Exercises I4

Example 4.1

(a) is there a sufficient statistic for θ?

I use the normalization property of the pdf:

$1 = \sum_{x=1}^{\infty} c_\theta \theta^x = c_\theta(\frac{1}{1-\theta} - 1)$

and solving the equation we get $f(x;\theta) = (1-\theta)\theta^{x-1}$ a geometric rv with parameter $1-\theta$.

$f(\vec{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) = (1-\theta)^n \theta^{\sum_{i=1}^{n} x_i} \theta^{-n}$

if we choose $h(x) = 1$ and $g(\overline{x};\theta) = \frac{(1-\theta)^n}{\theta^n}\theta^{n\overline{x}}$

we see how $f(\vec{x};\theta) = h(x)g(\overline{x};\theta)$ which means that $\overline{x}$ is a sufficient statistic because of the factorisation criterion. $\square$

(c) method of moments and asymptotic bias:

Since X_i is a geometric rv with parameter $1-\theta$ we have that $\mathbb{E}[X] = \frac{1}{1-\theta}$, solving the equation for the 1st moment we find:

$\hat{\theta}_{MM} = \frac{\overline{X}-1}{\overline{X}}$

The asymptotic distribution of $\overline{X}$ can be computed using CLT

$\frac{\overline{X}-\mathbb{E}[X_i]}{\sqrt{\frac{\text{var}(X_i)}{n}}} \dot{\sim} \mathcal{N}(0,1)$ we get that $\overline{X} \dot{\sim} \mathcal{N}(\frac{1}{1-\theta}, \frac{\theta}{n(1-\theta)^2})$

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It is not an unbiased estimator because $\mathbb{E}_\theta[\hat{\theta}_{MM}]$

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(d) MLE

$L(\theta) = \frac{(1-\theta)^n}{\theta^n}\theta^{\sum\limits_{i=1}^{n} x_i} = (1-\theta)^n\theta^{n(\overline{x}-1)}$
$L'(\theta) = -n(1-\theta)^{n-1}\theta^{n(\overline{x}-1)} + (1-\theta)^n n(\overline{x}-1)\theta^{n(\overline{x}-1)}$
we solve $L'(\hat{\theta}) = 0$ and get $\hat{\theta} = \frac{\overline{x}-1}{\overline{x}}$, the same estimator obtained by the method of moments.

For the sample $(8, 2, 3, 1)$ $\overline{x} = \frac{8+2+3+1}{4} = 3.5$, $\hat{\theta} = 0.71428$

.............................distribution TODO........

(e) The answer is the same as in (d) because the sample is iid.

Example 4.2

$f(x, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$ pdf of X_i

$L(\sigma) = \frac{1}{\sigma^n(2\pi)^{n/2}}e^{-\frac{\sum\limits_{i=1}^{n} x_i^2}{2\sigma^2}}$

$\ell(\sigma) = \ln(\frac{1}{\sigma^n(2\pi)^{\frac{n}{2}}}) - \frac{\sum\limits_{i=1}^{n} x_i^2}{2\sigma^2} =$

$= -n\ln(\sigma(2\pi)^{\frac{1}{2}}) - \frac{\sum\limits_{i=1}^{n} x_i^2}{2\sigma^2} =$

$= -n\ln(\sigma) - (2\pi)^{\frac{1}{2}} - \frac{\sum\limits_{i=1}^{n} x_i^2}{2\sigma^2}$

$\ell'(\sigma) = \frac{-n}{\sigma} + \frac{\sum\limits_{i=1}^{n} x_i^2}{\sigma^3}$

solve $\ell'(\hat{\sigma}) = 0$ we get

$\hat{\sigma} = \sqrt{\frac{\sum\limits_{i=1}^{n} x_i^2}{n}}$

and since MLE is `equivariant` then $\hat{\psi} = \ln(\hat{\sigma})$.

Example 4.3

(a) MLE of $\psi = \frac{\sigma}{\mu}$ and its distribution:

$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ pdf of X_i

$L(\mu, \sigma) = \frac{1}{\sigma^n (2\pi)^{n/2}} e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$

$\ell(\mu, \sigma) = -n \ln(\sigma\sqrt{2\pi}) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$

$$\nabla\ell(\mu, \sigma) = [\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2}, \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3}]^\top$$

Solving $\nabla\ell(\mu, \sigma) = \vec{0}$ we get:

$\begin{cases} \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} \\ \hat{\sigma} = \overline{x^2} - \overline{x}^2 \end{cases}$

and for the `equivariant` property $\hat{\psi} = \frac{\hat{\sigma}}{\hat{\mu}} = \frac{\overline{x^2} - \overline{x}^2}{\overline{x}}$

..............distribution...................

