KINEMATIC MODELS OF MOBILE ROBOTS

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- Kinematic Models of Systems with Pfeffien constraints:
  - $q \in C$
  - $C = R^2$
  - $\begin{pmatrix} a_1^T(q) \\ \dots \\ a_k^T(q) \end{pmatrix} \dot{q} = A^T(q)\dot{q} = 0$
  - $\dot{q} \in N\left(A^T(q)\right)$ an $m = (n-k)$-dimensional space
  - $\{g_1(q), g_2(q), \dots, g_m(q)\}$ a basis for $N\left(A^T(q)\right)$
    - All admisible velocities can be obtained as linear combinations of the basis vectors.
    - $\dot{q} = \sum_{i=1}^{m} u_i g_i(q) = \left(g_1(q), g_2(q), \dots, g_m(q)\right) \begin{pmatrix} u_1 \\ \dots \\ u_m \end{pmatrix} = G(q)u$
    - $G(q)$: $[n \times m]$
    - u: $[mx1]$: The vector is independent of the model, they are **exhogenous**.
  - We end up with a dynamic system in which the vector u is the input and q is the state.
    - $\dot{q} = G(q)u$
    - This is a linear function of the inputs but not of the states so it is a nonlinear system.
    - **Driftless system**: the state does not change if no inputs are applied.
      - $u = 0 \rightarrow \dot{q} = 0$
  - Since the number of inputs is less than the number of states the system is **underactuated**.
  - If the system is found to be comtrollable then the kinematic constraint is NH.
- Kinematic Models of Systems withot constraints:
  - Any $\dot{q} \in$ C is possible
  - $\dot{q} = \sum_{i=1}^{n} u_i g_i$
    - u_i: constants
    - g_i: vectors
  - Using the canonical form (the system is represented as the set of time derivatives of the states):
    - $\dot{q} = u$
  - Since the number of inputs is the same as the number of states the system is **fully actuated**.
- **Unicycle**:
  - Vehicle with a single wheel that is orientable.
  - Same model as Rolling Disk:
    - $q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$
    - $C = SE(2)$
    - Constraints:
      - Pure rolling:
        - $\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix} \dot{q} = 0$
        - It is a Pfeffien constraint
        - Geometrically it means that $\begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix}$ is orthogonal to the velocity vector $\dot{q}$.
    - Admissible velocities: $N - k = 2$ Dimensional space.

- $\dot{q} \in \mathcal{N}(A^T(q))$
- $\mathcal{N}(A^T(q)) = \left[\begin{pmatrix} cos\theta \\ \sin(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$
- Drive vector field
  - $\begin{pmatrix} cos\theta \\ \sin(\theta) \\ 0 \end{pmatrix}$
- Steer vector field
  - $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- $\dot{q} = u_1 g_1(q) + u_2 g_2(q)$
- Canonincal representation:
  - $\dot{x} = u_1 \cos(\theta)$
  - $\dot{y} = u_1 \sin(\theta)$
  - $\dot{\theta} = u_2$
- The interpretation of the inputs is:
  - $u_1^2 = v^2 = \dot{x}^2 + \dot{y}^2$ is the **driving velocity**
  - $u_2 = \omega = \dot{\theta}$ is the **steering velocity**
- **Statically balanced** vehicles that are equivalent to the unicycle:
  - Differential drive robot.
  - Synchrodrive robot.

- **Bicycle**:
  - A vehicle with one fixed wheel and one orientable wheel.
  - Coordinates:
    - Rear wheel (x, y)
    - Front wheel (x_f, y_f)
    - Orientation θ
    - Orientation of steering wheel ϕ
  - Configuration space
    - $q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix}$
    - $C = R^2 \times SO(2)^2$
    - Locally $C = R^4$
  - Constraints: Pure rolling for both wheels.
    - Rear:
      - $\sin\theta \dot{x} - \cos\theta \dot{y} = 0$
    - Front:
      - $\sin(\theta + \phi)\, \dot{x}_f - \cos(\theta + \phi)\, \dot{y}_f = 0$
    - The two zero-motion lines meet at a point C that is called **instantaneous center of rotation**. Every point of the robot is instantaneously rotating around C. When $\phi = 0$, C goes to infinity and the bicylce moves in a straight line.
    - As Pfeffien constraints:
      - $x_f = x + l\cos\theta \rightarrow \dot{x}_f = \dot{x} - l\dot{\theta}\sin\theta$
      - $y_f = y + l\sin\theta \rightarrow \dot{y}_f = \dot{y} + l\dot{\theta}\cos\theta$
      - The front wheel constraints become:
      - $\sin(\theta + \phi)(\dot{x} - l\dot{\theta}\sin\theta) - \cos(\theta + \phi)(\dot{y} + l\dot{\theta}\cos\theta) = 0$
      - Use trigonometric identity
      - $A^T(q) = \begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta + \phi) & -\cos(\theta + \phi) & -l\cos\phi & 0 \end{pmatrix}$
      - $A^T(q)\dot{q} = 0$

- Rear-wheel drive
  - Drive velocity:
    - $v^2 = \dot{x}^2 + \dot{y}^2$
  - Basis for the nullspace
    - $g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{l} \\ 0 \end{pmatrix}$
    - $g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
  - $\dot{q} = g_1 u_1 + g_2 u_2$
  - Note that:
    - $u_1^2 = v^2 = \dot{x}^2 + \dot{y}^2$ is the **driving velocity**
    - $u_2 = \omega = \dot{\phi}$ is the **steering velocity**
  - Remark: What if $\phi = \frac{\pi}{2}$
    - We have a singularity in the kinematic model because the vector of the driving velocity goes to infinity. There is no motion possible if the driving velocity is on the rear wheel
- Front-wheel drive:
  - Drive velocity:
    - $v^2 = \dot{x_f}^2 + \dot{y_f}^2$
  - Basis for nullspace:
    - $g_1(q) = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{\sin\phi}{l} \\ 0 \end{pmatrix}$
      - Same as previous but multiplied by $\cos\phi$
    - $g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
  - $\dot{q} = g_1 u_1 + g_2 u_2$
  - Note that:
    - $u_1^2 = v^2 = \dot{x_f}^2 + \dot{y_f}^2$ is the **driving velocity**
    - $u_2 = \omega = \dot{\phi}$ is the **steering velocity**
  - Remark: What if $\phi = \frac{\pi}{2}$
    - The model does not have a singularity anymore. The vehicle will rotate around the contact point of the rear wheel. Note that the instantaneous point of rotation C is the contact point of the rear wheel.
- Nonholonomy: Is the Pfeffien constraint non-holonomic?
  - Since the system has two constraints, both of them have to be integrable at the same time.
  - Can we prove controllability constructively by exhibiting a maneuver that goes anywhere in C?
    - "Guessing" the right manouver is very complicated so it is better to have a systematic proof of controllability instead of a constructive proof.
- Mechanically balanced robots equivalent to the bicycle:
  - Tricycle:
    - Two rear fixed wheels (with differential if the actuation is done here)
    - One front orientable wheel

- Car-like:
  - Two rear fixed wheels
  - Two fron orientable wheels
    - They cannot be exactly parallel because there would be 2 ICR and there would be slippage. This is avoided with a mechanical device called **Ackermann steering** that makes the wheels slightly not parallel so that there is only one ICR
  - A differential in the axel where the robot is actuated.

