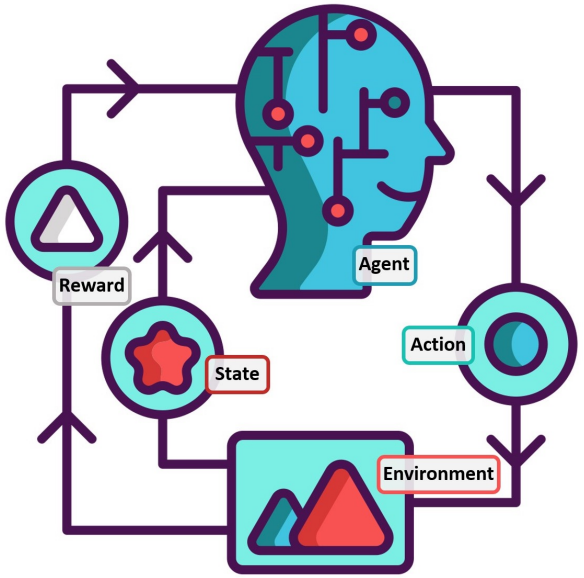
AMCO
ARTIFICIAL INTELLIGENCE, MACHINE LEARNING AND CONTROL RESEARCH GROUP

Reinforcement Learning
2025/2026

RL2
Reinforcement Learning Research Lab @ UniPD

Lecture #08
Off-Policy Monte Carlo & Temporal Difference Learning



Gian Antonio Susto

Announcement before starting: this week's lab

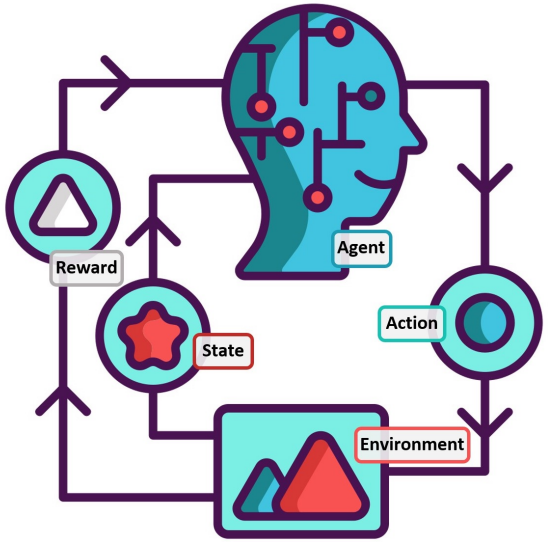
- Friday 14:30 -> Room Me
- Lab on Monte Carlo (MC) methods: windy grid-world environment

Announcement before starting: this week's lab

- Friday 14:30 -> Room Me
- Lab on Monte Carlo (MC) methods: windy grid-world environment
- Blackjack lab from previous year available on-line
- Send 'kudos' to Niccolò Turcato

niccolo.turcato@phd.unipd.it
https://www.linkedin.com/in/niccol%C3%B2-t-3411b1118/

Recap: Monte Carlo Methods



- Model-free approaches for the ‘full’ RL problem: MC methods exploits the fact that expectations can be approximated with averages
- On policy approaches

Prediction Lecture #05

Control Lecture #06 (GPI on state-action functions, Exploring Starts, ε-soft policies)

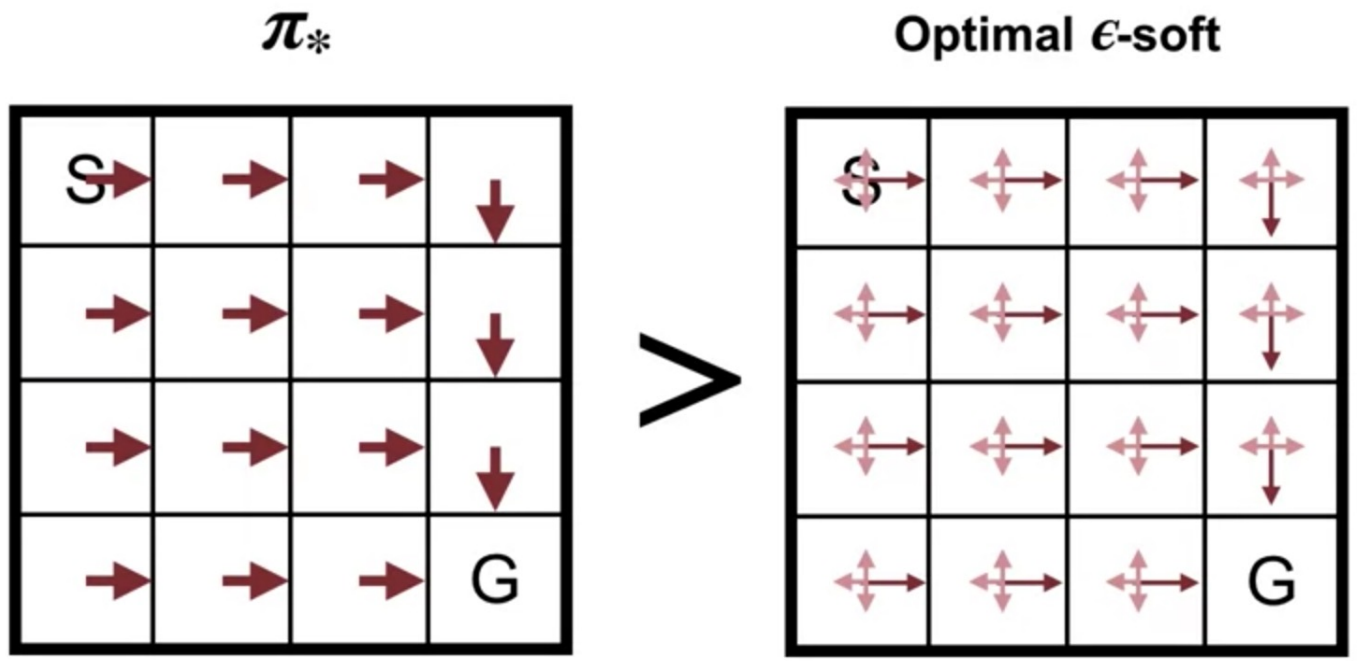
- Off-policy approaches

Prediction Lecture #07-08

Control Optional material on the book (Section 5.7)

Recap: in Control ε-soft policies are not optimal!

- It can be shown (policy improvement theorem) that the previous algorithm allow improvement (optional, see the book)
- ε-soft policies are not optimal!
- However, they may work quite well and may represent a feasible option when ES cannot be applied (which is true in many cases!)



Can we do better?

A. White, M. White 'Sample-based Learning Methods'

Recap: on-policy vs. off-policy

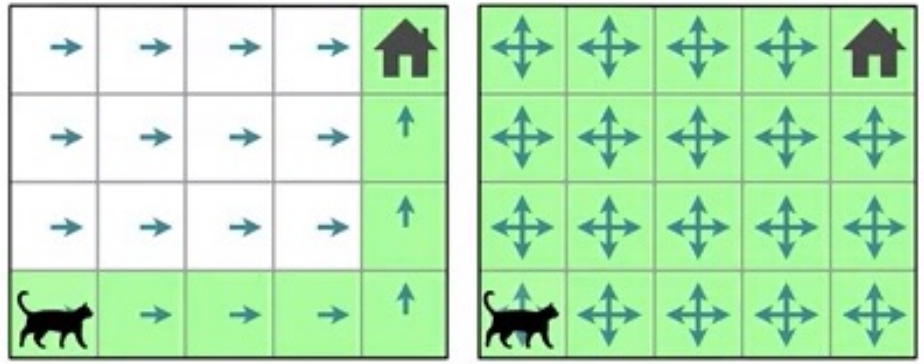
- What we have seen so far are on-policy learning
- On-policy ('learning on the job'): learn about policy π from experience sampled from policy π
- Off-policy ('look over someone's shoulder'): learn about policy π (target policy) from experience sampled from policy b (behaviour policy)
- In off-policy we use the behaviour policy to collect data from the environment in order to evalute an optimal policy

Why using a behaviour policy?

- We can leave the exploration to the behaviour policy and reach optimal policies!
- We may want to gain experience from observing other agents (or humans)
- Re-use experienc generated from old policies $\pi_1, \pi_2, \pi_3, \ldots, \pi_{t-1}$
- We can learn multiple policies while following one policy

Target policy

Behaviour policy

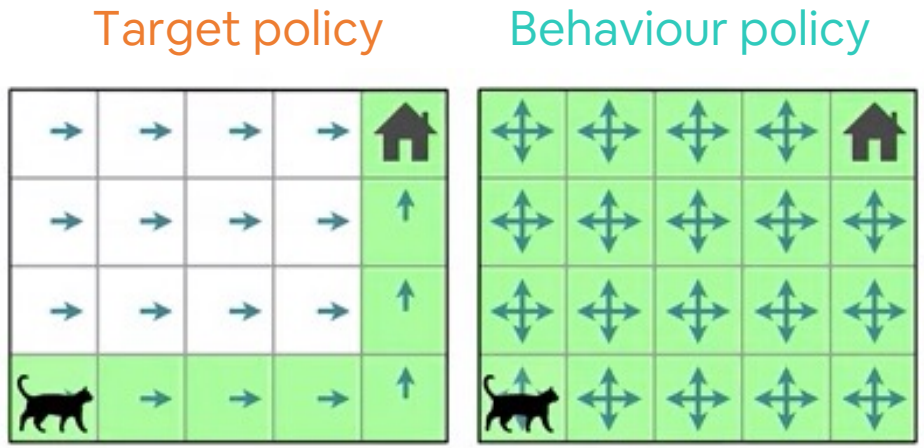


A. White, M. White 'Sample-based Learning Methods'

Recap: on-policy vs. off-policy

Things I have to consider to make off-policy work

- [Hypothesis #01] Knowledge of behaviour policy (and of target policy, obvious)
- [Hypothesis #02] Coverage
- Importance Sampling



Recap: Off-policy learning - Coverage

- If we resort to off-policy learning, we must ensure coverage:

$$\text{if } \pi(a|s) > 0 \Longrightarrow b(a|s) > 0$$

- Which means that we must ensure that the behaviour policy take explore the (state, action) pair that have non-zero probabilty in the target policy
- If this do not happen, the agent cannot learn the correct action value for that state because it never observed samples of what would happen.

Recap: Off-policy learning – Importance Sampling

DEFINITION. The *expectation* of a discrete random variable X taking the values $a_1, a_2, \ldots$ and with probability mass function p is the number

$$\mathrm{E}[X] = \sum_i a_i \mathrm{P}(X = a_i) = \sum_i a_i p(a_i).$$

We can consider this as a new random variable

$$\mathbb{E}_\pi[X] = \sum_{x \in X} x\pi(x) = \sum_{x \in X} x\rho(x)b(x) = \mathbb{E}_b[X\rho(x)]$$

- We have a way to 'correct' the expectation if we draw samples from a different policy
- To actually use this with data, we exploit MC (a weighted sampled average)

$$\mathbb{E}_\pi[X] \sim \frac{1}{n} \sum_{i=1,\ldots,n} x_i \rho(x_i)$$

Importance Sampling Ratio:

$$\rho(x) = \frac{\pi(x)}{b(x)}$$

Prediction: The Bandits Example

- 2-arm bandits: I already have a policy π that chooses a_1 with 75% probability and a_2 with 25%
- I'm looking over the shoulder of another agent that has 50% probabilty for each action (policy b)
- Initial estimations $q_\pi(a_i) = q_b(a_i) = 0$

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- Episode #01: a_2 -> +1

$q_b(a_2) = 1$, $q_\pi(a_2) = 1 * 25/50 = 0.5$

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- Episode #02: a_1 -> -1

$q_b(a_1) = -1$, $q_\pi(a_1) = -1 * 75/50 = -1.5$

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- Episode #02: a_1 -> -1

$q_b(a_1) = -1$, $q_\pi(a_1) = -1 * 75/50 = -1.5$

- Episode #03: a_1 -> +2

$q_b(a_1) = 0.5(-1+2) = 0.5$, $q_\pi(a_1) = 0.5*(-1+2) * 75/50 = 0.75$

Importance Sampling Ratio:

$$\rho(x) = \frac{\pi(x)}{b(x)}$$

Prediction: Off-policy learning

- In RL we have to deal with sequences!
- We use returns generated from b to evaluate π, we cannot directly compute v_π as the average of the returns seen!
- We have to correct each return thanks to importance sampling
- Given a starting state S_t the probability of the subsequent state-action trajectory under policy π is

$$\begin{aligned}
&\Pr\{A_t, S_{t+1}, A_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim \pi\} \\
&\quad = \pi(A_t|S_t)p(S_{t+1}|S_t, A_t)\pi(A_{t+1}|S_{t+1}) \cdots p(S_T|S_{T-1}, A_{T-1}) \\
&\quad = \prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k),
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Probability of a trajectory in an environment under policy π

Prediction: Off-policy learning

- The relative probability of the trajectory under the target and behaviour policy (the importance-sampling ratio) is

$$\rho_{t:T-1} \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k,A_k)}{\prod_{k=t}^{T-1} b(A_k|S_k)p(S_{k+1}|S_k,A_k)} =$$

Probability of a trajectory in an environment under policy π

Probability of a trajectory in an environment under policy b

- We can now exploit returns obtain with b to estimate v_π with MC

$$\mathbb{E}[\rho_{t:T-1} G_t \mid S_t = s] = v_\pi(s)$$

- We can consider importance-sampling ratios as ‘weights’

Prediction: Off-policy learning

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Probability of a trajectory in an environment under policy π

Probability of a trajectory in an environment under policy b

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Monte Carlo off-policy Prediction

Input: a policy π to be evaluated

Initialize

$V(s) \in$

Retur

Loop fo

Gene

$G \leftarrow$

Loop

G

A

V

$W \leftarrow W \dfrac{}{b(A_t \mid S_t)}$

$T-1, A_{T-1}, R_T$

0

Algorithm 1 Off-policy Monte Carlo prediction algorithm for estimating $V \approx v_\pi$

Initialize:
  $V(s) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$
  $C(s) = 0$ (accumulating weights), for all $s \in \mathcal{S}$
  π as the target policy and b as the behavior policy (where $b(a|s) > 0$ if $\pi(a|s) > 0$)
Loop forever (for each episode):
  Generate an episode using b: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$
  $G \leftarrow 0$
  $W \leftarrow 1$
  For $t = T-1$ down to 0:
    $W \leftarrow W \cdot \frac{\pi(A_t|S_t)}{b(A_t|S_t)}$
    If $W = 0$ then:
      Break
    $G \leftarrow \gamma G + R_{t+1}$
    $C(S_t) \leftarrow C(S_t) + W$
    $V(S_t) \leftarrow V(S_t) + \frac{W}{C(S_t)}(G - V(S_t))$

Monte Carlo off-policy Prediction

Differences w.r.t. on-policy

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, **arbitrarily, for all** $s \in S$

$Returns(s) \leftarrow$ **an empty list, for all** $s \in S$

Loop forever (for each episode):

Generate an episode following b: $S_0, A_0, R_1, S_1 \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$ $W \leftarrow 1$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$

$G \leftarrow \gamma W G + R_{t+1}$

Append G **to** $Returns(S_t)$

$V(S_t) \leftarrow$ **average**$(Returns(S_t))$

$$W \leftarrow W \frac{\pi(A_t \mid S_t)}{b(A_t \mid S_t)}$$

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We are using a different policy for data generation

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We are correcting for the weights given by the Importance Sampling

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We are correcting for the weights given by the Importance Sampling

We are computing the 'weights' incrementally, backwards for efficiency!

$$\rho_{t:T-1} \doteq \prod_{k=t}^{T-1} \frac{\pi(A_k \mid S_k)}{b(A_k \mid S_k)}$$

$$= \rho_t \rho_{t+1} \rho_{t+2} \cdots \rho_{T-2} \rho_{T-1}$$

$\longleftarrow$

$W_1 \leftarrow \rho_{T-1}$

$W_2 \leftarrow \rho_{T-1}\rho_{T-2}$

$W_3 \leftarrow \rho_{T-1}\rho_{T-2}\rho_{T-3}$

Monte Carlo off-policy Prediction

Differences w.r.t. on-policy

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Initialize:

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$Returns(s) \leftarrow$ **an empty list, for all** $s \in S$

Loop forever (for each episode):

Generate an episode following b: $S_0, A_0, R_1, S_1 \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$ $W \leftarrow 1$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$; **while** $W \neq 0$

$G \leftarrow \gamma W G + R_{t+1}$

Append G **to** $Returns(S_t)$

$V(S_t) \leftarrow$ **average**$(Returns(S_t))$

$$W \leftarrow W \frac{\pi(A_t \mid S_t)}{b(A_t \mid S_t)}$$

Additional check to improve efficiency

Off-policy Control Monte Carlo

- To apply off-policy MC control, keep in mind that coverage may change over time:

$$\pi(a|s) > 0 \Longrightarrow b(a|s) > 0$$

$$\pi'(a|s) > 0 \Longrightarrow b(a|s) > 0 \text{ (?)}$$

Off-policy Control Monte Carlo

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- In practice off-policy MC control is not effective! Over many steps, off-policy estimations are really high-variance and slow to converge!

