Automata, Languages and Computation

Chapter 7 : Properties of Context-Free Languages
Part II

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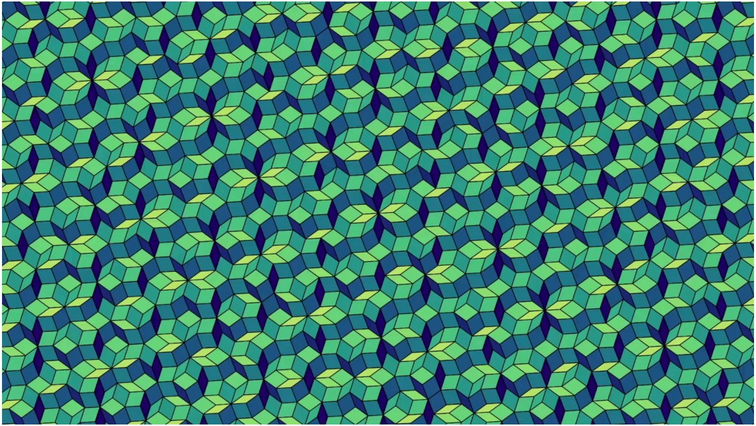
Properties of Context-Free Languages



1. Pumping lemma for CFLs : similar to regular languages
2. Closure properties for CFL : some of the closure properties of regular languages also hold for CFLs
3. Computational properties for CFLs : we can efficiently implement previous transformations for CFGs and PDAs
4. Decision problems for CFLs : we can test emptiness and membership; equivalence and other problems are undecidable

Pumping lemma for CFLs

In each sufficiently long string of a CFL we can find two substrings "next to each other" that

- can be eliminated
- can be iterated (synchronously)

still resulting in strings of the language

This property can be used to prove that some languages are not CFL

Parse trees

all'esame
anche dimostrazione

Theorem Let G be some CFG in CNF. Let T be a parse tree for a string $w \in L(G)$. If the longest path in T has n arcs, then $|w| \leqslant 2^{n-1}$

Proof By induction on $n \geqslant 1$

Base $n = 1$. T has one leaf and one inner node (root), and represents a derivation $S \Rightarrow a$. We have $|w| = 1 \leqslant 2^{n-1} = 2^0 = 1$

Parse trees

Induction $n > 1$. T's root uses a production $S \rightarrow AB$, and we can write $S \Rightarrow AB \overset{*}{\Rightarrow} w = uv$, where $A \overset{*}{\Rightarrow} u$ and $B \overset{*}{\Rightarrow} v$

We are using factorization here

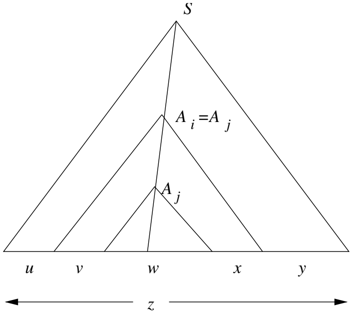
No path under the subtree rooted at A or B can have length greater than $n - 1$. By the inductive hypothesis we have $|u| \leqslant 2^{n-2}$ and $|v| \leqslant 2^{n-2}$

We can conclude that $|w| = |u| + |v| \leqslant 2^{n-2} + 2^{n-2} = 2^{n-1}$ $\square$

Pumping lemma for CFLs

Theorem Let L be some CFL. There exists a constant n such that, if $z \in L$ and $|z| \geqslant n$, we can factorize $z = uvwxy$ under the following conditions :

- $|vwx| \leqslant n$
- $vx \neq \epsilon$
- $uv^iwx^iy \in L$, for each $i \geqslant 0$



