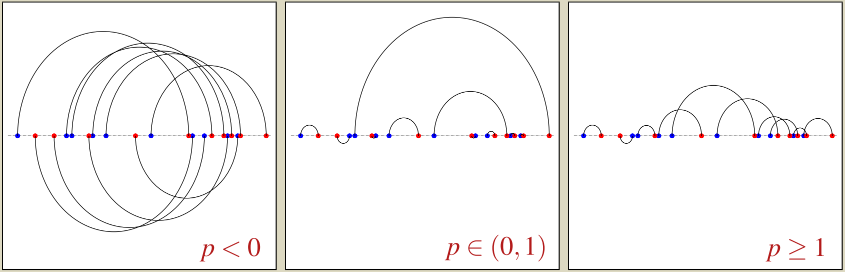
2020 Ambrosio–Glaudo (JEP): Refinement on the remainder term (among other things).

2021 Benedetto–Caglioti-Caracciolo–**D'A**–Sicuro–Sportiello (J. Stat. Phys.): Among other things, **exact formula** for $\lim_{n\to\infty}[E_\Omega(n) - E_{\Omega'}(n)]$, where Ω, Ω' are two manifolds (see this video contribution for a discussion of the latter results in the light of **Weyl's law**).

2021 Caracciolo *et al.*: Excitations are compatible with a loop-erased self-avoiding random walk process.

$d = 1$: Properties of the solution

For any ν, $p = 0$ and $p = 1$ separate **three qualitative regimes**:



If $(b_1, \ldots, b_n)$ and $(r_1, \ldots, r_n)$ are sorted in natural order π_{opt} is:

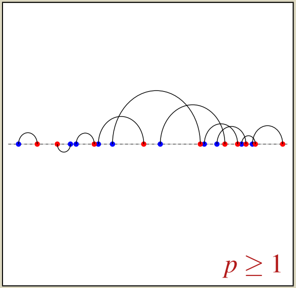
- **Cyclical** for $p < 0$: $\pi_{\text{opt}} = i + k \pmod{n}$ (Caracciolo–**D'A**–Sicuro 2017);
- **Non-crossing** for $p \in (0, 1)$: Intervals "covered by edges" are either disjoint or one is included into the other (McCann 1999);
- **Ordered** for $p \geq 1$: π_{opt} is the identity permutation.

$\Omega = [0,1]$ (or $\mathbb{R}, \mathbb{R}^+, \ldots$), $\mathscr{D} = |\,|$ and $p \geq 1$

Take $B_{j+1} \geq B_j$, $R_{j+1} \geq R_j$, for $j = 1, \ldots, n-1$.

Optimality + **(strict) convexity** + (strict) monotonicity of $\mathscr{D}^p$

$$\Downarrow$$



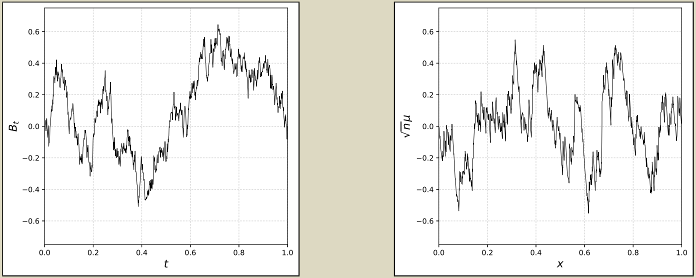
$$\mathscr{H}_{\mathrm{opt},(n,p)}\Big|_{p\geq 1} = \sum_{i=1}^{n} |B_i - R_i|^p.$$

$\Longrightarrow$ stairway to the **Brownian world!**

Brownian Bridge for $\Omega=[0,1]$, $\mathscr{D}=|\cdot|$, $p\geq 1$.

