UNIVERSITÀ DEGLI STUDI DI PADOVA

The frequency domain

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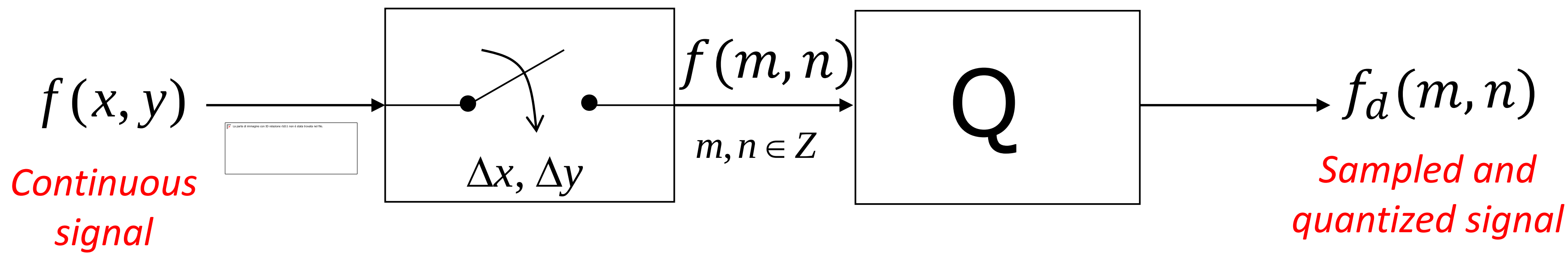
Agenda

- Recall: the Fourier transform
- Signal sampling and reconstruction

Sampling and quantization



- Sampling:

$$f(m,n) \triangleq f(m\Delta x, n\Delta y)$$

  - Δx and Δy sampling period along *x* and *y* axis

- Quantization:

$$f_d(m,n) = Q[f(m,n)]$$

Fourier transform

- Signals and sampling can be effectively described using a mathematical tool: the Fourier transform
- Fourier transform pair
  - Fourier transform
  - Inverse Fourier transform
- Define a transform space
- Under some (ideal) conditions: no loss
- A fast algorithm is available – FFT (Fast Fourier Transform)

Fourier transform

- The Fourier transform enables frequency analysis & filtering
- The transform moves to another domain, causing a change in the independent variable
  - Commonly: from time to time frequency

$$t \rightarrow f$$

  - Computer vision: from space to space frequency

$$x \rightarrow f_x$$
$$y \rightarrow f_y$$

Fourier transform in 1D

- Consider a 1D example (typical signal in time)
- Fourier transform

$$F(\mu) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi\mu t} dt$$

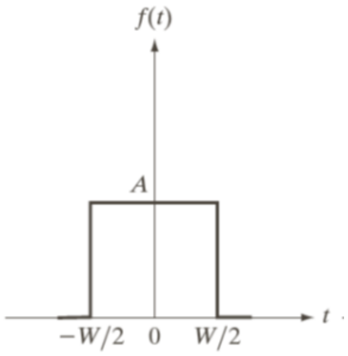
- Inverse transform

$$f(t) = \int_{-\infty}^{+\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Fourier transform in 1D

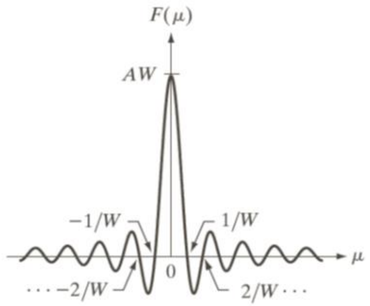
- Consider the rect function

$$f(t) = \begin{cases} A & if \; -\dfrac{W}{2} < t < \dfrac{W}{2} \\ 0 & elsewhere \end{cases}$$



