SINGULAR SOLUTIONS

$$\dot{x} = f(x, u, t)$$

$$x(t_i) = x_i$$

$$J = \int_{t_i}^{t_f} L(x, u, t)\, dt$$

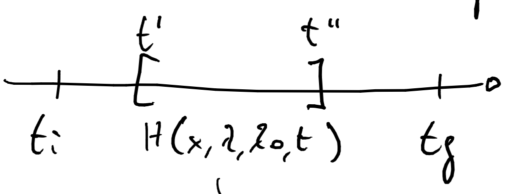
We write the Hamiltonian

$$H = \lambda_0 L + \lambda^T f = \lambda_0 L(x,u,t) + \lambda^T(t) f(x,u,t)$$

<u>Definition</u>: Let $(x^\circ, u^\circ, t_f^\circ)$ be an optimal solution of the above problem, and $\lambda_0^\circ, \lambda^\circ$ the corresponding multipliers.

The solution is <u>singular</u> if $\exists$ a subinterval $[t', t'']$, $t'' > t'$ in which the Hamiltonian

$H(x^\circ(t), w, \lambda_0^\circ, \lambda^\circ(t), t)$ is <u>independent</u> from at least one component of w in $[t'; t'']$



In this interval the Hamiltonian does not depend on the control (<u>Bad thing</u>)

<u>Idea</u>: $\exists$ a subset in which the Hamiltonian does not depend on the control

$$\frac{\partial H}{\partial u} = 0^T \implies \underline{\text{Singularity depends on } f \text{ and } L}$$

The idea is to find a cost index J that avoid

To do that we split the Hamiltonian in 2 parts, one depending on the state and one on the control.

Theorem: Assume the Hamiltonian of the form:

$$H(x,u,\lambda_0,\lambda,t) = H_1(x,\lambda_0,\lambda,t) + H_2(x,\lambda_0,\lambda,t)\,N(x,u,\lambda_0,\lambda,t)$$

Let (x^*, u^*, t_f^*) be an extremum and λ_0^*, λ^* the multipliers such that $N(x^*, \omega, \lambda_0^*, \lambda^*, t)$ is dependent on any component ω in any subinterval $[t_i, t_f]$

A nec. & suff. condition for (x^*, u^*, t_f^*) to be a singular extremum is that $\exists$ a subinterval $[t', t''] \subset [t_i, t_f]$, $t'' > t'$ such that
$H_2(x, \lambda_0, \lambda, t) = 0$, $\forall t \in [t', t'']$

example

$\dot{x} = Ax + Bu \qquad |u| \le 1$

$x(t_i) = x_i$

$J = \int_{t_i}^{t_f} 1 \cdot dt$ $\rightsquigarrow$ minimum time problem

Hamiltonian: $H = \underbrace{\overbrace{\lambda_0 1}^{\mathcal{L}} + \lambda^T A x}_{H_1} + \underbrace{\lambda^T B}_{H_2}\underbrace{u}_{N}$

$H_1 \to$ does not depend on u

$\lambda^T B \rightsquigarrow$ If in a subinterval this quantity $= 0$, we have a singular solution

$\downarrow$

this never occurs with a special hypothesis

~ The linear minimum time optimal control

OC of a linear sys with:

- fixed initial & final state
- constrained control
- cost index equal to the lenght of the time interval

<u>Problem</u>: $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $|u_j(t)| \leq 1$, $j = 1, 2, \dots, p$

$$\underbrace{A(t) \in C^{n-2},\ B(t) \in C^{n-1}}_{\text{at least } C^1 \text{ class}} \quad \forall t \in \mathbb{R}$$

$$x(t_i) = x_i \ , \ x(t_f) = 0$$

The <u>aim</u> is to determine the <u>final instant</u> $t_f \in \mathbb{R}$ and the <u>control</u> $u^\circ \in \bar{C}^0(\mathbb{R})$ → continuous almost everywhere and the <u>state</u> $x^\circ \in \bar{C}^1(\mathbb{R})$, minimizing:

$$J(t_f) = \int_{t_i}^{t_f} \underbrace{\mathcal{L}}_{\mathcal{L}=1}\, dt = t_f - t_i$$