- Controllability of **Nonlinear Driftless** systems:
  - $\dot{x} = \sum_{i=1}^{n} u_i g_i(x)$
    - u_i: inputs
    - $g_i(x)$: vector fields
  - The system is controllable if there exists a history of inputs over an interval of time such that the state x is driven from any configuration $x(0)$ to the desired configuration at a given time $x(T)$.
  - **Lie Bracket** operation:
    - $[g_i, g_j](x) = \frac{\partial g_j}{\partial x} g_i - \frac{\partial g_i}{\partial x} g_j$
    - Example:
      - $g_1 = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}$
      - $g_2 = \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix}$
      - $[g_i, g_j](x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \sin x_3 \\ -x_1 \cos x_3 \\ \cos x_3 \end{pmatrix}$
    - Interpretation: it is the result of a particular maneuver
      - $\dot{x} = g_1(x)u_1 + g_2(x)u_2$
      - Steps:
        1) $u_1 = 1, u_2 = 0 \quad t \in [0, \epsilon)$
        2) $u_1 = 0, u_2 = 1 \quad t \in [\epsilon, 2\epsilon)$
        3) $u_1 = -1, u_2 = 0 \quad t \in [2\epsilon, 3\epsilon)$
        4) $u_1 = 0, u_2 = -1 \quad t \in [3\epsilon, 4\epsilon)$
      - It can be shown that:
        - $x(4\epsilon) = x_0 + \epsilon^2 [g_1(x_0), g_2(x_0)] + o(\epsilon^3)$
      - This means that the total displacement is in the direction of the Lie bracket of the two vectors if ϵ is small enough so that the higher order terms are ignored.
  - **Accessibility Distribution** of a system:
    - $g_1(x), \dots, g_m(x)$
    - $\Delta = span\{g_1(x), \dots, g_m(x)\}$
      - Δ is a distribution associated to the collection of vectors. It is a range space that changes with x
    - $\Delta_A = \left\{g_1(x), \dots, g_m(x), [g_i, g_j](x), \dots, \left[g_k, [g_i, g_j]\right](x), \dots\right\}$
      - The directions can be expanded up to n which is the space that we are working on.
    - The system is controllable iff $\dim(\Delta_A) = n$ at any value of x.
      - **Accessibility rank test** or
      - **Lie Algebra rank condition** or
      - **Chow's theorem**
    - Application: **Controllability of the Unicycle**:
      - Kinematic model of the unicycle

- $\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega$

- Controllable iff $rank(g_1, g_2, [g_1 . g_2], [g_1, [g_1, g_2]], \ldots) = 3$ because the system is 3D
  - $[g_1, g_2] = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$
    - The resulting vector of the maneuver is in the direction of the zero motion line of the unicycle.
  - $rank\begin{pmatrix} cos\,\theta & 0 & sin\,\theta \\ sin\,\theta & 0 & -cos\,\theta \\ 0 & 1 & 0 \end{pmatrix} = 3$ Because the determinant is non-zero
  - So the unicycle is controllable.

- Application: **Controllability of the Bicycle**:
  - Kinematic model of the **rear wheel drive**
    - $\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{l} \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \omega$
  - Controllable iff $rank(g_1, g_2, [g_1 . g_2], [g_1, [g_1, g_2]], \ldots) = 4$ because the system is 4D
    - $[g_1, g_2] = g_3 = -\begin{pmatrix} 0 \\ 0 \\ \frac{1}{l\cos^2\phi} \\ 0 \end{pmatrix}$
      - This motion represents a rotation of the vehicle on the spot. It is called **wriggle**. It is also linearly independent from the other vectors but we still need another vector.
    - $[g_1, g_3] = g_4 = \begin{pmatrix} -\frac{\sin\theta}{l\cos^2\phi} \\ \frac{\cos\theta}{l\,cos^2\,\phi} \\ 0 \\ 0 \end{pmatrix}$
      - This motion represents a **slide**.
    - $rank\begin{pmatrix} cos\,\theta & 0 & 0 & -\frac{sin\,\theta}{l\,cos^2\,\phi} \\ sin\,\theta & 0 & 0 & \frac{cos\,\theta}{l\,cos^2\,\phi} \\ \frac{tan\,\phi}{l} & 0 & \frac{1}{l\,cos^2\,\phi} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 4$
      - $det = \frac{1}{l^2\cos^4\phi}$
  - Kinematic model of the **front wheel drive**.
    - $\dot{q} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{\sin\phi}{l} \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \omega$
  - Controllable iff $rank(g_1, g_2, [g_1 . g_2], [g_1, [g_1, g_2]], \ldots) = 4$ because the system is 4D

- $[g_1, g_2] = g_3 = \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ -\frac{\cos\phi}{l} \\ 0 \end{pmatrix}$
- $[g_2, g_3] = g_4 = \begin{pmatrix} -\frac{\sin\theta}{l} \\ \frac{\cos\theta}{l} \\ 0 \\ 0 \end{pmatrix}$
- The four vectors are clearly linearly independent so $rank(g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], \dots) = 4$

