UNIVERSITÀ DEGLI STUDI DI PADOVA

# Filtering in the frequency domain

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# Agenda



- Filtering in the frequency domain
- Low-pass and high-pass filters
- Band-reject filters
- Ideal, Butterworth, Gaussian filters
- Denoising in frequency

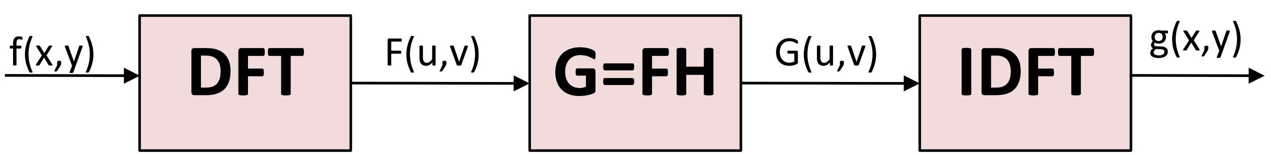
# Filters in frequency

- Filters can be defined in frequency
- What are the advantages of filtering in frequency?

# Filters in frequency



- Filters can be defined in frequency
- What are the advantages of filtering in frequency?
- Remainder of this unit: see many filters and their effect on the images



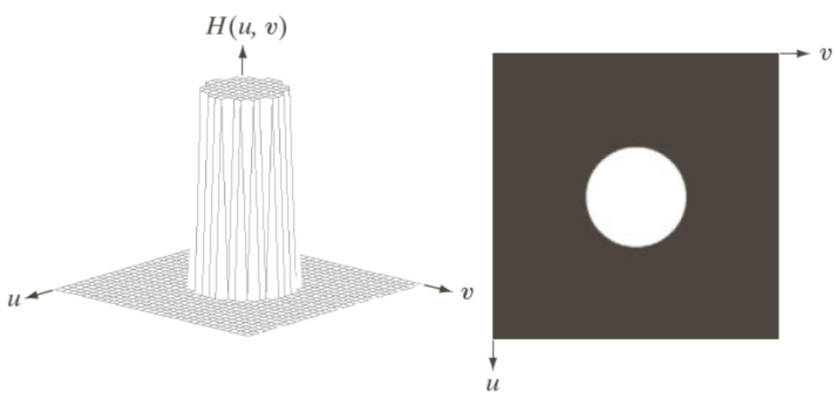
# Filters

- Lowpass filters – smoothing
  - Ideal
  - Butterworth
  - Gaussian
- Highpass filters – sharpening
  - Ideal
  - Butterworth
  - Gaussian
- Selective filters
  - Band-reject
  - Notch

# Ideal lowpass filter



- Expressed in tems of a distance D from the center
- The zero-frequency is in the middle of the figure



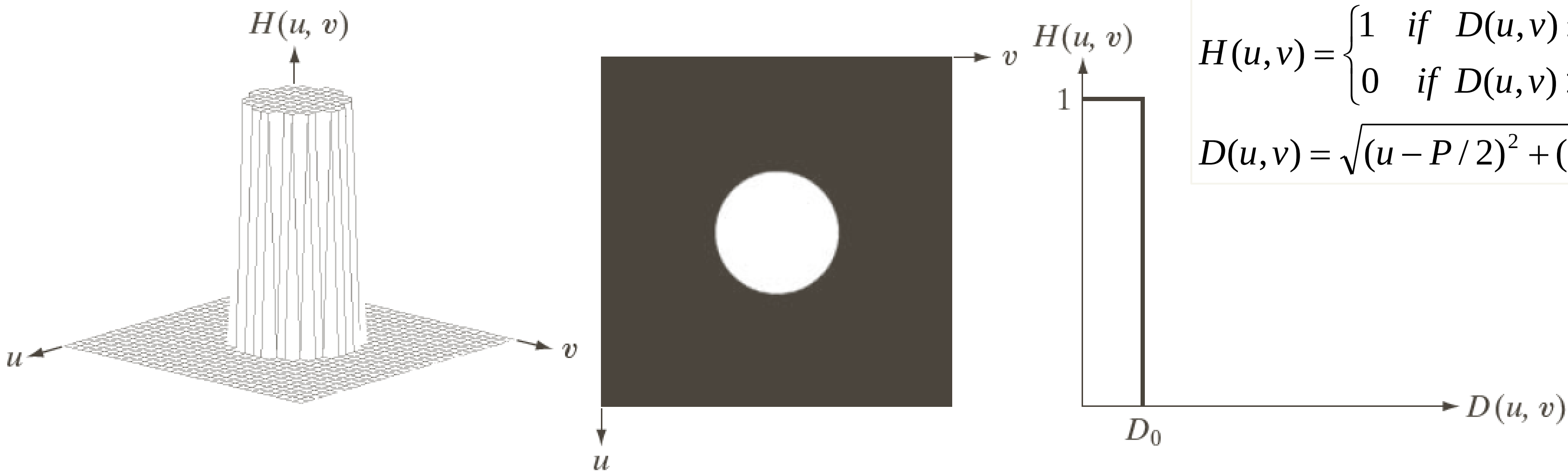

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

$$D(u,v)=\sqrt{(u-P/2)^2+(v-Q/2)^2}$$

# Ideal lowpass filter



- Ideal lowpass in frequency




$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

$$D(u,v)=\sqrt{(u-P/2)^2+(v-Q/2)^2}$$

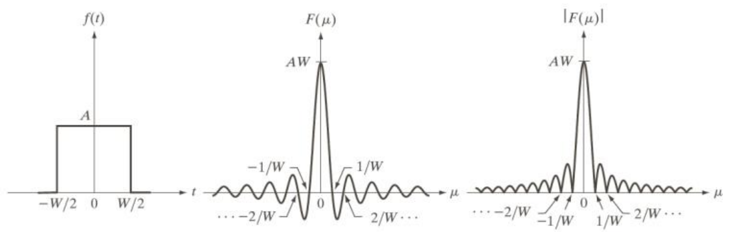
a b c

**FIGURE 4.40** (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

# Ideal lowpass filter

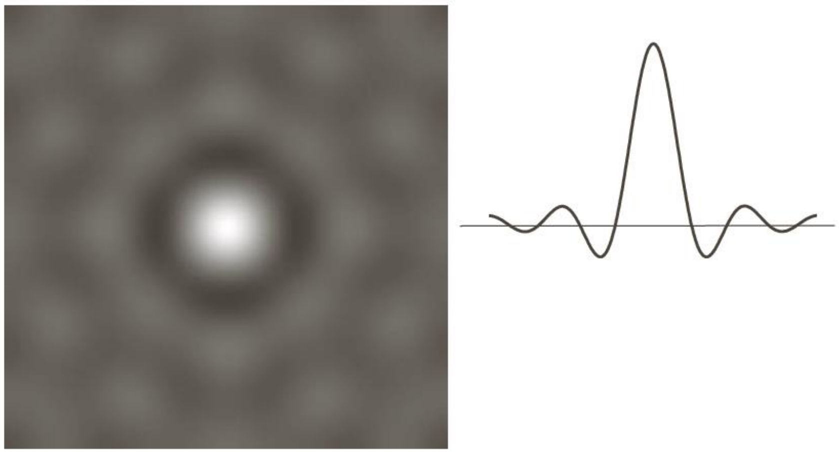
- Ideal lowpass in space
  - Observe the ripples around the main central component
- The filter is bound into a region
  - Corresponds to padding to 0 outside the filter window

