25. High gain output feedback

Linear systems: it is well known that:

Any minimum phase plant with $r=1$ can be stabilized by means of an high gain from the output.

In fact, recalling the concept of Evans locus:

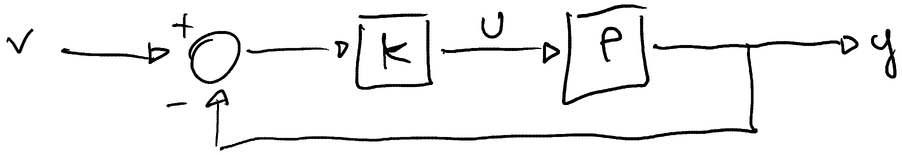
$$P(s) = k \cdot \frac{N(s)}{D(s)} \qquad k \text{ real}$$

the closed loop system is $W(s) = \dfrac{k N(s)}{D(s) + k N(s)}$

And in order to analyze the roots of the closed loop it is sufficient to study the polynomial

$$D_W(s) = D(s) + k N(s)$$

with k varying.



the control input $u = -ky + kv$ can be simplified as $u = -ky$ for large values of k.

From the state space point of view this fact can be formalized as follows:

$$P(s) = \frac{b_0 + b_1 s + \dots + b_{n-1} s^{n-1}}{s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & \dots & & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} b_0 & \dots & b_{n-1} \end{pmatrix}$$

$$u = -ky \qquad (A - BC) = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -(a_0 + kb_0) & \dots & & -(a_{n-1} + kb_{n-1}) \end{pmatrix}$$

if $(b_0, \dots, b_{n-1})$ is Hurwitz, then for any $(a_0, \dots, a_{n-1})$ and k large enough $(A + kB)$ is Hurwitz, therefore the closed-loop system is stable.

Non Linear systems: the analogous of the previous concept for the non linear case is the following:

Prop: Consider $\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$

with $f(0)=0$, $h(0)=0$ and relative degree $r=1$.

Suppose the zero dynamics stable in its first approximation i.e. the matrix Q has negative real part eigenvalues.

$$Q = \left(\frac{\partial q(z,\eta)}{\partial \eta} \right)_{(z,\eta)=(0,0)} \qquad \sigma(Q) \subset \mathbb{C}^-$$

Consider $\begin{cases} \dot{x} = f(x) + g(x)u \\ u = -k\,h(x) \end{cases}$ with $\begin{cases} k>0 & \text{if } L_g h(0) > 0 \\ k<0 & \text{if } L_g h(0) < 0 \end{cases}$

then $\exists\, k^*$ such that for all k satisfying $|k| > k^*$ $x=0$ is LAS

$r > 1$:

Assume that the given system is minimum phase in its first approximation, $\sigma(Q) \subset \mathbb{C}^-$, and has $r > 1$.

Choose a dummy output $w = k(x)$ such that $r_w = 1$

with $k(x) = L_f^{r-1} h(x) + \alpha_{r-2} L_f^{r-2} + \dots + \alpha_1 L_g h(x) + \alpha_0 h(x)$

where $\alpha_0, \dots, \alpha_{r-2}$ are real numbers.

$\underline{\alpha} = (\alpha_0, \dots, \alpha_{r-2})$ is Hurwitz.

then

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ w = k(x) \end{cases}$$

having relative degree $r_w = 1$ has zero dynamics which is AS in its first approximation:

$$w = \alpha_0 z_1 + \alpha_1 z_2 + \dots + z_r$$

and by imposing $w=0$, looking at the zero dynamics

$$z_r = -\alpha_0 z_1 - \dots - \alpha_{r-2} z_{r-1}$$

one obtains

$$S: \begin{cases} \dot{z}_1 = z_2 \\ \dots \end{cases}$$

$$S: \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_{r-1} = -\alpha_0 z_1 - \dots - \alpha_{r-2} z_{r-1} \\ \dot{\eta} = q(z_1, \dots, z_{r-1}, (-\alpha_0 z_1 - \dots - \alpha_{r-2} z_{r-1}), \eta) \end{cases}$$

$$\frac{\partial S}{\partial z \partial \eta} = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & 0 \\ & & & 1 \\ -\alpha_0 & \cdots & & -\alpha_{r-2} \end{matrix} & 0 \\ \hline \forall & \left.\frac{\partial q}{\partial \eta}\right|_{(0,0)} \end{array}\right)$$

$$\Rightarrow \sigma(Q_K) = \sigma(Q) \cup \sigma\begin{pmatrix} 0 & 1 & & \\ & & & 1 \\ -a_0 & \cdots & & -a_{r-2} \end{pmatrix} \text{ which is Hurwitz.}$$

$u = -k(x) = -k(\alpha_0 h + \alpha_1 L_g h + \dots + L_f^{r-1} h)$ provides AS from the state and it is obviously feasible.

