UNIVERSITÀ DEGLI STUDI DI PADOVA

Geometric transformations

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Agenda

- Geometric transformations
- Example of geometric transformations expressed in homogeneous coordinates

Transforming pixels

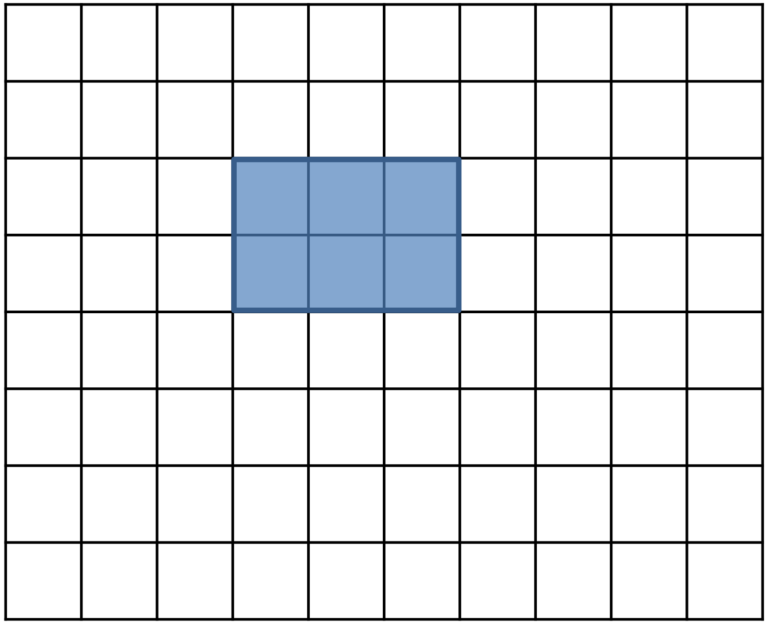
- Image processing: transforming pixels
- We already analyzed several methods for modifying the pixels of an image
  - Guess which ones?

Spatial operations

- Many different ways of transforming an image
- Single-pixel operations
  - Intensity transform, histogram equalization, …
  - The output value of each pixel depends on the pixel initial value
- Local operations
  - Linear and non-linear filters
  - The output value depends on the initial values of the pixel + its neighbors
- Geometric transforms
  - Scaling, rotation, …
  - "Moving" points