<u>Theorem</u>: <u>Nec conditions</u> for (x^*, u^*, t_f^*) to be an optimal solution are that $\exists$ a constant $\lambda_0^* \geq 0$ and a n-dimensional function $\lambda^* \in \bar{C}^1[t_i, t_f]$ not simultaneously null and such that:

$$\dot{\lambda}^* = -A^T \lambda^* = -\frac{\partial H}{\partial x}^T$$

$$\lambda^{*T} B w \geq \lambda^{*T} B u^* \quad \forall w \in \mathbb{R}^p : |w_j| \leq 1, \ j = 1, 2 \dots p$$

→ Pontryagin principle

Possible discontinuities in λ^* can appear only in the points in which u^* has a discontinuity.

Moreover we have the fixed final state $x(t_f)=0$ but we don't have t_f, therefore we have the trasversality condition

$$H\Big|_{t_f^*}^{*} = 0$$

Proof

The Hamiltonian associated to the problem is:

$$H = \lambda_0 + \lambda^T A x + \lambda^T B u$$

Applying the minimum principle

$$\dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T$$

$$H(\dots w \dots) \geq H(\dots u \dots) \quad \forall |w| \leq 1$$

and applying the trasversality conditions, the theorem is proved

End

~ Strong controllability

It's a new hypothesis.

When we had $\dot{x}(t) = A x(t) + B u(t)$, a system in the steady state case, with A and B constant matrices, for the controllability we had

rank $\{(B \ AB \ \dots \ A^{n-1}B)\}$

Now we have $A(t)$ and $B(t)$ so it's not possible anymore.

<u>Strong controllability</u> corresponds to the controllability in any instant <u>ti</u>, in any <u>time interval</u> and by any component of the <u>control vector</u>

Let us indicate with $b_j(t)$, the j-th column of $B(t)$

$$B(t) = \begin{pmatrix} | & | & \\ b_1(t) & b_2(t) & \dots \\ | & | & \end{pmatrix}$$

j-th column $\rightarrow b_2(t) = b_j^{(1)}(t)$

$$G_j(t) = \begin{pmatrix} | & | & \\ b_j^{(1)}(t) & b_j^{(2)}(t) & \dots \\ | & | & \end{pmatrix} \rightsquigarrow$$ If the det $\neq 0$ $\forall t \geq t_i$ the strong controllability is <u>guaranteed</u>

$$\begin{cases} b_j^{(1)}(t) = b_j(t) \\ \vdots \\ b_j^{(k)}(t) = \dot{b}_j^{(k-1)}(t) - A(t)b_j^{(k-1)}(t) \end{cases}$$

$k = 2, 3, \dots, n$

- In the <u>NON steady state</u> case is a <u>suff</u> condition for strong controllability
- In the <u>steady state</u> case is <u>nec & suff</u> condition and may be written as usual:

det $\{(b_j \ \ A b_j \ \dots \ A^{n-1} b_j)\} \neq 0 \quad \forall j = 1, \dots, p$

~ Characterization of the optimal solution

Theorem: Consider the minimum time optimal problem

$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$

$x(t_i) = x_i \qquad x(t_f) = 0 \qquad |u_j(t)| \leq 1$

$J = \int_{t_i}^{t_f} dt = t_f - t_i$

If the strong controllability condition is satisfied
$(\det\{G\} \neq 0)$

If $\exists$ the solution:

1) It's non singular
2) u_j is a bang-bang solution: every component of the optimal solution is piecewise constant assuming only the extreme values ± 1

3) The number of switches (discontinuity instants) is limited $(t_j < \infty)$

Proof

1) By contradiction

If the solution were singular, from the Pontryagin principle

$H = \underbrace{\lambda_0 1 + \lambda^T A x}_{H_1} + \underbrace{\lambda^T B}_{H_2} \underbrace{u}_{N}$

$\rightarrow \lambda^T B w \geq \lambda^T B u \quad \forall w : |w_j| \leq 1$

The singularity exists when $\lambda^T B = H_2 = 0$, so when

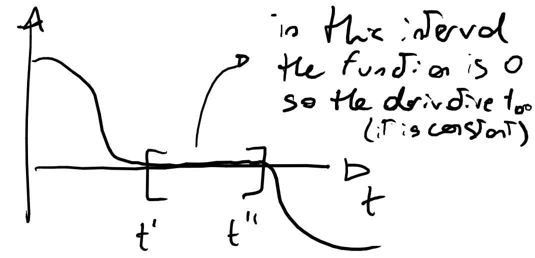
$\lambda^T(t)\, b_j(t) = 0$ in $[t', t''] \subset [t_i, t_f^\circ]$