- **Chained Forms**:
  - They are an "almost" cannonical form for mobile robots. A canonical form is of great interest for solving planning and control plobems with efficient, systematic procedures.
  - $(2, n)$ chained form is a nonlinear, driftless, 2-input system. Where n is the number of states
    - State:
      - $\dot{z} = \gamma_1(z)v_1 + \gamma_2(z)v_2$
    - Equations:
      - $\dot{z}_1 = v_1$
      - $\dot{z}_2 = v_2$
      - $\dot{z}_3 = z_2 v_1$
      - …
      - $\dot{z}_n = z_{n-1} v_1$
  - Using:
    - $\gamma_1 = \begin{pmatrix} 1 \\ 0 \\ z_2 \\ \dots \\ z_{n-1} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}$
  - We get that the repeated Lie Bracket:
    - $\gamma_{k+2} = ad_{\gamma_1}^k \gamma_2 = \begin{pmatrix} 0 \\ \dots \\ (-1)^k \\ \dots \\ 0 \end{pmatrix}$
    - Where $(-1)^k$ is the $(k+2)$-th component
  - This implies that the system is **controllable**, because the accessibility distribution has dimension n
    - $\Delta_A = span\{\gamma_1, \gamma_2, \dots, \gamma_n\}$
  - In general, it may be possible to transform a generic 2-input driftless system into chained form via coordinate and input transformations:
    - $\dot{q} = g_1(q)u_1 + g_2(q)u_2$
    - $z = \alpha(q)$
      - z: new coordinates
      - q: old coordinates
      - α: transformation function
    - $v = \beta(q)u$
      - v: new input
      - u: original inputs
      - β: transformation function
    - To compute the transformations we need to define the distributions:
      - $\Delta_0 = span\{g_1, g_2, ad_{g_1} g_2, \dots, ad_{g_1}^{n-2} g_2\}$
      - $\Delta_1 = span\{g_2, ad_{g_1} g_2, \dots, ad_{g_1}^{n-2} g_2\}$

- $\Delta_2 = span\{g_2, ad_{g_1}g_2, \dots, ad_{g_1}^{n-3}g_2\}$
- There exist two scalar functions $h_1(q)$ and $h_2(q)$ that satisfies
  - $dh_1 \cdot \Delta_1 = 0$
  - $dh_1 \cdot g_1 = 1$
  - $dh_2 \cdot \Delta_2 = 0$
  - $dh_2 \cdot g_2 = 1$
- So the transformation would be:
  - $z_1 = h_1$
  - $z_2 = L_{g_1}^{n-2}h_2$
  - $z_{n-1} = L_{g_1}h_2$
  - $z_n = h_2$
- Where L_{g_1} is the Lie derivative of a scalar function
  - $L_g\alpha(x) = \dfrac{\partial\alpha}{\partial x}g(x)$
  - $\alpha(x)$: Scalar function
- Example: Unicycle
  - $\dot{q} = \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}u_1 + \begin{pmatrix}0\\0\\1\end{pmatrix}u_2$
  - Coordinate Transformation:
    - $z_1 = \theta$
    - $z_2 = x\cos\theta + y\sin\theta$
    - $z_3 = x\sin\theta - y\cos\theta$
  - Input transformation
    - $u_1 = v_2 + z_3v_1$
    - $u_2 = v_1$
  - Chained form:
    - $\dot{z}_1 = \dot{\theta} = u_2 = v_1$
    - $\dot{z}_2 = \dot{x}\cos\theta + \dot{y}\sin\theta - (x\sin\theta - y\cos\theta)\dot{\theta} = u_1 - z_3u_2 = v_2$
    - $z_3 = \dot{x}\sin\theta - \dot{y}\cos\theta + (x\cos\theta + y\sin\theta)\dot{\theta} = z_2u_2 = z_2v_1$
- Example: Bicycle
  - $q = \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\\ \dfrac{\tan\phi}{l}\\ 0\end{pmatrix}u_1 + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}u_2$
  - Coordinate Transformation:
    - $z_1 = x$
    - $z_2 = \dfrac{1}{l}\sec^3\theta\tan\phi$
    - $z_3 = \tan\theta$
    - $z_4 = y$
  - Input transformation
    - $u_1 = \dfrac{v_1}{\cos\theta}$
    - $u_2 = -\dfrac{3}{l}v_1\sec\theta\sin^2\phi + \dfrac{1}{l}v_2\cos^3\theta\cos^2\phi$
  - Chained form
    - $\dot{z}_1 = v_1$
    - $\dot{z}_2 = v_2$
    - $\dot{z}_3 = z_2v_1$
    - $\dot{z}_4 = z_3v_1$
- Kinematic models of WMRs with 2 inputs and $n \le 4$ states can **always** be put in chained form.

- For $n \geq 5$: maybe
  - Remarkable case: n-trailer system
    - Regardless of the number of trailers, the system can be put in chained form if the **hooking distance** is zero.
      - Hooking distance: distance from the rear wheel of the vehicle to the hooking point with the trailer.
    - If the distance is non-zero and the number of trailers is more than one then the system cannot be put in chained form.
- Control and planning can be generalized using chained form.

• Kinematic model of mobile robots

- kinematic models
- unicycle
- bicycle
- controllability of nonlinear systems
- chained form

<u>wrap up</u>

a robot subject to k kinematic constraints

$$k \left\{ \underbrace{A(q)}_{n} \dot{q} = 0 \quad (*) \right.$$

$q \in C$ n-dim

2 possibilities	Local mobility	Global mobility
$(*)$ is holonomic	restricted	restricted $q \in C$, q must satisfy the integral of $(*)$, a <u>subset</u> of C of dim $n-k$
$(*)$ is nonholonomic	restricted	unrestricted $q \in C$, no limitation of q

$\boxed{\dot{q} \in N(A^T(q)), \text{ a linear subspace of } C \text{ of dim } n-k}$

<u>kinematic models of WMRs</u>

$q \in C$, n-dimensional

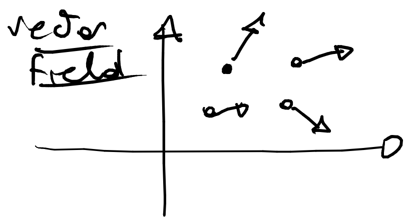
subject to $\quad k \left\{ \underbrace{A^T(q)}_{n} \dot{q} = 0 \right.$

we want to express in a kinematic model

i.e. $\dot{q} \in N(A^T(q)) \quad (n-k)$ dim

$\{ g_1(q), \ldots, g_{n-k}(q) \}$ a basis of $N(A^T(q))$

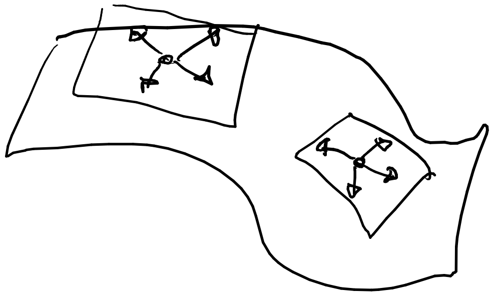
vector fields



$$\dot q \in N(A^T(q)) \iff \boxed{\dot q = \sum_{j=1}^{m=n-k} g_i(q) \underbrace{v_j}_{\text{scalar coeff.}}} \qquad (\#)$$