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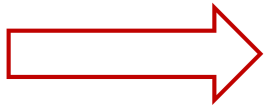
- In practice off-policy MC control is not effective! Over many steps, off-policy estimations are really high-variance and slow to converge!
- However, with Temporal Difference learning, we'll make off-policy work also on practice!

Monte Carlo methods: Exam

$\mathcal{P}, \mathcal{R}$
known



Data,
experience

True Expectations

Sample Averages

- All the content of Chapter 5 are Exam material, beside:

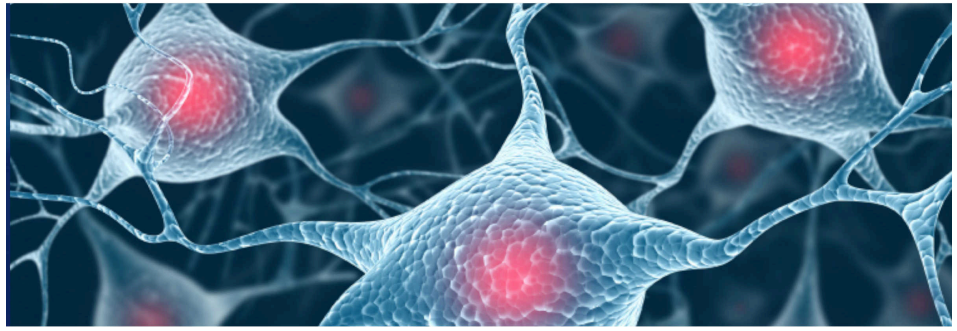
i. it can be considered optional the policy improvement theorem for ε-soft policies (in section 5.4)

ii. Sections 5.5 & 5.6 are related to Importance sampling: mandatory topic, but consider the simplified version presented in (this) class

iii. Sections 5.7, 5.8 and 5.9 can be skipped/optional (Section 5.7 talks about off-policy MC control)

- Pay particular attention to the algorithms!
- (Slides are always exam material – unless differently stated)

Temporal-difference Learning (Chapter 6)



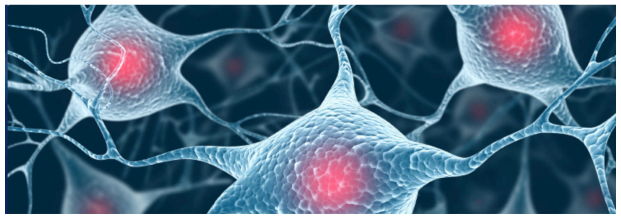
Intro to Temporal-difference (TD) Learning

From Chapter 6: ‘

If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning’

Intro to Temporal-difference (TD) Learning

- TD learning is another approach to model-free RL (for both prediction and control)
- TD learning applies bootstrapping, it works for incomplete episode (ie. It works also with continuous episodes)
- An historical result in RL is TD-gammon, a computer program developed by Gerald Tesauro (IBM) in 1992, that exploited TD-learning to play backgammon
- TD learning has still wide applications in RL and in other fields (in Neuroscience for example the firing of dopamine follow TD learning principles)
- We will see how TD learning is a general framework that also include Monte-Carlo Methods: we will start considering TD(0) and then the general framework



Limits of Monte Carlo approaches

1. MC only works on episodic tasks

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(However, with low γ values for example, approximations of the returns can be achieved)

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots$$

Limits of Monte Carlo approaches

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(However, with low γ values for example, approximations of the returns can be achieved)

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots$$

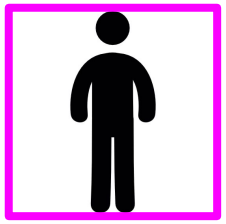
2. MC does not exploit available information as soon as they are collected: we are not using the rewards – that are informative! – as soon as they are available!

Limits of Monte Carlo approaches

‘Coming home in 20 mins’

Total Elapsed time:
0 min

Limits of Monte Carlo approaches



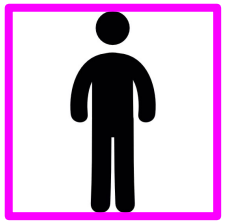
‘Coming home in 20 mins’

Sudden 20 mins call

Total Elapsed time:
20 min

Monte Carlo

Limits of Monte Carlo approaches



‘Coming home in 20 mins’

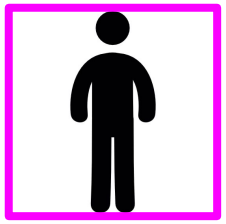
Sudden 20 mins call

Total Elapsed time:
30 min

Traffic: 10 mins more

Monte Carlo

Limits of Monte Carlo approaches



‘Coming home in 20 mins’

Sudden 20 mins call

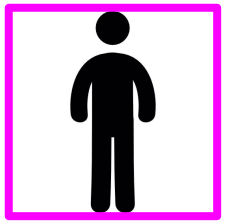
Total Elapsed time:
35 min

Traffic: 10 mins more

Monte Carlo

Gasoline: 5 mins more

Limits of Monte Carlo approaches



‘Coming home in 20 mins’

Sudden 20 mins call

Traffic: 10 mins more

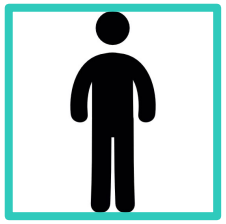
Gasoline: 5 mins more

Driving: 20 mins

Total Elapsed time:
55 min

Monte Carlo

Limits of Monte Carlo approaches



‘Coming home in 20 mins’

Sudden 20 mins call

‘Coming home in 20 mins’

Total Elapsed time:
55 min

Traffic: 10 mins more

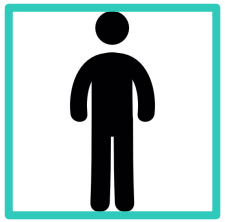
‘Coming home in 20 mins’

Gasoline: 5 mins more

‘Coming home in 20 mins’

Driving: 20 mins

Limits of Monte Carlo approaches



‘Coming home in 20 mins’

Sudden 20 mins call

‘Coming home in 20 mins’

Total Elapsed time:
55 min

Traffic: 10 mins more

‘Coming home in 30 mins’

Temporal Difference

Gasoline: 5 mins more

‘Coming home in 25 mins’

Driving: 20 mins

Recall from Lecture #02: Incremental Estimations are recurrent in RL

$$NewEstimate \leftarrow OldEstimate + StepSize \left[Target - OldEstimate \right]$$

We can apply this to MC methods:

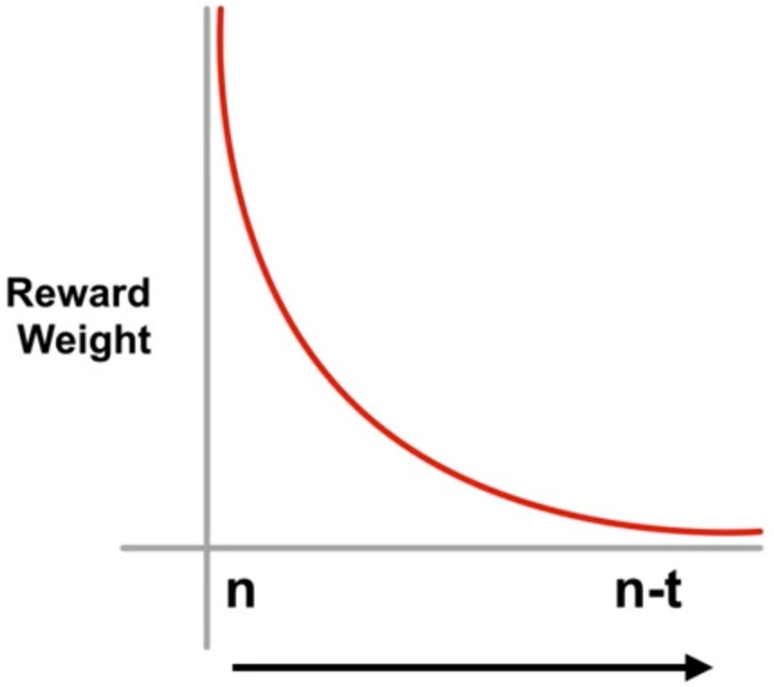
- Version ‘exact mean’

$$N(S_t) \leftarrow N(S_t) + 1$$

$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)} \left(G_t - V(S_t) \right)$$

- Version ‘step size’ (better for non-stationary problems)

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$



Prediction: Monte Carlo vs. TD-learning

The main difference between MC and TD is

- MC methods wait for the actual return G_t (that is available at the end of the episode) to update the estimation of $v_\pi(s)$

$$V(S_t) \leftarrow V(S_t) + \alpha\,(G_t - V(S_t))$$

Prediction: Monte Carlo vs. TD-learning

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$$V(S_t) \leftarrow V(S_t) + \alpha\,(G_t - V(S_t))$$

- TD(0) uses the immediate reward (that is available after taking one action) to update the estimation of the so-called TD target

$$V(S_t) \leftarrow V(S_t) + \alpha\,(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

Prediction: Monte Carlo vs. TD-learning

The main difference between MC and TD is

- MC methods wait for the actual return G_t (that is available at the end of the episode) to update the estimation of $v_\pi(s)$

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$$V(S_t) \leftarrow V(S_t) + \alpha\,(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

The TD target can be seen as the 'best' estimation of G_t at time t+1:

$$G_t = R_{t+1} + \gamma G(S_{t+1}) \sim R_{t+1} + \gamma V(S_{t+1})$$

Prediction: Monte Carlo vs. TD-learning

The main difference between MC and TD is

- MC methods wait for the actual return G_t (that is available at the end of the episode) to update the estimation of $v_\pi(s)$

$$V(S_t) \leftarrow V(S_t) + \alpha\,(G_t - V(S_t))$$

- [illegible] available after taking one action) to update the estimation of the so-called TD target

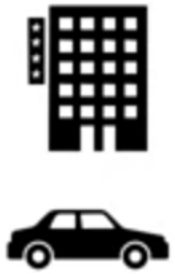
This quantity is called the *TD error*

$$V(S_t) \leftarrow V(S_t) + \alpha\,(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

The TD target can be seen as the 'best' estimation of G_t at time t+1:

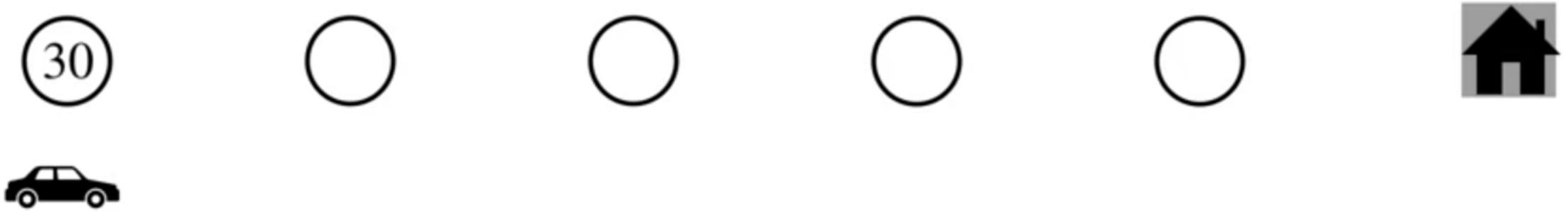
$$G_t = R_{t+1} + \gamma G(S_{t+1}) \sim R_{t+1} + \gamma V(S_{t+1})$$