Pumping lemma for CFLs

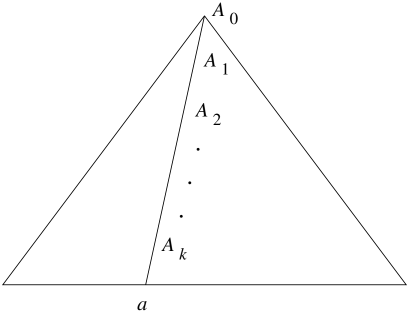
Proof Let G be some CFG in CNF such that $L(G) = L \smallsetminus \{\epsilon\}$. Let m be the number of variables of G. We choose $n = 2^m$

Let $z \in L$ such that $|z| \geqslant n$

From a previous theorem, the parse tree for z must have some path of length greater than m, otherwise we would get $|z| \leqslant 2^{m-1} = n/2$

Pumping lemma for CFLs

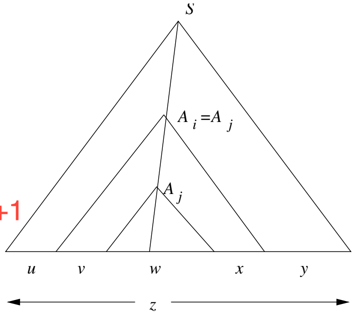
Consider all occurrences of variables in a path of length $k+1$, where $k \geqslant m$



Pumping lemma for CFLs

Since G has only m variables, at least one variable occurs more than once in the path. Let us assume $A_i = A_j$, where $k - m \leqslant i < j \leqslant k$, that is, we choose A_i in the lower part of the path

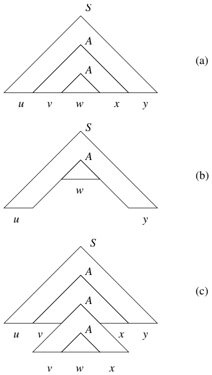
we choose the
i,j "close" to k.
in this window there
are m+1 indices
,paths in thesubtree
rooted in Ai
have max height of m+1



Pumping lemma for CFLs

We can then edit the parse tree in (a) in such a way that

- its yield becomes uv^0wx^0y, as shown in (b)
- its yield becomes uv^2wx^2y, as shown in (c)



Pumping lemma for CFLs

In the general case, we can edit the parse tree in (a) in such a way that its yield becomes uv^iwx^iy, for any $i \geqslant 0$

Since the longest path in the subtree rooted at A_i has length no longer than $m+1$, a previous theorem allows us to assert that $|vwx| \leqslant 2^m = n$ □

1. v and x must be close to each other
2. Since G is CNF, it has no eps rule the string vx cannot be empty

Example

Consider $L = \{0^i 1^i 2^i \mid i \geqslant 1\}$, and let n be the pumping lemma constant associated with L. We choose $z = 0^n 1^n 2^n$

For any factorization of z into $uvwxy$, with $|vwx| \leqslant n$ and v and x not both empty, we have that vwx cannot contain both 0 and 2, because the rightmost 0 and the leftmost 2 are $n+1$ places away one from the other

We therefore have the following cases:

- vwx does not contain 2; then vx has only 0 and 1; then uwy, which should be in L, has n occurrences of 2 but less than n occurrences of 0 or 1
- vwx does not contain 0; a similar reasoning as in the first case applies

Consequences of the pumping lemma

A CFL cannot **count** in more than two sequences

Example : $L = \{0^i 1^i 2^i \mid i \geqslant 1\}$

See previous slide

Try also to recognize L with a PDA

Consequences of the pumping lemma

A CFL cannot generate **crossing pairs**

Example : $L = \{0^i 1^j 2^i 3^j \mid i, j \geqslant 1\}$

Given n, we choose $z = 0^n 1^n 2^n 3^n$. Then vwx covers occurrences of at most two alphabet symbols. In all possible factorizations, the strings generated by iteration do not belong to L

Consequences of the pumping lemma

A CFL cannot generate **string copies**

Example : $L = \{ww \mid w \in \{0,1\}^*\}$

Given n, we choose $z = 0^n 1^n 0^n 1^n$. In all possible factorizations, the strings generated by iteration do not belong to L

