CALCULUS OF VARIATIONS & OPTIMAL CONTROL

Let's consider a dynamical system

$\dot{x} = f(x, u, t)$
$\quad \hookrightarrow C^2$ class

$u(t) \in \mathbb{R}^p$ control vector

$x(t) \in \mathbb{R}^n$ state vector

$x(t_i) = x_i$ known

Constraints:

$\chi(x(t_f), t_f) = 0$ class C^1 of dim $6 \delta \leq n+1$

$q(x, u, t) \leq 0$ class C^2 of dim β

$q_a = 0$ (active constraint) dim β_a

Norm: $\|(x, u, t_f)\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^{t_f} u(\tau) d\tau \right\| + \sup_t \|u(t)\| + |t_f|$

Cost index: $J(x, u, t_f) = \int_{t_i}^{t_f} L(x, u, t) dt$
$\quad \hookrightarrow C^2$ class

Goal: find t_f°, $u^\circ \in \bar{C}^0(\mathbb{R})$, $x^\circ \in \bar{C}^1(\mathbb{R})$ that satisfy the constraints and minimize J

Hamiltonian function (scalar)

$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 L(x, u, t) + \lambda^T(t) \overbrace{f(x, u, t)}^{\text{dynamical part of } x}$$

Theorem:

Let (x^*, u^*, t_f^*) be an admissible solution s.t.

$$rk\left\{ \frac{\partial \chi}{\partial(x(t_f), t_f)}\Big|^* \right\} = \sigma_f \qquad rk\left\{ \frac{\partial q_{active}}{\partial u}\Big|^* \right\} = \beta_a(t) \quad \forall t \in [t_i, t_f^*]$$

regularity

if (x^*, u^*, t_f^*) is a local minimum

$\exists\ \lambda_0^* \geq 0$, $\lambda^* \in \bar{C}^1[t_i, t_f^*]$, $\eta^* \in \bar{C}^0[t_i, t_f^*]$

not simultaneously null in $[t_i, t_f^*]$ such that:

- $\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T} - \frac{\partial q}{\partial x}\Big|^{*T} \eta^*$ <u>costate equation</u>

- $0 = \frac{\partial H}{\partial u}\Big|^{*T} + \frac{\partial q}{\partial u}\Big|^{*T} \eta^*$ <u>control equation</u>

- $\eta_j^*(t)\, q_j(x^*, u^*, t) = 0, \quad \eta_j^*(t) \geq 0, \ j = 1, 2, \dots \beta$

- $\lambda^*(t_f^*) = -\frac{\partial \chi}{\partial(x(t_f))}\Big|_{t_f^*}^{*T} \xi, \quad \xi \in \mathbb{R}^{\sigma_f}$
- $H\Big|_{t_f^*}^* = \frac{\partial \chi}{\partial t_f}\Big|_{t_f^*}^{*T} \xi$

$\Big\}$ trasversality conditions

The discontinuity of $\dot{\lambda}^*$ and η^* may occur only in the points $\bar{t}$ where u^* has a discontinuity and

$$H\Big|_{\bar{t}^-}^* = H\Big|_{\bar{t}^+}^*$$

Proof:

Rewrite the optimal control problem as a Lagrange problem and find the solution:

Introduce the new function v and the new variable z

$$v(t) = \int_{t_i}^{t_f} u(\tau)\,d\tau \qquad \dot v(t) = u(t) \qquad v(t_i) = 0$$

$\hookrightarrow \bar C^0$

$$z = \begin{pmatrix} x \\ v \end{pmatrix} \in \bar C^1(\mathbb{R}) \qquad \begin{cases} \text{if } v \text{ has a cusp } \int u \\ u \text{ has a jump} \end{cases}$$

admissible set (the same of before with $\dot v$ instead of u):

$$D = \Big\{ (z, t_f) \in \bar C^1(\mathbb{R}) \times \mathbb{R},\ z(t_i) = \begin{pmatrix} x_i \\ 0 \end{pmatrix},\ \chi(x(t_f), t_f) = 0,$$
$$v(t_f) \in \mathbb{R}^p,\ f(x(t), \dot v(t), t) - \dot x(t) = 0,\ q(x(t), \dot v(t), t) \le 0 \Big\}$$

