Università degli Studi di Trento

Department of Industrial Engineering

Mechanical Vibrations

Prof.: Bortoluzzi Daniele

Course Notes

Matteo Dalle Vedove
matteo.dallevedove@studenti.unitn.it

Academic Year 2021-2022
March 22, 2022

Contents

Introduction

The aim of this course is understanding how mechanical systems moves while subjected to external forces. Generally **dynamical systems** (mechanical, but also electrical, biological...) can so be regarded as *black box* where, for some given **inputs** $u(t)$ (independent variable) function of time, it produces an **output** $y(t)$ (dependent variable) that's still depend on time.

Vibrations are a subset of the all allowable motion of the system that are characterized by a confined motion respect to the equilibrium point with a certain periodicity.
Not all system are allowed to vibrate; from a formal point of view system vibrates if they are allowed to continuously reconvert potential energy (mainly elastic or gravitational) into kinetic. Vibration can be avoided by having *components* (in general) that allow to dissipate energy reducing so the vibrational motion respect to the equilibrium position.

The study of vibration relates to the study of the system by building it's **mathematical model** that's based both on **physical properties** as well as the related **accuracy**. The same system can be in fact modelled with different *depth* in terms of knowledge and detail: any addition in terms of physical phenomena (by increasing degrees of freedoms of the system, the number of parameters and equations) increases the complexity in the calculations (in simple words *we have to handle more stuff*) but also allows to create a model that better describes the reality. From an engineering point of view this is a trade-off between complexity and accuracy and such balance strictly depends on the goal of the study.

Dynamical system can be classified according to multiple orthogonal definitions such:

- the type of input can be deterministic (described by a analytical function) or random (where so a statistical description of the system is necessarily needed);
- depending on the **linearity** (or not) of the system. In practise no system is purely linear however such behaviour can be approximated as valid for *low displacement* from the equilibrium position where the two motions converges.

Chapter 1

Linear Systems with N Degrees of Freedom

1.1 Lumped parameter systems

1.1.1 1 Degree of Freedom

The idea of **lumped system** is to concentrate the **parameters** of the system, in particular:

- the **springs elements** where mostly of the elastic energy is stored (conservation of energy) and is assumed to be mass-less. Considering the **SCHEMATIC** of such element, we see that it's graphical representation present two connecting point that model the position on where the two ends of the springs are attached; those ends are subjected to forces F_1, F_2 and considering is center of gravity G we have that, for Newton

  $$\sum_i F_i = \cancelto{0}{m\ddot{x}_G} \qquad \Rightarrow \qquad F_1 = F_2 = F$$

  We have so proven that such lumped spring elements doesn't permit load unbalance to their ends; regarding the behaviour of the system, is we assume that's the spring it's linear we can consider the force as linearly proportional to displacement, hence

  $$F = k\big(x_2 - x_1\big) \tag{1.1}$$

  where k is the **stiffness** of the spring. When $x_2 - x_1 = 0$ we have that the spring absorbs no force;

- the **damper elements** describing the dissipative forces related to velocity; this element is still mass-less. Considering it's **SCHEMATIC**, the Newton equations determines

  $$\cancelto{0}{m\ddot{x}_G} = F_2 - F_1 \qquad \Rightarrow \qquad F_2 = F_1 = F$$

  hence, as for the spring, such element isn't allowed to have different forces transmitted through it's ends. If we consider a **linear dumper** model the generated force F can be regarded as

  $$F = c\big(\dot{x}_2 - \dot{x}_1\big) \tag{1.2}$$

  where c is the **viscous dumping** coefficient of the element;

- the **lumped mass** where all the mass of the system is concentrated and so contains the kinetic energy of the signal. Given x the center of the mass of weight m, using Newton's law we have that

  $$F = m\ddot{x} \tag{1.3}$$

AGGIUNGERE LE FIGURE

Real system aren't linear, however we can use the Taylor's series truncated to the first order to model every system as linear (within a certain range).

Damped Considering the system in **FIGURE** (FARE LA FIGURA) where the mass m is subjected to it's body-weight force mg (pointing downward) and an external force $f(t)$; setting to $y = 0$ the coordinate were the elastic force is equal to zero, we can rewrite it's equation of motion by disassembling the components and considering the forces that they generates:

$$\begin{aligned} m\ddot{y} &= F_{spring} + F_{dumper} + F_{weigth} + f(t) \\ &= -ky - c\dot{y} - mg + f(t) \end{aligned}$$

Such equation has a static solution, in fact if we consider $y_{st} = k \in \mathbb{R}$ (hence $\dot{y} = \ddot{y} = 0$) we have that

$$0 = -ky - mg \qquad \Rightarrow \qquad y_{st} = -\frac{mg}{k}$$

Considering now the scale x whose zero is for $y = -mg/k$ (the static solution of the differential equation), we have the substitution $x = y - y_{st}$ that determines the differential equation

$$\begin{aligned} m(\ddot{x} + \cancel{\ddot{y}_{st}}) &= -k(x + y_{st}) - c(\dot{x} + \cancel{\dot{y}_{st}}) - mg + f(t) \\ m\ddot{x} &= -kx - \cancel{(-mg)} - c\dot{x} - \cancel{mg} + f(t) \\ m\ddot{x} + c\dot{x} + kx &= f(t) \end{aligned} \tag{1.4}$$

In this equation the first term $m\ddot{x} + c\dot{x} + kx$ represent the **logic of the system**, while on the right hand side $f(t)$ we have the **external action**, the input of the system.