Reminder




a b c

**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.



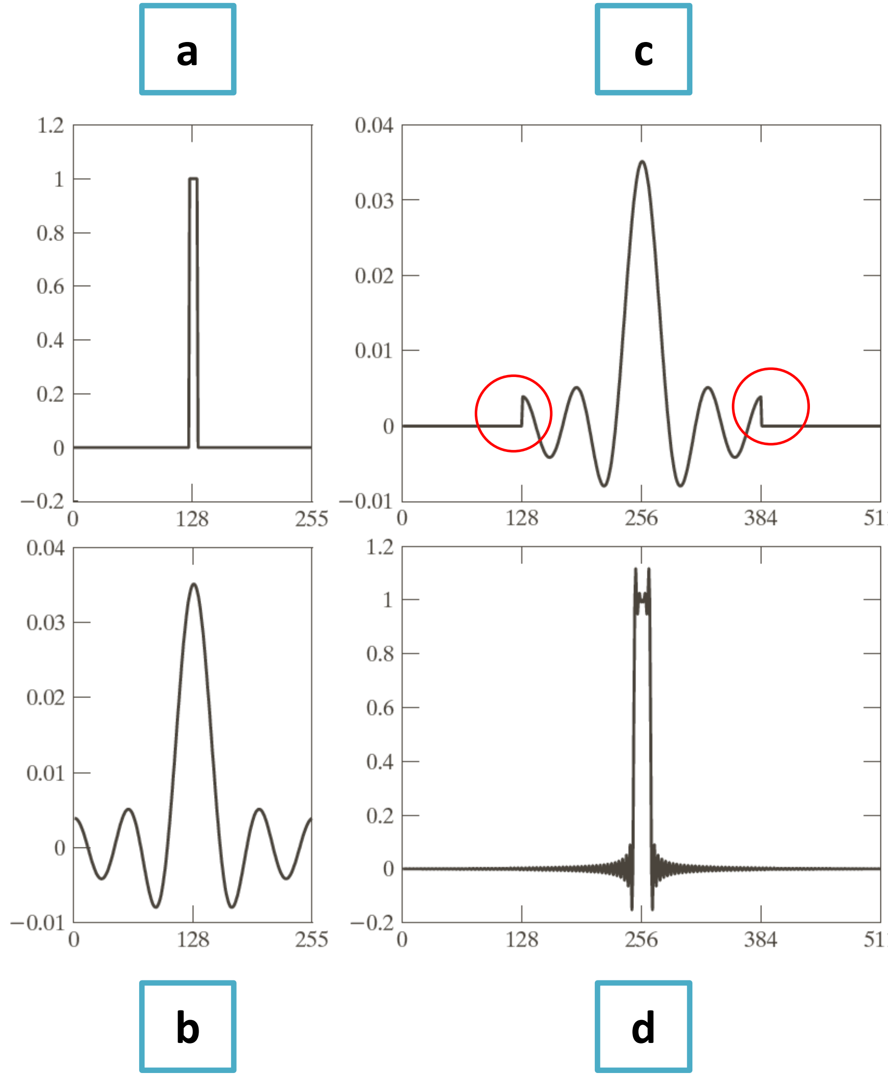
a b

**FIGURE 4.43** (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000 × 1000. (b) Intensity profile of a horizontal line passing through the center of the image.

# Ringing artifacts

a. Ideal LPF in frequency
b. Ideal LPF in space
c. Ideal LPF – padded to double length
    - Discontinuities generating the ringing artifacts
d. DFT of c.

Note the ringing artifacts caused by the padding of a function having a substantial non-zero tails

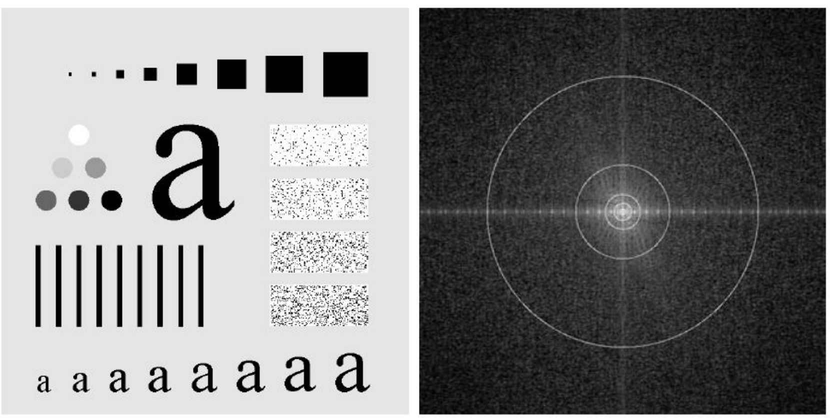



a c
b d

**FIGURE 4.34** (a) Original filter specified in the (centered) frequency domain. (b) Spatial representation obtained by computing the IDFT of (a). (c) Result of padding (b) to twice its length (note the discontinuities). (d) Corresponding filter in the frequency domain obtained by computing the DFT of (c). Note the ringing caused by the discontinuities in (c). (The curves appear continuous because the points were joined to simplify visual analysis.)