$\hookrightarrow \dot x = f(x, v, t) \to \mathcal{E}$ constraint

Cost function $J(z, t_f) = \int_{t_i}^{t_f} \mathcal{L}(x(t), v(t), t)\,dt$

Applying the necessary condition of the LP

$$\ell(x, \dot x, \dot v, \lambda_0, \lambda, \eta, t) = \lambda_0 \mathcal{L}(x, \dot v, t) + \lambda^T(t)\big[\overbrace{f(x, \dot v, t) - \dot x(t)}^{\mathcal{E}}\big] + \eta^T(t)\, q(x, \dot v, t) =$$
$$= H(x, \dot v, \lambda_0, \lambda, t) - \lambda^T \dot x + \eta^T q(x, \dot v, t)$$

Rank condition:

$$\operatorname{rk}\left\{ \frac{\partial(\mathcal{E}, q_a)}{\partial \dot z}\Big|^* \right\} = \operatorname{rk}\left\{ \begin{pmatrix} \frac{\partial \mathcal{E}}{\partial \dot x} & \frac{\partial \mathcal{E}}{\partial \dot v} \\ \frac{\partial q_a}{\partial \dot x} & \frac{\partial q_a}{\partial \dot v} \end{pmatrix}\Bigg|^* \right\} = \operatorname{rk}\left\{ \begin{pmatrix} -I & \frac{\partial \mathcal{E}}{\partial u}\Big|^* \\ 0 & \frac{\partial q_a}{\partial u}\Big|^* \end{pmatrix} \right\} = n + \beta_a(t)$$

- Euler - Lagrange

$$\begin{cases} \dfrac{\partial \ell}{\partial x}\Big|^{o} - \dfrac{d}{dt}\dfrac{\partial \ell}{\partial \dot{x}}\Big|^{o} = 0^T \quad (1) \\ \dfrac{\partial \ell}{\partial v}\Big|^{o} - \dfrac{d}{dt}\dfrac{\partial \ell}{\partial \dot{v}}\Big|^{o} = 0^T \quad (2) \end{cases}$$

$$\boxed{\frac{d}{dx} f(g(\dots(n(x)))) = \frac{df}{dg}\cdot\frac{dg}{dh}\cdot \dots \cdot \frac{dn}{dx}}$$

Recall: $\ell = H - \lambda^T \dot{x} + \eta^T q$

(1) $$\frac{\partial H}{\partial x} + \frac{\partial \eta^T q}{\partial x} - \frac{d}{dt}(-\lambda^T) = 0^T$$

$$\frac{\partial H}{\partial x} + \eta^T \frac{\partial q}{\partial x} = -\dot{\lambda}^T \Rightarrow \boxed{\dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T - \frac{\partial q}{\partial x}\Big|^T \eta}$$

costate equation

$$\frac{\partial \ell}{\partial v} = 0$$

(2) $$0 - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{v}} = 0^T$$

since $\frac{d}{dt}(\dots) = 0$ means that $(\dots)$ is constant

then $\frac{\partial \ell}{\partial \dot{v}} = c^T$ $\rightarrow$ We know from the transversality conditions that

$$\frac{\partial \ell}{\partial \dot{z}}\Big|_{t_f} = \xi^T \frac{\partial \chi}{\partial (z(t_f), t_f)}$$

$$\frac{\partial \ell}{\partial \dot{v}}\Big|_{t_f} = \xi^T \frac{\partial \chi}{\partial (v(t_f, t_f))} = 0$$

$$\boxed{c^T = 0}$$

$$\boxed{0 = \frac{\partial H}{\partial u}\Big|^{*T} + \frac{\partial q}{\partial u}\Big|^{*T} \eta^*}$$ Control equation

rewriting the E-L equation:

$$\frac{\partial \ell}{\partial \dot{z}} - \int_{t_i}^{t_f} \frac{\partial \ell}{\partial z}\, d\tau = c^T$$

$$-\lambda^T = \int_{t_i}^{t_f} \frac{\partial \ell}{\partial z}\, d\tau + c^T \rightsquigarrow \bar{C}^1$$ regularity of λ

(3) From the necessary condition of the LP

$$\eta_j^*(t)\, q_j(x^*, v^*, t) = 0 \qquad \text{(directly)}$$

(4) The second W-E condition of the LP

$$\underbrace{\left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^*}_{} = \ell - \left(\frac{\partial \ell}{\partial \dot z}\ \frac{\partial \ell}{\partial \dot v}\right)\begin{pmatrix}\dot x\\ \dot v\end{pmatrix} = H\Big|^* \text{ continuous}$$

$$H - \cancel{\lambda^T \dot x} + \cancel{\eta^T q} - \underbrace{\frac{\partial \ell}{\partial \dot x}}_{\lambda^T \dot x}\dot x - \cancel{\frac{\partial \ell}{\partial \dot v}}\dot v$$

