Soft Constraints

Outline

- Soft CSPs
  - C-semiring framework
  - Projection, Combination
  - Search

Soft constraints

- **Soft constraint:** **a classical constraint,** where

  each assignment of values to its variables

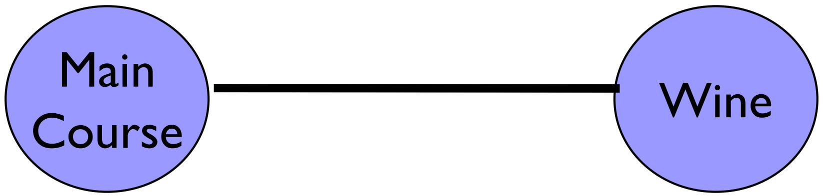
  has an associated <u>preference value</u> from a

  (totally or partially ordered) **set**

  - **This set** has **two operations**, which makes it **similar to a semiring**
  - **It** is called a **c-semiring**

- **Example**

D_{MC}= {Fish, Meat} D_{W}= {white, red}



(Fish, white) → 1 (Meat, white) → 0.3

(Fish, red) → 0.8 (Meat, red) → 0.7

Soft constraint satisfaction problems (soft CSPs)

- Soft CSP

  is a set of soft constraints over a set of variables

  based on a specific c-semiring $<A,+, \times, \mathbf{0}, \mathbf{1}>$:

- Solution of a Soft CSP: complete assignment
  - one value for each variable

- Preference value of a solution (global evaluation)
  - By combining (via x) the preference values of the partial assignments of the solution given by the constraints

A general framework for soft CSPs

It is based on a **c-semiring structure** that is a tuple **< A, +, x, 0, 1 >**

- A: **set** that specifies the **preference values** to be associated with each **tuple**, i.e., with each **assignment of values** of the variables

- **+**, ***x***: two semiring operations
  - + models **constraint projection**
  - *x* models **constraint combination**

- **0**: the **worst** preference value (**0** ∈A)
- **1**: the **best** preference values (**1** ∈A)

The c-semiring framework

- C-semiring **S = < A, +, x, 0, 1 >:**
  - **A set of preferences**
  - **+ additive operator**
    - ***induces the ordering $\leq_S$ over A*** defined as follows

      $a \leq_S b$ **iff** $a + b = b$

      (+ is idempotent, commutative, associative, unit element 0);
  - **x multiplicative operator**
    - ***combines preferences***

      (x is commutative, associative, unit element 1, absorbing element 0)
  - **0**: the **worst** preference value (**0** ∈A)
  - **1**: the **best** preference values (**1** ∈A)

Soft constraints

Variables $\{X_1,\ldots,X_n\}=X$
Domains $\{D(X_1),\ldots,D(X_n)\}=D$
C-semiring $<\mathbf{A},+,x,\mathbf{0},\mathbf{1}>$

- **Soft constraint:** a pair $c = < f, con >$ where
  - **Scope:** $con = \{X^c{}_1,\ldots, X^c{}_k\}$ subset of X
  - **Preference function:**

$$f\colon\ D(X^c{}_1)\ x\ldots x\ D(X^c{}_k) \rightarrow \mathbf{A}$$

$$(v_1,\ldots, v_k) \rightarrow \mathbf{p}$$

p is a preference value

- **Hard constraint:** a <u>soft constraint</u> where

  for each tuple $(v_1,\ldots, v_k)$

  $f\,(v_1,\ldots, v_k) = \mathbf{1}$ the tuple is allowed

  $f\,(v_1,\ldots, v_k) = \mathbf{0}$ the tuple is forbidden

Review: Soft constraint satisfaction problems

- Soft CSP

  is a set of soft constraints over a set of variables based on a specific c-semiring $<A, +, \times, \mathbf{0}, \mathbf{1}>$

- Solution of a Soft CSP: complete assignment
  - one value for each variable

- Preference value of a solution (global evaluation)
  - By combining (via x) the preference values of the partial assignments of the solution given by the constraints

Fuzzy constraint satisfaction problems (FCSPs)

- *Fuzzy c-semiring*: $< A = [0,1], + = max, \times = min, \mathbf{0} = 0, \mathbf{1} = 1 >$:
  - Preference values between 0 and 1
  - Higher values denote better preferences
    - 0 is the worst preference
    - 1 is the best preference
  - Combination is taking the smallest value

➔ optimization criterion = maximize the minimum preference

Pessimistic approach, useful in critical application (eg., space and medical settings)

[Fuzzy CSPs: Schiex UAI' 92, Ruttkay FUZZ-IEEE '94]

Example of FCSP

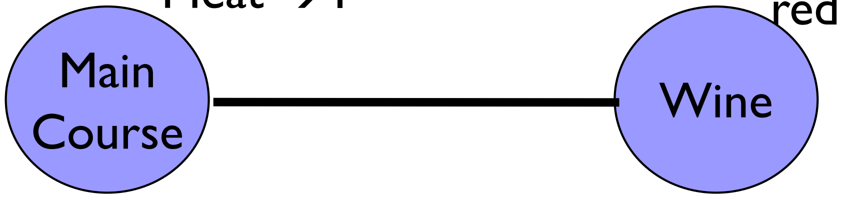
Fuzzy c-semiring

$S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$

$S_{FCSP} = \langle [0,1], max, min, 0, 1 \rangle$

Fish → 1
Meat → 1

white → 1
red → 1



(Fish, white) → 1

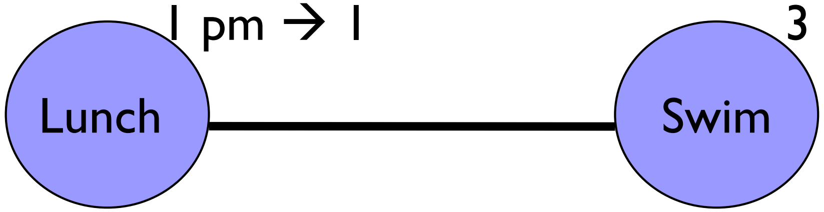
(Fish, red) → 0.8

(Meat, white) → 0.3

(Meat, red) → 0.7

12 pm → 1
1 pm → 1

2 pm → 1
3 pm → 1



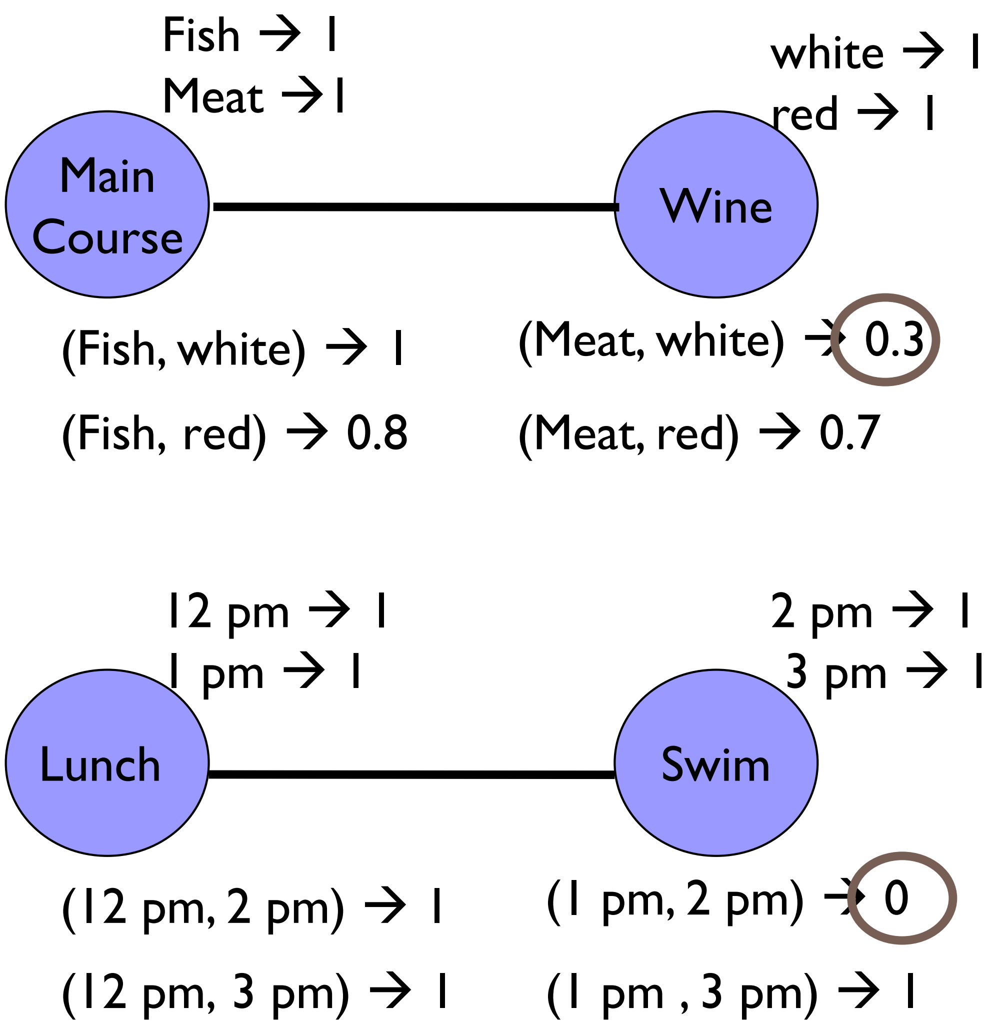
(12 pm, 2 pm) → 1

(12 pm, 3 pm) → 1

(1 pm, 2 pm) → 0

(1 pm , 3 pm) → 1

Example of FCSP



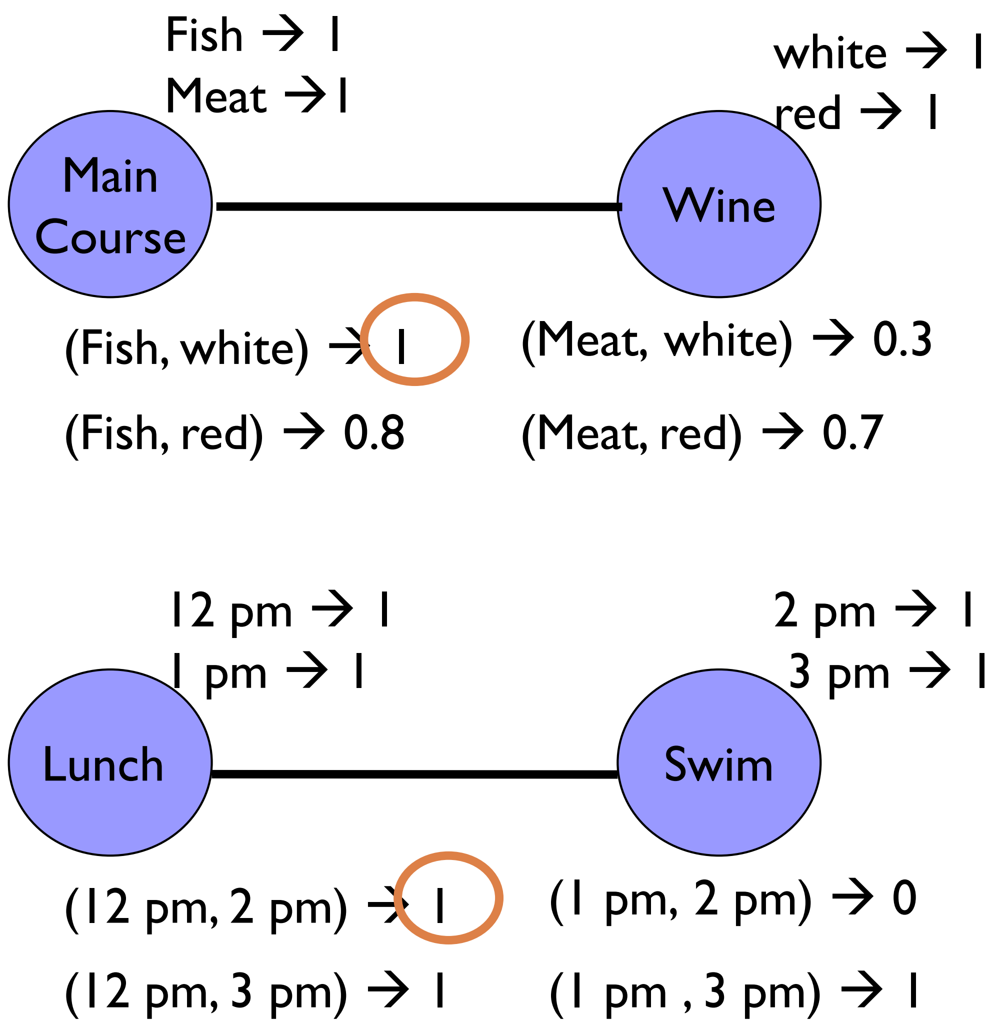
Fuzzy c-semiring

$S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$

$S_{FCSP} = \langle [0,1], max, min, 0, 1 \rangle$

Solution *S*	
Lunch=	1 pm
Main course =	meat
Wine=	white
Swim =	2 pm
pref(S)=min(0.3,0)=0	

