Automata, Languages and Computation

Chapter 4 : Properties of Regular Languages

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Lecture based on material originally developed by :
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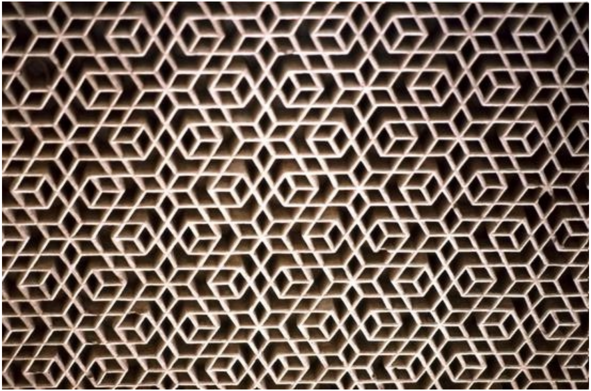
Properties of regular languages



1. Pumping Lemma : every regular language satisfies this property; useful to show that some languages are not regular
2. Closure properties : how to combine automata using specific operations
3. Decision problems : algorithms for the solution of problems based on automata/regex and their complexity
4. Automata minimization : reduce number of states to a minimum

Introduction to pumping lemma

Suppose $L_{01} = \{0^n 1^n \mid n \geqslant 1\}$ were a regular language

Then L_{01} must be recognized by some DFA A; let k be the number of states of A

Assume A reads 0^k. Then A must go through the following transitions :

fence notation

$0^k + 1$

(because of ε)

ϵ	p_0
0	p_1
00	p_2
$\dots$	$\dots$
0^k	p_k

By the **pigeonhole principle**, there must exist a pair i, j with $i < j \leqslant k$ such that $p_i = p_j$. Let us call q this state

Introduction to pumping lemma

q is the state we see twice
(after reading 0^i we are in q and for 0^j ($i \neq j$) we are again in q for pigeonhole)

Now you can **fool** A :

- if $\hat{\delta}(q, 1^i) \notin F$, then the machine will foolishly reject $0^i 1^i$
- if $\hat{\delta}(q, 1^i) \in F$, then the machine will foolishly accept $0^j 1^i$

⚠ these are different and we can only do one of the two correctly

In other words: state q would represent inconsistent information about the count of occurrences of 0 in the string read so far

Therefore A does not exists, and L_{01} is not a regular language

Pumping lemma for regular languages

n is different for each language

Theorem Let L be any regular language. Then $\exists n \in \mathbb{N}$ depending on L, $\forall w \in L$ with $|w| \geqslant n$, we can factorize $w = xyz$ with :

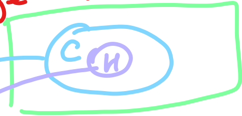
in the example before 0^i → 1 ← 0's

- $y \neq \epsilon$
- $|xy| \leqslant n$ ⟷ the cuts are in the "beginning of the string"
- $\forall k \geqslant 0,\ xy^k z \in L$

all languages

there are no such strings if the language is finite

not in the book if H then C H ⇒ C ≡ H ⊆ C



p. lemma

being regular

Pumping lemma for regular languages

We will do exercises like: Show that L is not regular proof: show that L does not have the p. lemma property Then L cannot be regular □

Proof

Suppose L is a regular language

Then L is recognized by some DFA A with, say, n states

Let $w = a_1 a_2 \cdots a_m \in L$ with $m \geqslant n$

Let $p_i = \hat{\delta}(q_0, a_1 a_2 \cdots a_i)$, for each $i = 0, 1, \ldots, n$

There exists $i < j \leqslant n$ such that $p_i = p_j$

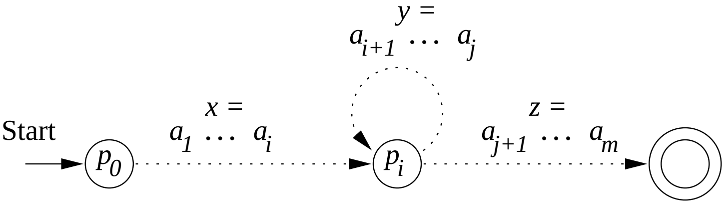
Be careful

P does NOT imply C, showing that p. lemma hold does not mean that it's a regular language

Pumping lemma for regular languages

Let us write $w = xyz$, where

- $x = a_1 a_2 \cdots a_i$
- $y = a_{i+1} a_{i+2} \cdots a_j$
- $z = a_{j+1} a_{j+2} \ldots a_m$