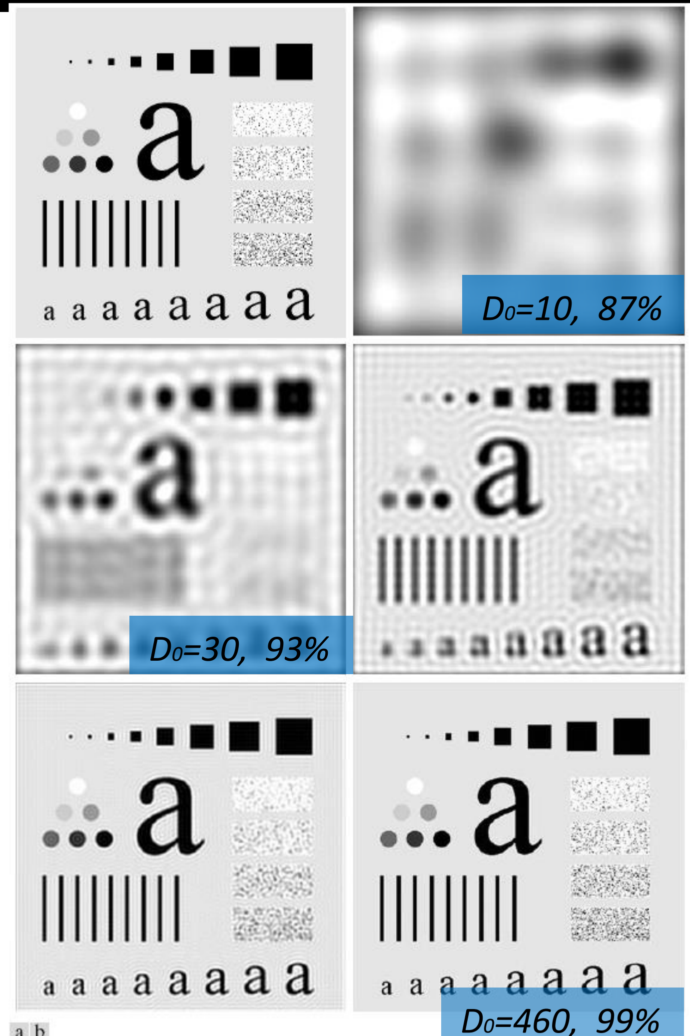
# ILPF – examples

- The ringing effect is noticeable



a b

**FIGURE 4.41** (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.




a b
c d
e f

**FIGURE 4.42** (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

# Coping with ringing effect

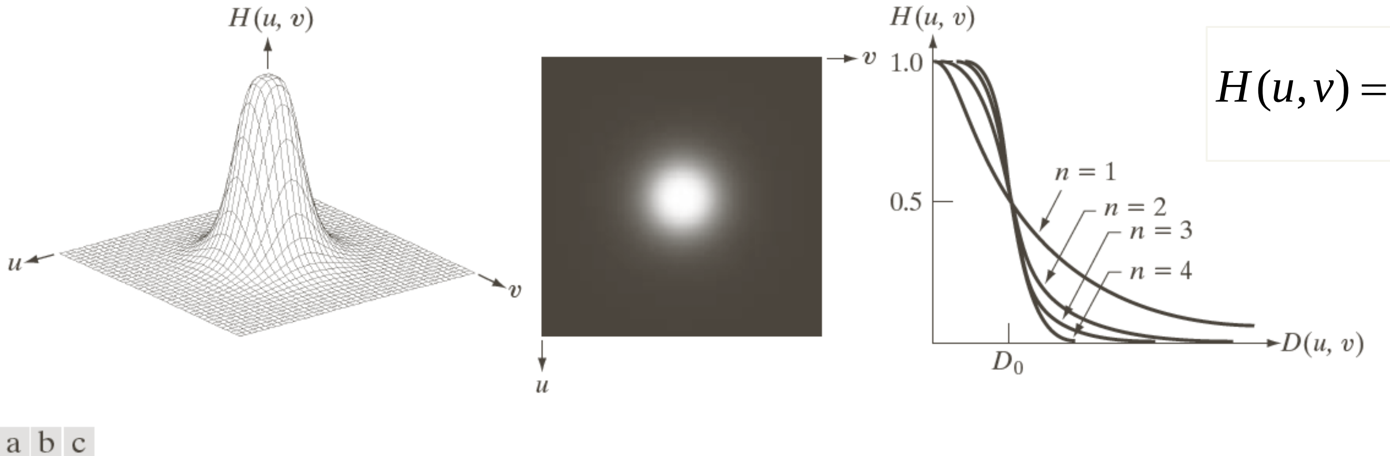


- The ringing effect causes poor image quality
- It is typical of ideal filters
  - Because of their strong transition
- Ringing effects can be reduced or eliminated by using filters showing smoother transitions
  - Butterworth filter
  - Gaussian filter

# Butterworth lowpass filter



- Butterworth in frequency
  - Tuned by means of the parameter $n$
  - Large values of $n$ cause this filter to be similar to the ILPF




$$H(u,v) = \frac{1}{1+(D(u,v)/D_0)^{2n}}$$

**FIGURE 4.44** (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

# Butterworth lowpass filter



- Butterworth in space

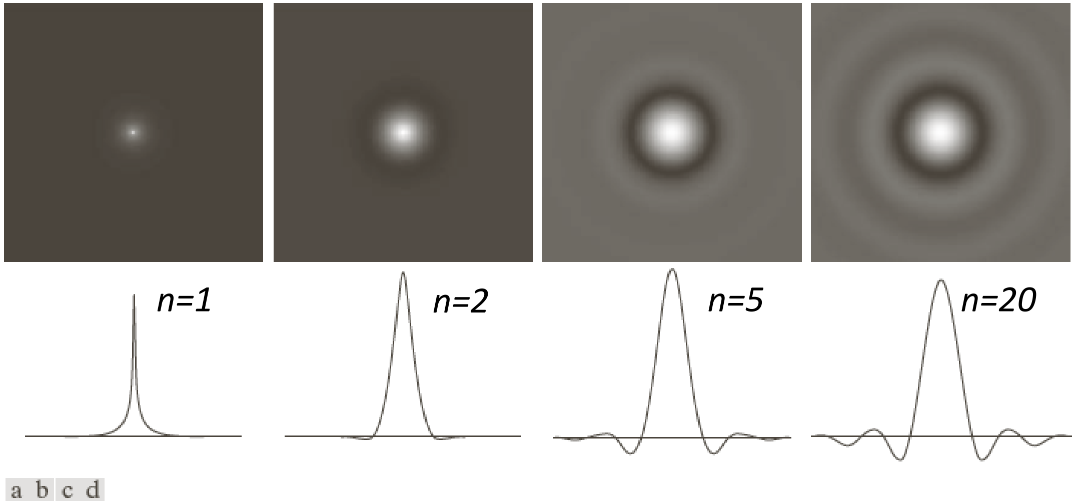



**FIGURE 4.46** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

*The ringing effect grows with the order of the filter*

Why?