Example of FCSP



Fuzzy c-semiring

$S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$

$S_{FCSP} = \langle [0,1], max, min, 0, 1 \rangle$

Solution *S*	
Lunch=	1 pm
Main course =	meat
Wine=	white
Swim =	2 pm

pref(S)=min(0.3,0)=0

Solution *S'*	
Lunch=	12 pm
Main course =	fish
Wine=	white
Swim =	2 pm

pref(S')=min(1,1)=1

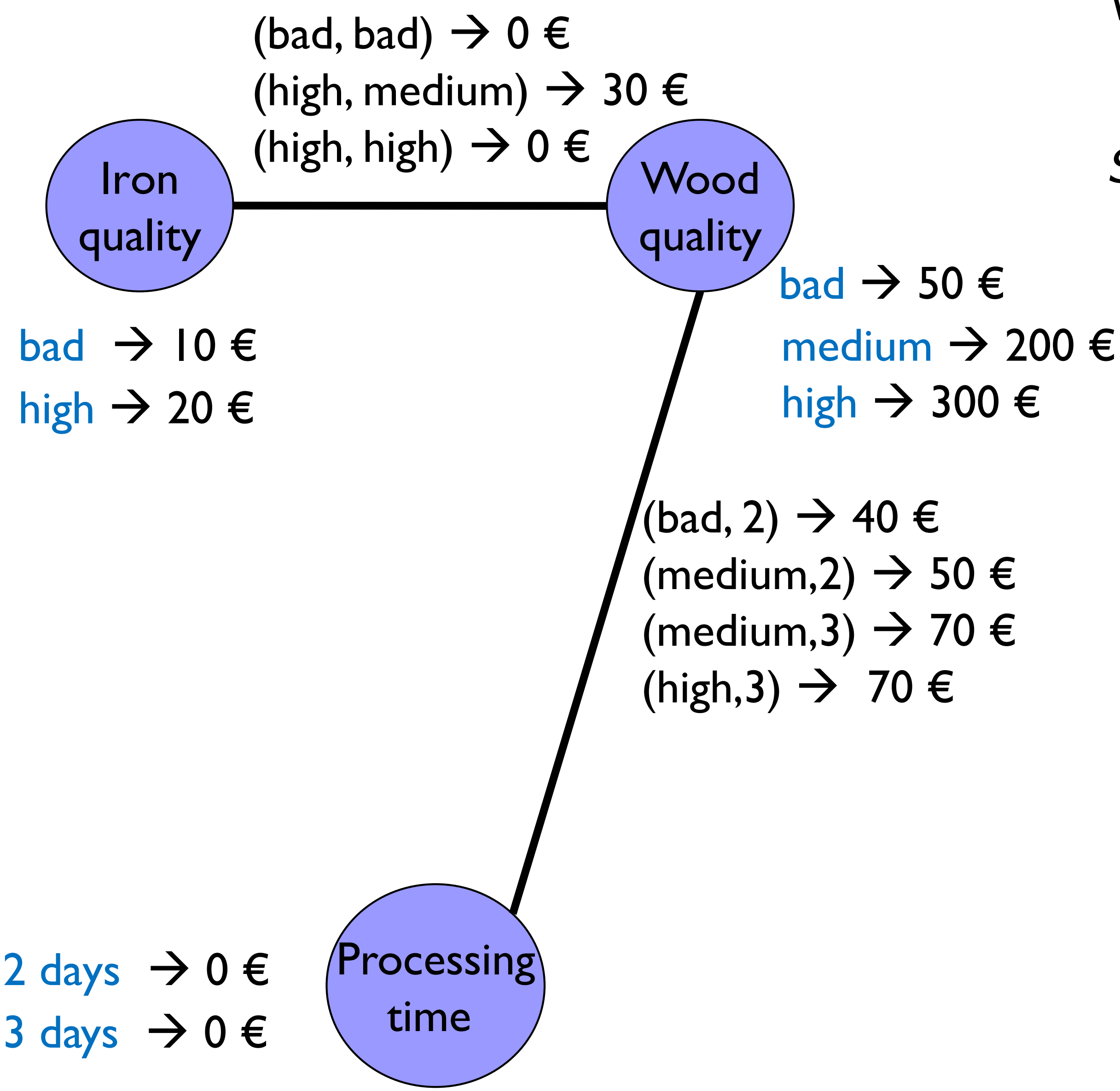
Weighted constraint satisfaction problems (WCSPs)

- *Weighted c-semiring* : $\langle N, + = min, \times = +, \mathbf{0} = +\infty, \mathbf{1} = 0 \rangle$:
  - Preference values are costs between 0 and $+\infty$
  - Lower costs are better
    - $+\infty$ is the worst cost
    - 0 is the best cost
  - Combination is taking the sum of the costs

→ optimization criterion = minimize the sum of costs

Useful in all settings where costs apply

Example of WCSP

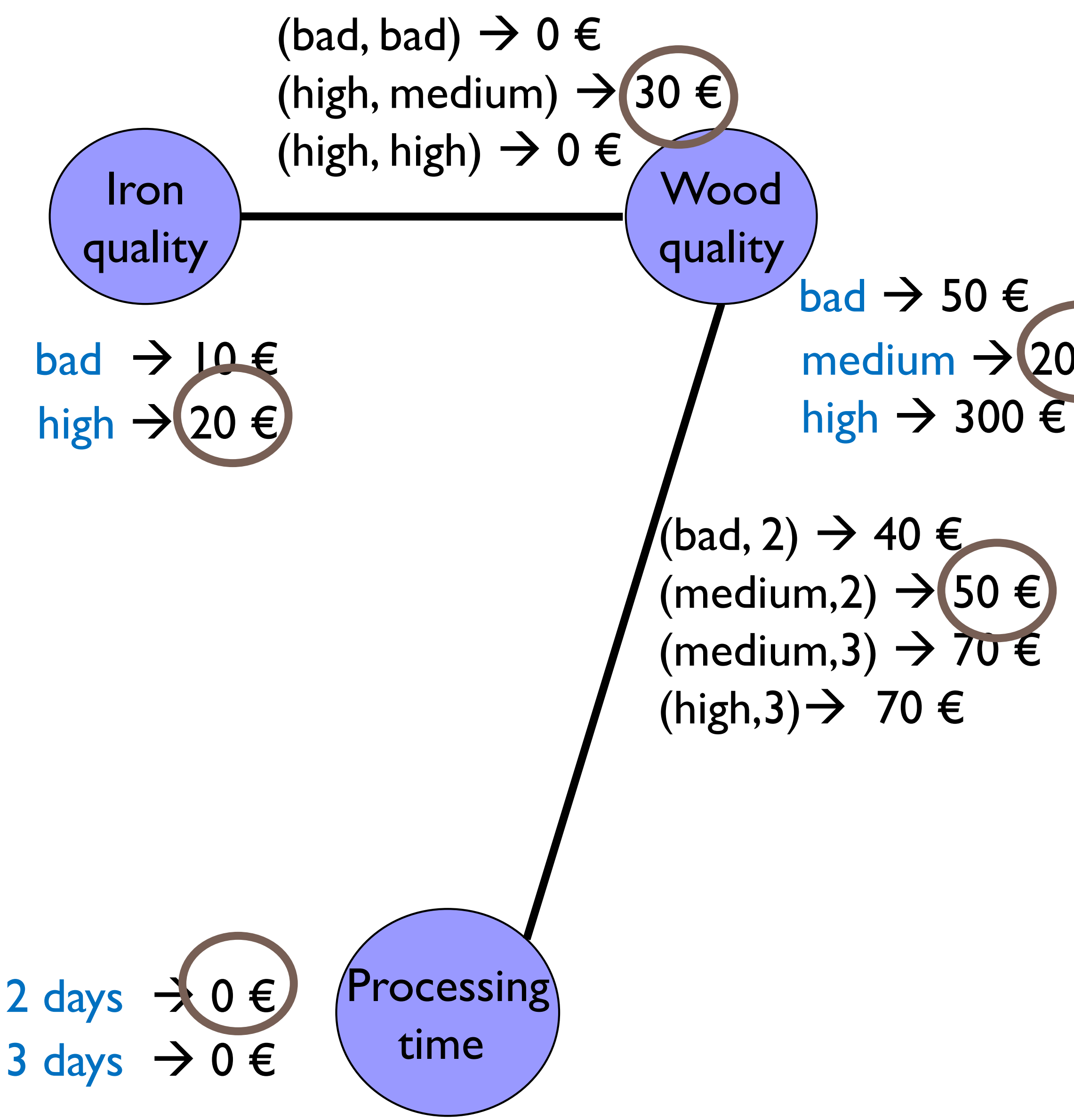


Weighted c-semiring

$$S = < A, +, \times, \mathbf{0}, \mathbf{1} >$$

$$S_{WCSP} = < [0, +\infty], min, +, +\infty, 0 >$$

Example of WCSP



Weighted c-semiring

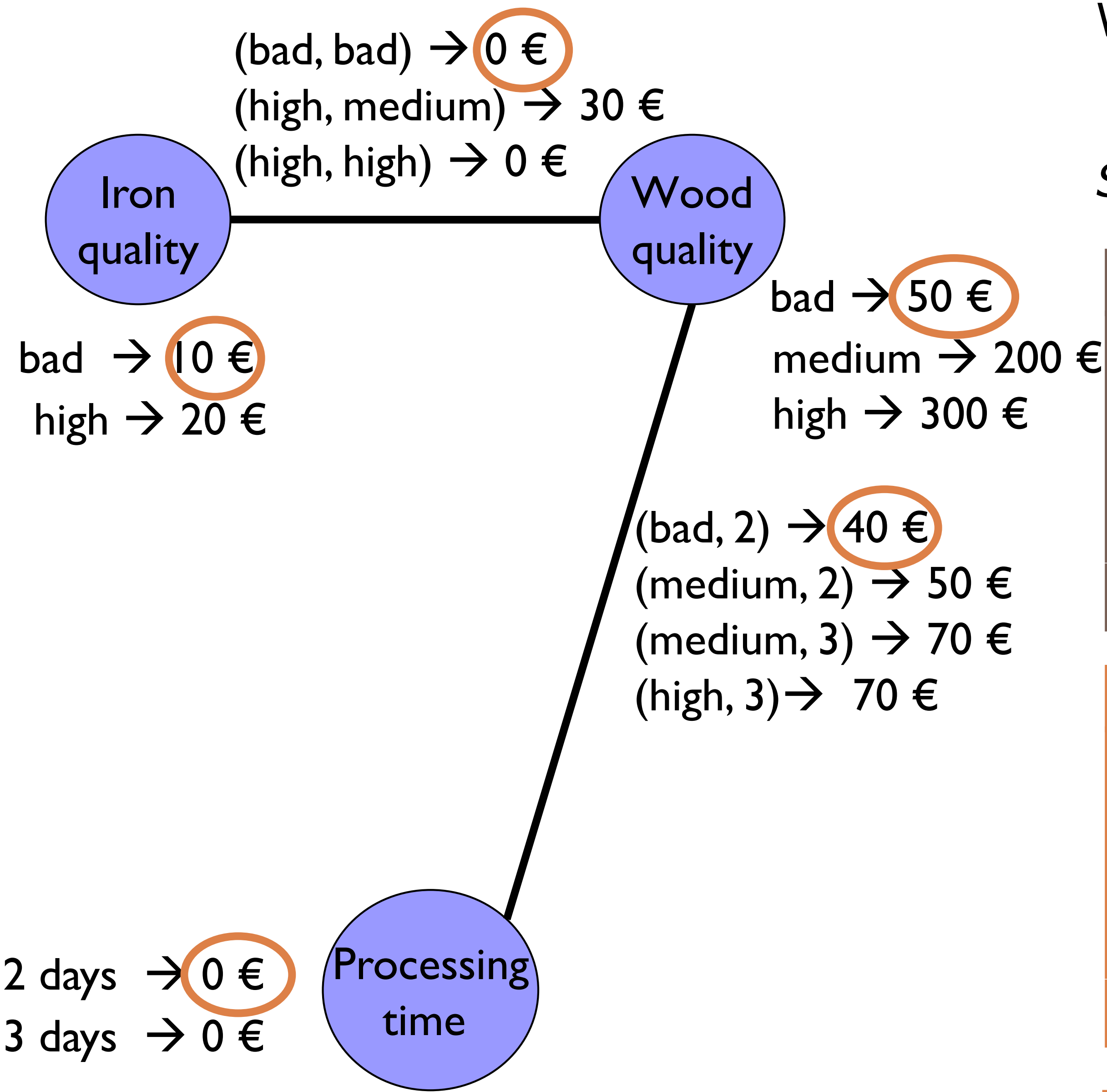
$$S = \langle A, +, \times, \mathbf{0}, \mathbf{1} \rangle$$

$$S_{WCSP} = \langle [0, +\infty], \min, +, +\infty, 0 \rangle$$

Solution *S*	
Iron=	high
Wood =	medium
Time=	2

pref(S)=20+30+200+50=**300**

Example of WCSP



Weighted c-semiring

$S \;=< A \;, \;\; + \;, \times \;, \mathbf{0}, \;\; \mathbf{1} >$

$S_{WCSP} = <[0,+\infty], min, +, +\infty, 0>$

Solution *S*	
Iron=	high
Wood =	medium
Time=	2

pref(S)=20+30+200+50=**300**

Solution *S'*	
Iron=	bad
Wood =	bad
Time=	2

pref(S')=10+50+40= **100**

S' is better than S

Solution ordering

- A soft CSP induces an ordering over the solutions, from the ordering $\leq_S$ of the c-semiring
  - If $\leq_S$ is a total order → total order over solutions (possibly with ties)
  - If $\leq_S$ is a partial order → total or partial order over solutions (possibly with ties)
- **Any ordering** can be obtained

