Exercise

Your friend has developed a new machine learning algorithm for binary classification (i.e., $y \in \{-1, 1\}$) with 0-1 loss and tells you that it achieves a generalization error of only 0.05. However, when you look at the learning problem he is working on, you find out that $\Pr_{\mathcal{D}}[y = 1] = 0.95$...

1) • Assume that $\Pr_{\mathcal{D}}[y = \ell] = p_\ell$. Derive the generalization error of the (dumb) hypothesis/model that *always* predicts ℓ.

2) • Use the result above to decide if your friend's algorithm has learned something or not.

Solution

1) We are considering the hypothesis/model: $h(\vec{x}) = \ell \quad \forall \vec{x} \in \mathcal{X}$

Generalization error of h:

$$L_{\mathcal{D}}(h) = \mathbb{E}_{(\vec{x},y)\sim\mathcal{D}}\left[\ell(h,(\vec{x},y))\right]$$

by def. of $L_{\mathcal{D}}(h)$

$$= 0 \cdot \Pr_{(\vec{x},y)\sim\mathcal{D}}\left[\ell(h,(\vec{x},y))=0\right] + 1 \cdot \Pr_{(\vec{x},y)\sim\mathcal{D}}\left[\ell(h,(\vec{x},y))=1\right]$$

by def. of $\mathbb{E}[\,]$ and 0-1 loss

$$= \Pr_{(\vec{x},y)\sim D}[\ell(h_1(\vec{x},y))=1]$$

by def. of 0-1 loss $\rightarrow$ $$= \Pr_{(\vec{x},y)\sim D}[h(\vec{x}) \neq y]$$

by def. of h $\rightarrow$ $$= \Pr_{(\vec{x},y)\sim D}[\ell \neq y]$$

by def. of prob. $\rightarrow$ $$= 1 - \Pr_{(\vec{x},y)\sim D}[y = \ell]$$

by def. of p_ℓ $\rightarrow$ $$= 1 - p_\ell$$

2) From point 1) above, the hypothesis $h(\vec{x})=1 \ \forall \vec{x} \in X$ has generalization error 0.05.

The "dumb" model ($h(\vec{x})=1 \ \forall \vec{x} \in X$) has generalization error as good as your friend's algorithm.

$\Rightarrow$ your friend's algorithm has not learned a relation between $\vec{x}$ and y; it has "learned" that

$$\Pr_{(\vec{x},y)\sim\mathcal{D}}[y=1] = 0.95.$$

(Such probability is trivial to "learn" with enough data)

$\Rightarrow$ your friend's algorithm has not learned something useful in terms of the ML task.

Exercise

Assume we have the following training set S, where $\mathbf{x} = [x_1, x_2] \in \mathbb{R}^2$ and $\mathcal{Y} = \{-1, 1\}$:
$S = \{([-3, 4], 1), ([2, -3], -1), ([-3, -4], -1), ([1, 1.5], 1)\}$.
Assume you decide to use $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ with
$h_1 = sign(-x_1 - x_2)$ # errors: $1+1+1+1 \Rightarrow L_S(h_1) = 4/4 = 1$
$h_2 = sign(-x_1 + x_2)$ # errors: $0+0+0+0 \Rightarrow L_S(h_2) = 0/4 = 0$
$h_3 = sign(x_1 - x_2)$
$h_4 = sign(x_1 + x_2)$
Your algorithm uses the ERM rule and the 0-1 loss.

- What model h_S is produced in output by your ML algorithm?
- Assume the realizability assumption holds. What can you say about the generalization error $L_{\mathcal{D}}(h_S)$ of h_S?

Solution

1) compute the training error for h_1, h_2, h_3, h_4. Report in output one of the hypotheses minimizing the training error.

Let's compute the training error $L_S(h_i)$ for $i = 1, \dots, 4$

i	$L_S(h_i)$
1	1
2	0
3	1
4	0

$\Rightarrow$ your ML algorith will report in output one between h_2 and h_4.

2) We know $L_S(h_S) = 0$.