↳ it is continuous

The transversality conditions of the LP imply:

$$\left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^{*T}_{t_f^*} = H^*_{t_f} = \frac{\partial \chi}{\partial t_f}\Big|^{*T} \int$$

$$\left(\frac{\partial \ell}{\partial \dot x}\right)^{*T}_{t_f^*} = -\lambda^{*T}(t_f^*) = \int^T \frac{\partial \chi}{\partial x(t_f)}\Big|^*_{t_f^*} \quad \text{does not depend on } v$$

End

If $\mathcal{L}, q, f$ do not depend on t (stationary problem)

$$\boxed{H\big|^* = c \ , \quad \forall t \in [t_i, t_f^*]}$$

~ Control Problem

$$\begin{cases} \dot{x} = f(x,u,t) \\ x(t_i) = x_i \end{cases} \qquad x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^p, \; f \in C^2$$

Constraints

$$\chi(x(t_f), t_f) = 0 \qquad \chi \in C^1(\mathbb{R}^{6f}) \qquad \mathrm{rk}\left\{ \left(\frac{\partial \chi}{\partial (x(t_f), t_f)} \right) \Big|^{*} \right\} = 6f \le n+1$$

$$\int_{t_i}^{t_f} h(x(t), u(t), t)\, dt = k \qquad h \in C^2(\mathbb{R}^6)$$

Cost functional : $J(x,u,t_f) = \int_{t_i}^{t_f} L(x,u,t)\, dt$

L → C^2 class

Hamiltonian:

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 L(x,u,t) + \lambda^T(t) f(x,u,t) + \rho^T h(x,u,t)$$

If (x^*, u^*, t_f^*) is a local minimum

$\exists\ \lambda_0^* \in \mathbb{R}$, $\lambda^* \in \bar{C}^1[t_i, t_f^*]$, $\rho^* \in \mathbb{R}$ not simultaneously null in $[t_i, t_f^*]$ s.t :

- $\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T}$ <u>costate equation</u>
- $0 = \frac{\partial H}{\partial u}\Big|^{*T}$ <u>control equation</u>
- $\lambda^*(t_f) = -\frac{\partial \chi}{\partial (x(t_f))}\Big|^{*T}_{t_f^*} \xi \;, \quad \xi \in \mathbb{R}^{6f}$
- $H\Big|^{*}_{t_f^*} = \frac{\partial \chi}{\partial t_f}\Big|_{t_f^*} \xi$

<u>transversality conditions</u>

- <u>discontinuity of $\dot{\lambda}^*$</u> may occur in t_k where u^* has a discontinuity and $H\Big|^{*}_{t_k^-} = H\Big|^{*}_{t_k^+}$

~ Control problem with linear system and convexity

$$\begin{cases} \dot{x} = A(t)x + B(t)u & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^p, A, B \in C^2 \\ x(t_i) = x_i \\ x(t_f) \in D_f \text{ fixed point of } \mathbb{R}^n \end{cases}$$

$q(x,u,t) \leq 0 \quad q \in C^2 \quad \text{rank}\left(\frac{\partial q_a}{\partial u}\Big|^*\right) = \beta_a(t) \; \forall t \in [t_i, t_f]$

$$J(x,u) = \underbrace{G(x(t_f))}_{\text{Boltza term}} + \int_{t_i}^{t_f} \mathcal{L}(x,u)\, dt \qquad \mathcal{L} \in C^2, G \in C^3$$

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 \mathcal{L}(x,u,t) + \lambda^T(t) f(x,u,t)$$

$\to (\lambda_0 = 1)$

(x^*, u^*) is a normal optimal solution if

- $\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T} - \frac{\partial q}{\partial x}\Big|^{*T} \eta^*$
- $0 = \frac{\partial H}{\partial u}\Big|^{*T} + \frac{\partial q}{\partial u}\Big|^{*T} \eta^*$
- $\eta_j^*(t)\, q_j(x^*, u^*, t) = 0 \qquad j = 1, 2, \dots, \beta$
- $\eta^*(t) \geq 0$

and if $D_f = \mathbb{R}^n$ $\quad \lambda^*(t_f) = \frac{dG}{dx(t_f)}\Big|^{*T}$