Kinematic Model

it express all the admissible (gen) velocities at q



(#) can be written as

$$\dot q = \begin{pmatrix} | & & | \\ g_1(q) & \cdots & g_m(q) \\ | & & | \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

$$\dot q = G(q)\, v$$

$n \times m$ → a tall matrix

it's a dynamical system $\left(\dot x = Ax + Bu \quad \dot x = \underbrace{f(x)}_{\text{drift}} + \sum_{i=1}^{p} g_i(x)\, u \right)$

with state q (n-dim) with inputs v (m-dim)

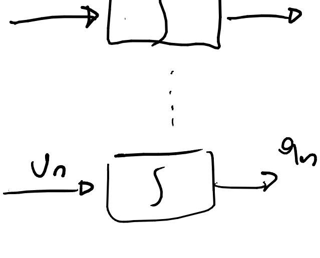
- nonlinear, driftless (no motion if $v=0$)

if a robot is <u>unconstrained</u>

$\dot q$ can be arbitrary

$$\dot q = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} v_2 + \dots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v_n$$

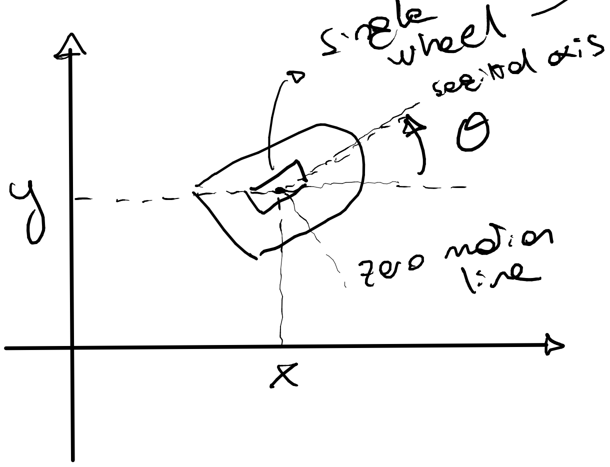
$$\dot q = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & & \vdots \\ \vdots & & 0 \\ 0 & & 1 \end{pmatrix} v = u$$



the kin. model of an unconstrained robot is a set of n simple integrators

Unicycle

balanced
it won't fall cause it is an ideal case



Same of the rolling coin

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \qquad n = 3$$

rolling without slipping $\rightarrow$ $\dot{x} \sin\theta - \dot{y} \cos\theta = 0$

$$\underbrace{\begin{pmatrix} \sin\theta & \cos\theta & 0 \end{pmatrix}}_{A^T(q)} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = 0$$

need two vectors

$\dot{q} \in N(A^T(q))$ a 2-dim space

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

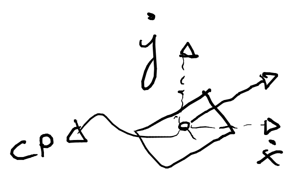
$$\dot{q} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{\text{drive vf}} v_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{steer vf}} v_2$$

kinematic model

$$\dot{q} = \underbrace{\begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix}}_{G(q)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{or} \quad \begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = v_2 \end{cases}$$

interpretation of v_1, v_2

$\dot{x}^2 + \dot{y}^2 = v_1 \rightarrow v_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2}$

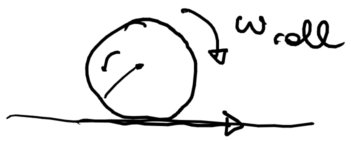


the magnitude (with sign) of the cartesian vel of the CP

$\downarrow$

v_1 driving velocity

from the side



$$\sqrt{\dot{x}^2 + \dot{y}^2} = r \cdot \omega_{roll}$$

$\dot{\theta} = u_2$ angular velocity of the wheel around the vertical axis

$\downarrow$

steering velocity

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{\text{drive vf}} \underbrace{v}_{\text{driving vel}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{steer vf}} \underbrace{\omega}_{\text{steering vel}}$$

v, ω : PSEUDOVELOCITIES ($\neq$ generalized velocities)

they are VELOCITY INPUTS!

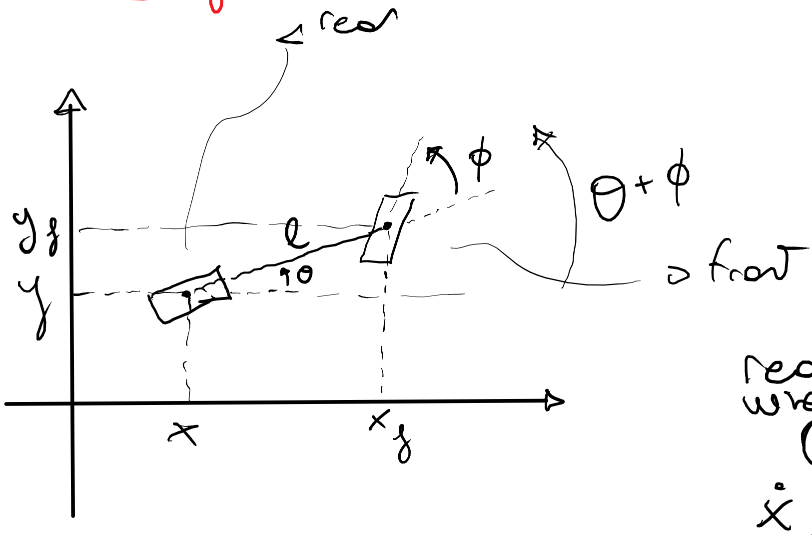
<u>holonomic or nonholonomic?</u>

is the <u>rolling without slipping</u> constraint H/NH?

we already know: it's NH

(<u>constructive</u> controllability)

Bicycle



$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \qquad n = 4$$

constraints? 2

rear wheel (1) $\boxed{\dot{x}\sin\theta - \dot{y}\cos\theta = 0}$

and in this case $\dot{x}, \dot{y}$ are $\dot{x}_f, \dot{y}_f$ and θ is $\theta+\phi$

front wheel (2) $\boxed{\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0}$

$$x_f = x + l\cos\theta$$
$$y_f = y + l\sin\theta \qquad \text{rigid body}$$

(2) becomes $(\dot{x} + l\sin\theta\,\dot{\theta})\sin(\theta+\phi) - (\dot{y} + l\cos\theta\,\dot{\theta})\cos(\theta+\phi) = 0$

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\left(\sin\theta\sin(\theta+\phi) + \cos\theta\cos(\theta+\phi)\right)$$

$$\boxed{\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\cos\phi = 0} \quad (2)$$

(1) + (2)

$$\underbrace{\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{pmatrix}}_{k=2\ \left\{ A^T(q) \right.,\ n=4} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}}_{\dot{q}} = 0$$

a basis for $N(A^T(q))$

drive vf

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/l \\ 0 \end{pmatrix}$$

in fact $A^T(q) \cdot g_1 = 0$

steer vf

$$g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/l \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

or

$$\dot{q} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ (\tan\phi)/l & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$ kinematic model of the bicycle

u_1, u_2 ?

