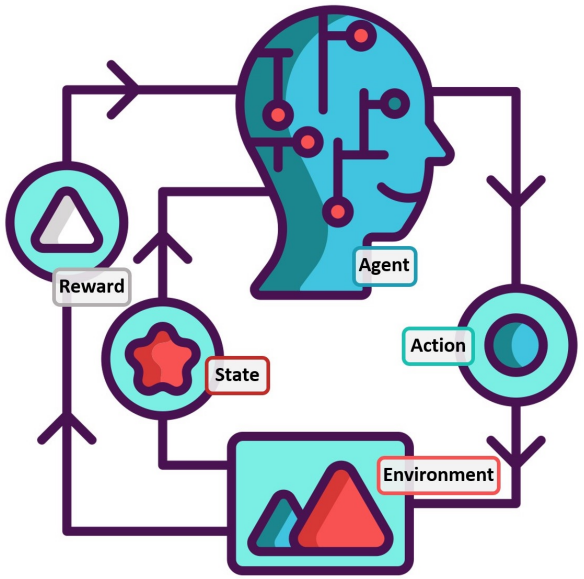
AMCO
ARTIFICIAL INTELLIGENCE, MACHINE LEARNING AND CONTROL RESEARCH GROUP

Reinforcement Learning
2025/2026

RL2
Reinforcement Learning Research Lab @ UniPD

Lecture #06
Dynamic Programming for MDPs & Monte Carlo

Gian Antonio Susto



Recap

In MDPs we have two problems (two goals to ‘solve the MDP’)

1. Prediction/evalution

2. Control

Recap

In MDPs we have two problems (two goals to 'solve the MDP')

1. Prediction/evalution (of a Policy)

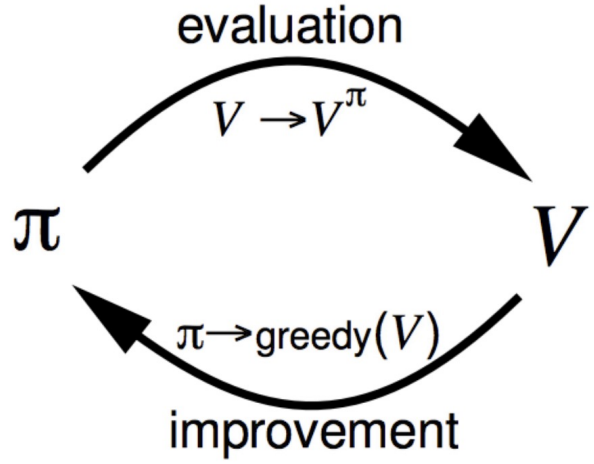
Understand how good a policy π is, ie. typically deriving the value function $v_\pi(s)$ and q_π

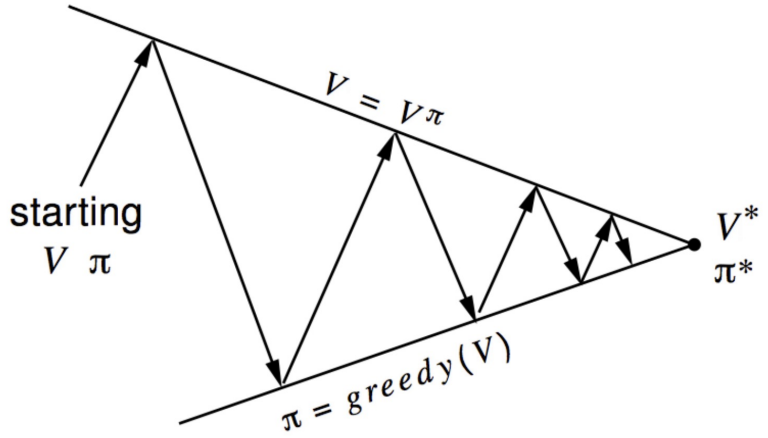
2. Control (Policy improvement)

Improve the current policy π/find the optimal policy typically deriving the value function $v^*(s)$ and the action-value function $q^*(s, a)$

Recap

Problem	Bellman Equation	Algorithm
Prediction	Bellman expectation equation	Iterative Policy Evaluation
Control	Bellman expectation equation + Greedy Policy Improvement	Policy Iteration





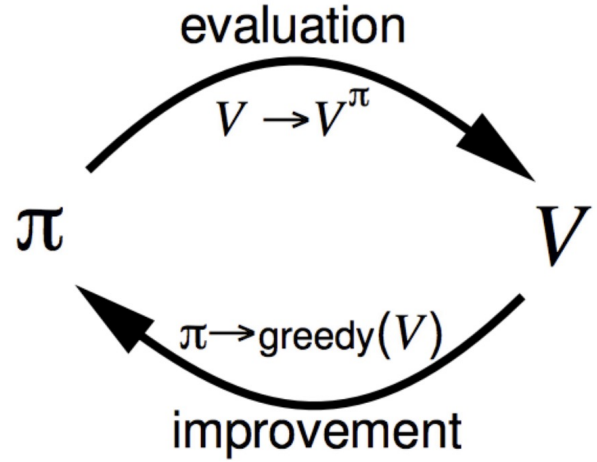
Control: Policy Iteration (Recap)

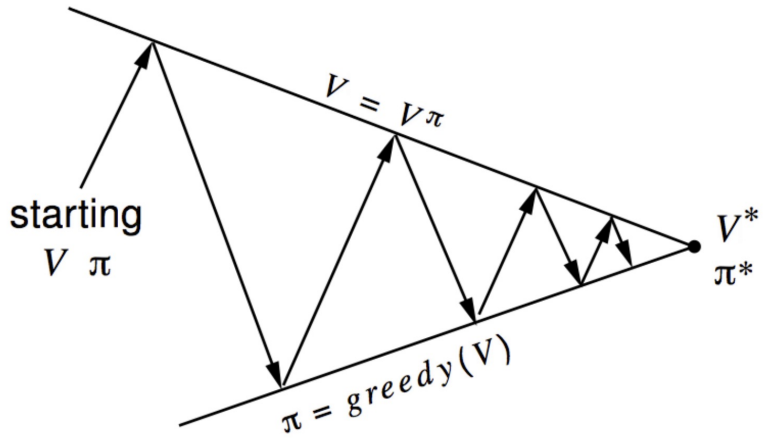
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2





Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$

- A policy $\pi' \geq \pi$ if $v_{\pi'}(s) \geq v_\pi(s)$ for all $s \subset S$
- The inequality holds because I'm maxing out

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$
- And improves the whole value function

$$
\begin{aligned}
v_\pi(s) \leq q_\pi(s, \pi'(s)) &= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}
$$

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$
- And improves the whole value function

$$
\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}
$$

After the next step, everything is equal (we are following π after the first step)

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedily: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\, q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$
- And improves the whole value function

$$
\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}
$$

I'm applying this again...

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$
- And improves the whole value function

$$v_\pi(s) \leq q_\pi(s, \pi'(s)) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right]$$

... and again!

$$\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right]$$

$$\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right]$$

$$\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)$$

Control: Policy Improvement Theorem

- Let's consider a deterministic policy π
- We can improve the policy by acting greedly: $\pi'(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ q_\pi(s, a)$

(that is, we pick the one action that provides the best value and then we follow π - π' is the greedy policy)

- This improves for one step: $q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$
- And improves the whole value function

$$
\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}
$$

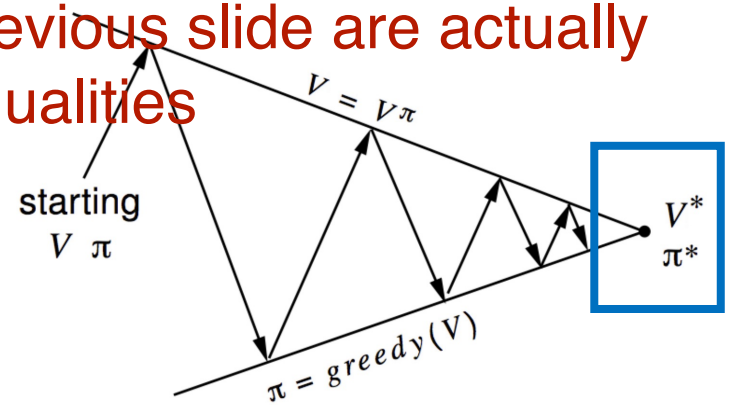
By acting greedly we are sure that we are not going to do worse... but do we reach optimality?

Control: Policy Improvement Theorem

- Let's consider the case in which improvement stops:

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

case where inequalities in previous slide are actually equalities

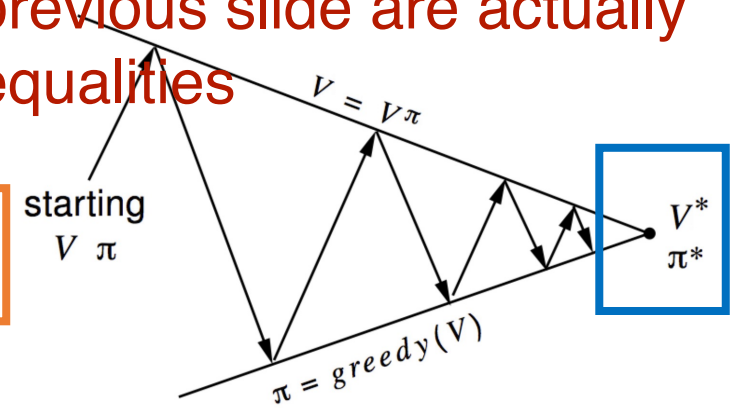


Control: Policy Improvement Theorem

- Let's consider the case in which improvement stops:

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

case where inequalities in previous slide are actually equalities



- But in this case, we have satisfied the Bellman optimality equation

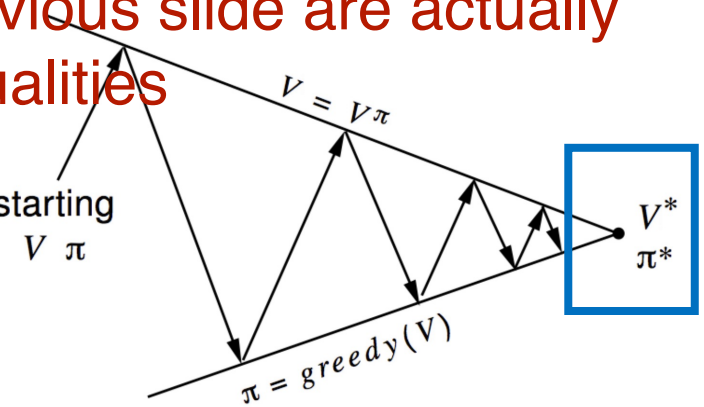
$$v_\pi(s) = \max_{a \in \mathcal{A}} q_\pi(s, a)$$

Control: Policy Improvement Theorem

- Let's consider the case in which improvement stops:

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

case where inequalities in previous slide are actually equalities



- But in this case, we have satisfied the Bellman optimality equation

$$v_\pi(s) = \max_{a \in \mathcal{A}} q_\pi(s, a)$$

- Therefore, we reached the optimal policy (π is optimal):

$$v_\pi(s) = v_*(s) \text{ for all } s \in \mathcal{S}$$

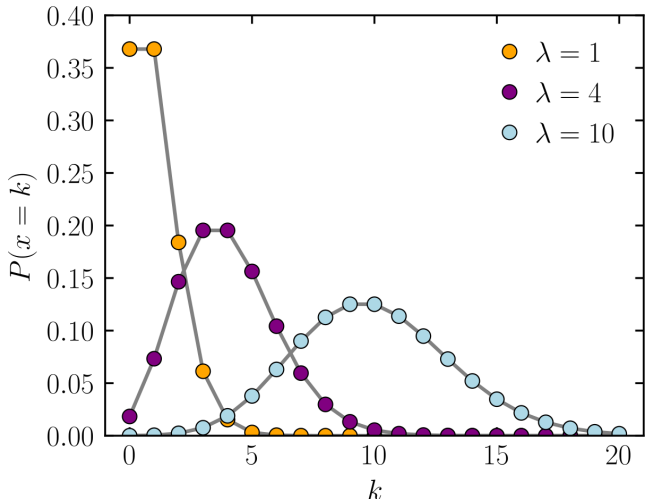
- Policy iteration actually solves MDPs!

Control: Example – Jack's Car Rental

- Jack's manage 2 locations for a nationwide car rental company
- If a customer arrives at one location and if he has cars available, he rents the car for $10
- Jack can move the cars overnight for $2 per car moved

Control: Example – Jack's Car Rental

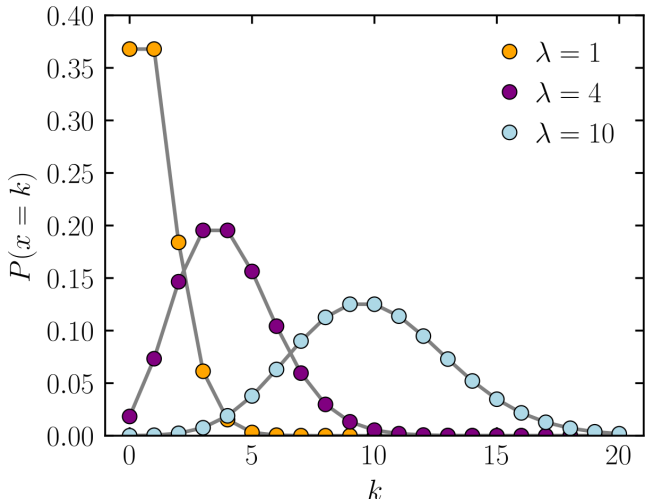
- Jack's manage 2 locations for a nationwide car rental company
- If a customer arrives at one location and if he has cars available, he rents the car for $10
- Jack can move the cars overnight for $2 per car moved
- The number n of cars requested and returned $\frac{\lambda^n}{n!}e^{-\lambda}$ are Poisson random variables
- 1° location: $\lambda = 3$ rental requests, $\lambda = 3$ returns
- 2° location: $\lambda = 4$ rental requests, $\lambda = 2$ returns
- No more than 20 cars can be at each location (additional cars are returned to the nationwide company) and no more than 5 cars can be moved overnight



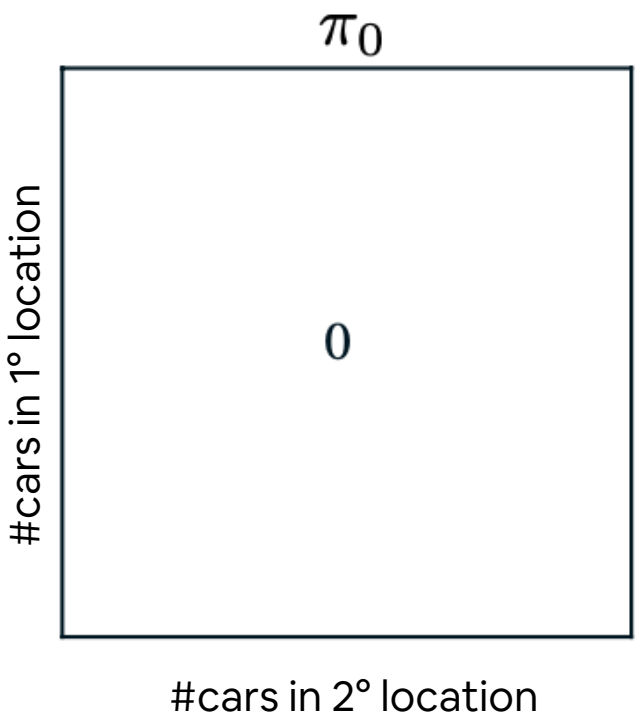
Control: Example – Jack's Car Rental

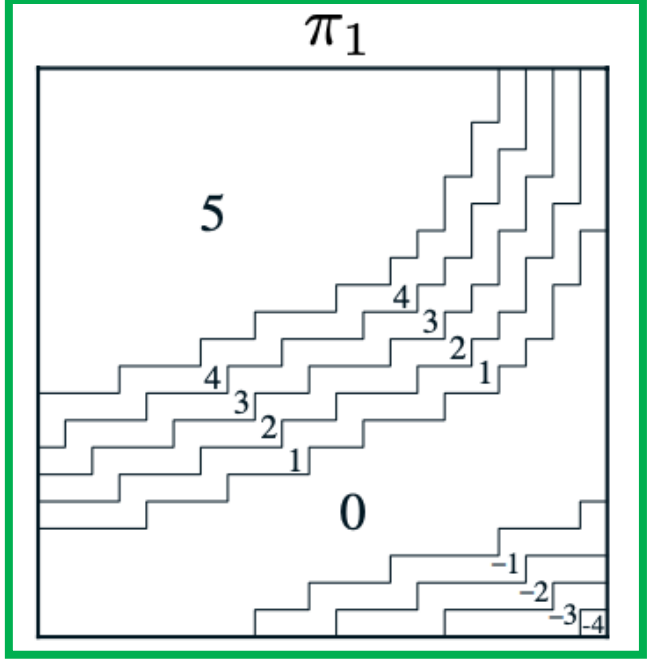
- We consider a discount rate $\gamma = 0.9$
- Time steps are days
- Actions are the net numbers of cars moved

- ions for a nationwide car
- t one location and if he has s the car for $10
- Jack can move the cars overnight for $2 per car moved
- The number n of cars requested and returned $\frac{\lambda^n}{n!}e^{-\lambda}$ are Poisson random variables
- 1° location: $\lambda = 3$ rental requests, $\lambda = 3$ returns
- 2° location: $\lambda = 4$ rental requests, $\lambda = 2$ returns
- No more than 20 cars can be at each location (additional cars are returned to the nationwide company) and no more than 5 cars can be moved overnight

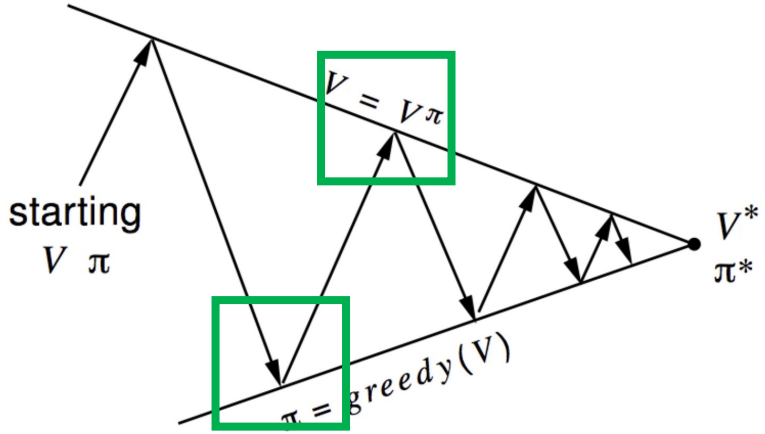


We start with the policy π_0 that moves zero cars. We then apply policy iteration