Properties

Equation 1.4 represent an **ordinary linear differential equation** in the spatial dimension $x(t)$ with **constant coefficients** for the differential terms. In the particular case presented, having a non-zero term on the right-hand side we have that the differential equation is **non-homogeneous**.

General method for solving ordinary linear differential equations with constant coefficients

Given the linear ordinary differential equation with constant coefficient

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u \tag{1.5}$$

where u represent the input of a system and y it's output. Such system requires to known all the past history of the input to build the output, however considering it's states x. In order to find the solution we have to consider the following hypothesis:

$$\begin{aligned} i) &\qquad u(t) = 0 \qquad \text{for } t < 0 \\ ii) &\qquad u(t)\text{is piecewise continuous} \\ iii) &\qquad \text{states of the systems are known for } t = 0 \end{aligned}$$

where the n states x_i are regarded as the output y and all it's derivative up to order $n-1$, hence

$$x_1 = y(0),\ x_2 = \dot{y}(0),\ \ldots,\ x_n = y^{(n-1)}(0)$$

If all this conditions are satisfied we can fined the solution $y(t)$ of the system as

$$y(t) = y_{homogeneous}(t) + y_{particular}(t) \tag{1.6}$$

Solution of the homogeneous The solution of the homogeneous is obtained by neglecting the input (right-hand side of the differential equation) and so considering only

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = 0$$

If we now consider each differentiation as a polynomial in a complex variable $\lambda \in \mathbb{C}$ we have that

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0\lambda^0 = 0$$

The polynomial $p(\lambda)$ has n roots that each can have a multiplicity μ_i; associated to each root we have an associated homogeneous solution described as

$$y_{h,1} = c_{11}e^{\lambda_1 t} + c_{12}te^{\lambda_1 t} + \cdots + c_{1\mu_1}t^{\mu_1 - 1}e^{\lambda_1 t}$$
$$y_{h,2} = c_{21}e^{\lambda_2 t} + c_{22}te^{\lambda_2 t} + \cdots + c_{2\mu_2}t^{\mu_2 - 1}e^{\lambda_2 t}$$

hence in general for N different roots we have

$$y_h(t) = \sum_{i=1}^{N}\sum_{j=1}^{\mu_i} c_{ij}t^{j-1}e^{\lambda_i t} \tag{1.7}$$

Particular solution To determine the solution it's necessary to find a function y that satisfy the whole differential equation.

Laplace transform approach

Solution of linear ordinary differential equation can be obtained considering the **Laplace transform**

$$\mathscr{L}\{f(t)\} = F(s) := \int_0^\infty f(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

that transform differential equation in t into algebraic equations in s, in fact it's proven that

$$\mathscr{L}\left\{f^{(n)}\right\} = s^n F(s) - \sum_{i=0}^{n-1} s^{n-i-1}f^{(i)}(0)$$

The idea is so to the find the solution in the domain of s and then find the solution in time by using the inverse Laplace transform

$$f(t) = \mathscr{L}^{-1}\{F(s)\} := \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}\,ds \qquad \text{with } j = \sqrt{-1} \tag{1.8}$$

This transformation is very useful due to the linearity property of $\mathscr{L}$, in fact

$$\mathscr{L}\{a\,f(t) + b\,g(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\} \qquad \forall a, b \in \mathbb{C}$$

Using techniques in time we seen that in such domain the output of the system is the combination of terms in the form $t^n e^{at}$ whose transform is

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \forall n \in \mathbb{N}, a \in \mathbb{C} \tag{1.9}$$

Applying this rule we can determine such common examples of Laplace transform as

$$\mathscr{L}\{1(t)\} = \frac{1}{s} \qquad a = n = 0$$
$$\mathscr{L}\{\sin(\omega t)\} = \mathscr{L}\left\{\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right\} = \frac{\omega}{s^2 + \omega^2} \qquad n = 0, a = \pm j\omega$$

Application of the Laplace transform Considering the differential equation 1.4 (page 2) defined as $m\ddot{x} + c\dot{x} + kx = f(t)$, we can find the solution in the s domain applying the Laplace transform on such equation, determining:

$$m\big(s^2X(s) - sx(0) - \dot{x}(s)\big) + c\big(sX(s) - x(0)\big) + kX(s) = F(s)$$
$$\big(ms^2 + cs - k\big)X(s) = msx(0) + m\dot{x}(0) + cx(0) + F(s)$$