Trick:

I derive many times $\lambda^T(t)\, b_j(t)$ and using $\dot{\lambda} = -A^T \lambda$:

$\lambda^T(t)\, b_j(t) = 0$

$\frac{d}{dt}$



$\dot{\lambda}\, b_j + \lambda \dot{b}_j = 0 \quad \forall t \in [t', t'']$

$\rightarrow -A\lambda\, b_j + \lambda \dot{b}_j$ → # Remember the matrix $G_j(t)$ of strong controllability

$\vdots$

$$\frac{d(\lambda^T(t)\, b_j(t))}{dt^i} = \lambda^{\circ T}(t)\, b_j^{(i+1)}(t) = 0 \qquad i = 1 \dots n-1$$

$$\rightarrow \lambda^{\circ T}(t)\, G_j(t) = 0 \qquad \forall t \in [t', t'']$$

On the other hand $\lambda^\circ(t)$ <u>must be $\neq 0$</u> $\forall t \in [t', t'']$ otherwise, from the <u>nec conditions</u> $\dot{\lambda} = -A^T\lambda$ it should be null in the whole interval and in particular $\lambda^\circ(t_f) = 0$ → impossible because $x(t_f) = 0$

and if $\lambda(t_f) = 0$ we have the transversality condition $H\big|_{t_f^*}^{*} = 0$ because t_f is not fixed, and this leads to $\lambda_0^\circ = 0$ in fact:

$$H = \lambda_0 1 + \underbrace{\lambda^T A x + \lambda^T B u}_{0 \text{ in this hypothesis}} \longrightarrow H = \lambda_0 1 \rightarrow \lambda_0 \text{ cannot be } 0 \,!!$$

<u>absurd</u>

Therefore, from the condition $\lambda^{\circ T}(t)\, G_j(t) = 0$, $\forall t \in [t', t'']$ it results that $\det\{G_j(t)\} = 0$, in fact from Cramer, if $\det\{G_j\} = 0$ we should have that $\lambda(t) = 0$ which is <u>not possible</u> from <u>strong controllability</u>

→ The solution is non singular

2) From the nonsingularity of the optimal solution, it results that the quantity

$\lambda^{\circ T}(t)\, b_j(t)$ can be <u>null</u> only on <u>isolated points</u>

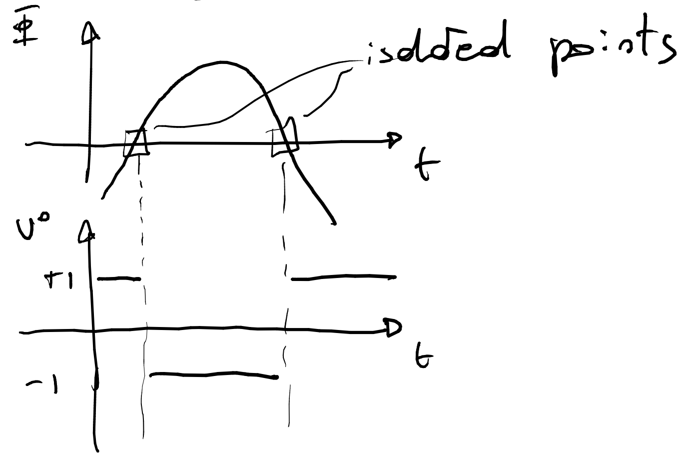
From the nec. condition : $\lambda^{\circ T} B w \geq \lambda^{\circ T} B v^\circ$

$$v^\circ(t) = \begin{cases} +1 \\ 0 \longrightarrow \text{only on isolated points} \\ -1 \end{cases}$$

$\lambda^{\circ T} B = \Phi$ = switching function $\rightarrow$ if $\Phi = 0$, $0 \cdot w \geq 0 \cdot v$ (I cannot say anything about the control)