Instances of semiring-based soft constraints

Each instance is characterized by a c-semiring <A, +, x, 0, 1>

- **Classical constraints**: < {0,1}, logical or, logical and, 0, 1>
  - Satisfy all constraints
- **Fuzzy constraints**: < [0,1], max, min, 0, 1>
  - Maximize the minimum preference
- **Weighted constraints:** < $R \cup +\infty$, min, +, $+\infty$, 0>
  - Minimize the sum of the costs
- **Probabilistic constraints**: < [0,1], max, x, 0, 1>
  - Maximize the joint probability
- **Multi-criteria problems**: Cartesian product of c-semirings
- …

Multi-criteria problems

- Main idea: one c-semiring for each criteria

- Given **n** c-semirings $S_i = <A_i, +_i, x_i, 0_i, 1_i>$, for $i=1,\ldots n$

  we can build the c-semiring S

  $$< <A_1,\ldots, A_n>, +, x, <0_1,\ldots,0_n>, <1_1,\ldots,1_n> >$$

  + and **x** obtained by ***pointwise application*** of $+_i$ and $\mathbf{x}_i$ on each c-semiring

- <u>Each assignment</u> is associated with a

  tuple of preference values $(p_1, \ldots, p_n)$

- The <u>ordering over the solutions</u> may be a partial order

Example of a multi-criteria problem (1)

The problem: choosing a route between two cities

- **Each piece** of highway has
  - a **preference** and
  - a **cost**

- **Two goals**: we want
  - **minimize the sum of the costs** and
  - **maximize the preference**

Each piece of highway has
a **preference** and a **cost**

Example of a multi-criteria problem (2)

- **C-semiring**: by **putting together**
  - one fuzzy c-semiring <[0,1], max, min, 0>
  - one weighted c-semiring <R+, min, +, +∞, 0>

- **Best solutions:**

  routes such that there is **no other route** with a **better semiring value**
  - <0.8, $10> is better than <0.7, $15>

- **Resulting order over solutions is partial:**
  - <0.6, $10> and <0.4, $5> are not comparable

Fundamental operations with soft constraints

c-semiring <A, +, x, 0, 1>

- **Projection:**

  eliminate one or more variables from a constraint **obtaining a new soft constraint preserving** the information on the remaining variables

- **Combination:**

  combine two or more soft constraints **obtaining a new soft constraint "synthesizing"** the information of the original ones

Projection

Variables $\{X_1,\ldots,X_n\}=X$
Domains $\{D(X_1),\ldots,D(X_n)\}=D$
C-semiring $<A,+,x,\mathbf{0},\mathbf{1}>$

- **Projection:**

  eliminate one or more variables from a constraint **obtaining a new soft constraint preserving** the information on the remaining variables

Formally:

Given a soft constraint $c=<f, con>$,

a set I of variables (I subset of X),

the projection of c over I is the constraint $c|_I=<f',con'>$ where

- $con' = I \cap con$
- $f'(t') = + \left(f(t) \right)$ over tuples of values t s.t. $t|_{I \cap con} = t'$

Projection: fuzzy example

S_{FCSP}=<[0,1],**max**,min,0,1>

If c=<f,con>, then $c|_l$ = <f', l ∩ con>
    f'(t') = + (f(t)) over tuples of values t s.t. $t|_{l \cap con}$ = t'

c =<f, {mc,w}>

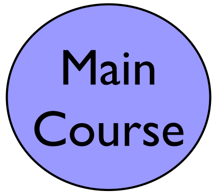
{Fish, Meat}   {white,red}

(Fish, white) → 1   (Meat, white) → 0.3

(Fish, red) → 0.8   (Meat, red) → 0.7

Projection of c over l = {mc}

$c|_l$



Fish → max(f(Fish,white), f(Fish,red))
= max(1,0.8) = 1

Meat → max(f(Meat, white), f(Meat, red))
= max(0.3,0.7)=0.7

Projection: weighted example

$S_{WCSP} = \langle [0,+\infty], \mathbf{min}, +, +\infty, 0 \rangle$

If $c=\langle f, con\rangle$, then $c\|_l = \langle f', l \cap con\rangle$ $f'(t') = +\,(\,f(t)\,)$ over tuples of values t s.t. $t\|_{l \cap con} = t'$

$c=\langle f, \{wq, pt\} \rangle$

{b,m,h} {2,3}

(b, 2) → 40 €
(b, 3) → +∞
(m,2) → 50 €
(m,3) → 70 €
(h,2) → +∞
(h,3) → 70 €

Projection of c over l={wq}

$c|_l$

b → min(f(b,2), f(b,3))=min(40,+∞)=40

m → min(f(m,2), f(m,3))=min(50,70)=50

h → min(f(h,2), f(h,3))=min(+∞,70)=70

Combination

c-semiring <A, +, x, 0, 1>

- **Combination:**

  combine two or more <u>soft constraints</u> obtaining <u>a new soft constraint</u> "synthesizing" the information of the original ones

Formally:

Given two soft constraints

$c1=\langle f1, con1\rangle$ and

$c2=\langle f2, con2\rangle$

then their combination is the constraint $c1 \times c2 = \langle f', con'\rangle$, where

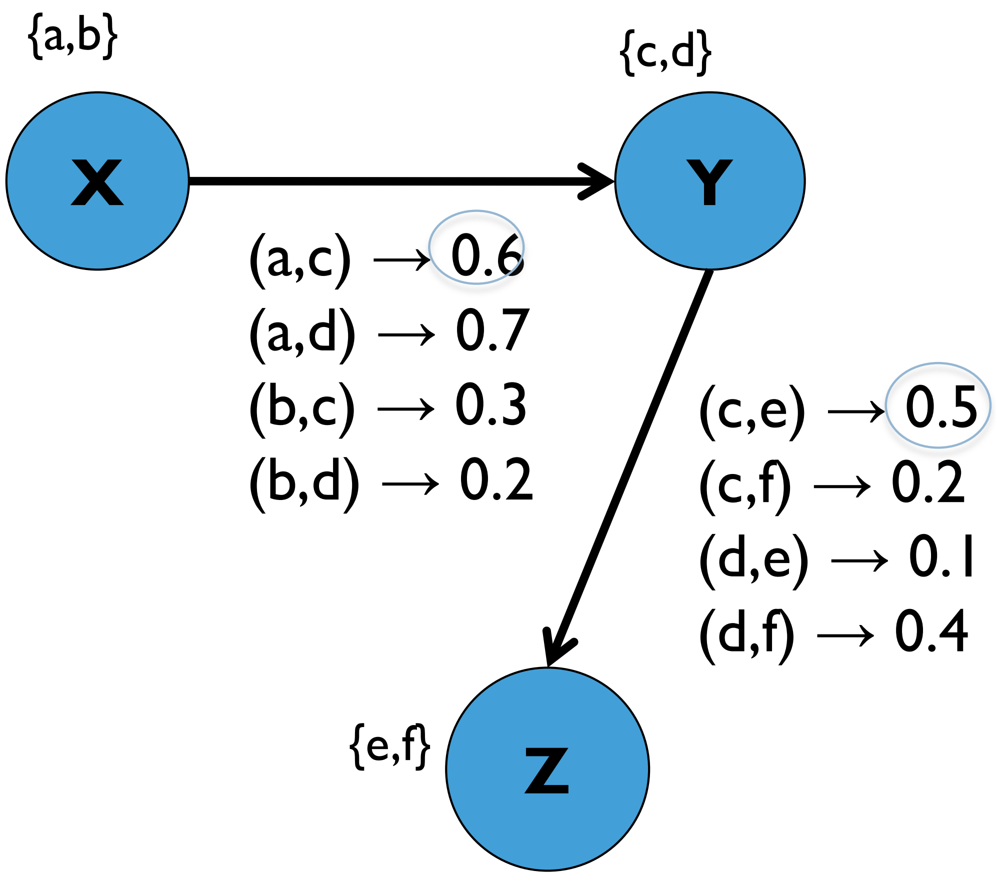
- $con' = con1 \cup con2$
- $f'(t) = f1(t|_{con1}) \times f2(t|_{con2})$

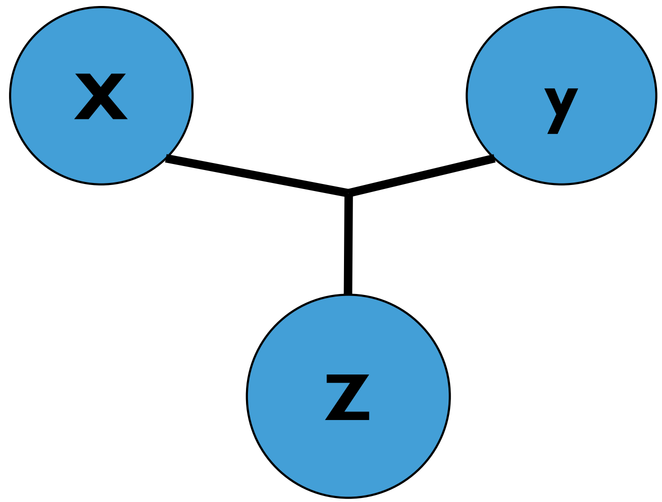
$S_{FCSP}=\langle [0,1], max, min, 0, 1 \rangle$

Combination: fuzzy example

If c1=<f1, con1> and c2=<f2, con2> then: c1 x c2 = <f, con1 ∪ con2>

- $f(t) = f1(t|_{con1}) \times f2(t|_{con2})$





(a,c,e) → min(0.6,0.5)= 0.5
(a,c,f) → min(0.6,0.2)= 0.2
(a,d,e) → min(0.7,0.1)= 0.1
(a,d,f) → min(0.7,0.4)= 0.4
(b,c,e) → min(0.3,0.5)= 0.3
(b,c,f) → min(0.3,0.2)= 0.2
(b,d,e) → min(0.2,0.1) = 0.1
(b,d,f) → min(0.2,0.4)= 0.2

$S_{WCSP}=\langle[0,+\infty],min,+,+\infty,0\rangle$

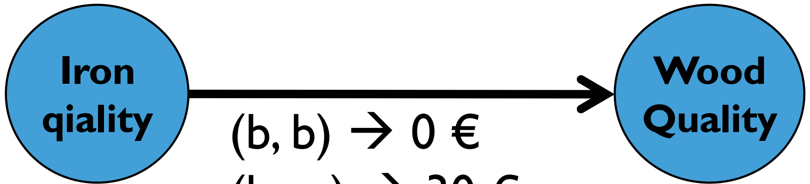
Combination: **weighted** example

If c1=<f1, con1> and c2=<f2, con2> then: c1 x c2 = <f, con1 ∪ con2>

- $f(t) = f1(t|_{con1}) \times f2(t|_{con2})$

{b ,h}

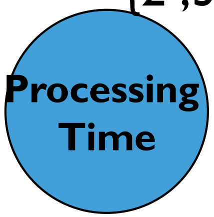
{b,m,h}



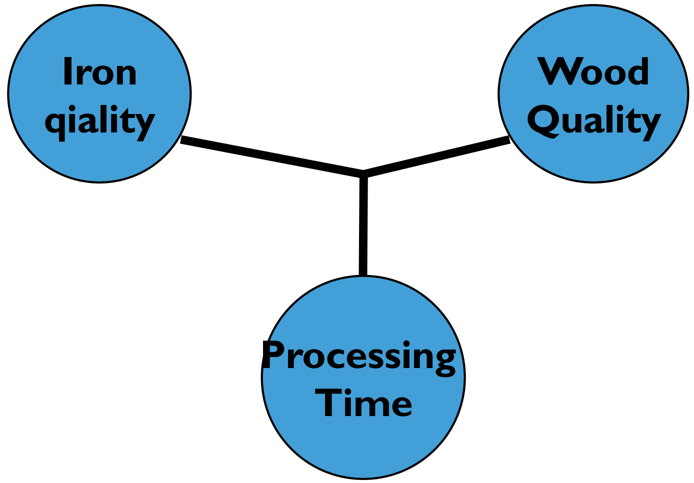
(b, b) → 0 €
(h, m) → 30 €
(h,h) → 0 €

Missing tuples in the constraint have cost $+\infty$

{2 ,3}



2 days → 20 €
3 days → 30€



(b,b,2) → 0+20 = 20
(h,m,3) →30+30= 60
....

