• Volterra series expansion

Considering the implicit representation (Input-affine)

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))\,u(t) \\ y(t) = h(x(t)) \end{cases}$$

$x(t) \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ analitic functions with scalar input and output

The explicit representation is the following

$$x(t) = \gamma_0(t, t_0; x_0) + \sum_{m \geq 0} \int_{t_0}^{t} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{m-1}} \gamma_m(t, \tau_1, \ldots, \tau_m, t_0; x_0)\, u(\tau_1) \ldots u(\tau_m)\, d\tau_1 \ldots d\tau_m$$

$$y(t) = W_0(t, t_0; x_0) + \sum_{m \geq 0} \int_{t_0}^{t} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{m-1}} W_m(t, \tau_1, \ldots, \tau_m, t_0; x_0)\, u(\tau_1) \cdots u(\tau_m)\, d\tau_1 \ldots d\tau_m$$

For the multi-input case, the system can be rewritten as:

$$\begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{p} g_i(x(t))\,u(t) \\ y(t) = h(x(t)) \end{cases}$$

The expressions of the explicit representations are VOLTERRA SERIES. They are the equivalent in the functional's field of the Taylor series expansion.
Any functional smooth enough can admit a series expansion
This expansion differs from Taylor's one for its ability to capture "memory" effects.
Output of a nonlinear system depends on the inputs at all other times, and Volterra series allow to separate each part depending on different power of $u(\cdot)$

- **Lie - derivative**

$$L_g h = \frac{\partial h}{\partial x} g$$

Lie series : $e^{L_g} = 1 + L_g + \frac{1}{2!} L_g^2 + \dots + \frac{1}{k!} L_g^k$

(note : Lie derivative satisfy the recursive property :

$$L_g^k h = \frac{\partial (L_g^{k} h)}{\partial x} g$$
)

- **Kernels computation (Picard expansion)**

$$\dot{y}(t) = \frac{\partial h}{\partial x}\Big|_{x(t)} \dot{x}(t) = \left(L_g + u(t) L_e\right) h \Big|_{x(t)}$$

integrating :

$$y(t) = y(t_0) + \int_{t_0}^{t} (L_g + u(\tau_1) L_e) h \Big|_{x(\tau_1)} d\tau_1$$

defining $\Lambda_1(x(\tau_1), u(\tau_1)) = (L_g + u(\tau_1) L_e) h \Big|_{x(\tau_1)} =$

$$= \Lambda_1(x_0, u(\tau_1)) + \int_{t_0}^{\tau_1} \frac{\partial \Lambda_1(x(\tau_2), u(\tau_1))}{\partial x}\Big|_{x(\tau_2)} \cdot \dot{x}(\tau_2)\, d\tau_2 =$$

$$= \Lambda_1(x_0, u(\tau_1)) + \int_{t_0}^{\tau_1} (L_g + u(\tau_2) L_e) \Lambda_1(\,; u(\tau_1))\Big|_{x(\tau_2)} d\tau_2$$

$$\Rightarrow y(t) = y_0 + \int_{t_0}^{t} \Lambda_1(x_0, u(\tau_1)) d\tau_1 + \int_{t_0}^{t}\int_{t_0}^{\tau_1} \Lambda_2(x(\tau_2), u(\tau_1), u(\tau_2)) d\tau_2 d\tau_1$$

with $\Lambda_2 = (L_g + u(\tau_2) L_e) \circ (L_g + u(\tau_1) L_e) h \Big|_{x(\tau_2)} =$

$$= (L_g^2 + u(\tau) L_e^2) h \Big|_{x(\tau)}$$

but

$$\Lambda_2(x(\tau_2), u(\tau_1), u(\tau_2)) =$$

$$= \Lambda_2(x_0, u(\tau_2), u(\tau_1)) + \int_{t_0}^{\tau_2} (L_g + u(\tau_3) L_e) \Lambda_2(\,; u(\tau_1), u(\tau_2))\Big|_{x(\tau_3)} d\tau_3$$

this consideration lead us to the most general form:

This consideration lead us to the most general form:

$$y(t) = h(x_0) + \int_{t_0}^{t} (L_f + u(\tau_1) L_g) h|_{x_0} \, d\tau_1 +$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau_1} (L_f + u(\tau_2) L_g) \circ (L_f + u(\tau_1) L_g) h|_{x_0} \, d\tau_2 \, d\tau_1 +$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau_1}\int_{t_0}^{\tau_2} (L_f + u(\tau_3) L_g) \circ (L_f + u(\tau_2) L_g) \circ (L_f + u(\tau_1) L_g) h|_{x_0} \, d\tau_3 \, d\tau_2 \, d\tau_1$$

...

$$\forall \; t \geq \tau_1 \geq \tau_2 \dots \geq \tau_m \geq t_0$$

using the previous expression, fixing $u(\tau_i) = 0 \; \forall i = 1 \dots m$ it's possible to compute the kernel of order zero

$$W_0(t, t_0; x_0) = h(x_0) + (t - t_0) L_f|_{x_0} + \frac{(t-t_0)^2}{2!} L_f^2 h|_{x_0} + \dots =$$

$$= \sum_{k \geq 0} \frac{(t-t_0)^k}{k!} L_f^k h|_{x_0} = e^{(t-t_0) L_f} h|_{x_0}$$

note: this is what we call, in linear systems, "free evolution"

and with some manipulations:

$$W_1(t, \tau_1, t_0; x_0) = \sum_{k_2, k_1 \geq 0} \frac{(\tau_1 - t_0)^{k_1}}{k_1!} \frac{(t - \tau_1)^{k_2}}{k_2!} L_f^{k_1} \circ L_g \circ L_f^{k_2} h|_{x_0} =$$

$$= e^{(\tau_1 - t_0) L_f} \circ L_g \circ e^{(t - \tau_1) L_f} h|_{x_0} \quad t \geq \tau_1$$

single input:

$$W_m(t, \tau_1, \dots, \tau_m, t_0; x_0) = \dots = e^{(\tau_m - t_0) L_f} \circ L_g \circ \dots \circ L_g \circ e^{(t - \tau_1) L_f} h|_{x_0}$$

