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Nonlinear Systems & Control
Part II
7/6/17

Student: ____________________

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1. Consider the linear time invariant (LTI) system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

Is it possible to assign the eigenvalues through a possibly dynamical output feedback? Justify the answer.

2. Consider the LTI and SISO system

$$\dot{x} = Ax + Bu + Dw$$
$$y = Cx$$

where $w \in \mathbb{R}$ denotes an arbitrary exogenous signal. Prove or disprove the following statement (a formal proof is not required): *The presence of unstable zeros prevents from the possibility of solving the disturbance decoupling problem* <u>*with stability*</u> *for any disturbance* w.

3. Given the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - \cos x_1 u \\ \dot{x}_3 &= x_2^2 + u \\ y &= \sin x_1 - x_3 \end{aligned}$$

a. Compute the feedback ensuring Input-Output Linearization;

b. Investigate on the stability of the zero dynamics;

c. Design a feedback ensuring asymptotic tracking of the reference signal $y_r(t) = \cos t$ and qualitatively discuss on the stability of the closed-loop system.

4. Design a feedback globally asymptotically stabilizing the origin of

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= (1 - x_2)^2 x_1 + (1 + x_1^2) u \end{aligned}$$

5. Prove the following statement: *It the dynamics $\dot{x} = f(x) + g(x)u$ $(u \in \mathbb{R})$ has a globally stable equilibrium at the origin with positive definite Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, then the system*

$$\dot{x} = f(x) + g(x)u$$
$$y = \mathrm{L}_g V(x)$$

is passive with storage function $V(x)$.

6. The Artestein-Sontag Theorem.

① Is it possible to assign the eigenvalues through a possibly dynamical output feedback

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \qquad n = 3$$

I don't have access to the state so from the output I should estimate the state and that's why I need an observer.

Since I need the estimate of the observer, I need full observability

$\rightarrow$ Arbitrarily assigning eigenvalues of $A - KC$

$\Leftrightarrow$ Solvable iff (A, C) observable

→ Arbitrarily assigning eigenvalues of $A-KC$
⟺ Solvable iff (A,C) observable

Check: $\rho\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \rho\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2 < 3$

not full rank, therefore there's an unobservable eigenvalue

Laplace

$$\begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda^2-1) = 0$$

$\lambda_1 = 1 \qquad \lambda^2 = 1 \Rightarrow \lambda_{23} = \pm 1$

or Sarrus + Ruffini

$\lambda^2(\lambda-1) - (\lambda-1) = 0$

$\lambda^3 - \lambda^2 - \lambda + 1$

$(\lambda-1)(\lambda^2-1) = 0$

$$\begin{array}{c|ccc|c} & 1 & -1 & -1 & 1 \\ 1 & & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$\lambda_1 = 1 \qquad \lambda_{23} = \pm 1$

observability of $\lambda = 1$

$$\rho\begin{pmatrix} A-\lambda I \\ C \end{pmatrix}\Bigg|_{\lambda=1} = \rho\begin{pmatrix} A-I \\ C \end{pmatrix} = \rho\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2 < 3 \text{ not full rank}$$

so $\lambda = 1$ is unobservable

$W(s) = C(sI-A)^{-1}B$

$$(sI-A)^{-1} = \left[\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]^{-1} =$$

$$= \begin{pmatrix} s & -1 & 0 \\ -1 & s & 0 \\ 0 & 0 & s-1 \end{pmatrix}^{-1}$$

algebraic complements $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$\begin{pmatrix} s(s-1) & s-1 & 0 \\ s-1 & s(s-1) & 0 \\ 0 & 0 & s^2-1 \end{pmatrix}^T$$

$\det(A) = (s-1)(s^2-1)$

$$= \frac{1}{(s-1)(s^2-1)} \begin{pmatrix} s(s-1) & s-1 & 0 \\ s-1 & s(s-1) & 0 \\ 0 & 0 & s^2-1 \end{pmatrix}$$

$$(s-1)(s^2-1) \quad \begin{pmatrix} 0 & 0 & s^2-1 \end{pmatrix}$$

So $$W(s) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s(s-1) & s-1 & 0 \\ s-1 & s(s-1) & 0 \\ 0 & 0 & s^2-1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} s-1 & s(s-1) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{s\cancel{(s-1)}}{\cancel{(s-1)}(s^2-1)} = \frac{s}{(s-1)(s+1)}$$

2 eigenvalues appears and one in $+1$ is missing. Therefore it is confirmed that the eigenvalue in $+1$ is unobservable and it has $Re[\,] > 0$, it means that I can't solve the reconstruction problem.