We have so a polynomial problem in s that can be solved determining

$$X(s) = \underbrace{\frac{msx_0 + m\dot{x}_0 + cx_0}{ms^2 + cs + k}}_{\text{homogeneous sol.}} + \underbrace{\frac{1}{ms^2 + cs + k}F(s)}_{\text{particular sol.}}$$

The first term is usually called **free response** while the second is the **forced response**; the term $\frac{1}{ms^2+cs+k}$ represent the **transfer function** of the system. From this expression we define the **natural angular frequency** ω_n and the **damping ratio** ζ defined as

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2c\omega_n} \tag{1.10}$$

This parameters allows to rewrite the response of the system as

$$X(s) = \frac{x_0 s + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}F(s) \tag{1.11}$$

In general the solution in the Laplace domain can be described as a sum of the homogeneous solution and the particular one as rational polynomial in the form

$$X(s) = \frac{A(s)}{B(s)} + \frac{N(s)}{D(s)}U(s)$$

We refer to the roots of the nominators $A(s), N(s)$ as **zeros** while the one of the denominators $B(s), D(s)$ are the **poles**.

Returning to the example, the poles p_i of the homogeneous term in equation 1.11 can be simply calculates as

$$p_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of the damping ratio ζ the poles can be real ($\zeta > 1$) or complex conjugated (for $\zeta < 1$). In particular the cases are

- if $\zeta = 0$ we have the poles in $p_{1,2} = \pm j\omega_n$ that are purely imaginary and each pole has a multiplicity $\mu = 1$;
- if $0 < \zeta < 1$ the poles are $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$ and so are complex conjugated. The magnitude of such complex root is

$$|p_{1,2}| = \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)} = \omega_n$$

- if $\zeta = 1$ we have a double root (multiplicity $\mu = 2$) in $p_{1,2} = -\omega_n$;
- if $\zeta > 1$ the poles are purely real $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ and their multiplicity is unitary ($\mu = 1$).

Time response of the homogeneous Considering the homogeneous term, assuming the initial state as zeros ($x_0 = 0$) for simplicity we have

$$X_h(s) = \frac{\dot{x}_0}{s^2 + 2j\omega_n s + \omega_n^2}$$

Applying the partial fraction decomposition we can obtain a transfer function in the form (where we consider that all poles have multiplicity $\mu = 1$, so the case $\zeta = 1$ cannot be described by this expression):

$$X_h(s) = \frac{R_{1h}}{s - p_1} + \frac{R_{2h}}{s - p_2}$$

We can compute the residuals as

$$R_{1h} = [(s-p_1)X_h(s)]\big|_{s=p_1} = \frac{\dot{x}_0}{p_1-p_2}$$
$$R_{2h} = [(s-p_2)X_h(s)]\big|_{s=p_2} = \frac{\dot{x}_0}{p_2-p_1}$$

Considering the result of equation 1.9 we can consider the partial fraction decomposition with parameters $n=0$ and $a=p_1,p_2$. In particular considering the linearity we have that

$$x_h(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = R_{1h}t^0e^{p_1t} + R_{2h}t^0e^{p_2t} = R_{1h}e^{p_1t} + R_{2h}e^{p_2t}$$

Depending on the *type* of the poles (real or imaginary), the general behaviour of the system might differs and we have to use mathematical complex equation to determine the final expression in time. Furthermore if $0<\zeta<1$ we have complex conjugated poles in the form $p_{1,2} = -\zeta\omega_n \pm j\omega_d$ (with $\omega_d = \omega_n\sqrt{1-\zeta^2}$ the ***damped natural frequency***), hence the residuals becomes

$$R_{1h} = \frac{\dot{x}_0}{p_1-p_2} = \frac{\dot{x}_0}{2j\omega_d}\frac{-j}{-j} = -j\frac{\dot{x}_0}{2\omega_d} \qquad R_{2h} = j\frac{\dot{x}_0}{2\omega_d}$$

Observing that the residual are also complex conjugated ($R_{2h} = R_{1h}^*$) as the poles, we can write the full solution of the homogeneous in the time domain as

$$\begin{aligned} x_h(t) &= R_{1h}e^{p_1t} + R_{2h}e^{p_2t} \\ &= 2\frac{\dot{x}_0}{2\omega_d}e^{-\zeta\omega_nt}\cos\left(-\frac{\pi}{2}+\omega_dt\right) = \frac{\dot{x}_0}{\omega_d}e^{-\zeta\omega_nt}\sin(\omega_dt) \end{aligned} \tag{1.12}$$

This is a *pseudo-periodic* function with period $T = \frac{2\pi}{\omega_d}$.

