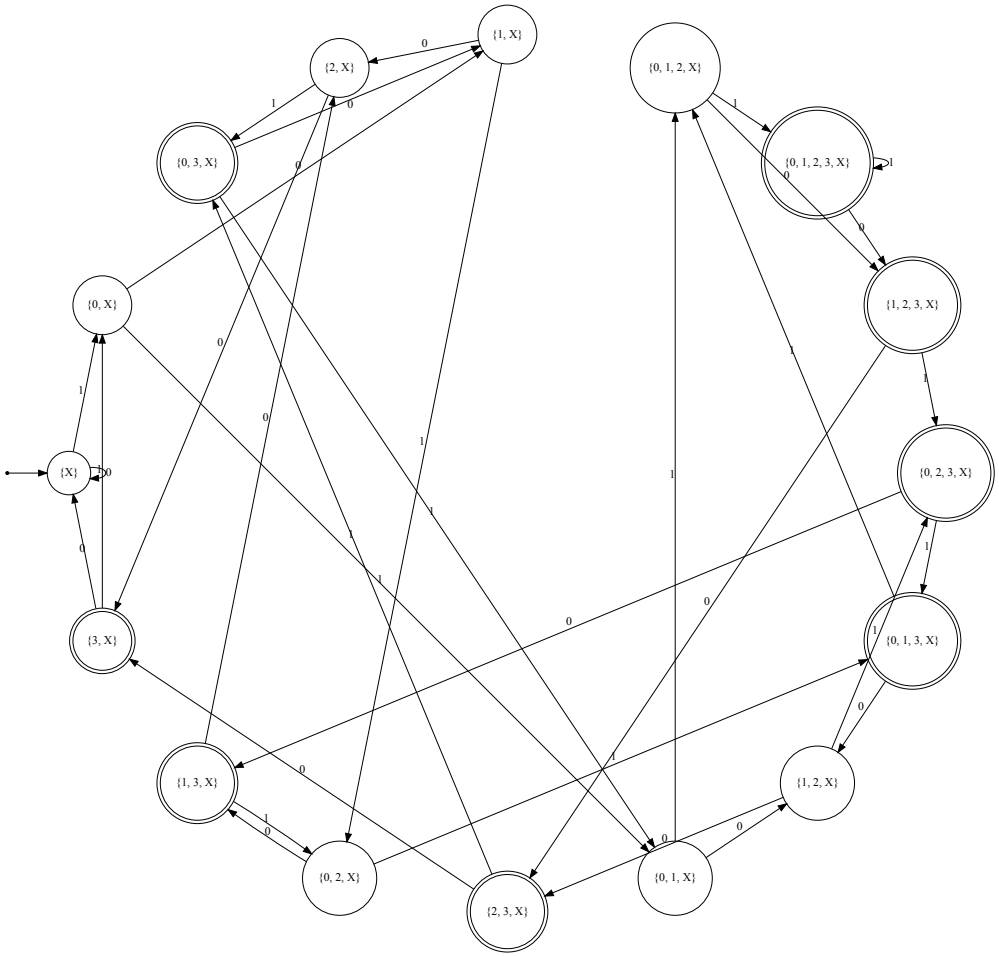
Automata, Languages and Computation

Chapter 2 : Finite Automata

Master Degree in Computer Engineering
University of Padua
Lecturer : Giorgio Satta

Lecture based on material originally developed by :
Gösta Grahne, Concordia University

Finite Automata



1. Deterministic finite automata : this is the simplest and most efficient type of FA
2. Nondeterministic finite automata : FAs with choices activating independent computations
3. Nondeterministic finite automata with ϵ-transitions : nondeterministic automata with special moves that do not consume the input

Deterministic finite automata

These devices read input from left to right

They can only store a quantity of information limited by a constant, using the important notion of **state**

They are easy to implement in a computer (table)

Deterministic finite automata

A **deterministic finite automaton** (DFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where :

- Q is a **finite** set of **states**
- Σ is the input symbol **alphabet**
- δ is a **transition** function $Q \times \Sigma \rightarrow Q$
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is a set of final states

Comment on the notion of determinism

Example

A DFA A that **accepts** the language of all strings of 0 and 1 containing the substring 01 :

$$L = \{x01y \mid x, y \in \{0,1\}^*\}$$

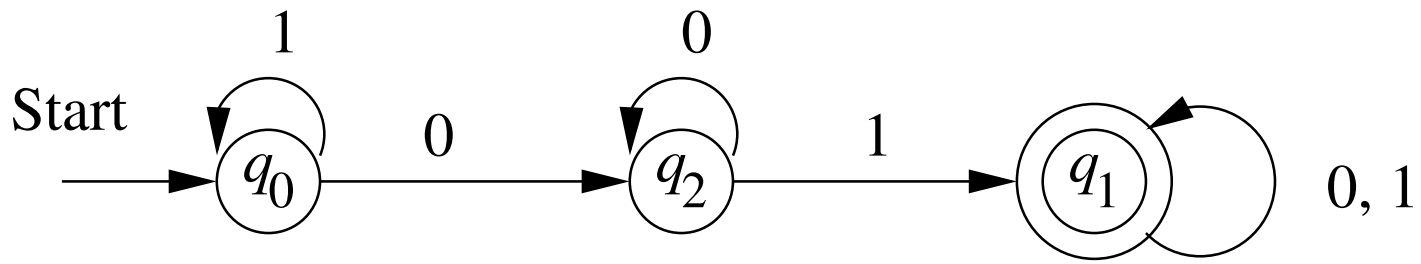
$Q = \{q_0, q_1, q_2\}$, $\Sigma = \{0,1\}$, $F = \{q_1\}$

δ function, specified by means of a **transition table** :

	0	1
$\rightarrow q_0$	q_2	q_0
$\star q_1$	q_1	q_1
q_2	q_2	q_1

Example

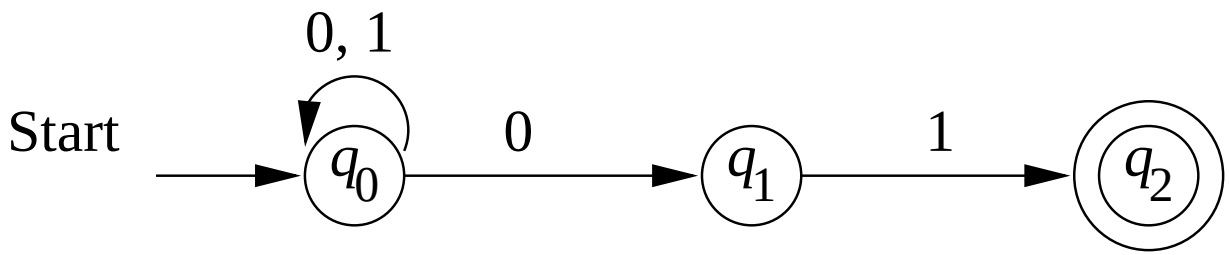
Our DFA A specified by means of a **transition diagram** :



What is the meaning of q_0, q_1, q_2?

Test

Does the following transition diagram correspond to a DFA ?



Acceptance

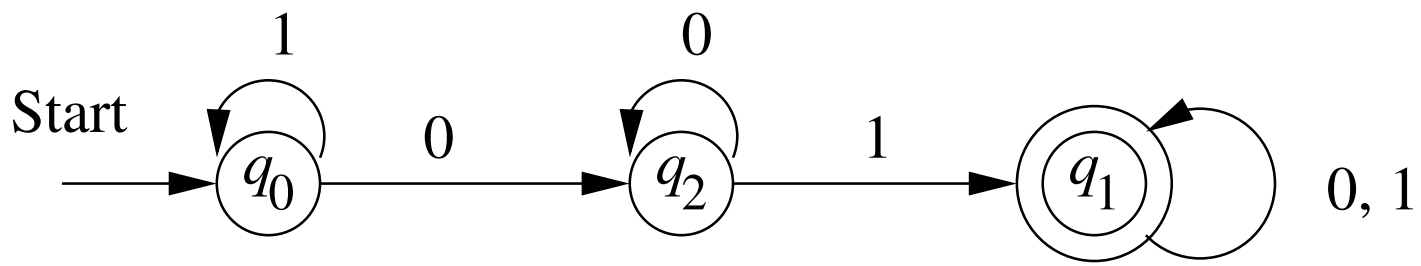
A DFA **accepts** a string $w = a_1 a_2 \cdots a_n$ if there is a path in the transition diagram that

- starts in the initial state
- ends in some final state
- has a sequence of transitions with labels $a_1 a_2 \cdots a_n$

Note that the above **is not** a mathematical definition

Test

Is the string 01101 accepted by the DFA below ?