Exercise

Using the pumping lemma, prove that the language

$$L = \{a^i b^j c^k \mid i, j \geqslant 0,\ k = \max\{i, j\}\}$$

is not context-free

Exercise

Solution Let us assume that L is a CFL; we will establish a contradiction. Let n be the pumping lemma constant associated with L

We choose $z = a^n b^n c^n \in L$ and analyze all possible factorizations $z = uvwxy$ with $|vwx| \leqslant n$ and $vx \neq \epsilon$, looking for a factorization that satisfies the pumping lemma

Exercise

$$z \;=\; \underbrace{a \cdots\cdots a}_{a \text{ block}}\underbrace{b \cdots\cdots b}_{b \text{ block}}\underbrace{c \cdots\cdots c}_{c \text{ block}}$$

We distinguish the following cases

- vwx is placed into the a block or into the b block
- vwx is placed into the c block
- vwx is placed across the a and b blocks, or else across the b and c blocks
    - v or x contain both a and b, or both b and c
    - v is placed into the a block and x is placed into the b block
    - v is placed into the b block and x is placed into the c block

Exercise

vwx is placed into the a block : consider the new string uv^kwx^ky with $k > 1$, which must belong to L

$\#_a$ (the number of a's) increases ($> n$), since $vx \neq \epsilon$, while $\#_c$ remains unchanged ($= n$) and equal to $\#_b$, that is, the minimum between $\#_a$ and $\#_b$

We therefore conclude that $uv^kwx^ky \notin L$ for $k > 1$

A similar reasoning applies to the case where vwx is placed into the b block

Exercise

vwx is placed into the c block : consider the new string $uv^k wx^k y$ with $k = 0$, which must belong to L

$\#_c$ decreases ($< n$), since $vx \neq \epsilon$, and is no longer equal to the maximum between $\#_a$ and $\#_b$, which is n, since the a block and the b block both remain unchanged

We therefore conclude that $uv^k wx^k y \notin L$ for $k = 0$

Exercise

vwx is placed across the a and b blocks or else across the b and c blocks

- v or x include both a and b : choosing $k = 2$, we break the structure $a^*b^*c^*$ and the new string doesn't belong to L
- v or x include both b and c : we use the same argument of the previous point
- v is placed into the a block and x is placed into the b block : choosing $k = 2$, increases $\#_a$ and/or $\#_b$ $(> n)$, while $\#_c$ remains unchanged $(= n)$ and therefore will not be equal to the maximum required; therefore the new string does not belong to L

Exercise

vwx is placed across the a and b blocks or else across the b and c blocks (continued)

- v is placed into the b block and x is placed into the c block
  - if $x \neq \epsilon$ we choose $k = 0$; $\#_c$ becomes smaller (and so does $\#_b$ if $v \neq \epsilon$) but $\#_a$ does not change, and provides the maximum value; therefore $uv^k wx^k y \notin L$ for $k = 0$
  - if $x = \epsilon$ we choose $k > 1$ so that $\#_b$ gets larger than $\#_a$, and $\#_c$ does not change; therefore $uv^k wx^k y \notin L$ for some appropriate $k > 1$

Exercise

In none of the possible cases we have been able to satisfy the pumping lemma: we have established a **contradiction**

We then conclude that language L is not CFL

Substitution

Assume two (finite) alphabets Σ and Δ, and a function

$$s : \Sigma \rightarrow 2^{\Delta^*}$$

maps a string
into a language

Let $w \in \Sigma^*$, with $w = a_1 a_2 \cdots a_n$, $a_i \in \Sigma$. We define

$$s(w) = s(a_1).s(a_2). \cdots .s(a_n)$$

and, for $L \subseteq \Sigma^*$, we define

maps a language
into a language

$$s(L) = \bigcup_{w \in L} s(w)$$

Function s is called a **substitution**

Example

Let $s(0) = \{a^n b^n \mid n \geqslant 1\}$ and $s(1) = \{aa, bb\}$

Then $s(01)$ is a language whose strings have the form $a^n b^n aa$ or $a^n b^{n+2}$, with $n \geqslant 1$

