1.1 $P(N=n) = \left(\frac{1}{2}\right)^n$

$P(H|N=n) = e^{-n}$

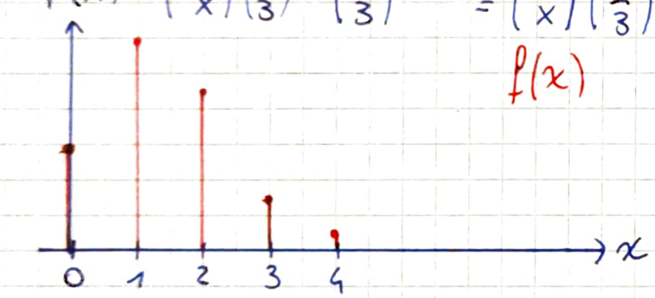
$$P(H) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{total prob.}} P(H|N=n)P(N=n) =$$

$$= \sum_{n=1}^{+\infty} e^{-n} \cdot \left(\frac{1}{2}\right)^n = \sum_{n=1}^{+\infty} \left(\frac{1}{2e}\right)^n = \frac{1}{1-\frac{1}{2e}} - 1 =$$

$$= \frac{2e}{2e-1} - 1 = \frac{2e-2e+1}{2e-1} = \frac{1}{2e-1} = 0.2254$$

1.2 $X \sim Bin(4, \frac{1}{3})$ CDF = ? Quantile = ? PDF = ?

$$f(x) = \binom{n}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{n-x} = \binom{4}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{4-x}$$



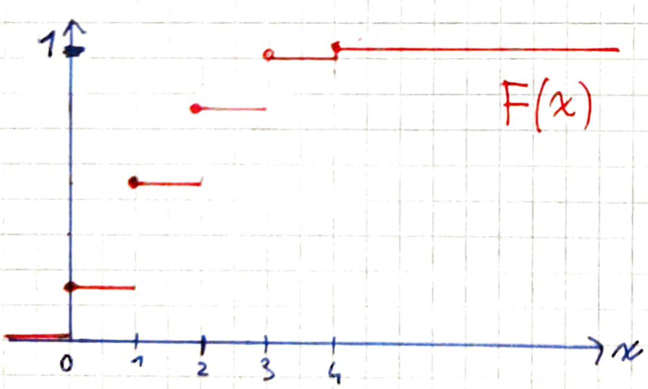
$f(0) = 1 \cdot 1 \cdot \left(\frac{2}{3}\right)^4 = 0.19753$