Let the **transport field** be $\mu_i := b_i - r_i$, for $i=1,\dots n$ and put $i = nt+\frac{1}{2}$. Then by **Donsker's Theorem**,

$$\sqrt{n}\,\mu_i \xrightarrow[weakly]{} B_t, \text{ the Brownian Bridge.}$$



Recall $\mathscr{H}_{\text{opt},(n,p)} = \sum_{i=1}^{n} |\mu_i|^p$. Then $E_{p,1}(n)\Big|_{p\geq 1} \underset{n\to\infty}{\sim} \mathbb{E}[B_t^p]\, n^{1-\frac{p}{2}}$.

(Boniolo–Caracciolo–Sportiello 2014, Caracciolo–Sicuro 2014, Caracciolo–**D'A**–Sicuro 2017)

$\Omega = \mathbb{S}_1$, $\mathscr{D} =$ arc-distance, $p=2$: The limit distribution of $\mathscr{H}_{\text{opt}}$

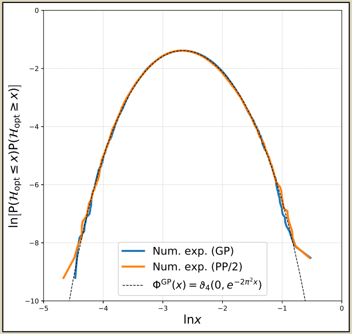
The cdf of $\mathscr{H}_{\text{opt},(n,2)}$ on $\mathbb{S}_1$ is **explicit**. This is because:

1. at $p=2$ we have **Parseval identity** for μ;
2. the mgf $\mathbb{E}\left[e^{-w\sum_{s\neq 0}|\hat{\mu}|_s^2}\right] = \prod_{s\geq 1} \frac{1}{1+\frac{w}{2\pi^2 s^2}}$ can be inverse-Laplace transformed (Watson 1961)!

We have (**D'A**, 2020):

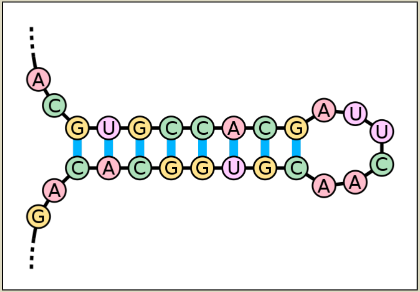
$$\mathbb{P}[\mathscr{H}_{\text{opt},(n,2)} \leq x] \underset{n\to\infty}{\to} \vartheta_4(0, e^{-2\pi^2 x})$$

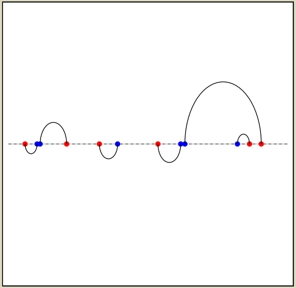
$$:= \sum_{s\in\mathbb{Z}} e^{i\pi s} e^{-2\pi^2 s^2 x}.$$



ERAPs at $p \in (0, 1)$: Further motivation in biology

Besides economics, interest due to the non-crossing property of the solution: Toy-model for the **secondary structure of RNA** (discarding pseudo-knots).



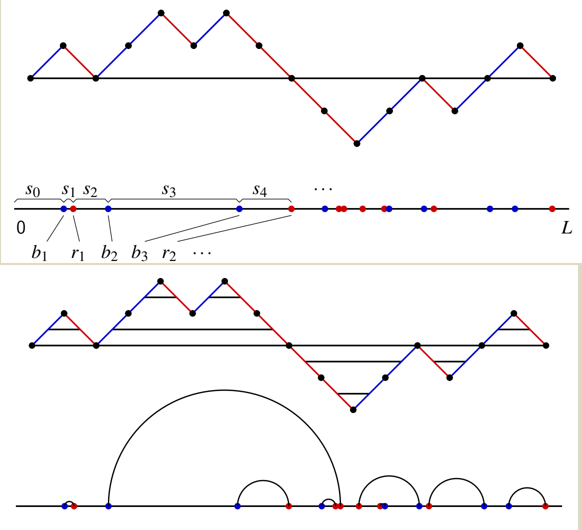


Despite this, poorly understood (McCann 1999).