Please, complete

Typical questions

- **Find an optimal solution**
  - Difficult

  (Ex.: branch and bound + constraint propagation)

- **Is t an optimal solution?**
  - Difficult

  (we have to find the optimal preference level)

- **Is t better than t' ?**
  - Easy: Linear in the number of constraints

  (compute the two pref. levels and compare them)

How to find optimal solutions

- Classical constraints:
  - Search
    - systematic (backtracking)
    - local
  - Constraint propagation
  - …
- Is it possible to extend/adapt these techniques to soft constraints?

Systematic search: Branch and Bound

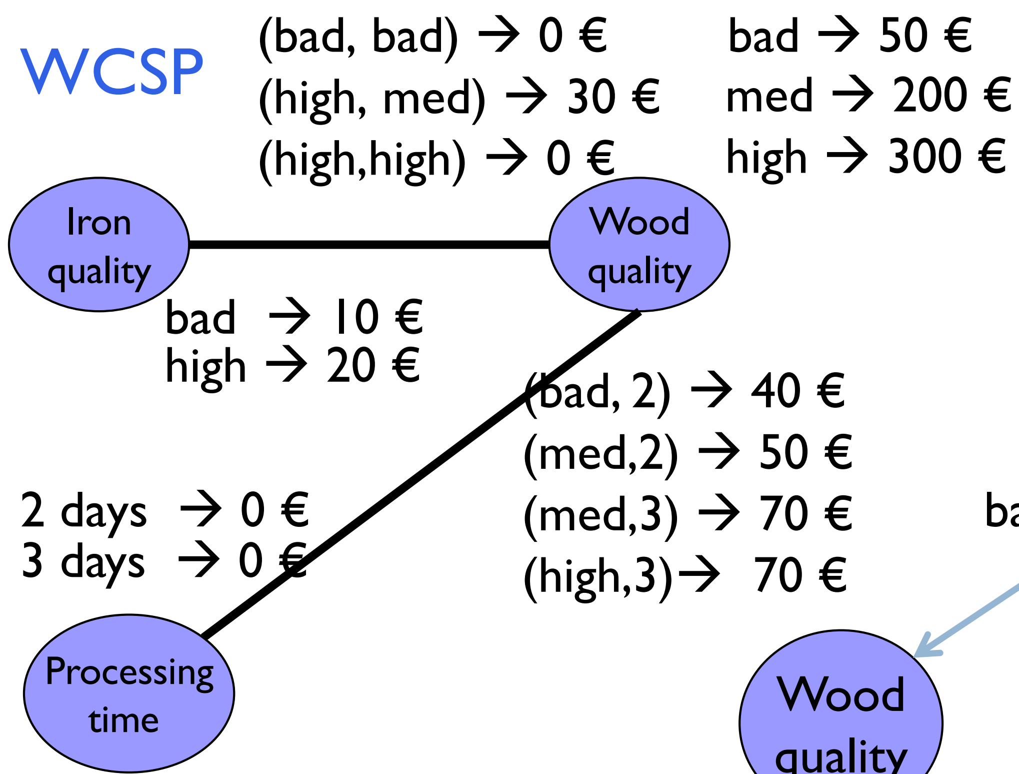
- Backtracking → **Branch and Bound**

- Main idea:
  - visit each assignment that may be a solution
  - skip only assignments that are shown to be dominated by others

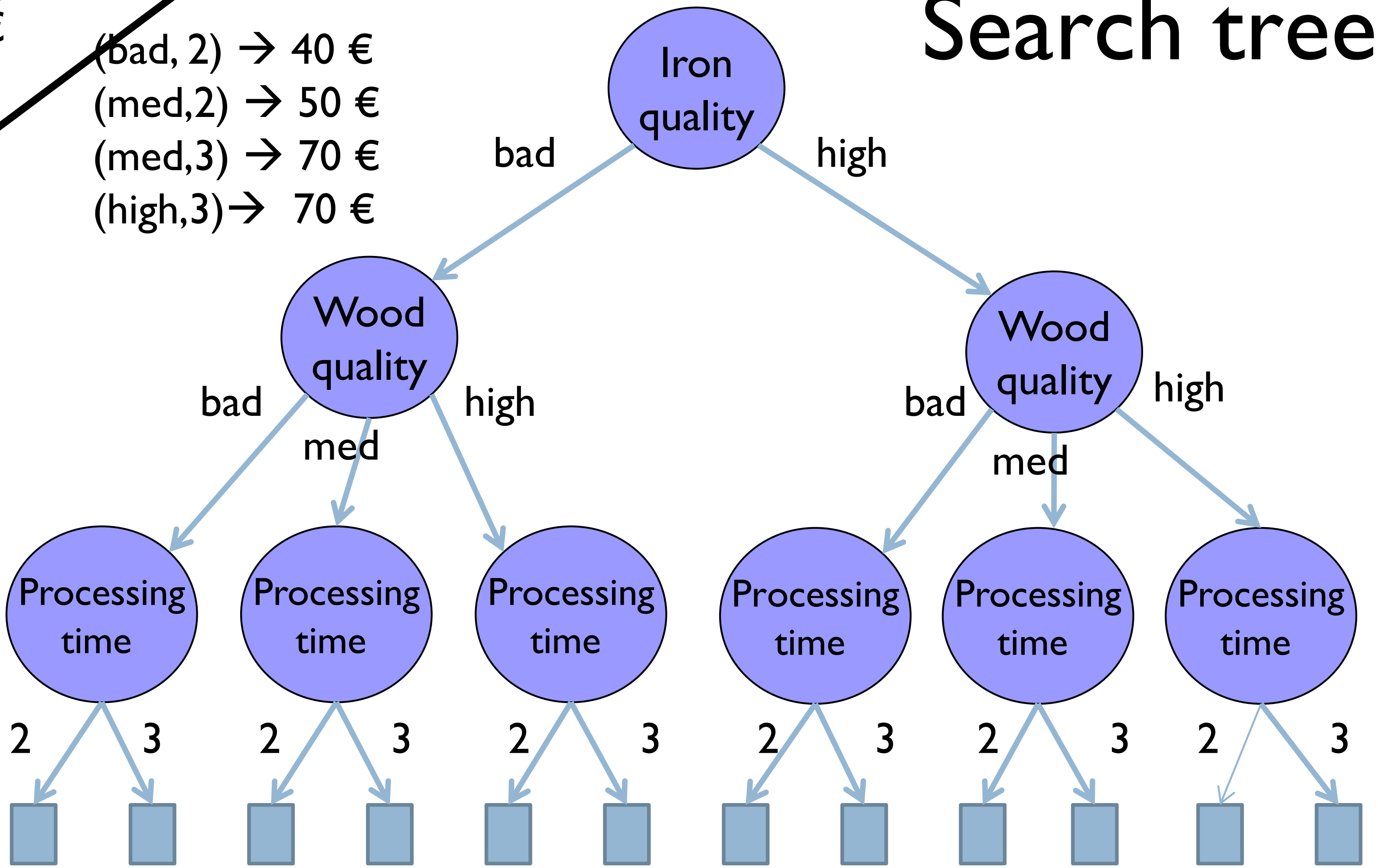
- **Search tree** to represent the space of all assignments

WCSP

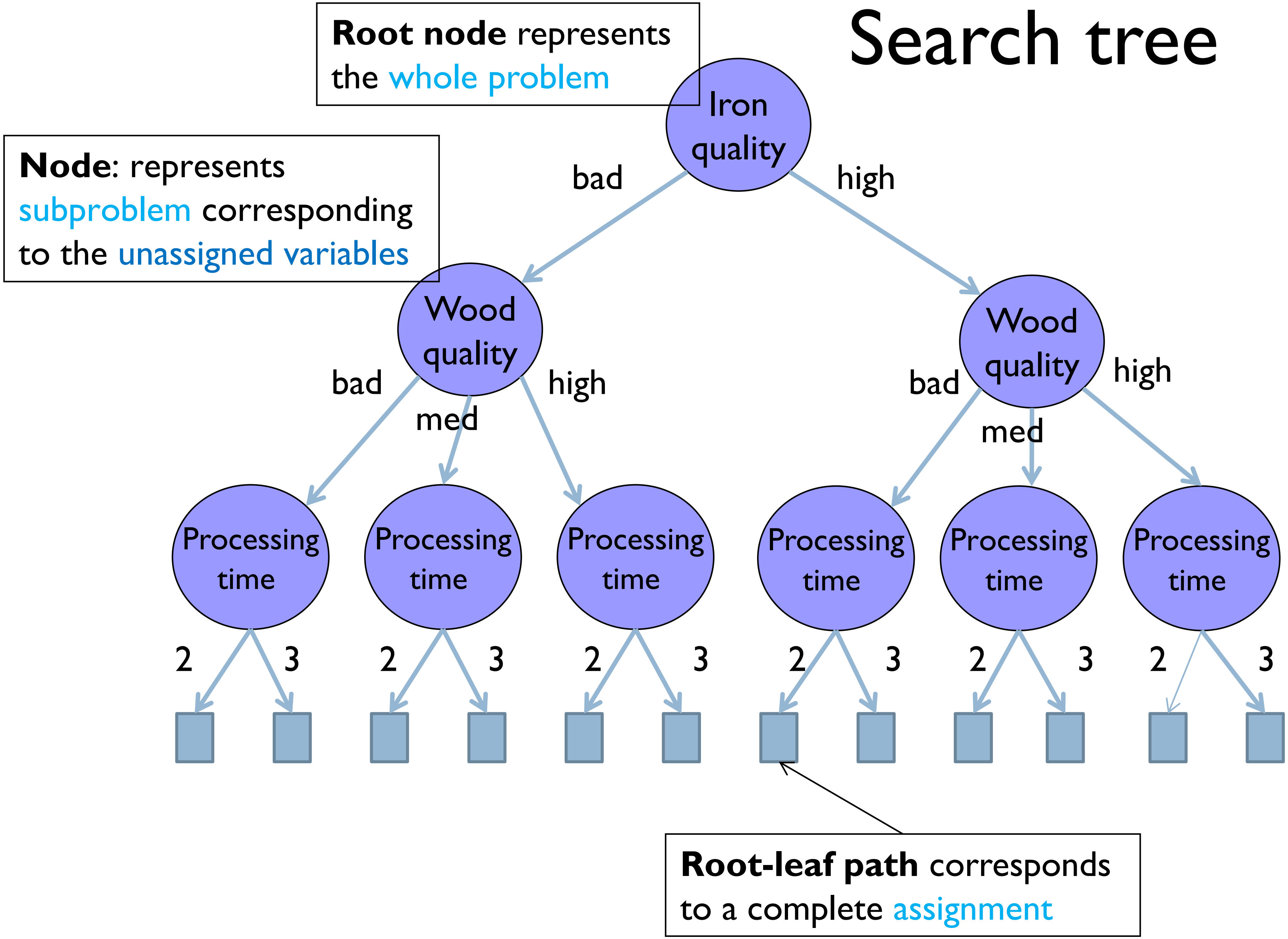
$S_{WCSP}=<[0,+\infty],min,+,+\infty,0>$



Search tree



Search tree


Root node represents the whole problem
Node: represents subproblem corresponding to the unassigned variables
Iron quality
bad
high
Wood quality
bad
med
high
Processing time
2
3
Root-leaf path corresponds to a complete assignment