$f(1) = 4 \cdot \frac{1}{3}\left(\frac{2}{3}\right)^3 = 0.39506$

$f(2) = 0.29630$

$f(3) = 0.09877$

$f(4) = 0.01235$



$F(0) = f(0) = 0.19753$

$F(1) = f(0) + f(1) = 0.59259$

$F(2) = 0.88889$

$F(3) = 0.98766$

$F(4) = 1$

$Q(p) = \inf\{x : F(x) \geq p\}$ $p \in (0,1)$

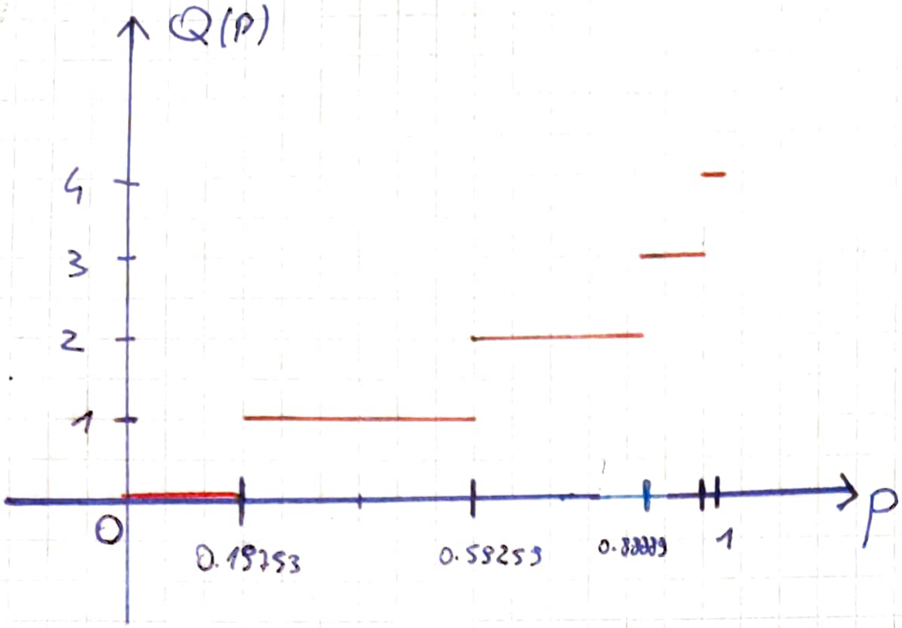
for $p \in (0, 0.19753]$ $Q(p) = 0$

$p \in [0.19753, 0.59259]$ $Q(p) = 1$

$p \in [0.59259, 0.88889]$ $Q(p) = 2$

$p \in [0.88889, 0.98766]$ $Q(p) = 3$

$p \in (0.98766, 1)$ $Q(p) = 4$



1.3 $f(x) = k e^{-\lambda x} \cdot \mathbb{1}_{x \geq 0}$

i.

$$1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} k e^{-\lambda x}\,dx = k \int_0^{+\infty} e^{-\lambda x}\,dx = -\frac{k}{\lambda} e^{-\lambda x}\Big|_0^{+\infty} =$$

$$= \frac{k}{\lambda} \rightarrow \boxed{k = \lambda}$$

ii.

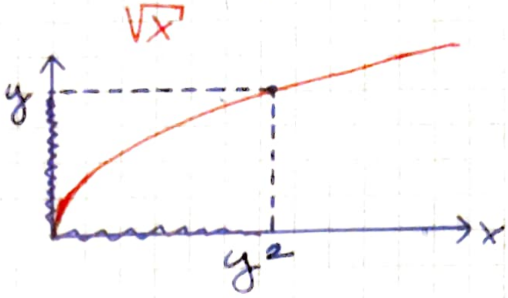
$$P(1 \leq X \leq 2) = \int_1^2 \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_1^2 = e^{-\lambda} - e^{-\lambda \cdot 2}$$

iii. ??? R was supposed to be X

1.4 $Y = \sqrt{X}$ $\quad f_x(x) = \lambda e^{-\lambda x}$

$B_y = \{x : g(x) \leq y\}$

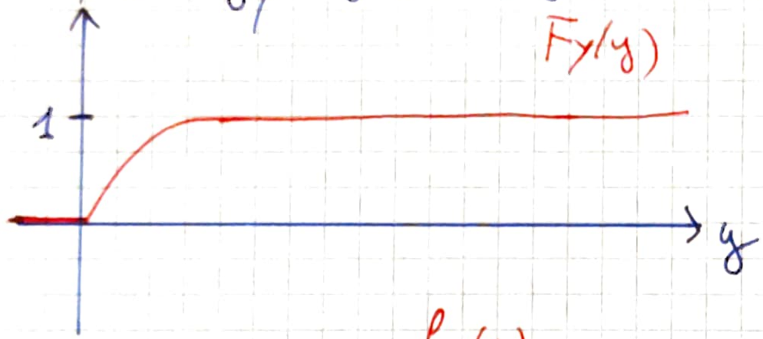
$B_y = \{x : x \leq y^2\}$

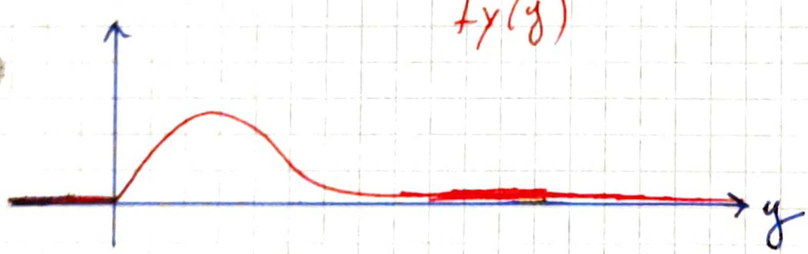


$$F_Y(y) = P(B_y) = \int_{B_y} f(x)\,dx = \int_0^{y^2} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{y^2} =$$
$$= 1 - e^{-\lambda y^2} \text{ for } y \geq 0$$