Evidently, $xy^k z \in L$, for any $k \geqslant 0$ $\square$

Example

Let Σ be some alphabet, and let $w \in \Sigma^*$, $a \in \Sigma$. We write $\#_a(w)$ to denote the **number of occurrences** of a in w

We define

$$L_{eq} = \{w \mid w \in \{0,1\}^*, \ \#_0(w) = \#_1(w)\}$$

In words, L_{eq} is the language whose strings have an equal number of 0's and 1's

Use the pumping lemma to show that L is not regular

Example

Proof Suppose L_{eq} were regular. Then $L(A) = L_{eq}$ for some DFA A

Let n be the number of states of A and let $w = 0^n 1^n \in L(A)$

By the pumping lemma we can factorize $w = xyz$ with

- $|xy| \leqslant n$,
- $y \neq \epsilon$

and state that, for each $k \geqslant 0$, we have $xy^k z \in L(A)$

we cannot cut where there is a 1 because we would have $|xy| > n$

$$w = \underbrace{000\cdots}_{x}\underbrace{\cdots 00}_{y}\underbrace{\cdots 0111\cdots 11}_{z}$$

⚠ It is not enough to show there exists a cut where the p.lemma is not true, we need to show the p.lemma is false for every cut (because the lemma says ∃)

Example

For $k = 0$ we have $xz \in L(A)$

This is a **contradiction**, since $|y| \geqslant 1$ and then xz has fewer 0's than 1's

We therefore conclude that $L(A) \neq L_{eq}$ $\square$

Comment of the if-then formulation of the pumping lemma: many students wrongly state that if the pumping lemma holds, then the language must be regular

⚠ if we choose $w = 101010\ldots$ it's impossible to disprove the p. lemma, it's important to choose the right w!

Example

Proof (alternative) We can see the application of the pumping lemma as a game between two players

Player P2 states that L_{eq} is regular, and player P1 wants to establish a **contradiction**

- P2 picks n (number of states of DFA, if it exists)
- P1 picks string $w = 0^n 1^n \in L_{eq}$, with $|w| \geqslant n$
- P2 picks a factorization $w = xyz$, with $|xy| \leqslant n$, $y \neq \epsilon$ and $xy^k z \in L_{eq}$ (assuming L_{eq} is regular)
- P1 picks k such that $xy^k z \notin L$, which is a violation of the pumping lemma. Specifically, P1 picks $k = 0$: $xz \notin L_{eq}$, since y contains just 0's, $y \neq \epsilon$, and thus $\#_0(xz) < \#_1(xz) = n$
- P1 concludes that L_{eq} cannot be regular □

Example

Let $L_{pr} = \{1^p \mid p \text{ prime}\}$. Using the pumping lemma, show that L_{pr} is not regular

Proof Let n be as in the pumping lemma, and let $p \geqslant n + 2$ be some prime number. Thus $1^p \in L_{pr}$

By the pumping lemma we can write $w = xyz$ with

- $|xy| \leqslant n$,
- $y \neq \epsilon$

such that, for each $k \geqslant 0$, we have $xy^k z \in L(A)$

Example

Let $|y| = m \geqslant 1$

$$w = \overbrace{\underbrace{111\cdots}_{x}\ \underbrace{\cdots 1}_{\substack{y \\ |y|=m\geqslant 1}}\ \underbrace{1111\cdots 11}_{z}}^{p}$$

Choose $k = p - m$, so that $xy^{p-m}z \in L_{pr}$ and then $|xy^{p-m}z|$ is a prime number

Example

We can write $|xy^{p-m}z| = |xz| + (p-m)|y| = p - m + (p-m)m = (1+m)(p-m)$

Let us verify that none of the two factors is a 1 :

- $y \neq \epsilon$, thus $1 + m > 1$
- $m = |y| \leqslant |xy| \leqslant n$, $p \geqslant n+2$, thus $p - m \geqslant n + 2 - m \geqslant n + 2 - n = 2$

We have derived a **contradiction** $\square$

Exercise

For a string w, we write w^R to denote the **reverse** of w. Example: $01011^R = 11010$ and $(w^R)^R = w$

Consider the language

$$L = \{ww^R \mid w \in \{0,1\}^*\}$$

Using the pumping lemma, show that L is not regular

Closure properties of regular languages

Let L and M be regular languages over Σ. Then the following languages are all regular