Extended transition function

The δ transition function can be **extended** to function $\hat{\delta}$ defined over state and string pairs (as opposed to state and alphabet symbol pairs)

Base $\hat{\delta}(q, \epsilon) = q$

Induction $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$

x e' una stringa, a un simbolo

Example

Given a string w over the alphabet Σ and a symbol a, let $\#_a(w)$ denote the number of **occurrences** of a in w

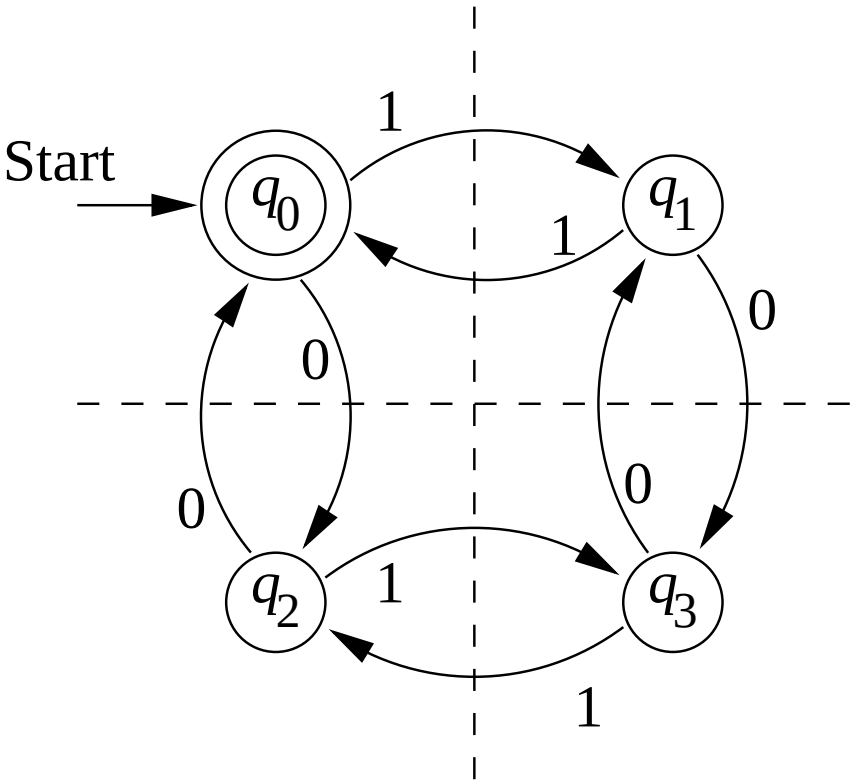
Specify a DFA A accepting all and only the strings in the following language

$$L = \{w \mid w \in \{0,1\}^*,\ \#_0(w) \text{ even},\ \#_1(w) \text{ even}\}$$

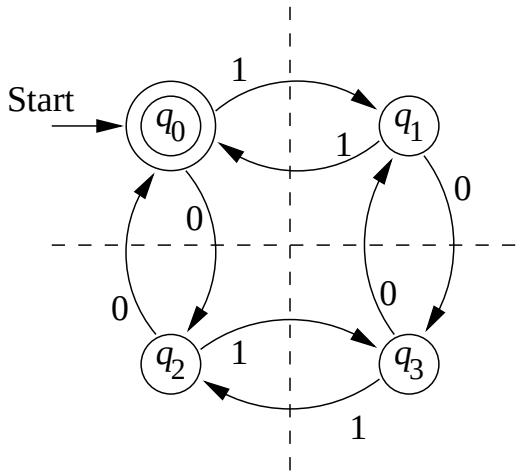
In words, L contains all and only the binary strings with an even number of 0's and an even number of 1's

What is the shortest string in L? Are there strings in L with odd length?

Example



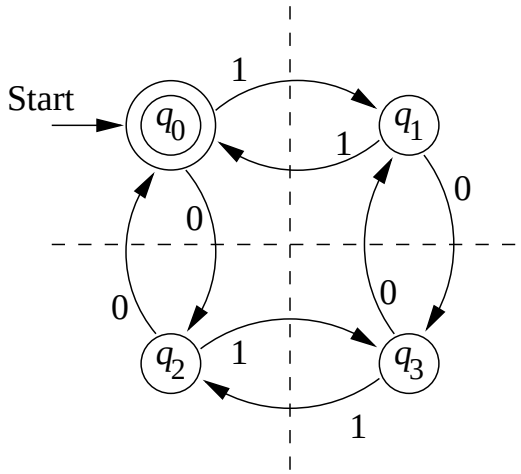
Example



States have the following meaning

- q_0: $\#_0(w)$ and $\#_1(w)$ even
- q_1: $\#_0(w)$ even, $\#_1(w)$ odd
- q_2: $\#_0(w)$ odd, $\#_1(w)$ even
- q_3: $\#_0(w)$ and $\#_1(w)$ odd

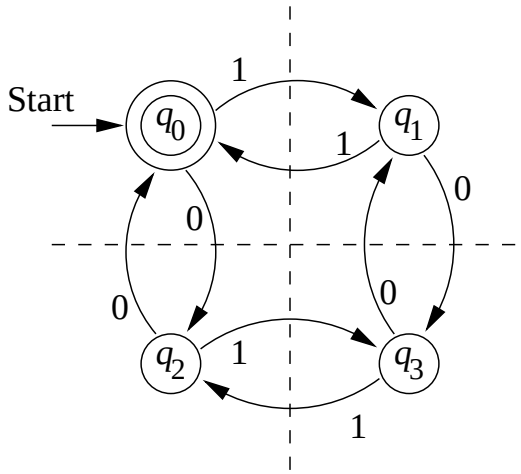
Example



Tabular representation of the DFA

	0	1
$\rightarrow \star q_0$	q_2	q_1
q_1	q_3	q_0
q_2	q_0	q_3
q_3	q_1	q_2

Example



Is string $w = 0101$ accepted by A ?

- $\hat{\delta}(q_0, \epsilon) = q_0$
- $\hat{\delta}(q_0, 0) = \delta(\hat{\delta}(q_0, \epsilon), 0) = \delta(q_0, 0) = q_2$
- $\hat{\delta}(q_0, 01) = \delta(\hat{\delta}(q_0, 0), 1) = \delta(q_2, 1) = q_3$
- $\hat{\delta}(q_0, 010) = \delta(\hat{\delta}(q_0, 01), 0) = \delta(q_3, 0) = q_1$
- $\hat{\delta}(q_0, 0101) = \delta(\hat{\delta}(q_0, 010), 1) = \delta(q_1, 1) = q_0 \in F$

Language recognized by a DFA

The language **recognized** by DFA A is

$$L(A) = \{w \mid \hat{\delta}(q_0, w) \in F\}$$

The languages accepted by the class of DFAs are called **regular languages**

$\{L \mid L = L(A), A \text{ is DFA}\}$

- a language is a set of strings
- a class of languages is a set of languages

Notational conventions

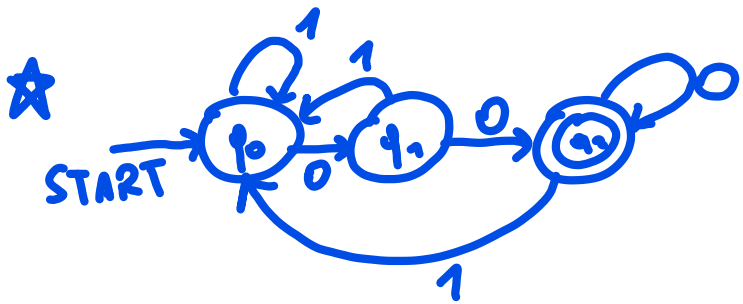
Commonly used notation for DFAs

- $a, b, c, \ldots$ alphabet symbols
- u, v, w, x, y, z strings over input alphabet
- $p, q, r, s, q_0, q_1, q_2, \ldots$ states

Test

Specify DFAs for the following languages over the alphabet $\{0, 1\}$:

- set of all strings ending in 00
- set of all strings with three consecutive 0's
- set of all strings with 011 as a substring
- set of all strings that start or end (or both) with 01



Exercise

Consider the language L of strings over the alphabet $\{0, 1\}$ with **exactly** one occurrence of string 00

Carry out the following points :

- draw the transition diagram of a DFA A such that $L(A) = L$
- state the meaning of each of A's states (i.e. for each state of A describe the strings leading to it)

Hint: define a "failure state" that can never reach any final state

Nondeterministic finite automata

These automata accept only regular languages

Easier to design than DFAs

Later on we will see several examples of this fact

Very useful for implementing the **search** for a pattern in a text

Nondeterministic finite automata

A nondeterministic finite automaton can **simultaneously** be in different states

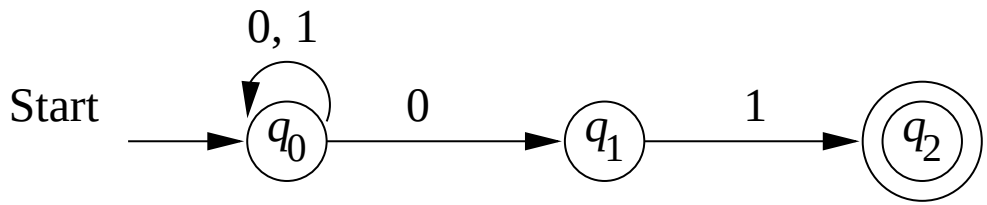
The automaton **accepts** if at least one final state is reached at the end of the scan of the input string

Equivalently, in a given state the automaton can **guess** which next state will lead to acceptance

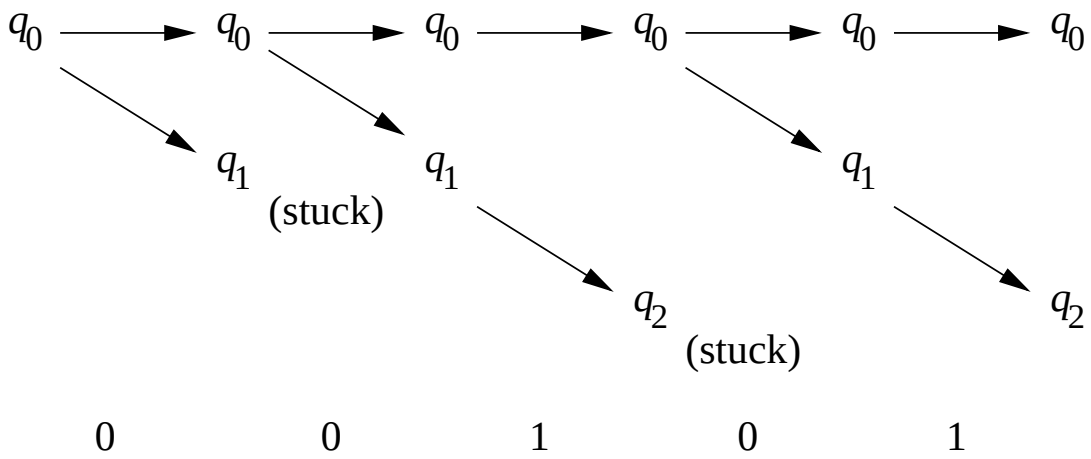
This interpretation is not in the textbook

Example

Nondeterministic automaton N accepting all and only the strings ending in 01



Simultaneous computations of N on input string 00101



Nondeterministic finite automaton

A **nondeterministic finite automata** (NFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where :

- Q is a **finite** set of **states**
- Σ is the **alphabet** of input symbols
- δ is a **transition** function $Q \times \Sigma \rightarrow 2^Q$, where 2^Q is the set of all subsets of Q (power set)
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

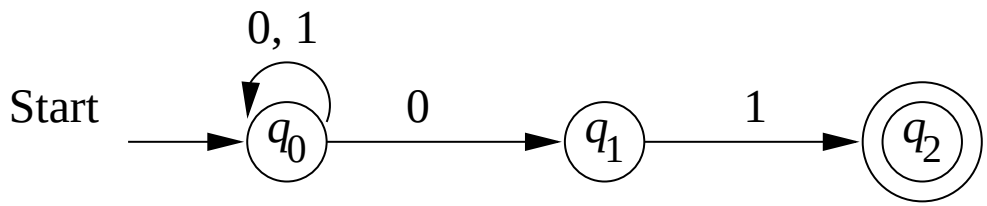
if $|Q| = n$ then $|2^Q| = 2^n$

insieme delle parti

the output of δ is a set, not an element (state) for NFAs

Example

The transition diagram



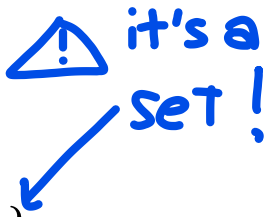
represents the nondeterministic automaton

$$A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

with transition function δ

	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\emptyset$	$\{q_2\}$
$\star q_2$	$\emptyset$	$\emptyset$

Extended transition function $\hat{\delta}$



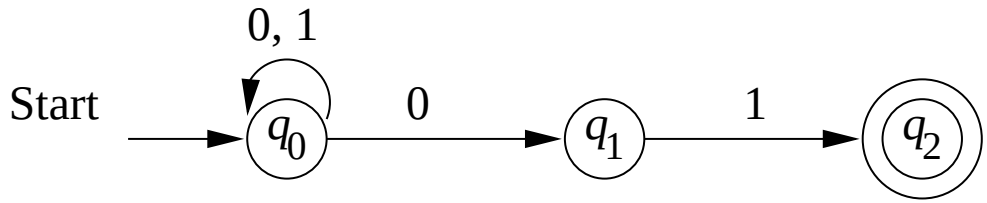
Base $\hat{\delta}(q, \epsilon) = \{q\}$

Induction

$$\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$$

Notice the difference with the case of DFA in the induction part. Can you explain this?

Example



(This automata recognises all strings that end in 01)

Computation of $\hat{\delta}(q_0, 00101)$

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$
- $\hat{\delta}(q_0, 0010) = \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00101) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$

this computation halts

this computation accepts

Accepted language for NFA

The **accepted** language for an NFA A is

set of final states

$$L(A) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset\}$$

In words, $L(A)$ is the set of all strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** final state. This amounts to say that at least one computation for w leads to acceptance

Test

Consider the following language over $\Sigma = \{0, 1\}$

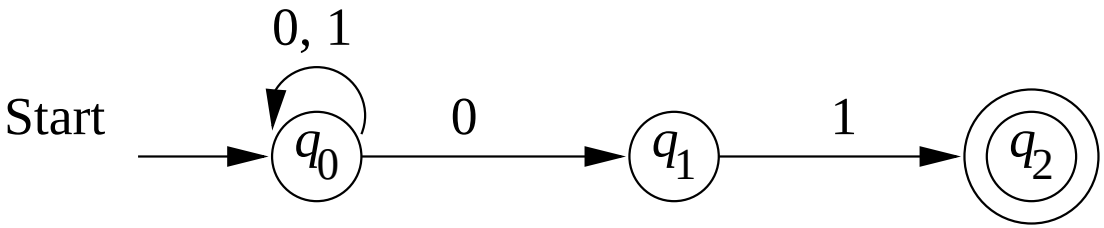
$$L = \{w \mid w = x1ab,\ x \in \Sigma^*,\ a, b \in \Sigma\}$$

Informally, L is the set of all strings with 1 as **third to last** symbol

Specify a NFA A such that $L(A) = L$

Exercise with solution

Show that the NFA



accepts the language $L = \{x01 \mid x \in \Sigma^*\}$

NO, WE SKIPPED IT

We prove the following three statements using **mutual induction**:

(i) $q_0 \in \hat{\delta}(q_0, w) \Leftrightarrow w \in \Sigma^*$

(ii) $q_1 \in \hat{\delta}(q_0, w) \Leftrightarrow w = x0,\ x \in \Sigma^*$

(iii) $q_2 \in \hat{\delta}(q_0, w) \Leftrightarrow w = x01,\ x \in \Sigma^*$

Exercise with solution

Base If $|w| = 0$ then $w = \epsilon$, and statement (i) follows from the definition of $\hat{\delta}$. As for statements (ii) and (iii), both hand sides hold false for ϵ

Induction Assume $w = xa$, with $a \in \{0, 1\}$, $|x| = n$, and assume statements (i)–(iii) hold true for x.