Let $L = L(\mathbf{0}^*)$. Then $s(L)$ is a language whose strings have the form

$$a^{n_1} b^{n_1} a^{n_2} b^{n_2} \cdots a^{n_k} b^{n_k},$$

with $k \geqslant 0$ and with $n_1, n_2, \ldots, n_k$ positive integers

Substitution

Next theorem is used later to prove several closure properties for CFL in a unified way and through very simple proofs

Theorem Let L be a CFL defined over Σ and let s be a substitution defined on Σ such that, for each $a \in \Sigma$, $s(a)$ is a CFL. Then $s(L)$ is a CFL

Proof Let $G = (V, \Sigma, P, S)$ be a CFG generating L and, for each $a \in \Sigma$, let $G_a = (V_a, T_a, P_a, S_a)$ be a CFG generating $s(a)$

Substitution

We construct a CFG $G' = (V', T', P', S)$ with

$$V' = (\bigcup_{a \in \Sigma} V_a) \cup V$$
$$T' = \bigcup_{a \in \Sigma} T_a$$
$$P' = (\bigcup_{a \in \Sigma} P_a) \cup P_R$$

where P_R is obtained from P by replacing each occurrence of a in any right-hand side with symbol S_a

Substitution

We prove $L(G') = s(L)$

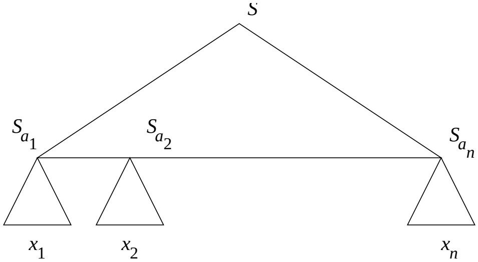
(Part $\supseteq$) Let $w \in s(L)$. Then there exists a string $x \in L$ such that

$$x = a_1 a_2 \cdots a_n$$

Furthermore, there exist strings $x_i \in s(a_i)$, such that $w = x_1 x_2 \cdots x_n$

Substitution

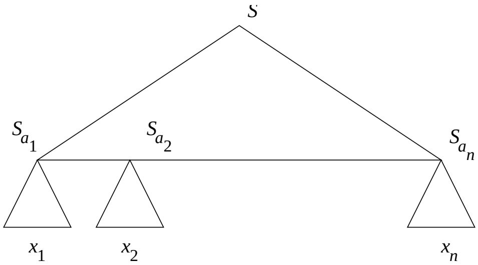
The associated parse tree for G' must have the form



We can then generate $S_{a_1} S_{a_2} \cdots S_{a_n}$ in G', and then generate $x_1 x_2 \cdots x_n = w$. Therefore $w \in L(G')$

Substitution

(Part $\subseteq$) Let $w \in L(G')$. Then the parse tree for w must have the form



Substitution

We can remove the subtrees at the bottom, and get a parse tree with yield

$$S_{a_1} S_{a_2} \cdots S_{a_n}$$

corresponding to a string $a_1 a_2 \cdots a_n \in L(G)$

We must also have $w \in s(a_1 a_2 \cdots a_n)$, and thus $w \in s(L)$ $\square$

Applications of the substitution theorem

Theorem The CFLs are closed under the following operations

- union
- concatenation
- Kleene closure ($*$) and positive closure ($+$)
- homomorphism

Proof For each of the operators above, we define a specific substitution and we apply the previous theorem

Union : Given two CFLs L_1 and L_2, consider the CFL $L = \{1, 2\}$. and define $s(1) = L_1$, $s(2) = L_2$. We have $L_1 \cup L_2 = s(L)$, which still is a CFL