(b) sample size 10, $\overline{x} = 1.5$ and $\sum_{i=1}^{n}(x_i - \overline{x})^2 = 3$

provide an approximation for $P(\hat{\psi} > 1 + \psi)$

$\hat{\mu} = 1.5, \hat{\sigma} = \frac{3}{n} = 0.3 \rightarrow \hat{\psi} = 0.2$

$P(\hat{\psi} > 1 + \psi) = P(\psi < -0.8)$

...... need the distribution.............

Example 4.4

$X_1, \ldots, X_n$ iid sample with $X_i \sim \mathcal{N}(\theta, \theta^2)$ with $\theta > 0$

(a) sufficient statistic for θ is $T = [\overline{x^2}, \overline{x}]$

$$f(\vec{x}, \theta) = \frac{1}{\theta^n (2\pi)^{n/2}} e^{-\frac{\sum\limits_{i=1}^{n} (x_i - \theta)^2}{2\theta^2}} =$$

$$= \frac{1}{\theta^n (2\pi)^{n/2}} e^{-\frac{\sum\limits_{i=1}^{n} (x_i^2 + \theta^2 - 2x_i\theta)}{2\theta^2}} =$$

$$= \frac{1}{\theta^n (2\pi)^{n/2}} e^{-\frac{n\overline{x^2} + n\theta^2 - 2n\theta\overline{x}}{2\theta^2}}$$

and for the likelihood factorisation criterion T is sufficient

(b) MLE of θ:

$f(x, \theta) = \frac{1}{\theta\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\theta^2}}$ pdf of X_i

$$L(\theta) = \frac{1}{\theta^n (2\pi)^{n/2}} e^{-\frac{\sum\limits_{i=1}^{n} (x_i - \theta)^2}{2\theta^2}}$$

$$\ell(\theta) = -n \ln\left(\theta\sqrt{2\pi}\right) - \frac{\sum\limits_{i=1}^{n} (x_i - \theta)^2}{2\theta^2}$$

$$= -n \ln\left(\theta\sqrt{2\pi}\right) - \frac{\sum\limits_{i=1}^{n} (x_i^2 + \theta^2 - 2x_i\theta)}{2\theta^2} =$$

$$= -n \ln\left(\theta\sqrt{2\pi}\right) - n\overline{x^2} - \frac{-n}{2} + \frac{n\overline{x}}{\theta}$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = -\frac{n}{\theta} - \frac{n\overline{x}}{\theta^2}$$

We solve $\frac{\partial \ell(\theta)}{\partial \theta} = 0$ and get

$\hat{\theta} = -\overline{x}$

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Example 4.5

$X_1, \dots, X_n$ iid random sample with $X_i \sim \mathrm{U}nif(\theta, 0)$
(i) Distribution of MLE of θ:
$f(x,\theta) = -\frac{1}{\theta}\mathbb{1}_{[\theta,0]}(x)$ pdf of X_i (θ is negative)

$$L(\theta) = \begin{cases} (-1)^n \frac{1}{\theta^n} & \text{if } x_{(1)} \geq \theta \\ 0 & \text{otherwise} \end{cases}$$

I cannot compute the log likelihood since the likelihood is 0 for some values of θ.

$L'(\theta)=)
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Example 4.6

X_1, X_2, X_3 iid sample with $X_i \sim \mathrm{B}er(\theta)$
(a) $S = \sum_{i=1}^{3} X_i$ is sufficient, prove it using the definition:
We need to check that the following $(\star)$ conditional probability does not depend on θ
$(\star) = P_\theta(\vec{X} = \vec{x} | T(\vec{X}) = t(\vec{x})) = \frac{f(\vec{x};\theta)}{q(t(\vec{x});\theta)}$
We have that the pdf of $\vec{X}$ is $f(\vec{x};\theta) = \theta^{x_1+x_2+x_3}(1-\theta)^{3-x_1-x_2-x_3}$
and since S is the sum of 3 independent Bernoulli rvs we have that the pdf of S is $q(x_1+x_2+x_3;\theta) = \binom{3}{x_1+x_2+x_3}\theta^{x_1+x_2+x_3}(1-\theta)^{3-x_1-x_2-x_3}$
which means that $(\star) = \frac{1}{\binom{3}{x_1+x_2+x_3}}$ that doesn't depend on θ. $\square$