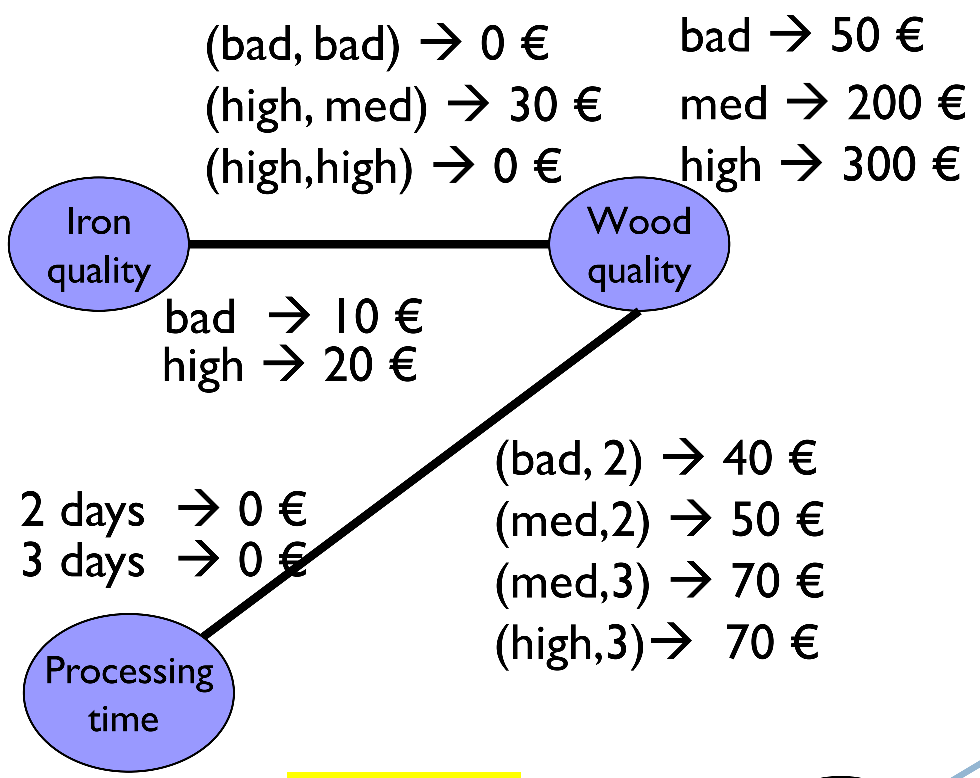
Systematic search : Branch and Bound

- **Lower bound (lb)**=
  preference of best solution so far (0 at the beginning)

- **Upper bound (ub) for each node**:
  upper bound to the **preference** of any assignment in the subtree rooted at the node

- If **ub** is **worst** than **lb** → **prune subtree**

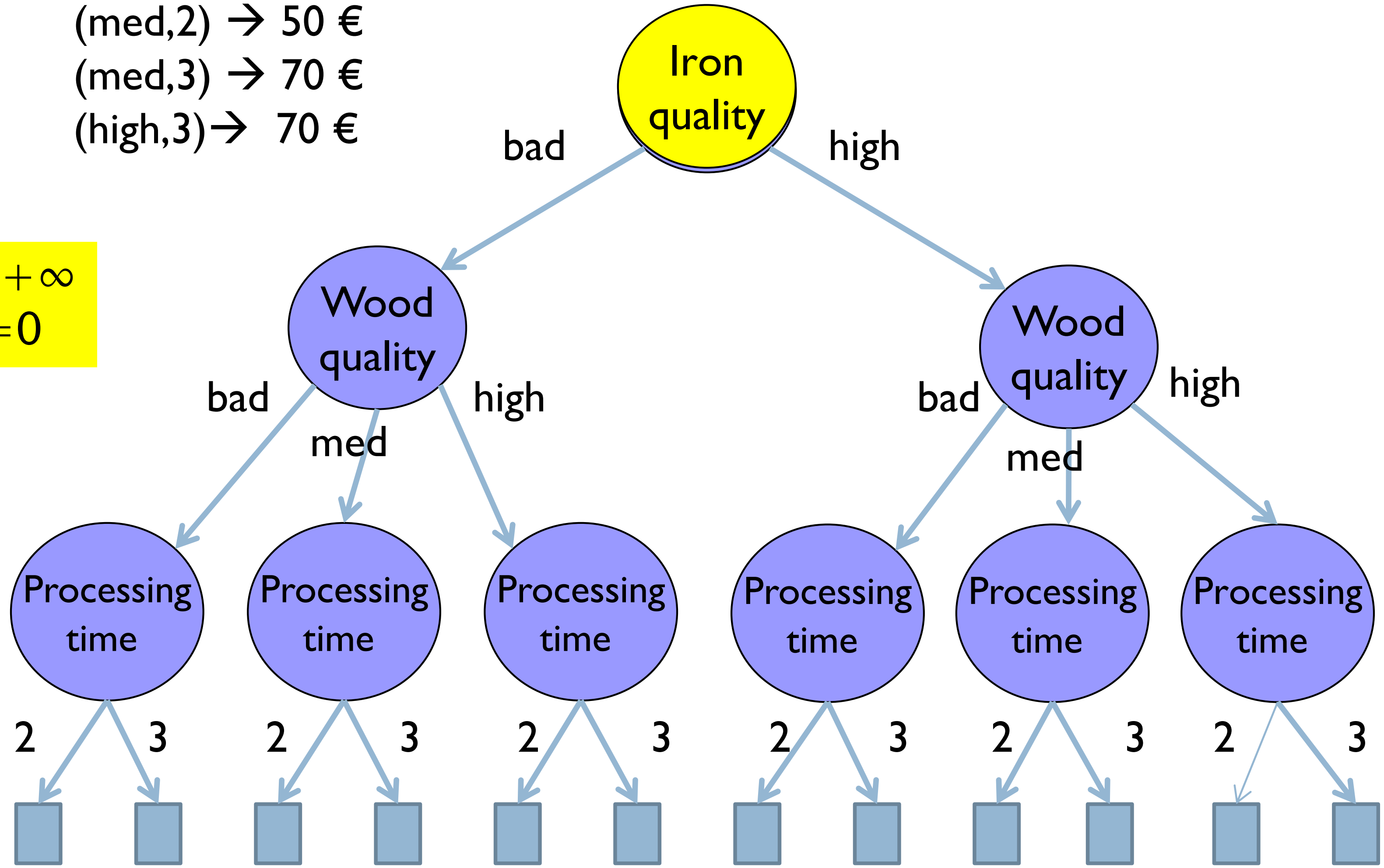
$S_{WCSP} = \langle [0, +\infty], min, +, +\infty, 0 \rangle$



lb = preference of best solution so far

ub = combination of preferences from constraints on assigned variables

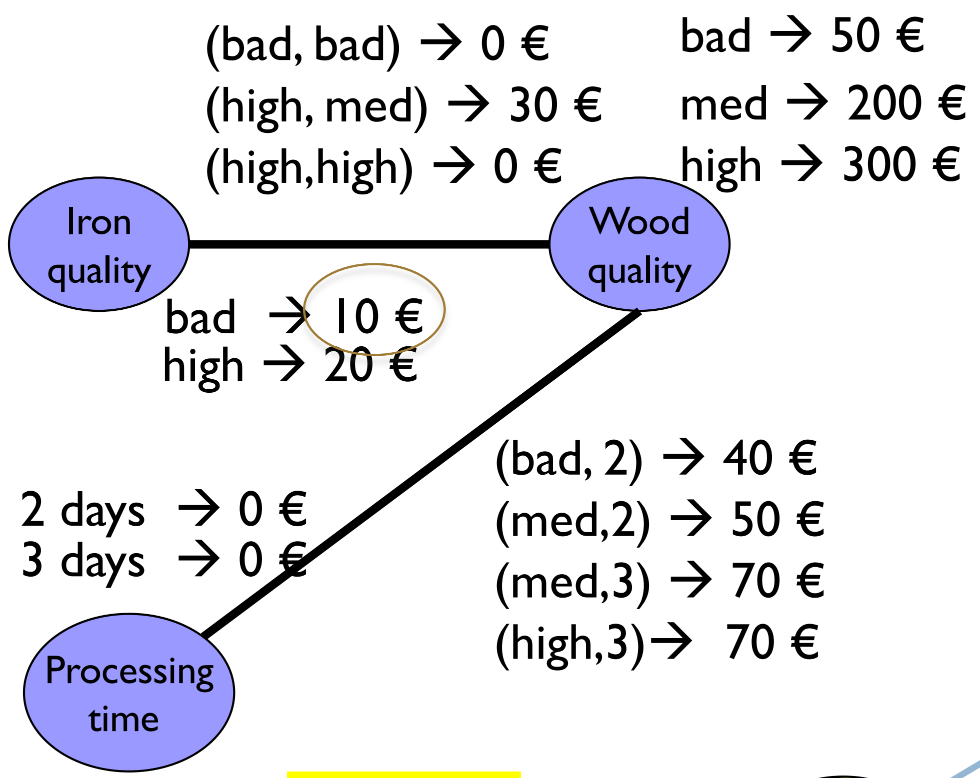
lb $= +\infty$
ub $= 0$



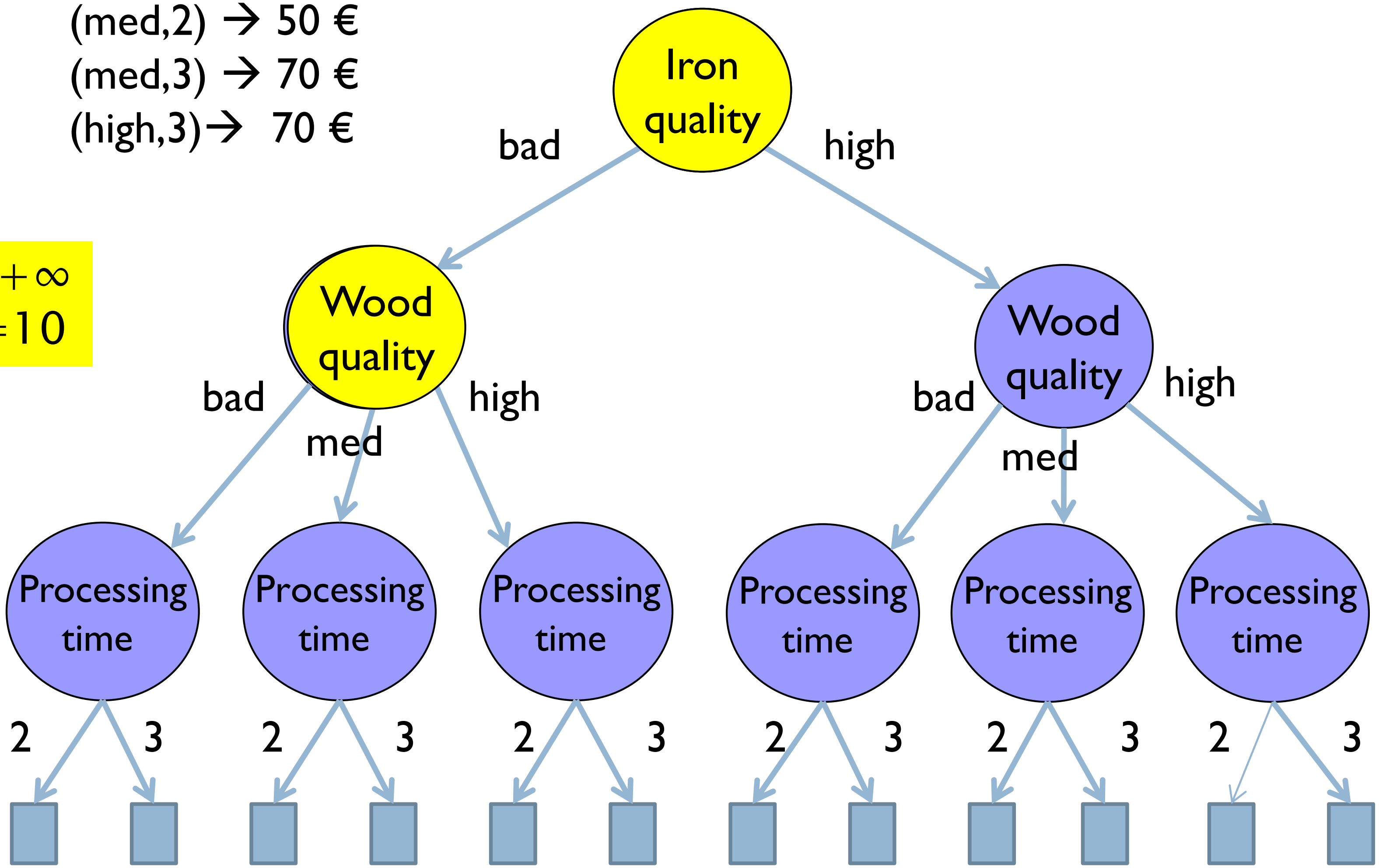
$S_{WCSP}=<[0,+\infty],min,+,+\infty,0>$