$u_1^2 = \dot{x}^2 + \dot{y}^2$ or $u_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2}$

(as before) so it's a good choice for REAR WHEEL DRIVE

$u_1 = v$ driving velocity

$u_2 = \dot{\phi} = \omega$ steering velocity

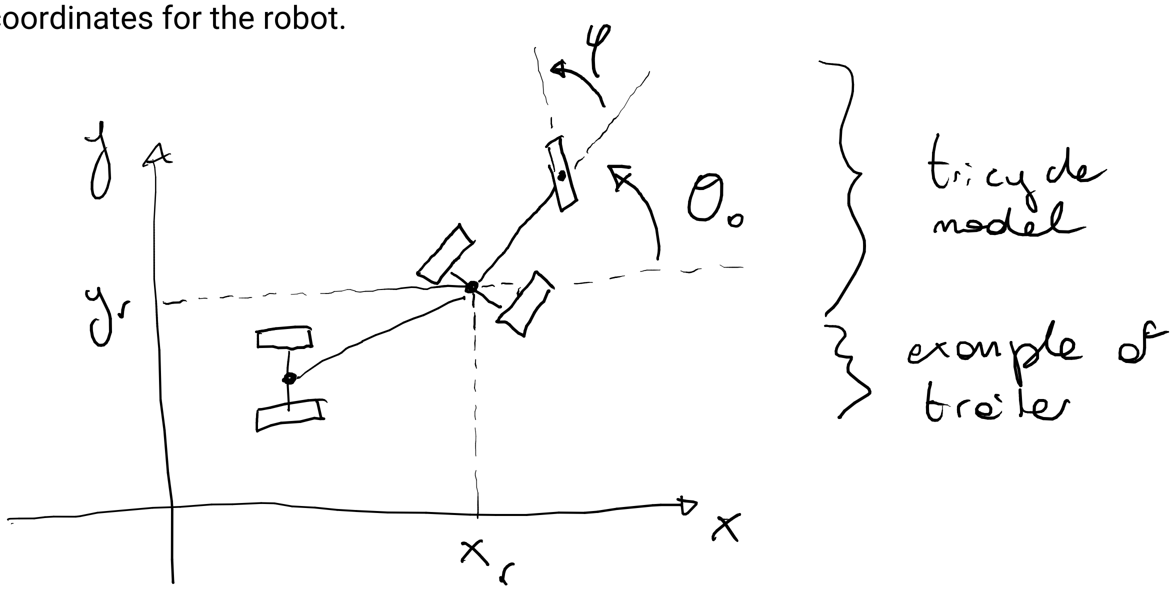
<u>exercise</u> : find a basis for the case of FRONT WHEEL DRIVE

Tip: <u>need</u> $\dot{x}_f^2 + \dot{y}_f^2 = u_1$

so try to modify the model of $\dot{q}$ in order to get u_1

Problems

Consider the mobile robot obtained by connecting N trailers to a rearwheel drive tricycle. Each trailer is a rigid body with an axle carrying two fixed wheels, that can be assimilated to a single wheel located at the midpoint of the axle, and is hinged to the midpoint of the preceding axle through a revolute joint. Find a set of generalized coordinates for the robot.



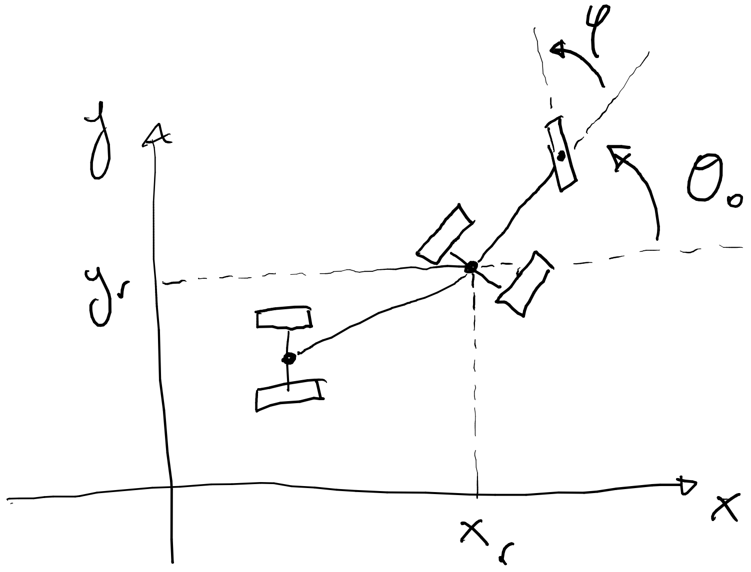
Once the configuration of the tricycle and of the first $i-1$ trailers are given $(i = 1 \dots N)$, the configuration of the i-th trailer is completely described by the its orientation θ_i w.r.t. the x axis.

Therefore the configuration vector for the complete vehicle is:

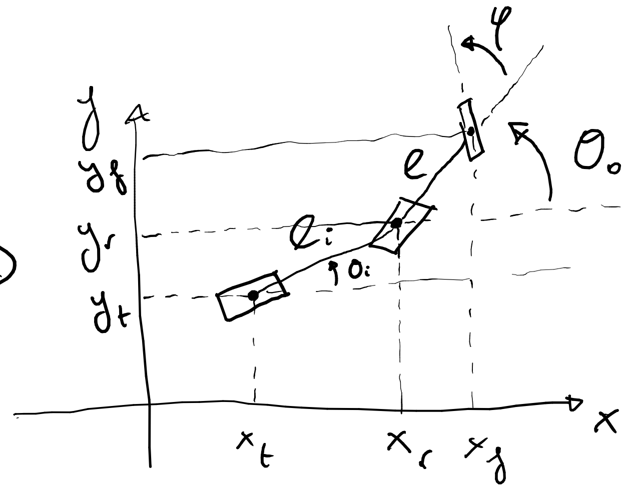
$$q = \begin{pmatrix} x \\ y \\ \theta_0 \\ \varphi \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix} \qquad C = \mathbb{R}^2 \times \underbrace{SO(2) \times \dots \times SO(2)}_{N+2}$$

Another possibility is to replace the absolute angle θ_i with the relative one $\theta_i - \theta_{i-1}$, to describe the orientation of each trailer

Derive the kinematic model of the tricycle robot towing N trailers considered in the previous Problem. Denote by "l" the distance between the front wheel and rear wheel axle of the tricycle, and by "l_i" the joint-to-joint length of the i-th trailer.