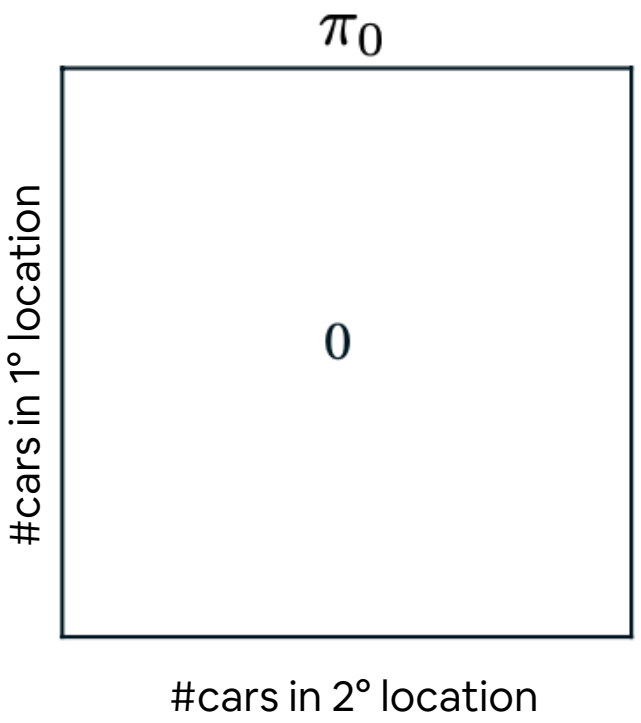


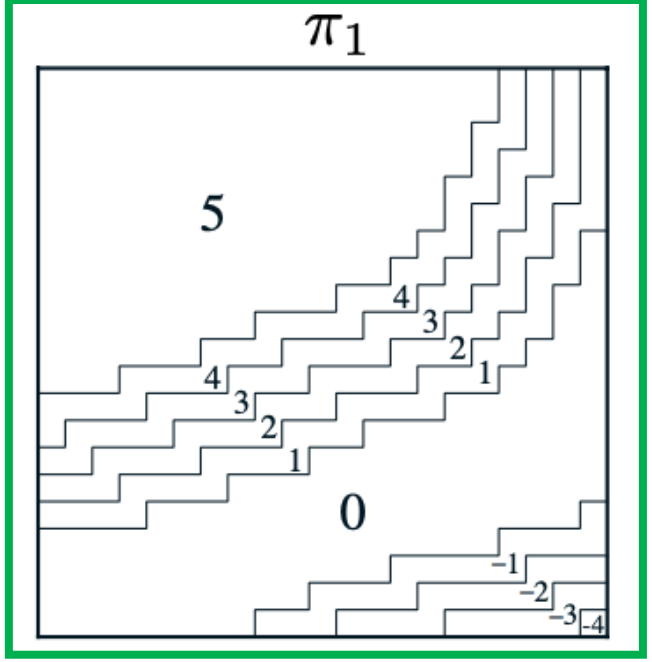


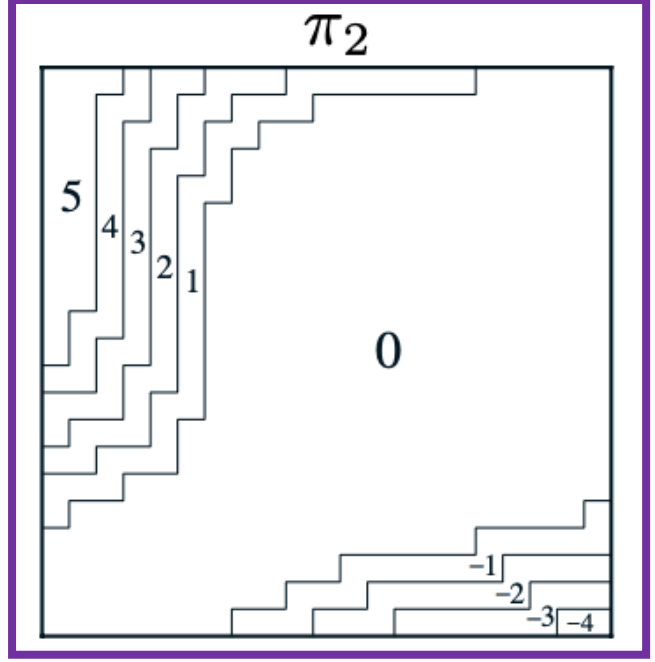
- 1° location:
$\lambda = 3$ rental requests,
$\lambda = 3$ returns
- 2° location:
$\lambda = 4$ rental requests,
$\lambda = 2$ returns

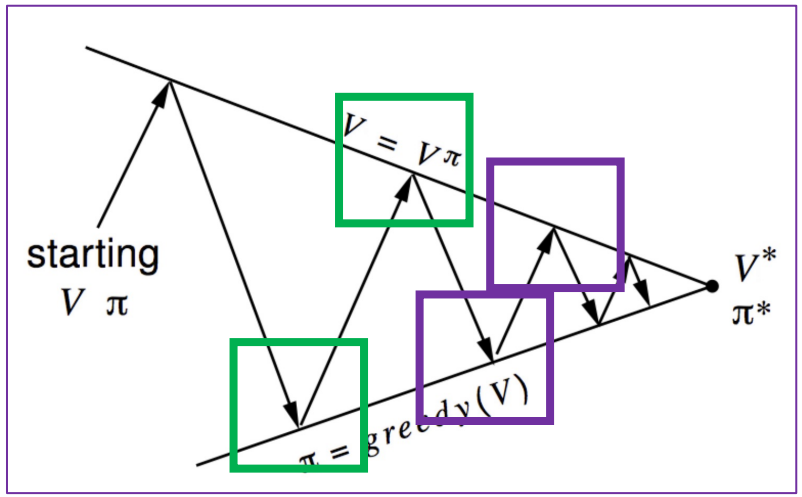


We start with the policy π_0 that moves zero cars. We then apply policy iteration

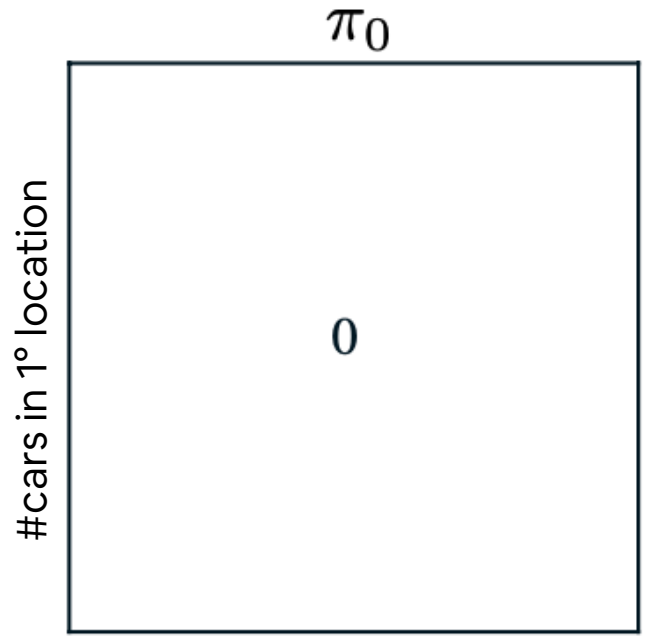


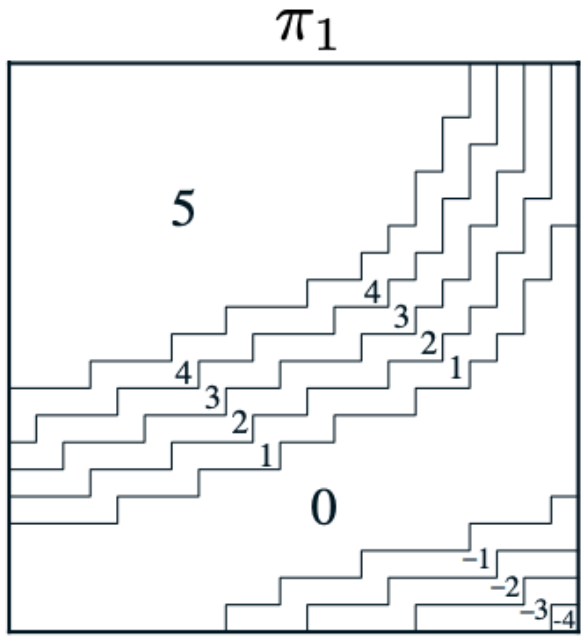


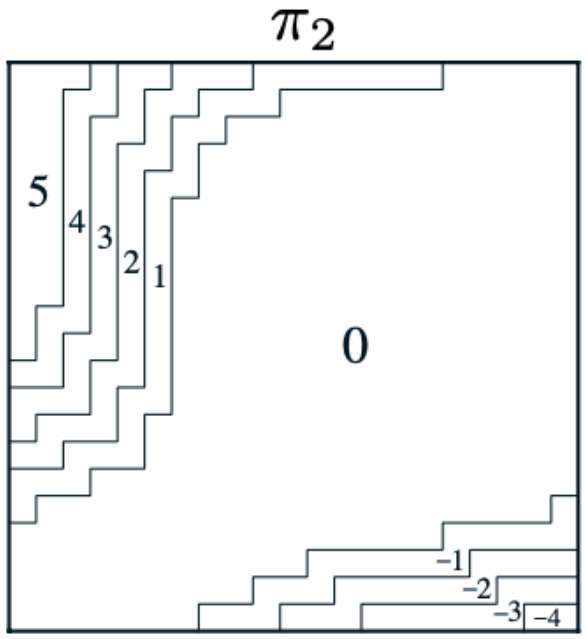




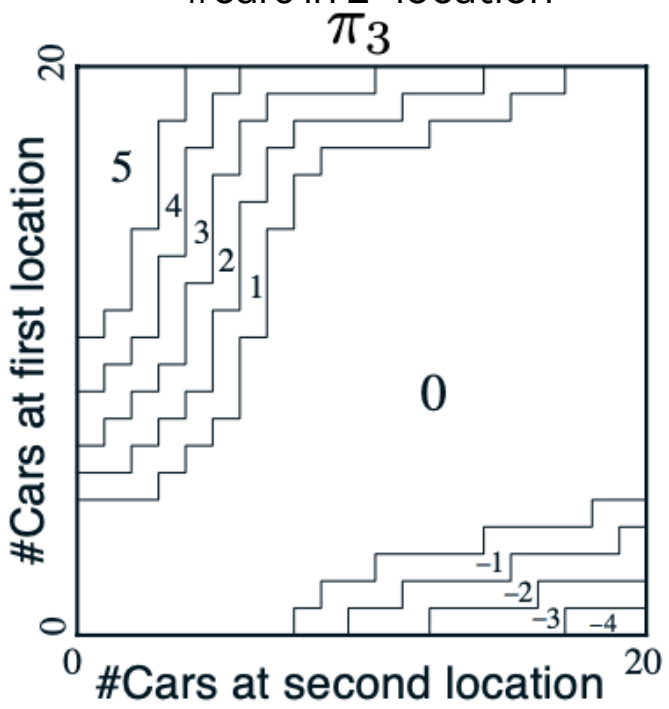
We start with the policy π_0 that moves zero cars. We then apply policy iteration



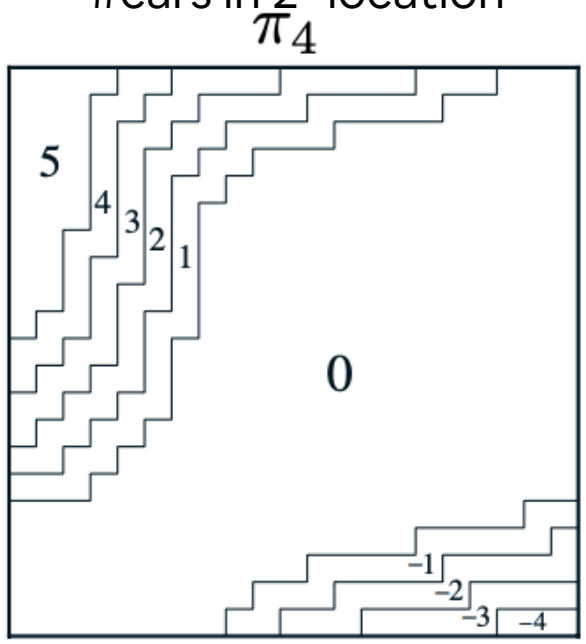


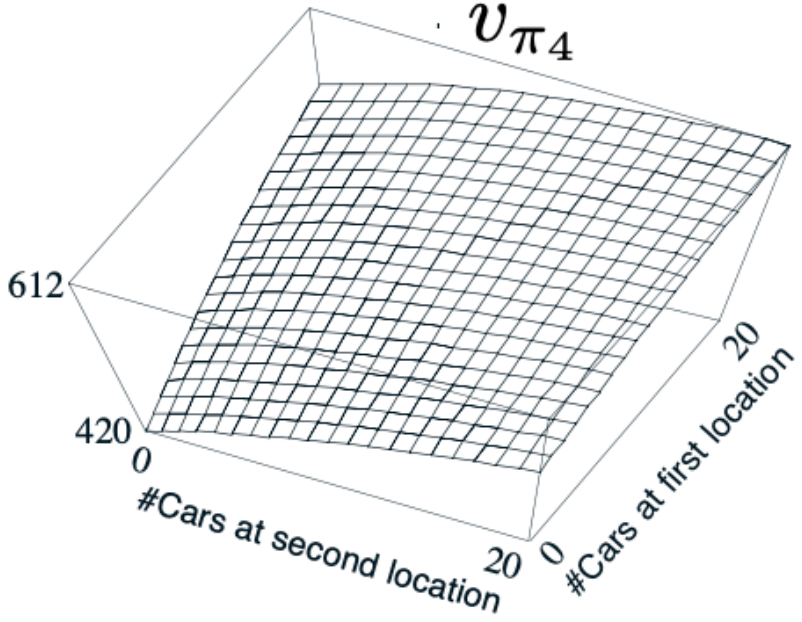


#cars in 2° location



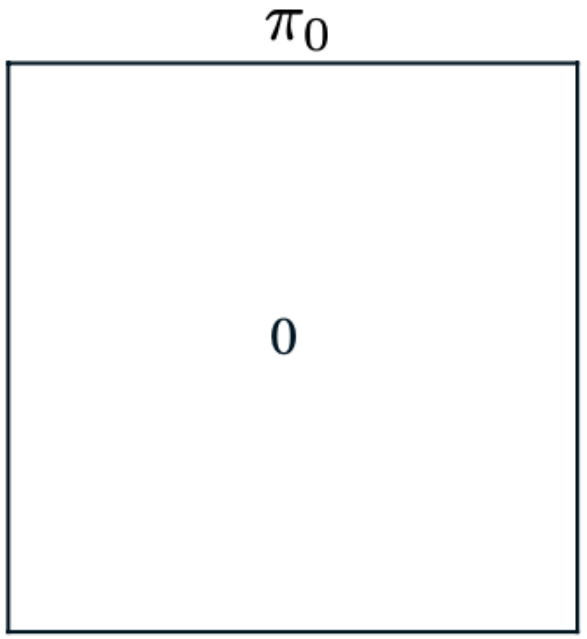
#cars in 2° location

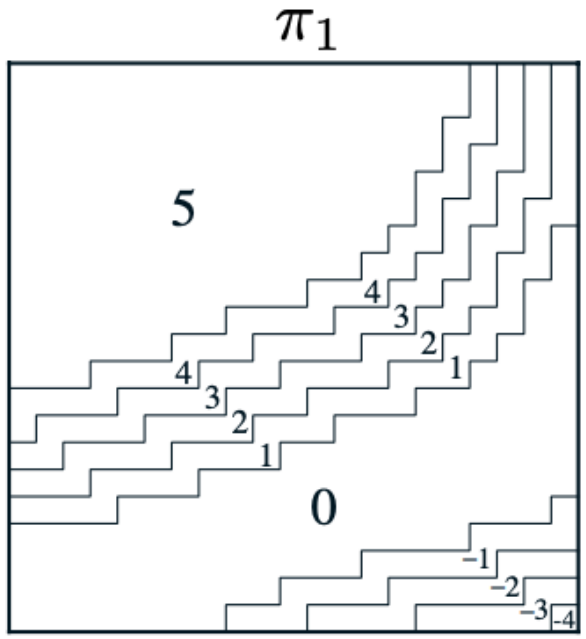


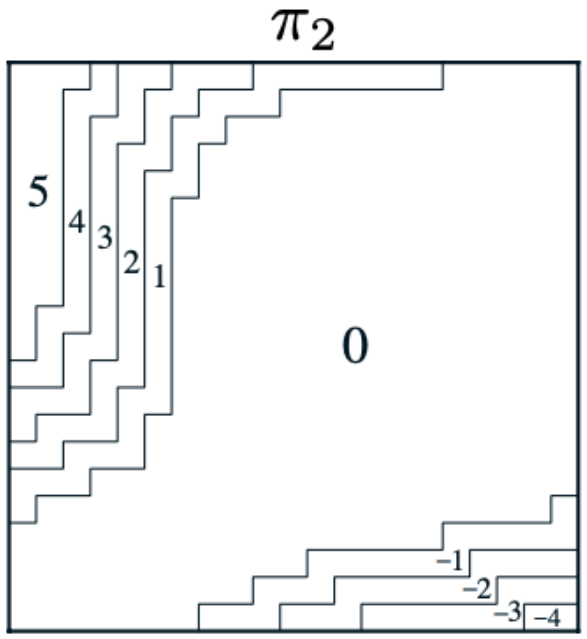


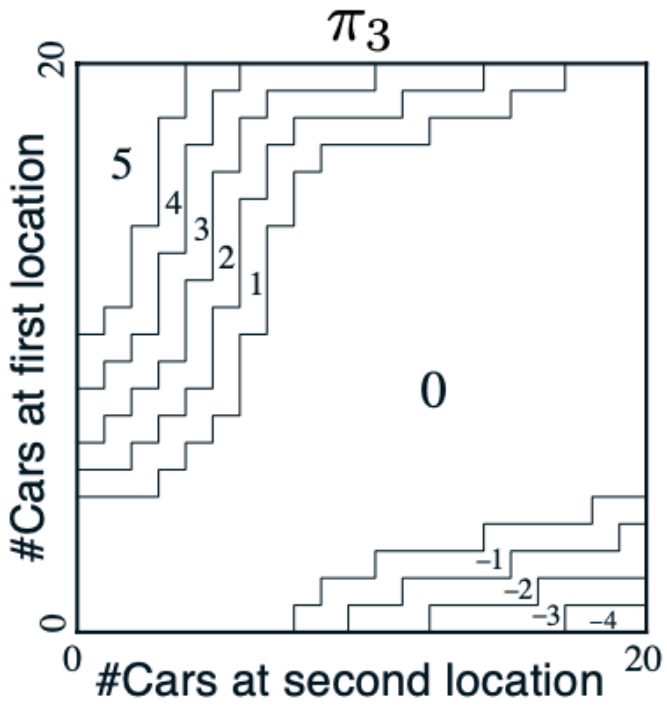
We start with the policy π_0 that moves zero cars. We then apply policy iteration