lb = preference of best solution so far

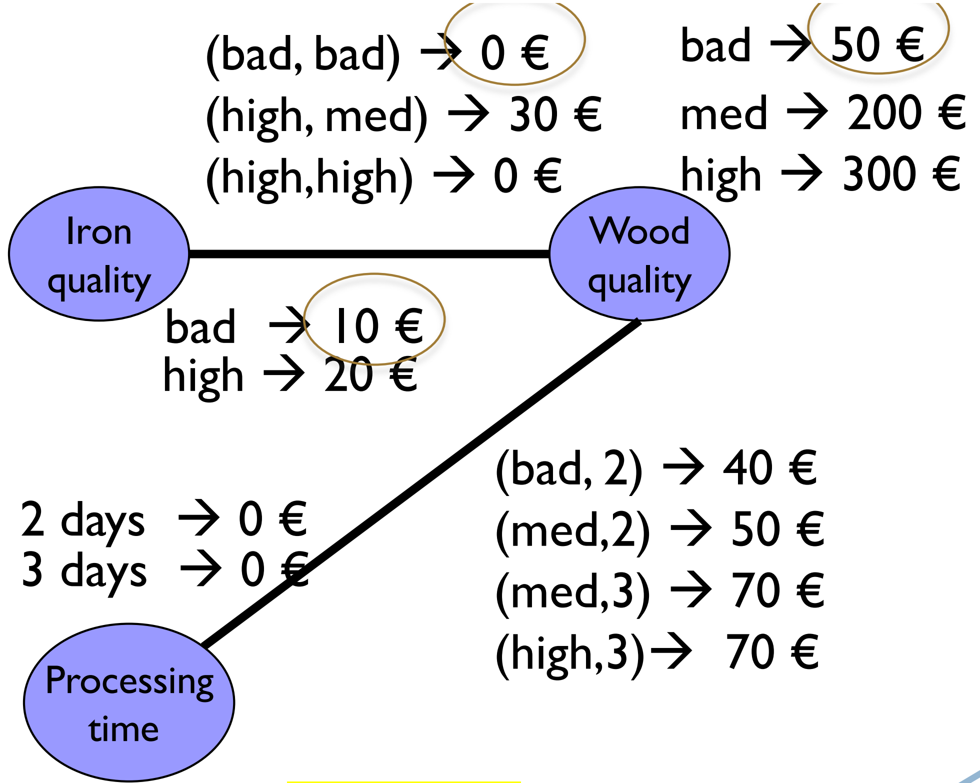
ub = combination of preferences from constraints on assigned variables



lb $=+\infty$
ub $=10$



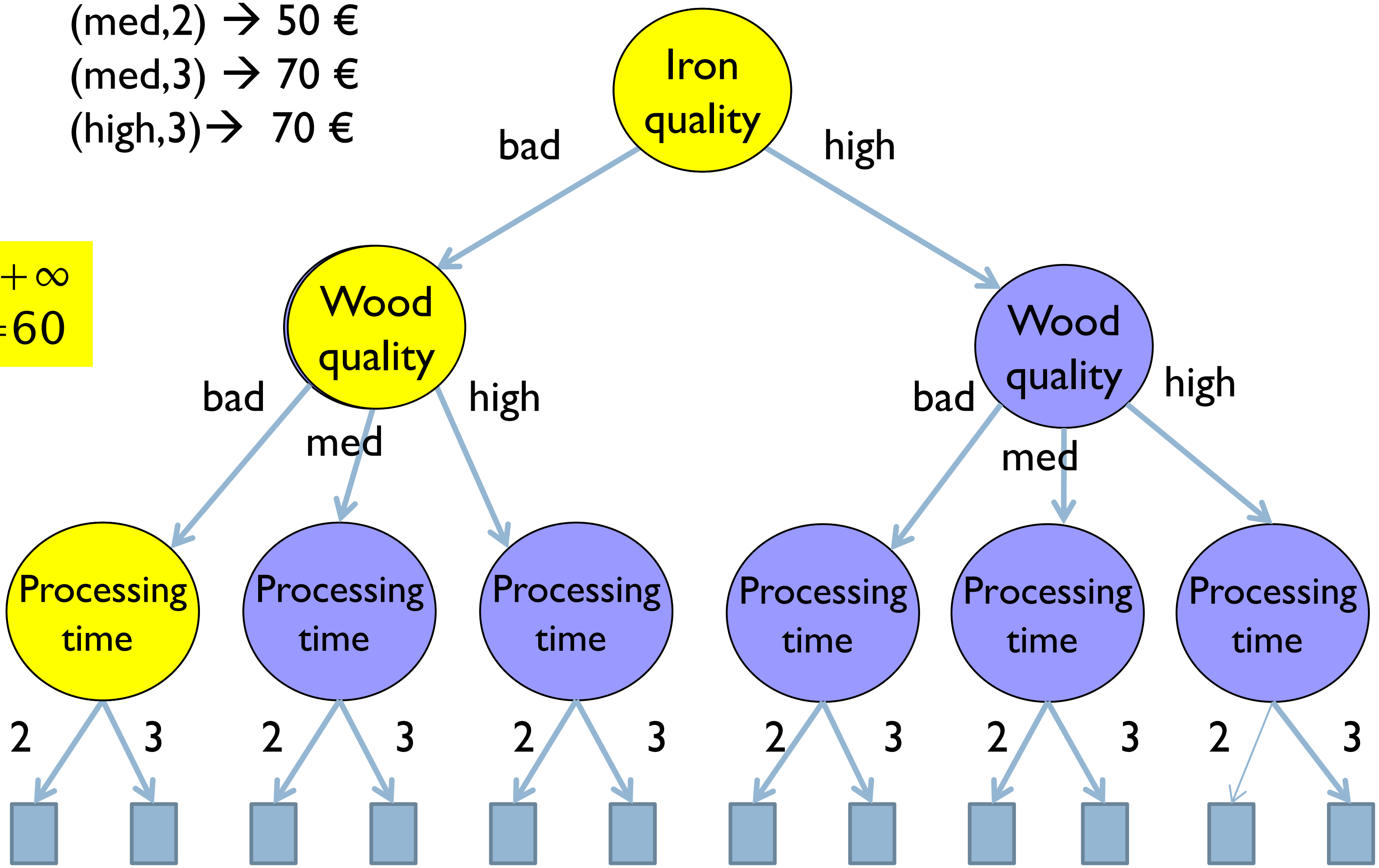
$S_{WCSP} = \langle [0,+\infty], min, +, +\infty, 0 \rangle$



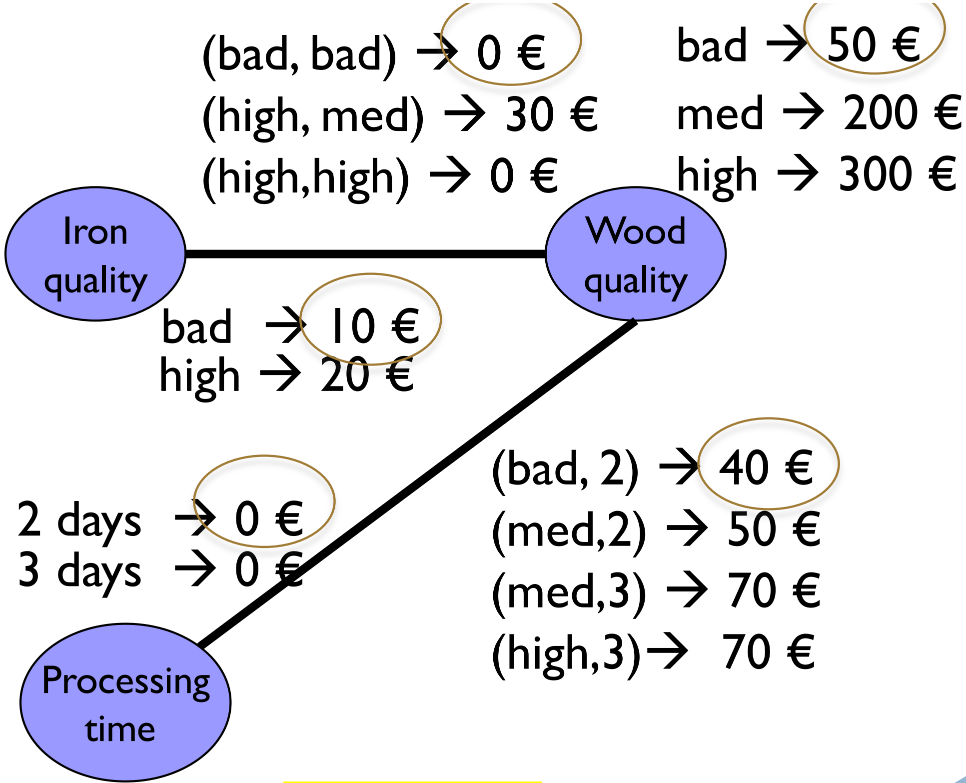
lb = preference of best solution so far

ub = combination of preferences from constraints on assigned variables

lb $=+\infty$
ub $=60$



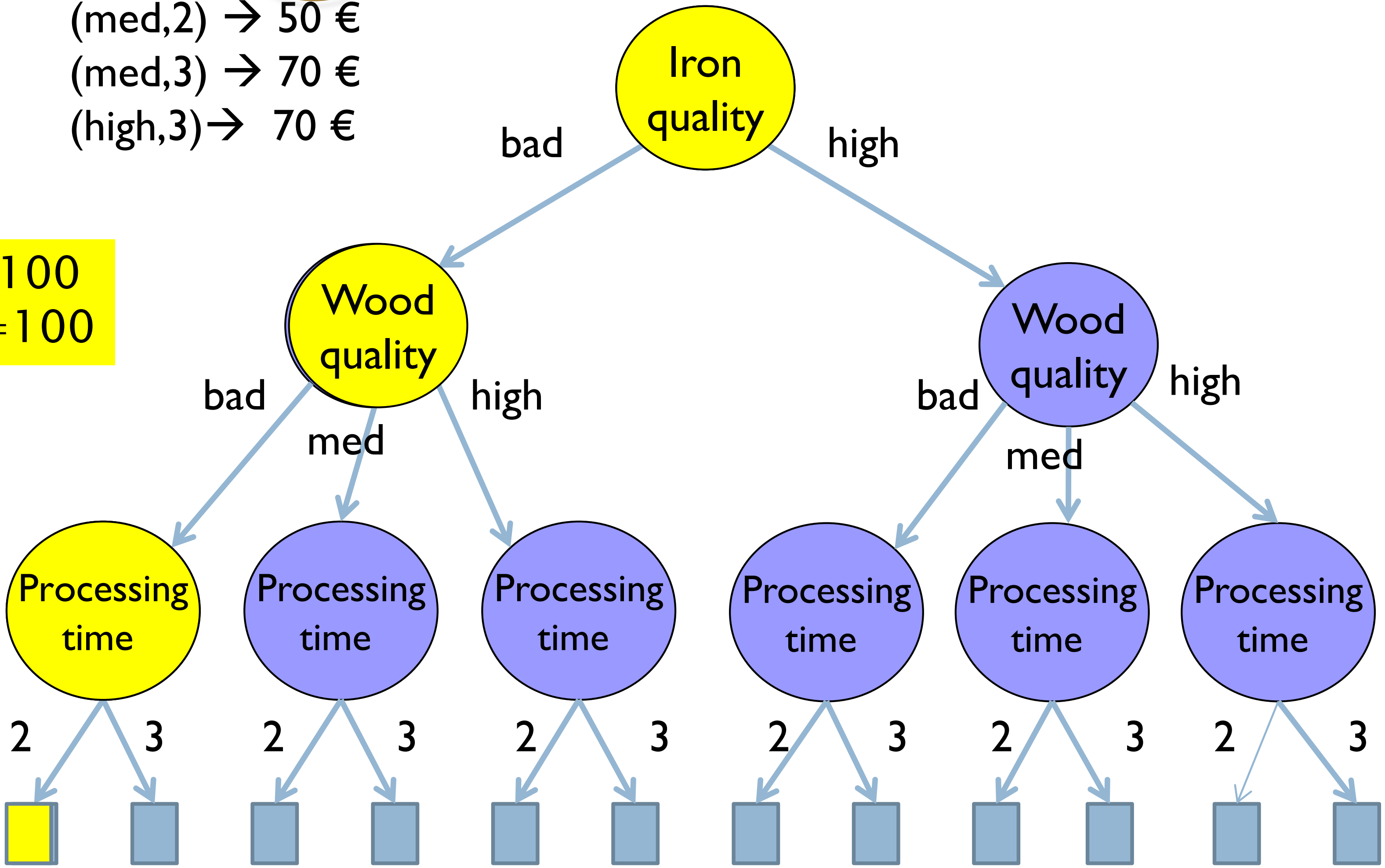
$S_{WCSP} = \langle [0,+\infty], min, +, +\infty, 0 \rangle$