$\Longrightarrow$



$$q = \begin{pmatrix} x \\ y \\ \theta_0 \\ \varphi \\ \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}$$

the N+2 kinematic constraints are:

$$\left.\begin{aligned} &\dot{x}_f \sin(\theta_0+\varphi) - \dot{y}_f \cos(\theta_0+\varphi) = 0 \\ &\dot{x}_r \sin\theta_0 - \dot{y}_r \cos\theta_0 = 0 \\ &\dot{x}_{t_i} \sin\theta_i - \dot{y}_{t_i} \cos\theta_i = 0 \end{aligned}\right\} \text{rolling without slipping}$$

$$\begin{cases} x_f = x_r + l\cos\theta_0 \\ y_f = y_r + l\sin\theta_0 \end{cases} \qquad \begin{cases} x_i = x_r - \sum_{j=1}^{i} l_j \cos\theta_j \\ y_i = y_r - \sum_{j=1}^{i} l_j \sin\theta_j \end{cases}$$

So the kinematic constraints become $\boxed{x_r, y_r = x, y}$

$$\dot{x}\sin(\theta_0+\varphi) - \dot{y}\cos(\theta_0+\varphi) - l\dot{\theta}_0\cos\varphi = 0$$

$$\dot{x}\sin\theta_0 - \dot{y}\cos\theta_0 = 0$$

$$\dot{x}\sin\theta_i - \dot{y}\cos\theta_i + \sum_{j=1}^{i} l_j\dot{\theta}_j\cos(\theta_i-\theta_j) = 0 \qquad i=1..N$$

The null space of the constraint matrix is spanned by

$$
\begin{pmatrix}
\cos\theta_0 & 0 \\
\sin\theta_0 & 0 \\
0 & 1 \\
\frac{1}{\ell}\tan\varphi & 0 \\
-\frac{1}{\ell_1}\sin(\theta_1-\theta_0) & 0 \\
-\frac{1}{\ell_2}\cos(\theta_1-\theta_0)\sin(\theta_2-\theta_1) & 0 \\
\vdots & \\
-\frac{1}{\ell_i}\left(\prod_{j=1}^{i-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_i-\theta_{i-1}) & 0 \\
\vdots & \\
-\frac{1}{\ell_N}\left(\prod_{j=1}^{N-1}\cos(\theta_j-\theta_{j-1})\right)\sin(\theta_N-\theta_{N-1}) & 0
\end{pmatrix}
$$

$$g_1(q) \qquad\qquad\qquad g_2(q)$$

The kinematic control system is then

$$\dot{q} = \underbrace{g_1(q)\,v}_{} + \underbrace{g_2(q)\,\omega}_{}$$

v → driving velocity

ω → steering velocity

Recap

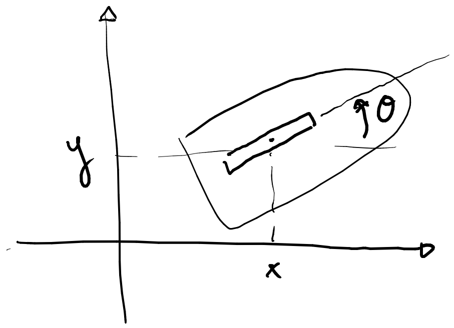
$q \in C \overset{\text{looks like}}{\simeq} \mathbb{R}^n$

$k\{A^T(q)\dot{q} = 0$ kinematic constraints

$\{g_1(q), \dots, g_{\overset{n-k}{m}}(q)\}$ basis of $\mathcal{N}(A^T(q))$

$$\underset{\text{state}}{\dot{q}} = \sum_{j=1}^{m} g_j(q)\,\underset{\text{control inputs}}{u_j} \quad \text{kinematic model}$$

Unicycle



$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

(*) rolling without slipping

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{q} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{\text{drive}} u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{steer}} u_2$$

$\dot{x}^2 + \dot{y}^2 = u_1^2$

or $u_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2} = v$ driving velocity

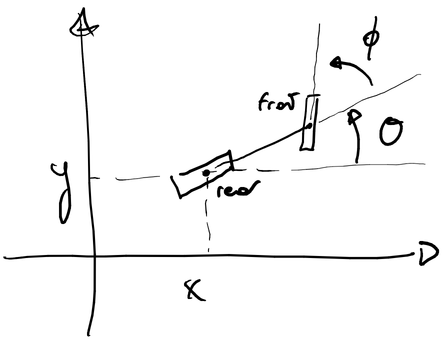
and $u_2 = \dot{\theta} = \omega$ steering velocity

Is this robot controllable? → constraint (*) non holonomic?

YES

by constructive controllability

Bicycle RWD



$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \qquad n = 4$$

Constraints:
- RWS front wheel
- RWS rear wheel

$\Rightarrow A^T(q)\dot{q} = 0 \quad \} \; k = 2$ (2 constraints)

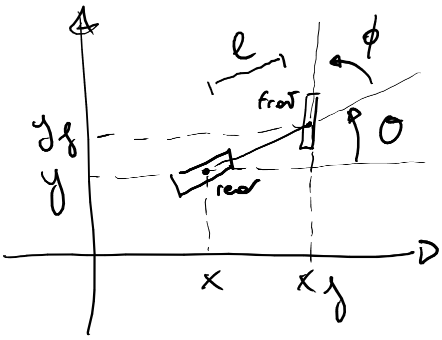
$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \end{pmatrix} \qquad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2 \qquad \text{RWD (rear wheel drive)}$$

$\dot{x}^2 + \dot{y}^2 = u_1^2 \qquad u_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2} = v$ driving velocity

$u_2 = \dot{\phi} = \omega$ steering velocity

Bicycle FWD



$$g_1(q) = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ (\sin\phi)/\ell \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$= \text{"old"}\, g_1 \cdot \cos\phi$