# Butterworth – example



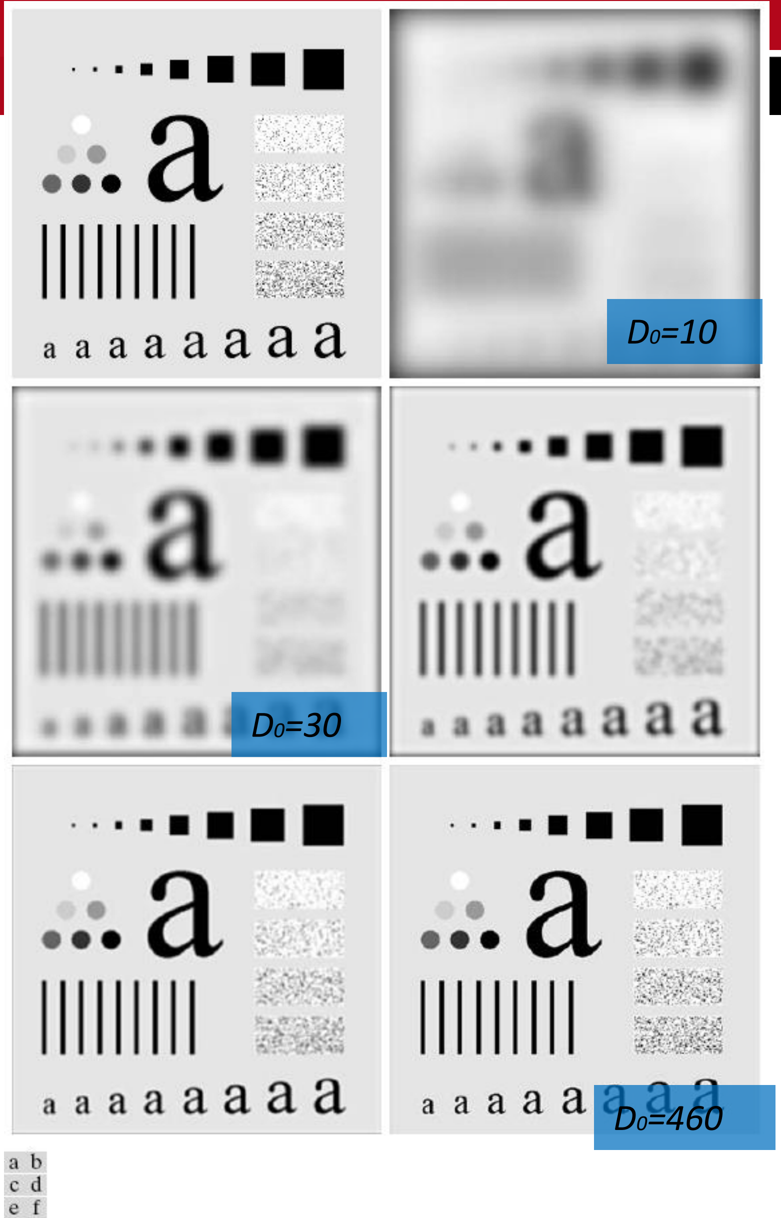



**FIGURE 4.45** (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

# Gaussian lowpass filter

- Gaussian filter in frequency
  - No ringing

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

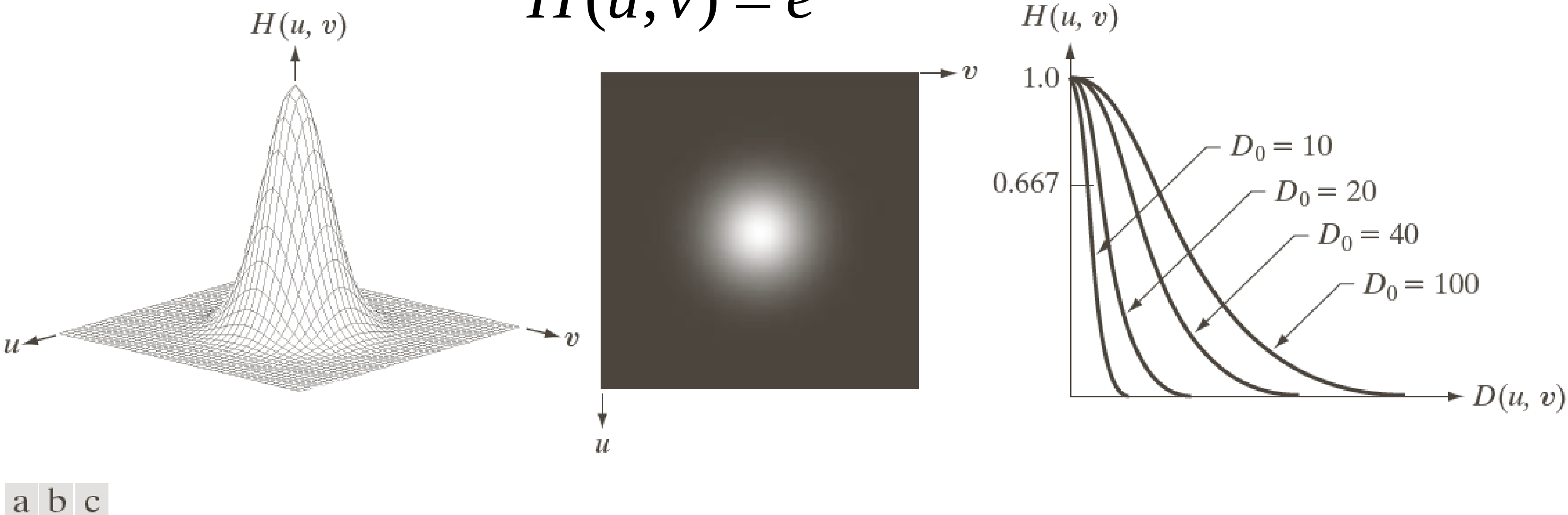



**FIGURE 4.47** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of $D_0$.

# Gaussian filter – example



- Filling gaps

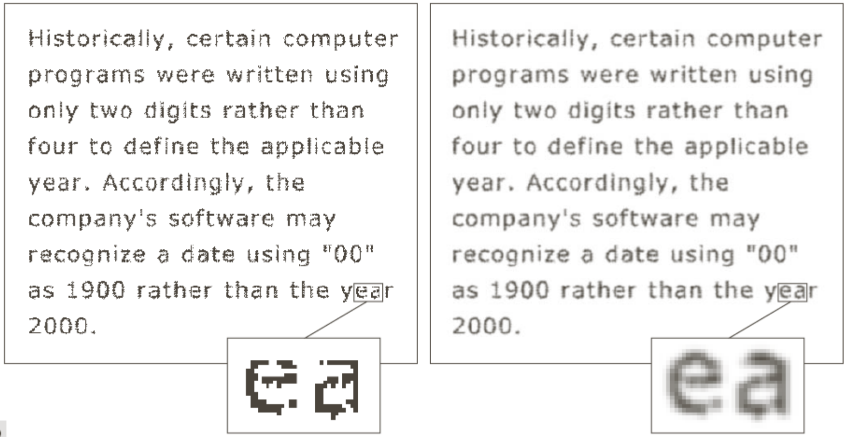



**FIGURE 4.49** (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

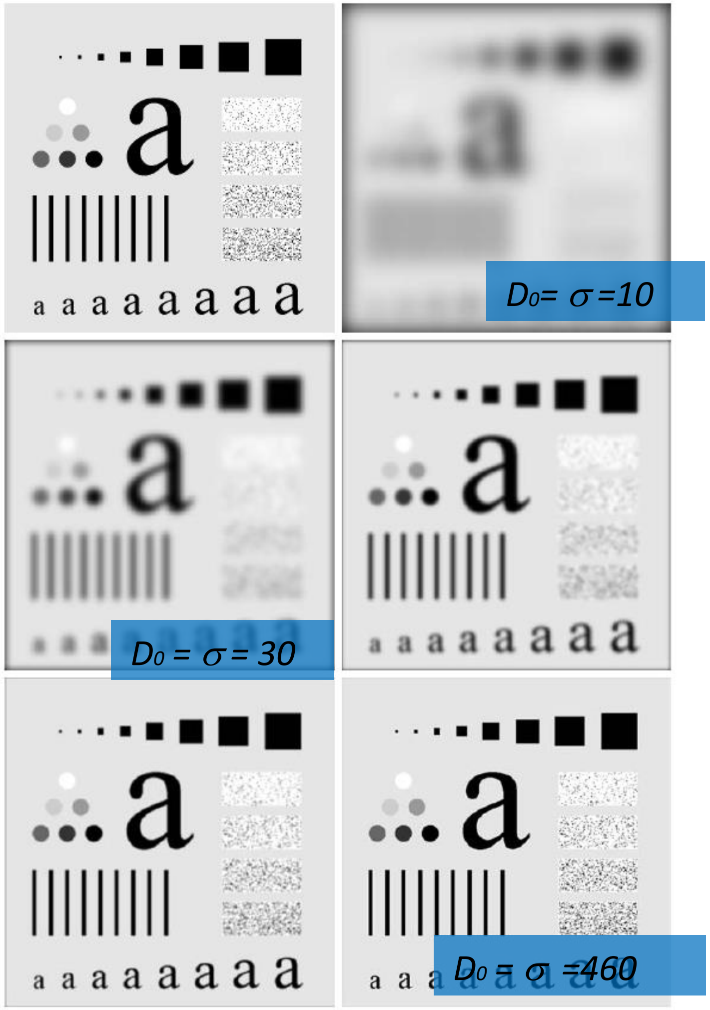



**FIGURE 4.48** (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

# Gaussian filter – example



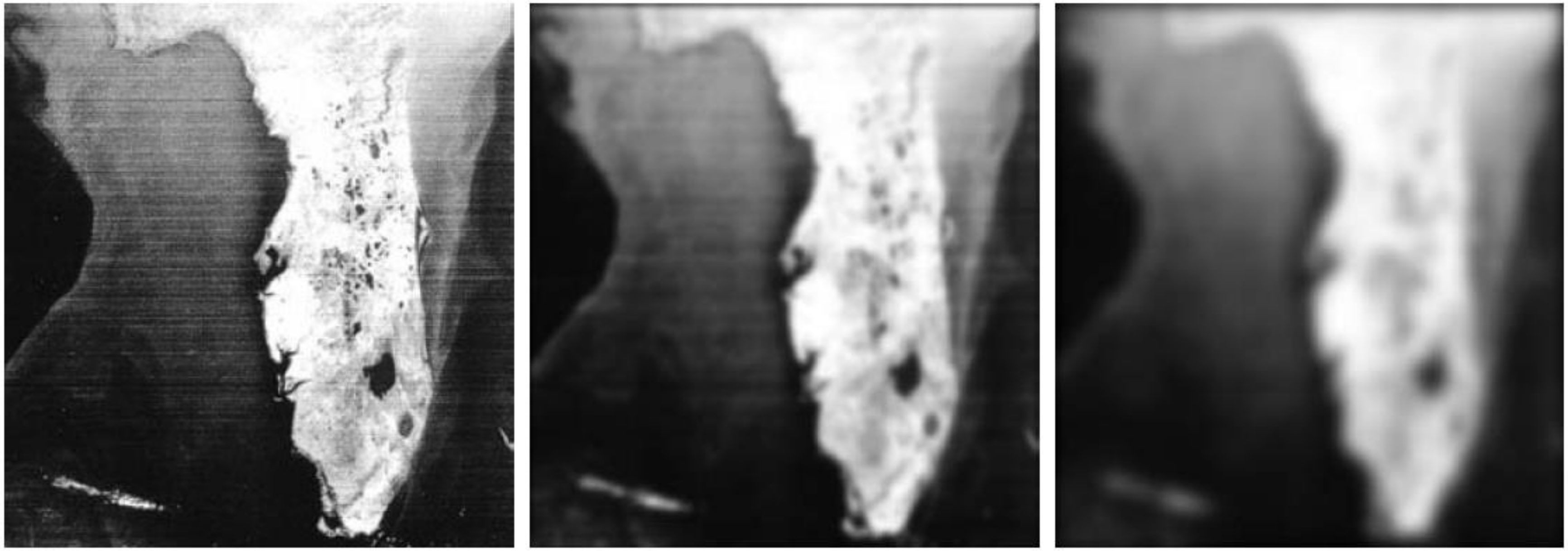
- Beautification


**FIGURE 4.50** (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

# Gaussian filter – example



- Reducing noise



a b c

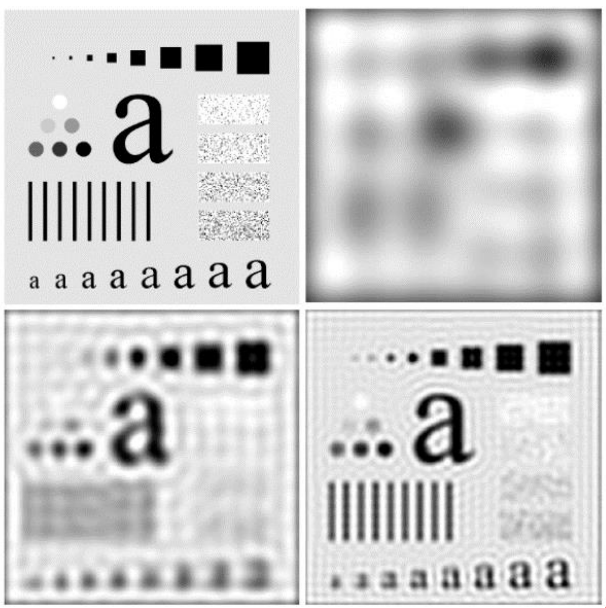
**FIGURE 4.51** (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

# Lowpass filter recap

**TABLE 4.4**

Lowpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$ | $H(u,v) = e^{-D^2(u,v)/2D_0^2}$ |



# Filters



- Lowpass filters – smoothing
  - Ideal
  - Butterworth
  - Gaussian
- Highpass filters – sharpening
  - Ideal
  - Butterworth
  - Gaussian
- Selective filters
  - Band-reject
  - Notch

# Filters



- Lowpass filters – smoothing
  - Ideal
  - Butterworth
  - Gaussian
- Highpass filters – sharpening
  - Ideal
  - Butterworth
  - Gaussian
- Selective filters
  - Band-reject
  - Notch

# Highpass filters

- A HP filter can be obtained from the corresponding LP:

$$H_{\mathrm{HP}}(u, v) = 1 - H_{\mathrm{LP}}(u, v)$$

- Features of LP and HP filters (e.g. ringing) are similar for a given type of filter
- Sharpening effect

# Highpass filters

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

**TABLE 4.5**
Highpass filters. $D_0$ is the cutoff frequency and $n$ is the order of the Butterworth filter.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$ | $H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$ | $H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$ |

