Deterministic finite automata
Nondeterministic finite automata
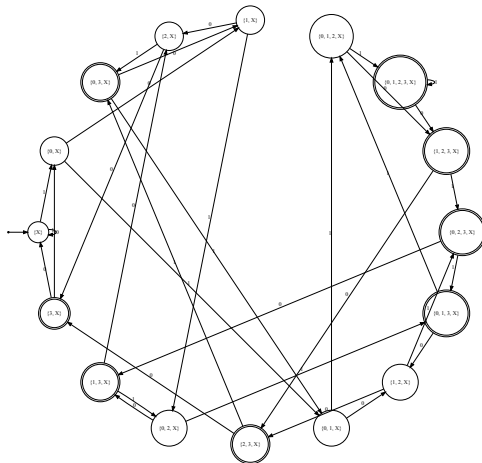Nondeterministic finite automata with ϵ-transitions

Automata, Languages and Computation

Chapter 2 : Finite Automata

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University of Padua
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Lecture based on material originally developed by :
Gösta Grahne, Concordia University

Deterministic finite automata
Nondeterministic finite automata
Nondeterministic finite automata with ε-transitions

Finite Automata

Deterministic finite automata
Nondeterministic finite automata
Nondeterministic finite automata with ϵ-transitions

1 Deterministic finite automata : this is the simplest and most efficient type of FA

2 Nondeterministic finite automata : FAs with choices activating independent computations

3 Nondeterministic finite automata with ϵ-transitions : nondeterministic automata with special moves that do not consume the input

Deterministic finite automata
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Nondeterministic finite automata with ϵ-transitions

Deterministic finite automata

These devices read input from left to right

They can only store a quantity of information limited by a constant, using the important notion of **state**

They are easy to implement in a computer (table)

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Deterministic finite automata

A **deterministic finite automaton** (DFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where :

- Q is a **finite** set of **states**
- Σ is the input symbol **alphabet**
- δ is a **transition** function $Q \times \Sigma \to Q$
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is a set of final states

Comment on the notion of determinism

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Example

A DFA A that **accepts** the language of all strings of 0 and 1 containing the substring 01 :
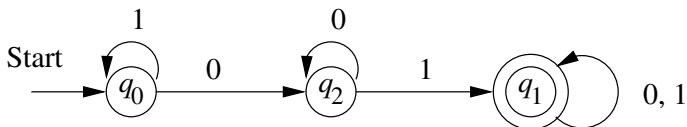
$$L = \{x01y \mid x, y \in \{0, 1\}^*\}$$

$Q = \{q_0, q_1, q_2\}$, $\Sigma = \{0, 1\}$, $F = \{q_1\}$

δ function, specified by means of a **transition table** :

	0	1
$\rightarrow q_0$	q_2	q_0
$\star q_1$	q_1	q_1
q_2	q_2	q_1

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Example

Our DFA A specified by means of a **transition diagram** :

What is the meaning of q_0, q_1, q_2?

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Test

Does the following transition diagram correspond to a DFA ?

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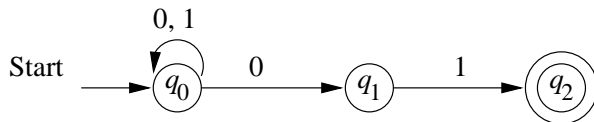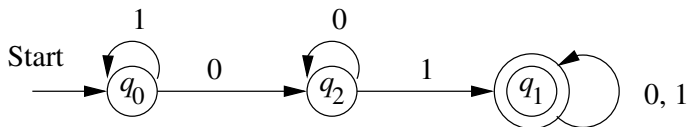
Acceptance

A DFA **accepts** a string $w = a_1 a_2 \cdots a_n$ if there is a path in the transition diagram that

- starts in the initial state
- ends in some final state
- has a sequence of transitions with labels $a_1 a_2 \cdots a_n$

Note that the above **is not** a mathematical definition

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Test

Is the string 01101 accepted by the DFA below ?

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Extended transition function

The δ transition function can be **extended** to function $\hat{\delta}$ defined over state and string pairs (as opposed to state and alphabet symbol pairs)

Base $\hat{\delta}(q, \epsilon) = q$

Induction $\hat{\delta}(q, xa) = \delta(\hat{\delta}(q, x), a)$

x e' una stringa, a un simbolo

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Example

Given a string w over the alphabet Σ and a symbol a, let $\#_a(w)$ denote the number of **occurrences** of a in w
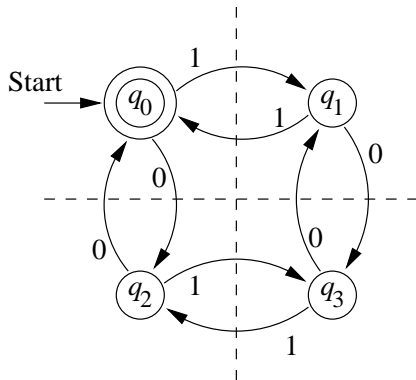
Specify a DFA A accepting all and only the strings in the following language

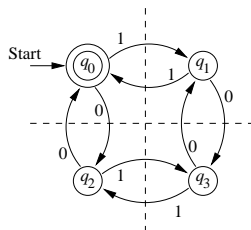$$L = \{w \mid w \in \{0,1\}^*,\ \#_0(w) \text{ even},\ \#_1(w) \text{ even}\}$$

In words, L contains all and only the binary strings with an even number of 0's and an even number of 1's

What is the shortest string in L? Are there strings in L with odd length?