(i) We know that $q_0 \in \hat{\delta}(q_0, x)$. From state q_0 we have transitions to q_0 for both 0 and 1

Exercise with solution

(ii) (*if*) w ends with 0. From (i) we know that $q_0 \in \hat{\delta}(q_0, x)$. There is a transition from q_0 to q_1 on symbol 0. Hence $q_1 \in \hat{\delta}(q_0, w)$
(*only if*) $q_1 \in \hat{\delta}(q_0, w)$. In order to reach q_1, the only possible transition is from q_0 upon reading 0. Thus $w = x0$

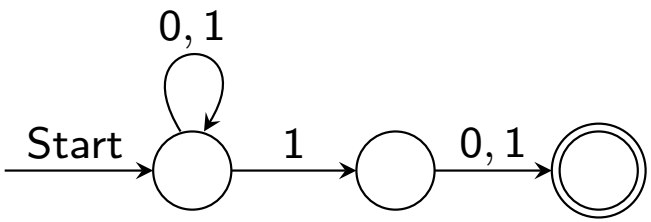
(iii) (*if*) w ends with 01, and thus $a = 1$ and x ends with 0. From (ii) we have $q_1 \in \hat{\delta}(q_0, x)$. From q_1 we can reach q_2 upon reading 1. Then $q_2 \in \hat{\delta}(q_0, w)$
(*only if*) $q_2 \in \hat{\delta}(q_0, w)$. In order to reach q_2, the only possible transition is from q_1 upon reading 1. From statement (ii) for x we have that x ends with 0. We then conclude that w ends with 01 □

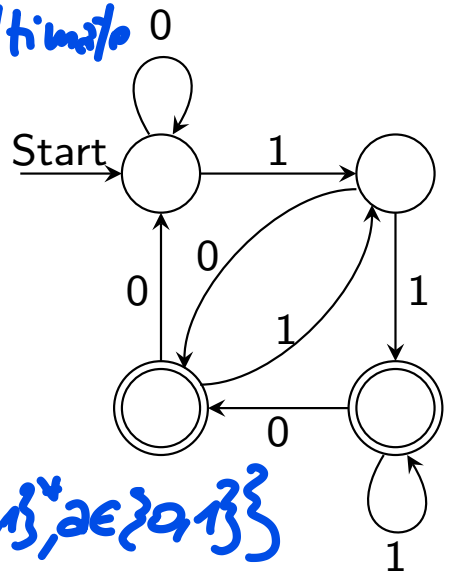
Equivalence for DFA and NFA

NFAs are **easier** than DFAs to "program", since nondeterminism makes it possible to simplify the structure of the automaton

Example : compare NFA and DFA accepting strings in $\{0,1\}^*$ with penultimate symbol 1

recognizes all strings with a 1 as penultimate symbol





$L = \{w \mid w \in \{0,1\}^*, w = x1a, x \in \{0,1\}^*, a \in \{0,1\}\}$

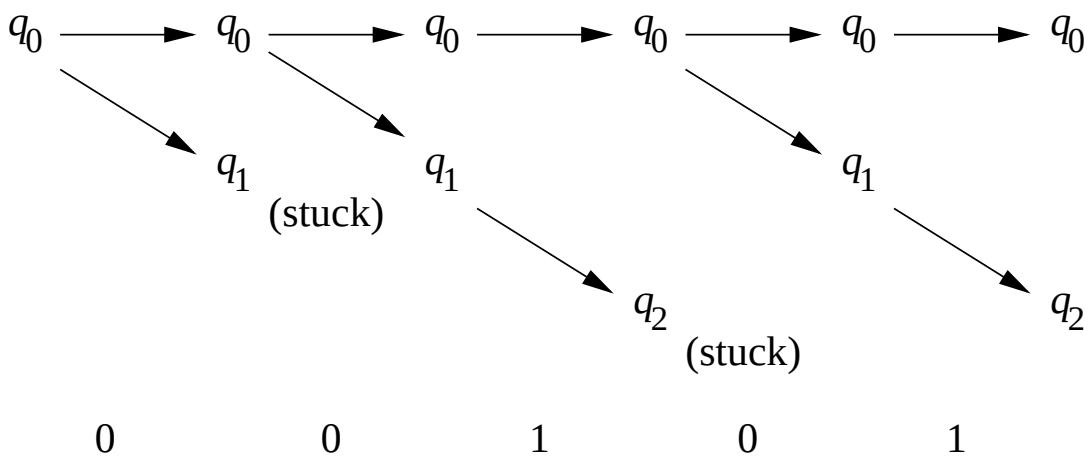
With an increase in the distance between 1 and the end of the string, the gap gets exponentially larger

Equivalence for DFA and NFA

Quite surprisingly, for every NFA N there exists some DFA D such that $L(D) = L(N)$. The proof involves the **subset construction**

Idea : build a state in D for every state set representing a "configuration" in a computation of N. The collection of all configurations is still a **finite set**

this is the automaton that recognizes strings that end in 01



Equivalence for DFA and NFA

Given an NFA

$$N = (Q_N, \Sigma, \delta_N, q_0, F_N)$$

the subset construction produces a DFA

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

such that $L(D) = L(N)$

Equivalence for DFA and NFA

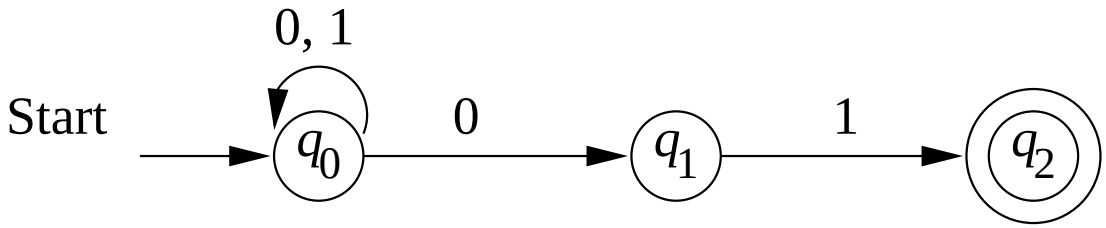
Subset construction :

- $Q_D = \{S \mid S \subseteq Q_N\}$
- $F_D = \{S \subseteq Q_N \mid S \cap F_N \neq \emptyset\}$
- For every $S \subseteq Q_N$ and $a \in \Sigma$,

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$$

Note : $|Q_D| = 2^{|Q_N|}$. Nonetheless, the large majority of states in Q_D turn out to be **garbage**, that is, they cannot be reached from the initial state

Example



Construction of δ_D :

	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$
$\star\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\star\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\star\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
$\star\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

some states are unreachable (e.g. $\{q_1\}, \{q_1, q_2\} \ldots$)

Example

Note : D states correspond to subsets of N states, but we could denote D states in any other way, for instance using the letters $A, B, \ldots, F$

	0	1
A	A	A
$\rightarrow B$	E	B
C	A	D
$\star D$	A	A
E	E	F
$\star F$	E	B
$\star G$	A	D
$\star H$	E	F

Equivalence for DFA and NFA

We can often avoid **exponential growth** of states in Q_D using a technique called **lazy evaluation** (or deferred evaluation)

State q of DFA A is **accessible** if there is at least one string w such that $\hat{\delta}_A(q_0, w) = q$

We build the transition table of D **only** for the accessible states of D

Equivalence for DFA and NFA

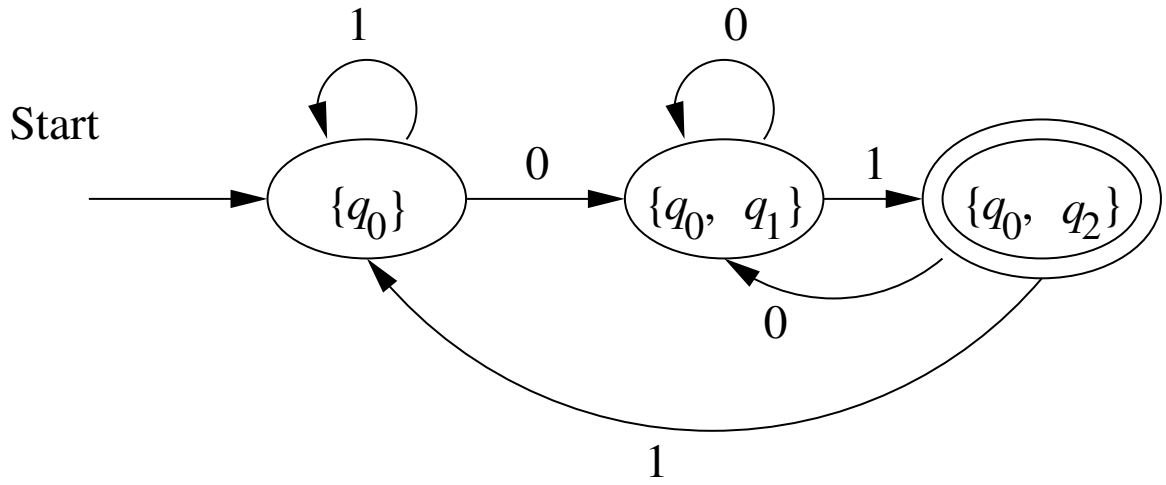
Construction of DFA D through lazy evaluation