IDEA: Try a canonical construction, an approximate solution sharing non-crossing property with $\pi_{\text{opt}} \rightarrow$ Dyck matchings!

The Dyck matching (Caracciolo–**D'A**–Erba–Sportiello 2020)

Construction:



The Dyck Conjecture at $p \in (0,1)$

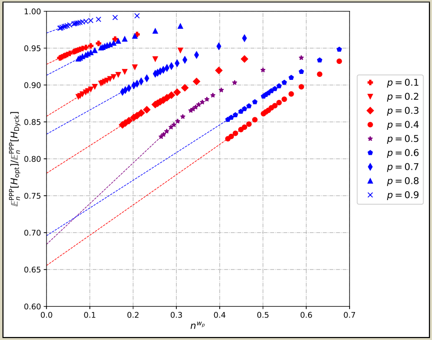
Expected energy of Dyck matchings grows asymptotically as

$$\mathbb{E}_n(\mathscr{H}_{\text{Dyck}}) \underset{n\to\infty}{\sim} \begin{cases} n^{1-p} & \text{if} \quad 0 \le p < \frac{1}{2} \\ \sqrt{n}\ln n & \text{if} \quad p = \frac{1}{2} \\ n^{\frac{1}{2}} & \text{if} \quad \frac{1}{2} < p \le 1 \end{cases} .$$

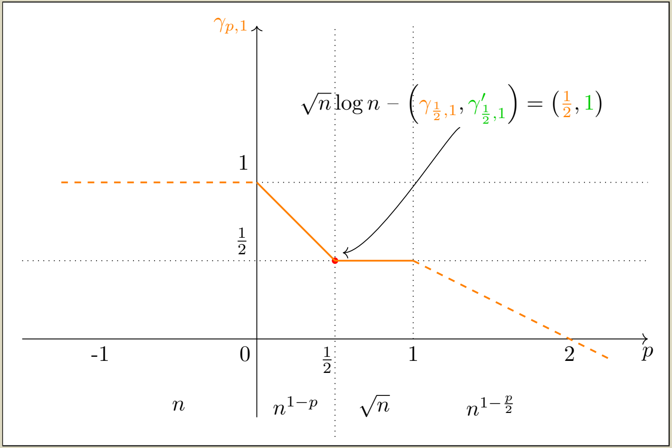
Conjecture (Caracciolo–**D'A**–Erba–Sportiello 2020):

$$\lim_{n\to\infty} \frac{\mathbb{E}_n(\mathscr{H}_{\text{opt}})}{\mathbb{E}_n(\mathscr{H}_{\text{Dyck}})} = k_p \,,$$

for some $k_p \in (0,1)$.
Upper bound implied by (Fournier–Guillin 2014).



Section of the Phase Diagram at $d = 1$



Product formula for number of solutions at $d = p = 1$ (Caracciolo–Erba–Sportiello 2021).

Beyond uniform disorder: Bulk and anomalous scaling

Recall that for $\nu = \mathbf{1}_{[0,1]}$ and $p \geq 1$ (Caracciolo *et al.* 2019):

$$E^{\mathrm{U}}_{p,1}(n) = n\frac{\Gamma\left(1+\frac{p}{2}\right)}{p+1}\frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}$$
$$= c_p n^{1-\frac{p}{2}}\left(1+o(1)\right) = c_p n^{\gamma_{p,1}}\left(1+o(1)\right).$$

Thus $E^{\mathrm{U}}_{p,1}(n)$ exhibits bulk scaling.

<u>What if ν is non-uniform?</u>

Definition. If $\gamma_{p,1} \neq 1-\frac{p}{2}$ or if $\gamma'_{p,1} \neq 0$, we say that $E_{p,1}(n)$ exhibits an **anomalous scaling**.