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Example

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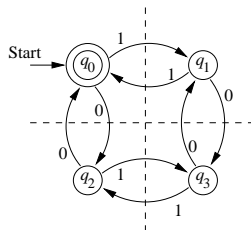Nondeterministic finite automata with ϵ-transitions

Example

States have the following meaning

- q_0: $\#_0(w)$ and $\#_1(w)$ even
- q_1: $\#_0(w)$ even, $\#_1(w)$ odd
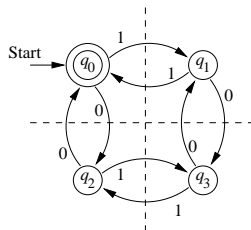- q_2: $\#_0(w)$ odd, $\#_1(w)$ even
- q_3: $\#_0(w)$ and $\#_1(w)$ odd

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Example

Tabular representation of the DFA

	0	1
$\rightarrow \star q_0$	q_2	q_1
q_1	q_3	q_0
q_2	q_0	q_3
q_3	q_1	q_2

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Example

Is string $w = 0101$ accepted by A ?

- $\hat{\delta}(q_0, \epsilon) = q_0$
- $\hat{\delta}(q_0, 0) = \delta(\hat{\delta}(q_0, \epsilon), 0) = \delta(q_0, 0) = q_2$
- $\hat{\delta}(q_0, 01) = \delta(\hat{\delta}(q_0, 0), 1) = \delta(q_2, 1) = q_3$
- $\hat{\delta}(q_0, 010) = \delta(\hat{\delta}(q_0, 01), 0) = \delta(q_3, 0) = q_1$
- $\hat{\delta}(q_0, 0101) = \delta(\hat{\delta}(q_0, 010), 1) = \delta(q_1, 1) = q_0 \in F$

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Language recognized by a DFA

The language **recognized** by DFA A is

$$L(A) = \{w \mid \hat{\delta}(q_0, w) \in F\}$$

The languages accepted by the class of DFAs are called **regular languages**

$$\{L \mid L = L(A), A \text{ is DFA}\}$$

- a language is a set of strings
- a class of languages is a set of languages

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Nondeterministic finite automata
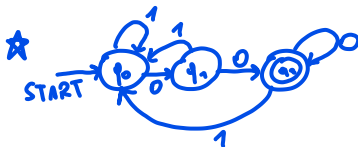Nondeterministic finite automata with ϵ-transitions

Notational conventions

Commonly used notation for DFAs

- $a, b, c, \ldots$ alphabet symbols
- u, v, w, x, y, z strings over input alphabet
- $p, q, r, s, q_0, q_1, q_2, \ldots$ states

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Test

Specify DFAs for the following languages over the alphabet $\{0, 1\}$:

- set of all strings ending in 00
  - set of all strings with three consecutive 0's
  - set of all strings with 011 as a substring
  - set of all strings that start or end (or both) with 01

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Exercise

Consider the language L of strings over the alphabet $\{0, 1\}$ with **exactly** one occurrence of string 00

Carry out the following points :

- draw the transition diagram of a DFA A such that $L(A) = L$
- state the meaning of each of A's states (i.e. for each state of A describe the strings leading to it)

Hint: define a "failure state" that can never reach any final state

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Nondeterministic finite automata

These automata accept only regular languages

Easier to design than DFAs

Later on we will see several examples of this fact

Very useful for implementing the **search** for a pattern in a text

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Nondeterministic finite automata

A nondeterministic finite automaton can **simultaneously** be in different states

The automaton **accepts** if at least one final state is reached at the end of the scan of the input string

Equivalently, in a given state the automaton can **guess** which next state will lead to acceptance

This interpretation is not in the textbook

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Example

Nondeterministic automaton N accepting all and only the strings ending in 01
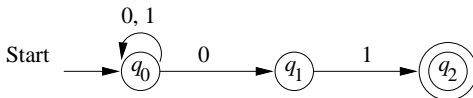
Simultaneous computations of N on input string 00101

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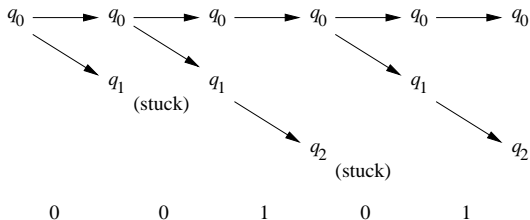
Nondeterministic finite automaton

A **nondeterministic finite automata** (NFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where :

