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Nonlinear Systems & Control
Part II
9/10/17

Student: ____________________

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1. Given a system with transfer function $W(s) = \frac{2s-1}{s^3+2s^2+s}$ and affected by disturbance $w \in \mathbb{R}$
   - **a.** Characterize, if any, all the action of disturbances that can be decoupled under feedback and compute the decoupling feedback;
   - **b.** discuss about the stability in closed-loop.
2. Given the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= \arctan x_3 + e^{x_1} u_2 + \sin x_2 u_1 \\ \dot{x}_3 &= -x_3^4 + x_1^2 + x_2 + u_1 \\ y_1 &= e^{x_1} x_2 \\ y_2 &= x_3 \end{aligned}$$

   - **a.** compute the relative degree;
   - **b.** compute, if any, the feedback solving the input/output linearization problem;
   - **c.** compute, if any, the feedback achieving input-output decoupling;
   - **d.** Discuss the stability of the closed-loop system.
3. Given a nonlinear system $\dot{x} = f(x) + g(x)u$ discuss about the maximal feedback linearizability problem.
4. Compute, a feedback yielding global asymptotic stability of the equilibrium of the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \sin {x_1}^2 + u \end{aligned}$$

① Characterize, if any, all the action of disturbance that can be decoupled under Fb and compute the decoupling fb.

$$W(s) = \frac{2s-1}{s^3+2s^2+s} \qquad W(s) = \frac{b_0 + b_1 s + \dots + b_m s^m}{a_0 + a_1 s + \dots + s^n} \qquad n = 3, \quad m = 1$$

$$b_0 = -1 \qquad a_0 = 0$$
$$b_1 = 2 \qquad a_1 = 1$$
$$a_2 = 2$$

$$A = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & \dots & \dots & -a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \Rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} b_0 & b_1 & \overset{\to m}{0} \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_2 - 2x_3 + u \\ y = -x_1 + 2x_2 \end{cases}$$

$\begin{cases} \dot{x}_3 = -x_2 - 2x_3 + u \\ y = -x_1 + 2x_2 \end{cases}$

$$r = \text{degree}(D(W(s))) - \text{degree}(N(W(s))) = 3 - 1 = 2$$

$$V^* = \ker\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix} = \ker\begin{pmatrix} C \\ CA \end{pmatrix} = \ker\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} -x_1 + 2x_2 = 0 \\ -x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} +x_1 = +2x_2 \Rightarrow x_2 = \frac{x_1}{2} \\ 2x_3 = x_2 \Rightarrow x_3 = \frac{1}{2}\frac{x_1}{2} = \frac{x_1}{4} \end{cases} \Rightarrow \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$V^* = \text{span}\left\{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}\right\}$$

Therefore I need D s.t. $\text{Im}\, D \subset \text{span}\left\{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}\right\}$ which means that

$\forall D$ s.t. $D_s = \begin{pmatrix} 4s \\ 2s \\ s \end{pmatrix}$ $s \in \mathbb{R}$, the DDP admits a solution

The Fb is $u = -\frac{1}{CAB}(CA^2 x - v) = F^* x + v$

$$F^* = \frac{-CA^r}{CA^{r-1}B} = \frac{-(0\ -2\ 3)}{2} = -\frac{1}{2}(0\ -2\ 3) = (0\ 1\ -\tfrac{3}{2})$$

- **Discuss about stability in closed-loop**

Coord. transf. $\begin{pmatrix} z \\ \hline \eta \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \hline \eta \end{pmatrix} x = \begin{pmatrix} C \\ CA \\ \hline T_2 \end{pmatrix} x$ with T_2 s.t. $T_2 B = 0$

$C = (-1\ 2\ 0)$
$CA = (0\ -1\ 2)$

$$T_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow T_2 = (0\ 1\ 0)$$

So $z_1 = (-1\ 2\ 0)x = -x_1 + 2x_2$ $\qquad x_1 = 2\eta - z_1$
$z_2 = (0\ -1\ 2)x = -x_2 + 2x_3$ $\qquad x_3 = \frac{z_2 - \eta_2}{2}$
$\eta = (0\ 1\ 0)x = x_2$

$$CA^2 = CA \cdot A = (0\ -1\ 2)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} = (0\ -2\ 3)$$

$CAB = (0\ -1\ 2)\begin{pmatrix} 0 \\ 0 \\ \ldots \end{pmatrix} = 2$

$$CA^2 = CA \cdot A = (0\ -1\ 2)\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} = (0\ -2\ 3)$$

$$CAB = (0\ -1\ 2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2$$

In the (z, η) coordinates the whole system can be seen as:

$$\dot{z} = \begin{pmatrix} z_2 \\ \vdots \\ b + au \end{pmatrix} \qquad u = \frac{-b(z,\eta)}{a(z,\eta)} + \frac{v}{a(z,\eta)}$$

$$\dot{\eta} = q(z,\eta) + k(z,\eta)\,w$$

$$\begin{cases} \dot{z}_1 = z_2 = -x_2 + 2x_3 \\ \dot{z}_2 = -b(z,\eta) + a(z,\eta)u = -CA^2x + CABu = \\ \quad = -(0\ -2\ 3)\,x + 2u = 2x_2 - 3x_3 + 2u = 2\eta - \frac{3}{2}(z_2 - \eta_2) + 2u \\ \dot{\eta} = \dot{x}_2 + \tilde{d}w = \frac{z_2 - \eta_2}{2} + \tilde{d}w = Q\eta + Sz + Dw \end{cases}$$

$$Q = \left.\frac{\partial q(0,\eta)}{\partial q}\right|_0 = -\frac{1}{2} \Rightarrow \text{stable zero dynamics}$$

$W(s) = \frac{2s-1}{s^3+2s^2+s}$ has an unstable zero so DDP with stability is not solvable.