Base $S = \{q_0\}$ is accessible in D

Induction If state S is accessible in D, then state $\delta_D(S, a)$ is also accessible in D, for every $a \in \Sigma$

Example

DFA D with only accessible states



In several **practical applications** D has about as many states as N

Equivalence for DFA and NFA

⚠ Important proof

Theorem Let D be the DFA obtained from an NFA N using the subset construction. Then $L(D) = L(N)$

Proof We first prove that, for every string $w \in \Sigma^*$, we have

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$$

← this is more general then $L(D)=L(N)$ so if it holds $\Rightarrow L(D)=L(N)$ holds

Check that both sides in the above equation are sets!

We use induction on $|w|$

Base $w = \epsilon$. The claim follows from the definition

Equivalence for DFA and NFA

Induction

$$\begin{aligned}
\hat{\delta}_D(\{q_0\}, xa) &= \delta_D(\hat{\delta}_D(\{q_0\}, x), a) && \text{definition of } \hat{\delta}_D \\
&= \delta_D(\hat{\delta}_N(q_0, x), a) && \text{induction} \\
&= \bigcup_{p \in \hat{\delta}_N(q_0, x)} \delta_N(p, a) && \text{definition of } \delta_D \\
&= \hat{\delta}_N(q_0, xa) && \text{definition of } \hat{\delta}_N
\end{aligned}$$

$L(D) = L(N)$ now follows from the definition of F_D $\square$

Equivalence for DFA and NFA

Theorem A language L is accepted by a DFA if and only if L is accepted by an NFA

Proof (If) Previous theorem

(Only if) Any DFA can be converted into an equivalent NFA by modifying δ_D into δ_N according to the following rule

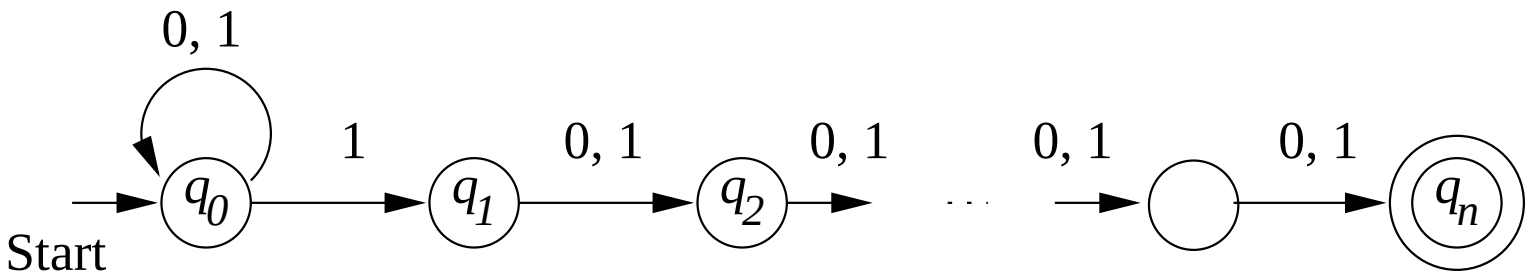
$$\textit{If } \delta_D(q, a) = p, \textit{ then } \delta_N(q, a) = \{p\}$$

By induction on $|w|$ one can show that $\hat{\delta}_D(q_0, w) = p$ if and only if $\hat{\delta}_N(q_0, w) = \{p\}$ □

Exponential growth of the state set

Theorem There exists an NFA N with $n+1$ states that has no equivalent DFA with less than 2^n states

Proof Let N be the NFA



$$L(N) = \{x1c_2c_3 \cdots c_n \mid x \in \{0,1\}^*, c_i \in \{0,1\}\}$$

Intuitively, an equivalent DFA must "remember" the last n symbols it has read

Those symbols might all be relevant for the final decision

Exponential growth of the state set

(proof by contradiction)

Suppose there exists a DFA D equivalent to N with fewer than 2^n states

There are 2^n binary strings of length n. Since D has fewer that 2^n states, there must be

- a state q,
- binary strings $a_1 a_2 \cdots a_n \neq b_1 b_2 \cdots b_n$,

such that

$$\hat{\delta}_D(q_0, a_1 a_2 \cdots a_n) = \hat{\delta}_D(q_0, b_1 b_2 \cdots b_n) = q$$

The above reasoning uses the so-called pigeonhole principle

if two strings (different) end up in the same state then they will have the same options later

Exponential growth of the state set

Since $a_1 a_2 \cdots a_n \neq b_1 b_2 \cdots b_n$, there exists i with $1 \leqslant i \leqslant n$ such that $a_i \neq b_i$; we assume $a_i = 1$ and $b_i = 0$ (the other case being symmetrical)

Case 1: $i = 1$; we have

$$\hat{\delta}_D(q_0, 1a_2 \cdots a_n) \in F$$
$$\hat{\delta}_D(q_0, 0b_2 \cdots b_n) \notin F$$

which is a contradiction

Exponential growth of the state set

Case 2: $i > 1$; since $\hat{\delta}_D(q_0, a_1 a_2 \cdots a_n) = \hat{\delta}_D(q_0, b_1 b_2 \cdots b_n)$ and D is deterministic, we have

$$\hat{\delta}_D(q_0, a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_n 0^{i-1}) =$$
$$\hat{\delta}_D(q_0, b_1 \cdots b_{i-1} 0 b_{i+1} \cdots b_n 0^{i-1})$$

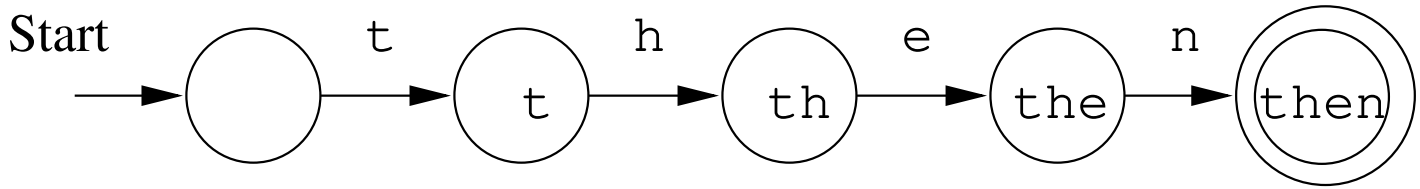
From the definition of L, we must have

$$\hat{\delta}_D(q_0, a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_n 0^{i-1}) \in F$$
$$\hat{\delta}_D(q_0, b_1 \cdots b_{i-1} 0 b_{i+1} \cdots b_n 0^{i-1}) \notin F$$

which is a contradiction □

the important thing is that they are in the same number not that they are the same

Partial DFA



This is not a DFA, since for some symbols in Σ transitions are not specified

A **partial** DFA has **at most** one outgoing transition for each state in Q and for each symbol in Σ

A partial DFA can be completed to a DFA if we add one non-accepting state having the status of a **trap state**, from which you cannot escape

Exercise with solution

Consider the NFA

$$N = (\{q_0, q_1\}, \{0, 1\}, \delta_N, q_0, \{q_1\}),$$

where $\delta_N(q_0, 0) = \{q_0, q_1\}$, $\delta_N(q_0, 1) = \{q_1\}$, $\delta_N(q_1, 0) = \varnothing$, $\delta_N(q_1, 1) = \{q_0, q_1\}$

- check whether strings $w_1 = 101$ and $w_2 = 0010$ are in $L(N)$, showing all steps in the computations
- construct the transition diagram of the DFA equivalent to N
- using a set-former, define the language accepted by the automaton; **suggestion**: this is easier if you look at the DFA

Exercise with solution

$w_1 = 101 \in L(A)$?