Prediction: TD(0) - Driving example



A. White, M. White 'Sample-based Learning Methods'

leave



0 **mins**
elapsed

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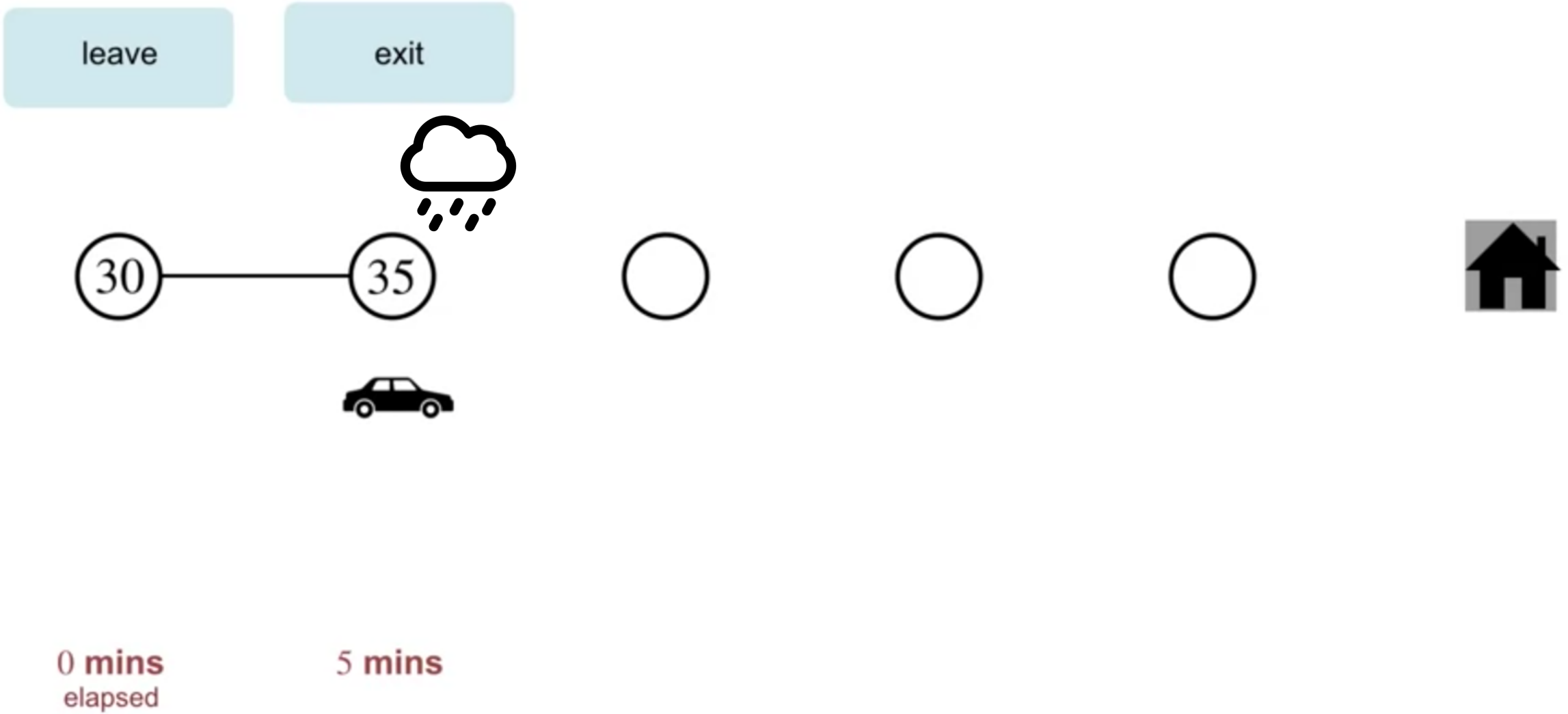
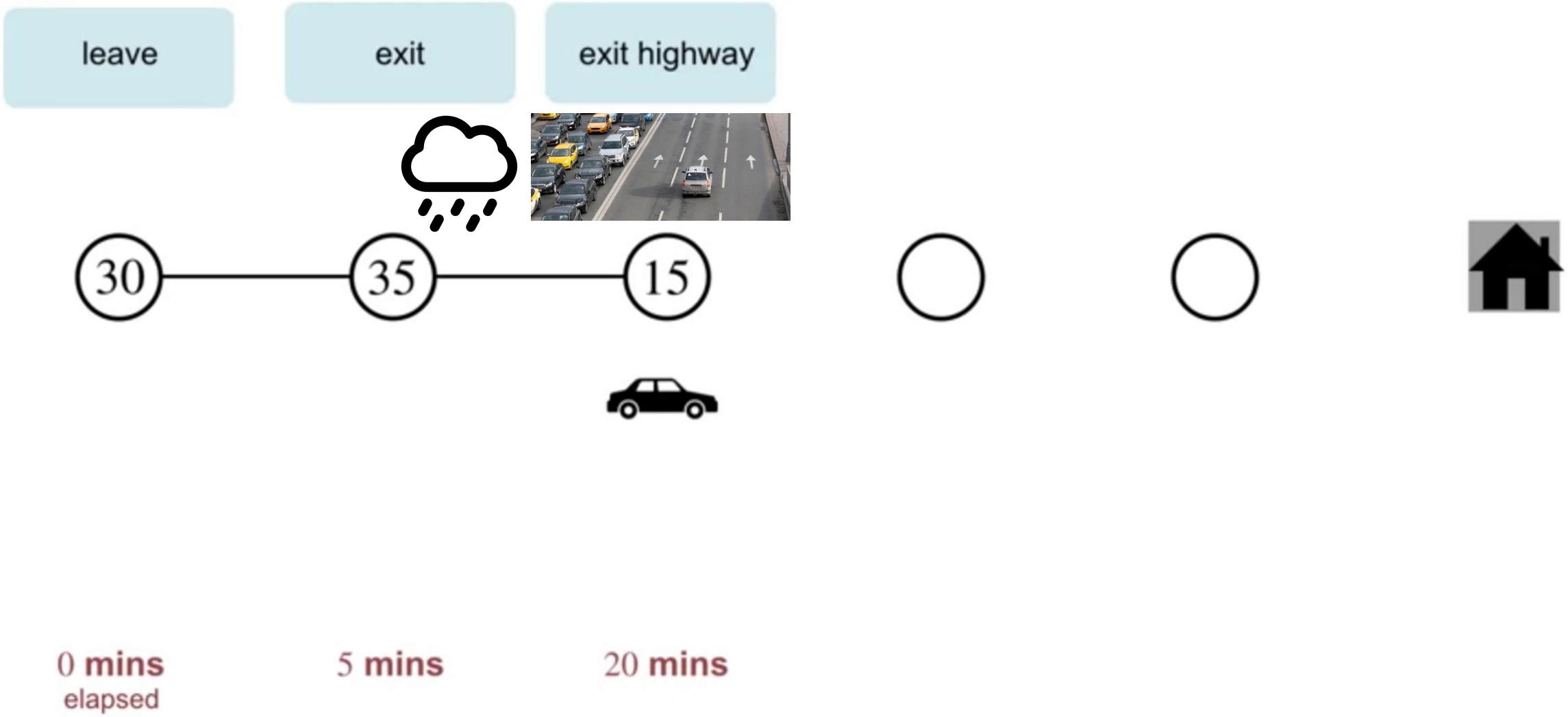
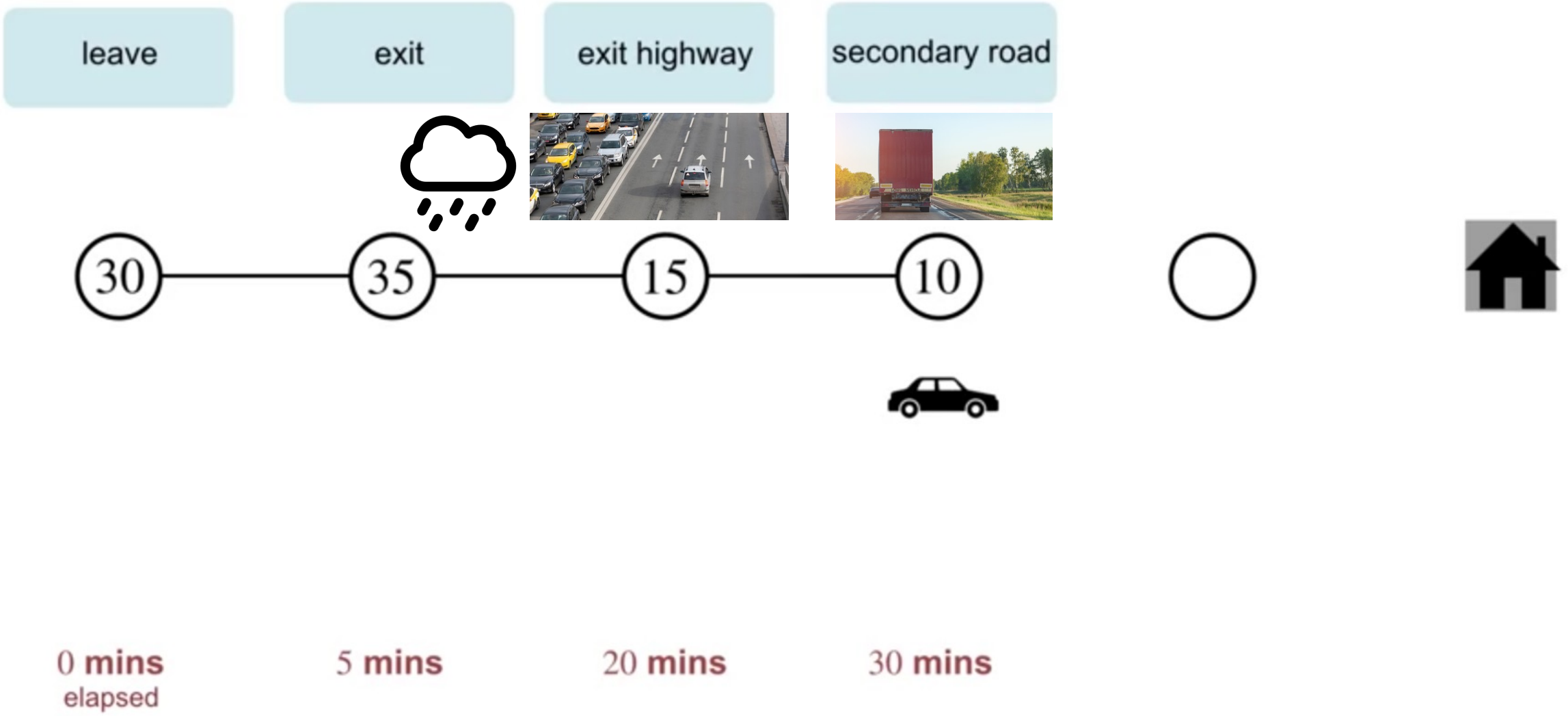


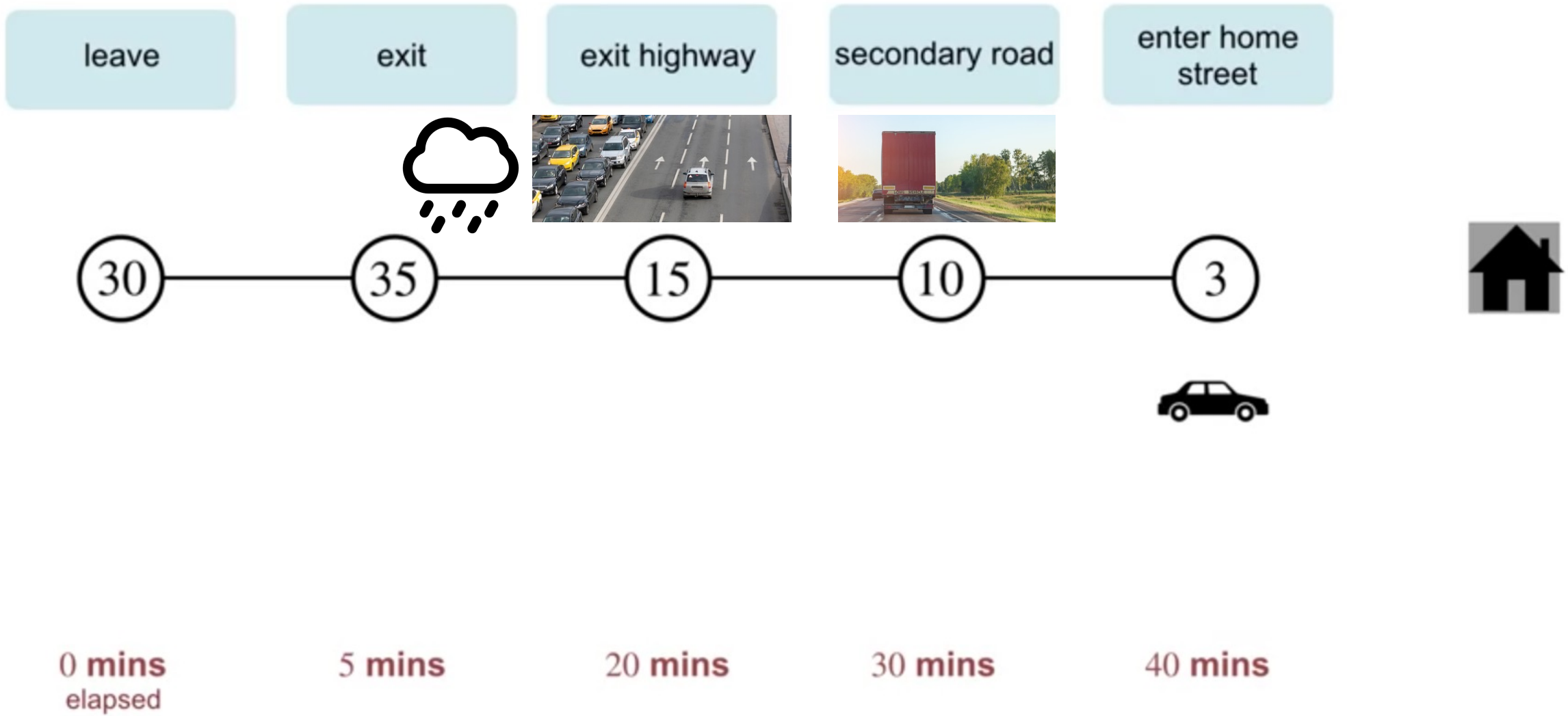
A. White, M. White 'Sample-based Learning Methods'



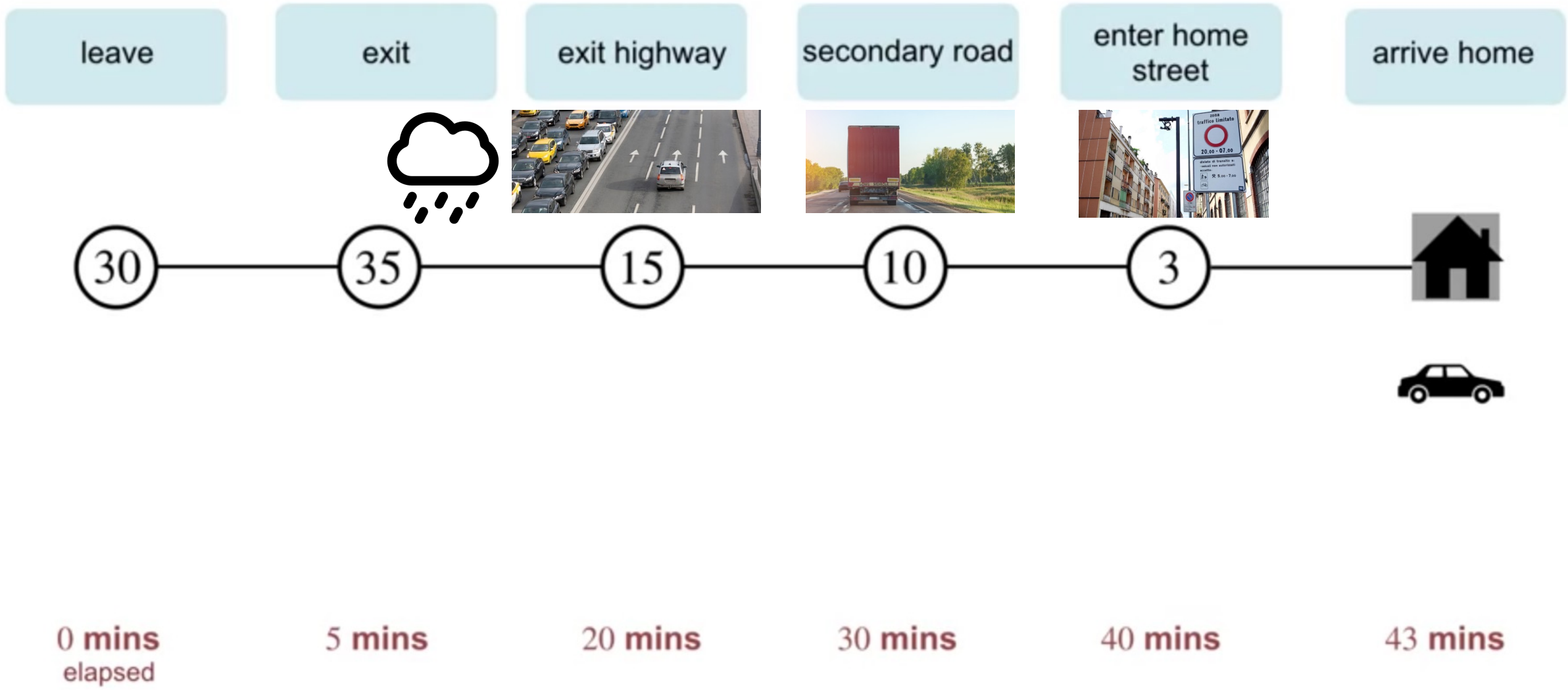
A. White, M. White 'Sample-based Learning Methods'