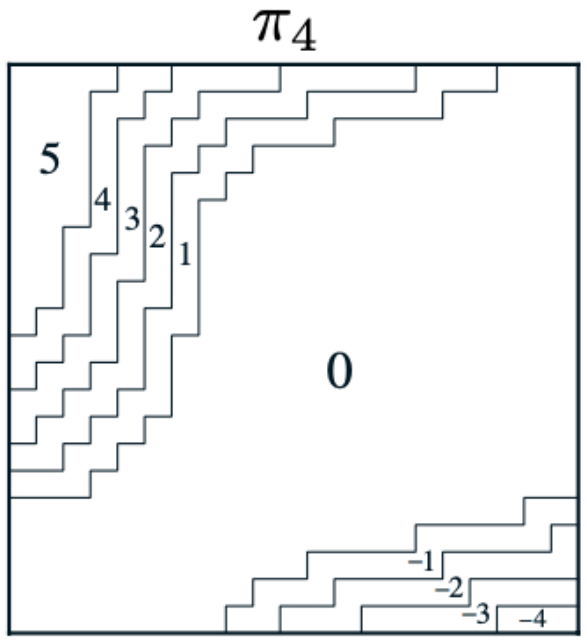
More on this on lab #3

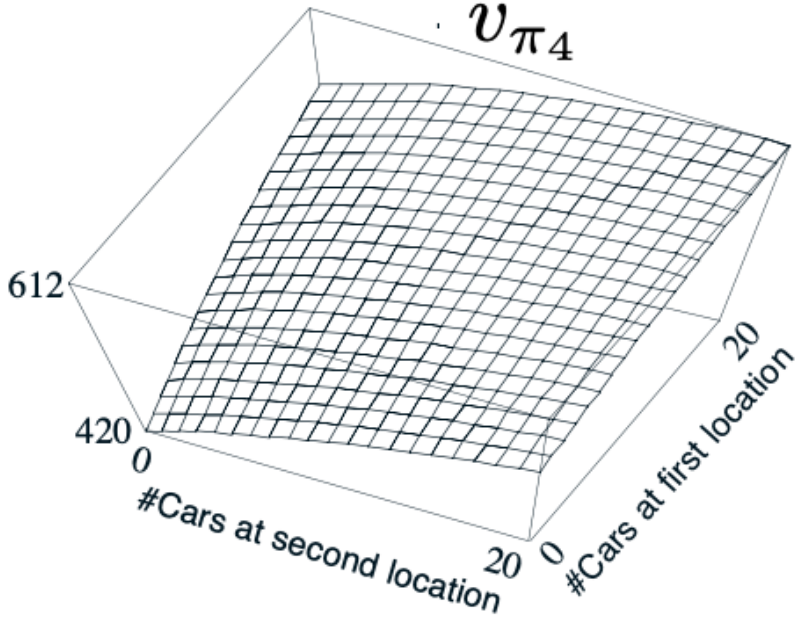










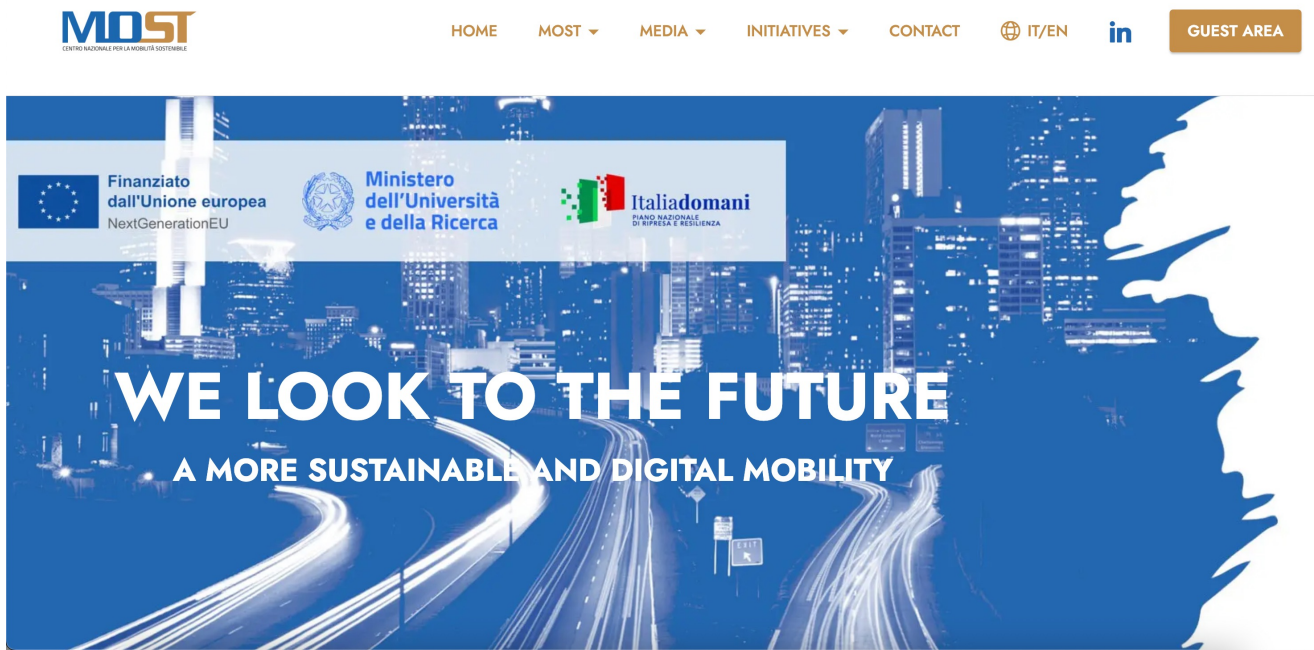


Control: Example – Jack's Car Rental

- We are dealing with similar problems for smart mobility tasks
- Optimization and fairness in micro-mobility services (ex. bike sharing)

https://arxiv.org/abs/2403.15780

https://www.centronazionalemost.it/eg/



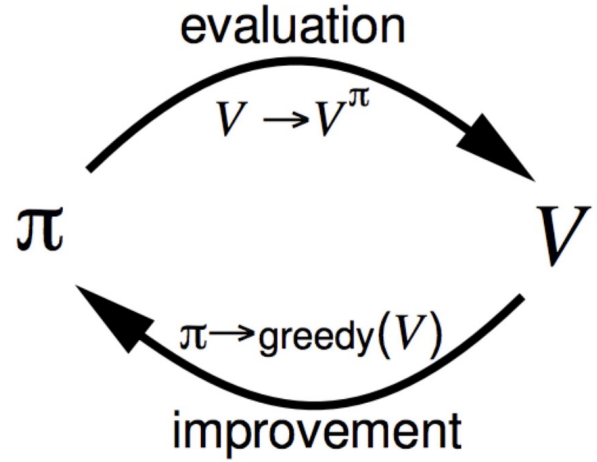
Control: Policy Iteration (Recap)

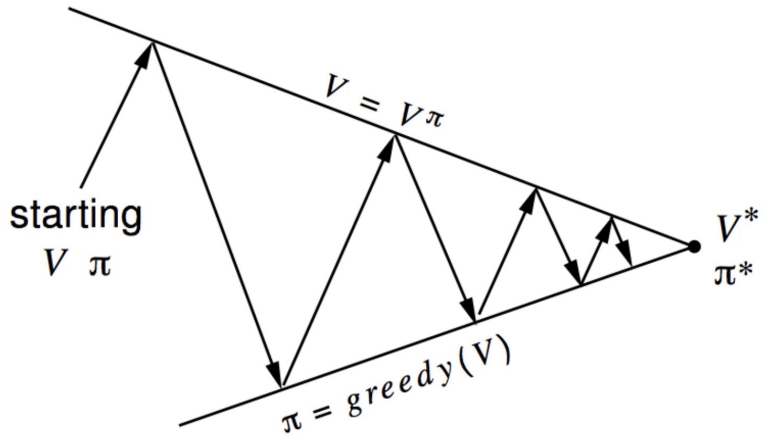
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2





Control: Policy Iteration (Recap)

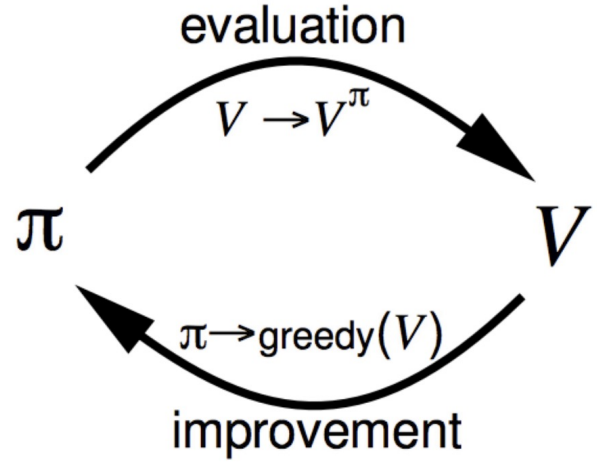
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

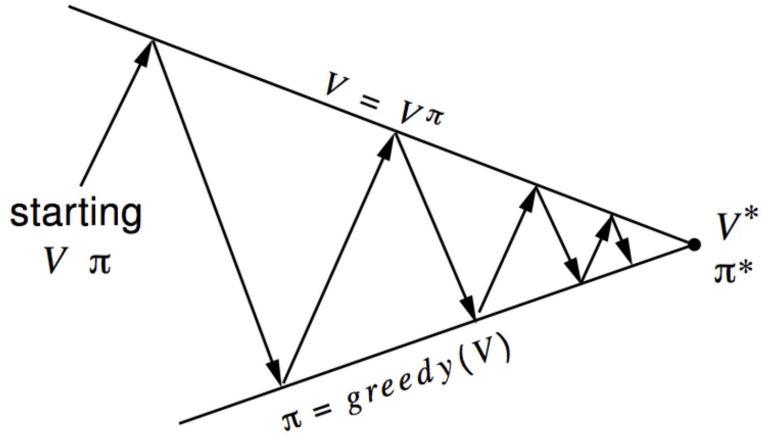
1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\left[r + \gamma V(s')\right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

This step can be quite slow… do we need the exact value of v_π?

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma V(s')\right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2





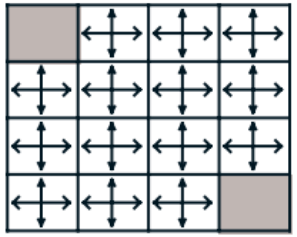
Recap

v_k for the random policy

greedy policy w.r.t. v_k

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

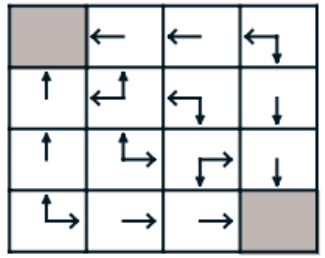
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

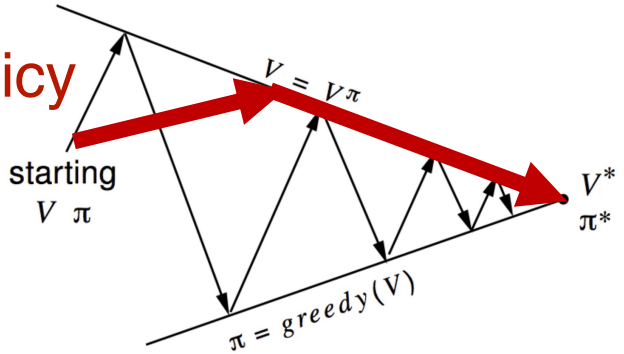
0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

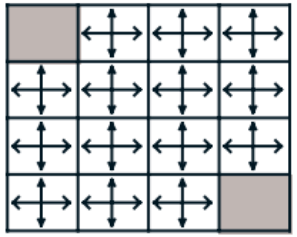
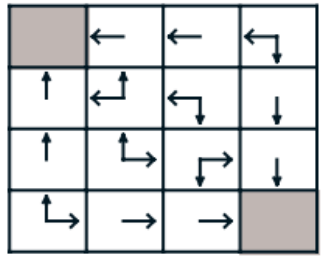
optimal policy



Recap

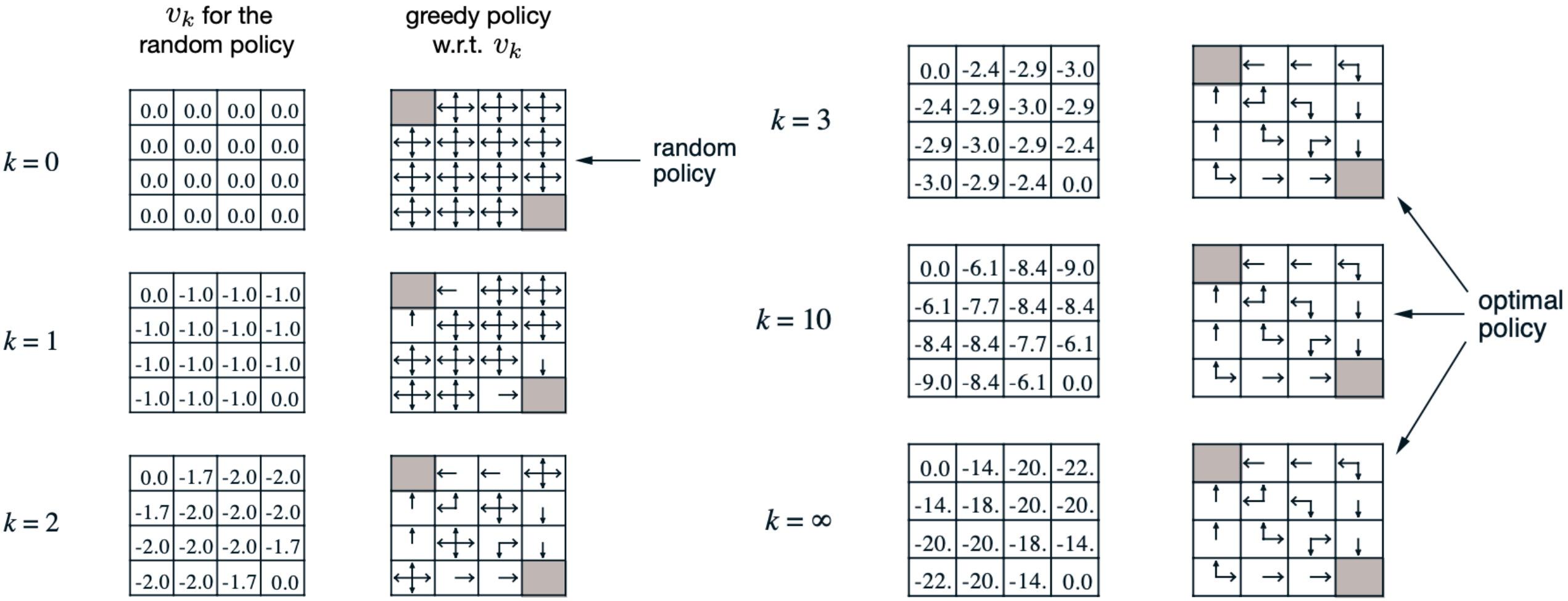
Just one iteration in this case:
evaluate the random policy -> gives us info->optimal policy



	v_k for the random policy	greedy policy w.r.t. v_k
$k = 0$	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0	 ← random policy
$k = 1$	0.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 0.0	
$k = 2$	0.0 -1.7 -2.0 -2.0 -1.7 -2.0 -2.0 -2.0 -2.0 -2.0 -2.0 -1.7 -2.0 -2.0 -1.7 0.0	
$k = 3$	0.0 -2.4 -2.9 -3.0 -2.4 -2.9 -3.0 -2.9 -2.9 -3.0 -2.9 -2.4 -3.0 -2.9 -2.4 0.0	
$k = 10$	0.0 -6.1 -8.4 -9.0 -6.1 -7.7 -8.4 -8.4 -8.4 -8.4 -7.7 -6.1 -9.0 -8.4 -6.1 0.0	
$k = \infty$	0.0 -14. -20. -22. -14. -18. -20. -20. -20. -20. -18. -14. -22. -20. -14. 0.0	optimal policy → 

knowing the real value of v_pi is not necessary to obtain the optimal policy

Recap



Control: Policy Iteration (Recap)

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\left[r + \gamma V(s')\right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

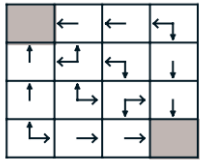
This step can be quite slow... do we need the exact value of v_π?

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma V(s')\right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

According to Policy Iteration, I should have to wait until having the true v_π

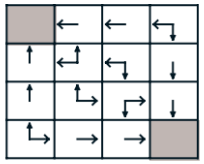
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



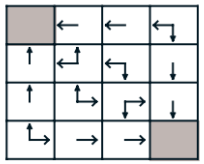
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



the idea is maybe its not necessary if we already have the optimal policy

Control: Policy Iteration (Recap)

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\left[r + \gamma V(s')\right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

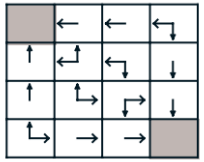
This step can be quite slow... do we need the exact value of v_π?

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma V(s')\right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

According to Policy Iteration, I should have to wait until having the true v_π

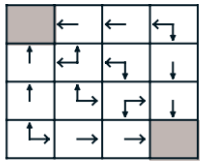
$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



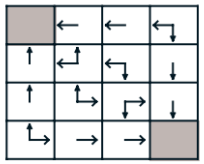
$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0



$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



When dealing with a RL problem we need to take into serious account efficiency!

Control: Policy Iteration (Recap)

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

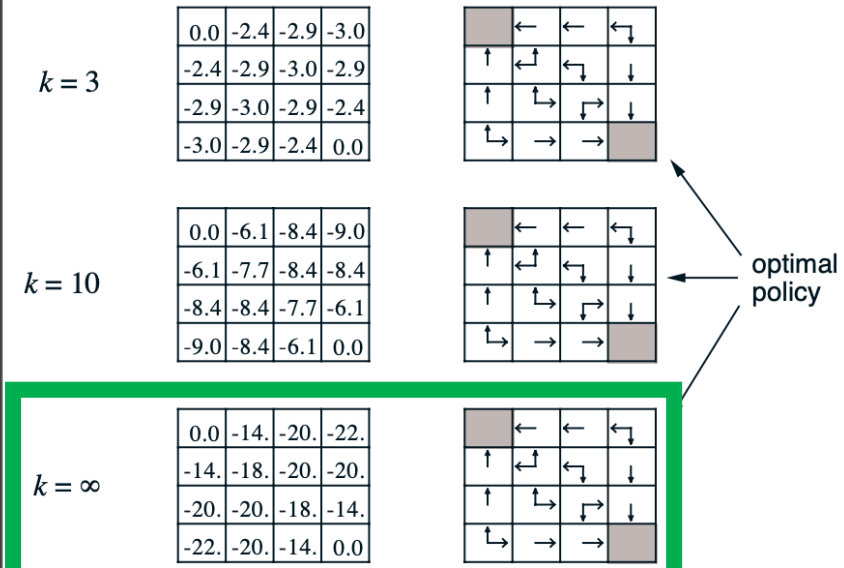
1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\left[r + \gamma V(s')\right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

This step can be quite slow... do we need the exact value of v_π?

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

According to Policy Iteration, I should have to wait until having the true v_π



We can truncate the policy evaluation step... in reality we can do that in many ways without losing the convergence guarantees of policy iteration!

Control: Value Iteration

We consider the case where the evaluation step is stopped after one update! -> This approach is called Value Iteration

Control: Value Iteration

We consider the case where the evaluation step is stopped after one update! -> This approach is called Value Iteration

The approach can be re-written as:

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big], \end{aligned}$$

for all states. We are combining policy evaluation and update in a single step!

Control: Value Iteration

We consider the case where the evaluation step is stopped after one update! -> This approach is called Value Iteration

The approach can be re-written as:

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big], \end{aligned}$$

for all states. We are combining policy evaluation and update in a single step!

We can show (but we won't) that this approach would reach v^*

Control: Value Iteration

We consider the case where the evaluation step is stopped after one update! -> This approach is called Value Iteration

The approach can be re-written as:

$$v_{k+1}(s) \doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a]$$

$$= \max_a \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_k(s')\Big],$$

This is an update rule!

for all states. We are combining policy evaluation and update in a single step!

We can show (but we won't) that this approach would reach v^*

Policy Evaluation (Prediction) vs. Value Iteration (Control)

Policy evaluation uses the Bellman Expectation Equation as an update rule

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_\pi(s')\right]$$

$$\begin{aligned} v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_k(s')\right] \end{aligned}$$

Value iteration uses the Bellman Optimality Equation instead!

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_*(s')\right]$$

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_k(s')\right], \end{aligned}$$

Policy Evaluation (Prediction) vs. Value Iteration (Control)

Policy evaluation uses the Bellman Expectation Equation as an update rule

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_\pi(s')\right]$$

$$\begin{aligned} v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_k(s')\right] \end{aligned}$$

Value iteration uses the Bellman Optimality Equation instead!