lb = preference of best solution so far

ub = combination of preferences from constraints on assigned variables

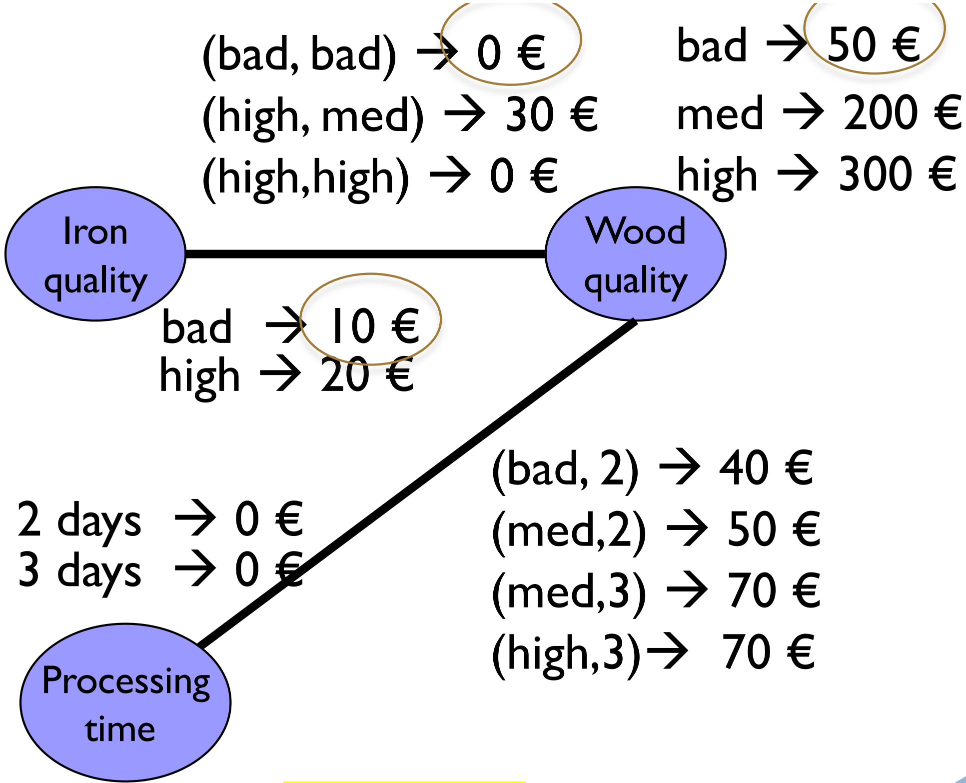
lb =100
ub =100



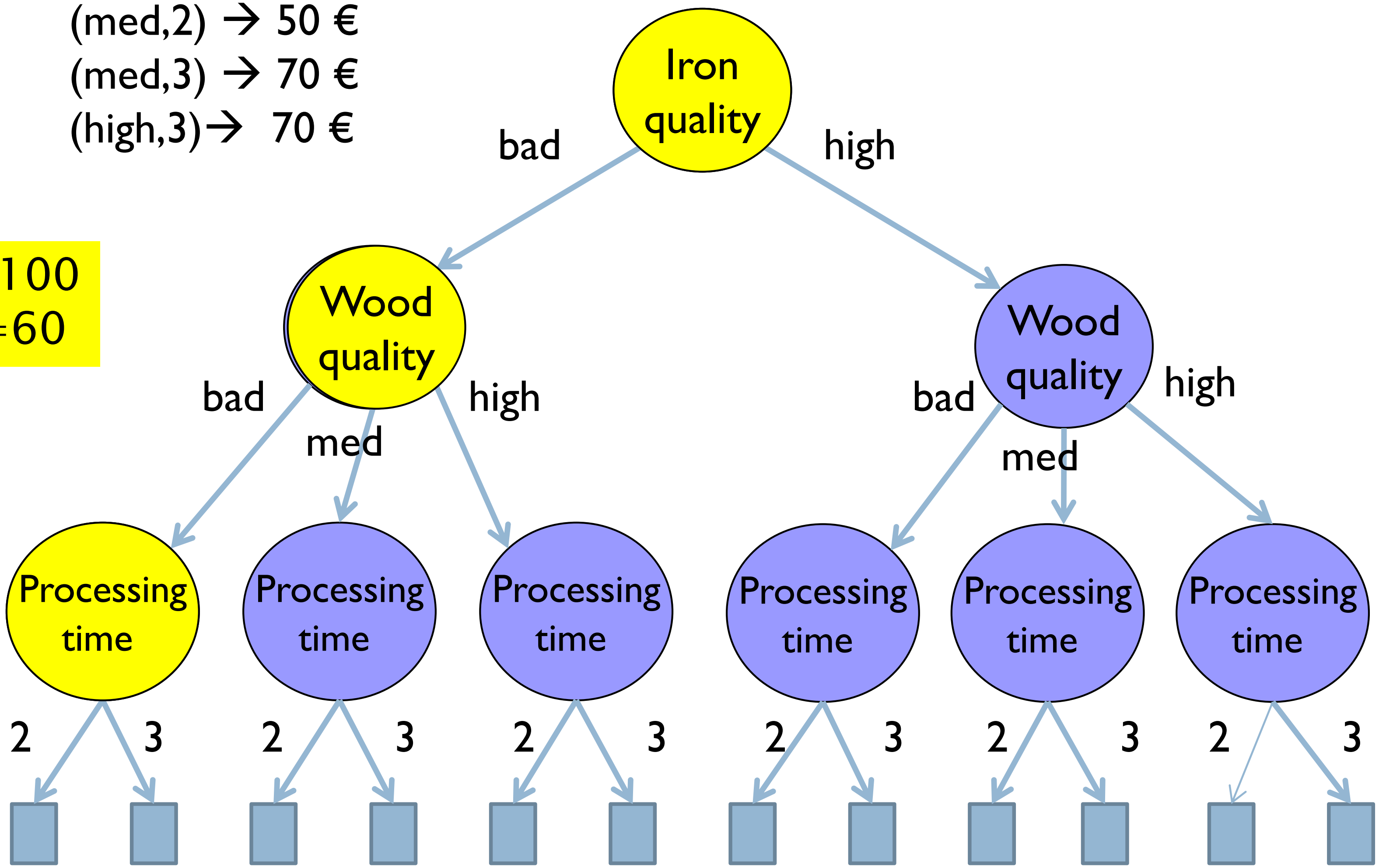
$S_{WCSP} = <[0, +\infty], min, +, +\infty, 0>$

lb = preference of best solution so far

ub = combination of preferences from constraints on assigned variables



lb =100
ub =60



$S_{WCSP}=<[0,+\infty],min,+,+\infty,0>$

lb = preference of best solution so far

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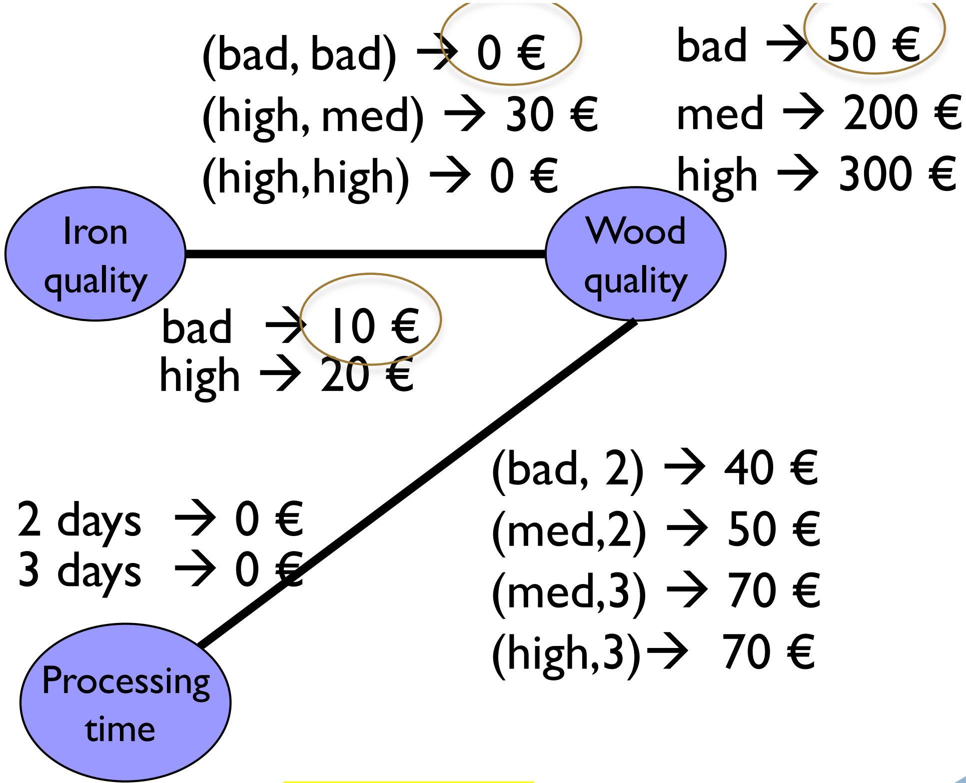
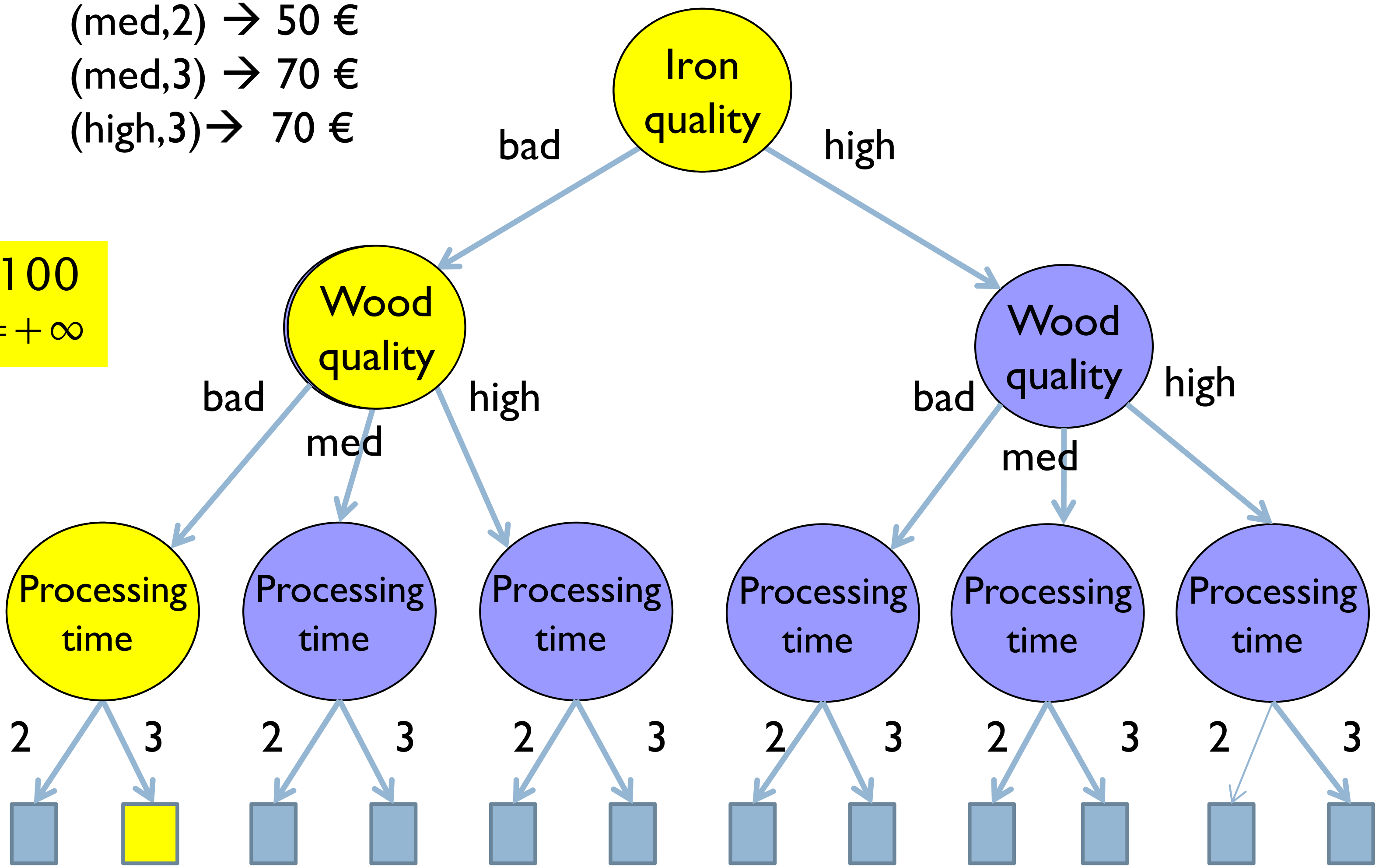