"Reduction to quadratures" in the bulk scaling

For R the cdf of ν, let $\Psi^{(\nu)} := \nu \circ R_\nu^{-1}$. Bobkov–Ledoux 2019 get

$$E_{p,1}^{(\nu)}(n) = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\int_0^1 \left[\frac{\sqrt{s(1-s)}}{\Psi^{(\nu)}(s)}\right]^p \mathrm{d}s\, n^{1-p/2} + o\left(n^{1-p/2}\right).$$

Caracciolo–**D'A**–Sicuro 2018: **Regularize the integral** inspired by cutoff regularization in Quantum Field Theory.

Example: $\nu(x) = e^{-x}$, $\Psi^{\exp}(s) = 1 - s$.

Cutoff method: Stop integration at distance $\frac{c}{n}$ away from the singularity:

$$E_{p,1}^{\exp}(n) = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\int_0^{1-c/n}\left(\frac{s}{1-s}\right)^{\frac{p}{2}} ds.$$

At $p = 2$, this gives:

$$E_{2,1}^{\exp}(n) = 2\ln n - 2\log c - 2 + o(1).$$

Exact result, $p = 2$ (Beta integrals):

$$E_{2,1}^{\exp}(n) = 2\sum_{k=1}^{n}\frac{1}{k} = 2\ln n + 2\gamma_E + o(1).$$

Rigorous approach to anomalous scaling

(**D'A**-Sportiello 2020-)

Guiding principle: Only the local properties of ν in a neighbourhood of the zero will determine the leading anomalous behaviour. So we decided to estimate contrib. of k-th edge at **fixed** n in

$$\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}|_{p\geq 1} = \sum_{k=1}^{n} |b_k - r_k|^p.$$

The general strategy:

1. **Binomial expand** $\mathscr{H}^{(p)}_{\mathrm{opt},(n,1)}|_{p\geq 1}$ for $p > 1$ even and fixed n;
2. Use **linearity** to compute expected contrib. of k-th edge;
3. **Asymptotic analysis** for $n \to \infty$ ($\implies$ local properties of ν);
4. **Analytic continuation** of result $\forall p \geq 1$.

More details on the general strategy

More precisely for a disorder ν, we have for Step 1-2:

$$E_{p,1}^{(\nu)}(n) = \sum_{k=1}^{n} \mathbb{E}\left[(b_k - r_k)^p\right] = \sum_{k=1}^{n}\sum_{q=0}^{p}\binom{p}{q}(-1)^{p-q} M_{n,k,q}^{(\nu)} M_{n,k,p-q}^{(\nu)},$$

where the ℓ-moment of the quantile function is

$$M_{n,k,\ell}^{(\nu)} = \langle R_\nu^{-1}(u)^\ell \rangle_{P_{n,k}}$$

and $\langle \dots \rangle_{P_{n,k}}$ denotes expectation w.r.t. $\mathrm{Beta}(k, n-k+1)$:

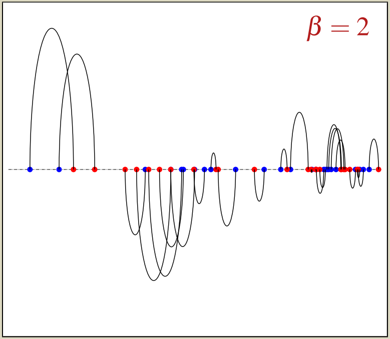
$$P_{n,k}(u)du := \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k} du.$$

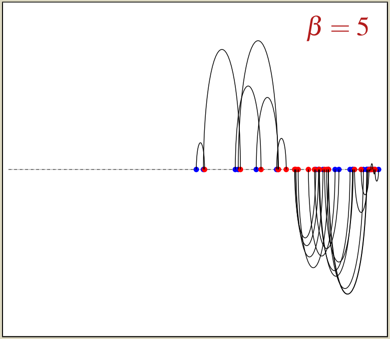
Example: Internal endpoint, algebraic zero $\nu_{\mathrm{fa},\beta}$