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 1) = \delta(q_0, 1) = \{q_1\}$
- $\hat{\delta}(q_0, 10) = \delta(q_1, 0) = \emptyset$, then $w_1 = 101 \notin L(A)$

$w_2 = 0010 \in L(A)$?

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_1\} \cup \{q_0, q_1\} = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 0010) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \emptyset = \{q_0, q_1\}$
- since $\{q_0, q_1\} \cap \{q_1\} \neq \emptyset$, $w_2 = 0010 \in L(A)$

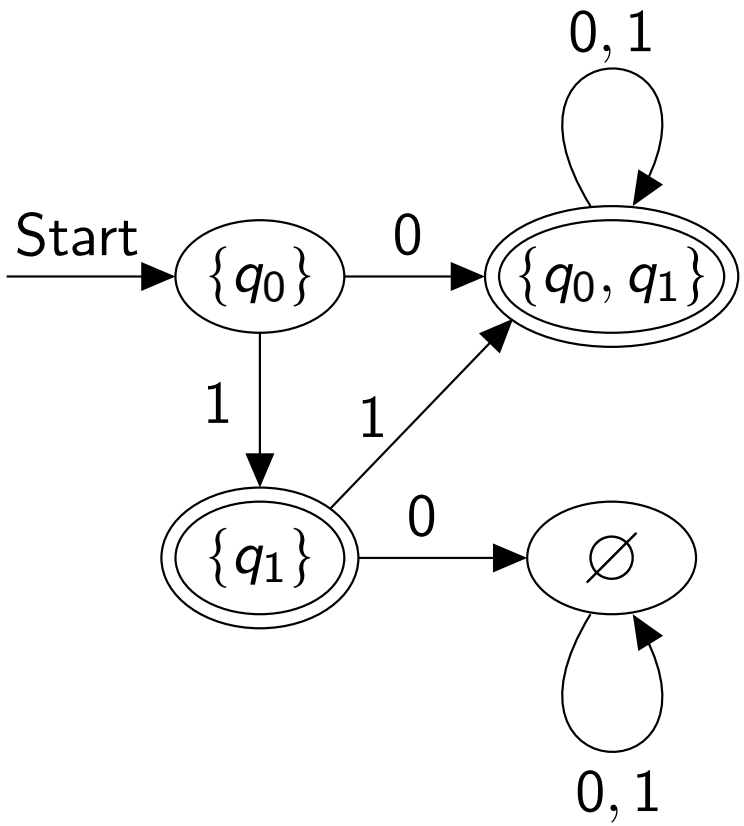
Exercise with solution

We construct the transition diagram of the equivalent DFA using the subset construction and lazy evaluation

- $\delta_D(\{q_0\}, 0) = \delta_N(q_0, 0) = \{q_0, q_1\}$
- $\delta_D(\{q_0\}, 1) = \delta_N(q_0, 1) = \{q_1\}$
- $\delta_D(\{q_0, q_1\}, 0) = \delta_N(q_0, 0) \cup \delta_N(q_1, 0)$
  $= \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\delta_D(\{q_0, q_1\}, 1) = \delta_N(q_0, 1) \cup \delta_N(q_1, 1)$
  $= \{q_1\} \cup \{q_0, q_1\} = \{q_0, q_1\}$
- $\delta_D(\{q_1\}, 0) = \delta_N(q_1, 0) = \varnothing$
- $\delta_D(\{q_1\}, 1) = \delta_N(q_1, 1) = \{q_0, q_1\}$
- $\delta_D(\varnothing, 0) = \delta_D(\varnothing, 1) = \varnothing$
- $\{q_0\}$ initial state, $\{q_0, q_1\}$ e $\{q_1\}$ final states

Exercise with solution

Graphical representation of the transition diagram



Exercise with solution

Using a set-former, define the language accepted by the automaton

$$\begin{aligned} L(A) \quad = \quad & \{w \in \{0,1\}^+ \mid w = 1 \text{ or } w = 0x \\ & \text{or } w = 11y,\ x, y \in \{0,1\}^*\} \end{aligned}$$

Exercises

Specify an NFA A for each of the following languages defined on the alphabet $\{0, 1\}$

- set of strings with two consecutive 0 or two consecutive 1
- set of strings such that at least one of the last three symbols is 1

NFA with ϵ-transitions

Extension of NFAs where transitions labelled with symbol ϵ are allowed; this means that the automaton can change state **without consuming** any of its input

They accept all and only the regular languages

Easier to design than NFAs

ϵ-NFA widely used in compilers and for search of patterns in a text

Example

A fractional number consists of

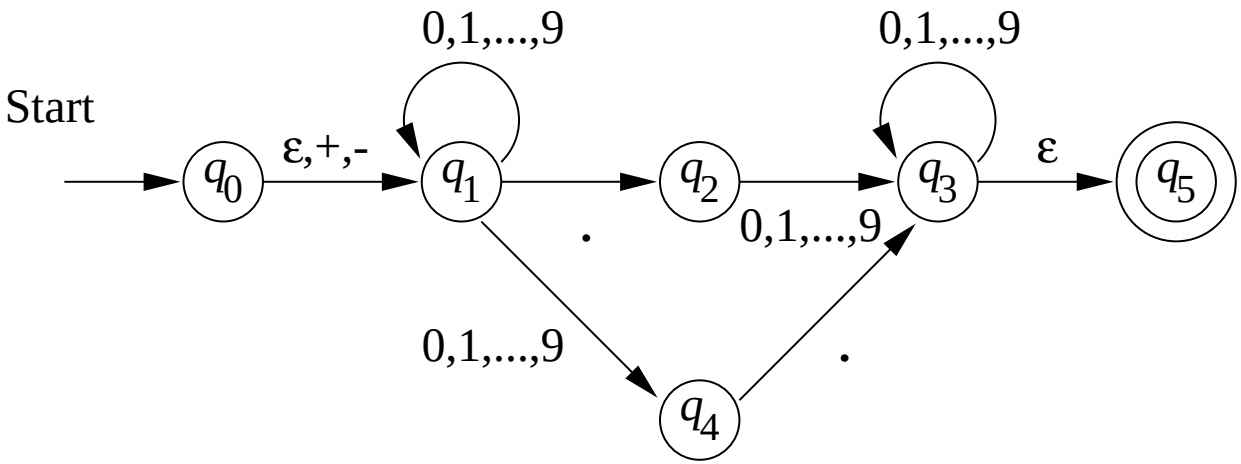
- $+$ or - sign, optional
- a first string of digits
- one decimal point
- a second string of digits

with the first or the second strings **optional**, but not both

This example comes from a lexical analyser in compiler theory

Example

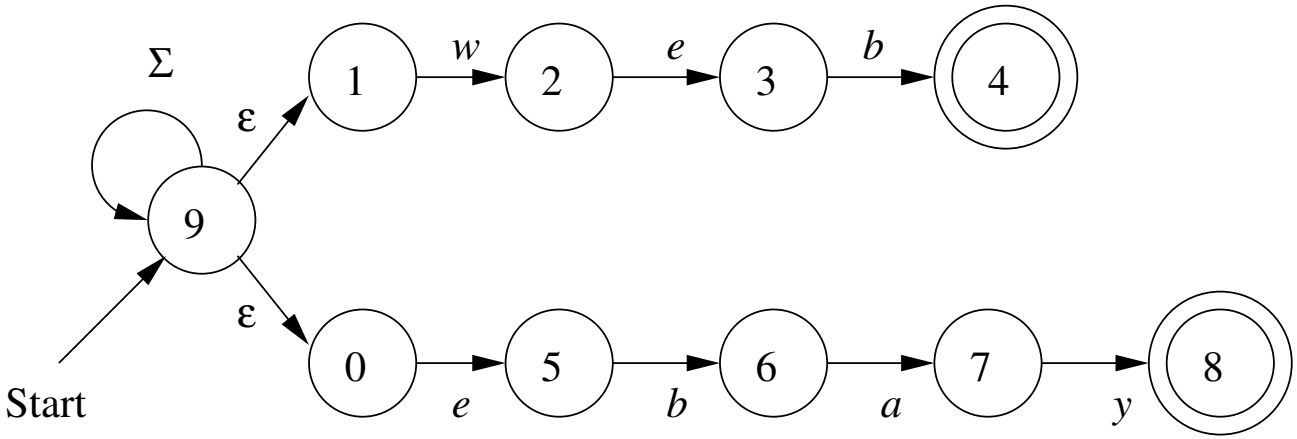
ϵ-NFA accepting fractional numbers



The ϵ-transition makes operators + and - optional

Example

ϵ-NFA accepting set of keywords $\{\texttt{ebay}, \texttt{web}\}$



The ϵ-transition makes it easy to combine several automata

NFA with ϵ-transitions

A **nondeterministic finite automaton with ϵ-transitions** (ϵ-NFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where