Applications of the substitution theorem

Concatenation : Given two CFLs L_1 and L_2, consider the CFL $L = \{1.2\}$ and define $s(1) = L_1$, $s(2) = L_2$. We thus have $L_1.L_2 = s(L)$, which still is a CFL

$*$ *and* $+$ *closures* : Given a CFL L_1, consider the CFL $L = \{1\}^*$ and define $s(1) = L_1$. We have $L_1^* = s(L)$, which still is a CFL. A similar argument holds for $+$

Homomorphism : Assume a CFL L and a homomorphism h, both over Σ. We define $s(a) = \{h(a)\}$ for each $a \in \Sigma$. We then have $h(L) = s(L)$, which still is a CFL □

Closure under string reverse

Theorem If L is a CFL, then so is L^R

Proof Assume L is generated by a CFG $G = (V, T, P, S)$. We build $G^R = (V, T, P^R, S)$, where

$$P^R = \{A \to \alpha^R \mid (A \to \alpha) \in P\}$$

Using induction on derivation length in G and in G^R, we can show that $(L(G))^R = L(G^R)$ (omitted) $\square$

CFL & intersection

$L_1 = \{0^n 1^n 2^i \mid n \geqslant 1,\ i \geqslant 1\}$ is a CFL, generated by the CFG

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow 0A1 \mid 01 \\ B &\rightarrow 2B \mid 2 \end{aligned}$$

$L_2 = \{0^i 1^n 2^n \mid n \geqslant 1,\ i \geqslant 1\}$ is a CFL, generated by the CFG

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow 0A \mid 0 \\ B &\rightarrow 1B2 \mid 12 \end{aligned}$$

$L_1 \cap L_2 = \{0^n 1^n 2^n \mid n \geqslant 1\}$ which is not a CFL

This was proved in a previous example

Intersection between CFL and regular language

Theorem Let L be some CFL and let R be some regular language. Then $L \cap R$ is a CFL

Proof Let L be accepted by the PDA

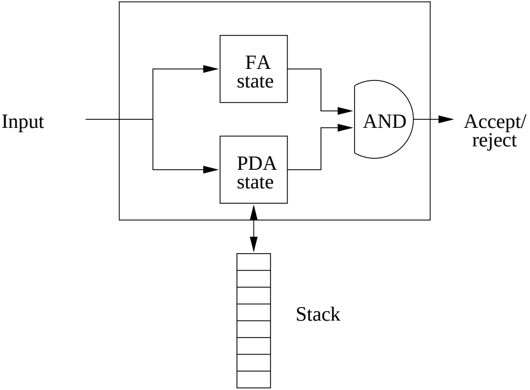
$$P = (Q_P, \Sigma, \Gamma, \delta_P, q_P, Z_0, F_P)$$

by final state, and let R be accepted by the DFA

$$A = (Q_A, \Sigma, \delta_A, q_A, F_A)$$

Intersection between CFL and regular language

We construct a PDA for $L \cap R$ based on the following idea



Intersection between CFL and regular language

We define

$$P' = (Q_P \times Q_A, \Sigma, \Gamma, \delta, (q_P, q_A), Z_0, F_P \times F_A)$$

where $(a \in \Sigma \cup \{\epsilon\})$

$$\delta((q,p),a,X) = \{((r,s),\gamma) \mid (r,\gamma) \in \delta_P(q,a,X), s = \hat{\delta}_A(p,a)\}$$

We can show (omitted) by induction on the number of steps in the computation $\overset{*}{\vdash}$ that

$$(q_P, w, Z_0) \underset{P}{\overset{*}{\vdash}} (q, \epsilon, \gamma)$$

if and only if

$$((q_P, q_A), w, Z_0) \underset{P'}{\overset{*}{\vdash}} ((q,p), \epsilon, \gamma), \text{ with } p = \hat{\delta}(q_A, w)$$

Intersection between CFL and regular language

(q, p) is an accepting state of P' if and only if

- q is an accepting state of P
- p is an accepting state of A

Therefore P' accepts w if and only if both P and A accept w, that is, $w \in L \cap R$ $\square$