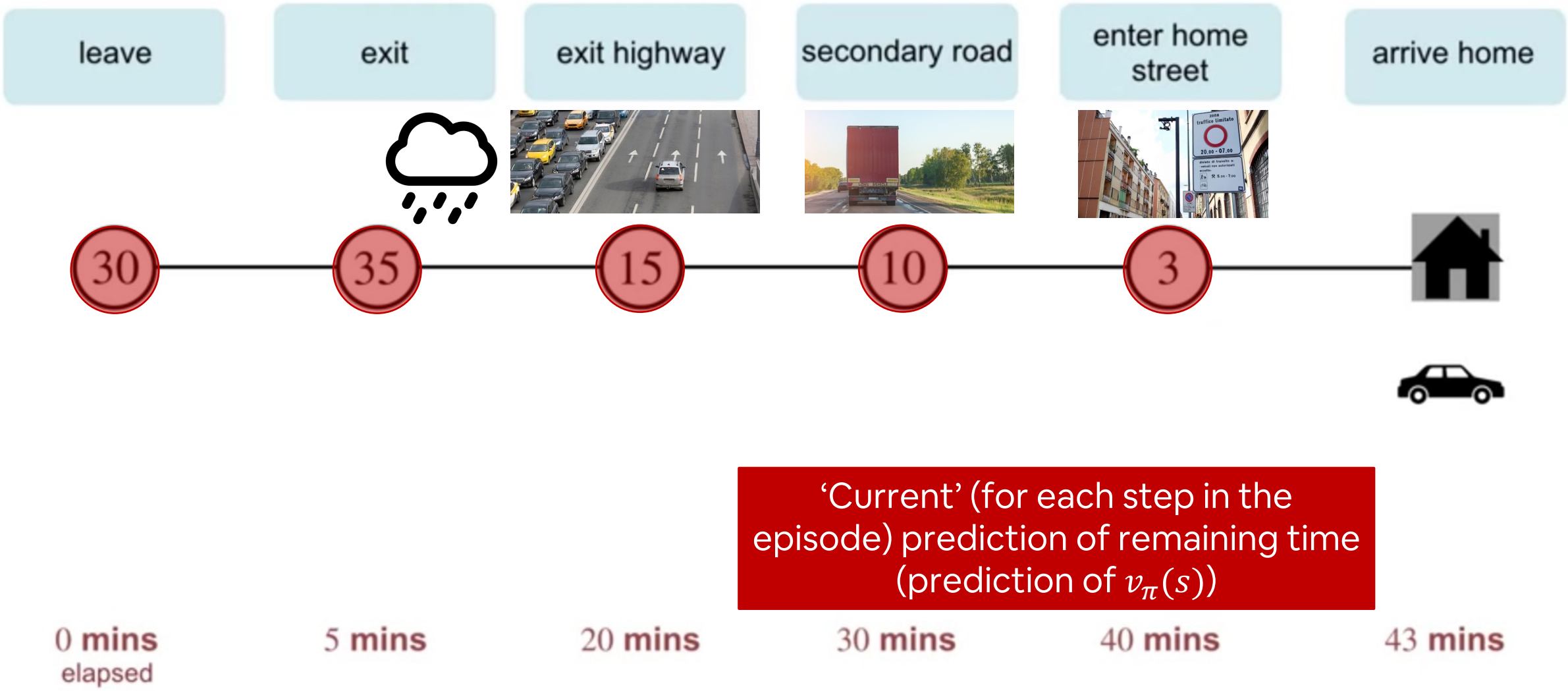
A. White, M. White 'Sample-based Learning Methods'



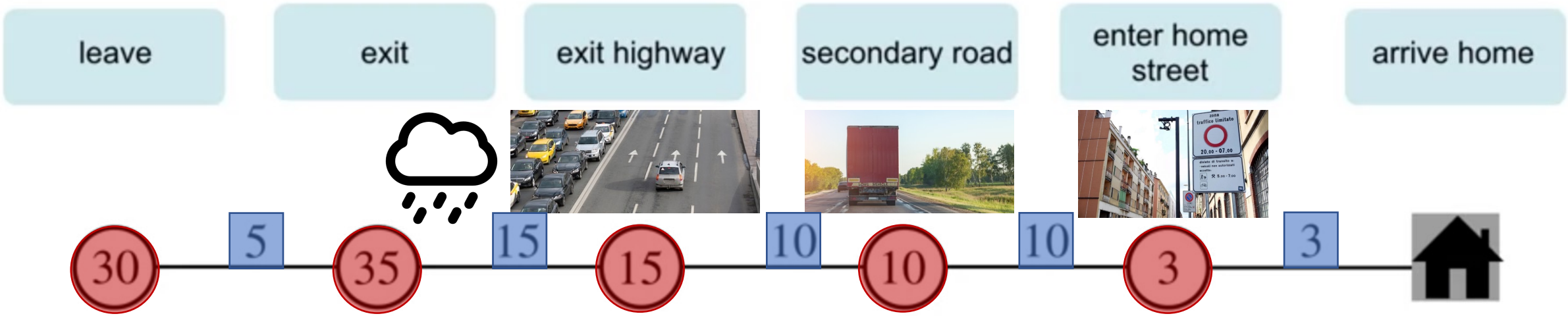
A. White, M. White 'Sample-based Learning Methods'



A. White, M. White 'Sample-based Learning Methods'



A. White, M. White 'Sample-based Learning Methods'



Rewards: the elapsed time for each steps

'Current' (for each step in the episode) prediction of remaining time (prediction of $v_\pi(s)$)

0 mins elapsed	5 mins	20 mins	30 mins	40 mins	43 mins

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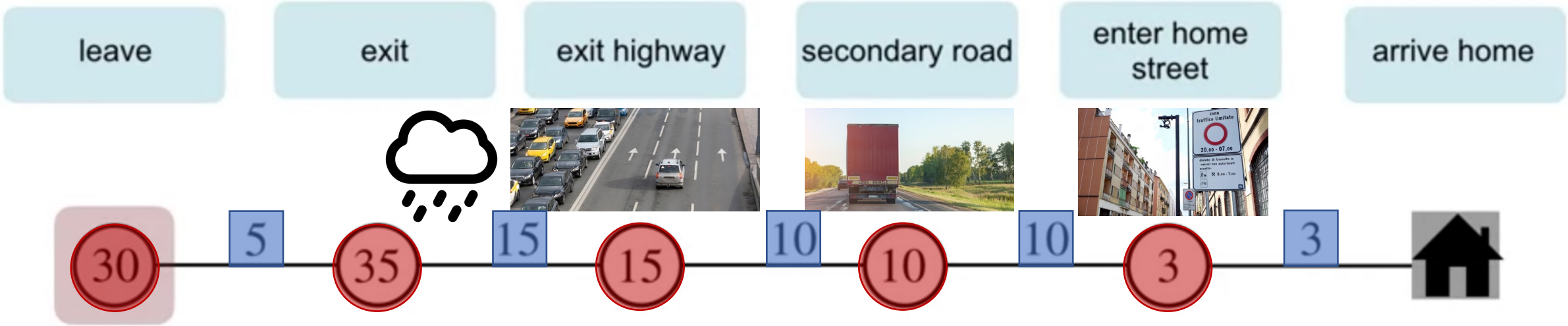

$$G_0 = 5 + 15 + 10 + 10 + 3 = 43$$

$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)] \qquad \alpha = 1 \quad \gamma = 1$$

$$V(\textbf{leave}) \leftarrow V(\textbf{leave}) + \alpha[G_0 - V(\textbf{leave})]$$
$$30 \qquad\qquad 43 \qquad\qquad 30$$

Monte Carlo

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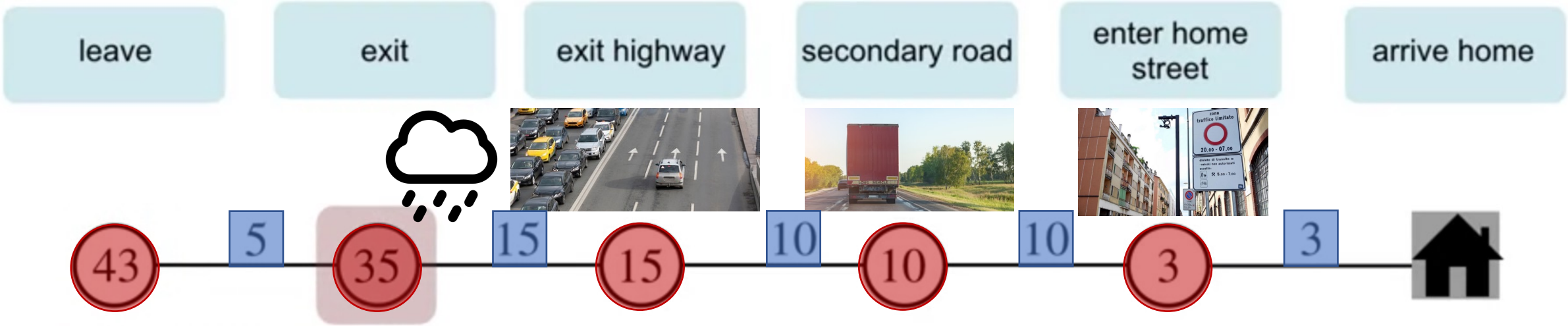

$$G_1 = 15 + 10 + 10 + 3 = 38$$

$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)]$$

$$\alpha = 1 \quad \gamma = 1$$

$$V(\textbf{exit}) \leftarrow V(\textbf{exit}) + \alpha[G_1 - V(\textbf{exit})]$$

Monte Carlo

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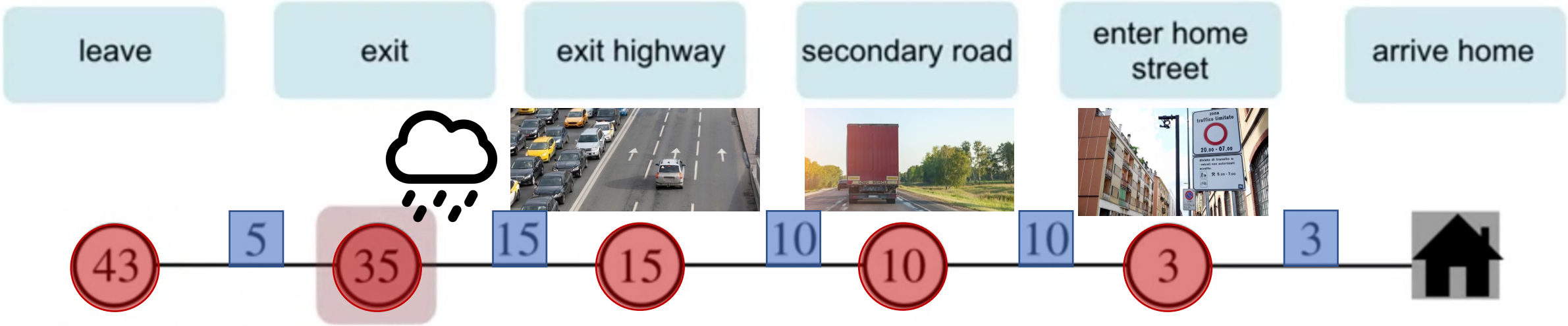


$$G_1 = 15 + 10 + 10 + 3 = 38$$

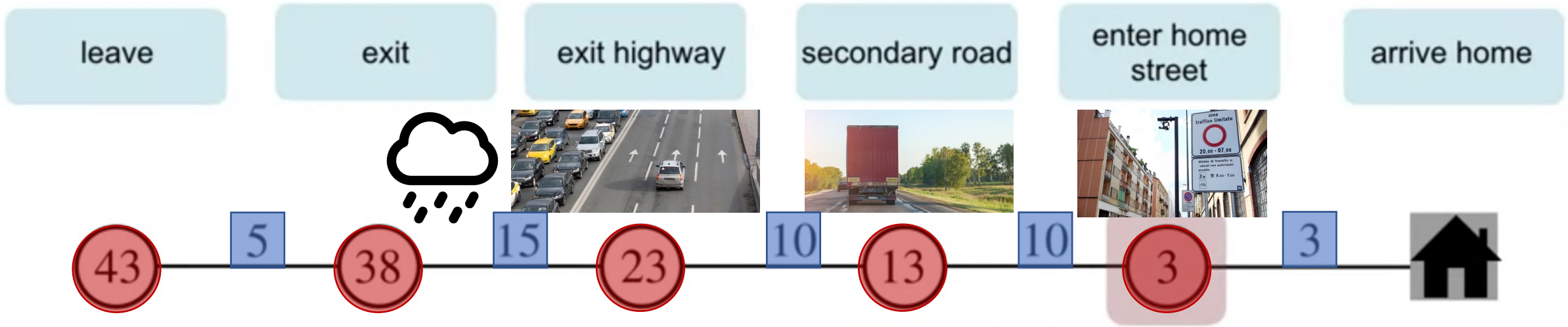
$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)]$$

$$\alpha = 1 \quad \gamma = 1$$

$$V(\textbf{exit}) \leftarrow V(\textbf{exit}) + \alpha[G_1 - V(\textbf{exit})]$$

Monte Carlo

A. White, M. White 'Sample-based Learning Methods'



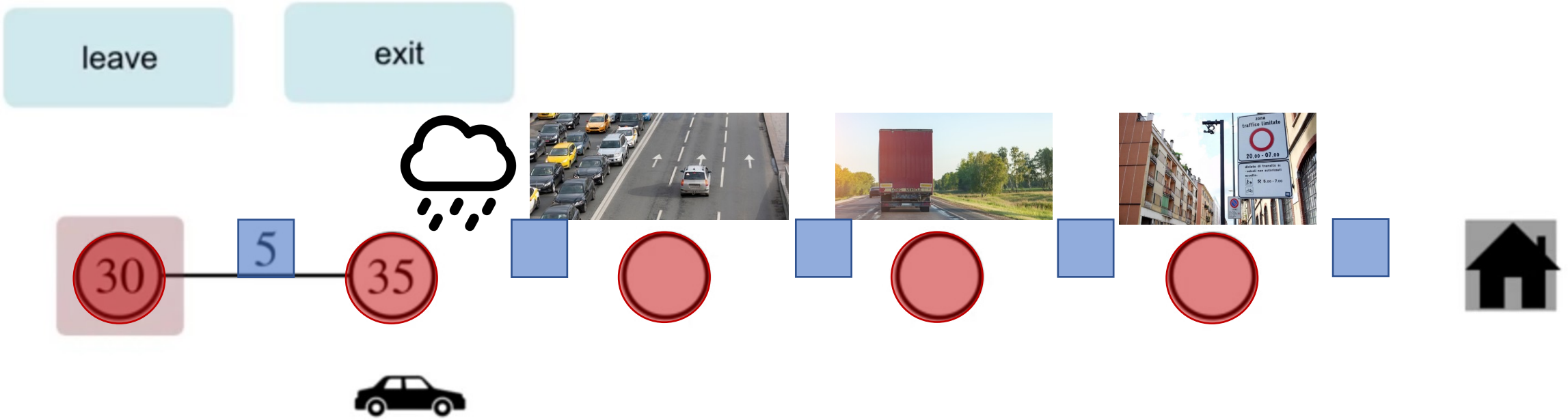
$$G_4 = 3$$

$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)] \qquad \alpha = 1 \quad \gamma = 1$$

$$V(\textbf{enter home st.}) \leftarrow V(\textbf{enter home st.}) + \alpha[G_4 - V(\textbf{enter home st.})]$$
$$3 \qquad\qquad\qquad 3 \qquad\qquad 3$$

Monte Carlo

A. White, M. White 'Sample-based Learning Methods'



$$V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) - V(S_t)]$$

$$\alpha = 1 \quad \gamma = 1$$

$$V(\textbf{leave}) \leftarrow V(\textbf{leave}) + \alpha\,[\, R_1 + \gamma\, V(\textbf{exit}) - V(\textbf{leave})\,]$$

$$30 \qquad 5 \qquad 35 \qquad 30$$

TD(0)