And since (A, C) is not fully observable I cannot assign eigenvalues arbitrarily.

If the unobservable eigenvalue was with $Re[\,] < 0$ I could have built the observer but due to the not fully observability of (A, C) I couldn't have assigned the eigenvalues.

If the question was on state feedback:

$\rho(B \ AB \cdots A^{n-1}B) = R = m < n$ non full rank

I would have had to find $\lambda \to |R - \lambda I|$ and then verify $\rho(A - \lambda I \,\vdots\, B)$.
At the end construct $W(s)$.

If the uncontrollable eigenvalue is $Re[\,] > 0$ I cannot do anything. If it's $Re[\,] < 0$ then the system is stabilizable but I cannot assign arbitrarily the eigen.

In order to assign them arbitrarily I need (A, C) fully observable for output Fb and (A, B) fully controllable for state Fb

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4 Design a fb globally asymptotically stabilizing the origin of:

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = (1-x_2)^2 x_1 + (1+x_1^2)\, u \end{cases}$$

$u = \frac{1}{(1+x_1^2)} \left(v - (1-x_2)^2 x_1 \right) \rightarrow$ In this way I rewrite the sys

$\Rightarrow \begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = v \end{cases}$ <u>backstepping</u>

Design the fb control $x_2 = \gamma(x_1)$ to stabilize the origin $x_1 = 0$

Choose a value $\gamma(x_1)$ s.t. $\dot{x}_1 = \gamma(x_1)$ is AS

$x_2 = \gamma(x_1) = x_1$

$\Rightarrow \dot{x}_1 = -x_1$

$V(x_1) = \frac{1}{2} x_1^2$, $\dot{V}(x_1) = x_1 \dot{x}_1 = -x_1^2 < 0$

So the origin of $\dot{x}_1 = -x_1$ is GAS

To backstep I use the change of variables

$\xi = x_2 - \gamma(x_1) = x_2 - x_1$

the sys now in the new coord is:

$\dot{x}_1 = -x_1 - \xi$

$\dot{\xi} = v - \frac{\partial \gamma}{\partial x_1} \dot{x}_1 = v - x_2 = v + \xi + x_1$

Taking $W(x) = \frac{1}{2} x_1^2 + \frac{1}{2} \xi^2$ I get

$\dot{W} = x_1 \dot{x}_1 + \xi \dot{\xi} = -x_1^2 - x_1 \xi + \xi (v + \xi + x_1) =$

$= -x_1^2 + \xi (v - \cancel{x_1} + \xi + \cancel{x_1}) = -x_1^2 + \xi (v - \xi)$

taking $v = -2\xi$ yelds $\dot{W} = -x_1^2 - \xi^2 < 0 \Rightarrow$ so the origin is GAS

alternative

1. $x_2 = \gamma(x_1)$ s.t. $\dot{V}_1 < 0$

$V_1(x_1) = \frac{1}{2} x_1^2 \Rightarrow \dot{V}_1(x_1) = \dot{x}_1 x_1 = -x_1 x_2$

setting $\gamma(x_1) = x_1$ I obtain

$\dot{V}(x_1) = -x_1^2 < 0$ GAS

setting $\gamma(x_1) = x_1$ I obtain
$\dot{V}(x_1) = -x_1^2 < 0$ GAS

② $e = x_2 - \gamma(x_1)$
$e = x_2 - x_1 \longrightarrow x_2 = e + x_1$

$$\begin{cases} \dot{x}_1 = -e - x_1 \\ \dot{e} = (1-e-x_1)^2 x_1 + (1+x_1^2)u + x_2 \end{cases}$$

Setting $u = \dfrac{-\left[(1-e-x_1)^2 x_1 - x_2\right] + v}{(1+x_1^2)}$

I obtain
$$\begin{cases} \dot{x}_1 = -e - x_1 \\ \dot{e} = v \end{cases}$$

$V_2(x_1, e) = \frac{1}{2}(x_1^2 + e^2) \Rightarrow \dot{V}_2 = \dot{x}_1 x_1 + \dot{e} e = -e x_1 - x_1^2 + e v$