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_*(s')\right]$$

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_k(s')\right], \end{aligned}$$

Instead of averaging over all actions, we just consider the 'best'

Control: Value Iteration

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation

Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Inizialization

Loop:

| $\Delta \leftarrow 0$

| Loop for each $s \in \mathcal{S}$:

| $v \leftarrow V(s)$

| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that

$\pi(s) = \text{argmax}_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

Control: Value Iteration

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \text{argmax}_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

Updates: we are also considering here the 'in place' implementation

Control: Value Iteration

Value Iteration, for estimating $\pi \approx \pi_*$

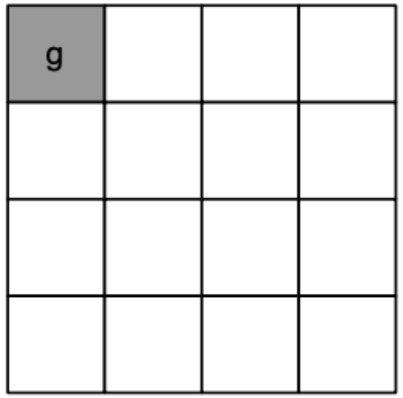
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Termination criteria

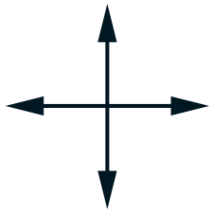
Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \text{argmax}_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

Control: Value Iteration on Small Gridworld



Problem

Nonterminal states: white box
Terminal states: grey boxes



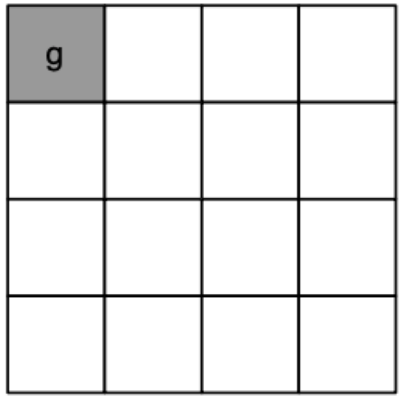
actions

$$\mathcal{A} = \{\texttt{up}, \texttt{down}, \texttt{right}, \texttt{left}\}$$

$$R_t = -1$$

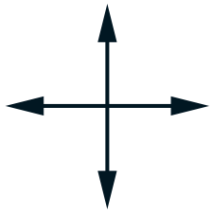
on all transitions

Control: Value Iteration on Small Gridworld



Problem

Nonterminal states: white box
Terminal states: grey boxes



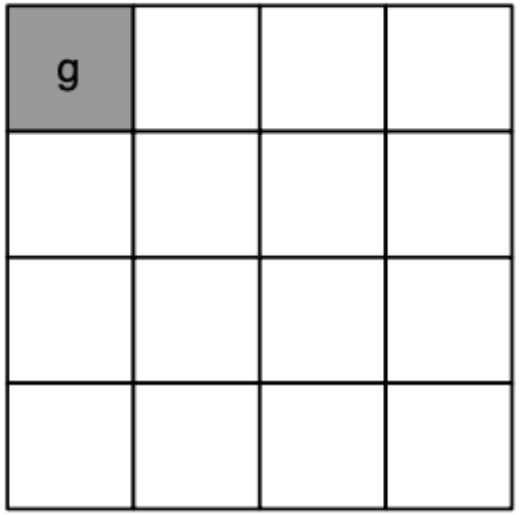
actions

$$\mathcal{A} = \{\texttt{up}, \texttt{down}, \texttt{right}, \texttt{left}\}$$

The book has another example ‘the gambler problem’: take a look!

$$R_t = -1$$

on all transitions



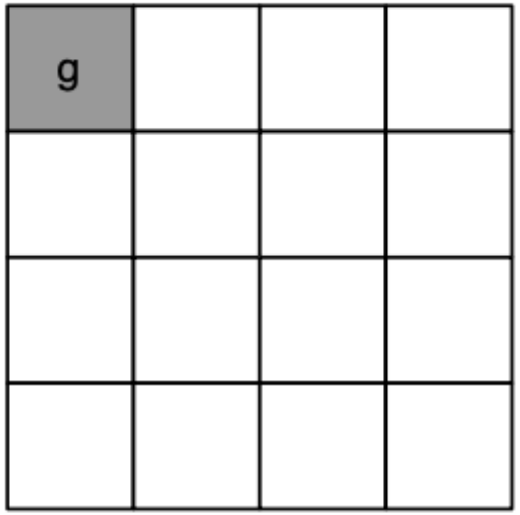
Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

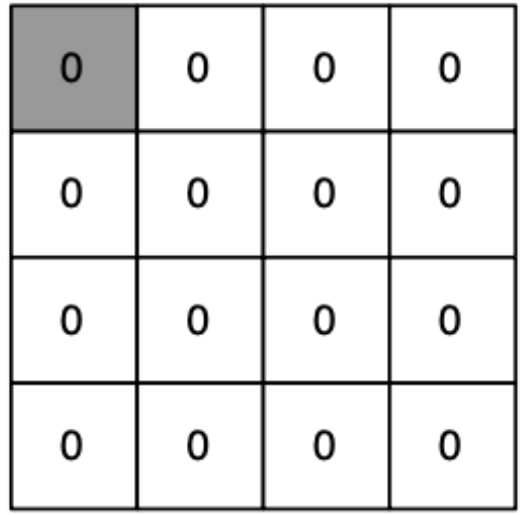
V_1

I start considering a random uniform policy

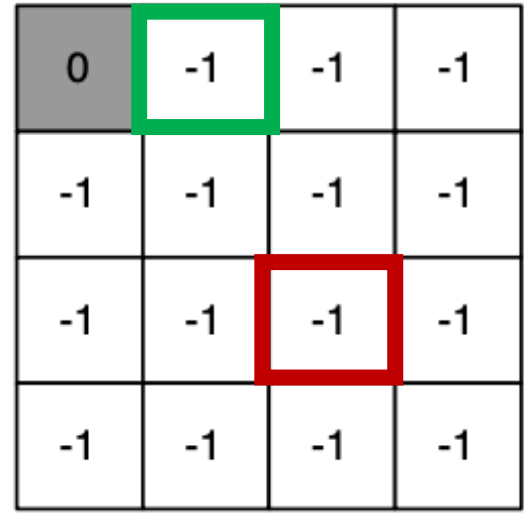
$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big],
\end{aligned}
$$



Problem

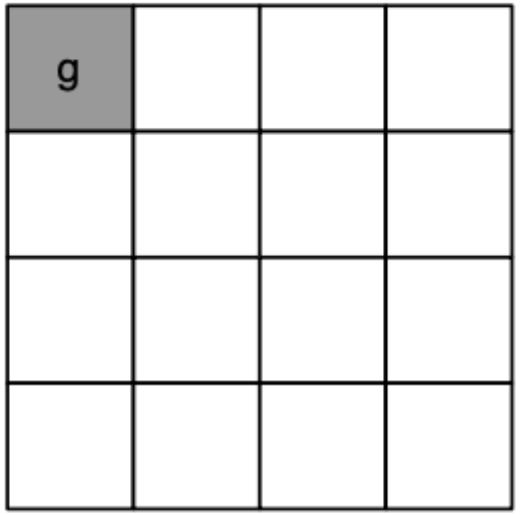


V_1

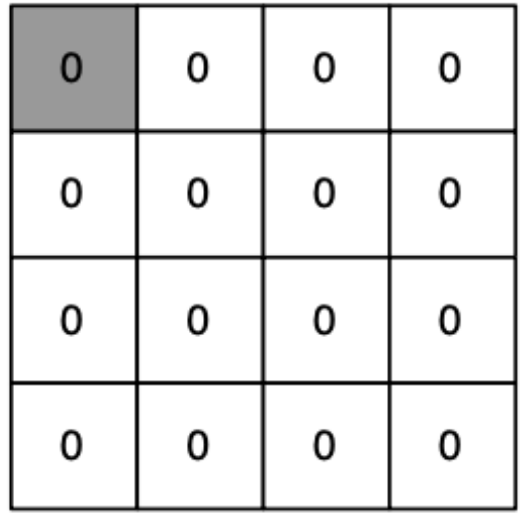


V_2

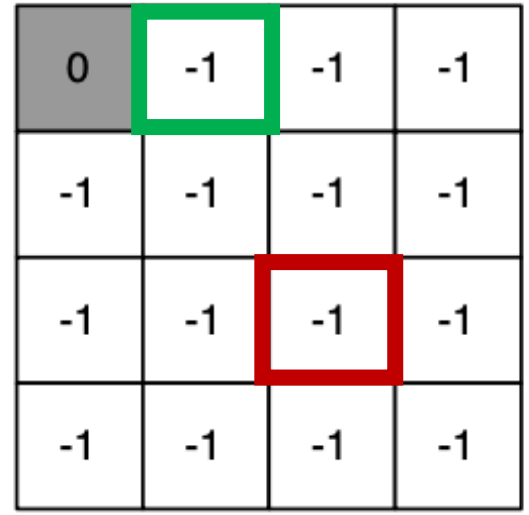
$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big],
\end{aligned}
$$



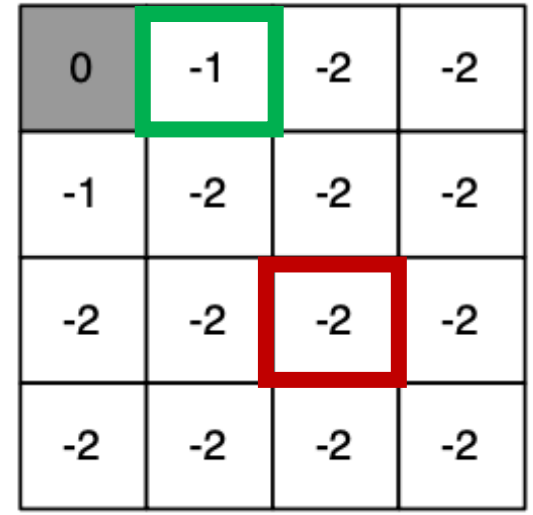
Problem



V_1



V_2



V_3

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big],
\end{aligned}
$$

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big],
\end{aligned}
$$

g			

This may work also for problems with non-terminal states!

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

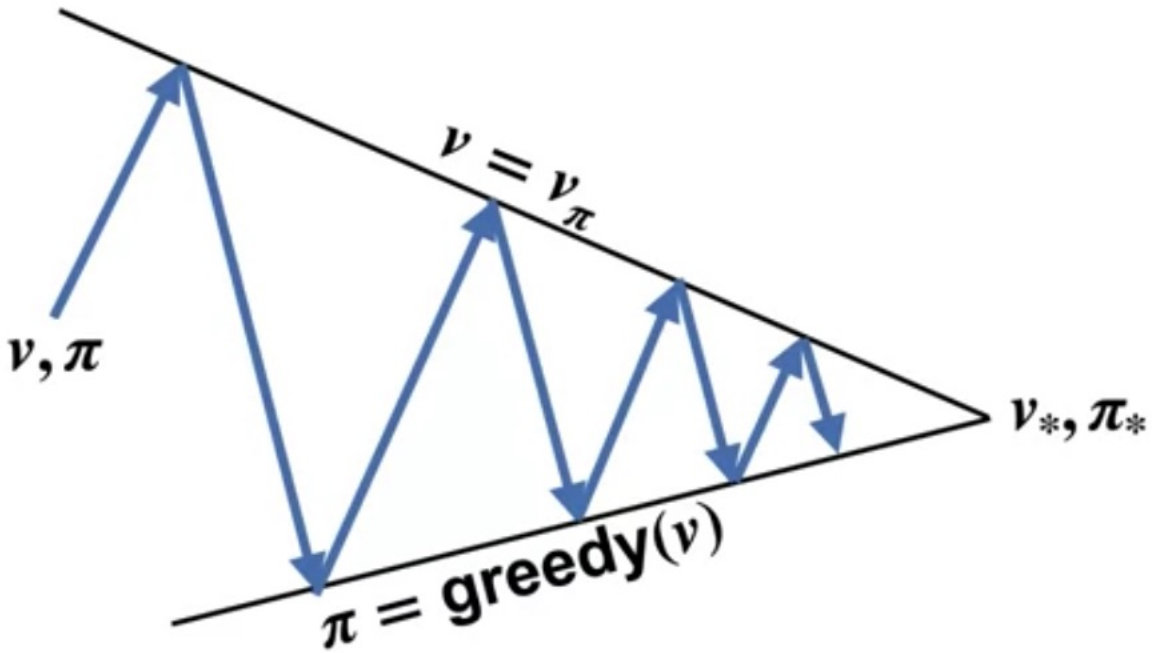
V_6

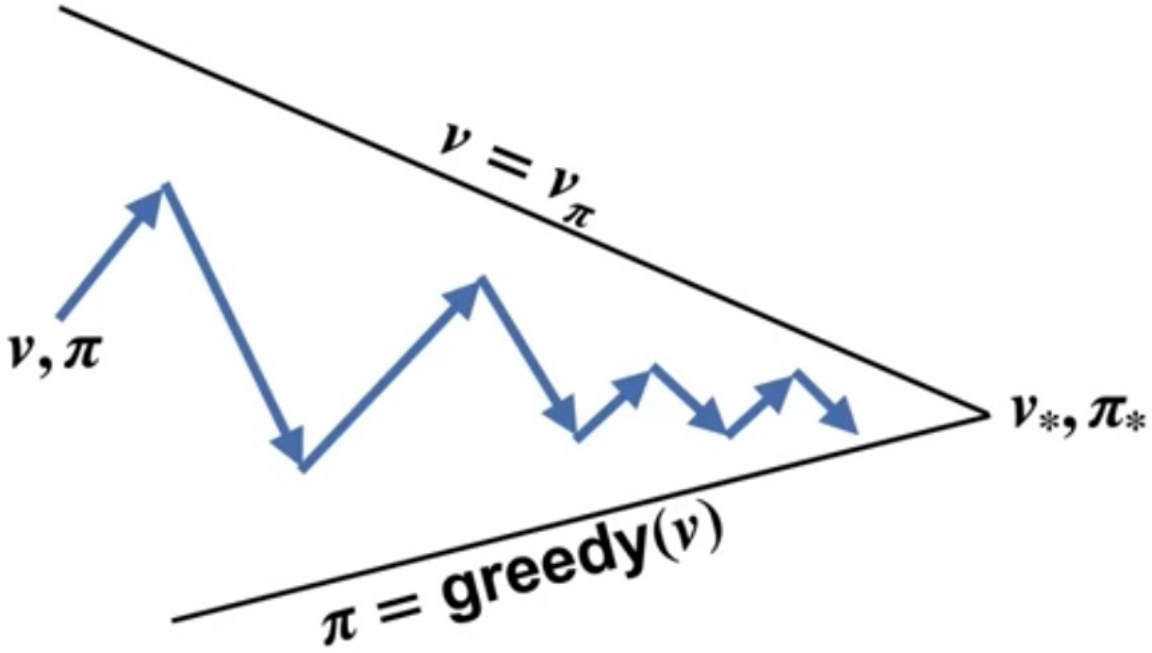
0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Control:

Policy Iteration vs. Value Iteration

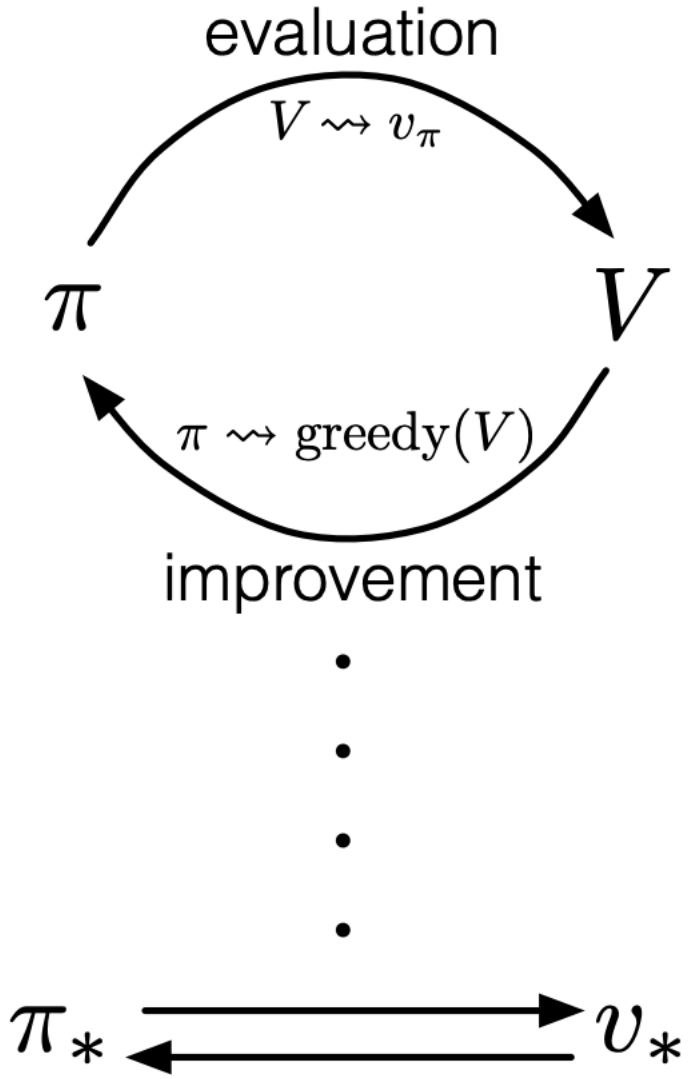




A. White, M. White 'Fundamentals of Reinforcement Learning'

Generalized Policy Iteration (GPI)

- We may end up with strategies that are hybrid version of Policy Iteration+Value Iteration
- GPI 'unifies' classical DP methods and include all the approaches where the procedure of evaluating and improving a policy interact



(Synchronous) DP Algorithms Summary

Problem	Bellman Equation	Algorithm
Prediction	Bellman expectation equation	Iterative Policy Evaluation
Control	Bellman expectation equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman optimality equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v^*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions n states
- Could also apply to action-value function $q_\pi(s,a)$ or $q^*(s,a)$, but in that case the complexity would be $O(m^2n^2)$ per iteration

(Synchronous) DP Algorithms Summary

Problem	Bellman Equation	Algorithm
Prediction	Bellman expectation equation	Iterative Policy Evaluation
Control	Bellman expectation equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman optimality equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v^*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions n states
- Could also apply to action-value function $q_\pi(s,a)$ or $q^*(s,a)$, but in that case the complexity would be $O(m^2n^2)$ per iteration
- How can I speed things up? Asynchronous DP

Asynchronous DP

- All the seen approaches have always a ‘for all state’ procedure: for such reason they are called synchronous (all states are backed up in parallel)

Asynchronous DP

- All the seen approaches have always a ‘for all state’ procedure: for such reason they are called synchronous (all states are backed up in parallel)
- Asynchronous DP methods give us the freedom to update states in any order (backs up states individually, in any order)

Asynchronous DP

- All the seen approaches have always a ‘for all state’ procedure: for such reason they are called synchronous (all states are backed up in parallel)
- Asynchronous DP methods give us the freedom to update states in any order (backs up states individually, in any order)
- In asynchronous DP for each selected state, apply the appropriate backup: in other words, they asynchronous DP methods are in-place approaches that are not organized in terms of systematic sweeps of the state set

Asynchronous DP

- All the seen approaches have always a ‘for all state’ procedure: for such reason they are called synchronous (all states are backed up in parallel)
- Asynchronous DP methods give us the freedom to update states in any order (backs up states individually, in any order)
- In asynchronous DP for each selected state, apply the appropriate backup: in other words, they asynchronous DP methods are in-place approaches that are not organized in terms of systematic sweeps of the state set
- Asynchronous DP methods can significantly reduce computation: If the state set is very large, then even a single sweep can be prohibitively expensive!

Asynchronous DP Example: Policy Evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V'(s) \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\left[r + \gamma V'(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Synchronous update implementation

I am using a buffer to have systematic, ordered updates

Asynchronous DP Example: Policy Evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

'In place' update implementation

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.
- Asynchronous DP methods still converge if all states continue to be selected

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.
- Asynchronous DP methods still converge if all states continue to be selected
- Other examples:

1. We can select the states to which we apply updates to improve the algorithm's rate of progress in a particular state.

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.
- Asynchronous DP methods still converge if all states continue to be selected
- Other examples:

1. We can select the states to which we apply updates to improve the algorithm's rate of progress in a particular state.
2. Or we might even try to skip updating some states entirely if they are not relevant to optimal behavior

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.
- Asynchronous DP methods still converge if all states continue to be selected
- Other examples:

1. We can select the states to which we apply updates to improve the algorithm's rate of progress in a particular state.
2. Or we might even try to skip updating some states entirely if they are not relevant to optimal behavior
3. We may collect trajectories and see the states that are used by a policy and just update those

Asynchronous DP

- Asynchronous DP approaches update the values of states in any order whatsoever, using whatever values of other states happen to be available.
- Asynchronous DP methods still converge if all states continue to be selected
- Other examples:

1. We can select the states to which we apply updates to improve the algorithm's rate of progress in a particular state.
2. Or we might even try to skip updating some states entirely if they are not relevant to optimal behavior
3. We may collect trajectories and see the states that are used by a policy and just update those
4. [Prioritized Sweeping – Chapter 8] Or we can try to order the updates to let value information propagate from state to state in an efficient way: some states may not need their values updated as often as others.