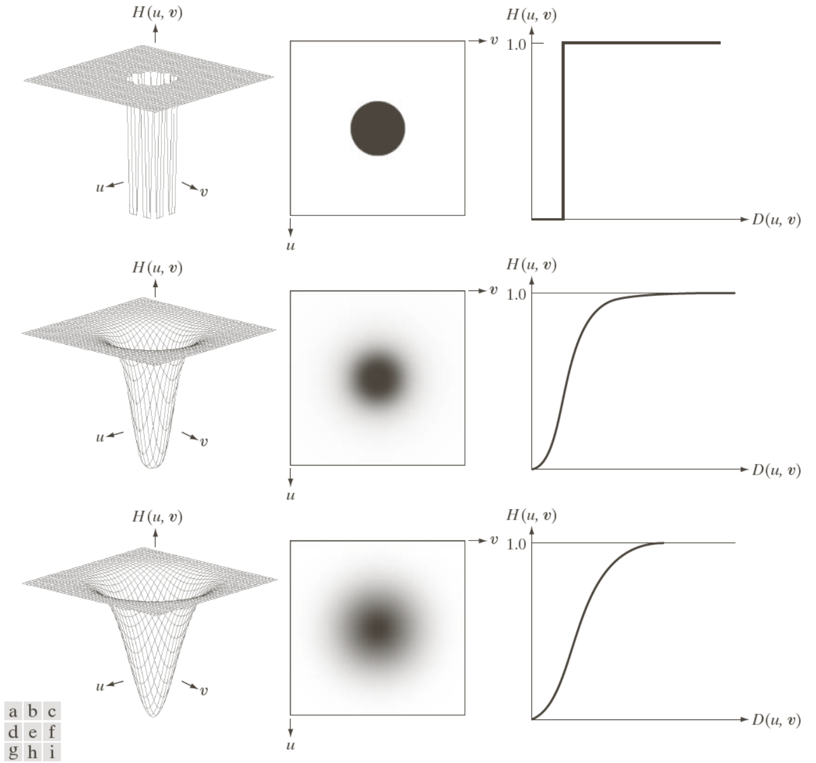



**FIGURE 4.52** Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

# Highpass filter in the spatial domain



- Spatial domain: ringing is present as in lowpass filtering

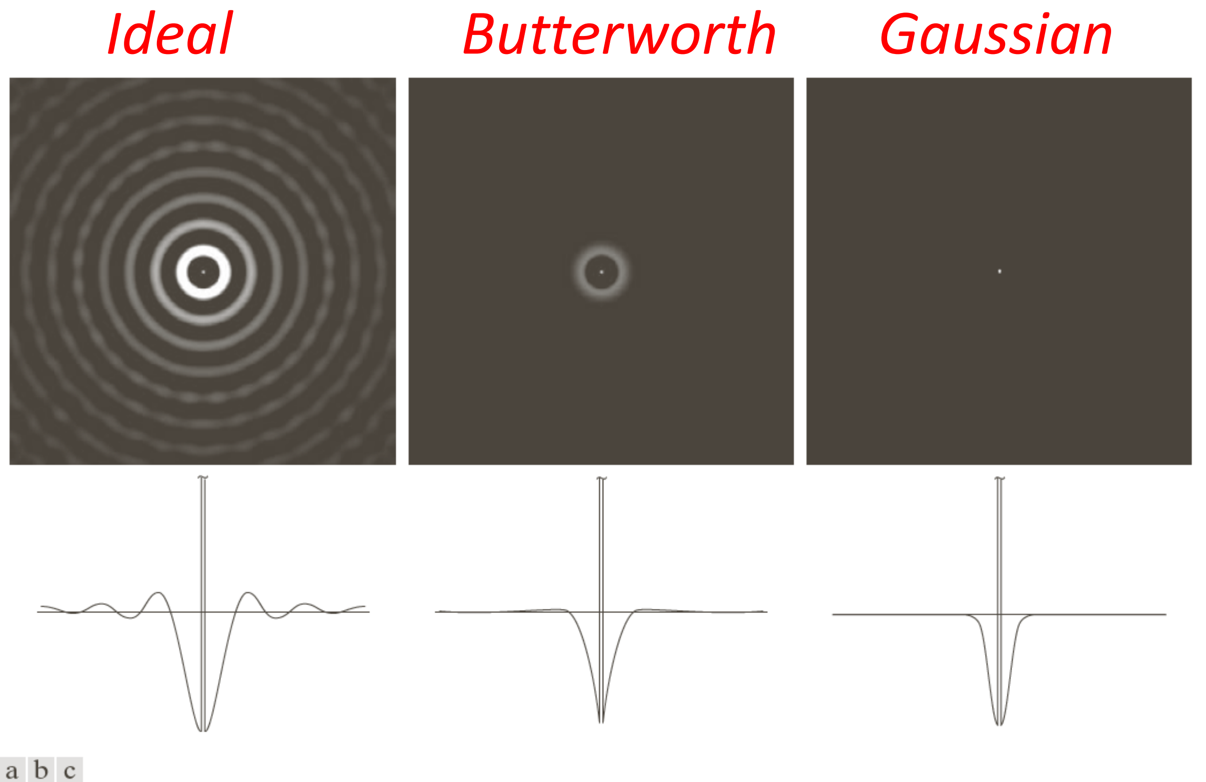



**FIGURE 4.53** Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

# Ideal highpass



a b c

**FIGURE 4.54** Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and $160$.

$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

# Butterworth highpass



a b c

**FIGURE 4.55** Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

$$H(u, v) = \frac{1}{1 + [D_0/D(u, v)]^{2n}}$$

# Gaussian highpass



a b c

**FIGURE 4.56** Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$

# Filters



- Lowpass filters – smoothing
  - Ideal
  - Butterworth
  - Gaussian
- Highpass filters – sharpening
  - Ideal
  - Butterworth
  - Gaussian
- Selective filters
  - Band-reject
  - Notch

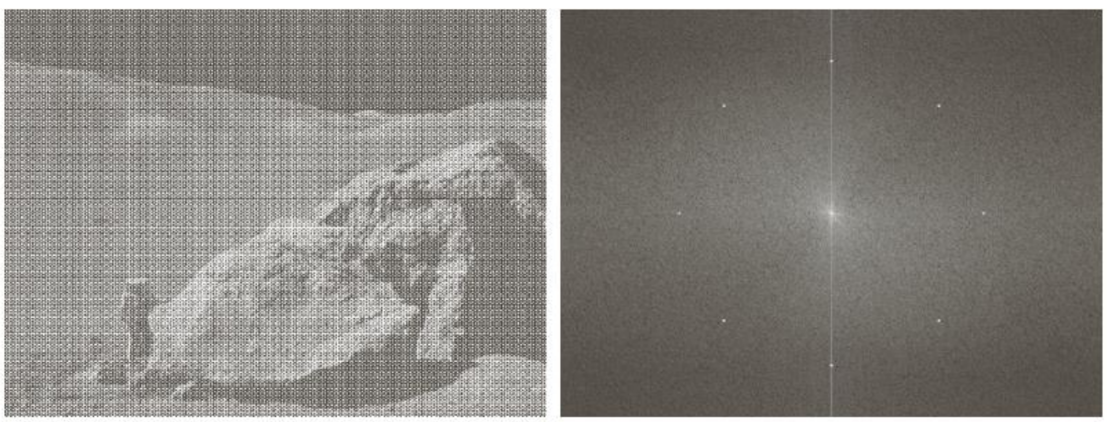
# Filters

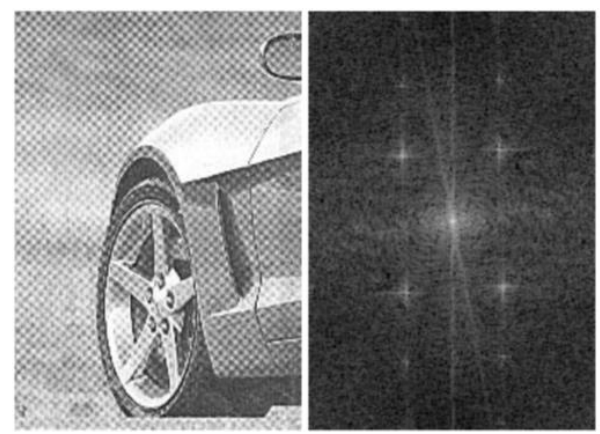


- Lowpass filters – smoothing
  - Ideal
  - Butterworth
  - Gaussian
- Highpass filters – sharpening
  - Ideal
  - Butterworth
  - Gaussian
- Selective filters
  - Band-reject
  - Notch