- Union: $L \cup M$
- Intersection: $L \cap M$
- Complement: $\overline{L} = \Sigma^* \setminus L$
- Difference: $L \setminus M$
- Reversal: $L^R = \{w^R \mid w \in L\}$
- Kleene closure: L^*
- Concatenation: $L.M$
- Homomorphism: $h(L) = \{h(w) \mid w \in L\}$
- Inverse homomorphism: $h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\}$

Closure under union

Theorem For any regular languages L e M, $L \cup M$ is regular

Proof Let E and F be regular expressions such that $L = L(E)$ and $M = L(F)$. Then $L \cup M$ is generated by $E + F$, and is regular by definition $\square$

Closure under concatenation and Kleene

The proof of closure under union is rather **immediate**, since regular expressions use the union operator

Similarly, we can immediately prove the closure under

- concatenation
- Kleene operator

Closure under complement

Theorem If L is a regular language over Σ, then so is $\overline{L} = \Sigma^* \setminus L$

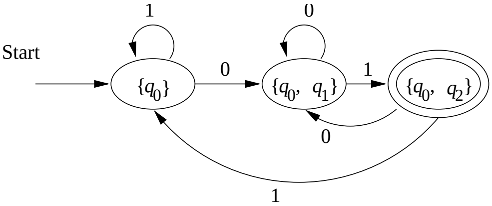
Proof Let L be recognized by a DFA

$$A = (Q, \Sigma, \delta, q_0, F).$$

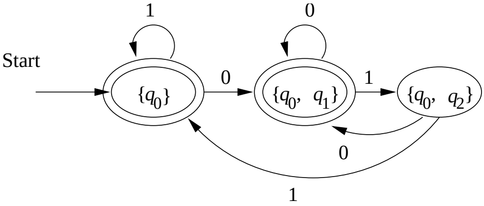
Let $B = (Q, \Sigma, \delta, q_0, Q \setminus F)$. Now $L(B) = \overline{L}$ $\square$

Example

Let L be recognized by the DFA



Then $\overline{L}$ is recognized by the DFA



Closure under intersection

Theorem If L and M are regular, then so is $L \cap M$

Proof By De Morgan's law, $L \cap M = \overline{\overline{L} \cup \overline{M}}$

We already know that regular languages are closed under complement and union $\square$

Intersection automaton

Proof (alternative) Let $L = L(A_L)$ and $M = L(A_M)$ for automata A_L and A_M with

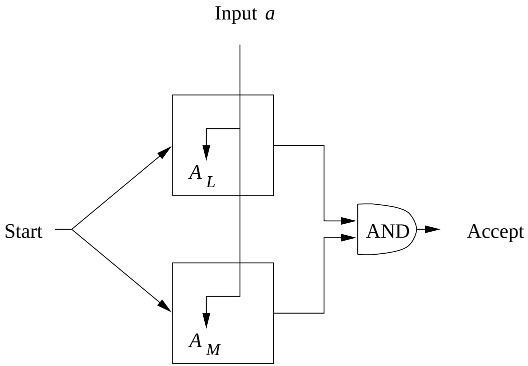
$$A_L = (Q_L, \Sigma, \delta_L, q_L, F_L)$$
$$A_M = (Q_M, \Sigma, \delta_M, q_M, F_M)$$

Without any loss of generality, we assume that both automata are deterministic

We shall construct an automaton that simulates A_L and A_M in parallel, and accepts if and only if both A_L and A_M accept

Intersection automaton

Idea : If A_L goes from state p to state s upon reading a, and A_M goes from state q to state t upon reading a, then $A_{L\cap M}$ will go from state (p,q) to state (s,t) upon reading a



Intersection automaton

Formally

$$A_{L \cap M} = (Q_L \times Q_M, \Sigma, \delta_{L \cap M}, (q_{L,0}, q_{M,0}), F_L \times F_M),$$

where

$$\delta_{L \cap M}((p, q), a) = (\delta_L(p, a), \delta_M(q, a))$$

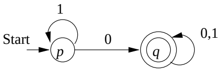
We can show by induction on $|w|$ that the book skips the next part

$$\hat{\delta}_{L \cap M}((q_{L,0}, q_{M,0}), w) = \left(\hat{\delta}_L(q_{L,0}, w), \hat{\delta}_M(q_{M,0}, w)\right)$$

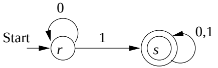
Then $A_{L \cap M}$ accepts if and only if A_L and A_M accept $\square$

Exercise

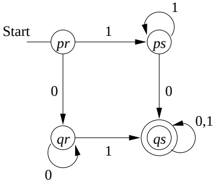
Build an automaton that accepts strings with at least one 0 and at least one 1. Let's build **simpler** automata and take the intersection