CL with feedback

$$A + BF^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\left(0\ 1\ -\frac{3}{2}\right) =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{7}{2} \end{pmatrix}$$

$$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = -\frac{7}{2}$$

Working on the system without unstable zero: $W' = \frac{1}{s^3+2s^2+1}$

If W' is fully observable there would be no disturbance

If W' is fully observable there would be no disturbance that can assure stability

$r = 2 \longrightarrow \rho\begin{pmatrix} C \\ CA \end{pmatrix} = \rho\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} = 2$ fully observable

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(2)
$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = \operatorname{arctg}(x_3) + e^{x_1} u_2 + \sin(x_2)\, u_1 \\ \dot{x}_3 = -x_3^4 + x_1^2 + x_2 + u_1 \\ y_1 = e^{x_1} x_2 \\ y_2 = x_3 \end{cases}$$

$$g_1 = \begin{pmatrix} 0 \\ s(x_2) \\ 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ e^{x_1} \\ 0 \end{pmatrix}$$

$$\frac{\partial h_1}{\partial x} = \begin{pmatrix} e^{x_1} x_2 & e^{x_1} & 0 \end{pmatrix}$$

$$dh_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

- **Vector relative degree**

$r_1 = 1$ $\quad L_{g_1} h_1 = \frac{\partial h_1}{\partial x} \cdot g_1 = e^{x_1} s x_2 \neq 0$

$L_{g_2} h_1 = \frac{\partial h_1}{\partial x} \cdot g_2 = e^{2x_1} \neq 0$

$r_2 = 1$ $\quad L_{g_1} h_2 = \frac{\partial h_2}{\partial x} \cdot g_1 = 1 \neq 0$

$L_{g_2} h_2 = \frac{\partial h_2}{\partial x} \cdot g_2 = 0$

$\dot{y}_2 = \dot{x}_3 = \dots + u_1$ ok

- **I/O decoupling problem**

decoupling matrix

$$A(x) = \begin{pmatrix} L_{g_1} h_1 & L_{g_2} h_1 \\ L_{g_1} h_2 & L_{g_2} h_2 \end{pmatrix} = \begin{pmatrix} e^{x_1} s(x_2) & e^{2x_1} \\ 1 & 0 \end{pmatrix}$$

$|A(x)| = -e^{2x_1} \neq 0$

So I can apply

$u = A^{-1}(x)\,(v - M(x))$

$$u = A^{-1}(x)\left(v - \mathcal{M}(x)\right)$$

$$\mathcal{M}(x) = \begin{pmatrix} L_f h_1 \\ L_f h_1 \end{pmatrix} \qquad f = \begin{pmatrix} -x_1^3 + x_2 \\ \arctan(x_3) \\ -x_3^4 + x_1^2 + x_2 \end{pmatrix}$$

$$L_f h_1 = \frac{\partial h_1}{\partial x} \cdot f = \begin{pmatrix} e^{x_1} x_2 & e^{x_1} & 0 \end{pmatrix} \begin{pmatrix} -x_1^3 + x_2 \\ \arctan(x_3) \\ * \end{pmatrix} =$$

$$= -e^{x_1} x_1^3 x_2 + e^{x_1} \arctan(x_3) + e^{x_1} x_2^2$$

$$= e^{x_1}\left(x_2^2 - x_1^3 x_2 + \arctan(x_3)\right)$$

$$L_f h_2 = \frac{\partial h_2}{\partial x} \cdot f = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * \\ * \\ -x_3^4 + x_1^2 + x_2 \end{pmatrix} = -x_3^4 + x_1^2 + x_2$$

$$A^{-1}(x) = \begin{pmatrix} e^{x_1} s(x_2) & e^{2x_1} \\ 1 & 0 \end{pmatrix}^{-1}$$

$$\boxed{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^{-1} = \frac{1}{\det} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}$$

$$A^{-1}(x) = \frac{1}{-e^{2x_1}} \begin{pmatrix} 0 & -e^{2x_1} \\ -1 & e^{x_1} s(x_2) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{1}{e^{2x_1}} & -\frac{1}{e^{x_1}} s(x_2) \end{pmatrix}$$

So $u = \begin{pmatrix} 0 & 1 \\ e^{-2x_1} & -e^{-x_1} s(x_2) \end{pmatrix} \left(v - \begin{pmatrix} L_f h_1 \\ L_f h_2 \end{pmatrix} \right)$

this is the feedback which achieves I/O decoupling and linearization

Zero Dynamics & stability

coord. transf.