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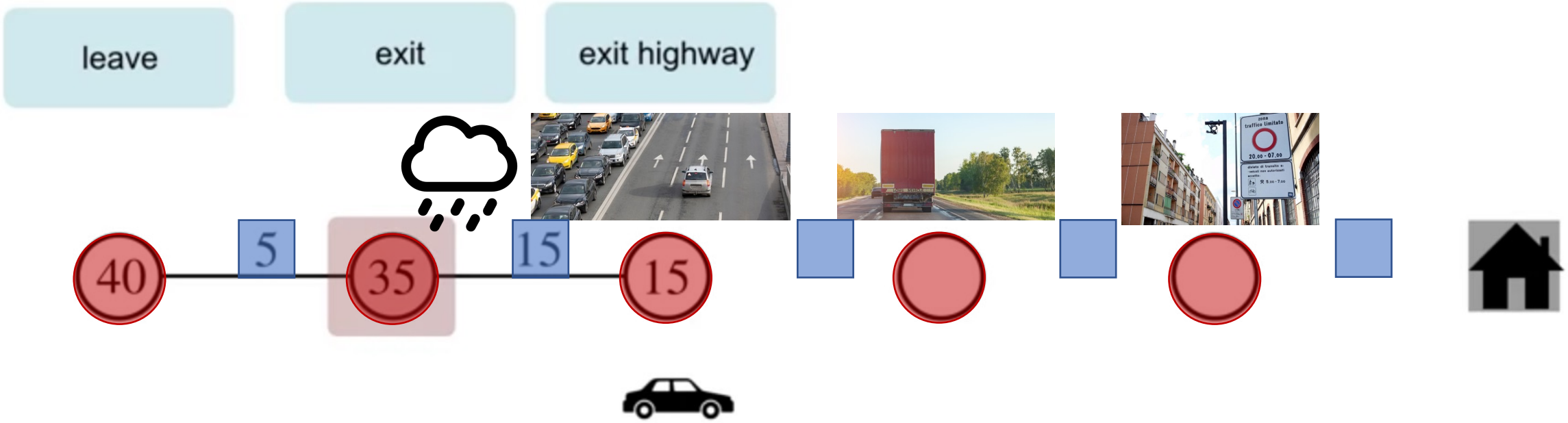


$$V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) - V(S_t)]$$

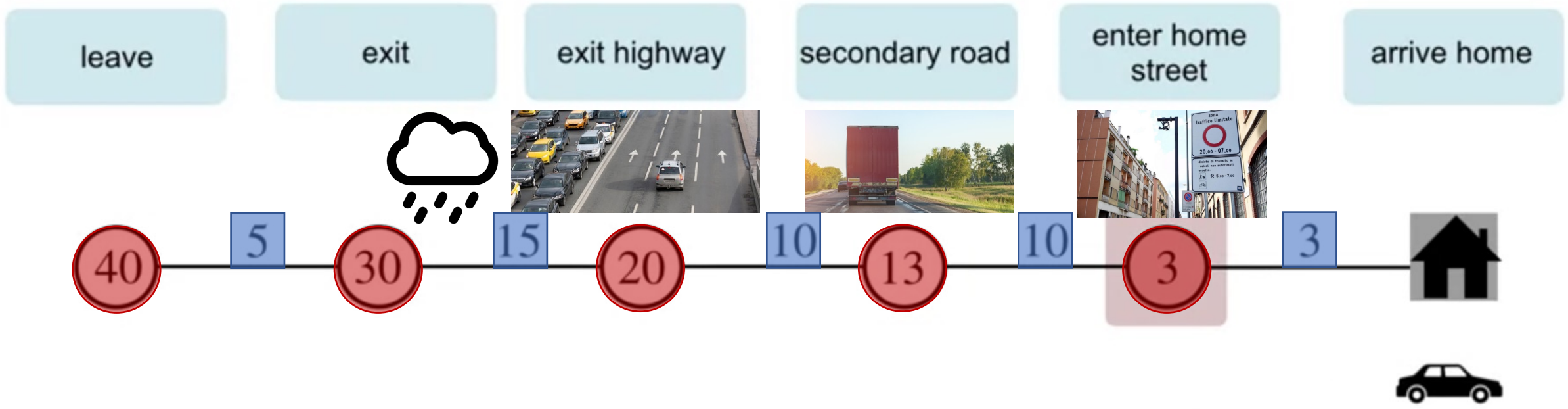
$\alpha = 1 \quad \gamma = 1$

$$V(\textbf{exit}) \leftarrow V(\textbf{exit}) + \alpha\,[\, R_2 + \gamma\, V(\textbf{exit highway}) - V(\textbf{exit})\,]$$

$$35 \qquad 15 \qquad 15 \qquad 35$$

TD(0)

A. White, M. White 'Sample-based Learning Methods'



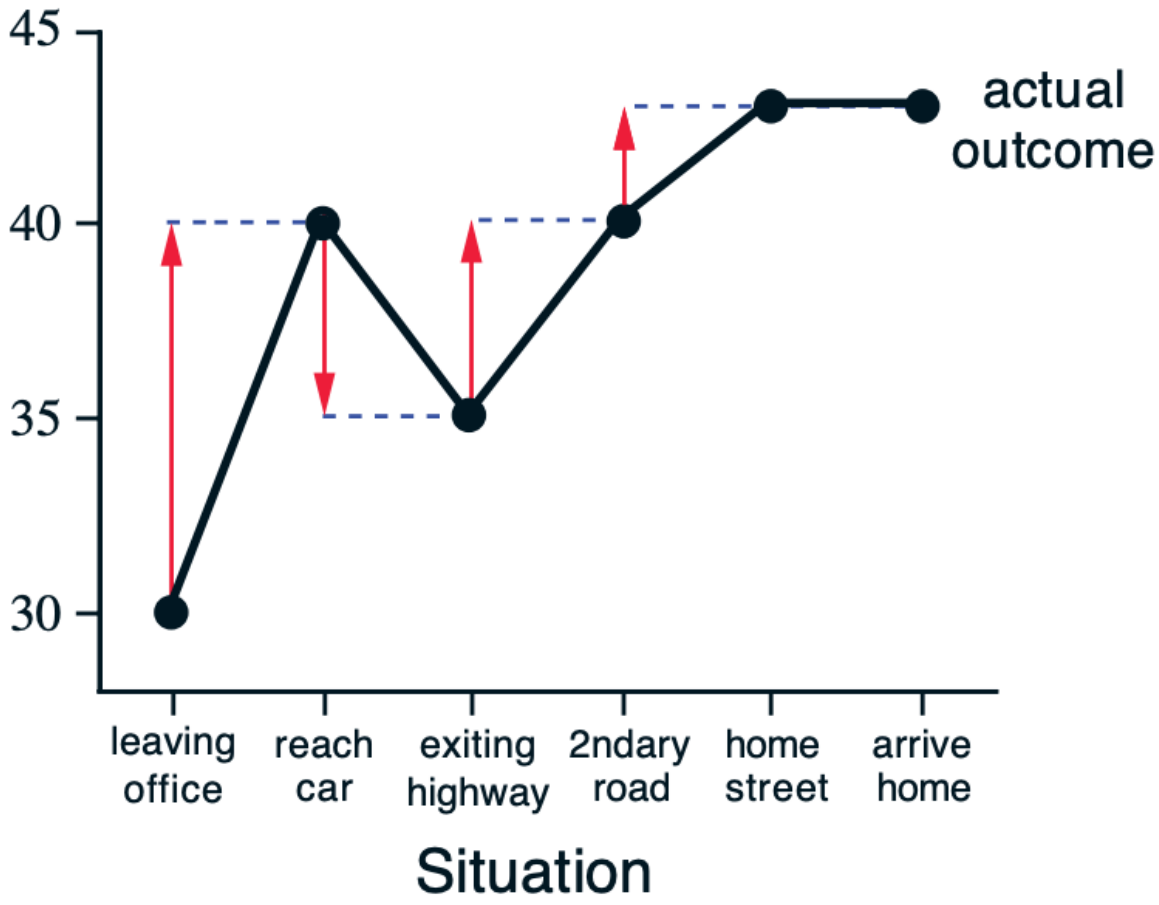
$$V(S_t) \leftarrow V(S_t) + \alpha[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)] \qquad \alpha = 1 \quad \gamma = 1$$

$$V(\textbf{enter home st.}) \leftarrow V(\textbf{enter home st.}) + \alpha[R_5 + \gamma V(\textbf{home}) - V(\textbf{arrive home st.})]$$
$$3 \qquad\qquad 3 \qquad 0 \qquad 3$$

TD(0)

A. White, M. White 'Sample-based Learning Methods'

Prediction: TD(0) - Driving example



Tabular TD(0) for estimating v_π

Input: the policy π to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\mathit{terminal}) = 0$

Loop for each episode:
    Initialize S
    Loop for each step of episode:
        $A \leftarrow$ action given by π for S
        Take action A, observe R, S'
        $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
        $S \leftarrow S'$
    until S is terminal

Tabular TD(0) for estimating v_π

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Algorithm parameter: step size $\alpha \in (0, 1]$

Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop for each episode:

  Initialize S

  Loop for each step of episode:

    $A \leftarrow$ action given by π for S

    Take action A, observe R, S'

    $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$

    $S \leftarrow S'$

  until S is terminal

We will see how the choice of α can have a major impact!

Tabular TD(0) for estimating v_π

Input: the policy π to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\mathit{terminal}) = 0$

Loop for each episode:
    Initialize S
    Loop for each step of episode:
        $A \leftarrow$ action given by π for S
        Take action A, observe R, S'
        $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
        $S \leftarrow S'$
    until S is terminal

TD-Learning

- TD methods update their estimates based on other estimates, ie. they bootstrap
- TD can learn before knowing the final outcome:

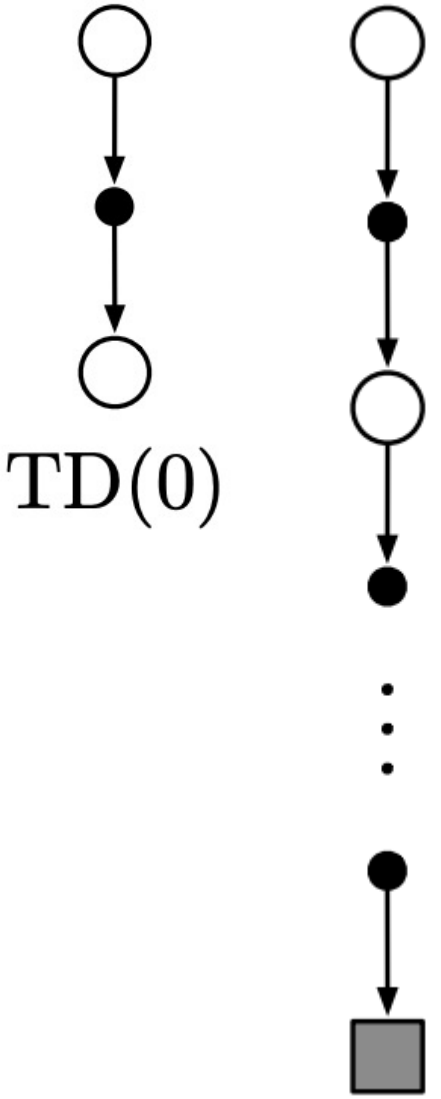
TD can learn online after every step / MC must wait until end of episode before return is known

- TD can learn without the final outcome

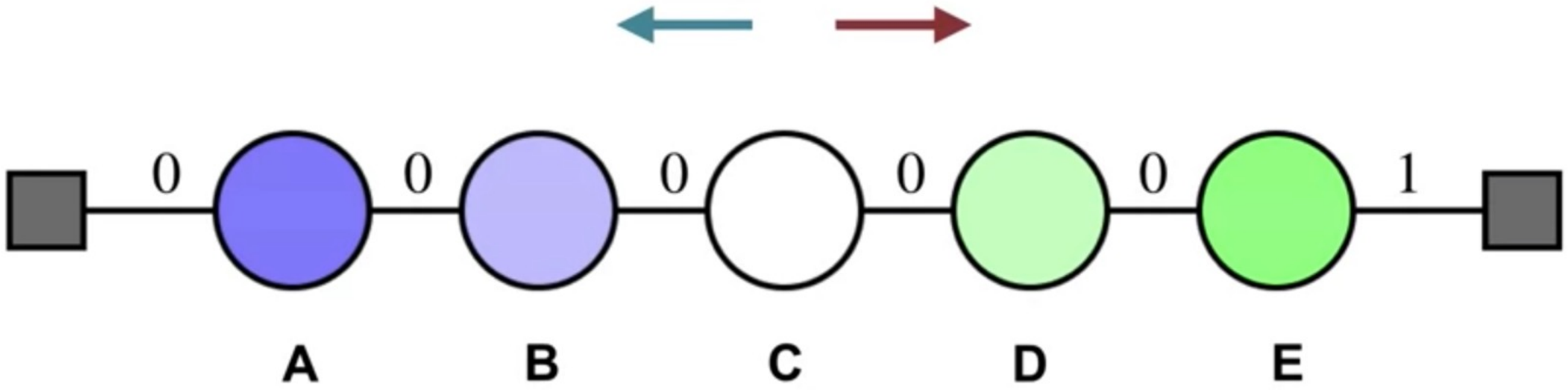
i. TD can learn from incomplete sequences / MC can only learn from complete sequences

ii. TD works in continuing (non-terminating) environments / MC only works for episodic (terminating) environments

- TD methods still converge to the right estimation of v_π
- Typically, TD approaches are faster to converge than MC!



Prediction: TD(0) – Random Walk



$$\pi(\,.\,|\,s) = 1/2 \quad \forall s \in \mathcal{S} \qquad\qquad \gamma = 1$$

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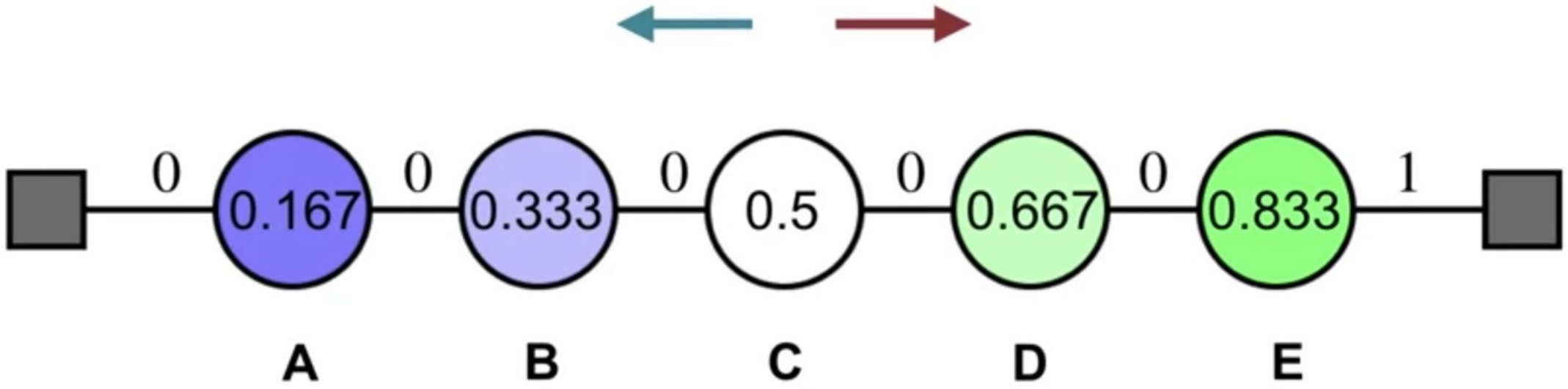
Prediction: TD(0) – Random Walk



$$\pi(\,.\,|\,s) = 1/2 \quad \forall s \in \mathcal{S} \qquad\qquad \gamma = 1$$

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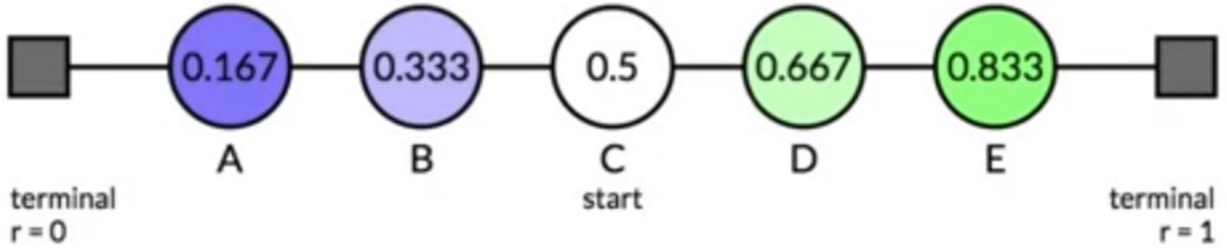
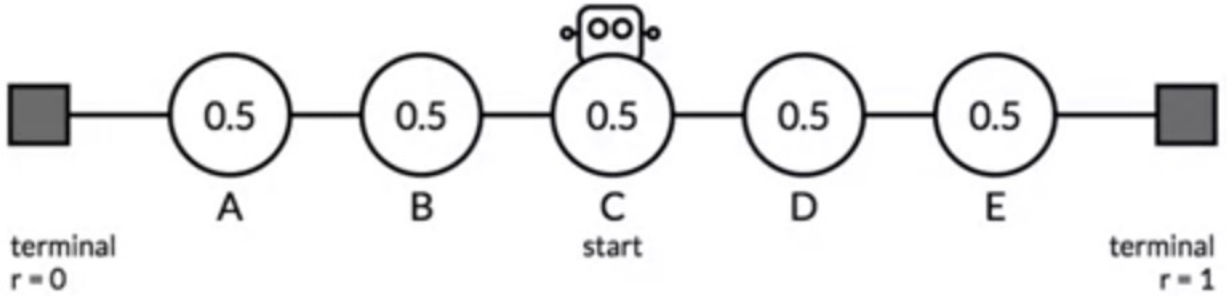
Episode #1