For what concerns the Picard expansion of the state, knowing that $L_f \, Id|_x = f(x)$ with Id identity function we can write:

$$x(t) = x_0 + \int_{t_0}^{t} \dot{x}(\tau) d\tau = x_0 + \int_{t_0} L_f \, Id|_{x(\tau)} d\tau$$

and with the same procedure we obtain

and with the same procedure we obtain

$$x(t) = \gamma_0(t,t_0,x_0) = e^{(t-t_0)L_g}\, Id\big|_{x_0}$$

• The EXCHANGE THEOREM

The application of $e^{(t-t_0)L_g}$ to h is equal to the multiplication of h with the application of $e^{(t-t_0)L_g}$ to the Identity function:

$$y(t) = e^{(t-t_0)L_g} h\big|_{x(t_0)} = h\left(e^{(t-t_0)L_g}\, Id\big|_{x(t_0)}\right)$$

$$\boxed{\begin{aligned} x(t) &= \gamma_0(t,t_0,x_0) = e^{(t-t_0)L_g}\, Id\big|_{x_0} \\ y(t) &= W_0(t,t_0,x_0) = e^{(t-t_0)L_g}\, h\big|_{x_0} \end{aligned}}$$

this theorem allows to find the expression of the kernels of the Volterra series

INPUT OUTPUT : $W_m(t,\tau_1,\dots,\tau_m,t_0;x_0) = e^{(\tau_m-t_0)L_g}\circ L_g\circ\dots\circ L_g\circ e^{(\tau_1-t_0)L_g} h\big|_{x_0}$

INPUT STATE : $\gamma_m(t,\tau_1,\dots,\tau_m,t_0;x_0) = e^{(\tau_m-t_0)L_g}\circ L_g\circ\dots\circ L_g\circ e^{(\tau_1-t_0)L_g} Id\big|_{x_0}$

Computation of successive kernels can be done using $\gamma_0(t,t_0;x_0)$, $W_0(t,t_0,x_0)$ and $g(x)$

we can write

$$W_1(t,\tau_1,t_0;x_0) = e^{(\tau_1-t_0)L_g}\circ L_g\circ e^{(t-\tau_1)L_g} h\big|_{x_0} \overset{\text{exchange th}}{=}$$

$$= L_g e^{(t-\tau_1)L_g} h\Big(\underbrace{e^{(\tau_1-t_0)L_g}\, Id\big|_{x_0}}_{\gamma_0(\tau_1,t_0;x_0)}\Big) =$$

$$= L_g e^{(t-\tau_1)L_g} h\big|_{x=\gamma_0(\tau_1,t_0;x_0)} =$$

$$= \frac{\partial W_0(t,\tau_1,x)}{\partial x}\cdot g\Big|_{\gamma_0(\tau_1,t_0,x_0)}$$

$$\gamma_1(t,\tau_1,t_0;x_0) = e^{(\tau_1-t_0)L_g} \circ L_\varepsilon \circ e^{(t-\tau_1)L_g}\, Id\Big|_{x_0} = \dots$$

$$W_2(t,\tau_1,\tau_2,t_0;x_0) = e^{(\tau_2-t_0)L_g} \circ L_\varepsilon \circ e^{(\tau_1-\tau_2)L_g} \circ L_\varepsilon \circ$$
$$\circ\, e^{(t-\tau_1)L_g}\, h\Big|_{x_0} \overset{(*)\ \text{exch.}}{=}$$
$$= L_\varepsilon e^{(\tau_1-\tau_2)L_g} \circ L_\varepsilon e^{(t-\tau_1)L_g} h\Big(\underbrace{e^{(\tau_2-t_0)L_g} Id\big|_{x_0}}_{\gamma_0(\tau_2,t_0;x_0)}\Big)$$
$$= L_\varepsilon e^{(\tau_1-\tau_2)L_g} \circ L_\varepsilon e^{(t-\tau_1)L_g} h\Big|_{x=\gamma_0(\tau_2,t_0;x_0)}$$
$$= \frac{\partial W_1(t,\tau_1,\tau_2;x)}{\partial x} \cdot \varepsilon\Big|_{x=\gamma_0(\tau_2,t_0,x_0)}$$

$\vdots$

$$W_n(t,\tau_1,\dots,\tau_n;x_0) = \frac{\partial W_{n-1}(t,\tau_1,\dots,\tau_n;x)}{\partial x} \cdot \varepsilon\Big|_{x=\gamma_0(\tau_n,t_0;x_0)}$$

The same procedure for γ_n

- **From explicit to implicit representation**

$$g(x) = \frac{\partial \gamma_0(t,t_0;x)}{\partial x}\Big|_{t=t_0}$$

$$\varepsilon(x) = \gamma_1(0,0,0;x)$$

$$h(x) = W_0(0,0;x)$$