- Q is a **finite** set of **states**
- Σ is the **alphabet** of input symbols
- δ is a **transition** function $Q \times \Sigma \to 2^Q$, where 2^Q is the set of all subsets of Q (power set) *insieme delle parti*
- $q_0 \in Q$ is the initial state
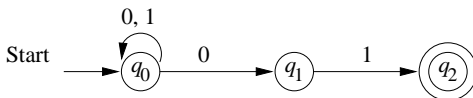- $F \subseteq Q$ is the set of final states

if $|Q| = n$
$|2^Q| = 2^n$

the output of δ is a set, not an element (state) for NFAs

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Nondeterministic finite automata
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Example

The transition diagram

represents the nondeterministic automaton

$$A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$$

with transition function δ

	0	1
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\varnothing$	$\{q_2\}$
$\star q_2$	$\varnothing$	$\varnothing$

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Extended transition function $\hat{\delta}$

it's a set!

Base $\hat{\delta}(q, \epsilon) = \{q\}$

Induction

different from $\hat{S}$ for DFAs

$$\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q, x)} \delta(p, a)$$

Notice the difference with the case of DFA in the induction part. Can you explain this?

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Example
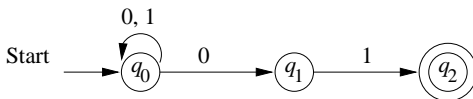
(this automata recognises all strings that end in 01)

Computation of $\hat{\delta}(q_0, 00101)$

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$
- $\hat{\delta}(q_0, 0010) = \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00101) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$

this computation fails

this computation succeds

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Accepted language for NFA

The **accepted** language for an NFA A is

set of final states

$$L(A) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

In words, $L(A)$ is the set of all strings $w \in \Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains **at least one** final state. This amounts to say that at least one computation for w leads to acceptance

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Test

HOME WORK

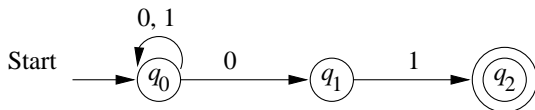Consider the following language over $\Sigma = \{0, 1\}$

$$L = \{w \mid w = x1ab, \ x \in \Sigma^*, \ a, b \in \Sigma\}$$

Informally, L is the set of all strings with 1 as **third to last** symbol

Specify a NFA A such that $L(A) = L$

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Exercise with solution

Show that the NFA

accepts the language $L = \{x01 \mid x \in \Sigma^*\}$

NO, WE SKIPPED IT

We prove the following three statements using **mutual induction**:

(i) $q_0 \in \hat{\delta}(q_0, w) \Leftrightarrow w \in \Sigma^*$

(ii) $q_1 \in \hat{\delta}(q_0, w) \Leftrightarrow w = x0, \ x \in \Sigma^*$

(iii) $q_2 \in \hat{\delta}(q_0, w) \Leftrightarrow w = x01, \ x \in \Sigma^*$

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Exercise with solution ~~No~~

Base If $|w| = 0$ then $w = \epsilon$, and statement (i) follows from the definition of $\hat{\delta}$. As for statements (ii) and (iii), both hand sides hold false for ϵ

Induction Assume $w = xa$, with $a \in \{0, 1\}$, $|x| = n$, and assume statements (i)–(iii) hold true for x.

(i) We know that $q_0 \in \hat{\delta}(q_0, x)$. From state q_0 we have transitions to q_0 for both 0 and 1

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Exercise with solution

(ii) (*if*) w ends with 0. From (i) we know that $q_0 \in \hat{\delta}(q_0, x)$. There is a transition from q_0 to q_1 on symbol 0. Hence $q_1 \in \hat{\delta}(q_0, w)$

(*only if*) $q_1 \in \hat{\delta}(q_0, w)$. In order to reach q_1, the only possible transition is from q_0 upon reading 0. Thus $w = x0$

(iii) (*if*) w ends with 01, and thus $a = 1$ and x ends with 0. From (ii) we have $q_1 \in \hat{\delta}(q_0, x)$. From q_1 we can reach q_2 upon reading 1. Then $q_2 \in \hat{\delta}(q_0, w)$

(*only if*) $q_2 \in \hat{\delta}(q_0, w)$. In order to reach q_2, the only possible transition is from q_1 upon reading 1. From statement (ii) for x we have that x ends with 0. We then conclude that w ends with 01 □

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Equivalence for DFA and NFA

NFAs are **easier** than DFAs to "program", since nondeterminism makes it possible to simplify the structure of the automaton

Example : compare NFA and DFA accepting strings in $\{0, 1\}^*$ with penultimate symbol 1

recognises all strings with a 1 as penultimate symbol
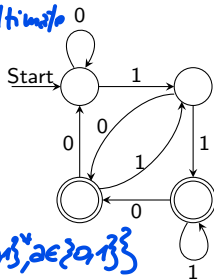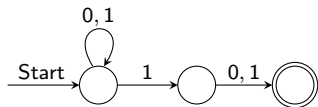
$L = \{w \mid w \in \{0,1\}^*, w = x 1 a, x \in \{0,1\}^*, a \in \{0,1\}\}$