Target / Exact Values



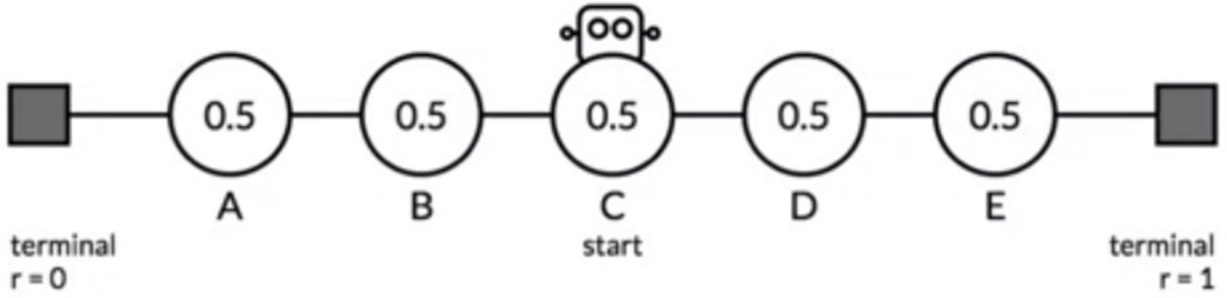
Updates using TD Learning

$$V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) \; - V(S_t)]$$



In the following:
$\alpha = 0.5$

Updates using Monte Carlo



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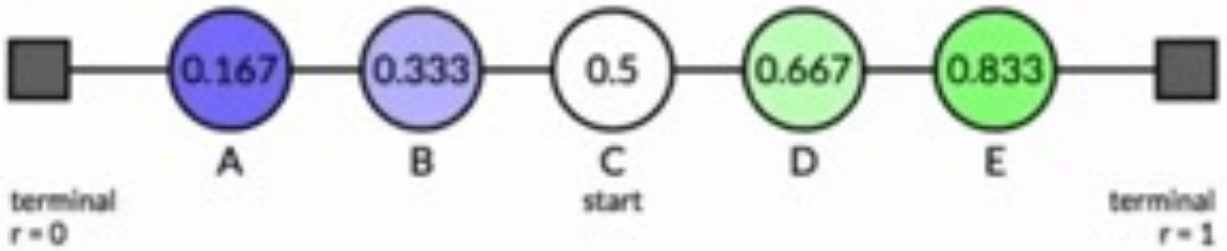
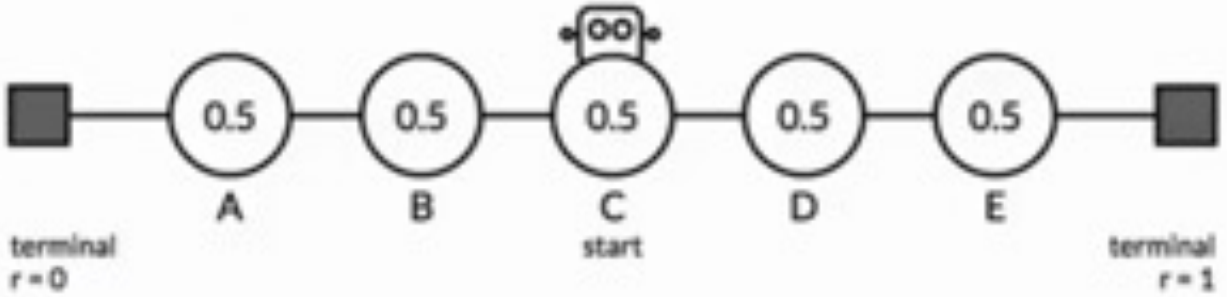
Episode #1

Target / Exact Values

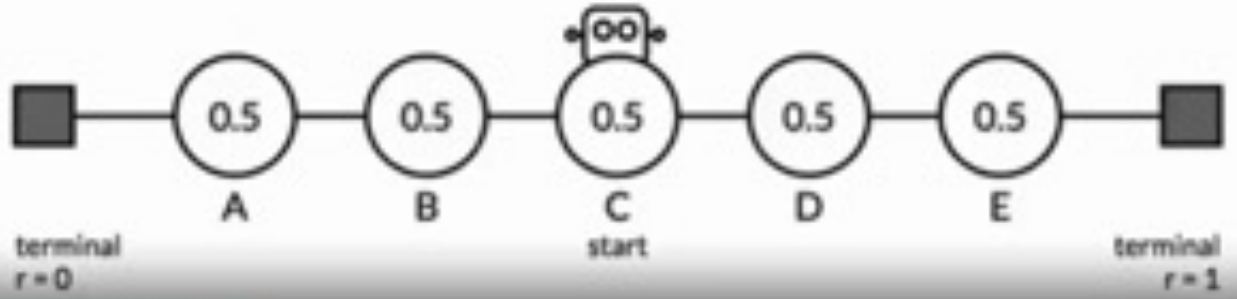


Updates using TD Learning

$$V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) \, - V(S_t)]$$



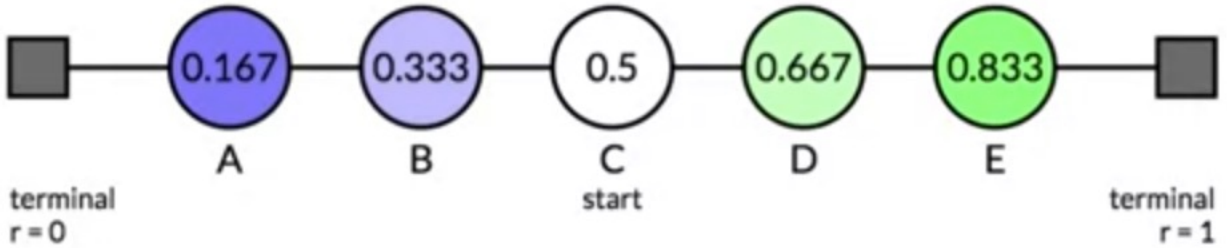
Updates using Monte Carlo



A. White, M. White 'Sample-based Learning Methods'

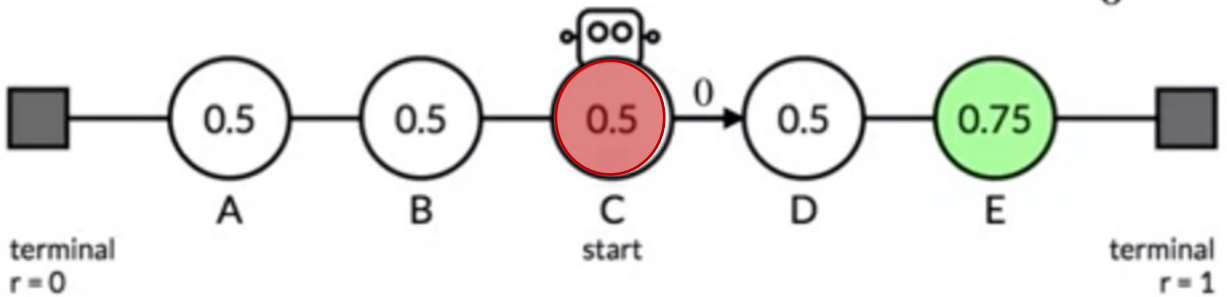
Episode #1

Target / Exact Values

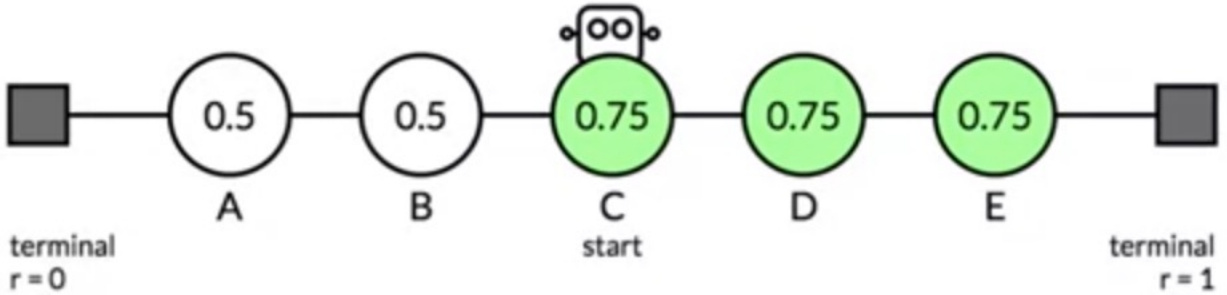


Updates using TD Learning

$$V(S_t) \leftarrow V(S_t) + \alpha[\underset{0}{R_{t+1}} + \gamma \underset{0.5}{V(S_{t+1})} - \underset{0.5}{V(S_t)}]$$



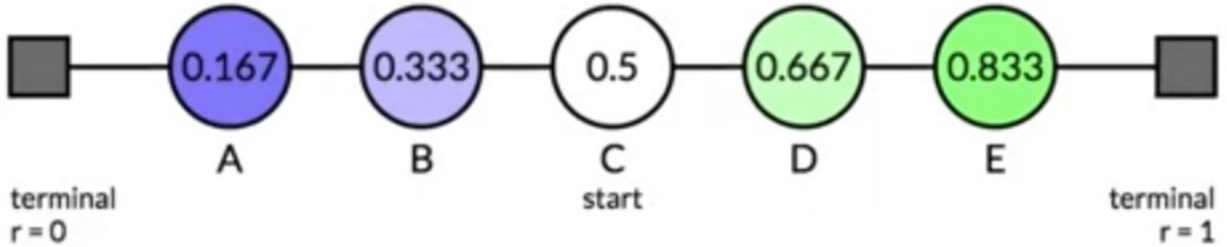
Updates using Monte Carlo



A. White, M. White 'Sample-based Learning Methods'

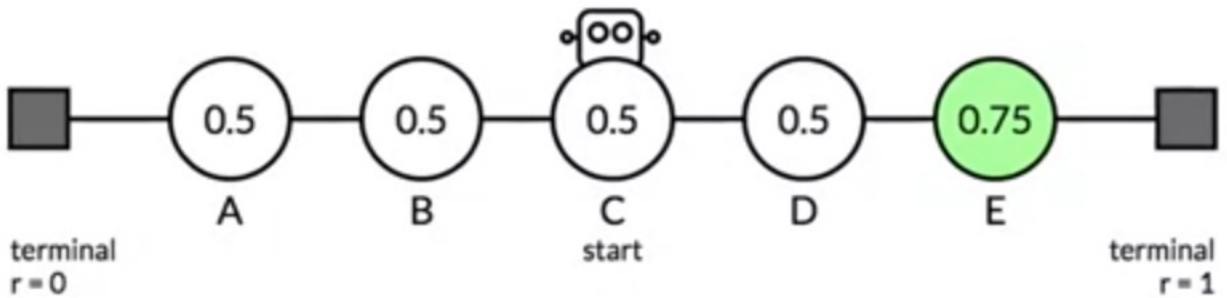
Episode #1

Target / Exact Values



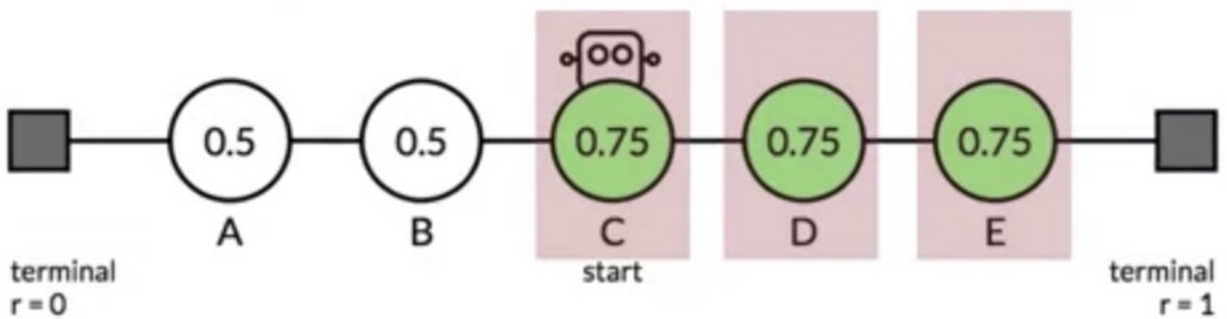
Updates using TD Learning

$$V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) \, - V(S_t)]$$



Updates using Monte Carlo

$$V(S_t) \leftarrow V(S_t) + \alpha[G_t - V(S_t)]$$



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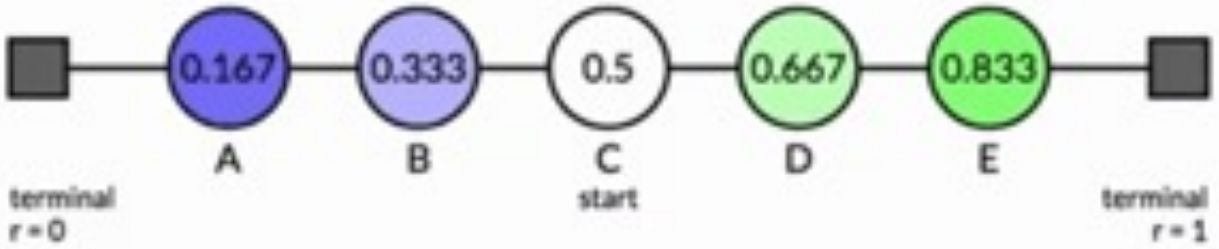
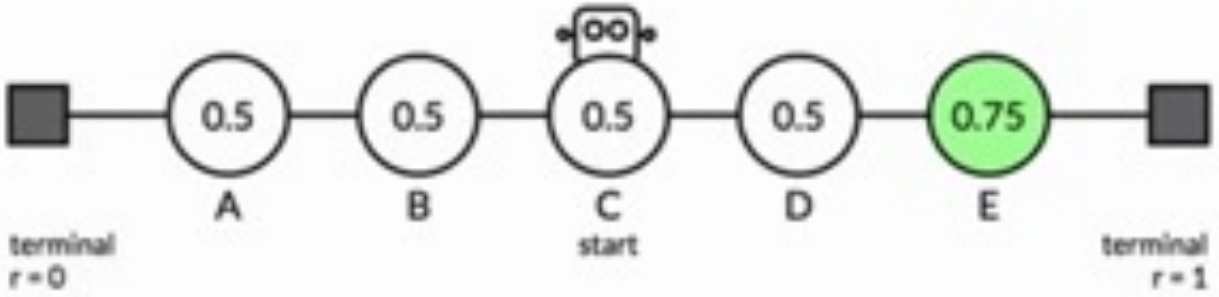
Episode #2