An example of Asynchronous DP method: Prioritized Sweeping (chapter 8)

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

Dynamic Programming: Exam

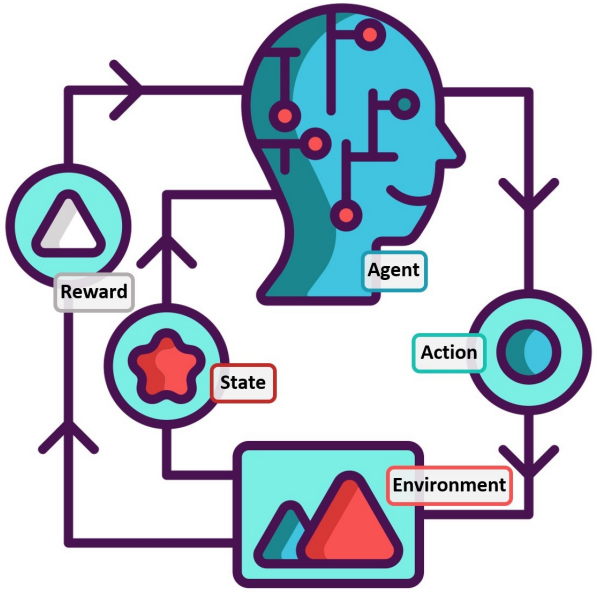
- All the content of Chapter 4 are Exam material!
- Pay particular attention to the algorithms!

Monte Carlo Methods (Chapter 5)

Some problems that can be formalized in a RL fashion (from Lecture #01)

- Autonomous agents (self-driving cars, drones, robots...)
- Games
- HVAC (Heating, Ventilating, Air Conditioning) energy optimization
- Trading and Portfolio management
- Online advertising & Recommendation systems (news, items, ...)
- Healthcare, biology

....

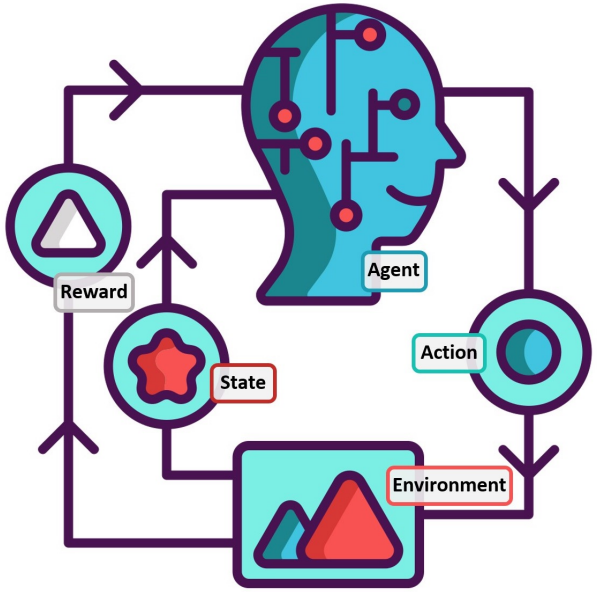


Some problems that can be formalized in a RL fashion (from Lecture #01)

- Autonomous agents (self-driving cars, drones, robots...)
- Games
- HVAC (Heating, Ventilating, Air Conditioning) energy optimization
- Trading and Portfolio management
- Online advertising & Recommendation systems (news, items, ...)
- Healthcare, biology

....

All of these problems can be formalized as MDPs ($\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$), but defining an MDP may be impossibile!
We can have an unknown environment behaviour!

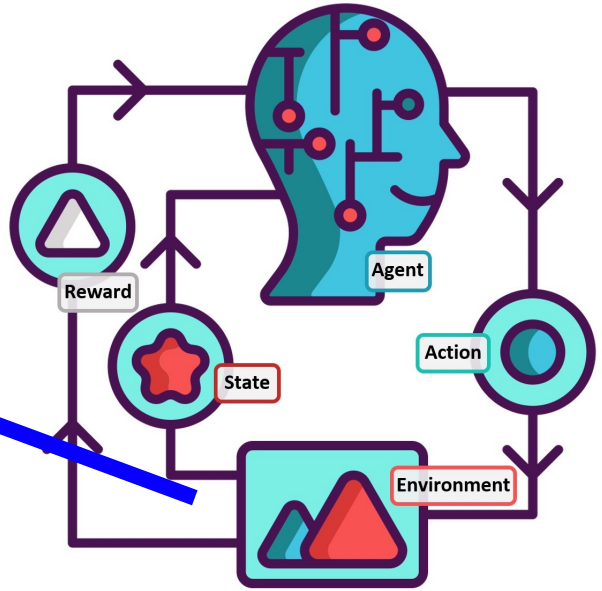


Some problems that can be formalized in a RL fashion (from Lecture #01)

- Autonomous agents (self-driving cars, drones, robots…)
- Games
- HVAC (Heating, Ventilating, Air Conditioning) energy optimization
- Trading and Portfolio management
- Online advertising & Recommendation systems (news, items, …)
- Healthcare, biology

….

All of these problems can be formalized as MDPs ($\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$), but defining an MDP may be impossibile!
We can have an unknown environment behaviour!



Monte Carlo Methods

- Monte Carlo (MC) Methods are simple, but effective, approaches to deal with the 'full' RL problem (without knowledge of the MDP's $\mathcal{P}$ $\mathcal{R}$)
- MC methods only require experience: sample sequences of states, actions and rewards (real world data gathered by interacting with the environment)
- MC will be our first 'model-free' approach (for both prediction and control)
- With MC methods we require episodic tasks

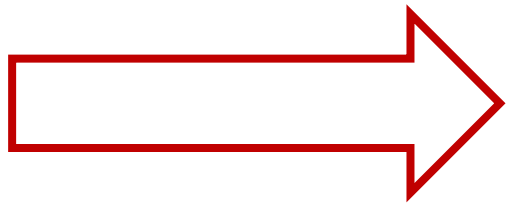
Monte Carlo Methods

- Monte Carlo (MC) Methods are simple, but effective, approaches to deal with the 'full' RL problem (without knowledge of the MDP's $\mathcal{P}$ $\mathcal{R}$)
- MC methods only require experience: sample sequences of states, actions and rewards (real world data gathered by interacting with the environment)
- MC will be our first 'model-free' approach (for both prediction and control)
- With MC methods we require episodic tasks

We will only look at prediction today!
Control on next lecture!

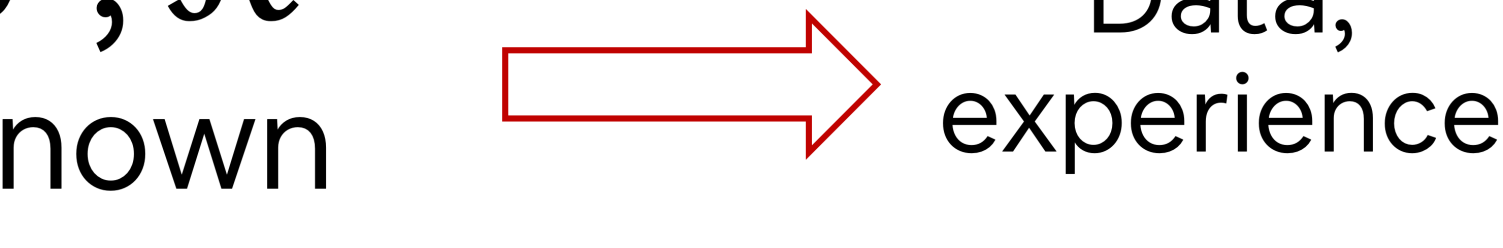
Pay attention, from now onwards in this course:

$\mathcal{P}, \mathcal{R}$ known



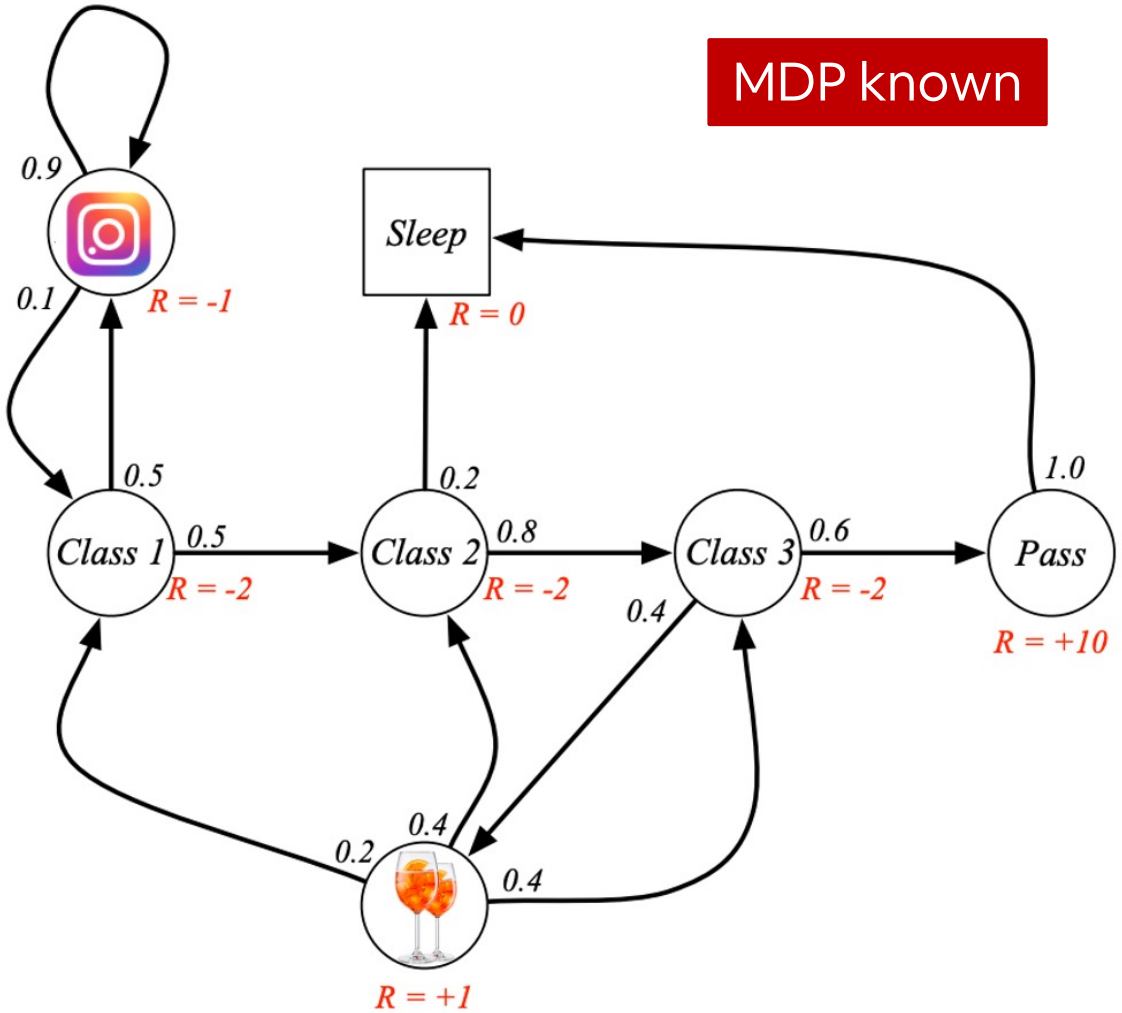
Data, experience

$\mathcal{P}, \mathcal{R}$
known

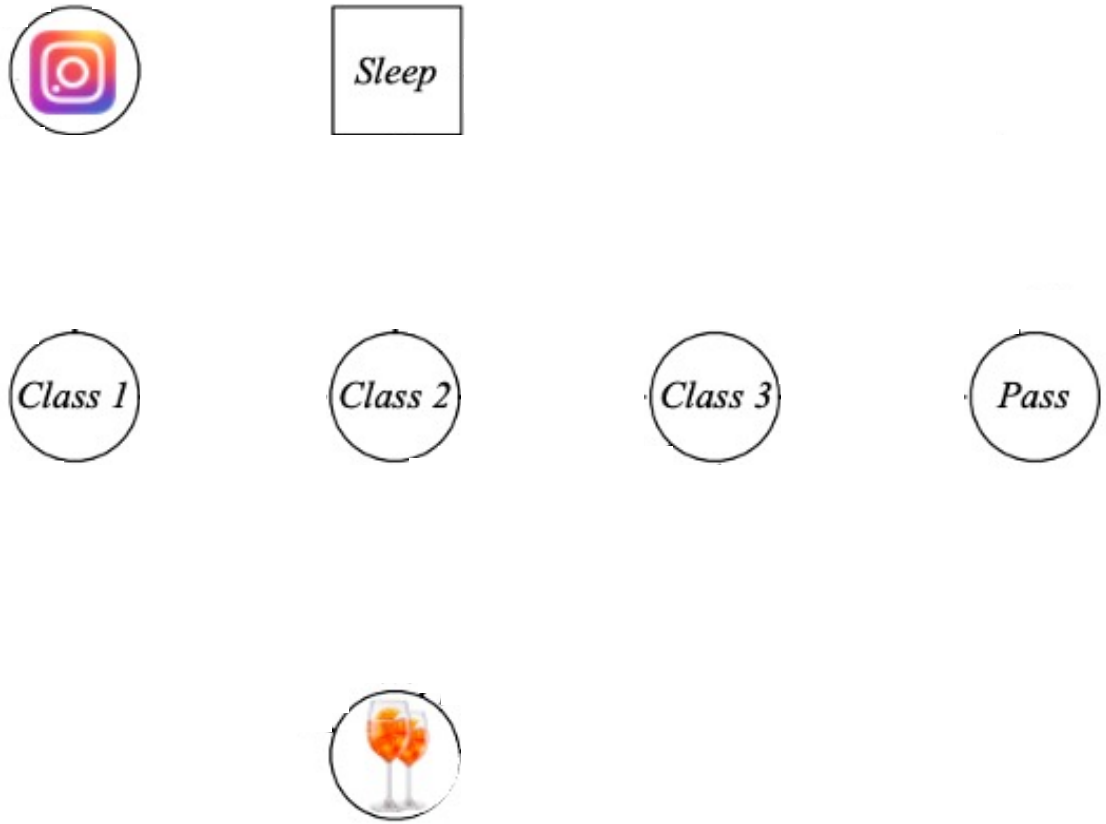


Data,
experience

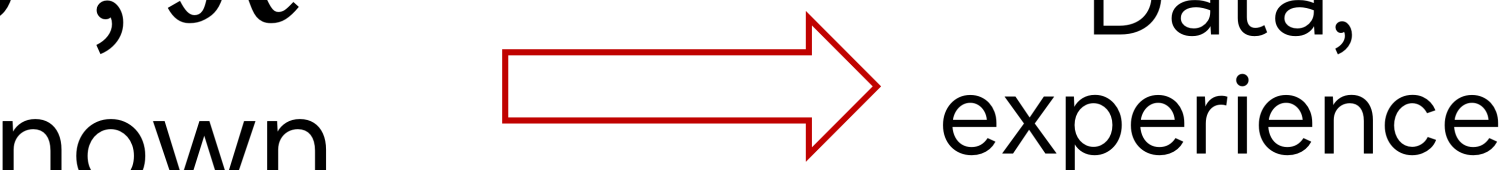
MDP known



MDP unknown

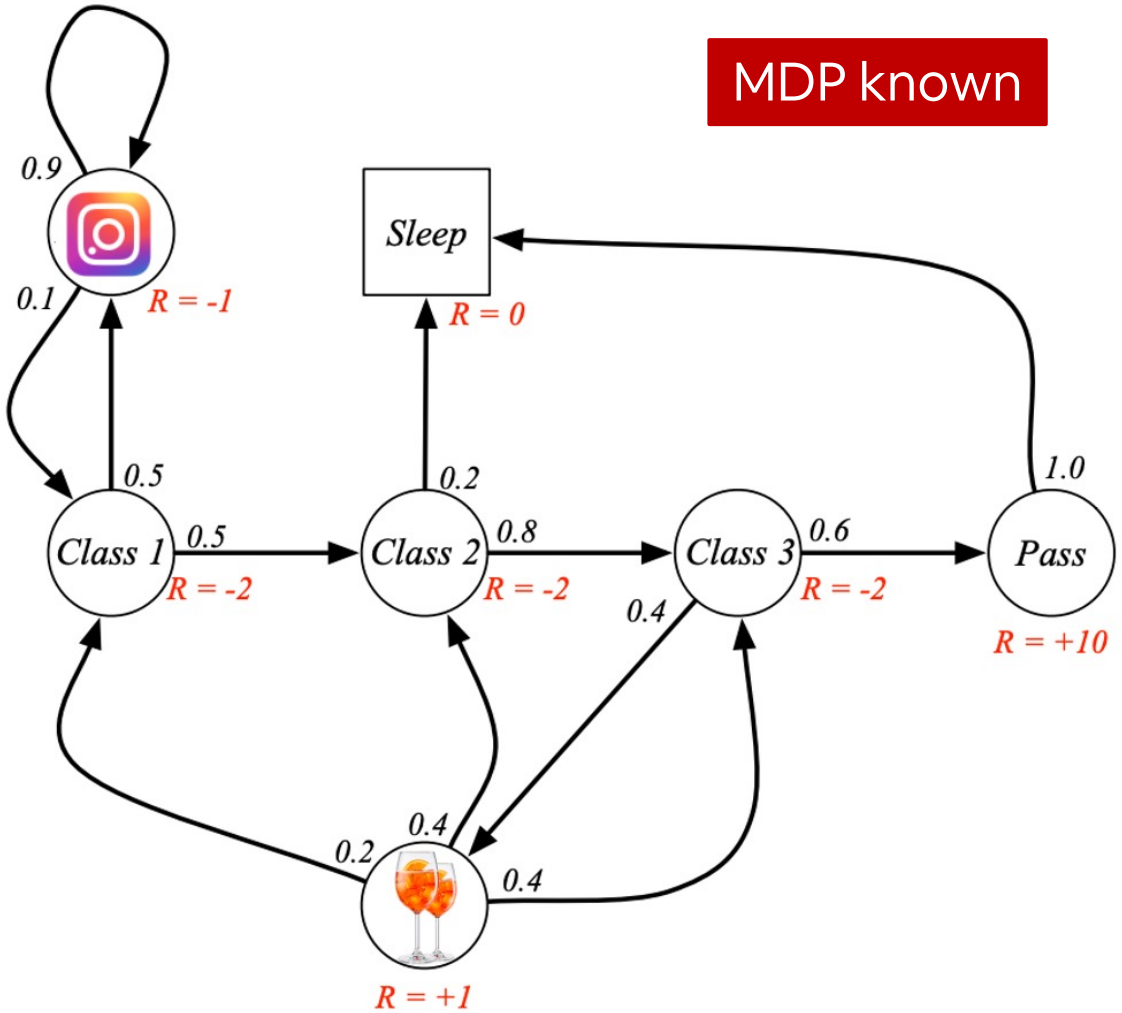


$\mathcal{P}, \mathcal{R}$ known

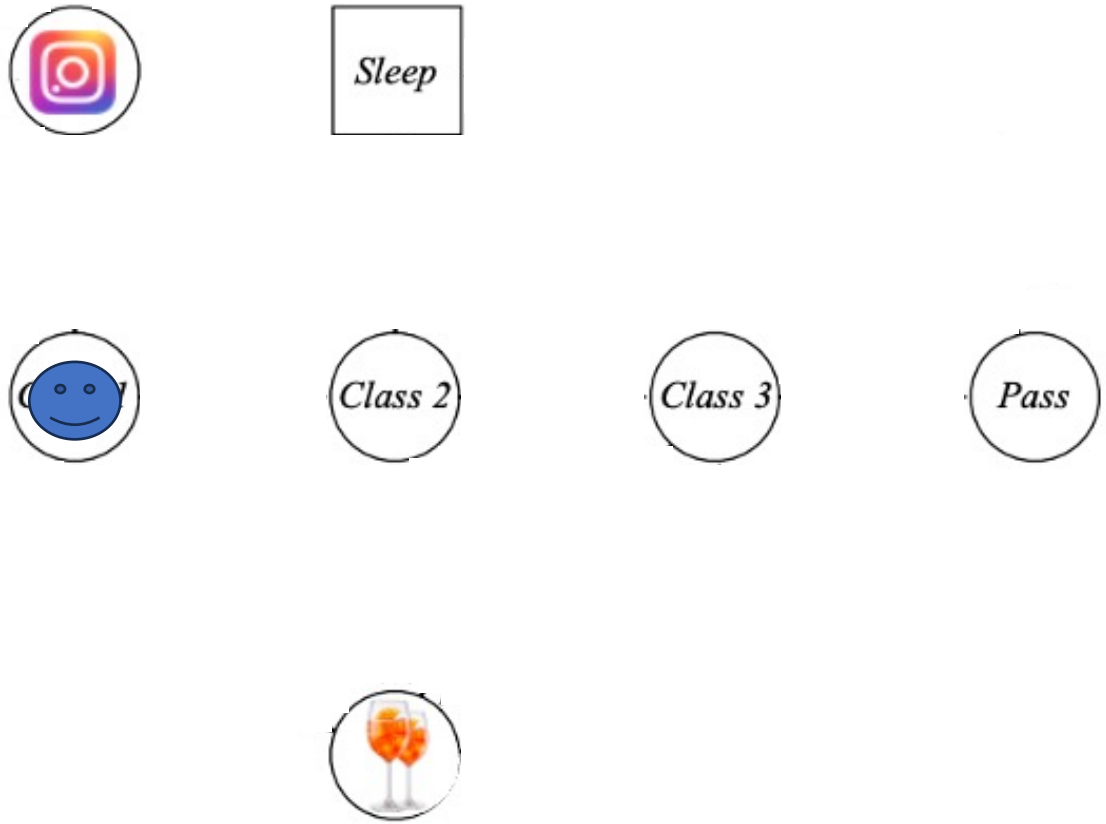


Data, experience

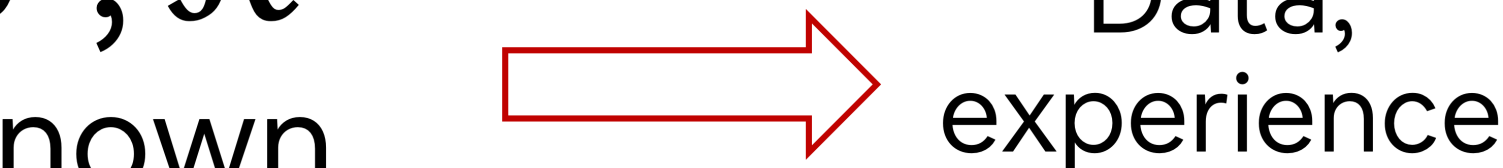
MDP known



MDP unknown

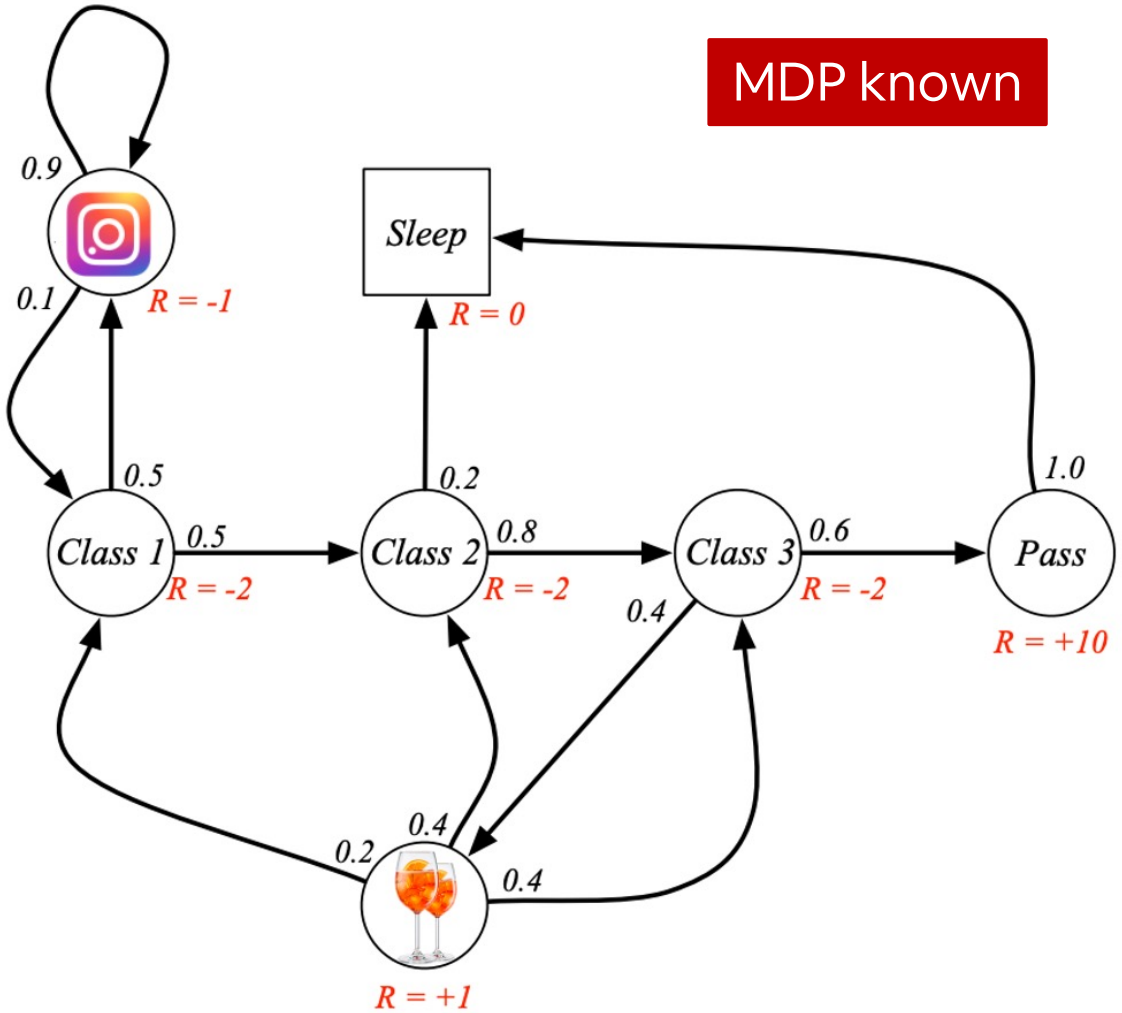


$\mathcal{P}, \mathcal{R}$ known

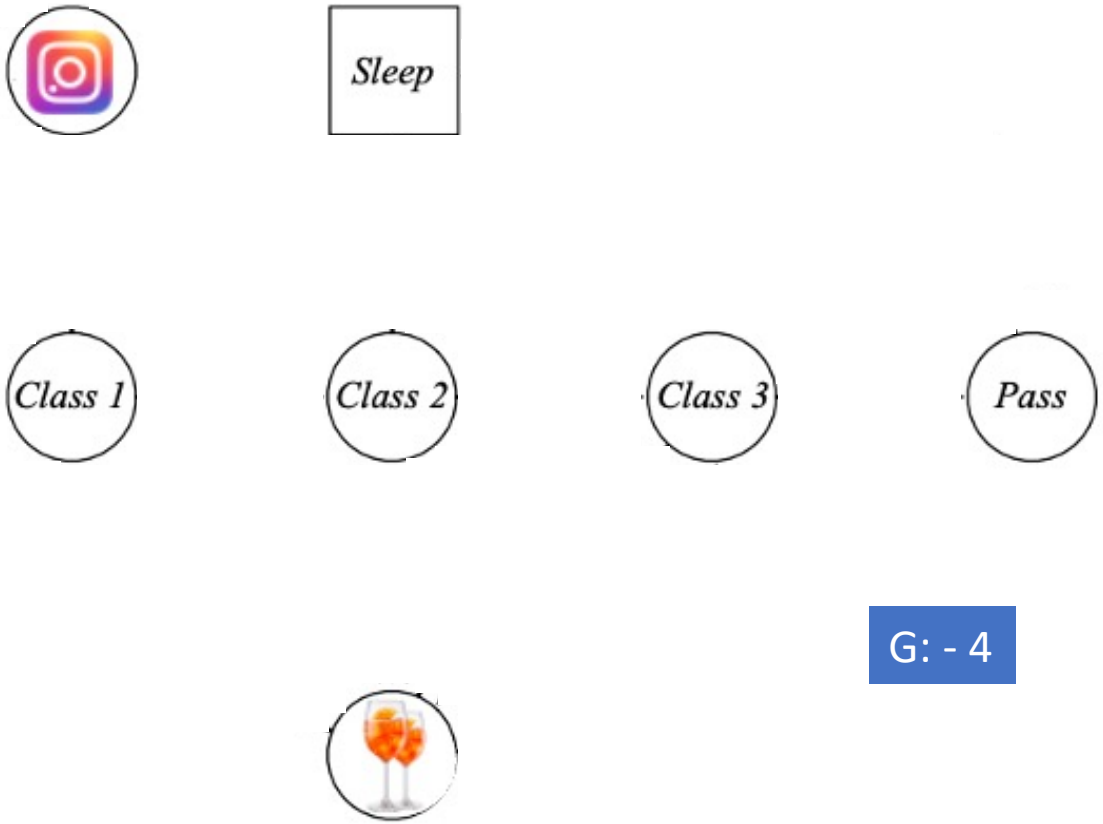


Data, experience

MDP known



MDP unknown



G: - 4

Monte Carlo Methods

- The term ‘Monte Carlo’ is often used more broadly for any estimation method that relies on repeated random sampling
- For example: estimating the average sum of 12 dice rolls

Sum of Faces	Probability
12	?
13	?
14	?
…	
71	?
72	?

Samples

39	38	49	36	33	47	37	49	41	40	44	44
35	43	47	28	46	52	47	45	43	43	42	43
40	32	29	48	43	49	48	39	42	35	44	48
36	32	41	49	34	54	37	42	37	38	42	50
39	37	45	49	44	34	38	50	35	36	42	45

A. White, M. White ‘Fundamentals of Reinforcement Learning’

Monte Carlo Methods

- The term ‘Monte Carlo’ is often used more broadly for any estimation method that relies on repeated random sampling
- For example: estimating the average sum of 12 dice rolls

Sum of Faces	Probability
12	?
13	?
14	?
...	
71	?
72	?

Samples

39 38 49 36 33

35 43 47 28 46 52 47 45 43 43 42 43

40 32 29 48 43 49 48 39 42 35 44 48

36 32 41 49 34 54 37 42 37 38 42 50

39 37 45 49 44 34 38 50 35 36 42 45

Average = 41.57

MC methods estimate expectations by averaging over a large number of random samples

A. White, M. White ‘Fundamentals of Reinforcement Learning’

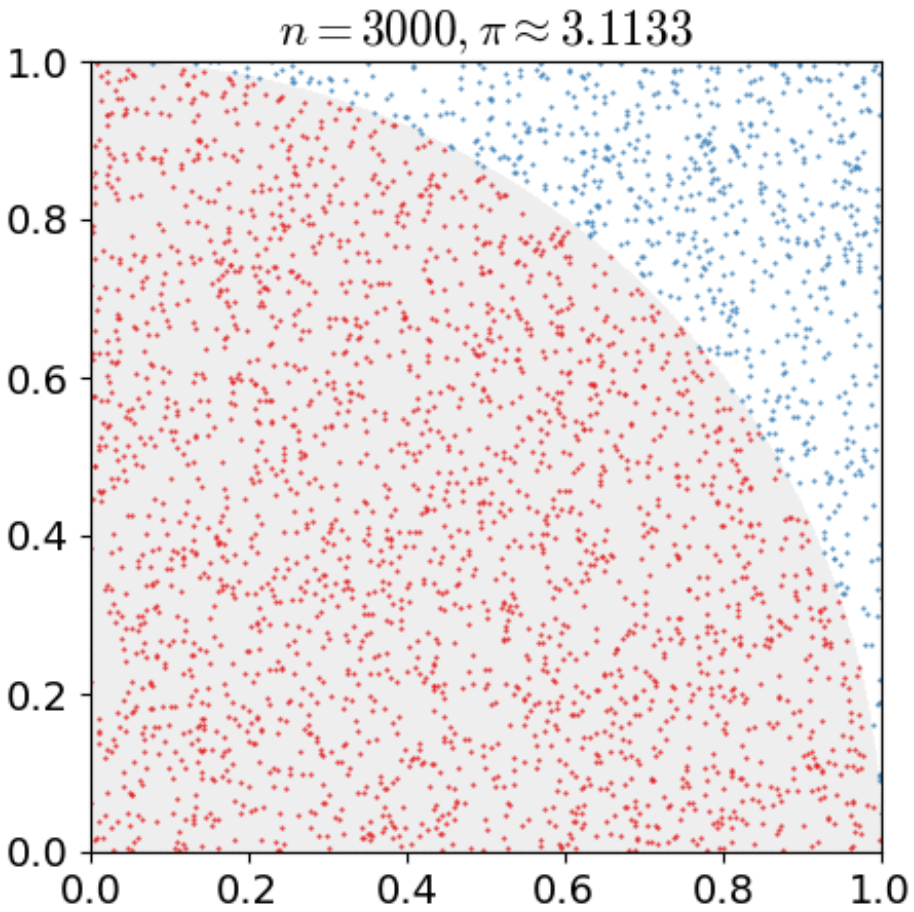
Expectations in RL

- In RL we are looking for estimation state value functions (and action-value functions) that are defined as expectations

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

- If we have episodes, we can see returns (we have data)!
- We can approximate the 'true' expected returns with the average of the returns seen if we get many data collected!



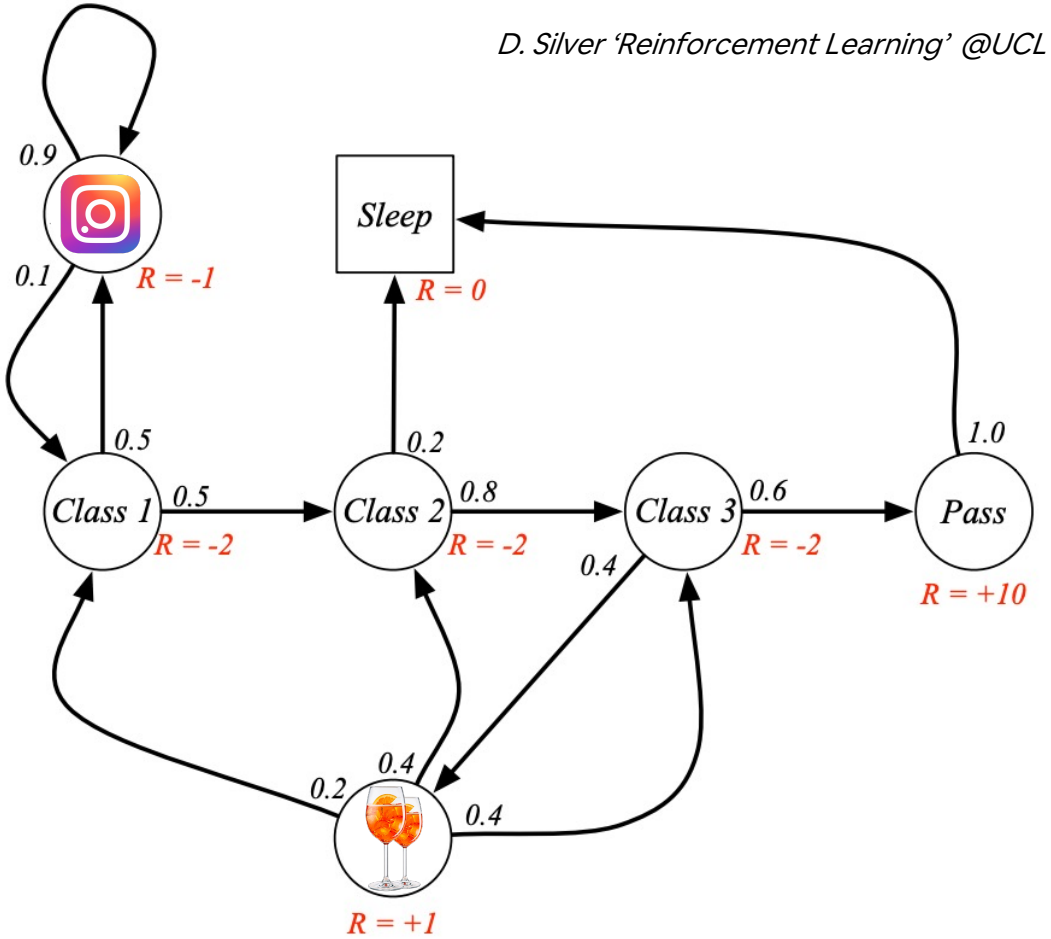
Prediction: Let's see an example!

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(+10)Pass(0)Sleep

Let's estimate the value function for state C1 with MC methods:

D. Silver 'Reinforcement Learning' @UCL



- For simplicity I am showing you a MRP, but the concept holds also for the MPD
- I don't know the MDP (the graph is not available!)

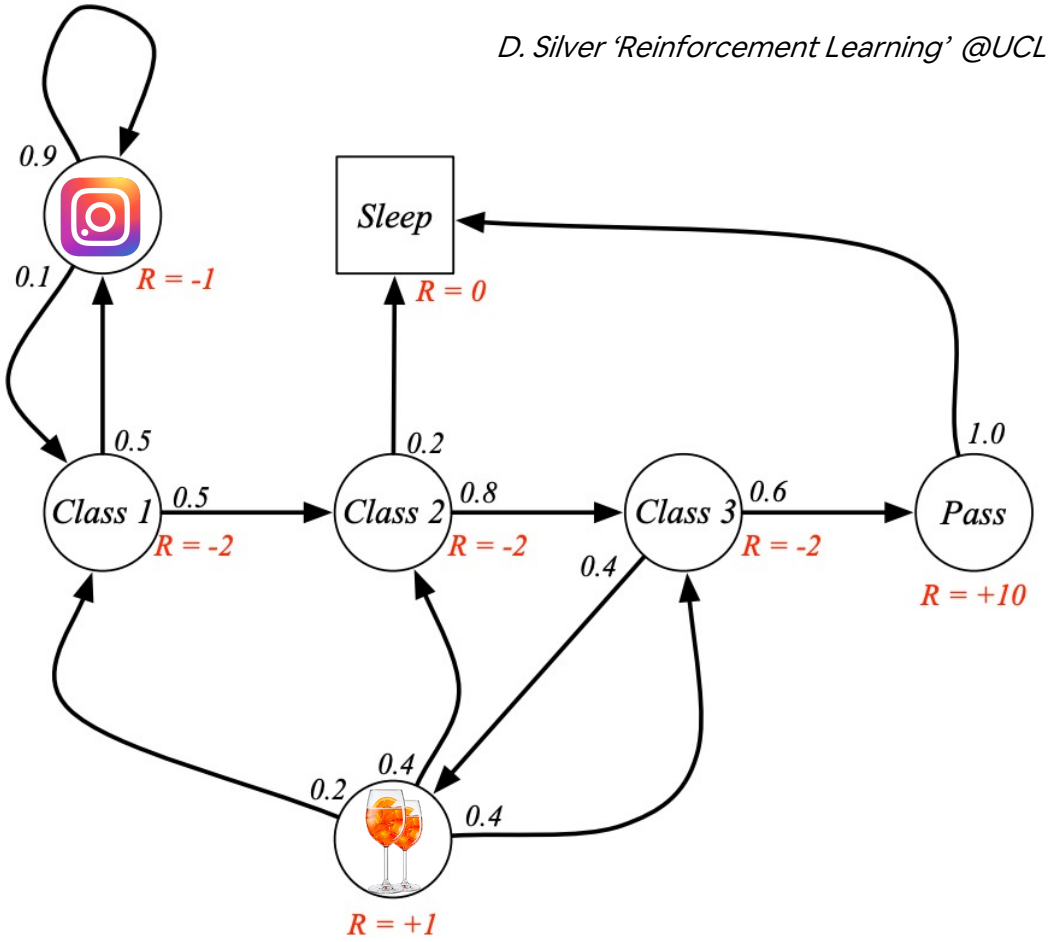
Prediction: Let's see an example!

