Exercises I1

Exercise 1.1

Since $X \perp Y$ $f_{XY}(x,y) = f_X(x) f_X(y) = e^{-x-y} \mathbb{1}_{x \geq 0, y \geq 0}$

$Z = (Z_1, Z_2)$
$Z_1 = \frac{X}{X+Y}$
$Z_2 = X + Y$
$g^{-1}(z_1, z_2) = (z_1 z_2, z_2 - z_1 z_2)$

$J(z) = \begin{pmatrix} z_2 & z_1 \\ -z_2 & 1 - z_1 \end{pmatrix}$
$|\det J| = |z_2| = \underbrace{z_2}_{\geq 0}$
$f_{Z_1 Z_2}(z_1, z_2) \neq 0$ when

$$(\triangle) \begin{cases} z_1 z_2 \geq 0 \\ z_2 - z_1 z_2 \geq 0 \end{cases} \rightarrow \begin{cases} z_2 \geq 0 \\ 0 \leq z_1 \leq 1 \end{cases}$$

$f_{Z_1 Z_2}(z_1, z_2) = e^{-z_1 z_2 - (z_2 - z_1 z_2)} z_2 = z_2 e^{-z_2}$ when $(\triangle)$ else it's 0

Now we determine the PDFs of Z_1 and Z_2 using the marginal rules:
$f_{Z_1}(z_1) = \int_0^{+\infty} z_2 e^{-z_2} \, dz_2 = 1$
$f_{Z_2}(z_2) = \int_0^1 z_2 e^{-z_2} \, dz_1 = z_2 e^{-z_2}$
We have $Z_1 \perp Z_2$ because:
$f_{Z_1 Z_2}(z_1, z_2) = f_{Z_1}(z_1) f_{Z_2}(z_2)$
$\square$

Exercise 1.2

$X_n \sim Bin(n, \theta)$
$f_{X_n(k)} = \binom{n}{k}\theta^k(1-\theta)^{n-k}$
a. we need to prove that $\lim_{n\to\infty} P(|\frac{X_n}{n} - \theta| > \epsilon) = 0 \;\; \forall\epsilon > 0(\star)$
$\mathbb{E}(X_n) = \frac{n\theta}{n} = \theta$ from expectation of Bin and linearity
$\sigma^2_{X_n} = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}$ from variance of Bin and the property of variance ($\mathrm{v}ar(\frac{X}{\alpha}) = \frac{1}{\alpha^2}\mathrm{v}ar(X)$)
we can use Chebychev inequality:
$0 \leq P(|\frac{X_n}{n} - \theta| > \epsilon) \leq \frac{\theta(1-\theta)}{n\epsilon^2}$, the left hand side of this inequality is equal to 0 and the right hand side converges to 0 for $n \to \infty$ so we proved $(\star)$. $\square$
b. Since in a. we proved that $X_n/n \xrightarrow{\mathrm{P}} \theta$ if we take $g(x) = 1 - x$ we have from the properties of convergence in probability that $g(X_n/n) \xrightarrow{\mathrm{P}} g(\theta) = 1 - \theta$. $\square$
c. Since we proved b. and a. we can use the property of convergence in probability that states that if
$X_n \xrightarrow{P} X,\; Y_n \xrightarrow{P} Y \text{ then } X_nY_n \xrightarrow{P} XY$.
We apply this property with $\frac{X_n}{n}$ and $1 - \frac{X_n}{n}$ and we get
$\frac{X_n}{n}(1 - \frac{X_n}{n}) \xrightarrow{P} \theta(1-\theta)$. $\square$