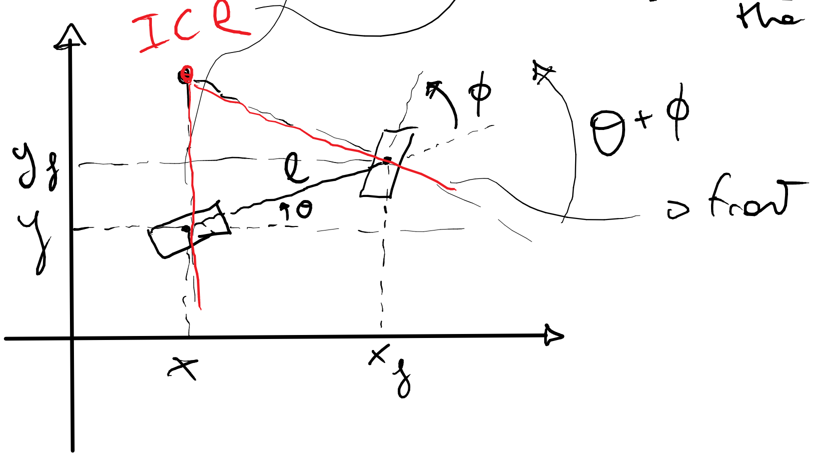
with this choice

$$\dot{x}_F^2 + \dot{y}_F^2 = u_1^2 = v^2$$

$$\begin{cases} x_F = x + \ell\cos\theta \\ y_F = y + \ell\sin\theta \end{cases}$$

$$\dot{q} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ (\sin\phi)/\ell \\ 0 \end{pmatrix} \underbrace{u_1}_{v} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underbrace{u_2}_{\omega} \qquad \text{FWD (front wheel drive)}$$

ICR



any point of the robot is rotating around the ICR

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \qquad n=4$$

$$\dot{q} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\sin\phi \\ (\sin\phi)/\ell \\ 0 \end{pmatrix} \underbrace{v_1}_{v} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underbrace{v_2}_{\omega} \qquad \text{FWD}$$

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \end{pmatrix} \underbrace{v_1}_{v} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underbrace{v_2}_{\omega} \qquad \text{RWD}$$

$v_1 = \pm\sqrt{\dot{x}^2+\dot{y}^2} = v$ driving velocity

$v_2 = \dot{\phi} = \omega$ steering velocity

What happens when $\phi = \frac{\pi}{2}$? Front wheel orthogonal to the sagittal axis (No PROBLEM for FWD case)

related to a MECHANICAL JAM

$g_1(q)$ diverges for RWD $\rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \end{pmatrix}$ the tangent diverges to ∞ → the v.g. blows up

→ a SINGULARITY

2 zero motion lines : they meet at the Instantaneous Center of Rotation ICR

- Controllability of nonlinear driftless systems

$$(*)\quad \dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i$$

$g_i(x)$: vector fields

x state $\in \mathbb{R}^n$
u input $\in \mathbb{R}^m$

> **def:** (natural controllability)
> $(*)$ is controllable if, for any $x_i, x_f \in \mathbb{R}^n$, there exists a time T and an input $u|_{[0,T]}$ such that moving from x_i
> $$x(T) = x_f$$

Preliminary concepts:

- an operation between 2 vf's

$$[g_1(x), g_2(x)] = \underbrace{\frac{\partial g_2}{\partial x}}_{n\times n} \underbrace{g_1}_{n\times 1} - \underbrace{\frac{\partial g_1}{\partial x}}_{n\times n} \underbrace{g_2}_{n\times 1}$$

Lie Bracket of g_1, g_2 → the result is a new vectorfield

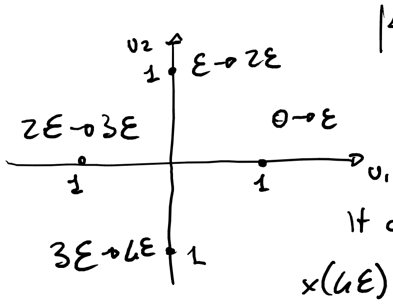
In general (it may, it may not) it is not a linear combination of g_1, g_2

TIP: if there

interpretation of $[g_1, g_2]$:

$\dot{x} = g_1(x)\, u_1 + g_2(x)\, u_2$ a particular control sequence

$$U = \begin{cases} u_1(t) = 1,\ u_2(t) = 0 & t \in [0, \varepsilon] \\ u_1(t) = 0,\ u_2(t) = 1 & t \in [\varepsilon, 2\varepsilon] \\ u_1(t) = -1,\ u_2(t) = 0 & t \in [2\varepsilon, 3\varepsilon] \\ u_1(t) = 0,\ u_2(t) = -1 & t \in [3\varepsilon, 4\varepsilon] \end{cases}$$



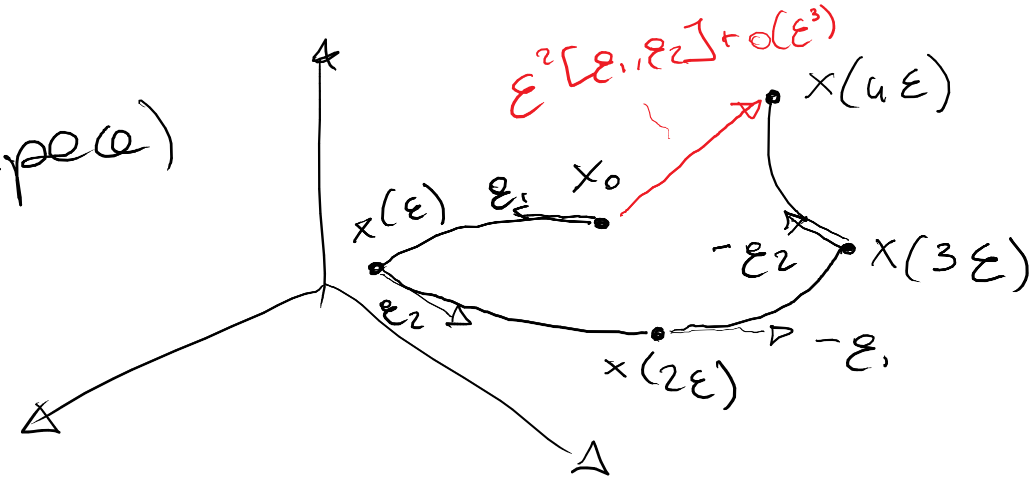
It can
$x(4\varepsilon) =$

It can be proven that the final state is

$$x(4\varepsilon) = x_0 + \varepsilon^2 [g_1(x), g_2(x)]\Big|_{x_0} + o(\varepsilon^3)$$

for $\varepsilon \to 0$, the displacement is eventually in the direction of $[g_1, g_2]$.

