Consequences of Weyl's law in low-dimensional Euclidean Random Assignment Problems

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"Statistical Properties of the Euclidean Random Assignment Problem", PhD Thesis of Université Paris-Saclay, 2020. `[tel-03098672]`

The Euclidean Random Assignment Problem (ERAP)

Let $\mathscr{B} = (B_1, \ldots, B_n)$ be blue and let $\mathscr{R} = (R_1, \ldots, R_n)$ be red points : n-samples of i.i.d. r.v. of pdf $\rho : \Omega \to \mathbb{R}$ ("disorder"), $(\Omega, \mathscr{D})$ is a metric space (mostly an **Euclidean** space with **Euclidean** distance $\mathscr{D}$). For $p \in \mathbb{R}$ and a permutation π, consider the *Hamiltonian*

$$\mathscr{H}(\pi) = \sum_{i=1}^{n} \mathscr{D}^p(B_i, R_{\pi(i)})$$

and the r.v. (ground state energy)

$$\mathscr{H}^{(p)}_{\text{opt},(n,d)} = \min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi) \quad (\pi_{\text{opt}} = \arg\min_{\pi \in \mathscr{S}_n} \mathscr{H}(\pi)).$$

Problem: understand the rate of $E_{p,d}(n) := \mathbb{E}[\mathscr{H}^{(p)}_{\text{opt},(n,d)}]$ as $n \to \infty$.

Three reasons for being interested in ERAPs

- **Spin Glasses** - ERAP is a toy model of spin-glass in finite dimension (frustration is due to trian. inequality) which is numerically simple (in comparison to e.g. Edwards–Anderson spin glass) (Mézard–Parisi 1988)
- **Optimal Transport** - ERAP is a Monge-Kantorovitch problem associated to empirical measures $\rho_{\mathscr{B}}$, $\rho_{\mathscr{R}}$:

$$\mathscr{H}_{\text{opt}} = nW_p^p(\rho_{\mathscr{B}}, \rho_{\mathscr{R}})$$

  where W_p is the p-Wasserstein distance (Villani 2009, Brezis 2018)
- **Computational Complexity Theory** - ERAP is a small (but crucial) modification of random TSP, but finding π_{opt} is **easy** (the assignment problem is P-complete).

ERAP: the phase diagram

We wish to study

$$E_{p,d}(n) := \mathbb{E}[\mathscr{H}^{(p)}_{\mathrm{opt},(n,d)}] \stackrel{?}{=} K_{p,d}\, n^{\gamma_{p,d}} (\ln n)^{\gamma'_{p,d}} (1+o(1))$$

as $n \to \infty$, depending on (p,d) and the choice of disorder.

Phase diagram: $(\gamma_{p,d}, \gamma'_{p,d})$ are expected to be "**universal**", i.e. largely independent on the microscopic details (which may affect the constant $K_{p,d}$).

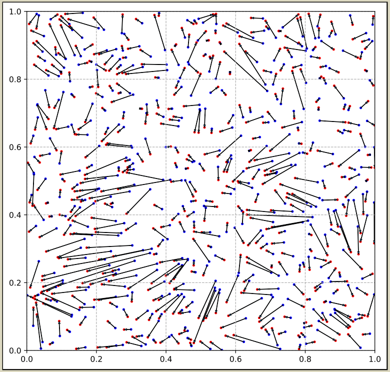
Remark: non-uniform disorder is more subtle!
Example: standard Gaussian disorder at $(p,d) = (2,1)$

$$E_{2,1}(n) \sim 2 \ln\ln n \qquad (\text{i.e. } \gamma_{2,1} = \gamma'_{2,1} = 0).$$

(Caracciolo–**D'A**–Sicuro 2019, Bobkov–Ledoux 2019 + Berthet–Fort 2020)

See also Benedetto–Caglioti 2020 for non-uniform case at $d = 2$.

ERAP at $d = 2$: an old problem

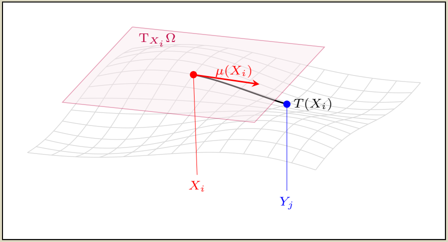


$(\gamma_{p,d}, \gamma'_{p,d}) = (\gamma_{\mathrm{LB}}, \frac{p}{2})$ if $p \geq 1$ (Ajtai–Komlós–Tusnády 1984)

Optimal assignment typically with a $O(\ln n)$–nearest-neighbor.