From the proof of Corollary 3.3 we know that:

$$\Pr\left[L_D(h_S) > \varepsilon\right] \leq \underbrace{|\mathcal{H}| e^{-\varepsilon m}}_{\delta}$$

, where m = # of samples in S

$\Rightarrow$ for a given δ, fix $\varepsilon = \frac{1}{m} \ln\left(\frac{|\mathcal{H}|}{\delta}\right)$

$\rightarrow \Pr\left[L_{\mathcal{D}}(h_S) \leq \frac{1}{m} \ln\left(\frac{|\mathcal{H}|}{\delta}\right)\right] \geq 1 - \delta$

For example, let's choose $\delta = 0.1$. We have that with probability ≥ 0.9, the following holds:

$$L_{\mathcal{D}}(h_S) \leq \frac{1}{4} \ln\left(\frac{4}{0.1}\right) \approx 0.75$$

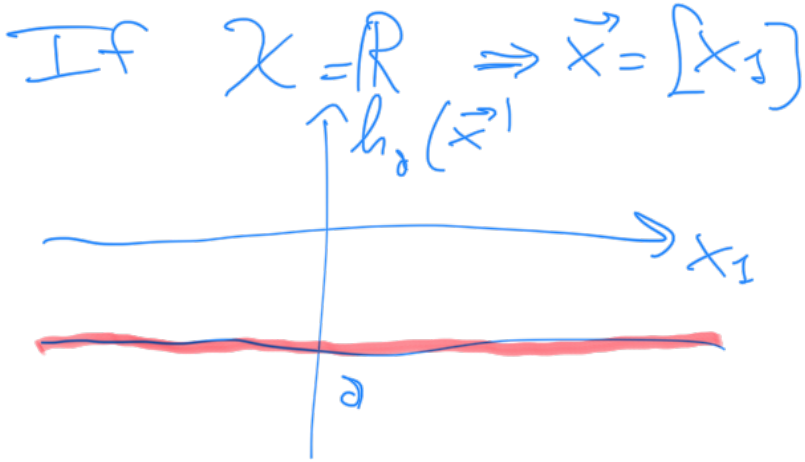
Exercise

Consider a linear regression problem, where $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \mathbb{R}$, with mean squared loss. The hypothesis set is the set of *constant* functions, that is $\mathcal{H} = \{h_a : a \in \mathbb{R}\}$, where $h_a(\mathbf{x}) = a$. Let $S = ((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m))$ denote the training set.

- Derive the hypothesis $h \in \mathcal{H}$ that minimizes the training error.
- Use the result above to explain why, for a given hypothesis $\hat{h}$ from the set of all linear models, the coefficient of determination $R^2 = 1 - \frac{\sum_{i=1}^{m}(\hat{h}(\mathbf{x}_i) - y_i)^2}{\sum_{i=1}^{m}(y_i - \bar{y})^2}$ where $\bar{y}$ is the average of the $y_i, i = 1, \ldots, m$ is a measure of how well $\hat{h}$ performs (on the training set).

$\mathcal{H} = \{ h_a : a \in \mathbb{R} \}$

$h_a(\vec{x}) = a \quad \forall \vec{x} \in \mathbb{R}$



·) Given $h_a \in \mathcal{H}$, the training error for such hypothesis is:

$$L_S(h_a) = \frac{1}{m} \sum_{i=1}^{m} \left(h_a(\vec{x}_i) - y_i \right)^2 = \frac{1}{m} \sum_{i=1}^{m} \left(a - y_i \right)^2$$