With an increase in the distance between 1 and the end of the string, the gap gets exponentially larger

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Equivalence for DFA and NFA

Quite surprisingly, for every NFA N there exists some DFA D such that $L(D) = L(N)$. The proof involves the **subset construction**

Idea : build a state in D for every state set representing a "configuration" in a computation of N. The collection of all configurations is still a **finite set**

this is the automaton that recognizes strings that end in 01

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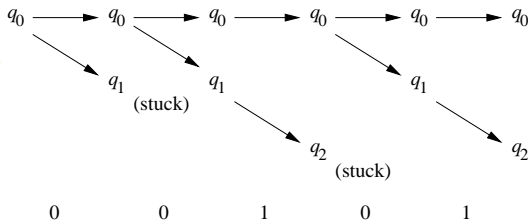
Equivalence for DFA and NFA

Given an NFA

$$N = (Q_N, \Sigma, \delta_N, q_0, F_N)$$

the subset construction produces a DFA

$$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$$

such that $L(D) = L(N)$

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Equivalence for DFA and NFA
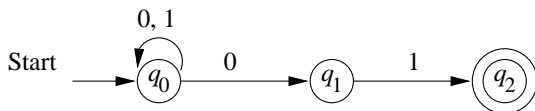
Subset construction :

- $Q_D = \{S \mid S \subseteq Q_N\}$
- $F_D = \{S \subseteq Q_N \mid S \cap F_N \neq \varnothing\}$
- For every $S \subseteq Q_N$ and $a \in \Sigma$,

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$$

Note : $|Q_D| = 2^{|Q_N|}$. Nonetheless, the large majority of states in Q_D turn out to be **garbage**, that is, they cannot be reached from the initial state

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Example

Construction of δ_D :

	0	1
$\emptyset$	$\emptyset$	$\emptyset$
$\rightarrow \{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_1\}$	$\emptyset$	$\{q_2\}$
$\star\{q_2\}$	$\emptyset$	$\emptyset$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\star\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\star\{q_1, q_2\}$	$\emptyset$	$\{q_2\}$
$\star\{q_0, q_1, q_2\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

some states

are

unreachable

$(e.g. \{q_1\}, \{q_1, q_2\} \ldots)$

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Example

Note : D states correspond to subsets of N states, but we could denote D states in any other way, for instance using the letters $A, B, \ldots, F$

	0	1
A	A	A
$\to B$	E	B
C	A	D
$\star D$	A	A
E	E	F
$\star F$	E	B
$\star G$	A	D
$\star H$	E	F

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Equivalence for DFA and NFA

We can often avoid **exponential growth** of states in Q_D using a technique called **lazy evaluation** (or deferred evaluation)

State q of DFA A is **accessible** if there is at least one string w such that $\hat{\delta}_A(q_0, w) = q$

We build the transition table of D **only** for the accessible states of D

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Equivalence for DFA and NFA

Construction of DFA D through lazy evaluation

Base $S = \{q_0\}$ is accessible in D
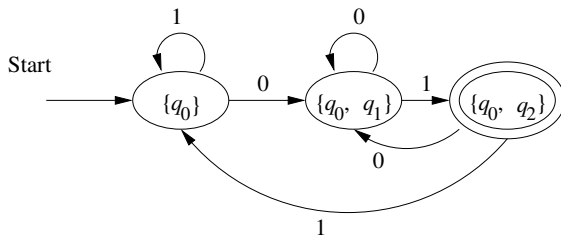
Induction If state S is accessible in D, then state $\delta_D(S, a)$ is also accessible in D, for every $a \in \Sigma$

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Example

DFA D with only accessible states

In several **practical applications** D has about as many states as N

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Equivalence for DFA and NFA

⚠ Important proof

Theorem Let D be the DFA obtained from an NFA N using the subset construction. Then $L(D) = L(N)$

Proof We first prove that, for every string $w \in \Sigma^*$, we have

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$$

← this is more
general
then $L(D) = L(N)$
so if it holds
⇒ $L(D) = L(N)$ holds

Check that both sides in the above equation are sets!

We use induction on $|w|$

Base $w = \epsilon$. The claim follows from the definition

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Equivalence for DFA and NFA

Induction

$$
\begin{aligned}
\hat{\delta}_D(\{q_0\}, xa) &= \delta_D(\hat{\delta}_D(\{q_0\}, x), a) && \text{definition of } \hat{\delta}_D \\
&= \delta_D(\hat{\delta}_N(q_0, x), a) && \text{induction} \\
&= \bigcup_{p \in \hat{\delta}_N(q_0, x)} \delta_N(p, a) && \text{definition of } \delta_D \\
&= \hat{\delta}_N(q_0, xa) && \text{definition of } \hat{\delta}_N
\end{aligned}
$$