The Caracciolo–Lucibello–Parisi–Sicuro approach

A (classical) field theory for general d and $p \geq 1$ (PRE 2014).



At $p = 2$, for a d-dimensional manifold Ω, the Lagrangian is

$$\mathscr{L}[\vec{\mu}, \phi] := \overbrace{\int_\Omega \frac{1}{2}\vec{\mu}^2(x)\nu_{\mathscr{B}}(\mathrm{d}x)}^{\text{Energy}} + \overbrace{\int_\Omega [\phi(x+\vec{\mu}(x))\nu_{\mathscr{B}}(x) - \phi(x)\nu_{\mathscr{R}}(\mathrm{d}x)]}^{\text{Transport constraint}}$$

$\nu_{\mathscr{B}(\mathscr{R})}$ is the "charge" density of blues (reds) and ϕ is a Lagrange multiplier

The Caracciolo–Lucibello–Parisi–Sicuro approach

If $|\vec{\mu}|$ is "small" when $n \to \infty$, one can use Taylor expansion in the "small parameter" $\varepsilon = |\nabla \cdot \vec{\mu}(x)|$. The linearized Lagrangian is

$$\mathscr{L}_{\mathrm{lin}}[\vec{\mu}, \phi] := \int_\Omega \left[\frac{1}{2}\vec{\mu}^2(x) + \vec{\mu}(x) \cdot \nabla\phi(x)\right] \mathrm{d}x + \int_\Omega \delta\nu(x)\phi(x)\,\mathrm{d}x.$$

The Euler-Lagrange eqs. at leading order in ε give Poisson eq. for ϕ with source $\delta\nu := \nu_{\mathscr{B}} - \nu_{\mathscr{R}}$

$$\Delta_\Omega \phi(x) = \delta\nu(x), \qquad -\Delta_\Omega = \text{Laplace-Beltrami op. on } \Omega$$

to be solved with Neumann bc on Ω (if $\partial\Omega \neq \varnothing$). Then $\vec{\mu} = -\nabla\phi$ and $E_\Omega = \int_\Omega |\vec{\mu}|^2$. Following Caracciolo–Lucibello–Parisi–Sicuro 2014, the energy writes

$$E_\Omega(n) = -2\,\mathrm{Tr}\,\Delta_\Omega^{-1}$$

which is **bad defined** !! $\implies$ Regularizations

The regularized spectral expansion and Weyl's law

A way of rewriting Caracciolo–Lucibello–Parisi–Sicuro's regularization is

$$E_{\Omega}(n) = -2\,\mathrm{Tr}\,\Delta_{\Omega}^{-1} \simeq 2 \sum_{\lambda \in \Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n^{2/d}(\log n)^{b}}\right)}{\lambda}$$

for an unknown cutoff function F **independent on Ω** (but possibly dependent on the local randomness of the ERAP), satisfying only $F(0) = 1$ and $\lim_{z\to\infty} F(z) = 0$.

The regularized spectral expansion and Weyl's law

Weyl's law (Ivrii 1980, Neumann b.c. case)

Let Ω be a d-dimensional manifold and $\Lambda(\Omega)$ be the spectrum of $-\Delta_\Omega$ with Neumann b.c. if $\partial\Omega \neq \varnothing$ without $\lambda = 0$. Let $\mathscr{N}_\Omega(\lambda)$ be the eigenvalue counting function. Then

$$\mathscr{N}_\Omega(\lambda) = \frac{\omega_d}{2\pi^d}|\Omega|\lambda^{\frac{d}{2}} + \frac{\omega_{d-1}}{4(2\pi)^{d-1}}|\partial\Omega|\lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})$$

- $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ = volume of standard d-ball;
- $|\Omega|$ d-dimensional volume of Ω;
- $|\partial\Omega|$ surface area of the boundary of Ω.