The switching function can be 0 only on isolated points

Therefore $v_j^\circ(t) = -\operatorname{sign}\{\lambda^{\circ T}(t)\, b_j(t)\}$ $j = 1, 2, \ldots, p$ $\forall t \in [t_i, t_f^\circ]$



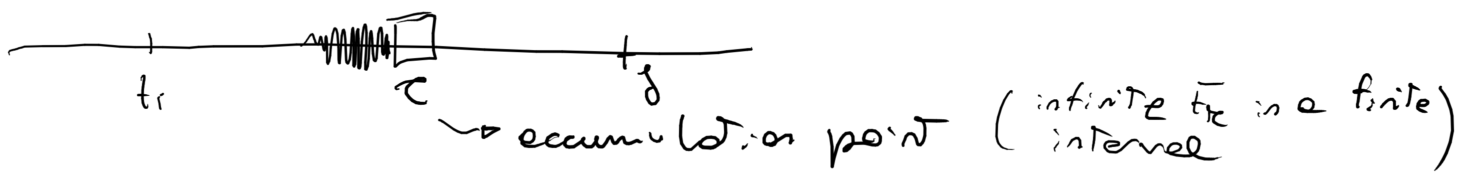
3) To demonstrate that the number of discontinuity instants is finite, we can proceed by <u>contradiction</u>:

Assume $\det\{b_j(t)\} = 0$ $\forall t \in [t', t'']$

which <u>contradicts</u> the hypothesis of strong controllability

In fact let us assume that at least for one component of the control $v_j^\circ(t)$ it is not true

Then an accumulation point $\tau \in [t_i, t_f^\circ]$ of switching instants $\bar{t}_k^{(j)}$ exists



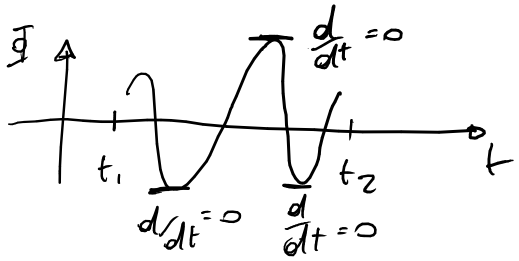
By continuity, given $u_j^\circ(t) = -\text{sign}\{\lambda^{\circ T}(t)\, b_j(t)\}$

$$\lambda^T(\bar{t}_k^{(j)})\, b_j(\bar{t}_k^{(j)}) = 0$$

$$\lambda^{\circ T}(\tau)\, b_j(\tau) = 0$$

Trick: for the continuity of $\lambda^{\circ T}(t)\, b_j(t) = 0$ between $\bar{t}_k^{(j)}$ and $\bar{t}_{k+1}^{(j)}$,

$\exists$ an instant $\tilde{t}_k^{(j)}$ in which $\dfrac{d}{dt}\left(\lambda^{\circ T}(t)\, b_j(t)\right) = 0$



the derivative $= 0$ when for example we have maxima and minima

Therefore, by continuity also in τ: $\dfrac{d}{dt}\left(\lambda^{\circ T}(t)\, b_j(t)\right)\Big|_\tau = 0$

and analogously,

$$\frac{d^i}{dt^i}\left(\lambda^{\circ T}(t)\, b_j(t)\right)\Big|_\tau = 0 \qquad i = 0, 1, \dots (n-1)$$

$$\Rightarrow \det\{G_j(t)\} = 0 \quad \forall t \in [t', t'']$$

<u>contradiction with SC hypothesis</u>

End

A control function that assumes only the limit values is called <u>bang-bang control</u> and the instants of discontinuity are called <u>commutation instants</u>

~ Uniqueness

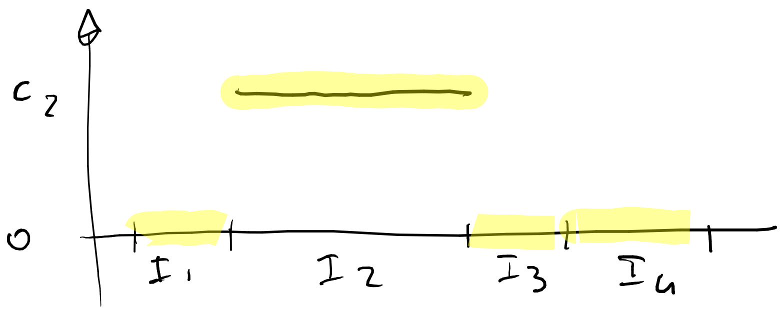
If the hypothesis of strong controllability is satisfied, if an optimal solution exists, it is unique