Time response of the particular solution Considering the particular solution that in the time domain is expressed as

$$X_p(s) = \frac{1/m}{s^2+2\zeta\omega_ns+\omega_n^2}F(s)$$

Considering the input as a unit step function of amplitude F_0, we have that it's transform is determined by

$$F(s) = \mathscr{L}\left\{F_0u(t)\right\} = F_0\frac{1}{s} \qquad \Rightarrow \qquad X_p(s) = \frac{F_0}{m}\frac{1}{s^2+2\zeta\omega_ns+\omega_n^2}\frac{1}{s}$$

The poles of the overall systems are so $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$, $p_3 = 0$, rewriting the particular solution with the related partial fraction decomposition

$$X_p(s) = \frac{F_0}{m}\frac{1}{(s-p_1)(s-p_2)}\frac{1}{s} = \frac{R_{1p}}{s-p_1} + \frac{R_{2p}}{s-p_2} + \frac{R_{3p}}{s}$$

where the decomposition in made on the assumption of unitary multiplicity ($\mu_i = 1$ hence $\zeta \neq 1$). Computing

$$R_{1p} = \left[(s-p_1)X_p(s)\right]\big|_{s=p_1} = \frac{F_0}{m}\frac{1}{(p_1-p_2)p_1} \qquad R_{2p} = \left[(s-p_2)X_p(s)\right]\big|_{s=p_2} = \frac{F_0}{m}\frac{1}{(p_2-p_1)p_2}$$
$$R_{3p} = \left[sX_p(s)\right]\big|_{s=0} = \frac{F_0}{m}\frac{1}{p_1p_2}$$

The particular solution in the time domain is so obtained with the inverse Laplace transform

$$x_p(t) = \mathscr{L}^{-1}\left\{\frac{R_{1p}}{s-p_1} + \frac{R_{2p}}{s-p_2} + \frac{R_{3p}}{s}\right\} = R_{1p}e^{p_1t} + R_{2p}e^{p_2t} + R_{3p}e^{p_3t}$$

In particular when we have $0 < \zeta < 1$ we have to complex conjugated poles $p_{1,2} = -\zeta\omega_n \pm j\omega_d$ (with $\omega_d = \omega_n\sqrt{1-\zeta^2}$) determines the residuals $R_{1p} = \frac{F_0}{m}\frac{1}{2j\omega_d}\frac{1}{-\zeta\omega_n + j\omega_d}$, $R_{2p} = \frac{F_0}{m}\frac{1}{-2j\omega_d}\frac{1}{-\zeta\omega_n - j\omega_d}$ and $R_{3p} = \frac{F_0}{m}\frac{1}{\omega_n^2} = \frac{F_0}{k}$; we so have that

$$\begin{aligned} x_p(t) &= |R_{1p}| e^{\mathbb{R}\mathrm{e}\{p_1\}t} \cos\left(\mathbb{I}\mathrm{m}\{p_1\}t + \arg\{R_{1p}\}\right) + R_{3p}e^{0t} \qquad\qquad \text{with } \zeta \in (0,1) \\ &= \frac{F_0}{m}\frac{1}{\omega_n^2\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos\left(\omega_d t - \frac{\pi}{2} - (\pi - \theta)\right) + \frac{F_0}{k} \\ &= \frac{F_0}{k}\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos\left(\omega_d t + \theta - \frac{3}{2}\pi\right) + \frac{F_0}{k} \\ &= \frac{F_0}{k}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t) \end{aligned}$$

Overall system response Hypothesising as initial condition $x_0 = 0$ and $\dot{x}_0 \neq 0$ with a damping $\zeta \in (0,1)$, then the overall response of the system in the Laplace domain can be decomposed in two parts: one associated to the transients (decaying exponential) and one relegated to the steady state response of the system:

$$X(s) = \underbrace{\frac{R_{1h}}{s-p_1} + \frac{R_{2h}}{s-p_2} + \frac{R_{1p}}{s-p_1} + \frac{R_{2p}}{s-p_2}}_{\text{transient}} + \underbrace{\frac{R_{3p}}{s}}_{\text{steady state}} \tag{1.13}$$

This relates to the following time response:

$$\begin{aligned} x(t) &= \left[\underbrace{\frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)}_{\text{homogeneous}} + \underbrace{\frac{F_0}{k}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right)}_{\text{particular}}\right] u(t) \\ &= \left[\underbrace{\frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t) - \frac{F_0}{k}\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)}_{\text{transient}} + \underbrace{\frac{F_0}{k}}_{\text{steady state}}\right] u(t) \end{aligned} \tag{1.14}$$

1.1.2 Frequency response of a system

We can analyse systems also considering inputs $u(t)$ that are sinusoidal, hence in the form $M\sin(\omega t)$ characterized by a transform

$$U(s) = M\frac{\omega}{s^2 + \omega^2}$$

Considering that the output in the Laplace domain is in the form $Y(s) = \frac{A(s)}{B(s)} + \frac{N(s)}{D(s)}U(s)$ where $N(s)/D(s) = G(s)$ is the **transfer function** of the system. Assuming that the roots of both $B(s)$ and $D(s)$ are all *stable* (meaning that doesn't have positive real part), we have the response that for a 2^{nd} order system can be regarded as

$$\begin{aligned} Y(s) &= \frac{A(s)}{B(s)} + G(s)M\frac{\omega}{s^2+\omega^2} \\ &= \frac{A(s)}{(s-p_1)(s-p_2)} + \frac{N(s)}{(s-p_1)(s-p_2)}M\frac{\omega}{(s-j\omega)(s+j\omega)} \\ &= \frac{R_{1h}}{s-p_1} + \frac{R_{2h}}{s-p_2} + \frac{R_{1p}}{s-p_1} + \frac{R_{2p}}{s-p_2} + \frac{R_{3p}}{s-j\omega} + \frac{R_{4p}}{s+j\omega} \end{aligned}$$