(a)



(b)



(c)

Closure under set difference

Theorem If L and M are regular languages, so is $L \setminus M$

Proof Observe that $L \setminus M = L \cap \overline{M}$

We already know that regular languages are closed under complement and intersection $\square$

Closure under reverse operator

$a \in \Sigma$
$x \in \Sigma^*$

$w^R = \begin{cases} \text{if } w = \varepsilon \text{ then } w^R = \varepsilon \\ \text{otherwise } w^R = a x^R \text{ s.t. } xa = w \end{cases}$

$ci\cancel{ax}$ $\quad o(cia)^R$

Theorem If L is regular, so is L^R

Proof Let L be recognized by FA A. Turn A into an FA for L^R by

- reversing all arcs
- make the old start state the new sole accepting state
- create a new start state p_0 such that $\delta(p_0, \epsilon) = F$, F the set of accepting states of old A □

Closure under reverse operator

Proof (alternative) Let E be a regular expression. We shall construct a regular expression E^R such that $L(E^R) = (L(E))^R$

We proceed by structural induction on E

Base If E is ϵ, $\emptyset$, or $\boldsymbol{a}$, then $E^R = E$ (easy to verify)

Closure under reverse operator

Induction

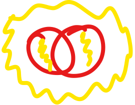
- $E = F + G$: We need to reverse the two languages. Then $E^R = F^R + G^R$
- $E = F.G$: We need to reverse the two languages and also reverse the order of their concatenation. Then $E^R = G^R.F^R$
- $E = F^*$:
  $w \in L(F^*)$ means $\exists k \ : \ w = w_1 w_2 \cdots w_k, \ w_i \in L(F)$
  then $w^R = w_k^R w_{k-1}^R \cdots w_1^R, \ w_i^R \in L(F^R)$
  then $w^R \in L(F^R)^*$
  Same reasoning for the inverse direction. Then $E^R = (F^R)^*$

Thus $L(E^R) = (L(E))^R$ $\square$

Test

State whether the following claims hold true, and motivate your answer

1. the intersection of a non-regular language and a finite language is always a regular language
2. the intersection of a non-regular language L_1 and an infinite regular language L_2 is never a regular language false
3. every subset of a non-regular language is a non-regular language false



1. $|R| < +\infty \Rightarrow |R \cap N| < +\infty$

2. $\underbrace{L(a^* b^*)}_{\text{regular}} \cap \underbrace{L_{>}}_{\text{not reg}} = \emptyset$ ↯

3. False take a finite subset (also $\emptyset$)

Superset and subset

Assume L is a regular language. We **cannot say anything** about languages L' and L'' with $L' \subset L$ and $L'' \supset L$

More precisely

- L' could be regular or non-regular
- L'' could be regular or non-regular

Often student gets confused about this, thinking that adding strings to L makes it 'more difficult' and removing strings from L makes it 'less difficult'. But this is **not true in general**

Homomorphisms

Let Σ and Δ be two alphabets. A **homomorphisms** over Σ is a function $h : \Sigma \to \Delta^*$

Informally, a homomorphism is a function which replaces each symbol with a string

Example : Let $\Sigma = \{0, 1\}$ and define $h(0) = ab$, $h(1) = \epsilon$; h is a homomorphism over Σ

Homomorphisms

We extend h to Σ^* : if $w = a_1 a_2 \cdots a_n$ then

$$h(w) = h(a_1)h(a_2)\cdots h(a_n)$$

Equivalently, we can use a **recursive** definition :

$$h(w) = \begin{cases} \epsilon, & \text{if } w = \epsilon; \\ h(x)h(a) & \text{if } w = xa,\ x \in \Sigma^*,\ a \in \Sigma. \end{cases}$$

Example : Using h from previous example on string 01001 results in *ababab*

Homomorphisms

For a language $L \subseteq \Sigma^*$

$$h(L) = \{h(w) \mid w \in L\}$$

Example : Let L be the language associated with the regular expression $\mathbf{10^*1}$. Then $h(L)$ is the language associated with the regular expression $(\boldsymbol{ab})^*$

Closure under homomorphism

Theorem Let $L \subseteq \Sigma^*$ be a regular language and let h be a homomorphisms over Σ. Then $h(L)$ is a regular language

Proof Let E be a regular expression generating L. We define $h(E)$ as the regular expression obtained by substituting in E each symbol $\boldsymbol{a}$ with $\boldsymbol{a_1 a_2 \cdots a_k}$, under the assumption that