- Its transform is evaluated as:

$$F(\mu) = AW \frac{\sin(\pi\mu W)}{\pi\mu W}$$



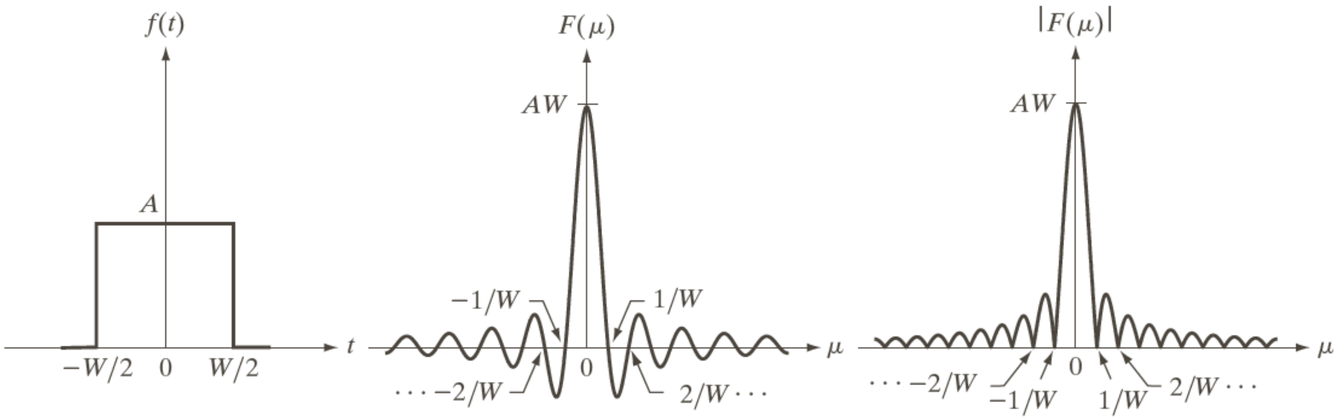
- This extends to infinity

Fourier transform in 1D

- The Fourier transform is generally complex
- It is common to deal with the magnitude of the transform AKA Fourier spectrum or frequency spectrum
  – A real quantity
- In our case:

$$|F(\mu)| = AT\left|\frac{\sin(\pi\mu W)}{\pi\mu W}\right|$$

- Example of a transformed signal
  - rect → sinc



Function | Transform | Spectrum

Signal sampling

Sampled signals

- Sampling is represented by a set of regular pulses (train of impulses)
  - Pulses are separated by a **sampling interval** ΔT
- The sampled signal is the multiplication between the original signal and the train of impulses

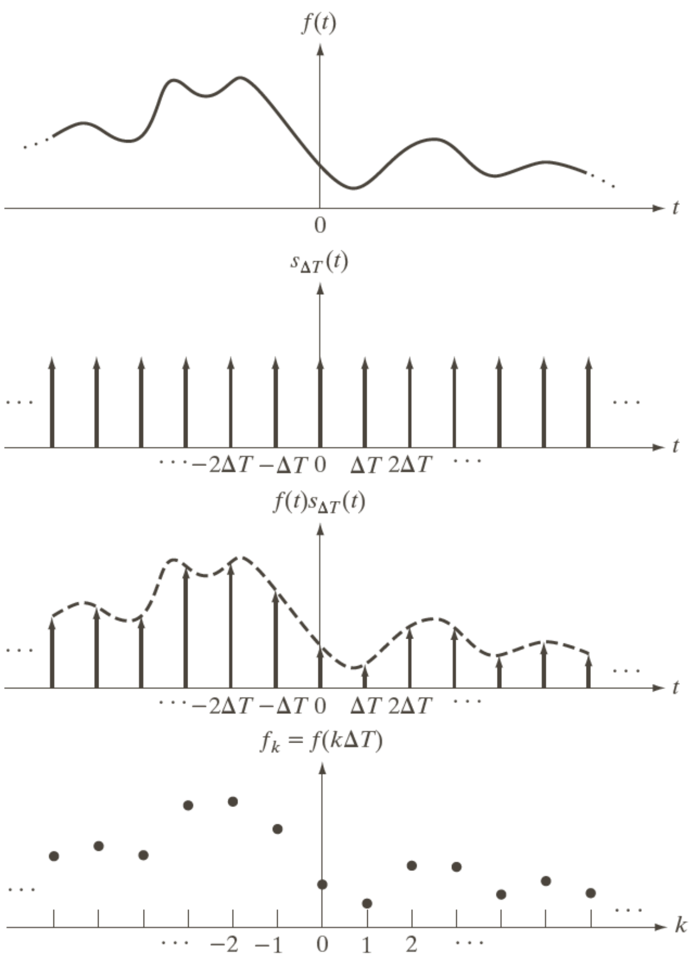


FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

Dirac delta function

- The impulses are expressed by means of the Dirac delta function
- Defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Dirac delta function