Consider the following pdf:

$$\nu_{\mathrm{fa},\beta}(x) = \beta x^{\beta-1}, \qquad x \in [0,1].$$

Example: Solutions at $n = 25$:





$$\text{Thus } R_{\mathrm{fa},\beta}(x) = x^{\beta} \implies R^{-1}_{\mathrm{fa},\beta}(u) = u^{\frac{1}{\beta}}.$$

A useful Lemma in asymptotic analysis

$$2 \quad M^{(\mathrm{fa}),\beta}_{k,n;\ell} = \left\langle \left(u^{\frac{1}{\beta}}\right)^{\ell} \right\rangle_{P_{n,k}} = \frac{\Gamma(k+\frac{\ell}{\beta})\Gamma(n+1)}{\Gamma(k)\Gamma(n+1+\frac{\ell}{\beta})}.$$

Lemma (see e.g. Whittaker–Watson, Tricomi)

As $n \to \infty$, for β finite, the function $\psi_n(\beta) := \ln\left[\frac{n^\beta \Gamma(n)}{\Gamma(n+\beta)}\right]$ has the series expansion

$$\psi_n(\beta) = \sum_{\substack{k\geq 1 \\ 0\leq \ell \leq k}} A_{k,l}\, n^{-k} \beta^{l+1} = \frac{\beta-\beta^2}{2n} + \frac{\beta - 3\beta^2 + 2\beta^3}{12n^2} + \cdots ,$$

where

$$A_{k,\ell} = (-1)^k B_{k-\ell} \frac{\Gamma(k)}{\Gamma(\ell)\Gamma(k-\ell+1)}, \qquad k \geq 1, \quad 0 \leq \ell \leq k,$$

for B_s the **s-th Bernoulli number**.

Separation of bulk and anomalous regimes

At leading order in $\frac{1}{n}$ the coefficient of β^p in

$$e^{\psi_n(\beta)} = \exp\left(-\frac{\beta^2}{2n}\sum_{\substack{k\geq 1\\ 0\leq l\leq k}}(-2A_{k,l})\left(\frac{\beta}{n}\right)^{2k-l-1}\left(\frac{\beta^2}{n}\right)^{-(k-l)}\right)$$

concentrates around the term with $k=l=1$ for large n (the minimum of $2k-l-1$ within the range $k\geq 1$, $0\leq l\leq k$).
This gives **two regimes** according to the magnitude of k/n (i.e. distance from the region of low density of points):

$$M^{(\mathrm{fa}),\beta}_{k,n;\ell} \underset{n\to\infty}{\sim} \begin{cases} \frac{\Gamma(k+\frac{\ell}{\beta})}{\Gamma(k)} n^{-\ell/\beta} & k \ \text{small } (\textbf{anomalous}) \\ \exp\left(-\frac{1}{2n}\left(\frac{\ell^2}{\beta}\right)\left(1-x^{-1}\right)\right)x^{\ell/\beta} & k = xn' \ (\textbf{bulk}) \end{cases}$$

Leading asymptotics in the bulk regime

Result

Lemma 1 (bulk regime, **D'A**–Sportiello 2020)

For the family of distributions $\nu_{\mathrm{fa},\beta} = \beta x^{\beta-1}$, $\beta \geq 1$,

$$E_p^{\beta}(n) \underset{n\to\infty}{\sim} b_{\beta,p}\, n^{1-p/2}, \qquad 2\beta + 2p - p\beta > 0,$$

with

$$b_{\beta,p} = \frac{1}{\beta^p} \frac{\Gamma(1+p)\,\Gamma\left(1 - \frac{p(\beta-2)}{2\beta}\right)}{\Gamma\left(2 + \frac{p}{\beta}\right)}.$$

Preliminaries: a useful (simple) Lemma

Lemma 0 (see **D'A**–Sportiello 2020)

Let $p \in \mathbb{N}$. Let $A(q) = a_p q^p + a_{p-1} q^{p-1} + \ldots + a_0$ be a polynomial of degree at most p. Then

$$\sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} A(q) = p!\, a_p .$$

Proof. 1. $A(q) = \sum_{k=0}^{p} b_k q(q-1)\cdots(q-k+1)$ ($\implies b_p = a_p$).