$L(D) = L(N)$ now follows from the definition of F_D $\qquad\qquad$ $\square$

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Equivalence for DFA and NFA

Theorem A language L is accepted by a DFA if and only if L is accepted by an NFA

Proof (If) Previous theorem

(Only if) Any DFA can be converted into an equivalent NFA by modifying δ_D into δ_N according to the following rule
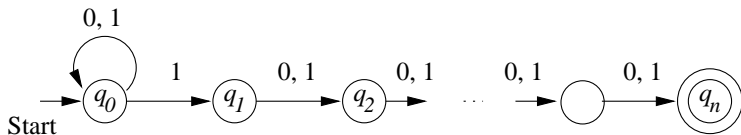   If $\delta_D(q, a) = p$, then $\delta_N(q, a) = \{p\}$

By induction on $|w|$ one can show that $\hat{\delta}_D(q_0, w) = p$ if and only if $\hat{\delta}_N(q_0, w) = \{p\}$ $\qquad\square$

Deterministic finite automata
Nondeterministic finite automata
Nondeterministic finite automata with ϵ-transitions

Exponential growth of the state set

⚠ **Theorem** There exists an NFA N with $n + 1$ states that has no equivalent DFA with less than 2^n states

Proof Let N be the NFA

$$L(N) = \{x1c_2c_3 \cdots c_n \mid x \in \{0,1\}^*, c_i \in \{0,1\}\}$$

Intuitively, an equivalent DFA must "remember" the last n symbols it has read

Those symbols might all be relevant for the final decision

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Exponential growth of the state set

(proof by contradiction)

Suppose there exists a DFA D equivalent to N with fewer than 2^n states

There are 2^n binary strings of length n. Since D has fewer that 2^n states, there must be

- a state q,
- binary strings $a_1 a_2 \cdots a_n \neq b_1 b_2 \cdots b_n$,

such that

$$\hat{\delta}_D(q_0, a_1 a_2 \cdots a_n) = \hat{\delta}_D(q_0, b_1 b_2 \cdots b_n) = q$$

The above reasoning uses the so-called pigeonhole principle

if two strings (different) end up in the same state then they will have the same options later

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Exponential growth of the state set

Since $a_1 a_2 \cdots a_n \neq b_1 b_2 \cdots b_n$, there exists i with $1 \leqslant i \leqslant n$ such that $a_i \neq b_i$; we assume $a_i = 1$ and $b_i = 0$ (the other case being symmetrical)

Case 1: $i = 1$; we have

$$\hat{\delta}_D(q_0, 1a_2 \cdots a_n) \in F$$
$$\hat{\delta}_D(q_0, 0b_2 \cdots b_n) \notin F$$

which is a contradiction

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Exponential growth of the state set

Case 2: $i > 1$; since $\hat{\delta}_D(q_0, a_1 a_2 \cdots a_n) = \hat{\delta}_D(q_0, b_1 b_2 \cdots b_n)$ and D is deterministic, we have

$$\hat{\delta}_D(q_0, a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_n 0^{i-1}) =$$
$$\hat{\delta}_D(q_0, b_1 \cdots b_{i-1} 0 b_{i+1} \cdots b_n 0^{i-1})$$
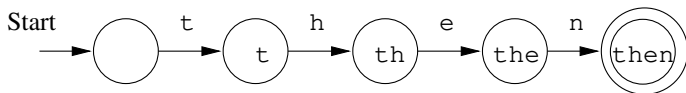
From the definition of L, we must have

$$\hat{\delta}_D(q_0, a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_n 0^{i-1}) \in F$$
$$\hat{\delta}_D(q_0, b_1 \cdots b_{i-1} 0 b_{i+1} \cdots b_n 0^{i-1}) \notin F$$

which is a contradiction $\qquad\square$

the important thing is that they are in the same number not that they are the same

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Partial DFA

This is not a DFA, since for some symbols in Σ transitions are not specified

A **partial** DFA has **at most** one outgoing transition for each state in Q and for each symbol in Σ

A partial DFA can be completed to a DFA if we add one non-accepting state having the status of a **trap state**, from which you cannot escape

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Exercise with solution

Consider the NFA

$$N \;=\; (\{q_0, q_1\}, \{0, 1\}, \delta_N, q_0, \{q_1\}),$$

where $\delta_N(q_0, 0) = \{q_0, q_1\}$, $\delta_N(q_0, 1) = \{q_1\}$, $\delta_N(q_1, 0) = \varnothing$, $\delta_N(q_1, 1) = \{q_0, q_1\}$

- check whether strings $w_1 = 101$ and $w_2 = 0010$ are in $L(N)$, showing all steps in the computations
- construct the transition diagram of the DFA equivalent to N
- using a set-former, define the language accepted by the automaton; **suggestion**: this is easier if you look at the DFA

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Exercise with solution

$w_1 = 101 \in L(A)$?

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 1) = \delta(q_0, 1) = \{q_1\}$
- $\hat{\delta}(q_0, 10) = \delta(q_1, 0) = \varnothing$, then $w_1 = 101 \notin L(A)$

$w_2 = 0010 \in L(A)$?