- Main features of the Dirac delta function:
  - Unit area

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

  - Sifting property

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$$

  - In a generic position

$$\int_{-\infty}^{+\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

- The delta function "selects" a specific point of the function
  - Expresses sampling

Moving to
the discrete domain

Dirac delta function – discrete

- When dealing with discrete variables the definition is simpler

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

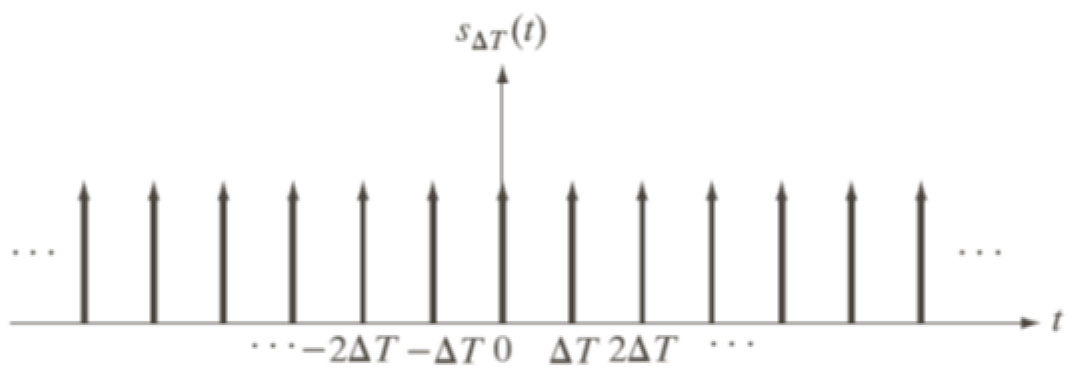
- This automatically satisfies

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

Dirac delta function – discrete

- An impulse train is expressed by a sum of delta functions

$$s_{\Delta T}(t) = \sum_{x=-\infty}^{\infty} \delta(t - n\Delta T)$$



Sampled signals

- Recall: the sampled signal is the multiplication between the original signal and the train of impulses
- A sampled signal is then expressed by

$$\tilde{f}(t) = \sum_{n=-\infty}^{+\infty} f(t)\delta(t - n\,\Delta T)$$

- A sampled signal is represented by a set of values

$$f_k = f(k\Delta T)$$

Effect of sampling

- Sampling: multiplication by a train of impulses
- Multiplication becomes convolution in the transformed domain

$$\tilde{F}(\mu) = F(\mu) * S(\mu)$$

  – Where S(μ) is the transform of the delta function

- The sampled signal then becomes:

$$\tilde{F}(\mu) = \int_{-\infty}^{+\infty} F(\tau) S(\mu - \tau) d\tau$$

Effect of sampling

- After some calculations:

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

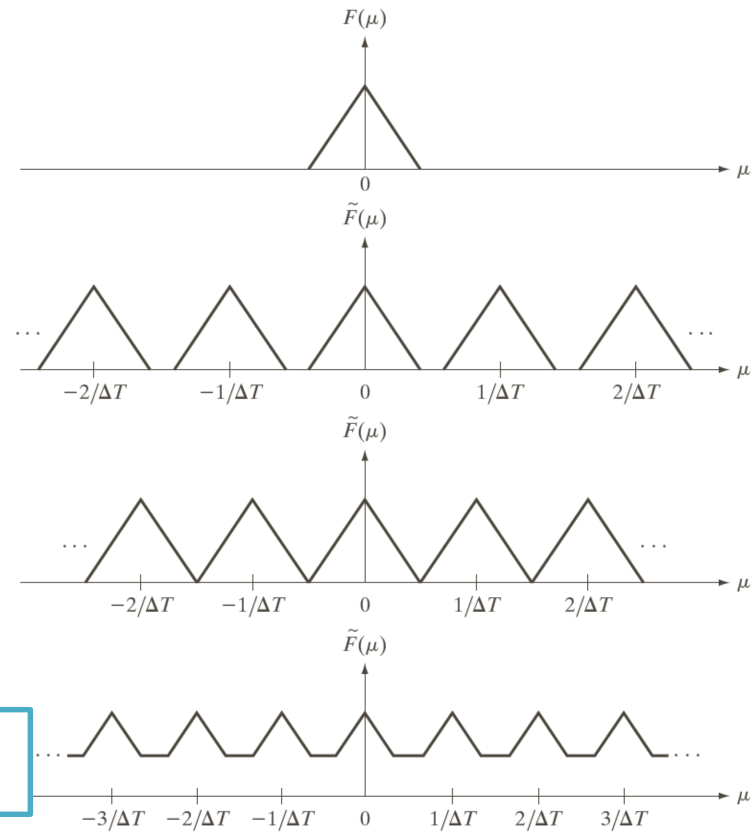
- This means that in the transformed domain the spectrum is **replicated**