~ Measurable function

Let $S(t_i, t_f)$ be the space of piecewise constant function defined as follows:

$$s(t) = \begin{cases} c_j & \forall t \in I_j \, , \; j = 1, 2, \dots, m \\ 0 & \forall t \notin \bigcup_{j=1}^{m} I_j \end{cases}$$

$$\{I_j\} \subset [t_i, t_f] \, , \quad I_j \cap I_k = \emptyset$$



Let $z(t)$ be a function s.t. $\exists$ a sequence $\{s^{(k)}\} \subset S(t_i, t_f)$ such that $\forall t \in [t_i, t_f]$, with the exception of isolated points, one has:

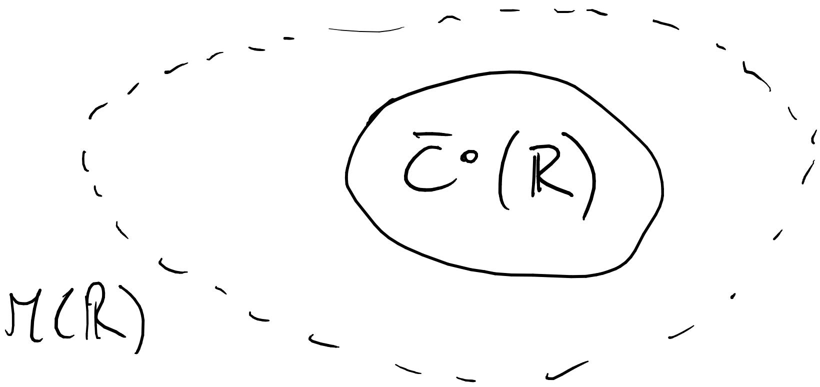
$$\lim_{k \to \infty} s^{(k)}(t) = z(t)$$

$z(t)$ is a measurable function, the limit of a piecewise constant function

The space $M(t_i, t_f)$ of measurable function is linear.

Remark

These results hold also to the case in which the control is in the space of measurable function defined on the space of real number $\mathbb{R}$



~ Existence of the Optimal Solution

Theorem: If the condition of strong controllability is satisfied and if an admissible solution exists:
- $\exists$ an optimal solution
- The solution is unique and non singular
- The control is bang-bang

Proof

The existence theorem is proved on the basis of the following results:

Theorem: Let assume that the control functions belong to the space of measurable functions. If $\exists$ an admissible solution, then the optimal solution exists

Trivial case: finite number of admissible solutions

We have for example $(t_1^\circ, u_1^\circ, x_1^\circ), (t_2^\circ, u_2^\circ, x_2^\circ), (t_3^\circ, u_3^\circ, x_3^\circ)$

Supposing t_2° is the lower bound of the sequence $t_1^\circ, t_2^\circ, t_3^\circ$, then $(t_2^\circ, u_2^\circ, x_2^\circ)$ is the solution of the minimum time problem

Non trivial case: ∞ admissible solutions

t_f° is the minimum extremum of the instants t_f corresponding to the admissible solutions $\rightarrow t_f^\circ = \inf\{t_f\}$

Can we find a minimum?

We build a sequence of admissible solutions

$\{(x^{(k)}, u^{(k)}, t_f^{(k)})\}$, $t_f^{(k)} \geq t_f^\circ$

↳ larger because t_f° is the minimum extremum of the sequence

$\lim_{k \to \infty} t_f^{(k)} = t_f^\circ$

Now consider the transition matrix of the system $\Phi(t, \tau)$, from the linear sys:

