12. Kernels of linear and bilinear systems

lunedì 8 giugno 2020 13:21

Linear

$$\begin{cases} \dot{x}(t) = f(x) + g(x)u \\ y(t) = h(x) \end{cases} \qquad f(x) = Ax \quad g(x) = B \text{ (constant)} \quad h(x) = Cx$$

$$W_0(t, t_0; x_0) = e^{(t-t_0)L_{Ax}} Cx\Big|_{x_0} = \qquad e^{L_\tau}\lambda = \sum_{k=0}^{\infty} \frac{L_\tau^k \lambda}{k!}$$

computation of Lie derivatives

$$L_f h(x) = h'(x) \cdot f(x)$$

$$L_f^k h(x) = \frac{\partial L_f^{k-1} h(x)}{\partial x} f(x) \quad \text{with } L_f^0 h(x) = h(x)$$

$$L_g L_f h(x) = \frac{\partial L_f h(x)}{\partial x} \cdot g(x)$$

$$= Cx\Big|_{x_0} + (t-t_0) L_{Ax} Cx\Big|_{x_0} + \frac{(t-t_0)^2}{2!} L_{Ax}^2 Cx\Big|_{x_0} + \dots$$

$$= Cx\Big|_{x_0} + (t-t_0) CAx\Big|_{x_0} + \frac{(t-t_0)^2}{2!} CA^2x\Big|_{x_0} \dots$$

$$= C\left[(t-t_0)A + \frac{(t-t_0)^2}{2!}A^2 + \dots\right]x_0 = Ce^{A(t-t_0)}x_0$$

$$\left[L_{Ax}^k C = CA^k x\right]$$

$$W_1(t, \tau_1, t_0; x_0) = e^{(\tau_1 - t_0)L_{Ax}} L_B \, e^{(t-\tau_1)L_{Ax}} Cx\Big|_{x_0} =$$

$$e^{(\tau_1 - t_0)L_{Ax}} L_B \cdot Ce^{A(t-\tau_1)}x\Big|_{x_0} =$$

$$e^{(\tau_1 - t_0)L_{Ax}} \cdot \underbrace{Ce^{A(t-\tau_1)}B}_{\text{constant w.r.t. } x}\Big|_{x_0} =$$

$$= Ce^{A(t-\tau_1)}B + 0 + 0 \dots = Ce^{A(t-\tau_1)}B$$

$$W_2(t, \tau_1, \tau_2, t_0; x_0) = e^{(\tau_2 - t_0)L_{Ax}} L_B \cdot e^{(\tau_1 - \tau_2)L_{Ax}} L_B \cdot$$

$$\cdot e^{(t-\tau_1)L_{Ax}} Cx\Big|_{x_0} =$$

$$= e^{(\tau_2 - t_0)L_{Ax}} L_B \, e^{(\tau_1 - \tau_2)L_{Ax}} \left(Ce^{A(t-\tau_1)}B\Big|_{x_0} \right)$$

$$= 0 \,/$$

$= 0 !$

$W_i = 0 \quad i \geq 2$

$$y(t) = W_0(t, t_0; x_0) + \int_{t_0}^{t} W_1(t, \tau, t_0; x_0)\, u(\tau)\, d\tau =$$

$$= Ce^{A(t-t_0)} x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)} B u(\tau) d\tau$$

- Bilinear

$$\begin{cases} \dot{x}(t) = f(x) + g(x) u \\ y(t) = h(x) \end{cases} \qquad f(x) = Ax \quad g(x) = Nx + B \qquad h(x) = Cx$$

$$W_0(t, t_0; x_0) = e^{(t-t_0) L_{Ax}} Cx \Big|_{x_0} = Ce^{A(t-t_0)} x_0$$

$$W_1(t, \tau_1, t_0; x_0) = e^{(\tau_1 - t_0) L_{Ax}} L_{Nx+B}\, e^{(t-\tau_1) L_{Ax}} Cx \Big|_{x_0} =$$

$$e^{(\tau_1 - t_0) L_{Ax}} L_{Nx+B} \left(Ce^{A(t-\tau_1)} x \right) \Big|_{x_0} =$$

$$e^{(\tau_1 - t_0) L_{Ax}} \left[Ce^{A(t-\tau_1)} (Nx + B) \right]$$

$$= Ce^{A(t-\tau_1)}(Nx + B) + (\tau_1 - t_0) L_{Ax} \left(Ce^{A(t-\tau_1)} (Nx+B) \right) +$$
$$+ \frac{(\tau_1 - t_0)^2}{2!} L_{Ax}^2 \left(Ce^{(t-\tau_1)}(Nx+B) \right) + \dots + \Big|_{x_0} =$$