$z = h_1 = e^{x_1} s(x_2) \longrightarrow x_2 = \arcsin\left(\frac{z}{e^{\eta}}\right)$

$\xi = h_2 = x_3$

η s.t. $\nabla\varphi \cdot g_1 = 0$ & $\nabla\varphi \cdot g_2 = 0$

$$\begin{pmatrix} \frac{\partial\varphi}{\partial x_1} & \frac{\partial\varphi}{\partial x_2} & \frac{\partial\varphi}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ s(x_2) \\ 1 \end{pmatrix} = 0$$

$$\left(\frac{\partial \varphi}{\partial x_1} \quad \frac{\partial \varphi}{\partial x_2} \quad \frac{\partial \varphi}{\partial x_3}\right)\begin{pmatrix} \\ s(x_2) \\ 1 \end{pmatrix} = 0$$

$$\frac{\partial \varphi}{\partial x_2} s(x_2) + \frac{\partial \varphi}{\partial x_3} = 0 \quad \longrightarrow \quad \eta = x_1 ?$$

$$\left(\frac{\partial \varphi}{\partial x_1} \quad \frac{\partial \varphi}{\partial x_2} \quad \frac{\partial \varphi}{\partial x_3}\right)\begin{pmatrix} 0 \\ e^{x_1} \\ 0 \end{pmatrix} = 0$$

$$\frac{\partial \varphi}{\partial x_2} e^{x_1} = 0 \quad \longrightarrow \quad \eta = x_1$$ verifies both conditions

under $U = A^{-1}(v - \mu) = \alpha(x) + \beta(x) v$

I obtain

$$\begin{cases} \dot{z} = v_1 \\ \dot{\xi} = v_2 \\ \dot{\eta} = \dot{x}_1 = -x_1^3 + x_2 = -\eta^3 + \arcsin\left(\frac{z}{e^{\eta}}\right) \end{cases}$$

zero dynamics:

$$Z^* = \{x \in \mathbb{R}^n : e^{x_1} x_2 = 0 \ \& \ x_3 = 0\} \Rightarrow \begin{matrix} v=0 \\ z=0 \end{matrix}$$

So I've $\dot{\eta} = -\eta^3$

$V(\eta) = \frac{1}{2}\eta^2$

$\dot{V}(\eta) = \eta\dot{\eta} = -\eta^4 < 0 \quad \forall \eta \neq 0$

So also the closed loop sys is AS

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(4) Compute a feedback yelding GAS

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \sin(x_1)^2 + u \end{cases}$$

Backstepping

(1) $\dot{x}_1 = -x_1^3 + x_2$

(1) $\dot{x}_1 = -x_1^3 + x_2$

Choosing $x_2 = \gamma(x_1)$ s.t. $\dot{V}(x_1) < 0$

$V(x_1) = \frac{1}{2}x_1^2 \Rightarrow \dot{V}(x_1) = \dot{x}_1 x_1 = -x_1^4 + x_2 x_1$

Picking $\gamma(x_1) = 0 \Rightarrow \dot{V}(x_1) = -x_1^4 < 0 \quad \forall x_1 \neq 0$ GAS

(2) $e_1 = x_2 - \gamma(x_1) = x_2$

No change of coordinates needed

$\dot{x}_1 = -x_1^3 + x_2$

$\dot{x}_2 = x_3$

Take x_3 as a virtual feedback

$x_3 = \gamma(x_1, x_2)$ s.t. $\dot{V}_2 < 0$

$V_2(x_1,x_2) = \frac{1}{2}x_1^2 + x_2^2 \Rightarrow \dot{V}_2 = \dot{x}_1 x_1 + \dot{x}_2 x_2 = -x_1^4 + x_2 x_1 + x_2 x_3$

Setting $\gamma(x_1,x_2) = -x_2 - x_1$

$\dot{V}_2 = \underset{<0}{-x_1^4} \underset{<0}{- x_2^2} < 0 \Rightarrow$ GAS

(3) $e_2 = x_3 - \gamma(x_1,x_2) = x_3 + x_2 + x_1$

change of coordinates $\quad x_3 = e - x_2 - x_1$

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = x_3 = e - x_2 - x_1 \\ \dot{x}_3 = \sin^2(x_1) + u \end{cases}$$

by choosing $u = -\sin^2(x_1) + v$

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = e - x_2 - x_1 \\ \dot{e} = v \end{cases}$$

$V_3(x_1,x_2,e) = \frac{1}{2}(x_1^2 + x_2^2 + e^2)$

$\dot{V}_3 = \dot{x}_1 x_1 + \dot{x}_2 x_2 + \dot{e}e = -x_1^4 + \cancel{x_2 x_1} + e x_2 - \cancel{x_1 x_2} - x_2^2 + e v$

Choosing $v = -x_2 - e$

One obtains

$$\dot{V}_3 = -x_1^4 - x_2^2 - e^2 < 0 \Rightarrow GAS$$