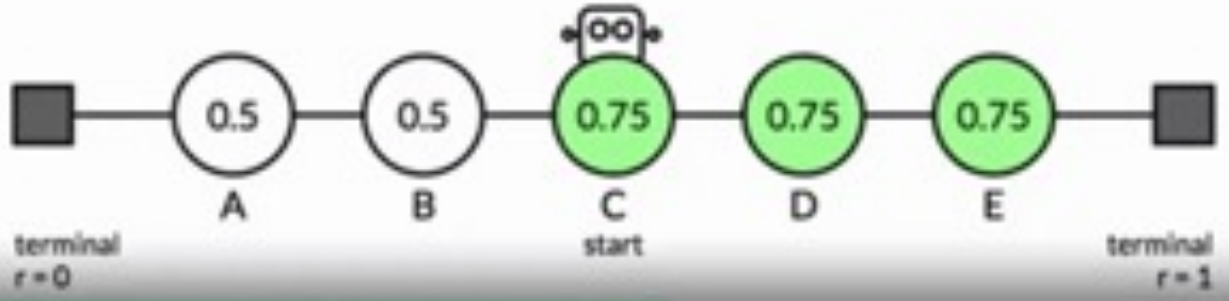
Target / Exact Values



Updates using TD Learning

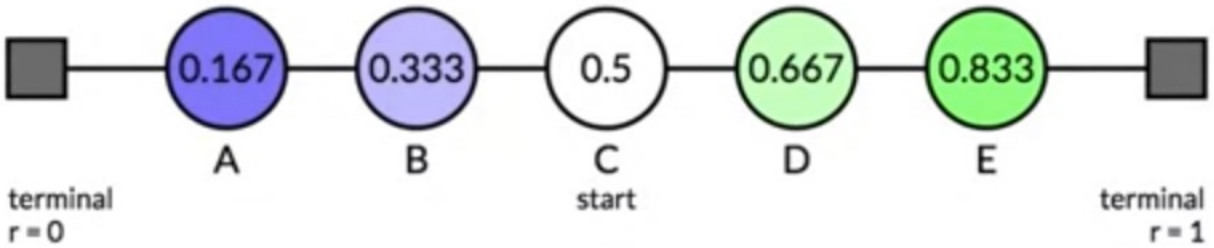


Updates using Monte Carlo

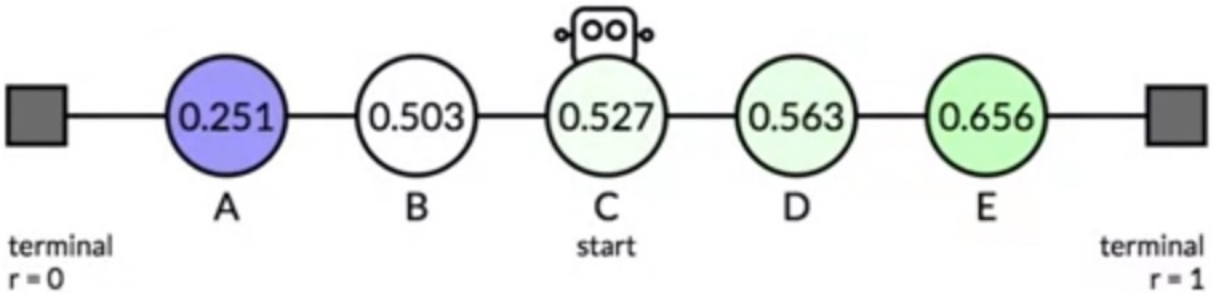


A. White, M. White 'Sample-based Learning Methods'

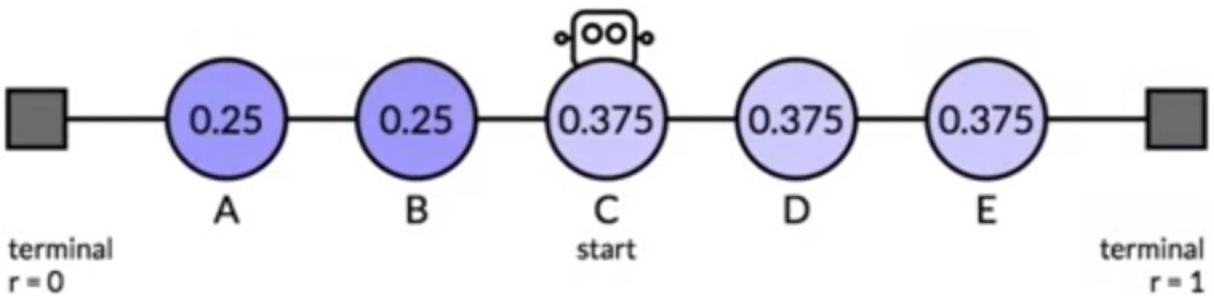
Target / Exact Values



Updates using TD Learning



Updates using Monte Carlo



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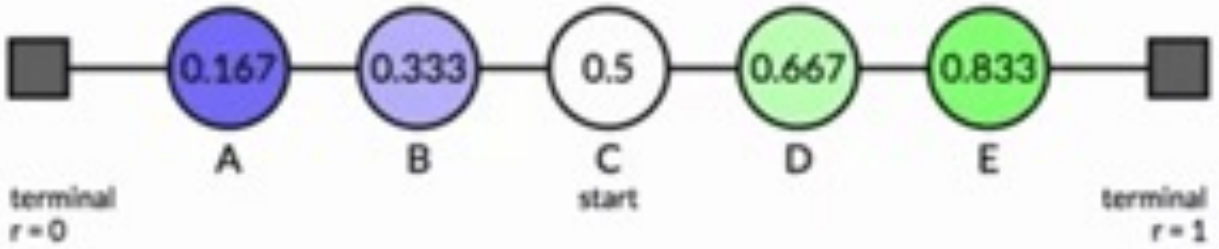
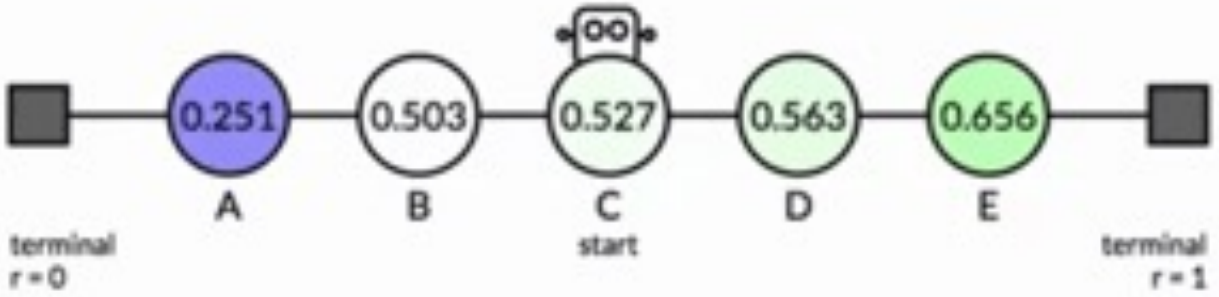
Additional Episodes

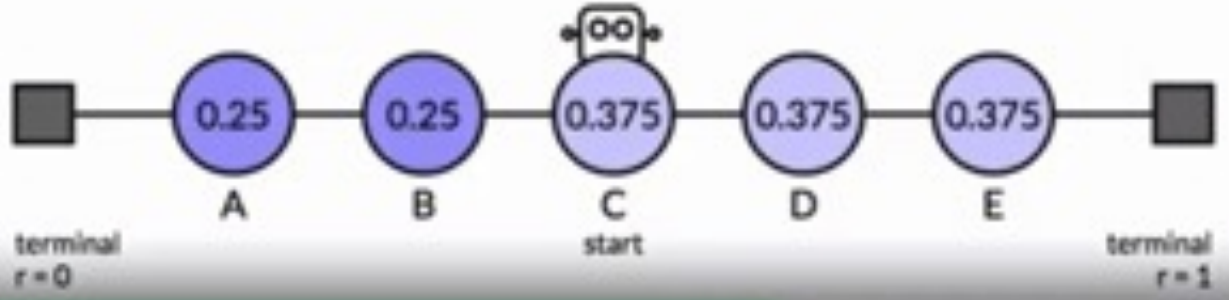
Target / Exact Values



Updates using TD Learning



Updates using Monte Carlo



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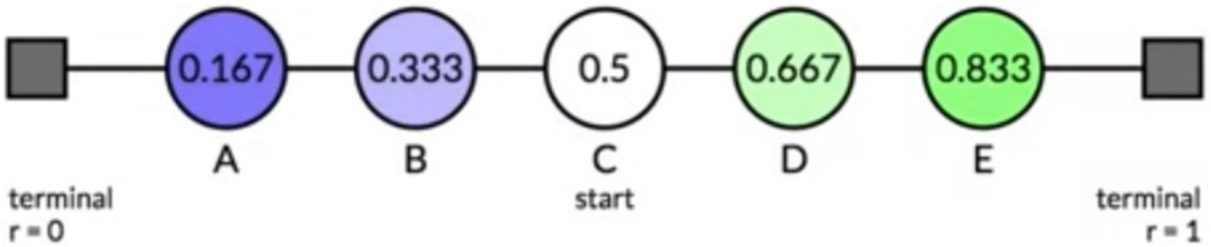
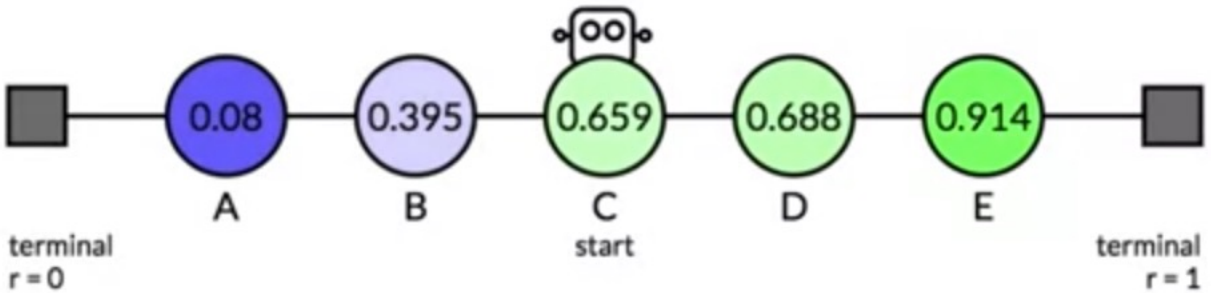
Additional Episodes

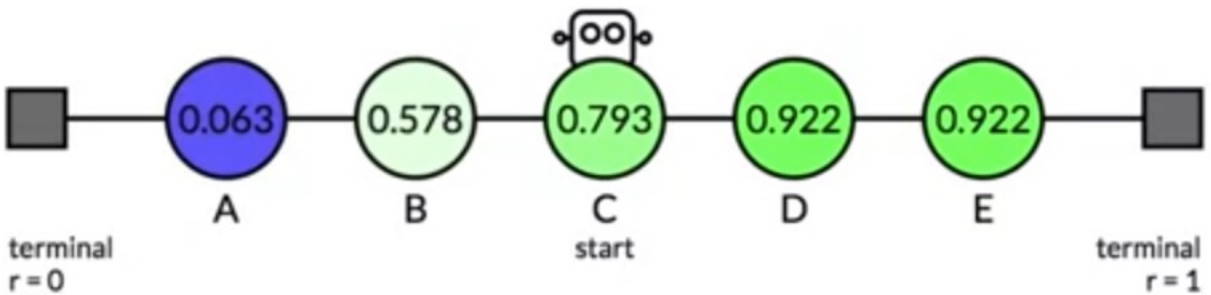
Target / Exact Values



Updates using TD Learning



Updates using Monte Carlo



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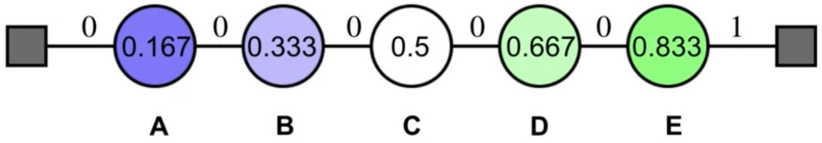
Prediction: TD(0) – Random Walk