Example: asymptotic energy differences at $d = 2$

The precise form of F is inessential as, for two manifolds Ω, Ω',

$$\lim_{n\to\infty} \left(E_\Omega(n) - E_{\Omega'}(n)\right) = 2 \lim_{n\to\infty} \left(\sum_{\lambda \in \Lambda(\Omega)} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda} - \sum_{\lambda \in \Lambda(\Omega')} \frac{F\left(\frac{\lambda}{n}\right)}{\lambda} \right)$$

$$= 2 \lim_{n\to\infty} \int_{0^+}^{\infty} F\left(\frac{\lambda}{n}\right) \frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

$$= 2 \lim_{n\to\infty} \int_{0^+}^{\infty} \mathrm{d}\lambda \left(\frac{F\left(\frac{\lambda}{n}\right)}{\lambda^2} - \frac{F'\left(\frac{\lambda}{n}\right)}{n\lambda} \right) \left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)$$

$$= 2 \int_{0^+}^{\infty} \frac{\mathrm{d}\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right)}{\lambda}$$

as $\left(\mathscr{N}_\Omega(\lambda) - \mathscr{N}_{\Omega'}(\lambda)\right) = O(\sqrt{\lambda} \ln \lambda)$ at $d = 2$ (and near the origin the integral is regularized by the spectral gap).

Explicit evaluation of energy differences

Benedetto-Caglioti-Caracciolo-**D'A**-Sicuro-Sportiello, JStatPhys 2021

Main result: even if the field theory is ill-posed, we can give a precise experimental (and predictive!) meaning to energy differences $E_\Omega(n) - E_{\Omega'}(n)$ through **regularization**. We did it in two ways :

- R_Ω or "**Robin mass**": integrals of the diagonal of Green's function for Poisson eq.;
- K_Ω or "**Kronecker mass**": expand spectral function $Z_\Omega(s)$ associated to $-\Delta_\Omega$ around the simple pole $s = 1$.

Remark 1: Robin and Kronecker masses satisfy (Morpurgo 2002)

$$\forall\Omega, \qquad R_\Omega - K_\Omega = \frac{\ln 2}{2\pi} - \frac{\gamma_E}{2\pi} = 0.0184511\dots$$

Remark 2: other regularizations are possible.

Example: square $\mathbb{R}$, 2-torus $\mathbb{T}$, Boy surface $\mathbb{B}$

Obtained from rectangle of aspect ratio ρ by appropriately gluing sides

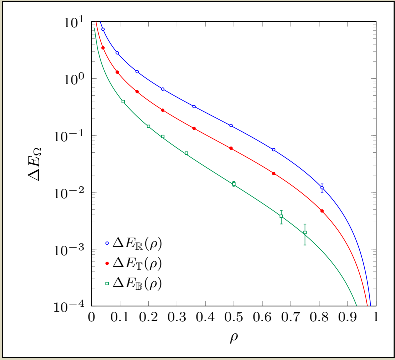
Energy shift w.r.t. manifold at aspect ratio $\rho = 1$:

$$\Delta E_\Omega(\rho) = 2(R_\Omega(\rho) - R_{\Omega'}(1)) = 2(K_\Omega(\rho) - K_{\Omega'}(1))$$

$$K_\mathbb{R}(\rho) = \frac{\gamma_\mathrm{E}}{2\pi} - \frac{\ln(4\pi^2\rho|\eta(i\rho)|^4)}{4\pi} + \frac{1}{2\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$

$$K_\mathbb{T}(i\rho) = \frac{\gamma_\mathrm{E} - \ln(4\pi\sqrt{\rho})}{2\pi} - \frac{1}{\pi}\ln|\eta(i\rho)|$$

$$K_\mathbb{B}(\rho) = \frac{\gamma_\mathrm{E}}{2\pi} - \frac{\ln(4\pi^2\rho)}{4\pi} - \frac{\ln\eta(i\rho)}{\pi} - \frac{1}{4\pi^2}\left(\rho + \frac{1}{\rho}\right)\zeta(2)$$



(See JStatPhys 2021 for more manifolds)

Coefficient of sub-leading $\log n$ correction at $(p,d)=(2,3)$

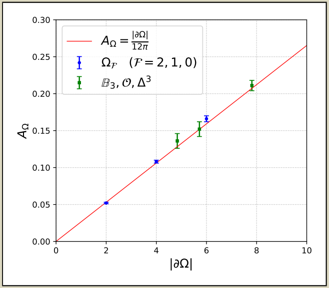
An analogous argument based on Weyl's law shows, for $(p,d)=(2,3)$ that

$$E_\Omega(n) = Kn^{\frac{1}{3}} + A_\Omega \log n + o(\log n),$$

for a universal (=independent on Ω) constant, and the **area term**

$$A_\Omega = \frac{|\partial\Omega|}{12\pi},$$

depending only on the surface area of the boundary of Ω (paper in preparation).



Thank you for your attention!