In order to verify if it is possible to realize it from the output:

Prop: $\dot{x} = f(x) - g(x) k(x) K$

which is AS in the first approximation.

Then, with T sufficiently small positive

$$\begin{cases} \dot{x} = f(x) - g(x)\xi \\ \dot{\xi} = \frac{1}{T}(-\xi + k(x) K \end{cases}$$

is AS in its first approximation at $(0,0)$

Proof:

$z = -\xi + k(x) k$

$\dot{x} = f(x) - g(x)(-z + k(x) K)$

$T\dot{z} = -T\dot{\xi} + Tk \frac{\partial k}{\partial x}(f - g(-z + k(x) K))$

$= -\cancel{T} \frac{1}{\cancel{T}} z + T b(z, x)$

$\cdots -\frac{1}{\cancel{T}} z + T b(z,x)$

$T\dot{z} = -z + T b(z,x)$

where $z \simeq 0$ putting in light the reduced dynamics (slow dynamics).

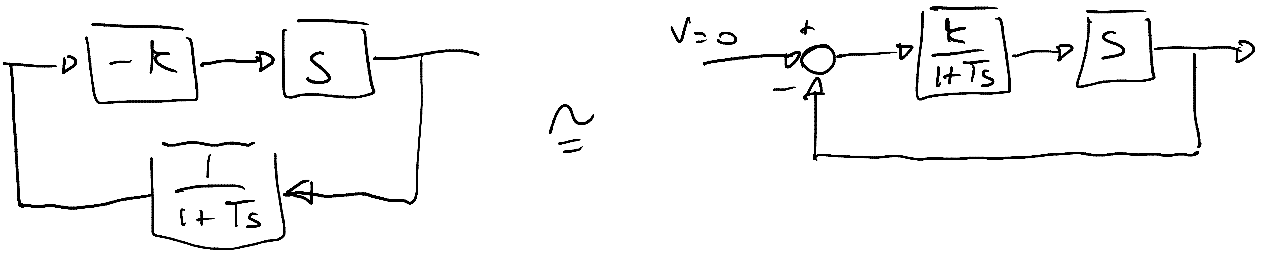
Since $\dot{x} = f - g\, k(x) K$, which is the original dynamics is AS in its first approximation, then also the system $\begin{cases} \dot{x} \\ \dot{\xi} \end{cases}$ is AS in its first approximation.

In conclusion:

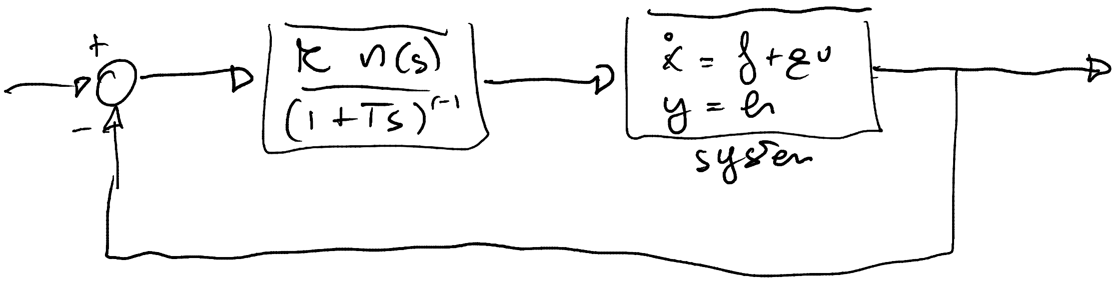
$$\begin{cases} \dot{x} = f(x) - g(x) u \\ u = k(x) \end{cases}$$

$$T\dot{\xi} + \xi = k(x) K \longrightarrow (Ts + 1)\, U(s) = -k\, Y(s)$$

$$\frac{U(s)}{Y(s)} = -\frac{k}{1+Ts}$$



and considering the above schemes:



where $n(s) = \alpha_0 + \alpha_1 s + \dots + s^{r-1}$ $\quad \underline{\alpha}$ Hurwitz

Theorem:

given S minimum phase in its first approximation with relative degree r. Asymptotic stability can be obtained through a dynamic high gain output feedback.

$$v(t) = \mathcal{L}^{-1}\left(-k \frac{n(s)}{(1+Ts)^{r-1}} Y(s)\right)$$