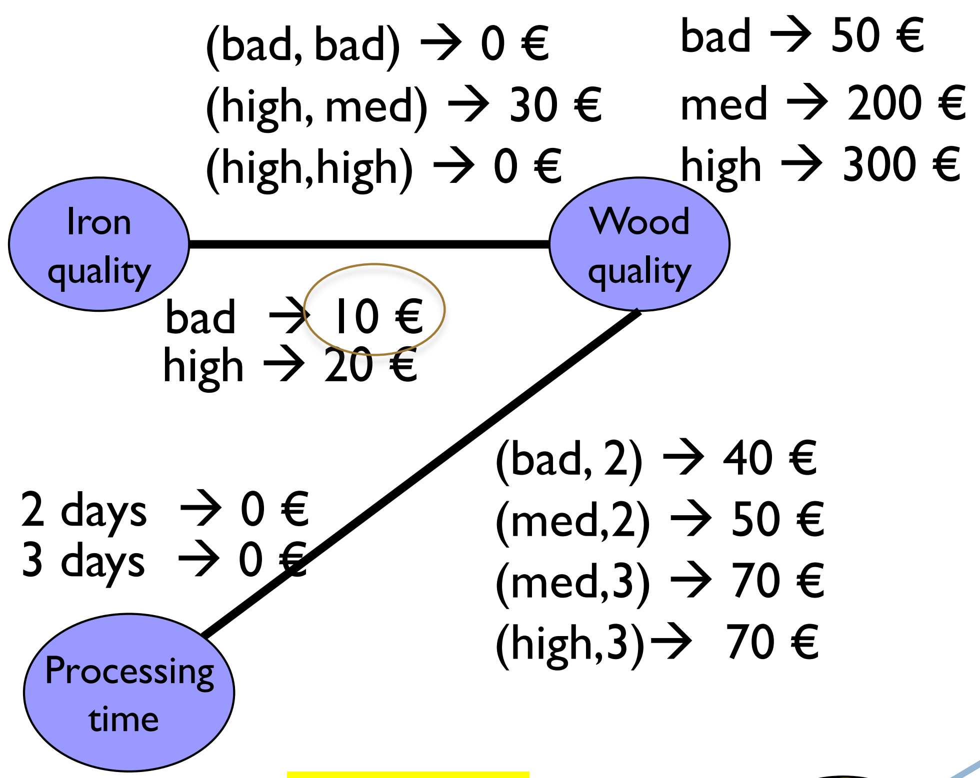
lb =100
ub =+∞



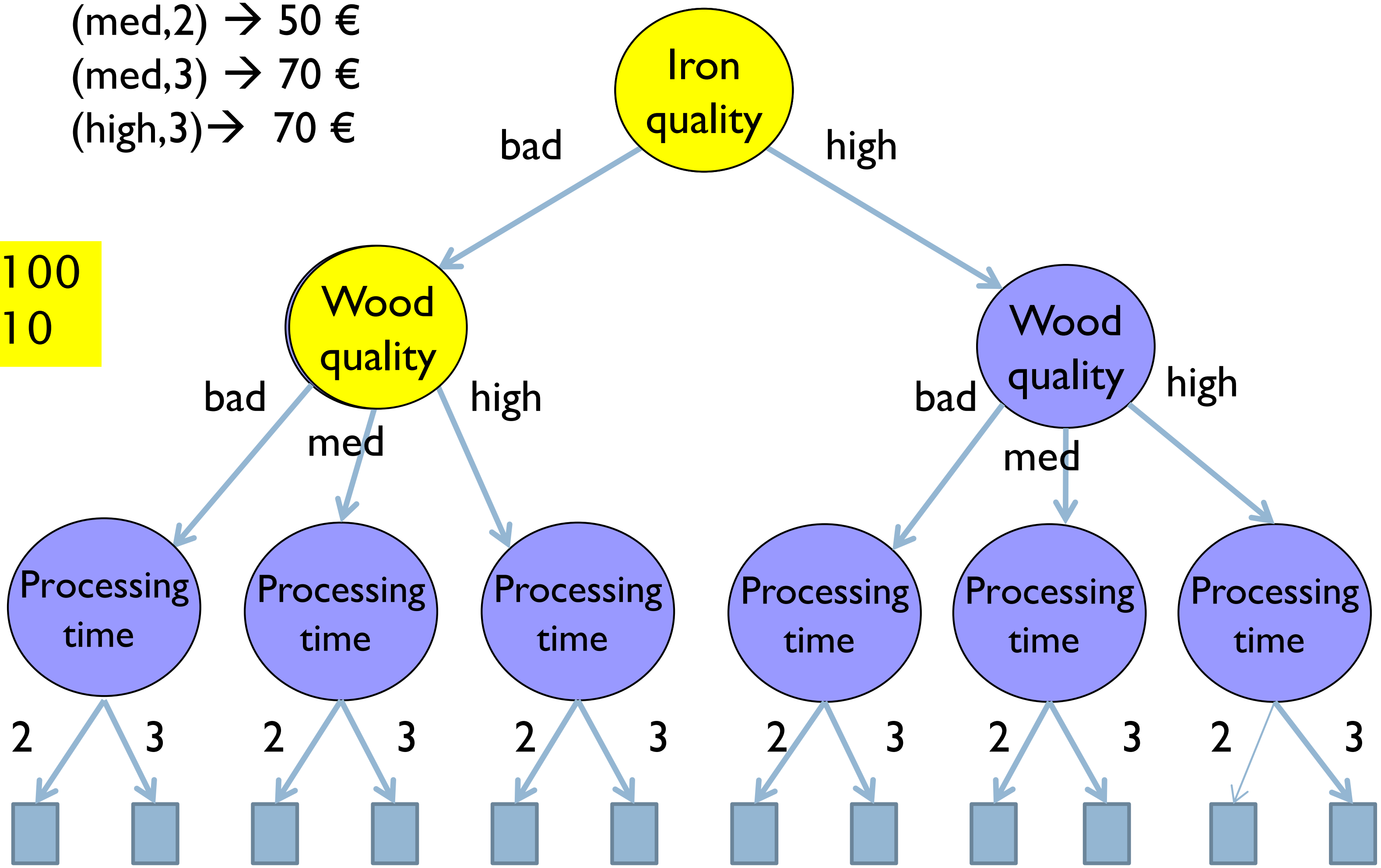
$S_{WCSP} = \langle [0, +\infty], \min, +, +\infty, 0 \rangle$

lb = preference of best solution so far

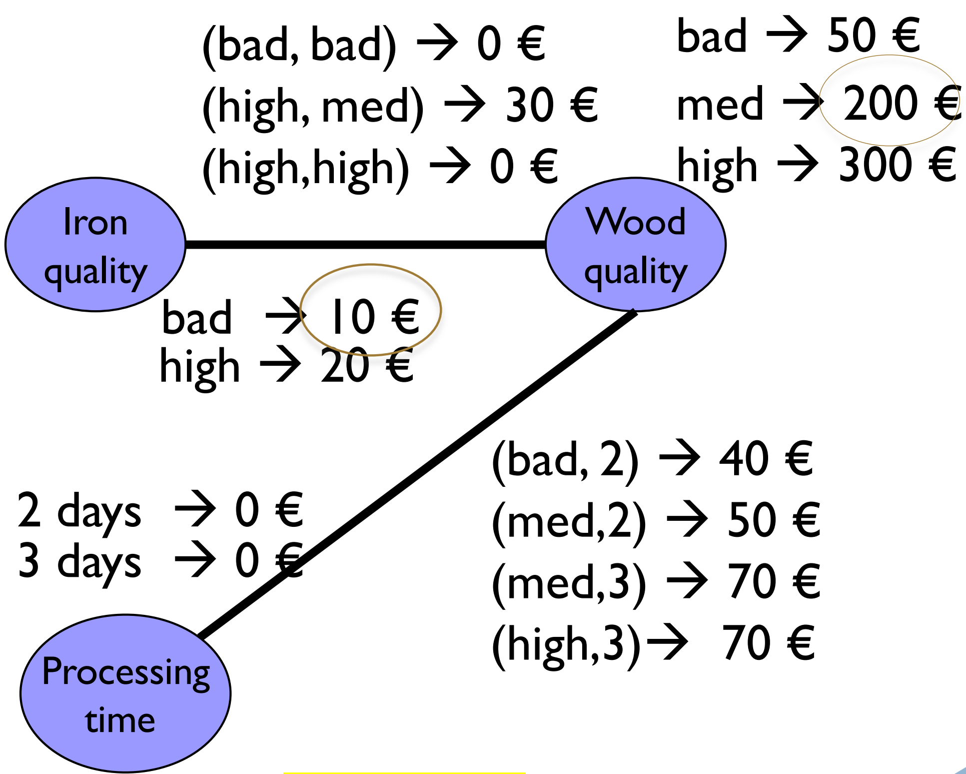
ub = combination of preferences from constraints on assigned variables



lb =100
ub=10



$S_{WCSP} = \langle [0, +\infty], min, +, +\infty, 0 \rangle$

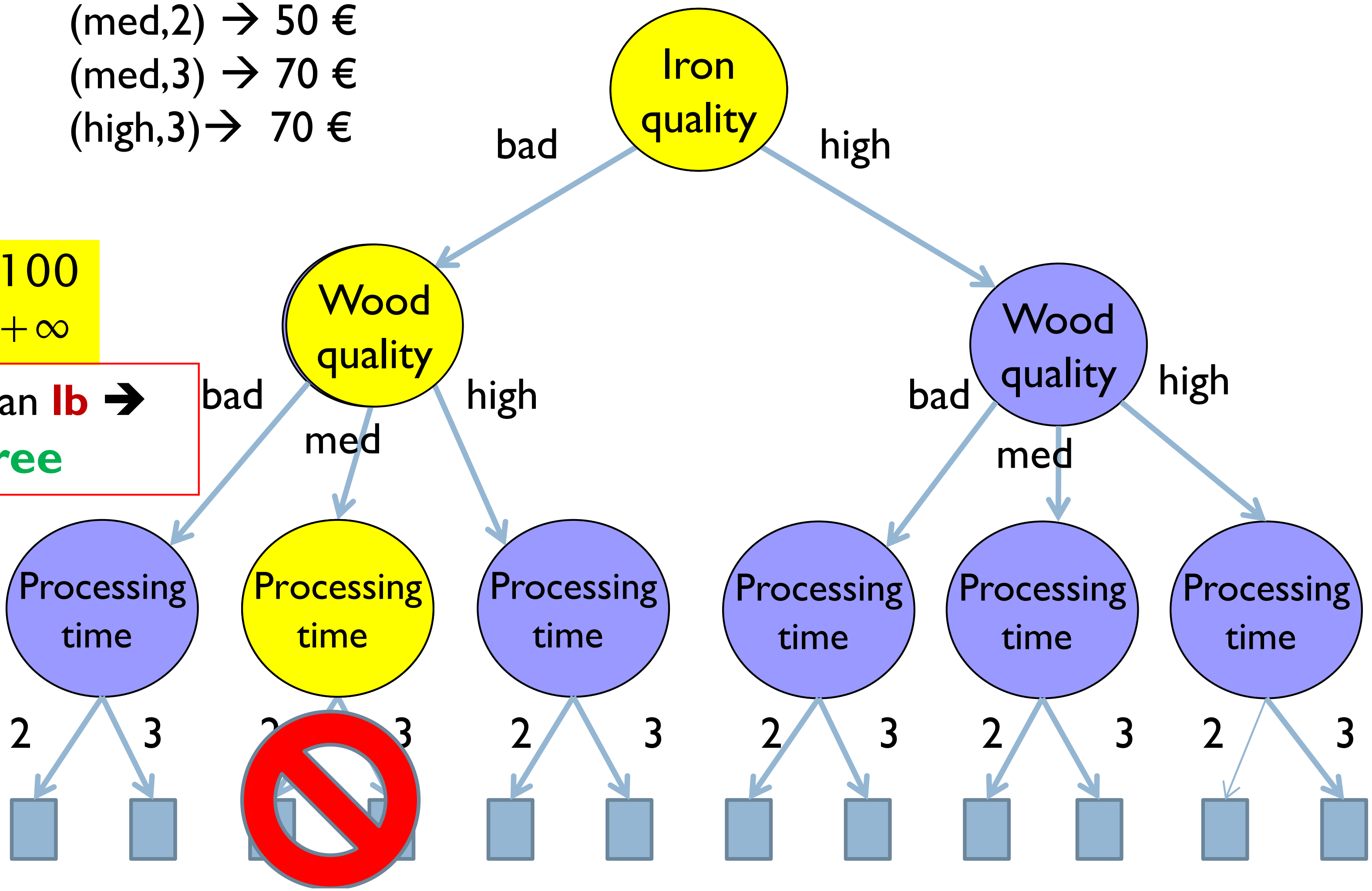


lb = preference of best solution so far

ub = combination of preferences from constraints on assigned variables

lb =100
ub=+∞

ub is worst than lb → **prune subtree**



Incomplete soft constraint problems

- SCSPs with some missing preferences
- New notions of optimal solutions
- Idea: interleaving search and preference elicitation

M. Gelain, M. S. Pini, F. Rossi, K. B. Venable, and T. Walsh. Elicitation strategies for soft constraint problems with missing preferences: Properties, algorithms and experimental studies. Artificial Intelligence, 174(3-4):270-294, 2010.