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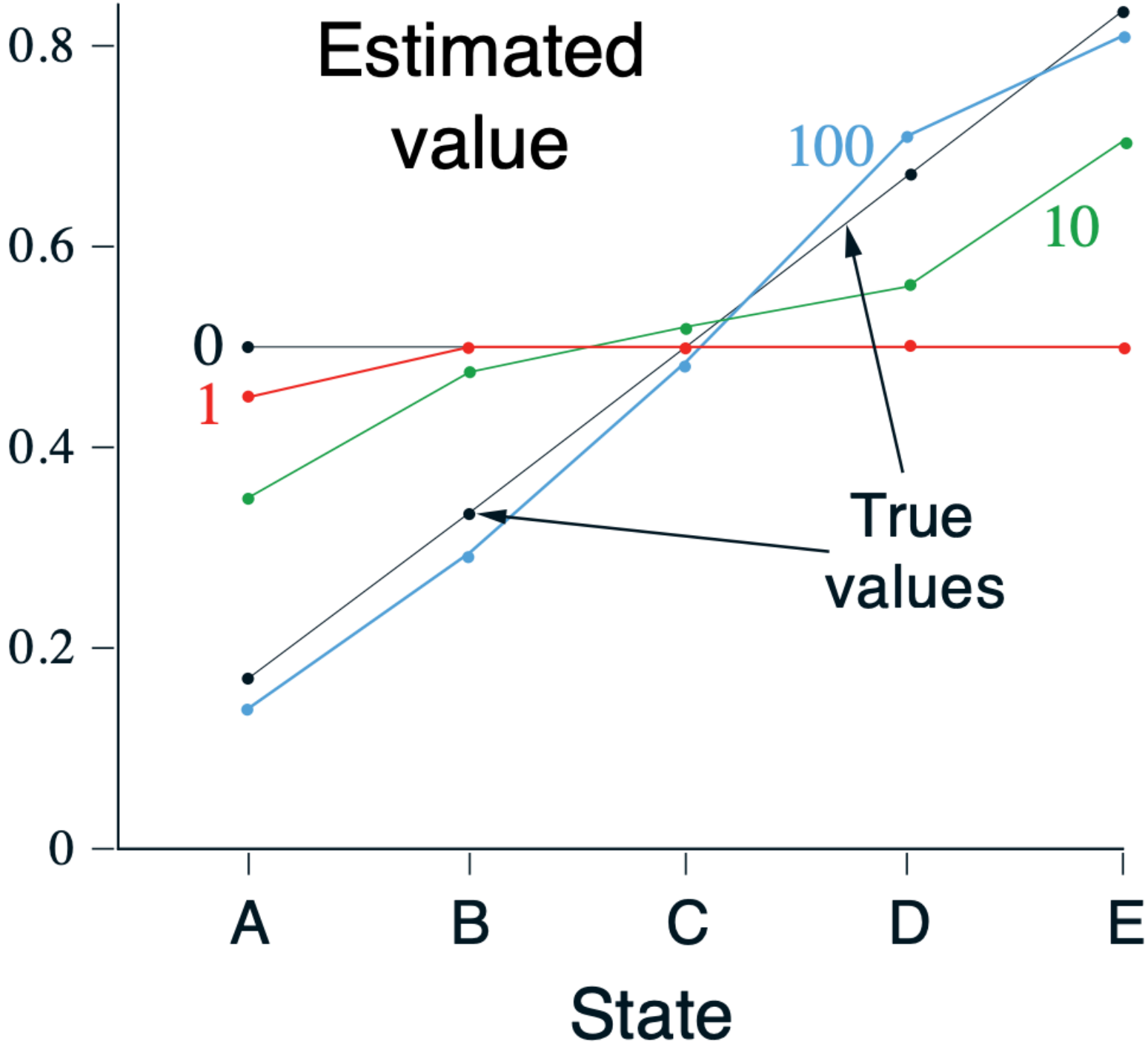
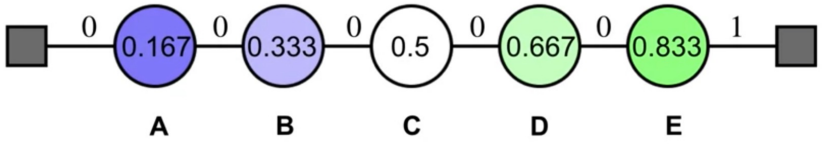
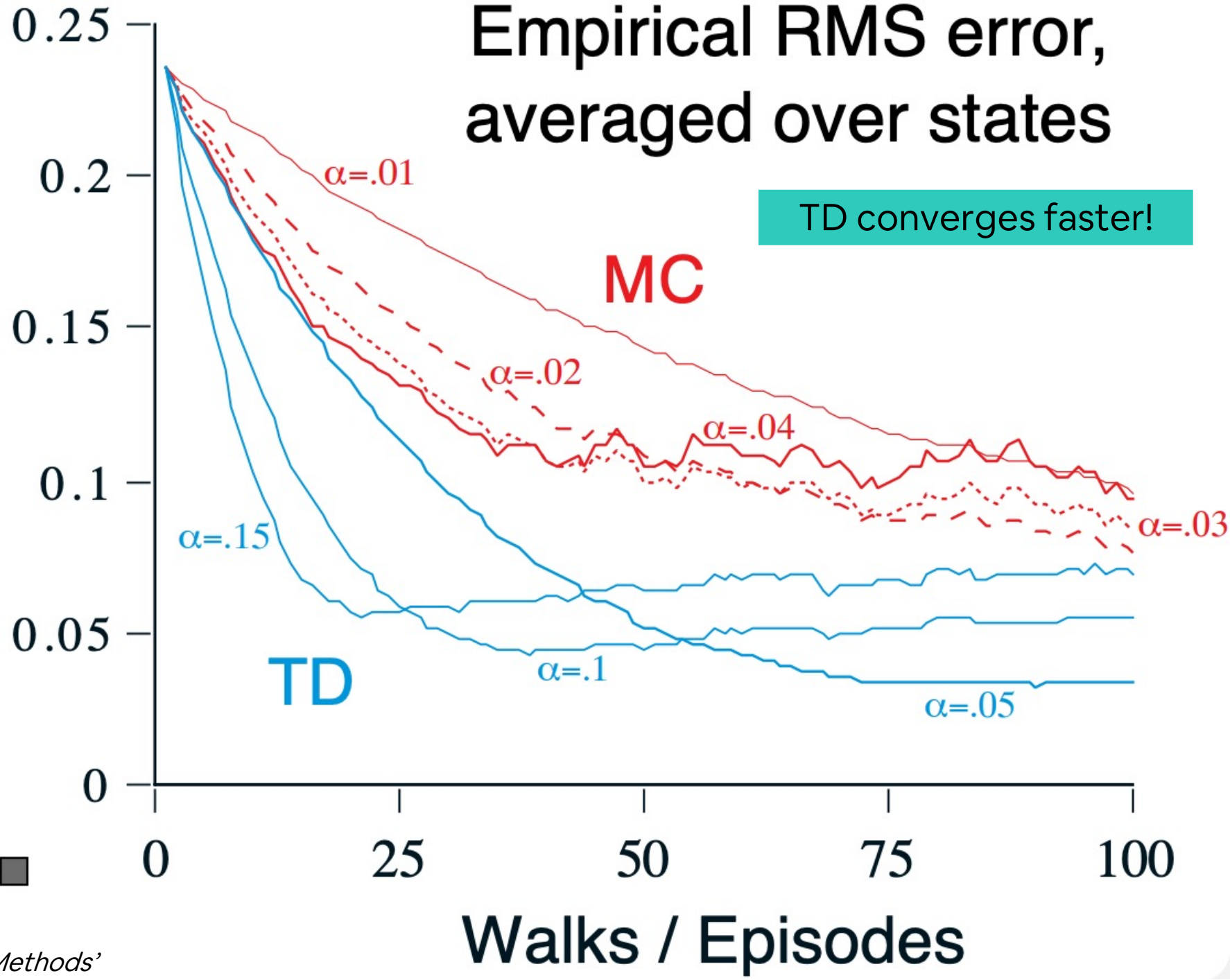


Prediction: TD(0) – Random Walk

Estimation error - Root Mean Square (RMS) – averaged over the 5 states, the averaged over 100 runs

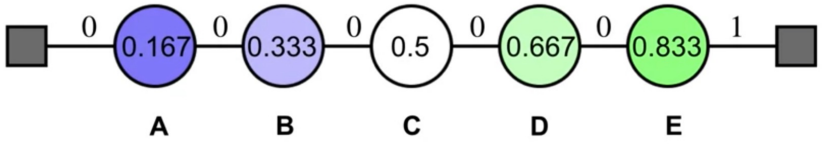


A. White, M. White 'Sample-based Learning Methods'



Prediction: TD(0) – Random Walk

Estimation error - Root Mean Square (RMS) – averaged over the 5 states, the averaged over 100 runs



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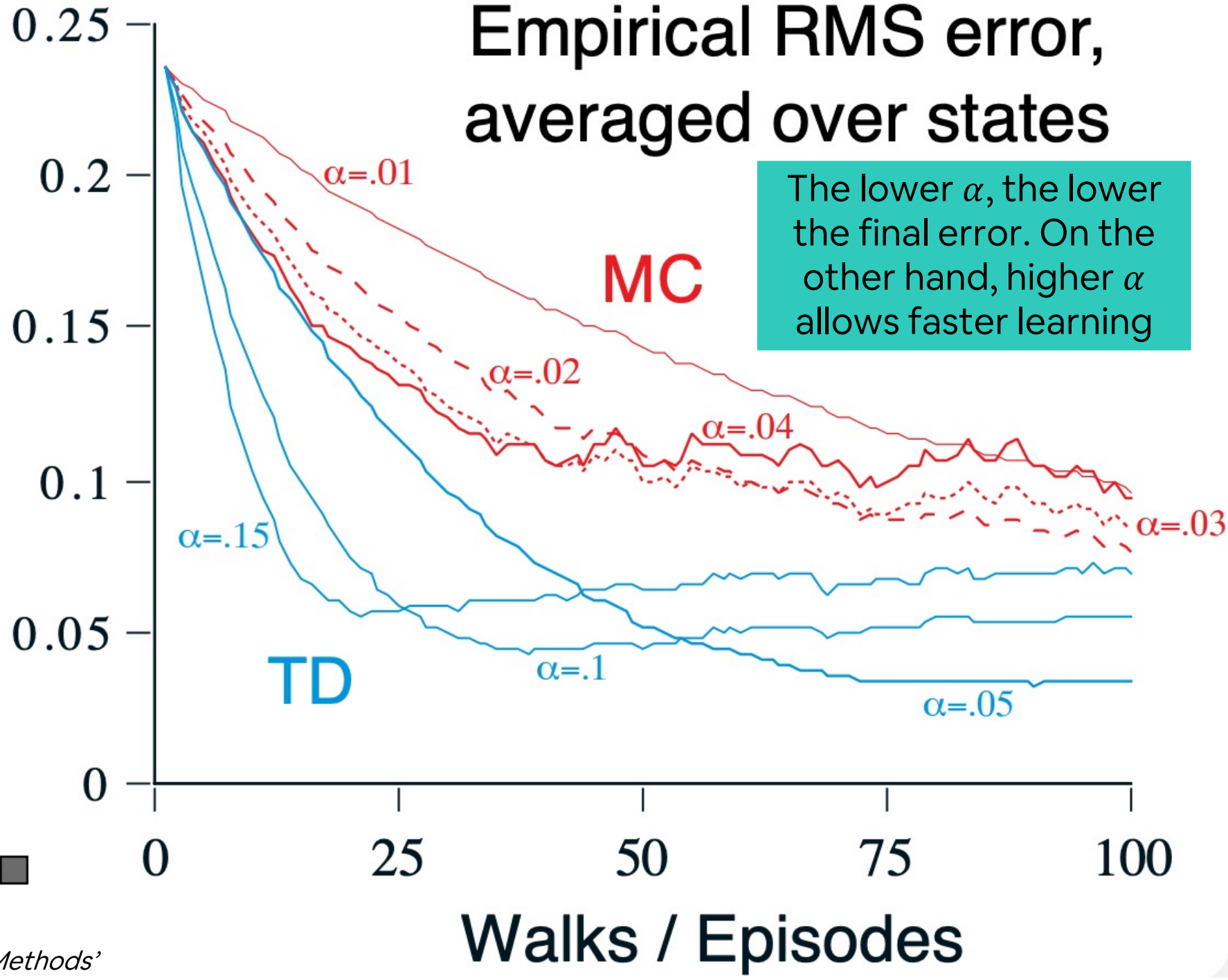
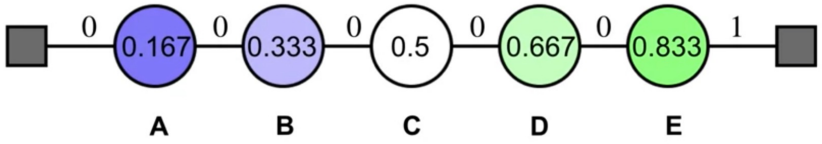


The lower α, the lower the final error. On the other hand, higher α allows faster learning

Prediction: TD(0) – Random Walk

Estimation error - Root Mean Square (RMS) – averaged over the 5 states, the averaged over 100 runs



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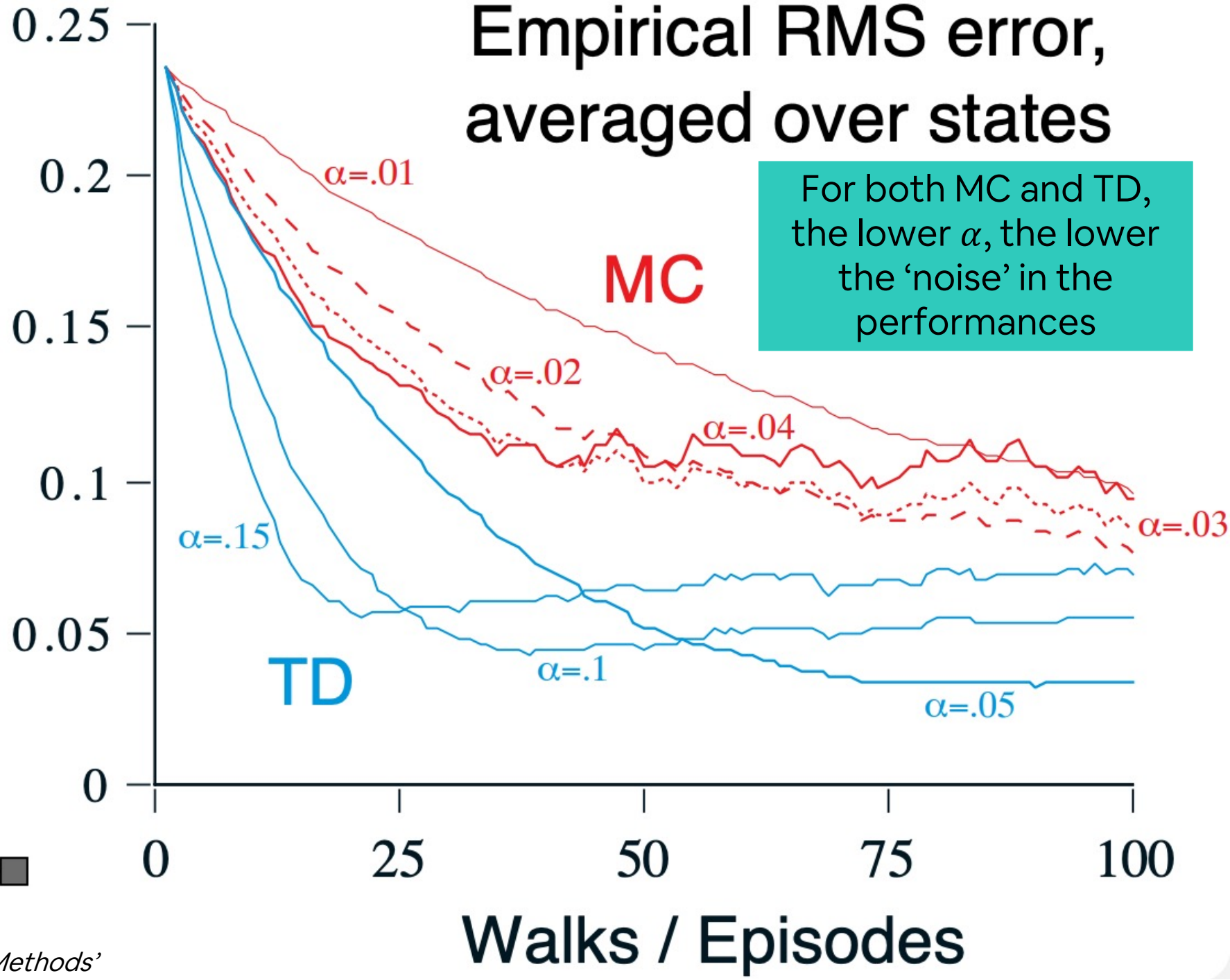

Bias/Variance Trade-off

- The return $G_t = R_{t+1} + \gamma R_{t+2} + \ ... + \gamma^{T-1} R_T$ is an unbiased estimate of $v_\pi(S_t)$
- True TD target $R_{t+1} + \gamma v_\pi(S_{t+1})$ (we don't know the true value function!) is an unbiased estimate of $v_\pi(S_t)$
- TD target $R_{t+1} + \gamma V(S_{t+1})$ (the target that we move to, it can be quite a wrong estimate!) is a biased estimate of $v_\pi(S_t)$
- TD target is much lower variance than the return:

1. Return depends on many random actions, transitions, rewards
2. TD target depends on one random action, transition, reward

Monte Carlo vs. TD(0)

- MC has high variance, zero bias

1. Good convergence properties (even with function approximation – Chapter 9, we will see this later in the course)
2. Not very sensitive to initial value
3. Very simple to understand and use

- TD has low variance, some bias

1. Usually more efficient than MC
2. TD(0) converges to $v_\pi(s)$ (but not always with function approximation)
3. More sensitive to initial values

Credits

- Image of the course is taken from C. Mahoney ‘Reinforcement Learning’ https://towardsdatascience.com/reinforcement-learning-fda8ff535bb6
- Many concepts and material for examples have been taken from A. White, M. White ‘Sample-based Learning Methods’
- The bias/variance trade-off was taken from D. Silver ‘Reinforcement Learning’ course @ UCL

AMCO
ARTIFICIAL INTELLIGENCE, MACHINE LEARNING AND CONTROL RESEARCH GROUP

Reinforcement Learning 2025/2026

RL2
Reinforcement Learning Research Lab @ UniPD

Thank you!
Questions?

Lecture #08
Off-Policy Monte Carlo & Temporal Difference Learning

Gian Antonio Susto