- $a \in \Sigma$
- $h(a) = a_1 a_2 \cdots a_k$, $k \geqslant 0$

We now prove the statement

$$L(h(E)) = h(L(E)),$$

using structural induction on E

Closure under homomorphism

Base $E = \epsilon$ or else $E = \emptyset$. Then $h(E) = E$, and $L(h(E)) = L(E) = h(L(E))$

$E = \boldsymbol{a}$ with $a \in \Sigma$. Let $h(a) = a_1 a_2 \cdots a_k$, $k \geqslant 0$. Then $L(\boldsymbol{a}) = \{a\}$ and thus $h(L(\boldsymbol{a})) = \{a_1 a_2 \cdots a_k\}$

The regular expression $h(\boldsymbol{a})$ is $\boldsymbol{a_1 a_2} \cdots \boldsymbol{a_k}$. Then $L(h(\boldsymbol{a})) = \{a_1 a_2 \cdots a_k\} = h(L(\boldsymbol{a}))$

Closure under homomorphism

Induction Let $E = F + G$. We can write

$$\begin{array}{rcll}
L(h(E)) & = & L(h(F+G)) & \\
 & = & L(h(F)+h(G)) & h \text{ defined over regex} \\
 & = & L(h(F)) \cup L(h(G)) & + \text{ definition} \\
 & = & h(L(F)) \cup h(L(G)) & \text{inductive hypothesis for } F, G \\
 & = & h(L(F) \cup L(G)) & h \text{ defined over languages} \\
 & = & h(L(F+G)) & + \text{ definition} \\
 & = & h(L(E)) &
\end{array}$$

Closure under homomorphism

Let $E = F.G$. We can write

$$
\begin{array}{rcll}
L(h(E)) & = & L(h(F.G)) & \\
 & = & L(h(F).h(G)) & h \text{ defined over regex} \\
 & = & L(h(F)).L(h(G)) & . \text{ definition} \\
 & = & h(L(F)).h(L(G)) & \text{inductive hypothesis for } F, G \\
 & = & h(L(F).L(G)) & h \text{ defined over languages} \\
 & = & h(L(F.G)) & . \text{ definition} \\
 & = & h(L(E)) &
\end{array}
$$

Closure under homomorphism

Let $E = F^*$. We can write

$$
\begin{array}{rcll}
L(h(E)) & = & L(h(F^*)) & \\
 & = & L([h(F)]^*) & h \text{ defined over regex} \\
 & = & \bigcup_{k \geqslant 0} [L(h(F))]^k & * \text{ definition} \\
 & = & \bigcup_{k \geqslant 0} [h(L(F))]^k & \text{inductive hypothesis for } F \\
 & = & \bigcup_{k \geqslant 0} \; h([L(F)]^k) & h \text{ definition over languages} \\
 & = & h(\bigcup_{k \geqslant 0} [L(F)]^k) & h \text{ definition over languages} \\
 & = & h(L(F^*)) & * \text{ definition} \\
 & = & h(L(E)) &
\end{array}
$$

$\square$

Conversion complexity

We can convert among DFA, NFA, ϵ-NFA, and regular expressions

What is the **computational complexity** of these conversions?

We investigate the computational complexity as a function of

- number of states n for an FA
- number of operators n for a regular expressions
- we assume $|\Sigma|$ is a constant

From ϵ-NFA to DFA

Suppose an ϵ-NFA has n states. To compute ECLOSE(p) we visit at most n^2 arcs. We do this for n states, resulting in time $\mathcal{O}(n^3)$

The resulting DFA has 2^n states. For each state S and each $a \in \Sigma$ we compute $\delta(S, a)$ in time $\mathcal{O}(n^3)$. In total, the computation takes $\mathcal{O}(n^3 \cdot 2^n)$ steps, that is, **exponential time**

If we compute δ just for the **reachable** states

- we need to compute $\delta(S, a)$ s times only, with s the number of reachable states
- in total the computation takes $\mathcal{O}(n^3 \cdot s)$ steps

Other conversions

From NFA to DFA : computation takes **exponential time**

From DFA to NFA :

- put set brackets around the states
- computation takes time $\mathcal{O}(n)$, that is, **linear time**

From FA to regular expression via state elimination construction: computation takes **exponential time**

Other conversions

From regular expression to ϵ-NFA :

- construct a tree representing the structure of the regular expression in time $\mathcal{O}(n)$
- at each node in the tree, we build new nodes and arcs in time $\mathcal{O}(1)$ and use **pointers** to previously built structure, avoiding copying
- grand total time is $\mathcal{O}(n)$, that is, **linear time**

Decision problems

In the problem instances below, languages L and M are expressed in any of the four representations introduced for regular languages

- $L = \emptyset$?
- $w \in L$?
- $L = M$?