By applying the reverse Laplace transform

$$y(t) = \underbrace{R_{1h}e^{p_1 t} + R_{2h}e^{p_2 t} + R_{1p}e^{p_1 t} + R_{2p}e^{p_2 t}}_{\text{same time constant } \tau \text{ of decay}} + R_{3p}e^{j\omega t} + R_{4p}e^{-j\omega t}$$

Neglecting the transient we have that the resulting output is a linear combination of exponential in the form $y(t)R_{3p}e^{j\omega t} + R_{4p}e^{-j\omega t}$; considering that $R_{4p} = R_{3p}^*$ we obtain that $y(t) = 2|R_{3p}|e^{0t}\cos(\omega t + \arg\{R_{3p}\})$ (for $t \gg \tau$). We can compute the residuals as

$$R_{3p} = \left[\cancel{(s - j\omega)} G(s) M \frac{\omega}{\cancel{(s - j\omega)}(s + j\omega)}\right]\Bigg|_{s=j\omega} = \frac{G(j\omega)M}{2j} \qquad R_{4p} = -\frac{G(-j\omega)M}{2j}$$

This results in the following steady state response in the time domain representing the so called **frequency response function**:

$$y_{ss}(t) = 2\frac{M}{2}|G(j\omega)|e^{0t}\cos\left(\omega t + \arg\{G(j\omega)\} - \frac{\pi}{2}\right) = MG(j\omega)\sin(\omega t + \varphi(j\omega)) \tag{1.15}$$

where $|G(j\omega)$ is the magnitude and $\varphi(j\omega)$ is the phase of the transfer function. Graphically a sinusoidal input is *amplified* by a factor $G(j\omega)$ (that so depends on the pulsation ω of the input) and *shifted* by a time $\frac{2\pi}{\varphi(j\omega)}$ (the peak of the output is *on the right* respect to the input in order to have causal systems). The frequency response function, given the transfer function $G(s)$, is computed considering $s = j\omega$; as example if we have

$$G(s) = \frac{1}{m}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \Rightarrow \qquad G(j\omega) = \frac{1}{m}\frac{1}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n}}$$

$$\Rightarrow \qquad |G(j\omega)| = \frac{1/k}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \qquad \varphi(j\omega) = -\arctan\left(\frac{2j\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Usually $|G(j\omega)|$ and $\varphi(j\omega)$ are represented in **Bode plots VEDERE APPUNTI**; defining the quality factor Q as the magnitude at the maximum frequency $k|G(j\omega_{peak})|$ and so

$$Q = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \xrightarrow{\zeta \text{ small}} \quad Q \approx \frac{1}{2\zeta}$$

Such approximation allows to experimentally determine the damping coefficient of the system; **VEDERE GRAFICO APPUNTI**

Convolution integral

Analysing systems in the Laplace domain we observed that the output $Y(s)$ is achieved by multiplying the transfer function $G(s)$ by the transform of the input $U(s)$ in case of zero initial conditions:

$$Y(s) = G(s)U(s)$$

If we determine an input $u(t)$ characterized by the transform $U(s)$, what will we observe is that the output matches the transfer function: $Y(s) = G(s)1 = G(s)$. Building the function

$$f(t,\tau) = \frac{1}{\tau}\Big[u_s(t) - u_s(t-\tau)\Big]$$

what we obtain is a rectangular pulse of amplitude $1/\tau$ and width τ (given $u_s(t)$ the unit step function); by pushing $\tau \to 0$, f tends to the **Dirac pulse/delta** $\delta(t)$, a distribution characterized by being zero $\forall t \neq 0$ where it evaluates at ∞; such function is characterized by having $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$ and has the important property $\int_{\infty}^{\infty} g(t)\delta(t)\, dt = g(0)$ for any function $g(t)$. To compute the transform of f respect to t we can use the linear property of the Laplace transform and we obtain

$$F(s,\tau) = \frac{1}{\tau}\frac{1}{s} - \frac{1}{\tau}\frac{1}{s}e^{-\tau s} = \frac{1}{\tau s}\left(1 - e^{-\tau s}\right) \quad \xrightarrow{\tau\to 0} \quad \lim_{\tau\to 0} \frac{1-e^{-\tau s}}{\tau s}$$

Using De L'Hopital rule what we obtain is that

$$F(s, \tau \to 0) = \lim_{\tau\to 0} \frac{se^{-\tau s}}{s} = 1$$

By determining the output of the system subjected to a Dirac delta as input, the output that we obtain is the **impulse response** of the system $g(t) = y(t)$.