- Q, Σ, q_0, and F are defined as for NFAs
- δ is a **transition** function $Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$, with 2^Q denoting the class of subsets of Q

Example

ϵ-NFA accepting fractional numbers

$$E = (\{q_0, q_1, \ldots, q_5\}, \{., +, -, 0, 1, \ldots, 9\}\ \delta, q_0, \{q_5\})$$

Transition function δ

	ϵ	+,-	.	$0, \ldots, 9$
$\rightarrow q_0$	$\{q_1\}$	$\{q_1\}$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\varnothing$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\varnothing$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_3	$\{q_5\}$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_4	$\varnothing$	$\varnothing$	$\{q_3\}$	$\varnothing$
$\star q_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Test

Specify an ϵ-NFA accepting the language of strings over $\{a, b, c\}$ with zero or more a's, followed by zero or more b's, followed by zero or more c's

ϵ-closure

Let us compute the **ϵ-closure** of a state q, written ECLOSE(q), adding all the states reachable from q itself through a sequence of one or more symbols ϵ

Needed later in the definition of $\hat{\delta}$ function

Base $q \in \text{ECLOSE}(q)$

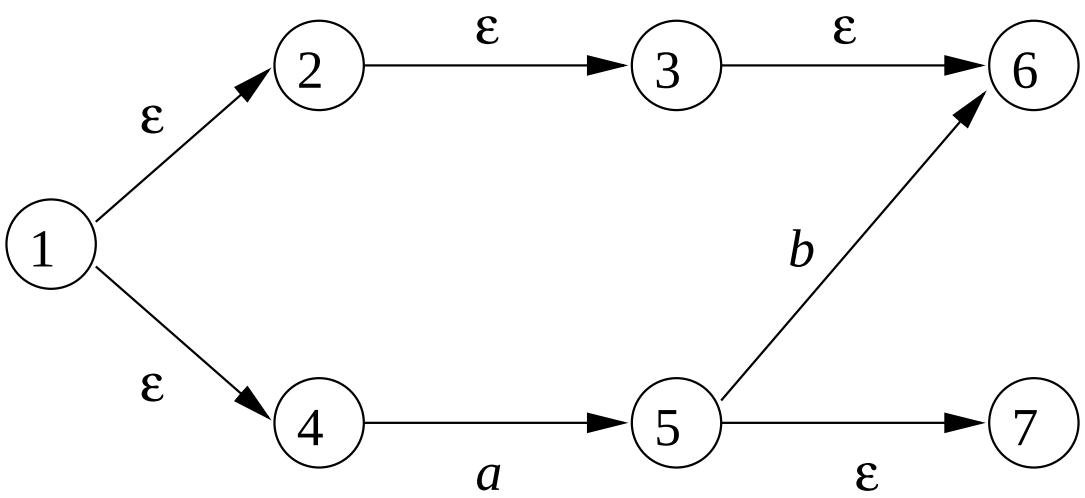
Induction $(p \in \text{ECLOSE}(q) \wedge r \in \delta(p, \epsilon)) \Rightarrow r \in \text{ECLOSE}(q)$

Extension to set of states S

$$\text{ECLOSE}(S) = \bigcup_{q \in S} \text{ECLOSE}(q)$$

Example

ϵ-NFA fragment



$$\text{ECLOSE}(1) = \{1, 2, 3, 4, 6\}$$

$$\text{ECLOSE}(\{4, 5\}) = \{4\} \cup \{5, 7\} = \{4, 5, 7\}$$

Extended transition function $\hat{\delta}$

Base $\hat{\delta}(q, \epsilon) = \text{ECLOSE}(q)$

Induction $\hat{\delta}(q, xa)$ is computed as

- $\{p_1, \ldots, p_k\} = \hat{\delta}(q, x)$
- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta(p_i, a)$
- $\hat{\delta}(q, xa) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

Note that processing of ϵ symbols is accounted for after the processing of each symbol in Σ

Example

We compute $\hat{\delta}(q_0, 5.6)$ for the ϵ-NFA accepting fractional numbers

$\hat{\delta}(q_0, \epsilon) = \text{ECLOSE}(q_0) = \{q_0, q_1\}$

Computation of $\hat{\delta}(q_0, 5)$:

- $\delta(q_0, 5) \cup \delta(q_1, 5) = \varnothing \cup \{q_1, q_4\} = \{q_1, q_4\}$
- $\text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_4) = \{q_1\} \cup \{q_4\} = \{q_1, q_4\} = \hat{\delta}(q_0, 5)$

Example

Computation of $\hat{\delta}(q_0, 5.)$:

- $\delta(q_1, .) \cup \delta(q_4, .) = \{q_2\} \cup \{q_3\} = \{q_2, q_3\}$
- $\text{ECLOSE}(q_2) \cup \text{ECLOSE}(q_3) = \{q_2\} \cup \{q_3, q_5\} = \{q_2, q_3, q_5\} = \hat{\delta}(q_0, 5.)$

Computation of $\hat{\delta}(q_0, 5.6)$:

- $\delta(q_2, 6) \cup \delta(q_3, 6) \cup \delta(q_5, 6) = \{q_3\} \cup \{q_3\} \cup \varnothing = \{q_3\}$
- $\text{ECLOSE}(q_3) = \{q_3, q_5\} = \hat{\delta}(q_0, 5.6)$

Accepted language for ϵ-NFA

The language **accepted** by ϵ-NFA $E = (Q, \Sigma, \delta, q_0, F)$ is

$$L(E) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

From ϵ-NFA to DFA

Given the ϵ-NFA

$$E = (Q_E, \Sigma, \delta_E, q_0, F_E)$$

we construct a DFA

$$D = (Q_D, \Sigma, \delta_D, q_D, F_D)$$

such that $L(D) = L(E)$

Construction details :

- $Q_D = \{S \mid S \subseteq Q_E,\ S = \text{ECLOSE}(S)\}$
- $q_D = \text{ECLOSE}(q_0)$
- $F_D = \{S \mid S \in Q_D,\ S \cap F_E \neq \emptyset\}$

From ϵ-NFA to DFA

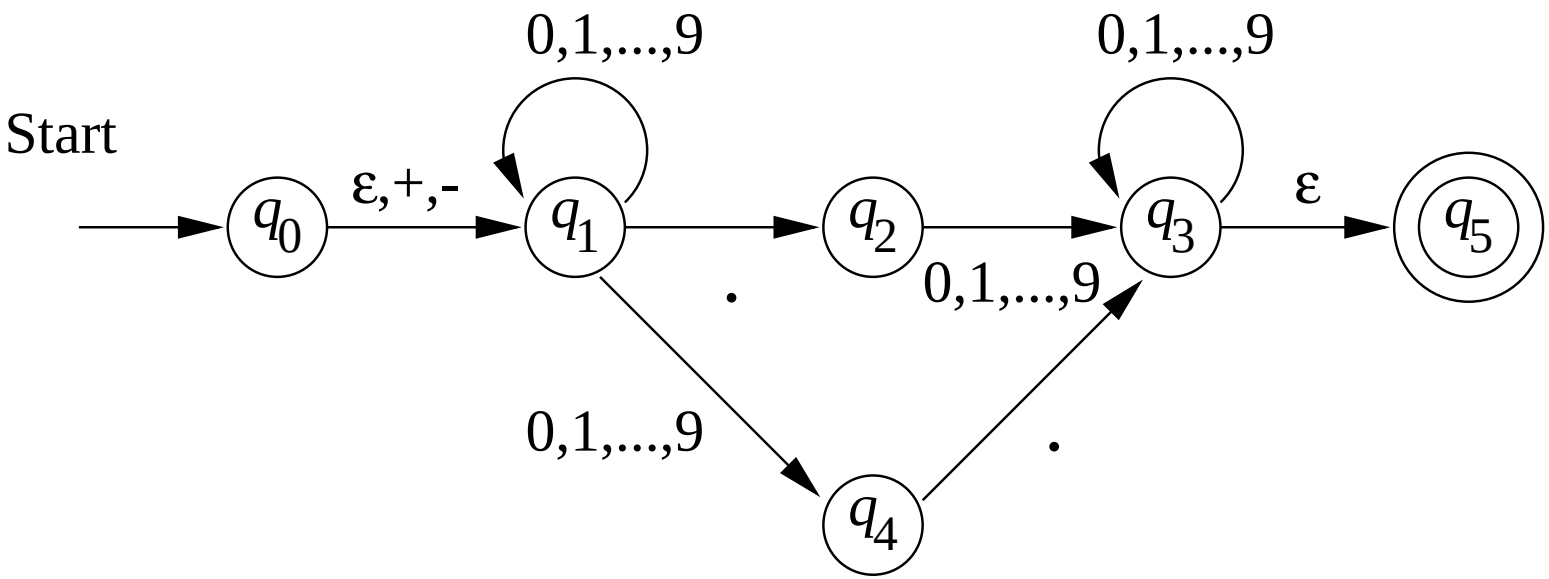
Construction details (cont'd)