- $\hat{\delta}(q_0, \epsilon) = \{q_0\}$
- $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_1\} \cup \{q_0, q_1\} = \{q_0, q_1\}$
- $\hat{\delta}(q_0, 0010) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- since $\{q_0, q_1\} \cap \{q_1\} \neq \varnothing$, $w_2 = 0010 \in L(A)$

Deterministic finite automata
Nondeterministic finite automata
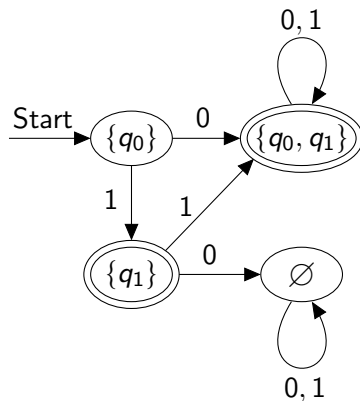Nondeterministic finite automata with ε-transitions

Exercise with solution

We construct the transition diagram of the equivalent DFA using the subset construction and lazy evaluation

- $\delta_D(\{q_0\}, 0) = \delta_N(q_0, 0) = \{q_0, q_1\}$
- $\delta_D(\{q_0\}, 1) = \delta_N(q_0, 1) = \{q_1\}$
- $\delta_D(\{q_0, q_1\}, 0) = \delta_N(q_0, 0) \cup \delta_N(q_1, 0)$
  $= \{q_0, q_1\} \cup \varnothing = \{q_0, q_1\}$
- $\delta_D(\{q_0, q_1\}, 1) = \delta_N(q_0, 1) \cup \delta_N(q_1, 1)$
  $= \{q_1\} \cup \{q_0, q_1\} = \{q_0, q_1\}$
- $\delta_D(\{q_1\}, 0) = \delta_N(q_1, 0) = \varnothing$
- $\delta_D(\{q_1\}, 1) = \delta_N(q_1, 1) = \{q_0, q_1\}$
- $\delta_D(\varnothing, 0) = \delta_D(\varnothing, 1) = \varnothing$
- $\{q_0\}$ initial state, $\{q_0, q_1\}$ e $\{q_1\}$ final states

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Exercise with solution

Graphical representation of the transition diagram

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Exercise with solution

Using a set-former, define the language accepted by the automaton

$$
\begin{aligned}
L(A) \quad = \quad & \{w \in \{0,1\}^+ \mid w = 1 \text{ or } w = 0x \\
& \text{or } w = 11y, \ x, y \in \{0,1\}^*\}
\end{aligned}
$$

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Exercises

Specify an NFA A for each of the following languages defined on the alphabet $\{0, 1\}$

- set of strings with two consecutive 0 or two consecutive 1
- set of strings such that at least one of the last three symbols is 1

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NFA with ϵ-transitions

Extension of NFAs where transitions labelled with symbol ϵ are allowed; this means that the automaton can change state **without consuming** any of its input

They accept all and only the regular languages

Easier to design than NFAs

ϵ-NFA widely used in compilers and for search of patterns in a text

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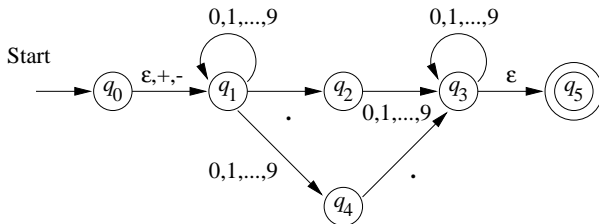
Example

A fractional number consists of

- $+$ or - sign, optional
- a first string of digits
- one decimal point
- a second string of digits

with the first or the second strings **optional**, but not both

This example comes from a lexical analyser in compiler theory

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Example
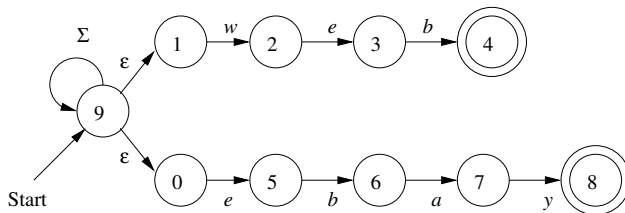
ϵ-NFA accepting fractional numbers

The ϵ-transition makes operators $+$ and $-$ optional

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Example

ϵ-NFA accepting set of keywords $\{\texttt{ebay}, \texttt{web}\}$

The ϵ-transition makes it easy to combine several automata

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NFA with ϵ-transitions

A **nondeterministic finite automaton with ϵ-transitions**
(ϵ-NFA) is a 5-tuple

$$A = (Q, \Sigma, \delta, q_0, F)$$

where

- Q, Σ, q_0, and F are defined as for NFAs
- δ is a **transition** function $Q \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^Q$, with 2^Q denoting the class of subsets of Q

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Example

ϵ-NFA accepting fractional numbers

$$E = (\{q_0, q_1, \ldots, q_5\}, \{., +, -, 0, 1, \ldots, 9\}\ \delta, q_0, \{q_5\})$$