Convolution theorem Considering the convolution theorem stating that

$$\mathscr{L}\left\{\int_0^\infty f_1(\theta) f_2(t-\theta)\, d\theta\right\} = F_1(s)F_2(s) \tag{1.16}$$

by applying the inverse transformation we of course get the result $\mathscr{L}^{-1}\{F_1(s)F_2(s)\} = \int_0^\infty f_1(\theta)f_2(t-\theta)\, d\theta$. Knowing so that in the Laplace domain $Y(s) = G(s)U(s)$ we obtain that the output of the system in the time domain can be thought as the convolution between impulse response and input:

$$y(t) = \mathscr{L}^{-1}\{G(s)U(s)\} = \int_0^\infty g(\theta)u(t-\theta)\, d\theta \tag{1.17}$$

In practise stable systems are characterized by an impulse response that present a decay with a certain time constant τ (and so has a finite *memory* on the history of the inputs) and by a computational point of view we don't have to compute the integral in the domain $(0, \infty)$, but $(0, t)$ is still ok.

Alternative formulation An alternative formulation of the convolution theorem is that the output can be regarded as

$$y(t) = \int_0^t u(\theta) g(\theta - t)\, d\theta$$

In practise this means that the convolution operator $*$ is commutative, in the sense that

$$y(t) = (u * g)(t) = (g * u)(t)$$

1.1.3 2 degrees of freedom

Considering now the case of a **two degrees of freedom system** as shown in **FIGURE DA FARE**, we can see that the dynamical equations of the system are

$$\begin{cases} m\ddot{x}_1 = -k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + f_2 \\ m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) - k_3x_2 - c_3\dot{x}_2 + f_2 \end{cases}$$

We can see that such system is linear regarding the positions x_i and it's derivative, in fact it can be considered in a matrix notation as

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}}_{\mathbf{C}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}}_{\mathbf{K}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{1.18}$$

where $\mathbf{M}, \mathbf{C}, \mathbf{K}$ are respectively the **mass**, **damping** and **stiffness** matrix. If we regard $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{f} = (f_1, f_2)$ the previous expression can be reduced to the form

$$\mathbf{M}\ddot{\boldsymbol{x}} + \mathbf{C}\dot{\boldsymbol{x}} + \mathbf{K}\boldsymbol{x} = \boldsymbol{f} \tag{1.19}$$

As in the analysis of 1 degree of freedom systems, the solution can be obtained by applying the Laplace transform on this expression that's now a system of linear differential equations, resulting in:

$$\mathbf{M}\Big(s^2\boldsymbol{X}(s) - s\boldsymbol{x}_0 - \dot{\boldsymbol{x}}_0\Big) + \mathbf{C}\Big(s\boldsymbol{X}(s) - \boldsymbol{x}_0\Big) + \mathbf{K}\boldsymbol{x}(s) = \boldsymbol{F}(s)$$

that solved for the unknown $\boldsymbol{X}(s)$ determines the solution

$$\boldsymbol{X}(s) = \underbrace{\overbrace{\left(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}\right)^{-1}}^{=\mathbf{G}(s)}\Big((\mathbf{M}s + \mathbf{C})\boldsymbol{x}_0 + \mathbf{M}\dot{\boldsymbol{x}}_0\Big)}_{\text{homogeneous solution}} + \underbrace{\left(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}\right)^{-1}\mathbf{F}(s)}_{\text{particular sol.}} \tag{1.20}$$

In this representation $\mathbf{G}(s)$ is the **transfer function**, a 2×2 matrix where each entry results in a rational polynomial in the complex variable s; such matrix can be regarded as the inverse of the matrix $\mathbf{D}$ defined as

$$\mathbf{D} = \mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2 s - k_2 \\ -c_s s - k_2 & m_2 s^2 + (c_2 + c_3)s + k_2 + k_3 \end{bmatrix}$$

$$\Rightarrow \qquad \mathbf{G}(s) = \mathbf{D}^{-1} = \frac{1}{\det \mathbf{D}} \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

Particular solution DA VEDERE

Homogeneous solution Considering now only the homogeneous term of the solution and assuming as initial states $\boldsymbol{x}_0 = (x_{10}, 0)$ and $\dot{\boldsymbol{x}}_0 = (0, 0)$, what we obtain is

$$\begin{aligned} X_{hom}(s) &= \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} m_1 s + c_1 & 0 \\ 0 & m_2 s + c_3 \end{bmatrix} \begin{pmatrix} x_{10} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} G_{11}(s)(m_1 s - c_1)x_{10} \\ G_{21}(s)(m_1 s + c_3)x_{10} \end{pmatrix} = \begin{pmatrix} X_{h1}(s) \\ X_{h2}(s) \end{pmatrix} \end{aligned}$$