(b) Show that $X_1 + 2X_2 + 3X_3$ is not sufficient:
The pdf of $\vec{X}$ is the same as in point (a)
$A \triangleq X_1$ is a Bernoulli rv, $B \triangleq 2X_2$ has a pmf $f_B(b) = \theta^b(1-\theta)^{2-b}$ and $C \triangleq 3X_3$ has a pmf $f_C(c) = \theta^c(1-\theta)^{3-c}$

functions of independent rvs are independent, which means A, B, C are independent rvs.
The rv $T = A + B + C$ takes values in $0, 1, 2, 3, 4, 5, 6$ and we have that:

A+B+C	A	B	C	Pr
0	0	0	0	$(1-\theta)^3$
1	1	0	0	$(1-\theta)^2\theta$
2	0	2	0	$(1-\theta)^2\theta$
3	1	2	0	$(1-\theta)\theta^2$
3	0	0	3	$(1-\theta)^2\theta$
4	1	0	3	$(1-\theta)\theta^2$
5	0	2	3	$(1-\theta)\theta^2$
6	1	2	3	θ^3

which means that the pmf of T is $$

Example 4.7

Compute the method of moments estimator for $X_1, \dots, X_n$ iid random sample with $X_i \sim \text{Unif}(0, \theta)$

Method of moments, we solve: $\mathbb{E}[Y] = \overline{Y}$
$\mathbb{E}[Y] = \frac{\theta}{2} = \overline{Y}$
which means that $\hat{\theta}_{MM} = 2\overline{Y}$
the bias of the method of moments estimator is $b(\theta; \hat{\theta}_{MM}) = \mathbb{E}_\theta(\hat{\theta}_{MM}) - \theta = 2\frac{\theta}{2} - \theta = 0$, it's unbiased.
To compute the MSE we also need $\text{var}(\hat{\theta}_{MM}) = \text{var}(2\overline{Y}) \underbrace{=}_{\text{prop. of } \overline{Y}} \frac{4\sigma^2}{n}$

so $\text{mse}(\theta; \hat{\theta}_{MM}) = \frac{4\sigma^2}{n}$.

For the MLE we have that $\hat{\theta} = Y_{(n)}$ is only asymptotically unbiased in fact:

$f_{Y_{(n)}}(t) = n(F(t))^{n-1} f(t) = n(\frac{t}{\theta})^{n-1} \frac{1}{\theta} \mathbb{1}_{[0,\theta]}$

$E_\theta(\hat{\theta}) = \int\limits_0^\theta t \; f_{Y_{(n)}}(t) \; \text{dt} =$

$= \int\limits_0^\theta n \frac{t^n}{\theta^n} \; \text{dt} =$

$= \frac{n}{n+1} \frac{\theta^{n+1}}{\theta^n} = \frac{n}{n+1} \theta$

$b(\theta; \hat{\theta}) = \mathbb{E}_\theta(\hat{\theta}) - \theta = \frac{n}{n+1}\theta - \theta = \frac{-1}{n-1}$,

$\text{var}(\hat{\theta}) = \text{var}(Y_{(n)}) \underbrace{=}_{\text{prop. of var}} \mathbb{E}[Y_{(n)}^2] - \mathbb{E}[Y_{(n)}]^2 =$

$(\mathbb{E}[Y_{(n)}^2] = \cdots = \frac{n}{n+2}\theta^2) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 =$

$= \theta^2 (\frac{n(n+1)^2 - n^3 - 2n^2}{(n+2)(n+1)^2}) =$
$= \theta^2 (\frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2}) =$
$= \theta^2 (\frac{n}{(n+2)(n+1)^2})$

$\text{mse}(\theta, \hat{\theta}) = \theta^2 (\frac{n}{(n+2)(n+1)^2}) + \frac{1}{(n-1)^2}$

We see that both estimators are consistent because the mse of both goes to 0 for $n \to \infty$

But the MM estimator is unbiased while the MLE is only asymptotically unbiased, also mse of MLE depends on θ while the mse of MM does not.

###TODO controlla se questa parte e' giusta!