Sampling and aliasing

- The spectrum is replicated
- Replicas can be at different distances
  - Depends on the **sampling period**
- Replicas can overlap
  - **Aliasing**

a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.



Aliasing

Sampling theorem

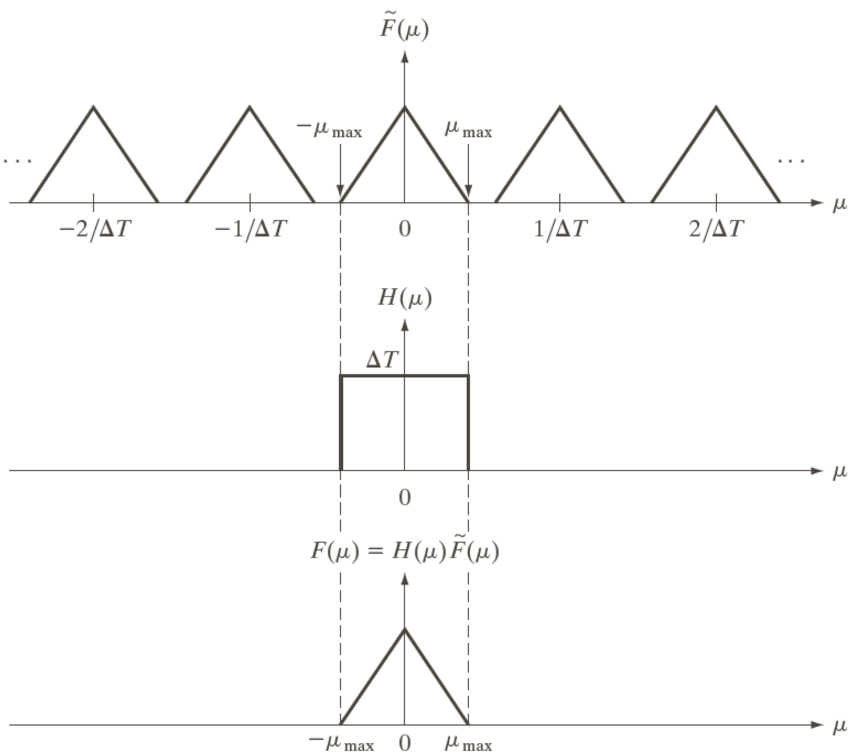
- The sampling theorem states the conditions under which we can reconstruct a signal
- Essentially: we shall be able to isolate a copy of $F(\mu)$
- This is possible if
  - The signal is band-limited (its spectrum is limited by $\mu_{\max}$) and:

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

- $2\mu_{\max}$ is known as the Nyquist rate

Signal reconstruction

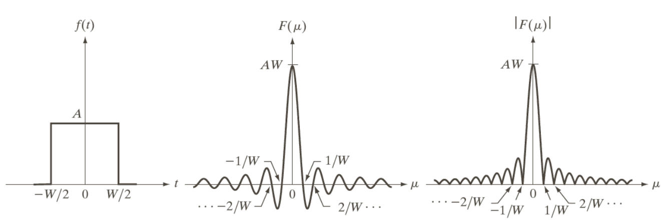
- To reconstruct a signal we need to isolate one repetition in frequency
  - Done by means of a rect function



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

Recall



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

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