2. $$\sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} A(q) = \sum_{k=0}^{p} b_k \sum_{q=k}^{p} \frac{p!}{q!(p-q)!} (-1)^{p-q} q(q-1)\cdots(q-k+1)$$

$$= \sum_{k=0}^{p} \frac{b_k\, p!}{(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} (-1)^{(p-k)-r} = \sum_{k=0}^{p} \frac{b_k\, p!}{(p-k)!} \delta_{p,k} = p!\, b_p . \qquad \blacksquare$$

Leading asymptotics in the bulk regime

Sketch of proof

1. Lemma 0 implies

$$E_{(\beta,p),n}(k) \underset{n\to\infty}{\sim} \frac{1}{\Gamma(p/2+1)} x^{p/\beta} \left(\frac{-1+x^{-1}}{n\beta^2}\right)^{p/2}, \qquad k = xn' \ (\mathbf{bulk}).$$

2. Put $\Lambda = \mathcal{O}(\sqrt{n})$. Then $E_p^{\beta,\mathrm{bulk}}(n) = \sum_{k=\Lambda}^{n} E_{(\beta,p),n}(k)$ can be transformed into an integral **without affecting the leading asymptotics**. This gives

$$E_p^{\beta,\mathrm{bulk}}(n) \underset{n\to\infty}{\sim} \int_{\Lambda/n}^{1} \mathrm{d}x\, x^{\frac{p}{\beta}-\frac{p}{2}} P(x)\, n^{1-p/2},$$

for a polynomial $P(x)$. **Remark**: convergence if $2\beta+2p-p\beta>0$;

3. Change of variables + standard Beta-Γ identity.

Leading asymptotics in the anomalous regime

Result

Lemma 2 (anomalous regime, D'A–Sportiello 2020).

For the family of distributions $\nu_{\mathrm{fa},\beta} = \beta x^{\beta-1}$, for $\beta \geq 1$,

$$E_p^{\beta,\mathrm{anom}}(n) \underset{n\to\infty}{\sim} a_{\beta,p}\, n^{-p/\beta} \ ,$$

with

$$a_{\beta,p} = \Gamma(-1-p/\beta) \sum_{q=0}^{p} \binom{p}{q} (-1)^q \frac{\Gamma(1+\frac{q}{\beta})\Gamma(1+\frac{p-q}{\beta})}{\Gamma(-\frac{q}{\beta})\Gamma(-\frac{p-q}{\beta})}.$$

Leading asymptotics in the anomalous regime

Sketch of proof

1. Recall that for $k = o(\sqrt{n})$ (**anomalous regime**)

$$\sum_{k=1}^{\Lambda} E_{(\beta,p),n}(k) \sim n^{-p/\beta} \sum_{k=1}^{\Lambda} \frac{1}{\Gamma^2(k)} \sum_{q=0}^{p} \binom{p}{q} (-1)^q \Gamma\left(k + \frac{q}{\beta}\right) \Gamma\left(k + \frac{p-q}{\beta}\right);$$

2. **Remark**: $\sum_k^{\Lambda}$ converges if $2\beta + 2p - p\beta < 0$. In this case we can take $\Lambda \to \infty$ (we "remove the infrared cutoff");

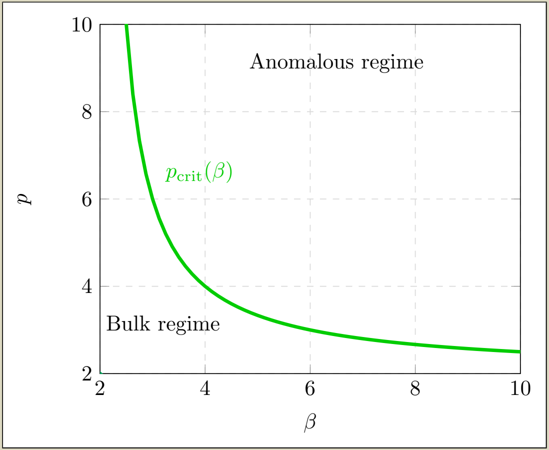
3. Recall the hypergeometric identity

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma^2(k)} = \Gamma(-1-a-b) \frac{\Gamma(a+1)}{\Gamma(-a)} \frac{\Gamma(b+1)}{\Gamma(-b)}.$$