Empty language

$L(A) \neq \emptyset$ for FA A if and only if at least one final state is **reachable** from the initial state of A

Algorithm for computing reachable states :

Base The initial state is reachable

Induction If q is reachable and there exists a transition from q to p, then p is reachable

Computation takes time proportional to the number of arcs in A, thus $\mathcal{O}(n^2)$

We already saw this idea in the lazy evaluation for translating NFA into DFA

Empty language

Given a regular expression E, we can decide $L(E) \stackrel{?}{=} \emptyset$ by structural induction

Base

- $E = \epsilon$ or else $E = \boldsymbol{a}$. Then $L(E)$ is non-empty
- $E = \emptyset$. Then $L(E)$ is empty

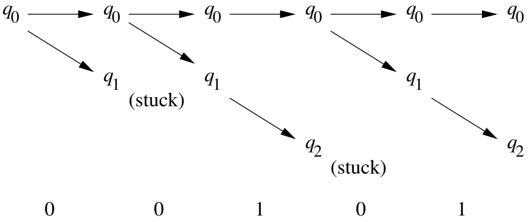
Induction

- $E = F + G$. Then $L(E)$ is empty if and only if both $L(F)$ and $L(G)$ are empty
- $E = F.G$. Then $L(E)$ is empty if and only if either $L(F)$ or $L(G)$ are empty
- $E = F^*$. Then $L(E)$ is not empty, since $\epsilon \in L(E)$

Language membership

We can test $w \in L(A)$ for DFA A by simulating A on w. If $|w| = n$ this takes $\mathcal{O}(n)$ steps

If A is an NFA with s states, simulating A on w requires $\mathcal{O}(n \cdot s^2)$ steps



Language membership

If A is an ϵ-NFA with s states, simulating A on w requires $\mathcal{O}(n \cdot s^3)$ steps

Alternatively, we can pre-process A by calculating ECLOSE(p) for s states, in time $\mathcal{O}(s^3)$. Afterwards, the simulation of each symbol a from w is carried out as follows

- from the current states, find the successor states under a in time $\mathcal{O}(s^2)$
- compute the ϵ-closure for the successor states in time $\mathcal{O}(s^2)$

This takes time $\mathcal{O}(n \cdot s^2)$

Language membership

If $L = L(E)$, for some regular expression E of length s, we first convert E into an ϵ-NFA with $2s$ states. Then we simulate w on this automaton, in $\mathcal{O}(n \cdot s^3)$ steps

Language membership

We can convert an NFA or an ϵ-NFA into a DFA, and then simulate the input string in time $\mathcal{O}(n)$

The time required by the conversion could be **exponential** in the size of the input FA

This method is used

- when the FA has small size
- when one needs to process several strings for membership with the same FA

Equivalent states

Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and let $p, q \in Q$. We define

$$p \equiv q \;\Leftrightarrow\; \forall w \in \Sigma^* \;:\; \hat{\delta}(p, w) \in F \text{ if and only if } \hat{\delta}(q, w) \in F$$

In words, we require p, q to have equal response to input strings, with respect to acceptance

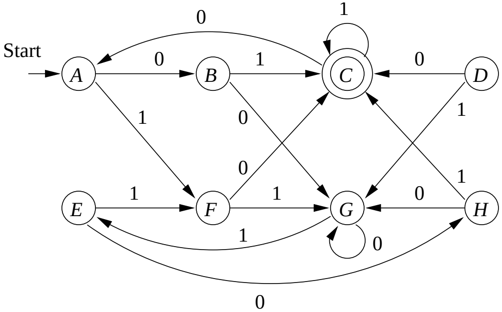
If $p \equiv q$ we say that p and q are **equivalent** states

If $p \not\equiv q$ we say that p and q are **distinguishable** states

Equivalently : p and q are distinguishable if and only if

$$\exists w \;:\; \hat{\delta}(p, w) \in F \text{ and } \hat{\delta}(q, w) \notin F, \text{ or the other way around}$$

Example



$\hat{\delta}(C, \epsilon) \in \mathcal{F},\ \hat{\delta}(G, \epsilon) \notin \mathcal{F} \ \Rightarrow \ C \not\equiv G$ ($\mathcal{F}$ finale states)