We can so compute separately the two components X_{h1}, X_{h2} of the solution. Knowing that the entry G_{11} is expressed as $\frac{d_{22}}{d_{11}d_{22} - d_{12}d_{21}}$, by applying all the substitutions we obtain a rational polynomial of degree 4:

$$X_{h1}(s) = \frac{(m_1 s + c_1)(m_2 s^2 + c_3 s + k_2 + k_3)x_{10}}{(m_1 s^2 + c_1 s + k_1 + k_2)(m_2 s^2 + c_3 s + k_2 + k_3) - k_2^2}$$

Such function has 4 roots p_1, p_2, p_3, p_4 of the denominator that can both be real or complex conjugated; assuming that such values are all distinct (meaning that the multiplicity of all poles is unitary) the resulting partial fraction decomposition is

$$X_{h1}(s) = \underbrace{\frac{R_{11}}{s - p_1} + \frac{R_{12}}{s - p_2}}_{X_{11}(s) \mapsto x_{11}(t)} + \underbrace{\frac{R_{13}}{s - p_3} + \frac{R_{14}}{s - p_4}}_{X_{13}(s) \mapsto x_{13}(t)}$$

where usually the underlined terms are considered together because in for **RIVEDERE DEFINIZIONE** critical the poles are complex conjugated and their combined time response results in an exponentially decaying sinusoidal signal. Similarly we can compute the response of the second component $X_{h2}(s)$ of the solution reaching the partial fraction expansion

$$X_{h2}(s) = \underbrace{\frac{R_{21}}{s - p_1} + \frac{R_{22}}{s - p_2}}_{X_{21}(s) \mapsto x_{21}(t)} + \underbrace{\frac{R_{23}}{s - p_3} + \frac{R_{24}}{s - p_4}}_{X_{23}(s) \mapsto x_{23}(t)}$$

This means that the homogeneous response (considering the preliminary assumption formulated) is in the form

$$X_{hom}(s) = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} e^{p_1 t} + \begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix} e^{p_2 t} + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} e^{p_3 t} + \begin{pmatrix} R_{14} \\ R_{24} \end{pmatrix} e^{p_4 t}$$

MANCA DA FINIRE LA LEZIONE DEL 21 MARZO

Free vibration and natural frequencies

Let's now consider the case where are dumping coefficients c_i are zero, hence the matrix $\mathbf{C}$ is identically null. Considering so the homogeneous response assuming as non-zero only x_{10}, the previous results simplifies and the rational polynomial has a bi-quadratic denominator (**RIVEDERE**):

$$X_{h1}(s) = \frac{s^3 + s\frac{k_2+k_3}{m_2}}{s^4 + \left(\frac{k_2+k_3}{m_2} + \frac{k_1+k_2}{m_1}\right) s^2 + \frac{k_1k_2+k_2k_3+k_3k_1}{m_1m_2}} x_{10}$$

In this case roots are purely imaginary and evaluates to $\pm j\omega_1$ and $\pm j\omega_2$: such values ω_i are the so called **natural frequencies** of the system, meaning that if the system is not subjected to any force, it will vibrate as a linear combination of sinusoidal functions pulsating at such frequencies. Having that the homogeneous response can be regarded as

$$X_{hom}(s) = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix} e^{p_1 t} + \begin{pmatrix} R_{12} \\ R_{22} \end{pmatrix} e^{p_2 t} + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} e^{p_3 t} + \begin{pmatrix} R_{14} \\ R_{24} \end{pmatrix} e^{p_4 t}$$

and so applying the Laplace transform the free response is made by the components

$$\begin{aligned} x_{h1}(t) &= 2|R_{11}| \cos\left(\omega_1 t + \arg\{R_{11}\}\right) + 2|R_{13}| \cos\left(\omega_1 t + \arg\{R_{13}\}\right) \\ x_{h2}(t) &= 2|R_{21}| \cos\left(\omega_2 t + \arg\{R_{21}\}\right) + 2|R_{23}| \cos\left(\omega_1 t + \arg\{R_{23}\}\right) \end{aligned}$$

In this case where there's no damping, is mathematically proven that the arguments of corresponding residuals $R_{1i} \leftrightarrow R_{2i}$ can only be in phase or in phase opposition. Considering $\arg\{R_{1i}\} = \alpha$ then it means that $\arg\{R_{2i}\}$ can only be α or $\alpha + \pi$. Considering that $\cos(\alpha+\pi) = -\cos\alpha$, we can rewrite the the time response as function of coefficients u_{ij} that now can be negative as

$$x_{hom} = \underbrace{\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \cos(\omega_1 t + \alpha)}_{\text{I mode}} + \underbrace{\begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} \cos(\omega_2 t + \beta)}_{\text{II mode}} \tag{1.21}$$

The underlined terms are the so called **natural modes** of vibrations of the system; in general any n degrees of freedom system results in n different natural frequencies with n vectors $\boldsymbol{u}_i \in \mathbb{R}^n$ associated to the modal responses. Such natural modes allows to fully describe the homogeneous response of the systems, in fact any free oscillation is a linear combination of linear modes. Considering in fact the starting systems model

$$\mathbf{M}\ddot{\boldsymbol{x}} + \mathbf{K}\boldsymbol{x} = \mathbf{0}$$

substituting the first mode of the system (respect to that is shown in equation 1.21) we determine

$$-\omega_1^2 \mathbf{M} \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \cos(\omega_1 t + \alpha) + \mathbf{K} \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} \cos(\omega_1 t + \alpha) = \mathbf{0}$$