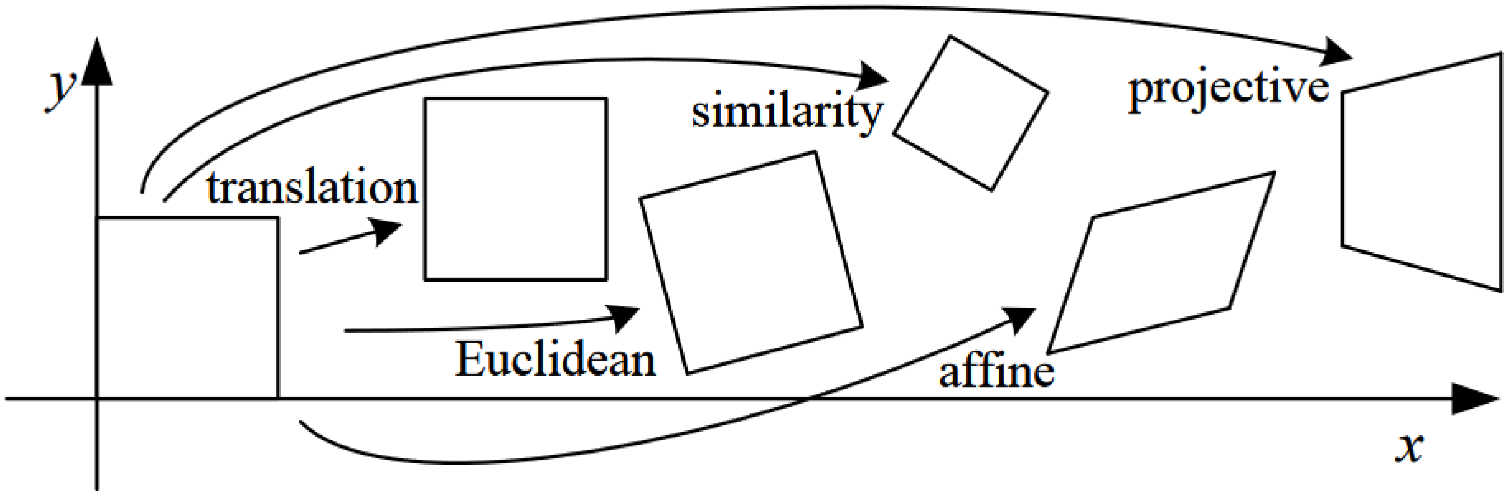
Geometric transforms

- A geometric transform is a modification of the spatial relationship among pixels
- Two steps
  - Coordinate transform $(x', y') = T\{(x, y)\}$
  - Image resampling
- Coord transformations work on **geometrical points**

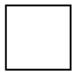
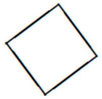
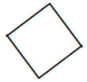
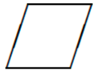
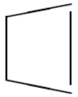


Planar transformations

- Overview of basic planar transformations



Planar transformations hierarchy

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

Planar transformations

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

- How to express a planar transformation?
- Simple example
  - Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Transformed point

Original point

Translation vector

Homogeneous coordinates in 2D

- Recap: points in 2D can be expressed in **homogeneous coordinates**
- To homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{w}x \\ \tilde{w}y \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

- From homogeneous coordinates

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{bmatrix}$$

Translation in hom coords

- Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Translation in hom coords

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Yielding

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + b_1 \\ y + b_2 \\ 1 \end{bmatrix}$$

Planar transformation hierarchy

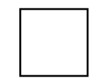
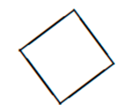
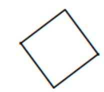
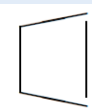
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

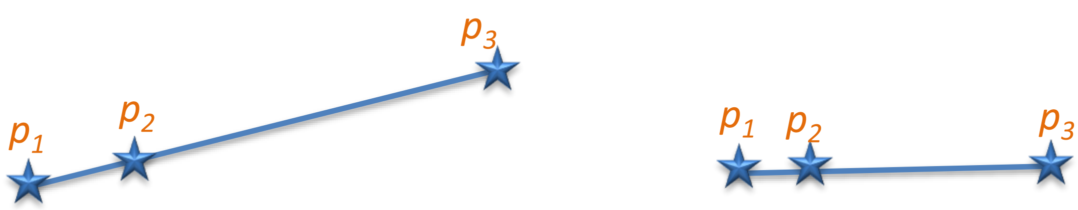
Table 2.1 Hierarchy of 2D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2×3 matrices are extended with a third $[\mathbf{0}^T \; 1]$ row to form a full 3×3 matrix for homogeneous coordinate transformations.

Affine transform

- Affine transform: a more generic transformation
- Linear transform followed by a translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Preserves
  - Point collinearity
  - Distance ratios along a line
    - Given p_1, p_1 and p_1 lying on a line, $\frac{|p_2 - p_1|}{|p_3 - p_2|} = k$ (constant)



Affine transform

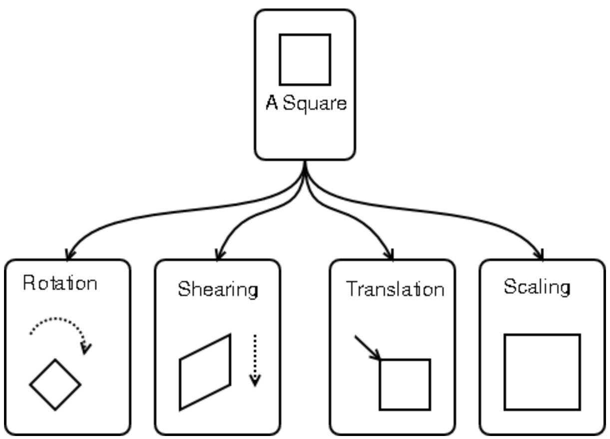
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boxed{A} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \end{matrix}} \end{bmatrix}$$