$$= Ce^{A(t-\tau_1)} B + Ce^{A(t-\tau_1)} Nx + (\tau_1 - t_0) Ce^{A(t-\tau_1)} NAx +$$
$$\frac{(\tau_1 - t_0)^2}{2!} Ce^{A(t-\tau_1)} NA^2 x$$

$$= Ce^{A(t-\tau_1)} \left[B + Nx + (\tau_1 - t_0) NAx + \frac{(\tau_1 - t_0)^2}{2!} NA^2 x \dots \right]$$

$$= Ce^{A(t-\tau_1)} B + Ce^{A(t-\tau_1)} Nx \left[1 + (\tau_1 - t_0) A + \frac{(\tau_1 - t_0)^2}{2!} A^2 \dots \right]$$

$$= Ce^{A(t-\tau_1)} B + Ce^{A(t-\tau_1)} N e^{A(\tau_1 - t_0)} x_0$$

$$W_2(t, \tau_1, \tau_2, t_0; x_0) = e^{(\tau_2 - t_0) L_{Ax}} L_{Nx+B}\, e^{(\tau_1 - \tau_2) L_{Ax}} L_{Nx+B} \cdot$$
$$\dots (t-\tau_1) L_{Ax}$$

$$W_2(t,\tau_1,\tau_2,t_0;x_0) = e^{(\tau_2-t_0)L_{Ax}} L_{Nx+B}\, e^{(\tau_1-\tau_2)L_{Ax}} L_{Nx+B} \cdot$$
$$\cdot\, e^{(t-\tau_1)L_{Ax}} C_x \big|_{x_0}$$
$$= e^{(\tau_2-t_0)L_{Ax}} L_{Nx+B}\left(W_1(t,\tau_1,\tau_2;x)\right)\Big|_{x_0}$$
$$= e^{(\tau_2-t_0)L_{Ax}} L_{Nx+B}\left(Ce^{A(t-\tau_1)}B + Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}x\right)\Big|_{x_0}$$
$$= e^{(\tau_2-t_0)L_{Ax}}\left(Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}(Nx+B)\right)\Big|_{x_0}$$
$$= Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}(Nx+B) +$$
$$+ (\tau_2-t_0)L_{Ax}\left(Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}(Nx+B)\right) \dots =$$
$$= Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}B + Ce^{A(t-\tau_1)}Ne^{(\tau_1-\tau_2)}Nx$$
$$+ (\tau_2-t_0)Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}NAx \dots =$$
$$Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)} = \tilde{\pi}$$
$$= \tilde{\pi}B + \tilde{\pi}Nx + \tilde{\pi}(\tau_2-t_0)NAx \dots =$$
$$= \tilde{\pi}B + \tilde{\pi}N\left[e^{A(\tau_2-t_0)}\right]x\Big|_{x_0} =$$
$$= Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}B +$$
$$+ Ce^{A(t-\tau_1)}Ne^{A(\tau_1-\tau_2)}Ne^{A(\tau_2-t_0)}x_0$$

$\vdots$

$$W_n(t,\tau_1,\dots\tau_n;x_0) = \left(Ce^{A(t-\tau_1)}N\dots Ne^{A(\tau_{n-1}-\tau_n)}\right)\cdot$$
$$\cdot\left(B + Ne^{A(\tau_n-t_0)}x_0\right)$$

$$\gamma_0(t,t_0;x_0) = e^{(t-t_0)L_g}\, Id\big|_{x_0}$$

$$e^{tL_g}\, Id\big|_x = \underbrace{Id\big|_x}_{x} + t\,\underbrace{L_g Id\big|_x}_{g(x)} \dots$$
$$e^{tL_g}h(x) = h(x) + tL_g h(x) \dots$$

if $f(x) = Ax \Rightarrow e^{t L_{Ax}} Id|_x = e^{At} x$

$$W_2(t, \tau_1, \tau_2, t_0; x_0) = \frac{\partial W_1(t, \tau_1, \tau_2, x)}{\partial x} \cdot \varepsilon(x) \Big|_{x = \gamma_0(\tau_2, t_0, x_0)}$$

$$\gamma_0(\tau_2, t_0, x_0) = e^{A(\tau_2 - t_0)} x_0$$

$$W_2(t, \tau_1, \tau_2, t_0; x_0) = C e^{A(t-\tau_1)} N e^{A(\tau_1 - \tau_2)} (Nx + B) \Big|_{x = \gamma_0(\tau_2, t_0, x_0)}$$

$$= C e^{A(t-\tau_1)} N e^{A(\tau_1 - \tau_2)} B +$$

$$+ C e^{A(t-\tau_1)} N e^{A(\tau_1 - \tau_2)} N e^{A(\tau_2 - t_0)} x_0$$