In order to obtain a true relationship independent on the time t we consider we have to solve the eigenvalue-eigenvector problem of the form

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \boldsymbol{u} = \mathbf{0}$$

We can in fact compute the characteristic polynomial as function of ω^2 and this determines the natural frequencies ω_1, ω_2 (having a polynomial in ω^2, each root ω_i^2 of the characteristic polynomial results in two opposite value ω_i and $-\omega_i$). Determined such parameters we can compute the eigenvectors $\boldsymbol{u}_i = (u_{1i}, u_{2i})$ associated to the system. Such vectors are proven to be orthogonal, in fact considering that they are solution of the system $-\omega_i^2 \mathbf{M}\boldsymbol{u}_i + \mathbf{K}\boldsymbol{u}_i = \mathbf{0}$, by firstly considering the first mode and premultiplying it by $\boldsymbol{u}_2^T$ results in the scalar

$$-\omega_1^2 \boldsymbol{u}_2^t \mathbf{M}\boldsymbol{u}_1 + \boldsymbol{u}_2^T \mathbf{K}\boldsymbol{u}_1 = 0$$

Knowing that a scalar transposed coincides with himself, applying the rules of the transposition of multiplication what we obtain is that

$$-\omega_1^2 \boldsymbol{u}_1^T \mathbf{M}^T \boldsymbol{u}_2 + \boldsymbol{u}_1^T \mathbf{K}^T \boldsymbol{u}_2 = 0$$

If we moreover consider that the matrices $\mathbf{M}, \mathbf{K}$ are mainly symmetric

$$(I) \qquad -\omega_1^2 \boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_2 + \boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_2 = 0$$

Considering instead now the values obtained using the second mode and premultiplying everything by $\boldsymbol{u}_1^T$ what we get is

$$(II) \qquad -\omega_2^2 \boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_2 + \boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_2 = 0$$

Computing the difference $(II) - (I)$ we obtain

$$(\omega_2^2 - \omega_1^2)\boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_2 + \cancel{\boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_2 - \boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_2} = 0$$

We so obtained that the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are representing orthogonal modes respect to the mass matrix $\mathbf{M}$, but it happens so also with respect to the stiffness matrix $\mathbf{K}$, in fact

$$\cancel{-\omega_1^2 \boldsymbol{u}_2^T \mathbf{M}\boldsymbol{y}_1} + \boldsymbol{u}_2^T \mathbf{K}\boldsymbol{u}_1 = 0 \qquad \Leftrightarrow \qquad \boldsymbol{u}_2^T \mathbf{K}\boldsymbol{u}_1 = 0$$

Regarding $\cos(\omega_1 t + \alpha)$ as a function $q_1(t)$ and similarly $\cos(\omega_2 t + \beta) = q_2(t)$, then the homogeneous response of the system can be described as a linear combination of orthogonal modes

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathbf{U}\boldsymbol{q} \tag{1.22}$$

where $\mathbf{U}$ is the **modal matrix**.

Considering now the more general case of the dynamical system modelled as $\mathbf{M}\ddot{\boldsymbol{x}} + \mathbf{K}\boldsymbol{x} = \boldsymbol{f}$, if we consider the states $\boldsymbol{x}$ as the linear combination of the modal responses what we obtain is that we have *decoupled* differential equations, in fact

$$\begin{aligned} \mathbf{M}(\boldsymbol{u}_1\ddot{q}_1 + \boldsymbol{u}_2\ddot{q}_2) + \mathbf{K}(\boldsymbol{u}_1 q_1 + \boldsymbol{u}_2 q_2) &= \boldsymbol{f} \\ \boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_1\ddot{q}_1 + \cancel{\boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_2\ddot{q}_2} + \boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_1 q_1 + \cancel{\boldsymbol{u}_1^T \mathbf{K}\boldsymbol{u}_2 q_2} &= \boldsymbol{u}_1^T \boldsymbol{f} \\ \boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_1\ddot{q}_1 + \omega_1^2 \boldsymbol{u}_1^T \mathbf{M}\boldsymbol{u}_1 q_1 &= \boldsymbol{u}_1^T \boldsymbol{f} \end{aligned}$$

Considering that each product $\boldsymbol{u}_i^T \mathbf{M}\boldsymbol{u}_i$ evaluates to a scalar b_i representing the so called **modal mass** of the system, but that also $\boldsymbol{u}_i^T \boldsymbol{f}$ evaluates to the **modal force** f_i, we obtain the *compact* formulation

$$b_1\ddot{q}_1 - \omega_1^2 b_1 q_1 = f_i$$

Appendix A

Recall: Complex Numbers