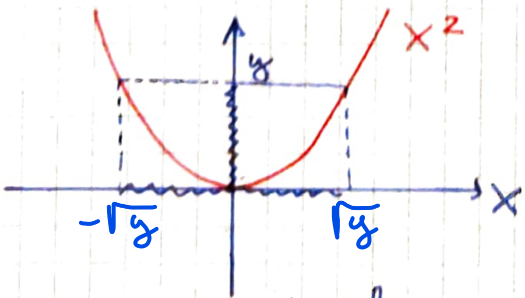
$$\Rightarrow F_Y(y) = (1 - e^{-\lambda y^2})\,\mathbb{1}_{x \geq 0}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = (2y e^{-\lambda y^2})\,\mathbb{1}_{x \geq 0}$$





1.5 $X \sim N(0,1)$ $\quad Y = X^2$ pdf = ? $\quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



$B_y = \{x : -\sqrt{y} \leq x \leq \sqrt{y}\}$

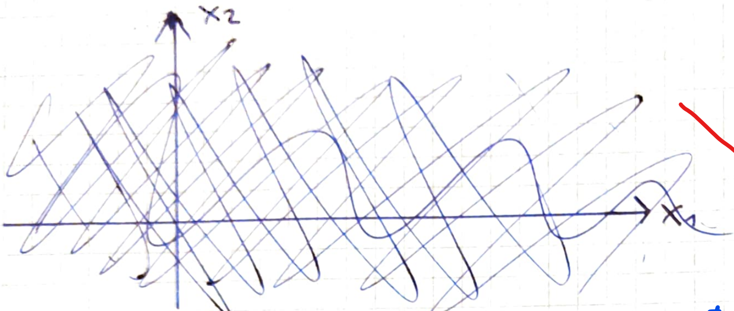
$$F_Y(y) = P(B_y) = \int_{B_y} f_X(x)\,dx = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) =$$
$$= 2\Phi(\sqrt{y}) - 1$$

take the derivative to have a p.d.f. .... $\frac{y^{\frac{1}{2}-1} e^{-y/2}}{\sqrt{2\pi}}$ □

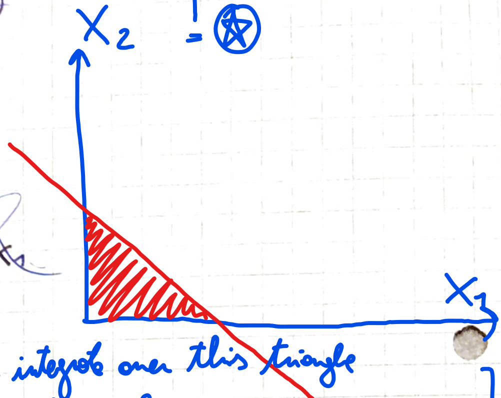
1.6 X_1 indep. X_2 $\qquad$ $f(x) = e^{-x} \mathbb{1}_{x \geq 0}$

$Y = X_1 + X_2$ $\quad$ pdf of $Y = ?$

$\cancel{f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)}$



$F_Y(y) = P(Y \leq y) \overset{!}{=} P(X_1 + X_2 \leq y) \overset{!}{=} ⊛$



i can integrate over this triangle

$f_Y(y) = (f_{X_1} * f_{X_2})(y)$ [The joint pdf that factorizes because X_1 is indep from X_2 i get the convolution]

$$f_{X_1} * f_{X_2}(y) = \int_{-\infty}^{+\infty} f_{X_1}(u) f_{X_2}(y-u)\, du = \int_0^y e^{-u} \cdot e^{u-y}\, du =$$

$f_{X_1}(u) = e^{-u} \mathbb{1}_{u \geq 0}$

$f_{X_2}(y-u) = e^{u-y} \cdot \underbrace{\mathbb{1}_{y-u \geq 0}}_{\mathbb{1}_{u \leq y}}$

$$= e^{-y} \int_0^y du = y e^{-y} \mathbb{1}_{y \geq 0}$$

$f_Y(y) = y e^{-y} \mathbb{1}_{y \geq 0}$

$$⊛ = \int_0^y \int_0^{y-x_1} \underbrace{e^{-x_1} \cdot e^{-x_2}}_{\text{factorized because its independent}} dx_2\, dx_1$$

$= \dots = 1 - e^{-y} - y e^{-y}$

if I take the derivative I get the same result