Exercises l5-l6

#statistics

Example 5.1

$1 = \sum_{x=1}^{\infty} c_\theta \theta^x = c_\theta(\frac{1}{1-\theta} - 1)$

and solving the equation we get $f(x;\theta) = (1-\theta)\theta^{x-1}$ a geometric rv with parameter $1-\theta$.

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(a) Wald confidence interval for θ of level $1-\alpha$, $\alpha = 0.9$ $\overline{x} = 10$ and $n = 30$.

$|W| = |\frac{\hat{\theta}-\theta_0}{\widehat{se}}|$

$L(\theta) = \frac{(1-\theta)^n}{\theta^n}\theta^{\sum_{i=1}^{n} x_i} = (1-\theta)^n \theta^{n(\overline{x}-1)}$
$\ell(\theta) = n\log(1-\theta) + n(\overline{x}-1)\log\theta$
$J(\theta) = \frac{n(\overline{x}-1)}{\theta^2} + \frac{n}{(1-\theta)^2}$
$\hat{\theta} = \frac{\overline{x}-1}{\overline{x}} = 0.9$
$J(\hat{\theta}) = 3333.3333$
$\widehat{se} = 1/\sqrt{\hat{J}} = 0.01732$

$R = \{\boldsymbol{X} : |W| \geq z_{1-\frac{\alpha}{2}}\} = \{\boldsymbol{X} : \theta_0 \leq \hat{\theta} - z_{1-\frac{\alpha}{2}}\widehat{se}\} \vee \theta_0 \geq \hat{\theta} + z_{1-\frac{\alpha}{2}}\widehat{se}\}$
$R^{\mathsf{c}} = \{\boldsymbol{X} : \hat{\theta} - z_{1-\frac{\alpha}{2}}\widehat{se} \leq \theta_0 \leq \hat{\theta} + z_{1-\frac{\alpha}{2}}\widehat{se}\}$

$z_{1-\frac{0.1}{2}} = 1.644854$

confidence interval of level $1-\alpha = 0.9$:

$CI = [0.871511, 0.928489]$

TODO CONTROLLA

(b) I would reject H_0 since $\theta_0 = 0.5$ is outside the confidence interval.

(c) The likelihood ratio test statistic is $\lambda(\boldsymbol{x}) = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{L(\theta_0)}{4.42 \cdot 10^{-43}}$

$R_\alpha(\theta_0) = \{\boldsymbol{x} : -2\log(\lambda(\boldsymbol{x})) > \chi^2_{1,1-\alpha}\}$

We can compute $-2\log(\lambda(\boldsymbol{x}))$ and $\chi^2_{1,1-\alpha}$ and plot them, then we take the values of θ for which the sample is in the acceptance region, this is done with the following R script:

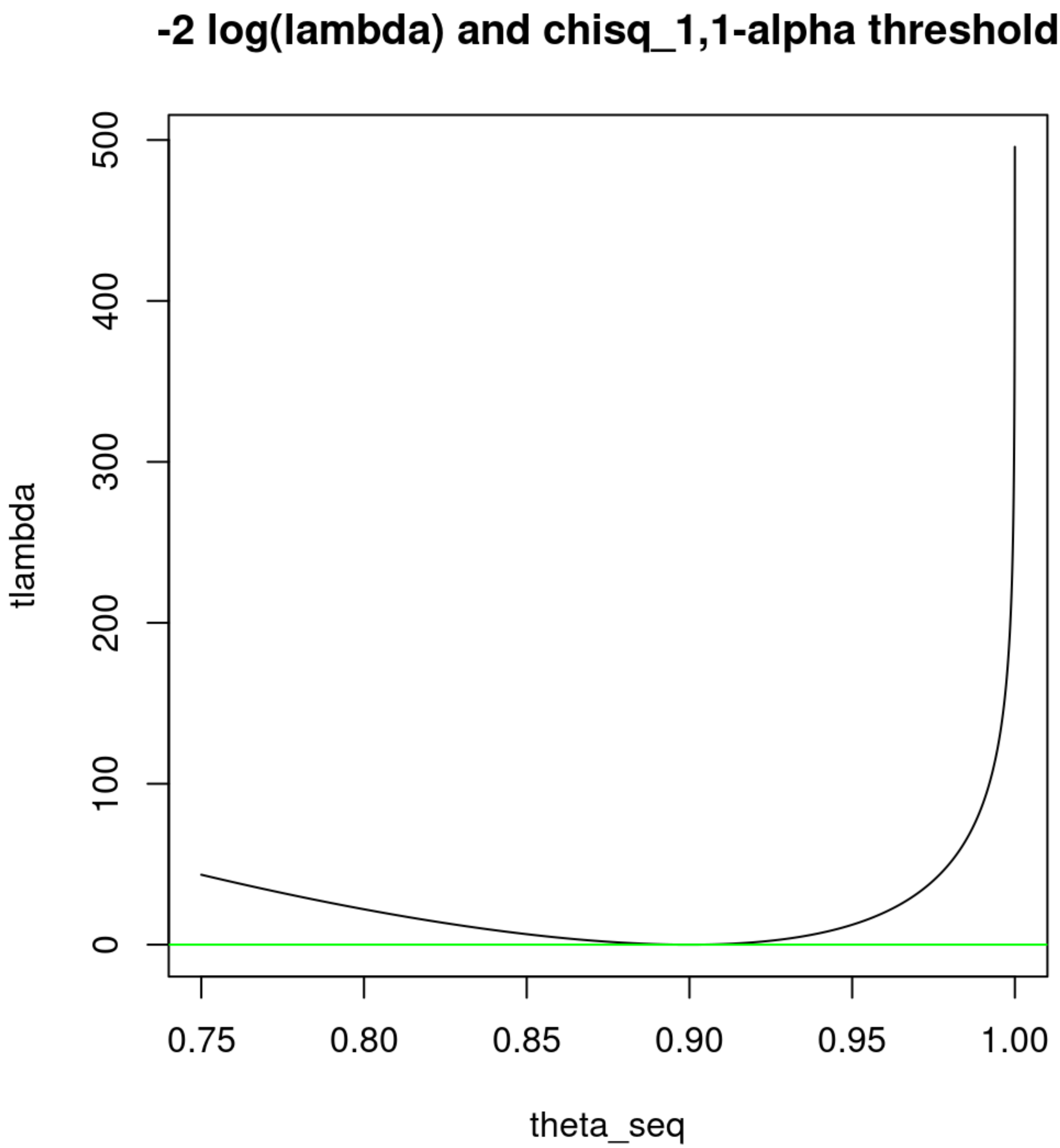
```
ell_f <-function(theta){
  n=30
  bar_x = 10
  n*log(1-theta)+n*(bar_x-1)*log(theta)
}
theta_seq = seq(0.75,1,0.00001)

alpha = 0.9
threshold = qchisq(p = 1-alpha,df = 1)
MLE = 0.9
tlambda = -2*ell_f(theta_seq)+2*ell_f(MLE)
plot(theta_seq,tlambda,main = '-2 log(lambda) and
chisq_1,1-alpha threshold',lwd=1,lty=1,type = 'l')
abline(h=threshold,col='green')
```

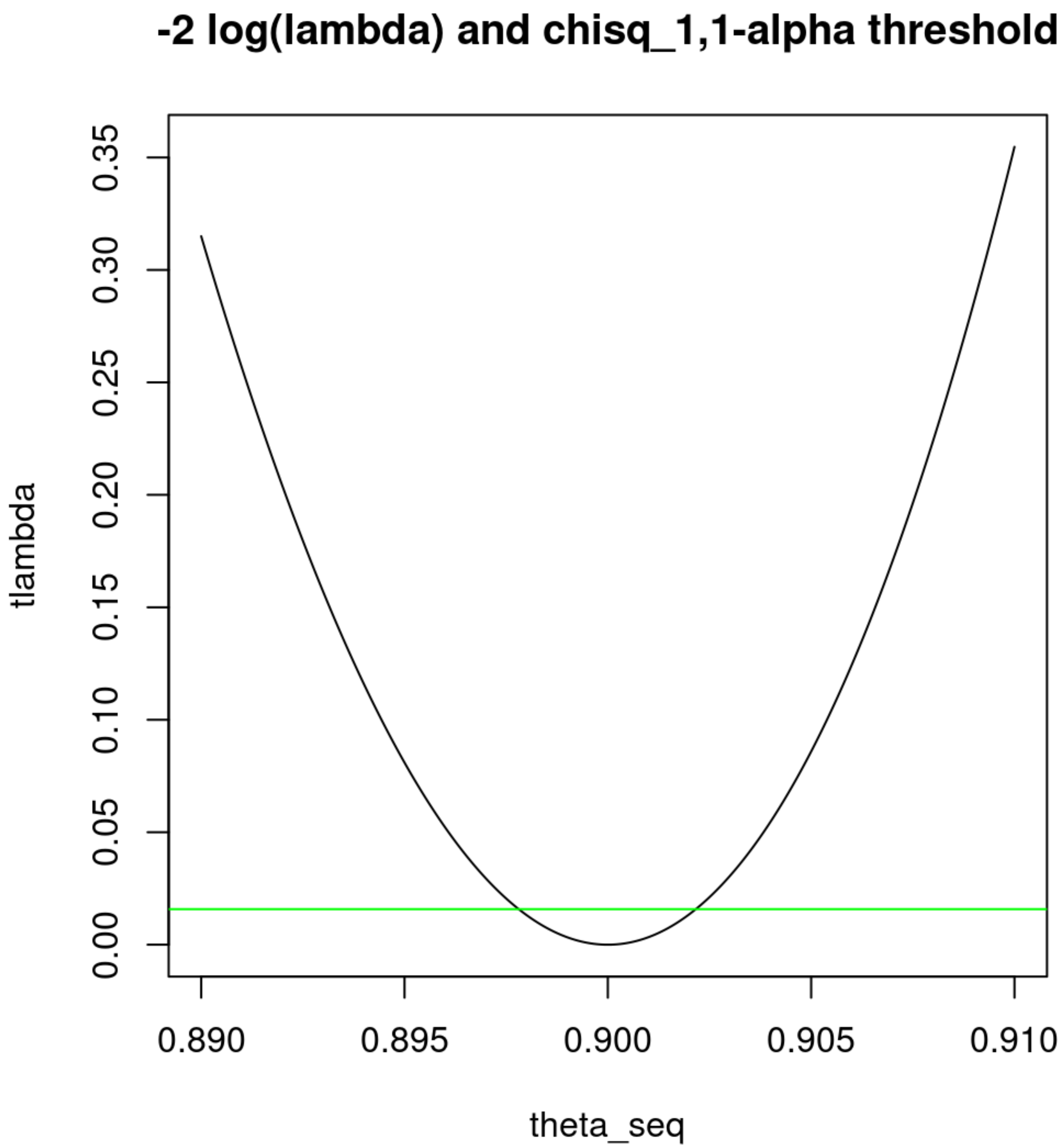
```
confidence_interval = theta_seq[tlambda<threshold]
sprintf('[ %f , %f
]',confidence_interval[1],confidence_interval[length(confidence_interval)])
```