# Noise in frequency domain

- Noise can affect the frequency domain
  - E.g. noise on some frequencies





# Selective filtering

- Operate on selected frequency bands
  - Band-pass filter
  - Band-reject filter
- Operate on selected frequency rectangles/areas
- Same profiles already discussed
  - Ideal
  - Butterworth
  - Gaussian

# Band-reject filters




a b c *Ideal* *Butterworth (n=1)* *Gaussian*

**FIGURE 5.15** From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

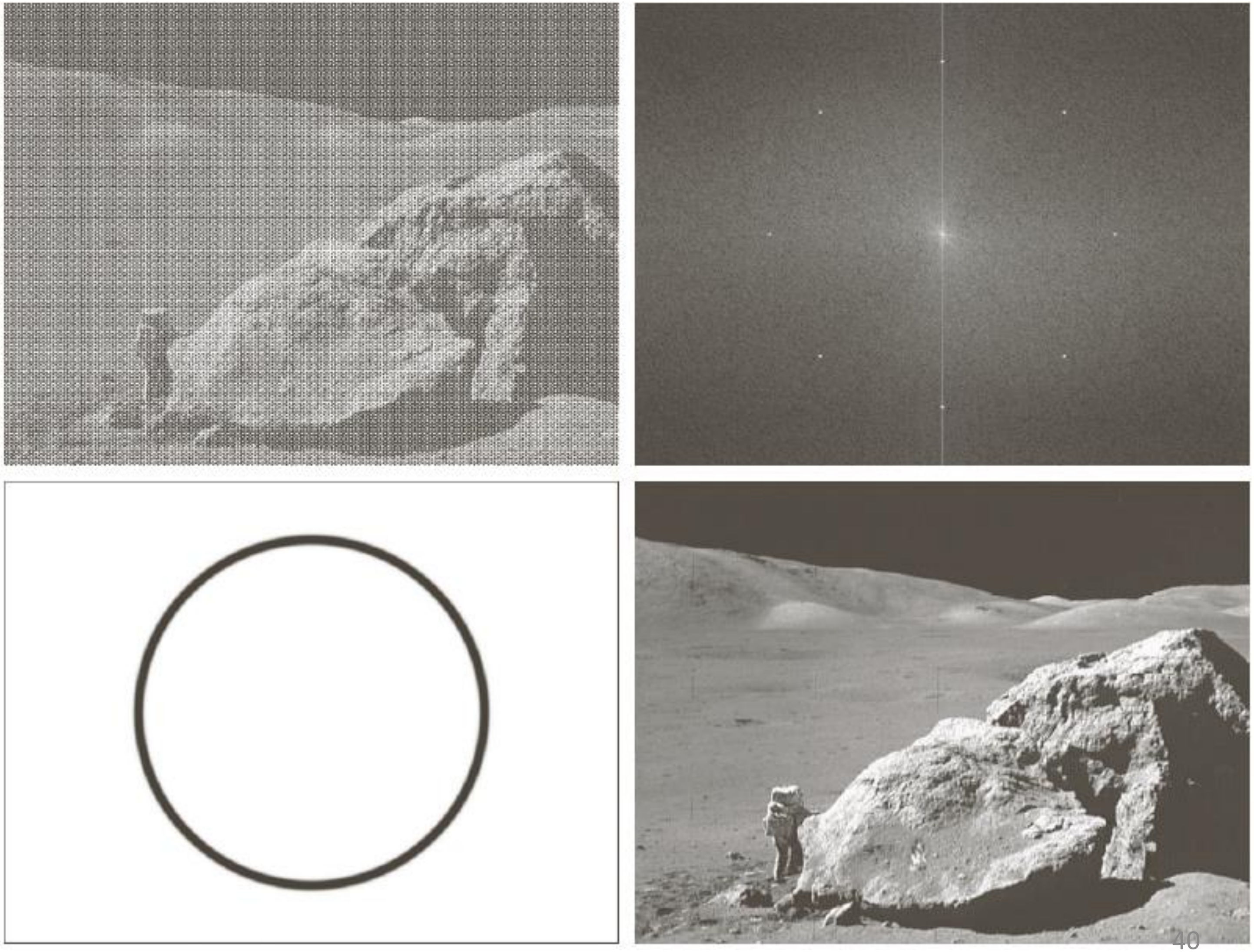
**TABLE 4.6**
Bandreject filters. $W$ is the width of the band, $D$ is the distance $D(u, v)$ from the center of the filter, $D_0$ is the cutoff frequency, and $n$ is the order of the Butterworth filter. We show $D$ instead of $D(u, v)$ to simplify the notation in the table.

| Ideal | Butterworth | Gaussian |
|---|---|---|
| $H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$ | $H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$ | $H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$ |