Transition function δ

	ϵ	+,-	.	$0, \ldots, 9$
$\rightarrow q_0$	$\{q_1\}$	$\{q_1\}$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\varnothing$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\varnothing$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_3	$\{q_5\}$	$\varnothing$	$\varnothing$	$\{q_3\}$
q_4	$\varnothing$	$\varnothing$	$\{q_3\}$	$\varnothing$
$\star q_5$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

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Test

Specify an ϵ-NFA accepting the language of strings over $\{a, b, c\}$ with zero or more a's, followed by zero or more b's, followed by zero or more c's

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ϵ-closure

Let us compute the **ϵ-closure** of a state q, written $\text{ECLOSE}(q)$, adding all the states reachable from q itself through a sequence of one or more symbols ϵ

Needed later in the definition of $\hat{\delta}$ function
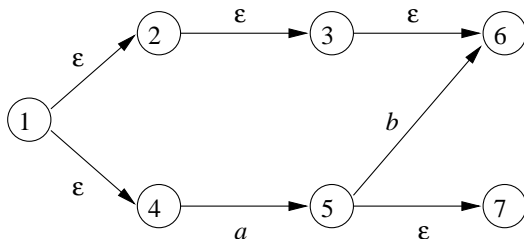
Base $q \in \text{ECLOSE}(q)$

Induction $(p \in \text{ECLOSE}(q) \land r \in \delta(p, \epsilon)) \Rightarrow r \in \text{ECLOSE}(q)$

Extension to set of states S

$$\text{ECLOSE}(S) = \bigcup_{q \in S} \text{ECLOSE}(q)$$

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Example

ϵ-NFA fragment

$$\text{ECLOSE}(1) = \{1, 2, 3, 4, 6\}$$
$$\text{ECLOSE}(\{4, 5\}) = \{4\} \cup \{5, 7\} = \{4, 5, 7\}$$

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Extended transition function $\hat{\delta}$

Base $\hat{\delta}(q, \epsilon) = \mathsf{ECLOSE}(q)$

Induction $\hat{\delta}(q, xa)$ is computed as

- $\{p_1, \ldots, p_k\} = \hat{\delta}(q, x)$
- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta(p_i, a)$
- $\hat{\delta}(q, xa) = \mathsf{ECLOSE}(\{r_1, \ldots, r_m\})$

Note that processing of ϵ symbols is accounted for after the processing of each symbol in Σ

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Example

We compute $\hat{\delta}(q_0, 5.6)$ for the ϵ-NFA accepting fractional numbers

$\hat{\delta}(q_0, \epsilon) = \text{ECLOSE}(q_0) = \{q_0, q_1\}$

Computation of $\hat{\delta}(q_0, 5)$:

- $\delta(q_0, 5) \cup \delta(q_1, 5) = \varnothing \cup \{q_1, q_4\} = \{q_1, q_4\}$
- $\text{ECLOSE}(q_1) \cup \text{ECLOSE}(q_4) = \{q_1\} \cup \{q_4\} = \{q_1, q_4\} = \hat{\delta}(q_0, 5)$

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Example

Computation of $\hat{\delta}(q_0, 5.)$:

- $\delta(q_1, .) \cup \delta(q_4, .) = \{q_2\} \cup \{q_3\} = \{q_2, q_3\}$
- $\text{ECLOSE}(q_2) \cup \text{ECLOSE}(q_3) = \{q_2\} \cup \{q_3, q_5\} = \{q_2, q_3, q_5\} = \hat{\delta}(q_0, 5.)$

Computation of $\hat{\delta}(q_0, 5.6)$:

- $\delta(q_2, 6) \cup \delta(q_3, 6) \cup \delta(q_5, 6) = \{q_3\} \cup \{q_3\} \cup \varnothing = \{q_3\}$
- $\text{ECLOSE}(q_3) = \{q_3, q_5\} = \hat{\delta}(q_0, 5.6)$

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Accepted language for ϵ-NFA

The language **accepted** by ϵ-NFA $E = (Q, \Sigma, \delta, q_0, F)$ is

$$L(E) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$$

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From ϵ-NFA to DFA

Given the ϵ-NFA
$$E = (Q_E, \Sigma, \delta_E, q_0, F_E)$$

we construct a DFA
$$D = (Q_D, \Sigma, \delta_D, q_D, F_D)$$

such that $L(D) = L(E)$

Construction details :

- $Q_D = \{S \mid S \subseteq Q_E, \ S = \text{ECLOSE}(S)\}$
- $q_D = \text{ECLOSE}(q_0)$
- $F_D = \{S \mid S \in Q_D, \ S \cap F_E \neq \varnothing\}$