that outputs:

```
"[ 0.897810 , 0.902160 ]"
```

which is the confidence interval for $\alpha = 0.9$.





(d) p-value:

p-value $= P_\theta(-2\log(\lambda) \geq -2\log(\lambda_{obs}))$

$\lambda_{obs} = 1$

Take the log on both sides and multiply by -2.

p-value $= P_\theta(\chi^2_1 \geq 0) = 1$.
TODO controlla

Example 5.2

Let $X_i \sim Poi(\theta_1), i = 1, \dots, m$ and $Y_j \sim Poi(\theta_2), j = 1, \dots, n$, with X_i, Y_jbeing independent for all i, j.

(a) Log-likelihood ratio test for $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$ at the level α. $\overline{y} = 6, \overline{x} = 2, m = 15, n = 10$, compute the test and get the p-value.

$T_n = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{S_x^2}{m} + \frac{S_y^2}{n}}}$
And we reject H_0 if $|T_n| > t_{\nu, 1-\frac{\alpha}{2}}$, with:
$\nu = \frac{(\frac{S_x^2}{m} + \frac{S_y^2}{n})^2}{\frac{S_x^4}{m^2(m-1)} + \frac{S_y^4}{n^2(n-1)}}$

TODO come si fa senza sample variance?????

Example 5.3

$X_i \sim \mathcal{N}(\mu, \sigma^2), i = 1, \dots, n$ both parameters are unknown
(a) $H_0 : \mu = 1$ against $H_1 : \mu \neq 1$ and observed sample with $\overline{x} = 2.1$ and $s^2 = 1.2$ determine n needed for $\beta(2) = 0.01$, (since $\mu = 2$ we are under H_1)

$\beta(2) = P_{\mu=2}(\boldsymbol{x} \notin R)$
if R is defined with a LRT we are in the case of a t-test,
$R = \{\boldsymbol{x} : |\frac{\sqrt{n}(\overline{x}-\mu_0)}{s}| \geq t_{n-1,1-\frac{\alpha}{2}}\}$
$\beta(2) = 1 - P_{\mu=2}(\text{reject } H_0) = P_{\mu=2}(\boldsymbol{x} \in R) = 0.01$
$\overline{x} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$
$T_n \sim \mathcal{N}(\frac{\sqrt{n}}{s}, \frac{\sigma^2}{s^2})$
$\beta(2) = 1 - 2\Phi\left(\left(t_{n-1,1-\frac{\alpha}{2}} - \frac{\sqrt{n}}{s}\right)\frac{s}{\sigma}\right)$
TODO come lo calcolo senza sapere sigma?????

Example 5.4

$X_1, \ldots, X_n$ iid random sample from $\mathrm{Unif}(0, \theta)$, $\theta > 0$. Construct a $1 - \alpha$ confidence interval for θ.

We use a LRT, we have that $L(\theta) = \frac{1}{\theta^n}$ if $x_{(n)} \leq \theta$ and 0 otherwise.
$R_\alpha(\theta_0) = \{\boldsymbol{x} : -2\log(\lambda(\boldsymbol{x})) > \chi^2_{1,1-\alpha}\}$
$\hat{\theta} = x_{(n)}$
$\lambda(\boldsymbol{x}) = x^n_{(n)} \cdot \frac{1}{\theta^n}$

$R_\alpha(\theta_0) = \{\boldsymbol{x} : e^{-2\log(\lambda(\boldsymbol{x}))} < e^{\chi^2_{1,1-\alpha}}\}$