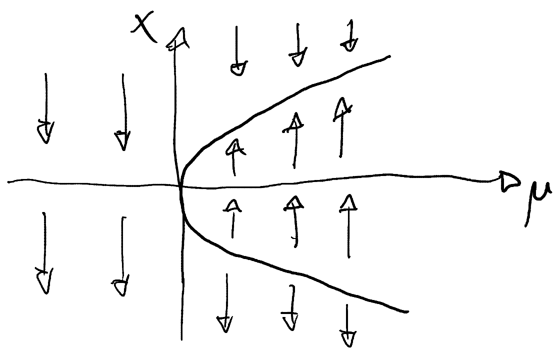
Biforcations are the change of behaviour of an equilibrium point when some parameters vary (structural perturbations).
Consider $\dot{x} = f(x,\mu)$, a system where μ is a parameter varying, and as μ varies, the equilibrium satisfies $f(x,\mu)=0$.
As μ varies, the solution $x^*(\mu)$ are smooth functions of μ so long that $\frac{\partial f}{\partial x}(x^*(\mu),\mu)$ does not have an eigenvalue in 0.

Consider the case when $f(x^*(\mu),\mu)=0$ and $\left|\frac{\partial f}{\partial x}(x^*(\mu),\mu)\right|=0$ so we have a change in the behaviour, the point is a non-hyperbolic point and it's possible to reduce its dynamics to the center manifold!

Consider now the monodimensional case: necessary conditions to have a biforcation are
$f(0,0)=0$ and $\frac{\partial f}{\partial x}(0,0)=0$

3 cases:

Saddle-node biforcation

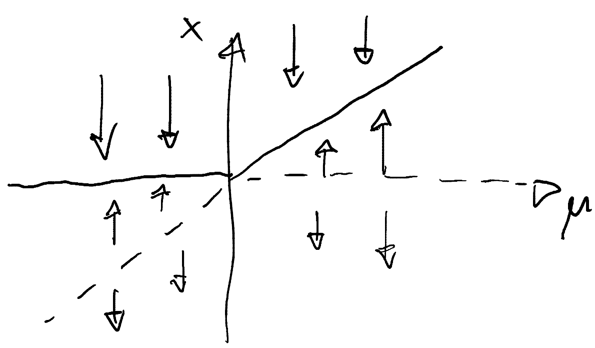


$\dot{x} = \mu - x^2 \Rightarrow \mu = x^2$ 2 equilibria

Suff. conditions

$\left.\frac{\partial f}{\partial \mu}\right|_{(0,0)} \neq 0 \qquad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$

Transcritical biforcation



$\dot{x} = \mu x - x^2 \Rightarrow$ equilibria $x=0 \quad x=\mu$

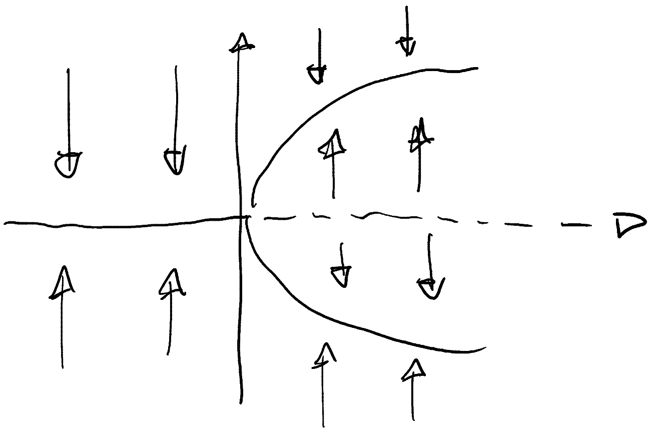
Suff. conditions

$\frac{\partial f}{\partial \mu}(0,0)=0 \qquad \frac{\partial^2 f}{\partial x^2}(0,0)=0 \qquad \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$

Pitchfork biforcation

$\dot{x} = \mu x - x^3 \Rightarrow$ equilibria $x=0 \quad x=\mu$

equilibria



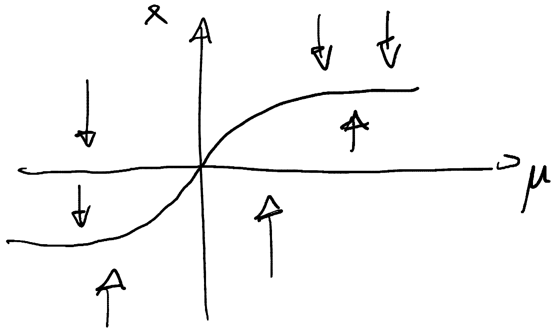
$$\dot{x} = \mu x - x^3 \Rightarrow x = 0 \quad x = \mu \quad x = -\mu$$

Suff. Conditions

$$\frac{\partial f}{\partial \mu}(0,0) = 0 \qquad \frac{\partial f^2}{\partial x^2}(0,0) = 0$$

$$\frac{\partial f^2}{\partial x \partial \mu}(0,0) \neq 0 \qquad \frac{\partial^3 f}{\partial x^3}(0,0) \neq 0$$

No bifurcation



No change of behavior of the equilibrium point

$$\dot{x} = \mu - x^3 \Rightarrow x^3 = \mu \text{ equilibria}$$

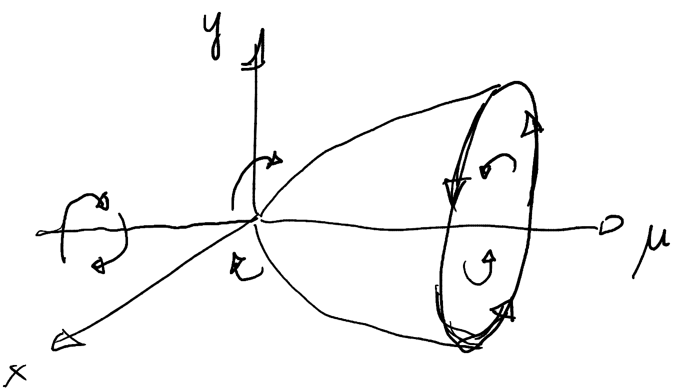
Hopf bifurcation ($\mathbb{R}^2$)

$$\begin{cases} \dot{x} = -\omega y + x(\mu - (x^2+y^2)) \\ \dot{y} = \omega x + y(\mu - (x^2+y^2)) \end{cases}$$

the linear part is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{so} \quad \lambda_1 = j\omega, \ \lambda_2 = -j\omega \quad \text{pure imaginary eigenvalues}$$

equilibria are in some sense orbits.



frontal view $\Rightarrow$