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

Let's estimate the value function for state C1 with MC methods:

[episode #1] +4

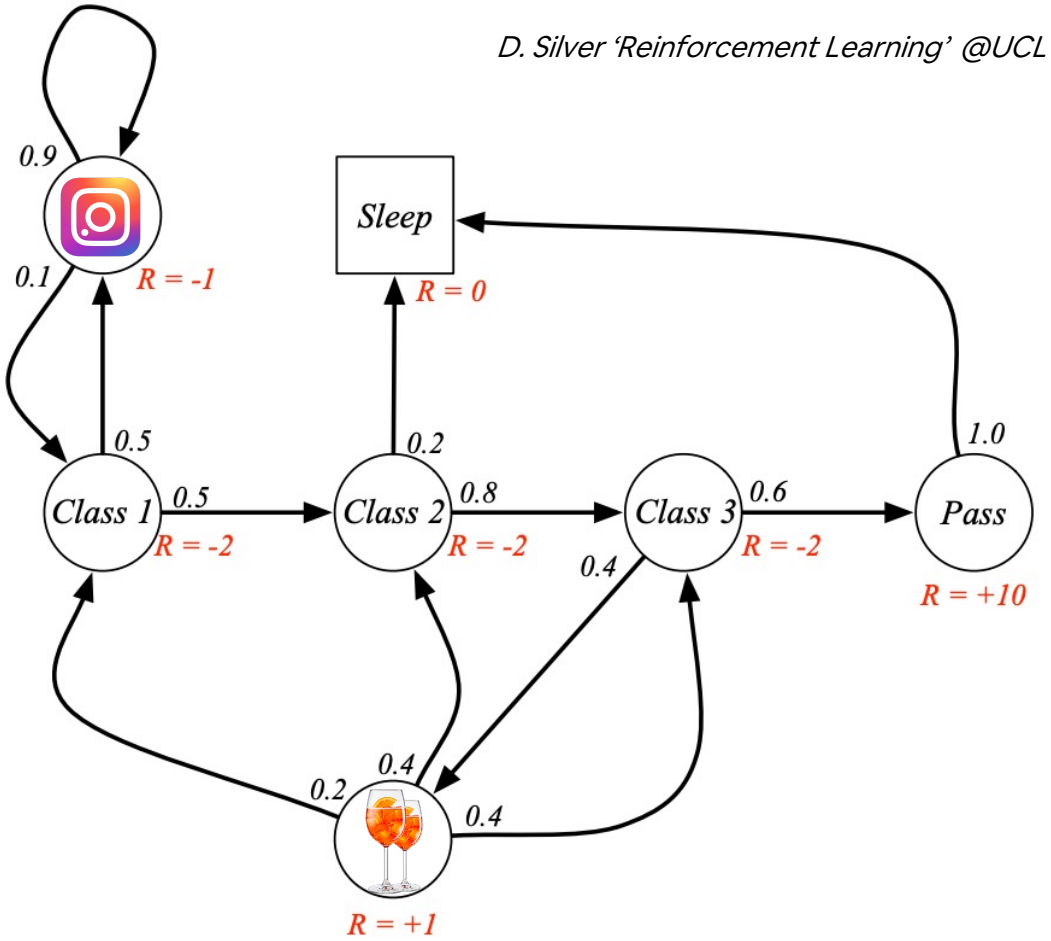


Prediction: Let's see an example!

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep



Let's estimate the value function for state C1 with MC methods:

[episode #1] +4

[episode #2] -8, -4

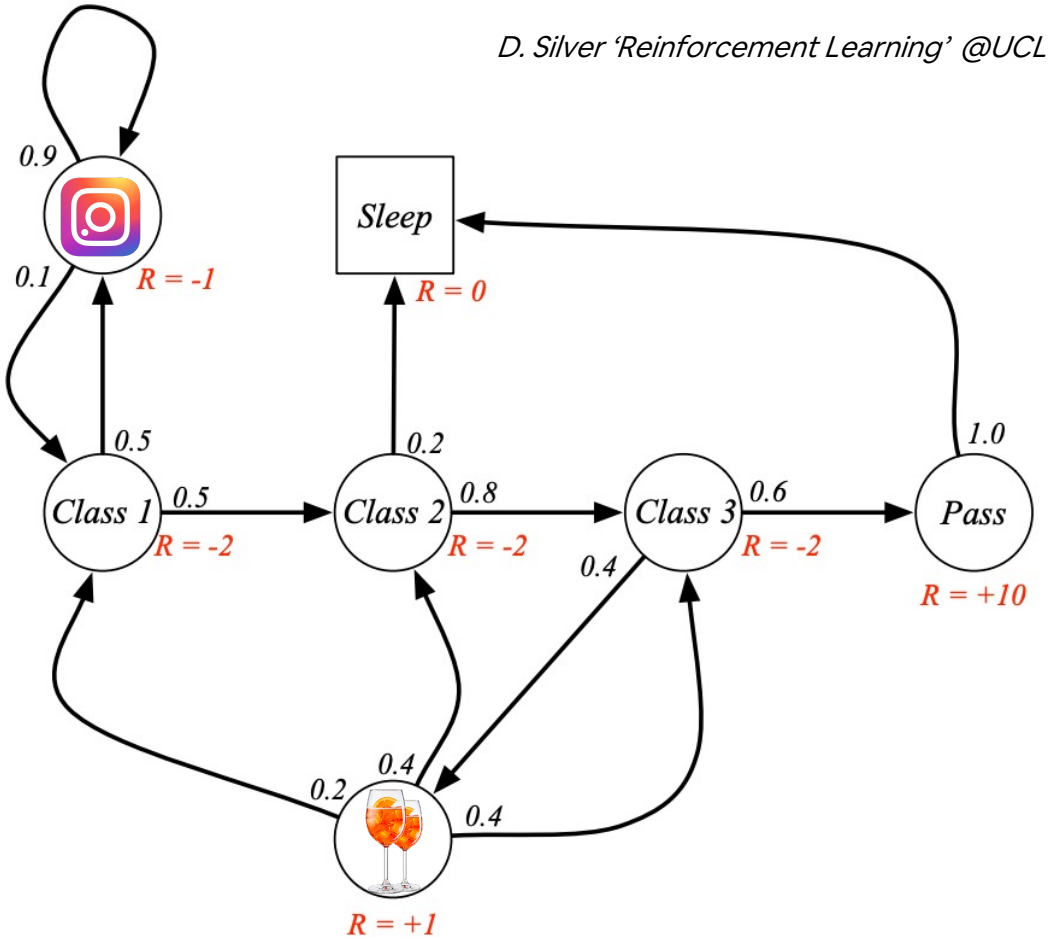
Prediction: Let's see an example!

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep

[#3] C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)C3(-2)Pass(+10)Sleep



Let's estimate the value function for state C1 with MC methods:

[episode #1] +4

[episode #2] -8, -4

[episode #3] +1

Prediction: Let's see an example!

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep

[#3] C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)C3(-2)Pass(+10)Sleep

[#4] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C1(-2)IG(-1)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)Sleep

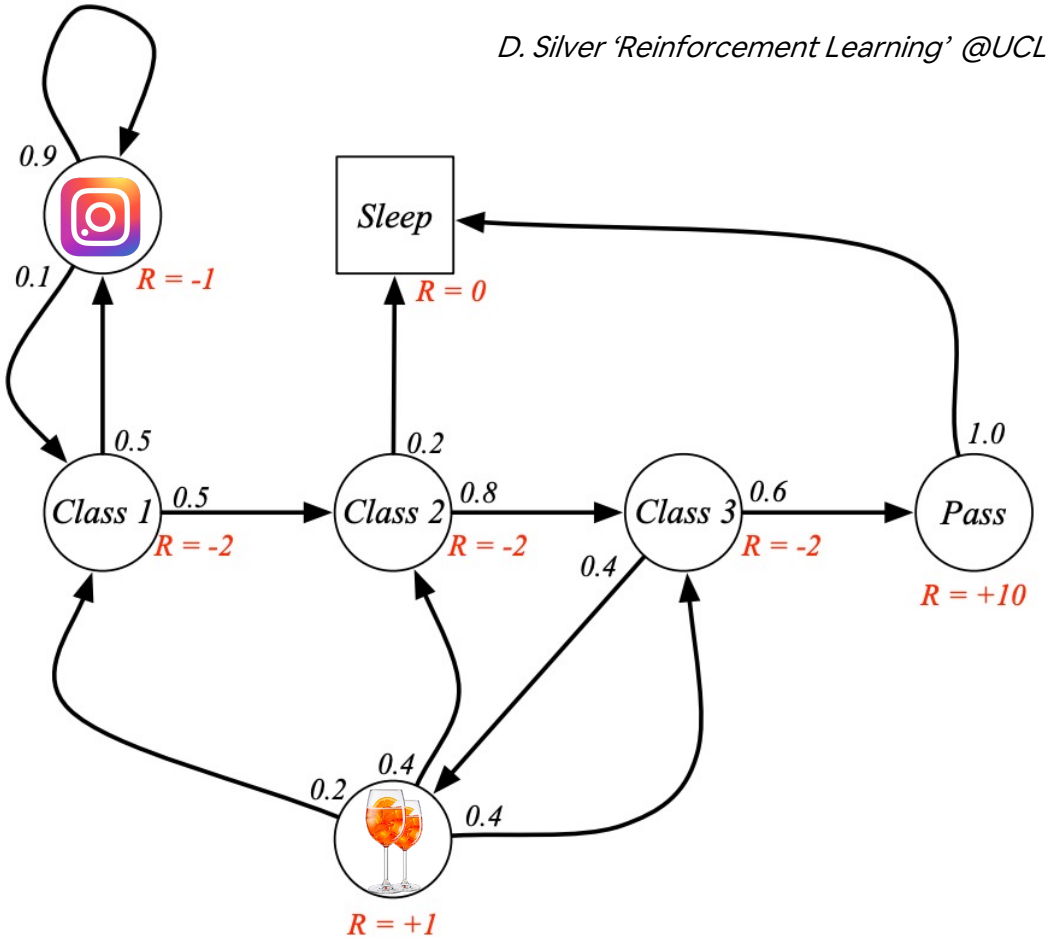
Let's estimate the value function for state C1 with MC methods:

[episode #1] +4

[episode #2] -8, -4

[episode #3] +1

[episode #4] -20, -16, -11, -7



Prediction: Let's see an example!

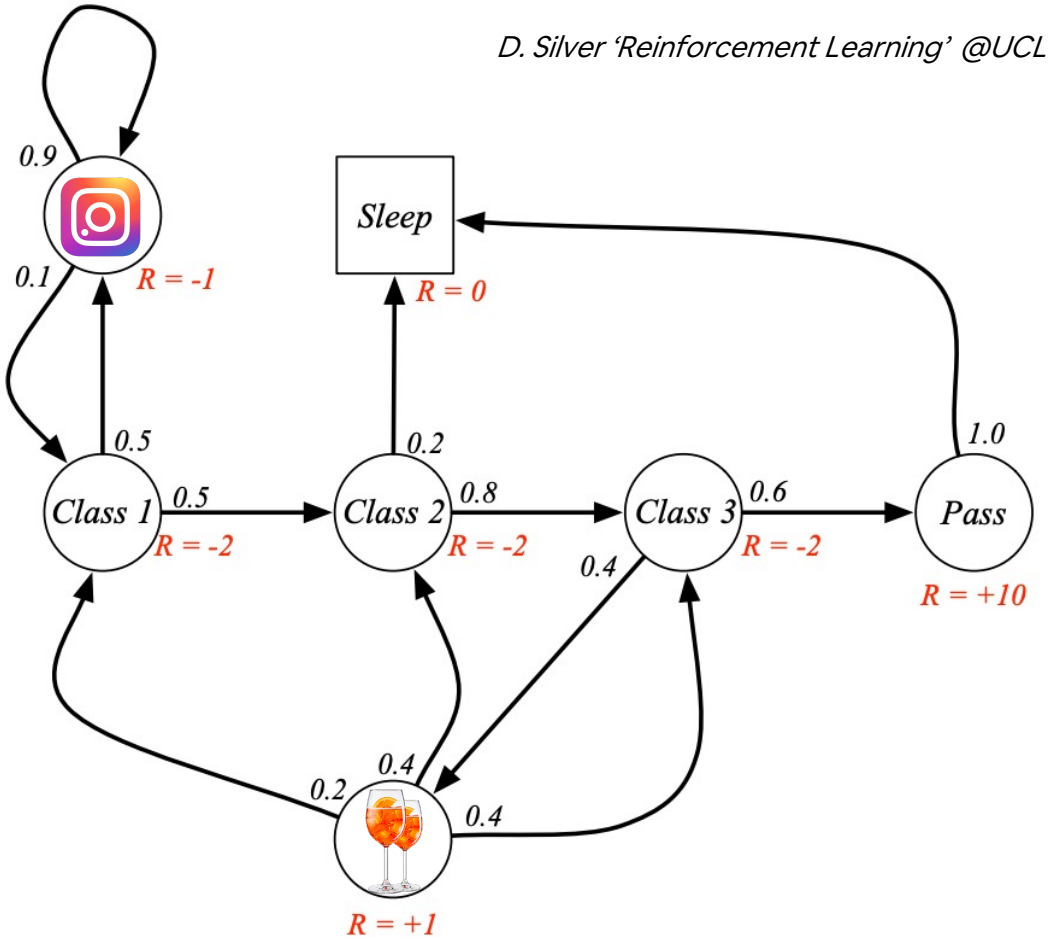
In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep
[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep
[#3] C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)C3(-2)Pass(+10)Sleep
[#4] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C1(-2)IG(-1)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)Sleep

Let's estimate the value function for state C1 with MC methods:

[episode #1] +4
[episode #2] -8, -4
[episode #3] +1
[episode #4] -20, -16, -11, -7

v_π = -7.625 = 0.125(4-8-4+1-20-16-11-7)

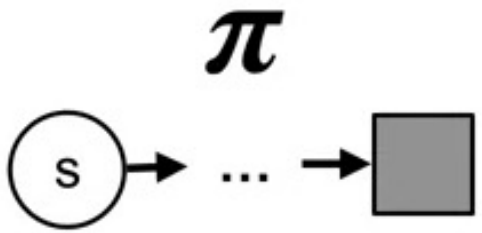


(Monte Carlo Methods vs Bandits)

Sample **rewards**

$$\frac{3+5+0+1}{4} = 1.5$$

Estimated value



Sample **returns**

$$\frac{3+5+0+1}{4} = 1.5$$

Estimated value

A. White, M. White 'Fundamentals of Reinforcement Learning'

Prediction: (First-visit) MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

We need to generate an estimation for each state!

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

Prediction: (First-visit) MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

We generate episode/experience

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

Prediction: (First-visit) MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

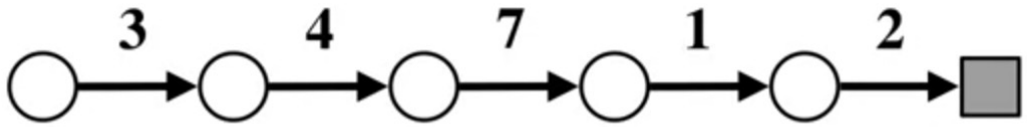
Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

We update the return for each of the 'visited' states by going backwards from the final time stamp in the episode... why?

Prediction: (First-visit) MC prediction – Computing returns efficiently



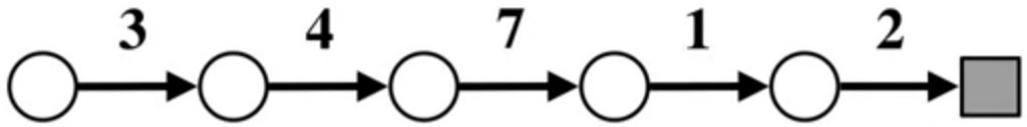
$\gamma = 0.5$

10 '+' operations

$$G_0 = R_1 + \gamma R_2 + \gamma^2 R_3 + \gamma^3 R_4 + \gamma^4 R_5 = 7$$
$$G_1 = R_2 + \gamma R_3 + \gamma^2 R_4 + \gamma^3 R_5 = 8$$
$$G_2 = R_3 + \gamma R_4 + \gamma^2 R_5 = 8$$
$$G_3 = R_4 + \gamma R_5 = 2$$
$$G_4 = R_5 = 2$$
$$G_5 = 0$$

A. White, M. White 'Fundamentals of Reinforcement Learning'

Prediction: (First-visit) MC prediction – Computing returns efficiently



$\gamma = 0.5$

$$G_0 = R_1 + \gamma G_1$$
$$G_1 = R_2 + \gamma G_2$$
$$G_2 = R_3 + \gamma G_3$$
$$G_3 = R_4 + \gamma G_4$$
$$G_4 = R_5 + \gamma G_5$$
$$G_5 = 0$$

5 '+' operations

10 '+' operations

$$G_0 = R_1 + \gamma R_2 + \gamma^2 R_3 + \gamma^3 R_4 + \gamma^4 R_5 = 7$$
$$G_1 = R_2 + \gamma R_3 + \gamma^2 R_4 + \gamma^3 R_5 = 8$$
$$G_2 = R_3 + \gamma R_4 + \gamma^2 R_5 = 8$$
$$G_3 = R_4 + \gamma R_5 = 2$$
$$G_4 = R_5 = 2$$
$$G_5 = 0$$

A. White, M. White 'Fundamentals of Reinforcement Learning'

Prediction: (First-visit) MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

Pay attention: we are just updating the states that have been visited!

Prediction: (First-visit) MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

A more efficient approach is to use incremental updates (we have already seen this)

Incremental Update

$NewEstimate \leftarrow OldEstimate + StepSize[Target - OldEstimate]$

Prediction: First-visit MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

Prediction: Every-visit MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

~~Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:~~

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

Prediction: First-visit vs Every Visit

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep

[#3] C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)C3(-2)Pass(+10)Sleep

[#4] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C1(-2)IG(-1)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)Sleep

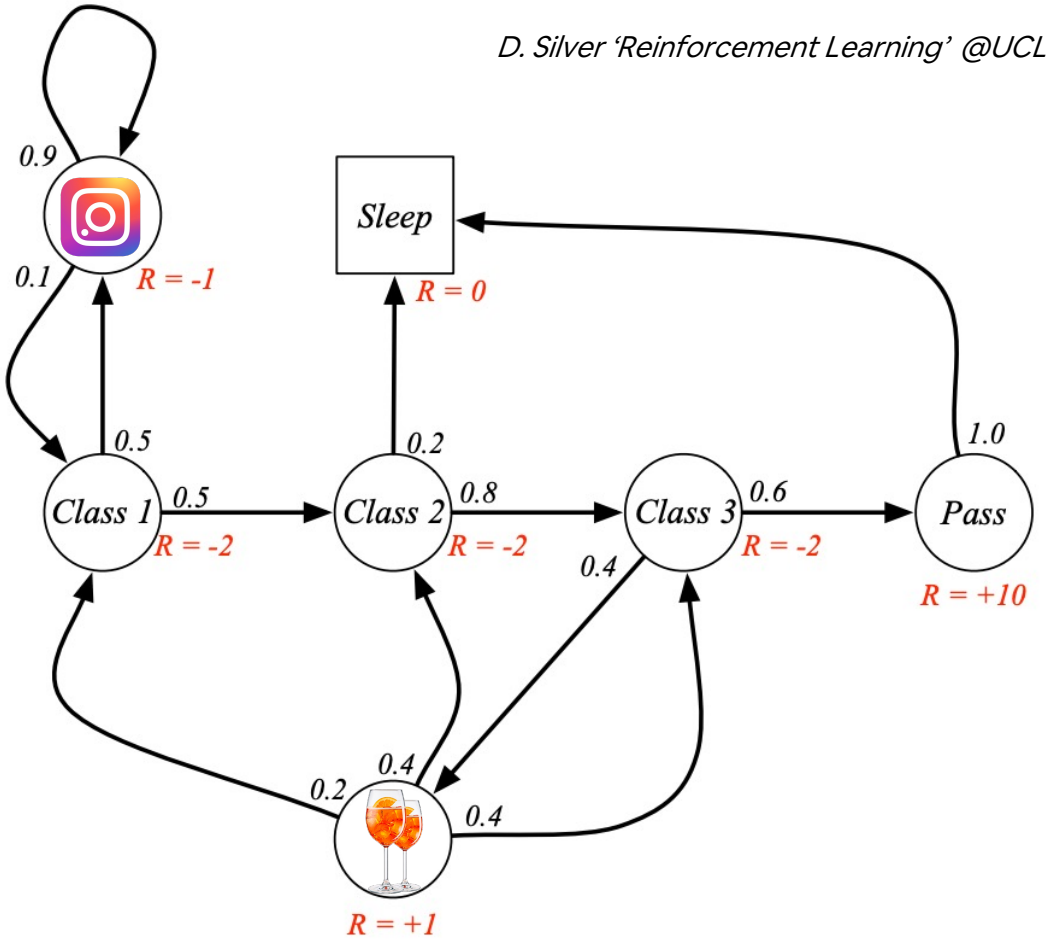
Let's estimate the value function for state C1 with MC methods:

Every-visit estimation v_π = -7.625 = 0.125(4-8-4+1-20-16-11-7):

[episode #1] +4 [episode #2] -8, -4

[episode #3] +1 [episode #4] -20, -16, -11, -7

D. Silver 'Reinforcement Learning' @UCL



Prediction: First-visit vs Every Visit

In the student MDP (with $\gamma = 1$), we draw some samples: following the depicted policy π

[#1] C1(-2)C2(-2)C3(-2)Pass(+10)Sleep

[#2] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)Sleep

[#3] C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)C3(-2)Pass(+10)Sleep

[#4] C1(-2)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C1(-2)IG(-1)IG(-1)IG(-1)C1(-2)C2(-2)C3(-2)Spritz(+1)C2(-2)Sleep

Let's estimate the value function for state C1 with MC methods:

Every-visit estimation v_π = -7.625 = 0.125(4-8-4+1-20-16-11-7):

[episode #1] +4 [episode #2] -8, -4

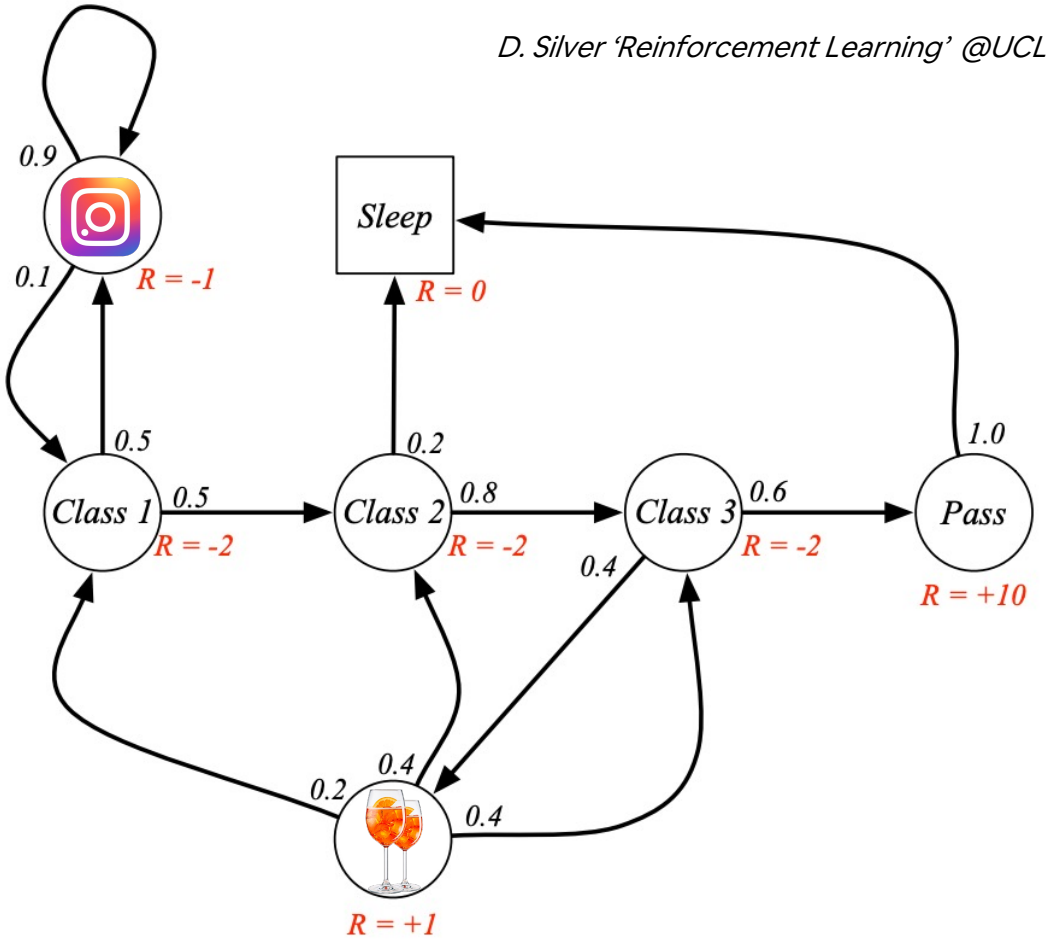
[episode #3] +1 [episode #4] -20, -16, -11, -7

First-visit estimation v_π = -5.75 = 0.25(4-8+1-20):

[return #1] +4 [return #2] -8

[return #3] +1 [return #4] -20

D. Silver 'Reinforcement Learning' @UCL



Prediction: Every-visit MC prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots,$

$G \leftarrow \gamma G + R_{t+1}$

~~Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:~~

Append G to $Returns(S_t)$

$V(S_t) \leftarrow$ average$(Returns(S_t))$

Every-visit MC don't consider the check for S_t having occured earlier in the episode.

We will focus on first-visit version, but later in the course every-visit version will be more relevant

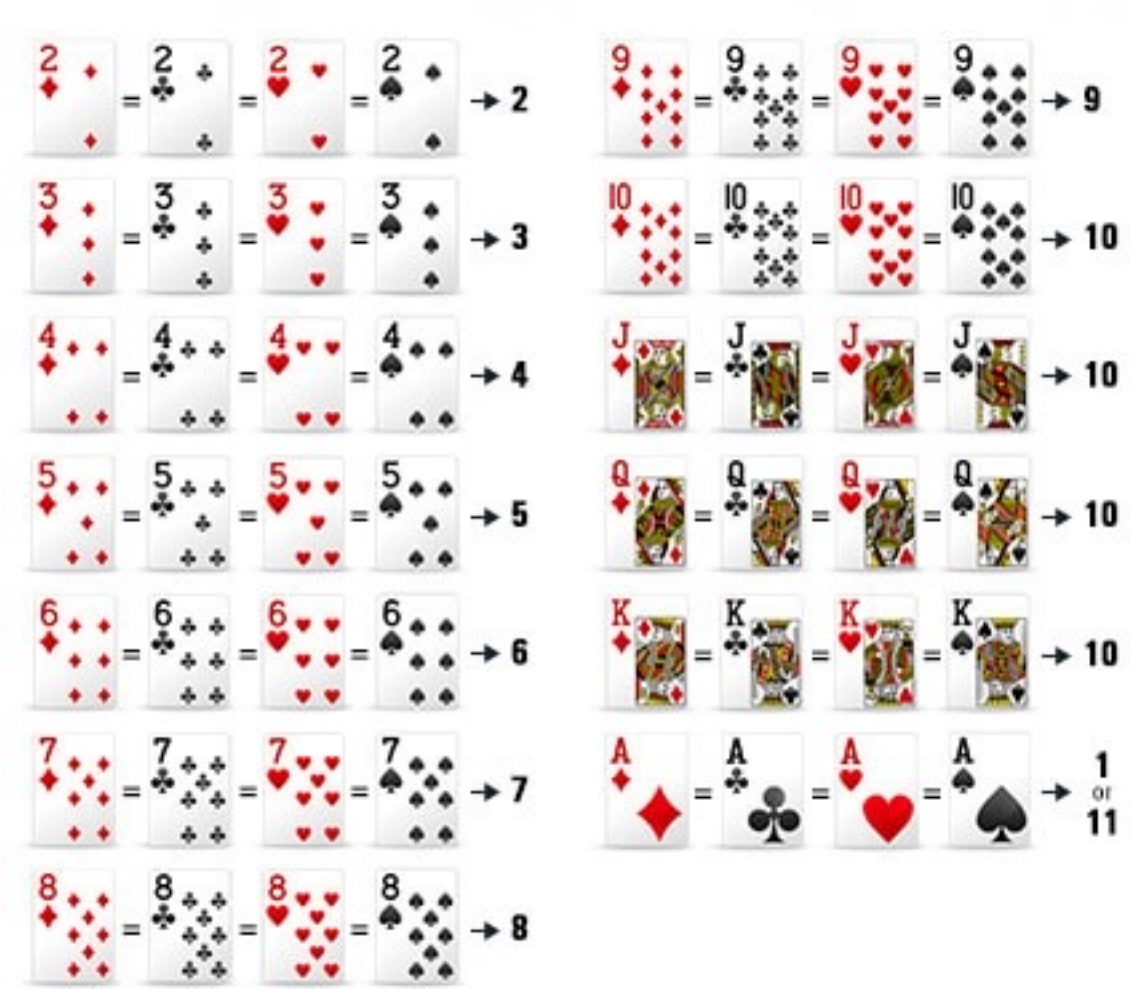
Prediction: MC prediction - Blackjack

Prediction: MC prediction - Blackjack

Pay attention: we are going to use Blackjack for the third Lab!

Prediction: MC prediction - Blackjack

- Card game played against a dealer
- Played with classic 52 deck
- Goal: collect cards so that their sum is as large as possible without exceeding 21



Prediction: MC prediction - Blackjack

- Game (episode) begins with 2 cards to both the player and the dealer
- One of the card of the dealer is faced down
- If the player starts with 21, he/she immediately win (unless the dealer has also 21, then it is a draw)

Prediction: MC prediction - Blackjack

- Game (episode) begins with 2 cards to both the player and the dealer
- One of the card of the dealer is faced down
- If the player starts with 21, he/she immediately win (unless the dealer has also 21, then it is a draw)
- Without 21 the player can request cards (hit): one card at each request is given

Prediction: MC prediction - Blackjack

- Game (episode) begins with 2 cards to both the player and the dealer
- One of the card of the dealer is faced down
- If the player starts with 21, he/she immediately win (unless the dealer has also 21, then it is a draw)
- Without 21 the player can request cards (hit): one card at each request is given
- If 21 is passed, the player lose

Prediction: MC prediction - Blackjack

- If the player stops requesting card (stick) then it is the dealer turn
- In our simplified case, the dealer has a fixed strategy: he sticks on any sum of 17 or greater, hits otherwise

Prediction: MC prediction - Blackjack

- If the player stops requesting card (stick) then it is the dealer turn
- In our simplified case, the dealer has a fixed strategy: he sticks on any sum of 17 or greater, hits otherwise
- We also consider a deck with replacement (no counting cards ☺)
- Formalization as an episodic task with rewards +1/-1/0 for winning/losing/drawing
- Undiscounted MDP

Prediction: MC prediction - Blackjack

- If the player stops requesting card (stick) then it is the dealer turn
- In our simplified case, the dealer has a fixed strategy: he sticks on any sum of 17 or greater, hits otherwise
- We also consider a deck with replacement (no counting cards ☺)
- Formalization as an episodic task with rewards +1/-1/0 for winning/losing/drawing
- Undiscounted MDP

Which can be the states?
Which can be the actions?

Prediction: MC prediction - Blackjack

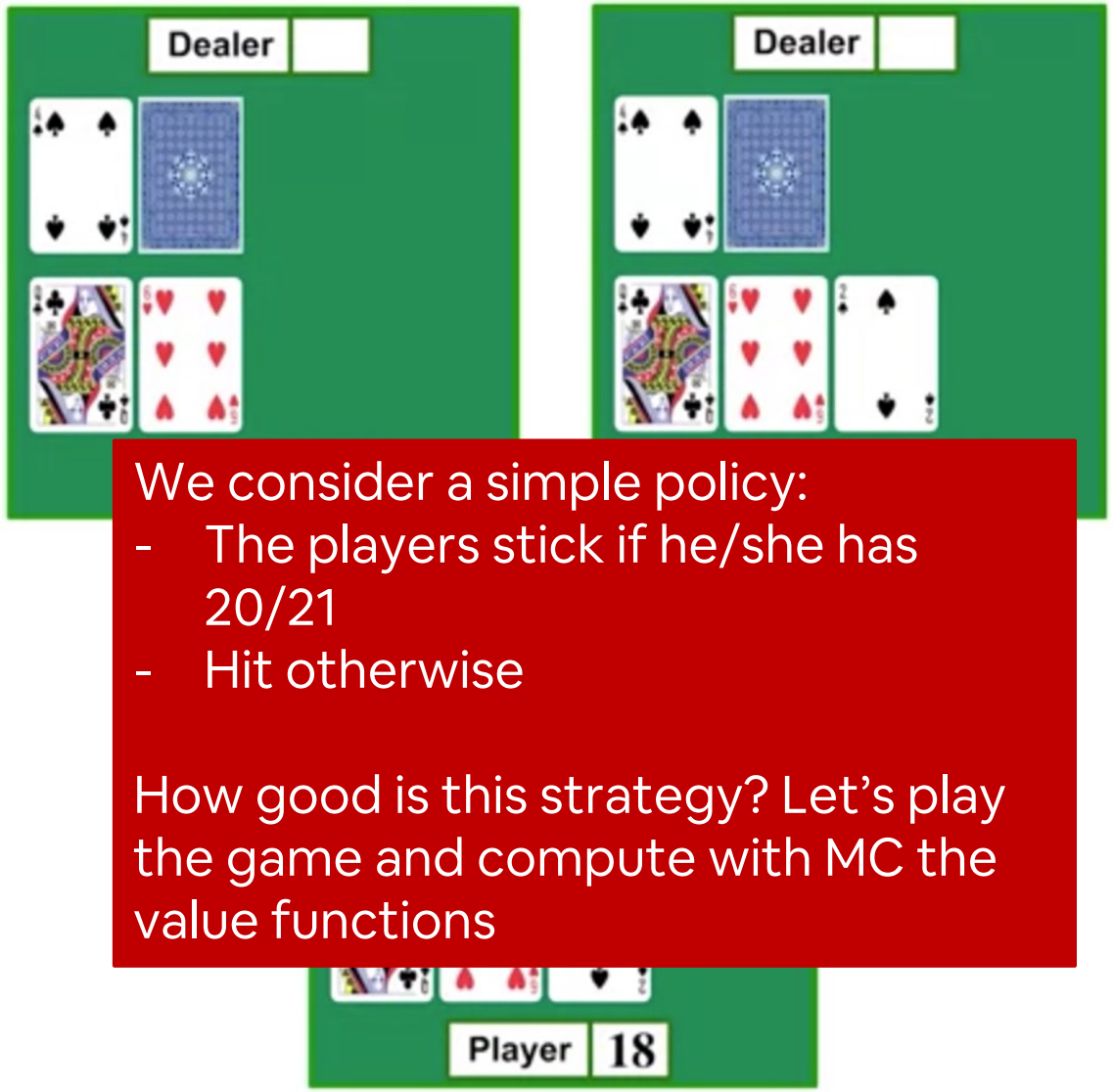
- If the player stops requesting card (stick) then it is the dealer turn
- In our simplified case, the dealer has a fixed strategy: he sticks on any sum of 17 or greater, hits otherwise
- We also consider a deck with replacement (no counting cards ☺)
- Formalization as an episodic task with rewards +1/-1/0 for winning/losing/drawing
- Undiscounted MDP, with 2 actions (hit and stick)
- For the player (agent) this is a problem with 200 states, depending on

1. Having or not a 'usable' ace (can be considered both as 1 or 11);
2. The player current sum (12-21);
3. The dealer's shown card (Ace-10)

Prediction: MC prediction - Blackjack

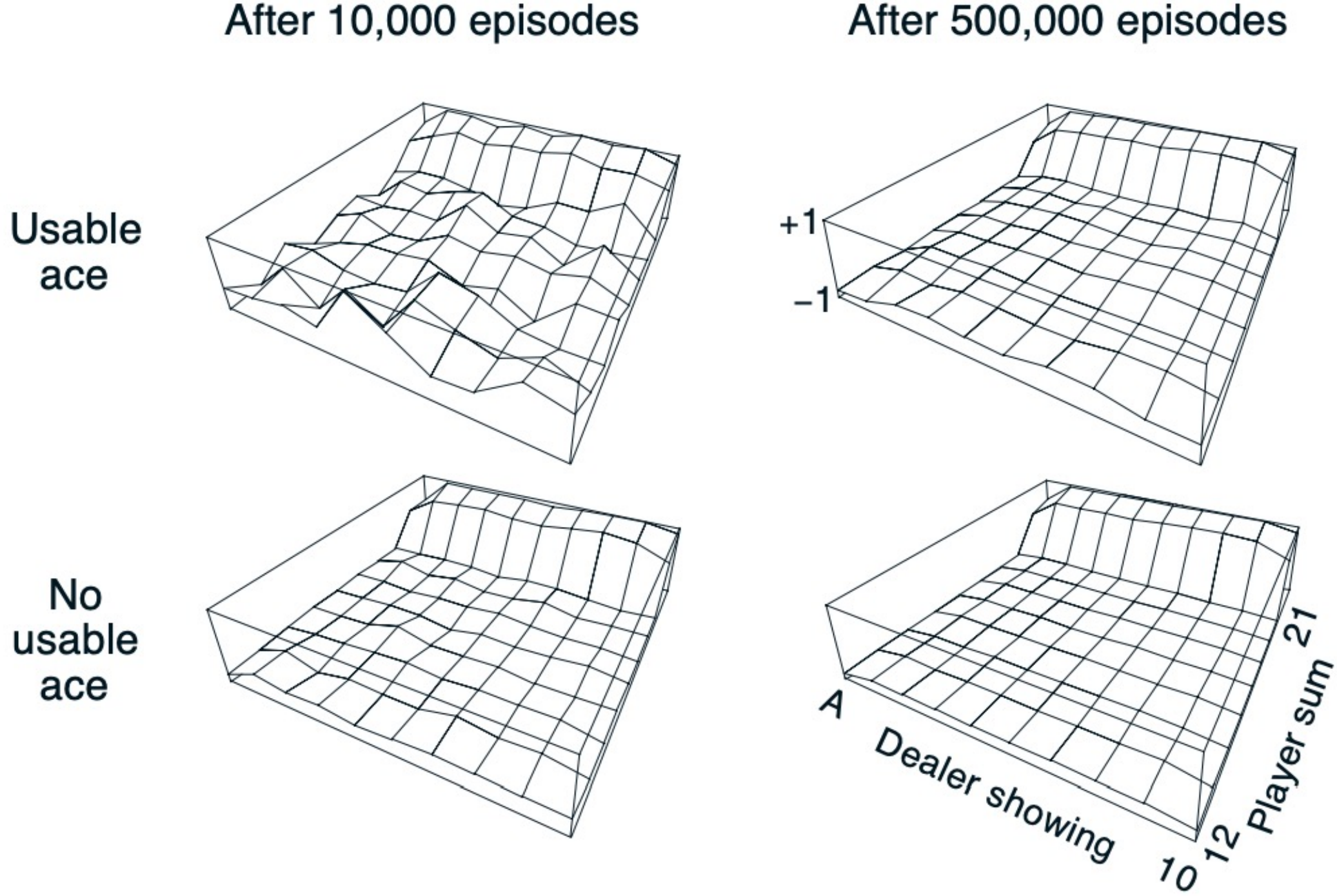
- If the player stops requesting card (stick) then it is the dealer turn
- In our simplified case, the dealer has a fixed strategy: he sticks on any sum of 17 or greater, hits otherwise
- We also consider a deck with replacement (no counting cards ☺)
- Formalization as an episodic task with rewards +1/-1/0 for winning/losing/drawing
- Undiscounted MDP, with 2 actions (hit and stick)
- For the player (agent) this is a problem with 200 states, depending on

1. Having or not a 'usable' ace (can be considered both as 1 or 11);
2. The player current sum (12-21);
3. The dealer's shown card (Ace-10)



We consider a simple policy:
- The players stick if he/she has 20/21
- Hit otherwise

How good is this strategy? Let's play the game and compute with MC the value functions


After 10,000 episodes
After 500,000 episodes
Usable ace
No usable ace
+1
−1
A
Dealer showing
10
12
Player sum
21

MC learning: final comments

- The same algorithm can be used for estimating q-value functions
- MC learns directly from experience: there's no need to keep a large model of the environment.
- MC methods can estimate the value of an individual state independently of the values of any other states. (In dynamic programming, the value of each state depends on the values of other states!)
- The computation needed to update the value of each state along the way doesn't depend in any way on the size of the MDP (which was the case for DP). Rather, it depends on the length of the episode!

Credits

- Image of the course is taken from C. Mahoney ‘Reinforcement Learning’ https://towardsdatascience.com/reinforcement-learning-fda8ff535bb6

Thank you!
Questions?

Lecture #06
Dynamic Programming for MDPs & Monte Carlo

Gian Antonio Susto