Computation of $\delta_D(S, a)$, $a \in \Sigma$ and $S \in Q_D$

- $S = \{p_1, \ldots, p_k\}$
- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\delta_D(S, a) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

Example

ϵ-NFA E



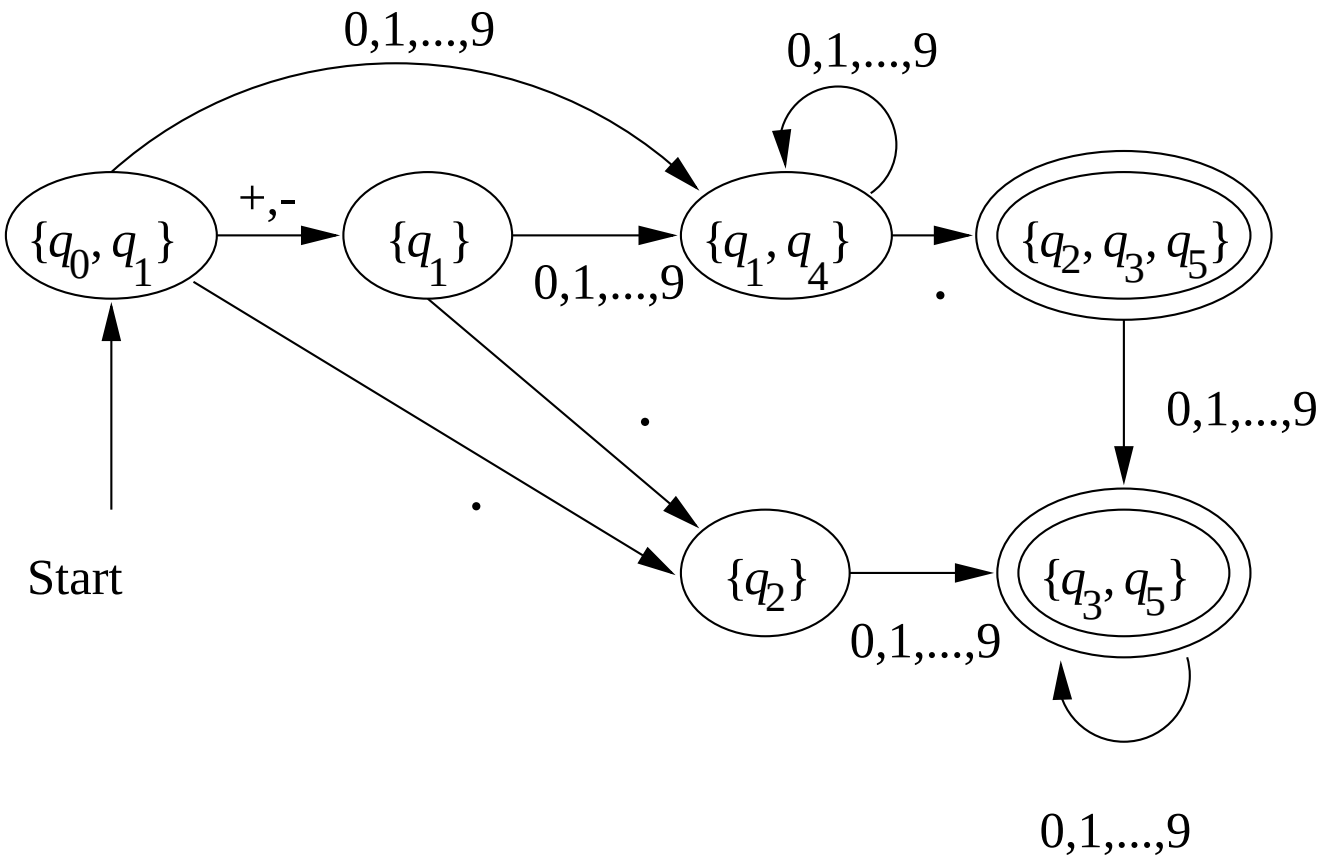
Example

Computation of some of the values of δ_D

- $\delta_D(\{q_0, q_1\}, +) = \text{ECLOSE}(\delta_E(q_0, +) \cup \delta_E(q_1, +)) = \text{ECLOSE}(\{q_1\}) = \{q_1\}$
- $\delta_D(\{q_1\}, 0) = \text{ECLOSE}(\delta_E(q_1, 0)) = \text{ECLOSE}(\{q_1, q_4\}) = \{q_1, q_4\}$
- $\delta_D(\{q_1, q_4\}, .) = \text{ECLOSE}(\delta_E(q_1, .) \cup \delta_E(q_4, .)) = \text{ECLOSE}(\{q_2, q_3\}) = \{q_2, q_3, q_5\}$
- $\delta_D(\{q_2, q_3, q_5\}, 0) = \text{ECLOSE}(\delta_E(q_2, 0) \cup \delta_E(q_3, 0) \cup \delta_E(q_5, 0)) = \text{ECLOSE}(\{q_3\} \cup \{q_3\} \cup \varnothing) = \text{ECLOSE}(\{q_3\}) = \{q_3, q_5\}$
- $\ldots$

Example

DFA D constructed from E; the DFA has been further simplified, omitting the trap state and all transitions leading to that state



Equivalence between ϵ-NFA and DFA

Theorem A language L is recognized by ϵ-NFA E if and only if L is recognized by DFA D

Proof

(If) Convert $\delta_D(q, a) = p$ into $\delta_E(q, a) = \{p\}$. Then add $\delta_E(q, \epsilon) = \varnothing$ for each state q of D

(Only if) Using our construction for D, we prove $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(q_D, w)$ by induction on $|w|$

Base $\hat{\delta}_E(q_0, \epsilon) = \text{ECLOSE}(q_0) = q_D = \hat{\delta}(q_D, \epsilon)$

Equivalence between ϵ-NFA and DFA

Induction Let $w = xa$. We show $\hat{\delta}_E(q_0, xa) = \hat{\delta}_D(q_D, xa)$ using the inductiove hypothesis $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(q_D, x)$

Let $\hat{\delta}_E(q_0, x) = \{p_1, \ldots, p_k\}$

From the definition of $\hat{\delta}_E$

- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\hat{\delta}_E(q_0, xa) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

Equivalence between ϵ-NFA and DFA

From the inductive hypothesis $\hat{\delta}_D(q_D, x) = \{p_1, \ldots, p_k\}$

Using the definition of D we compute $\delta_D(\{p_1, \ldots, p_k\}, a)$

- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\delta_D(\{p_1, \ldots, p_k\}, a) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

We can now write

$$\begin{aligned}
\hat{\delta}_D(q_D, xa) &= \delta_D(\hat{\delta}_D(q_D, x), a) \\
&= \delta_D(\{p_1, \ldots, p_k\}, a) \\
&= \text{ECLOSE}(\{r_1, \ldots, r_m\}) \\
&= \hat{\delta}_E(q_0, xa)
\end{aligned}$$

□