Other properties for CFLs

Theorem Let L, L_1, L_2 be CFLs and let R be a regular language. Then

- $L \setminus R$ is a CFL
- $\overline{L}$ may fall outside of CFLs
- $L_1 \setminus L_2$ may fall outside of CFLs

Proof
Operator $\setminus$ with REG : $\overline{R}$ is regular, $L \cap \overline{R}$ is CFL, and $L \cap \overline{R} = L \setminus R$

Other properties for CFLs

Complement operator : If $\overline{L}$ would always be a CFL, then we have that

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

would always be CFL, which is a contradiction

Operator $\setminus$ with CFL : Σ^* is a CFL. If $L_1 \setminus L_2$ would always be a CFL, then $\Sigma^* \setminus L = \overline{L}$ would always be a CFL, which is a contradiction $\square$

Test

Assert whether the following statements hold, and motivate your answer

- the intersection of a non-CFL L_1 and a CFL L_2 can be a non-CFL L1={a^n b^n c^n}, L2={a,b,c}*=sigma*
- the intersection of a non-CFL and a finite language is always a CFL

Computational properties for CFLs

We investigate the **computational complexity** for some of the transformations previously presented

We need these results to establish the efficiency of some decision problems which we will consider later

We denote with n the **length** of the entire representation of a PDA or a CFG (for more detailed results, we should instead distinguish between number of variables, number of stack symbols, etc.)

Computational properties for CFLs

The following conversions can be computed in time $\mathcal{O}(n)$

- conversion from PDA accepting by final state to PDA accepting by empty stack
- conversion from PDA accepting by empty stack to PDA accepting by final state
- conversion from CFG to PDA

Given a PDA of size n we can build an equivalent CFG in time (and space) $\mathcal{O}(n^3)$, using a **preliminary binarization** of the transitions of the autmaton

The construction of Chapter 6 (which we have not presented) requires exponential time

Conversion to CNF

We can compute in time $\mathcal{O}(n)$

- the set of reachable symbols $r(G)$
- the set of generating symbols $g(G)$
- the elimination of useless symbols from a CFG

Conversion to CNF

We can compute in time $\mathcal{O}(n)$ the set of nullable symbols $n(G)$

We can compute in time $\mathcal{O}(n)$ the elimination of ϵ-productions from a CFG, using a **preliminary binarization** of the grammar

We can compute in time $\mathcal{O}(n^2)$ the set of unary symbols $u(G)$ and the elimination of unary productions from a CFG

Conversion to CNF

We can compute in time $\mathcal{O}(n)$ the replacement of terminal symbols with variables (first transformation for CNF)

We can compute in time $\mathcal{O}(n)$ the reduction of production with right-hand side length larger than 2 (second transformation for CNF)

Given a CFG of size n, we can construct an equivalent CFG in CNF in time (and space) $\mathcal{O}(n^2)$

Emptiness test

Let G be some CFG with start symbol S. $L(G)$ is empty if and only if S is not generating

We can then test emptiness for $L(G)$ using the already mentioned algorithm for the computation of $g(G)$, running in time $\mathcal{O}(n)$

CFL membership

The **membership problem** for a CFL string is defined as follows

Given as input a string w, we want to decide whether $w \in L(G)$, where G is some **fixed** CFG

Note : G does not depend on w and is **not** considered part of the input for our problem. Therefore the length of G does not affect the running time of the problem

CFL membership

Assume G in CNF and $|w| = n$. Since the parse trees for w are binary, the number of internal nodes for each tree is $2n - 1$ (proof by induction)

We can therefore generate all the parse trees of G with $2n - 1$ nodes and test whether some tree yields w

There are more efficient algorithms that take advantage of **dynamic programming** techniques