$\dot{x} = A(t)\, x(t) + B(t)\, u(t)$

$\dot{x}^{(k)} = A(t)\, x^k(t) + B(t)\, u^k(t)$

↓ solution of the equation

$$x^{(k)}(t_f^{(k)}) = \phi(t_f^{(k)}, t_i)\, x_i + \int_{t_i}^{t_f^{(k)}} \phi(t_f^{(k)}, t)\, B(t)\, u^{(k)}(t)\, dt$$

$$x^{(k)}(t_f^\circ) = \phi(t_f^\circ, t_i)\, x_i + \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\, B(t)\, u^{(k)}(t)\, dt$$

$$\lim_{k\to\infty}\left[x^{(k)}(t_f^{(k)}) - x^{(k)}(t_f^{o})\right] = \lim_{k\to\infty}\underbrace{\left[\phi(t_f^{(k)},t_i) - \phi(t_f^{o},t_i)\right]}_{\to 0} x_i +$$

$$+ \lim_{k\to\infty}\int_{t_i}^{t_f^{o}}\underbrace{\left[\phi(t_f^{(k)},t) - \phi(t_f^{o},t)\right]}_{\to 0} B(t)\, u^{(k)}(t)\, dt \; +$$

$$+ \lim_{k\to\infty}\underbrace{\int_{t_f^{o}}^{t_f^{(k)}}\phi(t_f^{(k)},t)\, B(t)\, u^{(k)}(t)\, dt}_{\to 0} = 0$$

The first 2 limits goes to 0 cause ϕ is continuous and $t_f^{(k)} \to t_f^{o}$

The last one goes to 0 cause independently to the quantity in the integral, the extrema are from t_f^{o} to $t_f^{(k)}$, but $t_f^{(k)} \to t_f^{o}$ therefore due to the small interval the integral is 0.

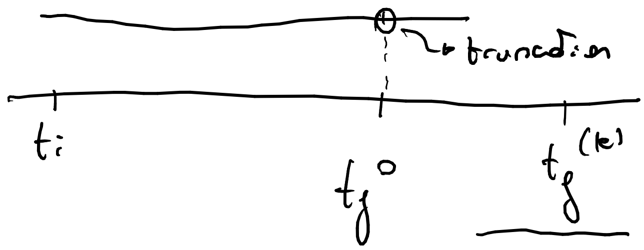
So we know that each $\boxed{x^{(k)}(t_f^{(k)}) = 0}$ because it is an admissible solution therefore:

$$\lim_{k\to\infty}\left[\underbrace{x^{(k)}(t_f^{(k)})}_{\to 0} - x^{(k)}(t_f^{o})\right] = 0$$

$$\Downarrow$$

$$\boxed{\lim_{k\to\infty} x^{(k)}(t_f^{o}) = 0}$$

Only the control now is missing.



Given the admissible solution $\{x^{(k)}, u^{(k)}, t_f^{(k)}\}$, we consider $\bar{u}^{(k)}$ as the truncation of $u^{(k)}$ in t_f^{o}

This function is limited and L_2, this means that if we do the integral $\int |f|^2 < \infty \rightarrow$ it is bounded, since $|u_j^{(k)}(t)| \leq 1$.

The sequence of functions $\{\bar{u}^{(k)}\} \in M$ of measurable function such that $|u_j(t)| \leq 1$, $\forall t \in [t_i, t_f^\circ]$
Therefore this sequence admits a subsequence, indicated still with $\{\bar{u}^{(k)}\}$, <u>weakly</u> converging to a function $u^\circ \in M$.

In synthesis:

$$\begin{tikzcd}
\text{from } \{u^{(k)}\} \arrow[r, "\text{truncation in } t_f^\circ"] & \{\bar{u}^{(k)}\} \text{ limited } L_2 \text{ fun } \int |f|^2 < \infty \arrow[d, "\text{find a subsequence}"] \\
& \{\bar{u}^{(k)}\} \arrow[r, "\text{weakly}"] & u^\circ
\end{tikzcd}$$

- Weak convergence of $\forall$ measurable function h

$$\lim_{k \to \infty} \int_{t_i}^{t_f^\circ} h^T(t)\, \bar{u}^{(k)}(t)\, dt = \int_{t_i}^{t_f^\circ} h^T(t)\, u^\circ(t)\, dt$$

the limit of the integral of h times the sequence, is equal to the integral of h times u°

Given x°, the evolution of the state corresponding to u°:

$$\lim_{k \to \infty} x^{(k)}(t_f^\circ) = \phi(t_f^\circ, t_i)x_i + \lim_{k \to \infty} \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\, B(t)\, \bar{u}^{(k)}(t)\, dt$$

$$= \phi(t_f^\circ, t_i)x_i + \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\, B(t)\, u^\circ(t)\, dt = \boxed{x^\circ(t_f^\circ)}$$

Therefore, given $\lim_{k\to\infty} x^{(k)}(t_f^0)=0$, we deduce $x^0(t_f^0)=0$

So we have found the optimal solution (x^0, u^0, t_f^0) with a control able to transfer the state to the origin.