$\hat{\delta}(A, 01) = C \in \mathcal{F},\ \hat{\delta}(G, 01) = E \notin \mathcal{F} \ \Rightarrow \ A \not\equiv G$

Example

We prove $A \equiv E$

$\hat{\delta}(A, 1) = F = \hat{\delta}(E, 1)$. Thus $\hat{\delta}(A, 1x) = \hat{\delta}(E, 1x) = \hat{\delta}(F, x)$, $\forall x \in \{0, 1\}^*$

$\hat{\delta}(A, 00) = G = \hat{\delta}(E, 00)$. Thus $\hat{\delta}(A, 00x) = \hat{\delta}(E, 00x) = \hat{\delta}(G, x)$, $\forall x \in \{0, 1\}^*$

$\hat{\delta}(A, 01) = C = \hat{\delta}(E, 01)$. Thus $\hat{\delta}(A, 01x) = \hat{\delta}(E, 01x) = \hat{\delta}(C, x)$, $\forall x \in \{0, 1\}^*$

State equivalence algorithm

We can compute distinguishable state pairs using the following recursive relation

Base If $p \in F$ and $q \notin F$, then $p \not\equiv q$

Induction If $\exists a \in \Sigma \, : \, \delta(p, a) \not\equiv \delta(q, a)$, then $p \not\equiv q$

We compute distinguishable states by backward propagation

State equivalence algorithm

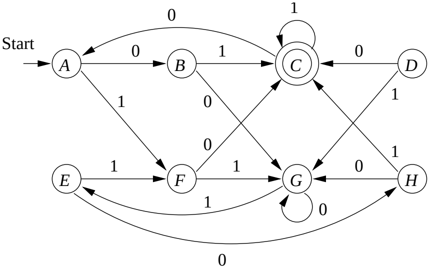
Apply the recursive relation using an **adjacency table** and the following dynamic programming algorithm

- initialize table with pairs that are distinguishable by string ϵ
- for all not yet visited pairs, try to distinguish them using one symbol string: if you reach a pair of **already** distinguishable states, then update table
- iterate until no new pair can be distinguished

Example

$$\exists a \in \Sigma \;:\; \delta(p, a) \not\equiv \delta(q, a)$$
$$\Rightarrow p \not\equiv q$$

	A	B	C	D	E	F	G
B	x						
C	x	x					
D	x	x	x				
E		x	x	x			
F	x	x	x		x		
G	x	x	x	x	x	x	
H	x		x	x	x	x	x



Correctness

Theorem If p and q are not distinguished by the algorithm, then $p \equiv q$

Proof
Suppose to the contrary that there is a *bad pair* $\{p, q\}$ such that

- $\exists w : \hat{\delta}(p, w) \in F, \ \hat{\delta}(q, w) \notin F$, or the other way around
- the algorithm does not distinguish between p and q

Each bad pair can be distinguished by some string w

We choose the bad pair p, q with the shortest distinguishing string w. Let $w = a_1 a_2 \cdots a_n$

Correctness

Now $w \neq \epsilon$, since otherwise the algorithm would distinguish p from q at the basis step. Thus $n \geqslant 1$

Let us consider states $r = \delta(p, a_1)$ and $s = \delta(q, a_1)$

r, s cannot be a bad pair, otherwise r, s would be identified by a string shorter than w

therefore the algorithm must have correctly discovered that r and s are distinguishable. But then the algorithm would distinguish p from q in the inductive part

We conclude that there are no bad pairs, and the theorem holds true □

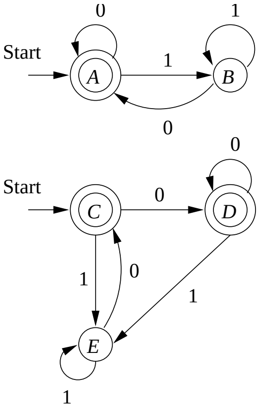
Regular language equivalence

Let L and M be regular languages (specified by means of some representation)

To test $L \stackrel{?}{=} M$:

- convert L and M representations into DFAs
- construct the union DFA (never mind if there are two start states)
- apply state equivalence algorithm
- if the two start states are distinguishable, then $L \neq M$, otherwise $L = M$

Example



Example

The state equivalence algorithm produces the table

B	x			
C		x		
D		x		
E	x		x	x
	A	B	C	D

We have $A \equiv C$, thus the two DFAs are equivalent

Both DFAs recognize language $L(\epsilon + (\mathbf{0} + \mathbf{1})^*\mathbf{0})$