CFL membership

Let $w = a_1 a_2 \cdots a_n$. We construct a triangular **parse table** where cell X_{ij} is set valued and contains all variables A such that

$$A \overset{*}{\underset{G}{\Rightarrow}} a_i a_{i+1} \cdots a_j$$

X_{15}				
X_{14}	X_{25}			
X_{13}	X_{24}	X_{35}		
X_{12}	X_{23}	X_{34}	X_{45}	
X_{11}	X_{22}	X_{33}	X_{44}	X_{55}
a_1	a_2	a_3	a_4	a_5

CFL membership

We **iteratively** construct the parse table, one row at a time and from bottom to top

First row is populated with the base case, while remaining rows are populated by the inductive case

Idea : $A \underset{G}{\overset{*}{\Rightarrow}} a_i a_{i+1} \cdots a_j$ if and only if

- for some production $A \rightarrow BC$
- for some integer k with $i \leqslant k < j$

we have $B \underset{G}{\overset{*}{\Rightarrow}} a_i a_{i+1} \cdots a_k$ and $C \underset{G}{\overset{*}{\Rightarrow}} a_{k+1} a_{k+2} \cdots a_j$

CFL membership

Base $X_{ii} \leftarrow \{A \mid (A \to a_i) \in P\}$

Induction We build X_{ij} for increasing values of $j - i \geqslant 1$

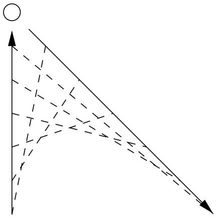
$X_{ij} \leftarrow X_{ij} \cup \{A\}$ if and only if there exist k, B, C such that

- $i \leqslant k < j$
- $(A \to BC) \in P$
- $B \in X_{ik}$ and $C \in X_{k+1,j}$

CFL membership

In the inductive case, to populate X_{ij} we need to check at most n pairs of previously built cells of the parse table

$$(X_{ii}, X_{i+1,j}),\ (X_{i,i+1}, X_{i+2,j}),\ \ldots,\ (X_{i,j-1}, X_{jj})$$



The operation above is related to vector convolution

CFL membership

We assume we can compute each check $B \in X_{ik}$ in time $\mathcal{O}(1)$. Then each set X_{ij} can be populated in time $\mathcal{O}(n)$

We need to populate $\mathcal{O}(n^2)$ sets X_{ij}

We summarize all of the previous observations by means of the following statement

Theorem The algorithm for the construction of the parse table computes all of the sets X_{ij} in time $\mathcal{O}(n^3)$. We then have $w \in L(G)$ if and only if $S \in X_{1n}$

Example

Let G be a CFG with productions

$$S \rightarrow AB \mid BC$$
$$A \rightarrow BA \mid a$$
$$B \rightarrow CC \mid b$$
$$C \rightarrow AB \mid a$$

and let $w = baaba$

$\{S,A,C\}$				
-	$\{S,A,C\}$			
-	$\{B\}$	$\{B\}$		
$\{S,A\}$	$\{B\}$	$\{S,C\}$	$\{S,A\}$	
$\{B\}$	$\{A,C\}$	$\{A,C\}$	$\{B\}$	$\{A,C\}$
b	a	a	b	a

Summary of decision problem for CFLs

We have presented **efficient** algorithms for the solution of the following decision problems for CFLs

- given a CFG G, test whether $L(G) \neq \emptyset$
- given a string w, test whether $w \in L(G)$ for a fixed CFG G

Undecidable decision problem for CFLs

In the next chapters we will develop a mathematical theory to prove the existence of decision problems that **no algorithm can solve**

Let us now anticipate some of these problems, concerning CFLs

- given a CFG G, test whether G is ambiguous
- given a representation for a CFL L, test whether L is inherently ambiguous
- given a representation for two CFLs L_1 and L_2, test whether the intersection $L_1 \cap L_2$ is empty
- given a representation for two CFLs L_1 and L_2, test whether $L_1 = L_2$
- given a representation for a CFL L defined over Σ, test whether $L = \Sigma^*$