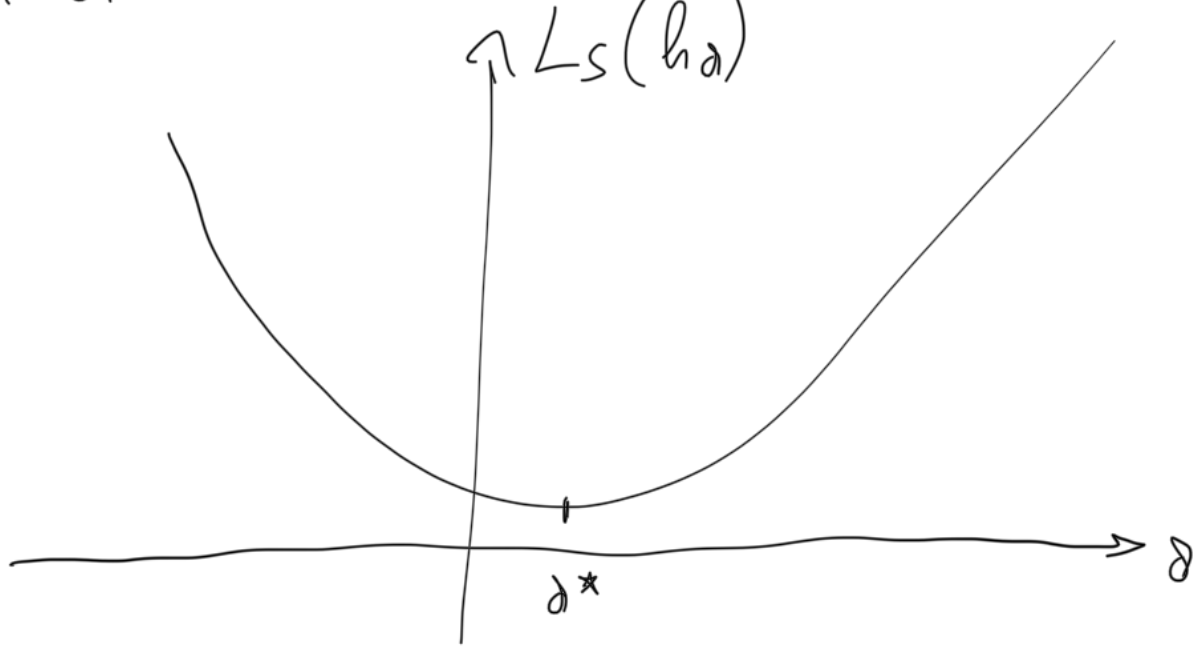
by def. of $L_S()$, h_a, and squared loss

by def. of h_a

Now, finding $h_a \in \mathcal{H}$ that minimizes the training error corresponds to find a that minimizes:

$$L_S(h_a) = \frac{1}{m} \sum_{i=1}^{m} \left(a - y_i \right)^2 = f(a) = \underbrace{(\ )}_{\geq 0} a^2 + (\ )\, a + (\ )$$

As a function of a



$\Rightarrow$ compute $\frac{d L_S(h_a)}{da}$ and find a s.t. $\frac{d L_S(h_a)}{da} = 0$

$$\frac{d L_S(h_a)}{da} = \frac{d}{da}\left(\frac{1}{m}\sum_{i=1}^{m}(a - y_i)^2\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\left(\frac{d}{da}\left((a - y_i)^2\right)\right)$$

$$= \frac{1}{m} \sum_{i=1}^{m} (2a - 2y_i)$$

$$= \frac{2}{m} \sum_{i=1}^{m} (a - y_i)$$

$$\frac{2}{m} \sum_{i=1}^{m} (a - y_i) = 0 \iff \sum_{i=1}^{m} (a - y_i) = 0$$

$$\iff \left(\sum_{i=1}^{m} a \right) - \left(\sum_{i=1}^{m} y_i \right) = 0$$

$$\iff \left(\sum_{i=1}^{m} a \right) = \left(\sum_{i=1}^{m} y_i \right) \iff m a = \sum_{i=1}^{m} y_i \Rightarrow a = \frac{\sum_{i=1}^{m} y_i}{m} = \bar{y}$$

- $R^2 = 1 - \left(\sum_{i=1}^{m} (\hat{h}(\vec{x}_i) - y_i)^2 \right) / \left(\sum_{i=1}^{m} (y_i - \bar{y})^2 \right)$ → (A)

(A) is the error of $\hat{h}$ (on the training set) relative to the error of the best "naive" predictor (which always predicts a constant with looking at $\vec{x}$)

$\Rightarrow 1 - (A) = R^2$ is a measure of how well $\hat{h}$ performs compared to the best naive predictor.

$-\infty \leq R^2 \leq 1$ → best possible model

When $R^2 < 0$ → model worse than naive