Bulk meets anomalous: The critical hyperbola

Equating leading exponents gives **the critical hyperbola**:

$$2\beta + 2p = p\beta, \qquad p \geq 1, \beta \geq 1.$$



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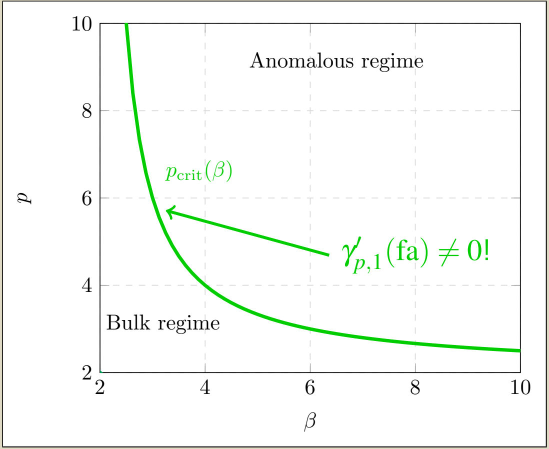


$\gamma'_{p,1} \neq 0$ along the critical hyperbola

Main result

Theorem 1 (**D'A**–Sportiello 2020)

For the family of distributions $\nu_{\mathrm{fa},\beta}(x) = \beta x^{\beta-1}$, for $\beta \geq 1$,

$$E_p^\beta(n) \underset{n\to\infty}{\sim} \frac{p!}{(p/2)!}\left(\frac{p-2}{2p}\right)^p n^{1-p/2} \log n, \qquad 2\beta + 2p = p\beta.$$

Thus <u>along the critical hyperbola</u>:

$$\gamma_{p,1} = 1 - \frac{p}{2}, \qquad \gamma'_{p,1} = 1$$

independently on β!

Matteo D'Achille (matteo.dachille@u-pec.fr)

$\gamma'_{p,1} \neq 0$ along the critical hyperbola

Sketch of proof

1. Here $E_{(n,p)} \sim E^{\text{bulk}}_{(n,p)} + E^{\text{anom}}_{(n,p)}$ at leading order in n. We need to keep a **finite cutoff** $1 \ll \Lambda(n) \ll n$ throughout the calculation.

2.

$$E^{\text{bulk}}_{(n,p)} = \int_{\Lambda/n}^{1} \mathrm{d}x \, \frac{1 + xR(x)}{x} \underset{n\to\infty}{\sim} \int_{\Lambda/n}^{1} \mathrm{d}x \, \frac{1}{x} + \int_{0}^{1} \mathrm{d}x \, R(x) \, ,$$

where $R(x)$ is a polynomial;

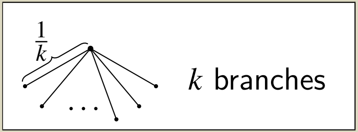
3. One proves (after some work involving several hypergeometric identities and the Gauss digamma theorem) that

$$E^{\text{anom}}_{(n,p)} = o(n^{1-p/2} \log n).$$

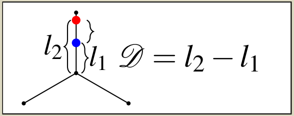
1. ERAPs on the k-star graph (W.i.p. with Caracciolo, Liu and Sportiello, 2021-)

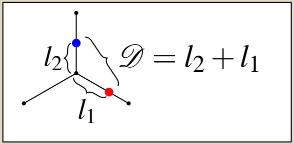
How we choose $((\Omega, \mathscr{D}), (\nu_{\mathscr{B}}, \nu_{\mathscr{R}}), p)$:

1. $\Omega = \mathbb{K}_{1,k}$ (i.e. “star” with k branches)



with distance $\mathscr{D}$:



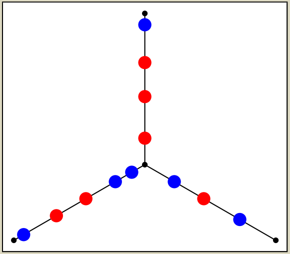


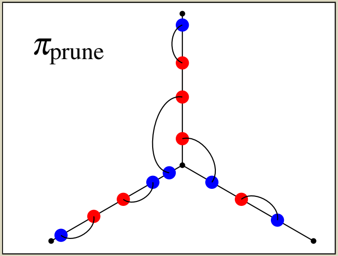
2. $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} = \nu$, $\nu \sim U[0, \frac{1}{k}] \otimes \mathscr{U}\{1, k\}$;
3. $p \geqslant 1$.

On the configuration π_{prune}

IDEA: Try a canonical construction, an approximate solution sharing local ordering property with $\pi_{\text{opt}} \to \underline{\pi_{\text{prune}}}$!

Example: $k = 3$ and $n = 6$





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On the configuration π_{prune}

Conjecture : $\mathbb{E}_\nu\left[\mathscr{H}_{\text{opt}}^{(p)}\right] \underset{n\to\infty}{\sim} \mathbb{E}_\nu\left[\mathscr{H}_{\text{prune}}^{(p)}\right], \qquad p \geq 1\,.$

Consequence: $\gamma_{p,1}^{(k)} = 1 - \frac{p}{2}$, $\gamma'^{(k)}_{p,1} = 0$.

And also explicit (and simple) formulas for limit constants!

Example: At $p = 2$

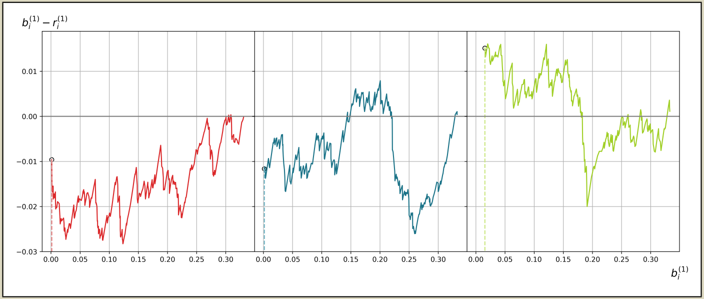
$$\lim_{n\to+\infty} \mathbb{E}\left[\mathscr{H}_{\text{opt}}^{(2)}\right] = \frac{3k-2}{3k^2}.$$

On the configuration π_{prune}

Example: At $p = 2$, $\forall k \geq 1$,

$$\lim_{n \to +\infty} \mathbb{E}\left[\mathscr{H}_{\text{opt}}^{(2)}\right] = \frac{3k-2}{3k^2}.$$

Sketch of proof: k independent Brownian Bridges (stopped at $\frac{1}{k}$).



2. Further implications of the Generalised Selberg Integral connection

Take $\Omega = [0,1]$, $\mathscr{D} = |\cdot|$ and $\nu_{\mathscr{B}} = \nu_{\mathscr{R}} := \nu = \mathbf{1}_{[0,1]}(x)$.
(Caracciolo *et al.* 2019) has shown, for integer ℓ,

$$\mathbb{E}[|b_k - r_k|^\ell] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}.$$

This implies the exact formula for $E_{p,1}\big|_{p\geq 1}(n)$ for $f = |\cdot|^p$.

Problem*: What other choices of $f = f(|\cdot|)$ imply a "nice expression" for $E(n)$ (i.e. not necessarily involving hypergeometric functions) after resummation?

*Question raised by N.Enriquez during a talk given that I delivered at CIRM Marseilles - Luminy in March 2021.

Thank you for your attention!