DFA minimization

Important application of the equivalence algorithm : given DFA as input, produces equivalent DFA with **minimum number of states**

Minimal DFA is **unique**, up to renaming of the states

Idea :

- eliminate states that are unreachable from the initial state
- merge equivalent states into an individual state

Example

B	x						
C	x	x					
D	x	x	x				
E		x	x	x			
F	x	x	x		x		
G	x	x	x	x	x	x	
H	x		x	x	x	x	x
	A	B	C	D	E	F	G

State partition based on the equivalence relation :
$\{\{A, E\}, \{B, H\}, \{C\}, \{D, F\}, \{G\}\}$

Example

B	x			
C		x		
D		x		
E	x		x	x
	A	B	C	D

State partition based on the equivalence relation : $\{\{A, C, D\}, \{B, E\}\}$

Transitivity

Theorem If $p \equiv q$ and $q \equiv r$, then $p \equiv r$

Proof

Suppose to the contrary that $p \not\equiv r$

- Then $\exists w$ such that $\hat{\delta}(p, w) \in F$ and $\hat{\delta}(r, w) \notin F$ or the other way around
- *Case* 1 : $\hat{\delta}(q, w)$ is accepting. Then $q \not\equiv r$
- *Case* 2 : $\hat{\delta}(q, w)$ is not accepting. Then $p \not\equiv q$

Therefore it must be that $p \equiv r$ $\square$

Relation $\equiv$ is reflexive, symmetric and transitive : thus $\equiv$ is an **equivalence relation**

We can talk about equivalence classes

DFA minimization

To minimize DFA $A = (Q, \Sigma, \delta, q_0, F)$, construct DFA $B = (Q/_{\equiv}, \Sigma, \gamma, q_0/_{\equiv}, F/_{\equiv})$, where

- elements of $Q/_{\equiv}$ are the equivalence classes of $\equiv$
- elements of $F/_{\equiv}$ are the equivalence classes of $\equiv$ composed by states from F
- $q_0/_{\equiv}$ is the set of states that are equivalent to q_0
- $\gamma(p/_{\equiv}, a) = \delta(p, a)/_{\equiv}$

DFA minimization

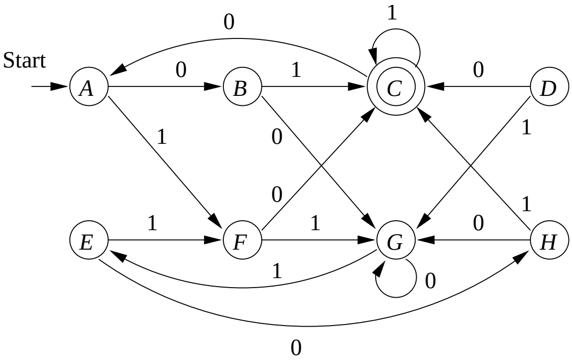
In order for B to be well defined we have to show that

$$\text{If } p \equiv q \text{ then } \delta(p, a) \equiv \delta(q, a)$$

If $\delta(p, a) \not\equiv \delta(q, a)$, then the equivalence algorithm would conclude that $p \not\equiv q$. Thus B is well defined

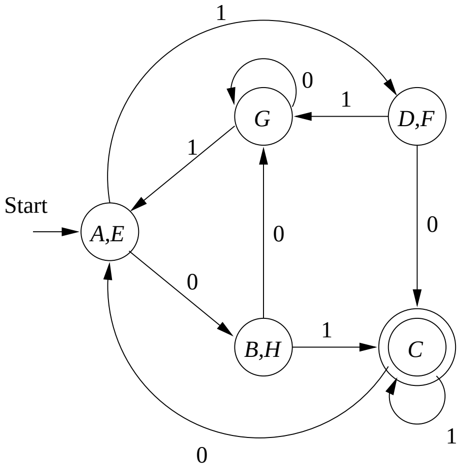
Example

Minimize



Example

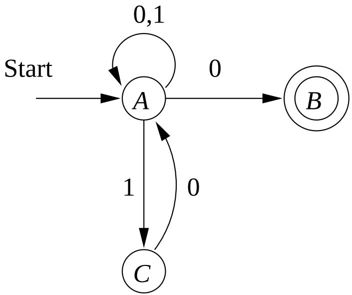
We obtain



Automata minimization

We **cannot** apply the algorithm to NFAs

Example : To minimize



we simply remove state C. However, $A \not\equiv C$