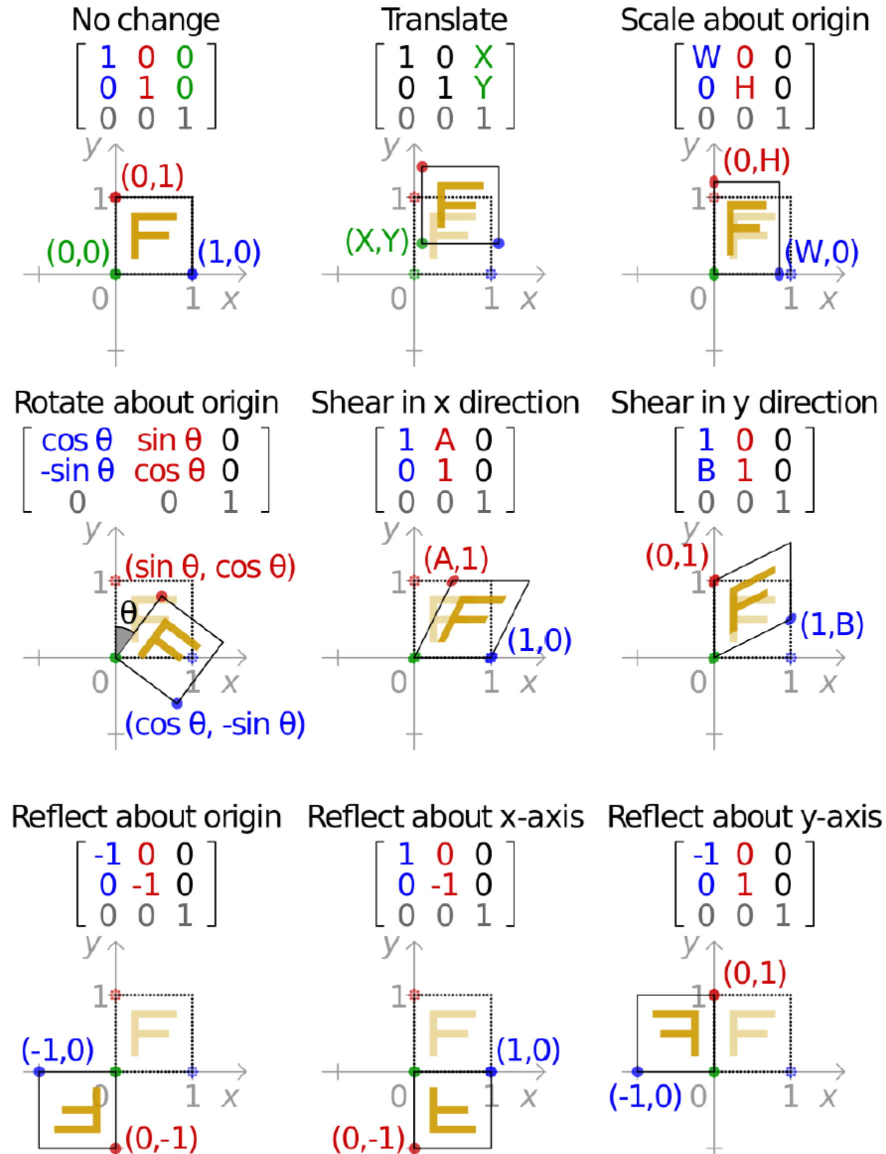
- Homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{matrix}} & \boxed{\begin{matrix} t_{13} \\ t_{23} \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Multiple operations combined into a single matrix multiplication

Affine transforms





Examples

Original

Scaled

Original

Rotated

Original

Horizontal shear

Original

Affine warp

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