$$H_{bp}(u,v) = 1 - H_{br}(u,v)$$

# Band-reject





a b
c d

**FIGURE 5.16** (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

# Notch filters



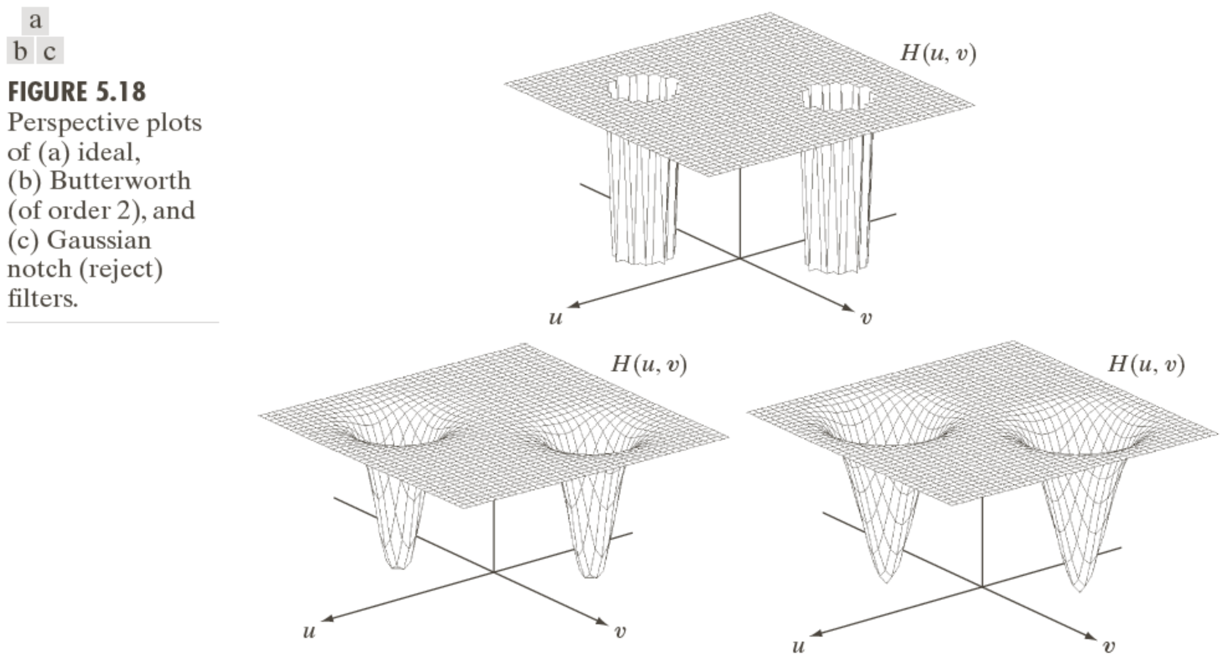
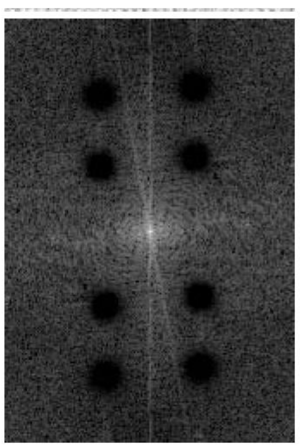



**FIGURE 5.18** Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

# Band-reject and notch – examples



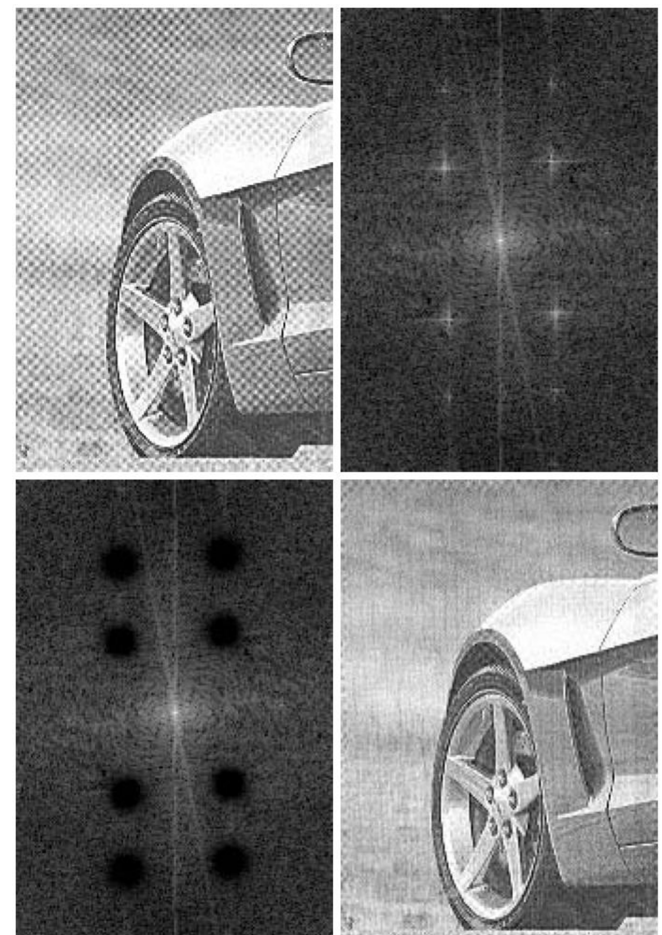


*Notch*

$$H_{NR}(u,v) = \prod_{k=1}^{4} \left[\frac{1}{1+\left[\frac{D_{0k}}{D_k(u,v)}\right]^{2n}}\right]\left[\frac{1}{1+\left[\frac{D_{0k}}{D_{-k}(u,v)}\right]^{2n}}\right]$$

a b
c d

**FIGURE 4.64** (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.



# General observations

# Filtering in frequency



- Several filters previously described perform operations that are similar to the filters in the space domain
  - Smoothing
  - Sharpening / edge highlighting
- What are the differences with respect to the filters in the spatial domain?

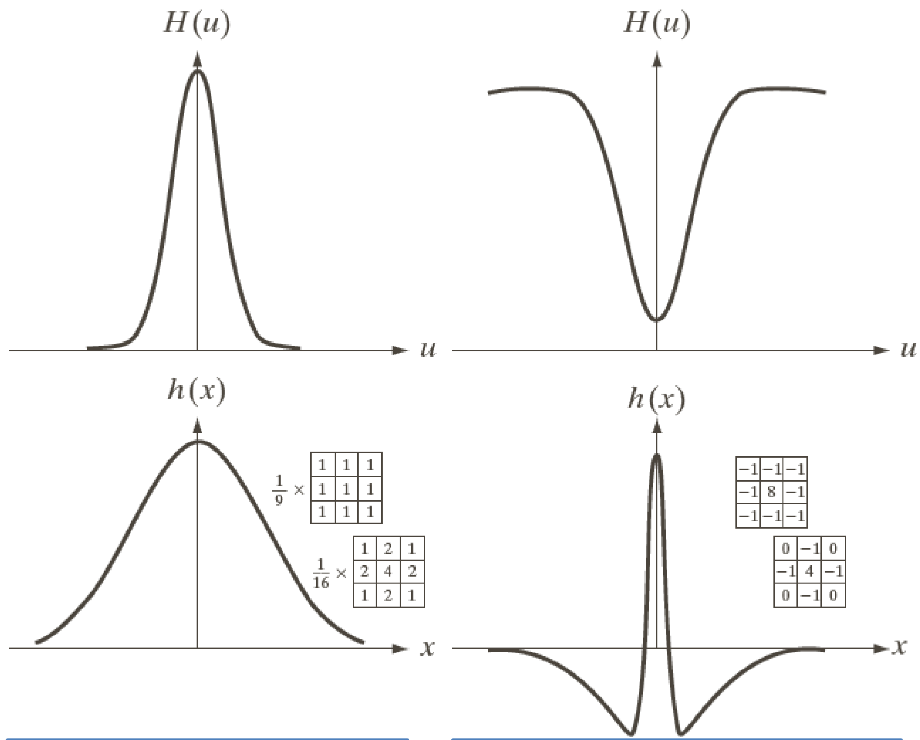
# Deriving the spatial mask



- Filtering in frequency lets us operate on frequencies
- The image is provided in the space domain:
  - Need to filter on the space domain: **convolution**
  - Need to evaluate the filters
- Filtering in frequency means to apply convolution using filters that implement in space operations defined in frequency

# Deriving the spatial mask

- The spatial mask can be obtained applying the inverse transform
  - Example: Gaussian filters




a c
b d

**FIGURE 4.37** (a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

Low-pass

High-pass

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# Filtering in the frequency domain

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