Setting $v = -e + x_1$

$\dot{V}_2 = -\cancel{e x_1} - x_1^2 + \cancel{e x_1} - e^2 < 0$

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③ u-y Feedback linearization

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \sin(x_1) - \cos(x_1) u \\ \dot{x}_3 = x_2^2 + u \\ y = \sin(x_1) - x_3 \end{cases} \qquad f = \begin{pmatrix} x_2 \\ \sin(x_1) \\ x_2^2 \end{pmatrix} \qquad g = \begin{pmatrix} 0 \\ -\cos(x_1) \\ 1 \end{pmatrix}$$

$dh = (\cos(x_1) \quad 0 \quad -1)$

$r = 1$ $\quad L_g h = \frac{\partial h}{\partial x} \cdot g = -1 \neq 0$

Coordinate transformation $\begin{pmatrix} z \\ \eta \end{pmatrix} \begin{matrix} r = 1 \\ n - r = 2 \end{matrix}$

$z = h = \sin(x_1) - x_3$

η_1 & η_2 s.t. $\nabla \varphi \cdot g = 0$

$$\begin{pmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ -\cos(x_1) \\ 1 \end{pmatrix} = 0$$

$$-\frac{\partial \varphi}{\partial x_2}\cos(x_1) + \frac{\partial \varphi}{\partial x_3} = 0$$

$$\eta_1 = x_1 \qquad \eta_2 = x_2 + x_3\cos(x_1)$$

$$\dot{z} = \cos(x_1)\cdot \dot{x}_1 - \dot{x}_3 = \cos(x_1)\cdot x_2 - x_2^2 + u$$

$$\dot{\eta}_1 = \dot{x}_1 = x_2$$

$$\dot{\eta}_2 = \dot{x}_2 - \sin(x_1)\dot{x}_1 x_3 + \dot{x}_3\cos(x_1) =$$

$$= \sin(x_1) - \cancel{\cos(x_1)u} - x_3x_2\sin(x_1) + x_2^2\cos(x_1) + \cancel{\cos(x_1)u}$$

setting $u = x_2^2 - \cos(x_1)x_2 + v$

I obtain

$$\begin{cases} \dot{z} = v \\ \dot{\eta}_1 = x_2 \\ \dot{\eta}_2 = \sin(x_1) - x_3x_2\sin(x_1) + x_2^2\cos(x_1) \end{cases}$$

So I obtaine I/O FL on the system

Zero dynamics

$$\eta_1 = x_1 \qquad \eta_2 = x_2 + x_3\cos(x_1)$$

$$z = \sin(x_1) - x_3 \longrightarrow z = 0,\ v = 0 \longrightarrow \sin(x_1) = x_3$$
$$x_3 = \sin(\eta_1)$$

$$x_2 = \eta_2 - \sin(\eta_1)\cos(\eta_1)$$

$$\dot{\eta}_1 = \eta_2 - \sin(\eta_1)\cos(\eta_1)$$

$$\dot{\eta}_2 = \sin(\eta_1) - \sin^2(\eta_1)\left(\eta_2 - \sin(\eta_1)\cos(\eta_1)\right) +$$
$$+ \left(\eta_2 - \sin(\eta_1)\cos(\eta_1)\right)^2\cos(\eta_1)$$

$$Q = \left.\frac{\partial q(0,\eta)}{\partial \eta}\right|_0 = \dots$$

At the end of this very complex derivative the dynamics should be unstable because one eigenvalue has $Re[\,] > 0$
then the closed loop sys is unstable / not stabilizable

Output tracking

$$z_R = \begin{pmatrix} y_R \\ \vdots \\ y^{(r)}_R \end{pmatrix} \overset{r=1}{=} y_R = \cos(t)$$

So the input is the same as before (v), but with

$$v = y_R^{(r)} - \sum_{i=1}^{r} c_i (z^i - y_R^i) = y_R^{(r)} - c_1 (z - y_R)$$

$$y_R^{(r)} = -\sin(t)$$

$$e = z - y_R$$

So $v = -\sin(t) - c_0 e$ with c_0 s.t. $c_0 + s = 0$
root $\in \mathbb{C}^-$

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(2) If all zeros are unstable DDP with stability is not solvable.
If zeros are combination of stable and unstable zeros I can cancel stable zeros only and DDP with stability is solvable iff
$\operatorname{Im} D \subset V_S^*$