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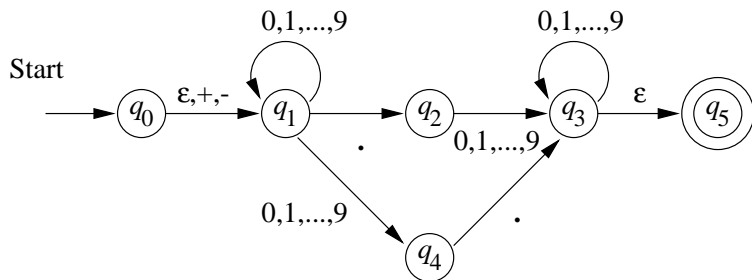
From ϵ-NFA to DFA

Construction details (cont'd)

Computation of $\delta_D(S, a)$, $a \in \Sigma$ and $S \in Q_D$

- $S = \{p_1, \ldots, p_k\}$
- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\delta_D(S, a) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

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Example

ϵ-NFA E

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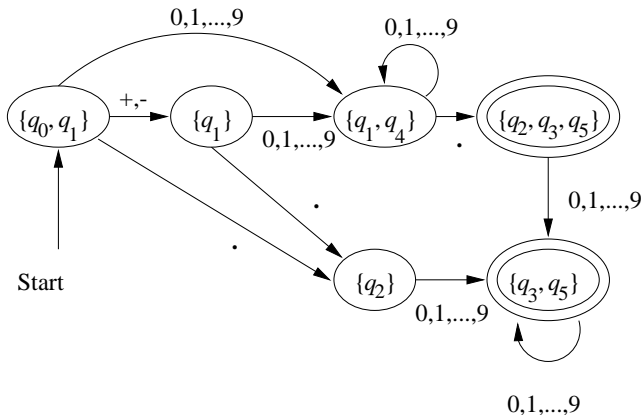Nondeterministic finite automata with ϵ-transitions

Example

Computation of some of the values of δ_D

- $\delta_D(\{q_0, q_1\}, +) = \text{ECLOSE}(\delta_E(q_0, +) \cup \delta_E(q_1, +)) = \text{ECLOSE}(\{q_1\}) = \{q_1\}$

- $\delta_D(\{q_1\}, 0) = \text{ECLOSE}(\delta_E(q_1, 0)) = \text{ECLOSE}(\{q_1, q_4\}) = \{q_1, q_4\}$

- $\delta_D(\{q_1, q_4\}, .) = \text{ECLOSE}(\delta_E(q_1, .) \cup \delta_E(q_4, .)) = \text{ECLOSE}(\{q_2, q_3\}) = \{q_2, q_3, q_5\}$

- $\delta_D(\{q_2, q_3, q_5\}, 0) = \text{ECLOSE}(\delta_E(q_2, 0) \cup \delta_E(q_3, 0) \cup \delta_E(q_5, 0)) = \text{ECLOSE}(\{q_3\} \cup \{q_3\} \cup \varnothing) = \text{ECLOSE}(\{q_3\}) = \{q_3, q_5\}$

- $\cdots$

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Example

DFA D constructed from E; the DFA has been further simplified, omitting the trap state and all transitions leading to that state

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Equivalence between ϵ-NFA and DFA

Theorem A language L is recognized by ϵ-NFA E if and only if L is recognized by DFA D

Proof   NO DIMOSTRAZIONE, solo enunciato

(If)   Convert $\delta_D(q, a) = p$ into $\delta_E(q, a) = \{p\}$. Then add $\delta_E(q, \epsilon) = \varnothing$ for each state q of D

(Only if)   Using our construction for D, we prove $\hat{\delta}_E(q_0, w) = \hat{\delta}_D(q_D, w)$ by induction on $|w|$

Base $\hat{\delta}_E(q_0, \epsilon) = \mathrm{ECLOSE}(q_0) = q_D = \hat{\delta}(q_D, \epsilon)$

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Equivalence between ϵ-NFA and DFA

Induction Let $w = xa$. We show $\hat{\delta}_E(q_0, xa) = \hat{\delta}_D(q_D, xa)$ using the inductiove hypothesis $\hat{\delta}_E(q_0, x) = \hat{\delta}_D(q_D, x)$

Let $\hat{\delta}_E(q_0, x) = \{p_1, \ldots, p_k\}$

From the definition of $\hat{\delta}_E$

- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\hat{\delta}_E(q_0, xa) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

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Equivalence between ϵ-NFA and DFA

From the inductive hypothesis $\hat{\delta}_D(q_D, x) = \{p_1, \ldots, p_k\}$

Using the definition of D we compute $\delta_D(\{p_1, \ldots, p_k\}, a)$

- $\{r_1, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta_E(p_i, a)$
- $\delta_D(\{p_1, \ldots, p_k\}, a) = \text{ECLOSE}(\{r_1, \ldots, r_m\})$

We can now write

$$
\begin{aligned}
\hat{\delta}_D(q_D, xa) &= \delta_D(\hat{\delta}_D(q_D, x), a) \\
&= \delta_D(\{p_1, \ldots, p_k\}, a) \\
&= \text{ECLOSE}(\{r_1, \ldots, r_m\}) \\
&= \hat{\delta}_E(q_0, xa)
\end{aligned}
$$

$\square$