End

Remark:

The existence of the optimal solution is guaranteed only for the couples (t_i, x_i) for which the admissible solution exists.

~ Minimum time problem for steady state system

$$\dot{x}(t) = \underset{\substack{\downarrow \\ \text{constant}}}{A}\, x(t) + \underset{\substack{\downarrow \\ \text{constant}}}{B}\, u(t)$$

In this case there's a result on the number of <u>commutation points</u> ⇝ I can say the maximum number of switching points

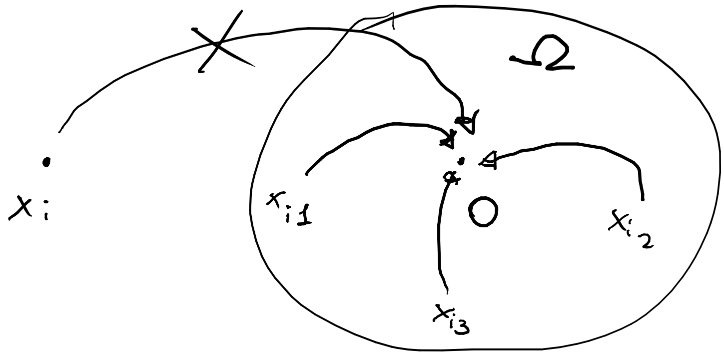
Given $x(t_i) = x_i$ $\quad x(t_f) = 0 \quad |u| \leq 1$

$$J = \int_{t_i}^{t_f} dt = t_f - t_i$$

<u>Theorem</u>: consider the control function belonging to the space of measurable function.

If the system is <u>controllable</u>

⇒ ∃ a neighbor Ω of the origin such that $\forall x_i \in \Omega$ there exists an optimal solution



Proof

If I can show in this case that ∃ an admissible solution, I have solved it due to the previous results.

Let us fix any $t_f > 0$. The fact that the system is controllable can be described by saying that we can find a control able to transfer the initial state $x_i \in \Omega$ to the origin at the instant t_f:

$$\dot{x} = Ax + Bu \qquad x(t_i) = x_i$$

$$x(t) = e^{A(t-t_i)}x_i + \int_{t_i}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

I want that $\underline{x(t_f) = 0}$, so $t = t_f$

must be 0

$$x(t_f) = e^{A(t_f-t_i)}x_i + \int_{t_i}^{t_f} e^{A(t_f-\tau)}Bu(\tau)\,d\tau = 0$$

Since A, B and t_f are fixed, we miss only u.

Assume:

$u(\tau) = B^T e^{-A^T\tau} v$ with $v \in \mathbb{R}^p$

substituting in the last integral:

$$-e^{-At_i}x_i = \underbrace{\left(\int_{t_i}^{t_f} e^{-A\tau}BB^T e^{-A^T\tau}\,d\tau\right)}_{\text{Gramian matrix}} v$$

In order to show the controllability, it is equivalent to say that the Gramian matrix is <u>non singular</u>.

From the previous considerations:

$$v = -(\text{Gramian})^{-1} e^{-At_i}x_i$$

substituting in $u(\tau)$:

$$u(\tau) = -B^T e^{-A^T\tau}(\text{Gramian})^{-1}e^{-At_i}x_i$$

This is an admissible control, and it exists if the initial state is sufficiently near the origin.

The control $v(t)$ to be admissible, must be $|v| \leq 1$ as known, and to respect this condition, since everything is give with exception of x_i, therefore choosing it very small, of course $v(t)$ could be ≤ 1

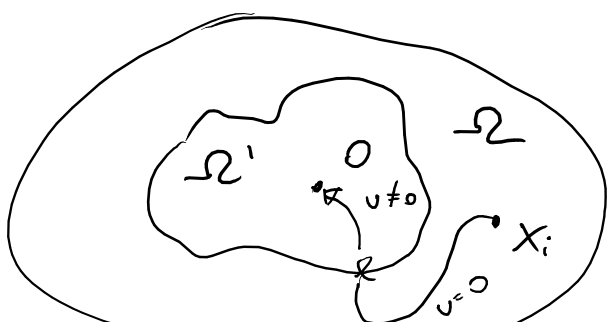
End

How can we avoid the constraint of being near the origin?
If the eigenvalues of A have negative real part ($Re\{\lambda\} < 0$), $\exists$ an optimal solution whatever the initial state is.

Proof

- Let Ω be a neighbour of the origin including the original state x_i for which an admissible solution exists.
- Let Ω' be a closed subset in Ω

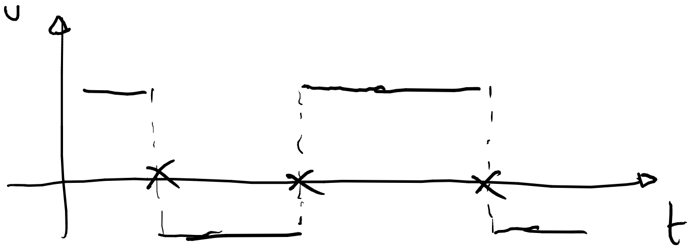
$\rightarrow$ From the asymptotic stability of the system it is possible to reach Ω' in a finite time with null input (free evolution), from any initial state



This probably isn't the optimal solution but I don't care because I need only an admissible one

End

- How many <u>commutation instants</u>?



If we have the controllability condition, if $Re\{\lambda\} \leq 0$, the number of commutation instants for any component of the control is $\leq n-1$ whatever the initial state is.

$\hookrightarrow$ dimension of A

Proof

Given the optimal control

$$u_j^\circ(t) = -\operatorname{sign}\{\lambda^{\circ T}(t)\, b_j\}, \quad j = 1, 2, \dots, p \qquad \forall t \in [t_i, t_f^\circ]$$

where the costate λ° in the steady state case is:

$$H = \lambda_0 \cdot 1 + \lambda^T A x + \lambda^T B u \quad \rightarrow \quad \dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T = -A^T \lambda$$

$$\Downarrow$$

$$\lambda^\circ(t) = e^{-A^T(t-t_i)} \lambda_i$$

Now:

$$\dot{x}(t) = A x(t) + B u(t) \qquad A \in \mathbb{R}^{n \times n} \rightarrow n \text{ eigenvalues}$$

denoting by α_k the eigenvalues and by m_k their multiplicity, the expression of the r-th component of λ° with $r = 1, \dots, n$ is:

$$\lambda_r^\circ(t) = \sum_{s=1}^{k} P_{rs}(t)\, e^{-\alpha_s (t-t_i)}$$

$\rightarrow$ polynomial function of degree $< m_s$

Now substituting $\lambda_r^\circ(t)$ in the expression of the bang-bang control we obtain:

$$u_j^\circ(t) = -\text{sign}\left\{\sum_{s=1}^{k}\underbrace{\left(\sum_{r=1}^{n} b_{jr}\, p_{rs}(t)\right)}_{p_{js}(t)} e^{-\alpha_s(t-t_i)}\right\} \qquad j = 1, \dots, p$$

$$= -\text{sign}\left\{\sum_{s=1}^{k} p_{js}(t)\, e^{-\alpha_s(t-t_i)}\right\} \qquad \forall t \in [t_i, t_f^\circ]$$

$p_{js}(t) \rightarrow$ polynomial function of degree $< m_s$

The optimal control is given by the sum of polynomial functions in which each of them has degree $< m_s$, so the argument of the sign function has at most $m_1 + m_2 + \dots + m_{k-1} = n-1$ real solutions

Therefore the control u_j° has at most $n-1$ roots

$$\begin{array}{c|c|}
\alpha_1 & m_1 \\
\alpha_2 & m_2 \\
\vdots & \vdots \\
\alpha_k & m_k
\end{array} \rightarrow \text{the sum of these is } n$$

End

Remark:

In the minimum time problem it is not possible to relate directly the control and the state esplicitly

$u^\circ(t) = -\text{sign}\{\lambda^T(t) B\}$ and <u>not</u>

$\cancel{u^\circ(t) = f(x^\circ(t))}$