$\mathbb{R}^n$
(state space)



this is a MANEUVER : result of a control sequence

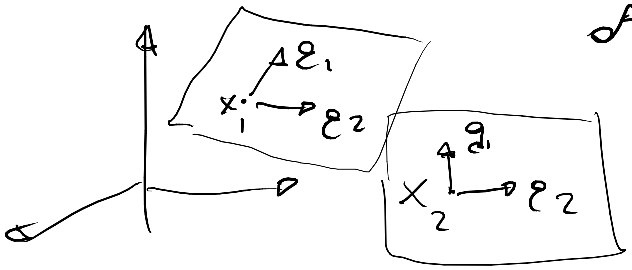
Now consider : **DRIFTLESS**

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i \qquad (*)$$

$\{g_1(x), \dots, g_m(x)\}$ input vf's

$\Delta = \text{span}\{g_1(x), \dots, g_m(x)\}$

a DISTRIBUTION (associated to each $x \in \mathbb{R}^n$, a linear space, the linear combinations of $g_1(x) \dots, g_m(x)$)



- $\Delta_A = \text{span}\{g_1(x), \dots, g_m(x), \underbrace{[g_1,g_2], [g_1,g_3] \dots}_{\text{first order brackets}},$

$$\underbrace{[g_1,[g_2,g_3]]}_{\text{second order brackets}}, \dots \}$$

ACCESSIBILITY DISTRIBUTION

(*) is controllable iff $\dim \Delta_A = n$ (Chow's Th.)

i.e. $\text{rank}\left(g_1 \dots g_m \; [g_1,g_2] \dots [g_1,[g_2,g_3]] \right) = n$

Back to mobile robots

- Unicycle

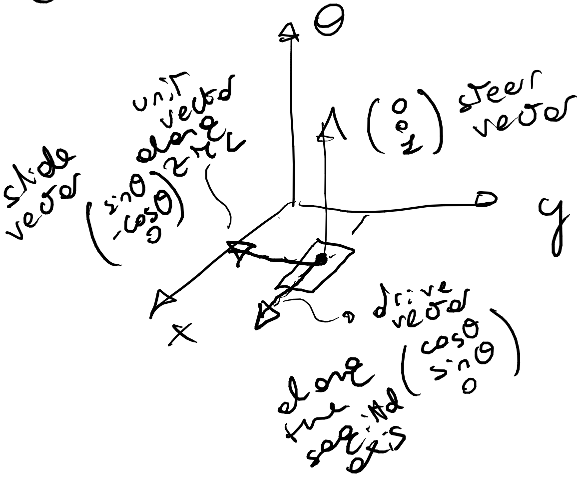
$$\dot{q} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{g_1} v + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} \omega$$

driver g_1 steer g_2

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} g_1 - \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$ Lie Bracket of our two input vfs

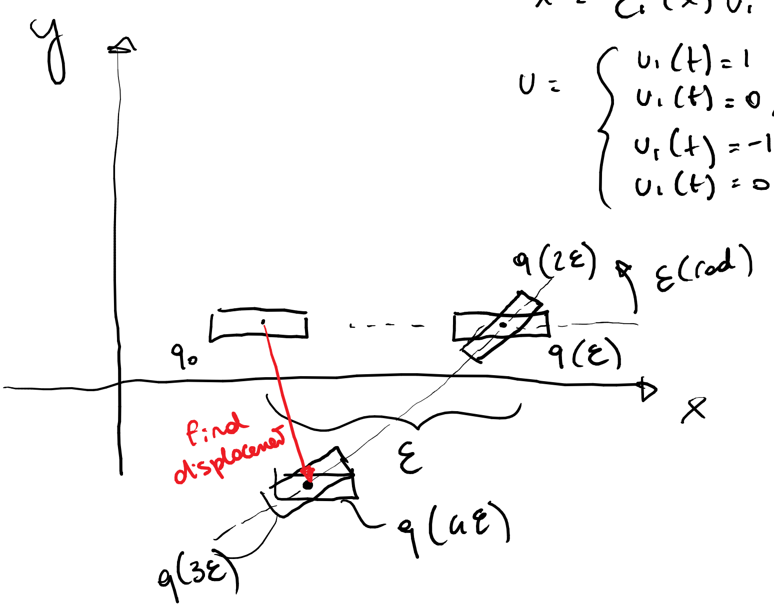
geometrical interpretation



The LB manouver

$$\dot{x} = g_1(x) u_1 + g_2(x) u_2$$ a particular control sequence

$$u = \begin{cases} u_1(t) = 1, \; u_2(t) = 0 & t \in [0, \varepsilon] \\ u_1(t) = 0, \; u_2(t) = 1 & t \in [\varepsilon, 2\varepsilon] \\ u_1(t) = -1, \; u_2(t) = 0 & t \in [2\varepsilon, 3\varepsilon] \\ u_1(t) = 0, \; u_2(t) = -1 & t \in [3\varepsilon, 4\varepsilon] \end{cases}$$



accessibility distribution :

$$\Delta_A = \text{span}\{g_1, g_2, [g_1, g_2], \dots\}$$

don't need them

$$\begin{array}{c} \text{rank} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{pmatrix} = \\ \begin{array}{ccc} \quad g_1 & \quad g_2 & \quad [g_1, g_2] \end{array} \end{array}$$

change of columns (rank does not change)

$$\text{rank} \left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = 3$$

the unicycle is controllable

(RWS is NH)

rolling without slipping

Bycicle

- RWD $\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \omega$

$$\qquad\qquad g_1 \qquad\qquad\qquad g_2$$

$$[g_1, g_2] \triangleq g_3 = -\begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{\ell\cos^2\phi} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{pmatrix}$$

"wriggle" vf

$$[g_1 \cdot [g_1, g_2]] = [g_1, g_3]$$

or

$$[g_2 \cdot [g_1, g_2]] = [g_2, g_3]$$

I choose $[g_1, g_3] \triangleq g_4 =$

$$= -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{\ell\cos^2\phi} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sin\theta}{\ell\cos^2\phi} \\ -\frac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{pmatrix}$$

slide vf

$$\Delta_A = \text{span}\{g_1, g_2, g_3, g_4, \dots\}$$

don't need them because we need 4 dim.

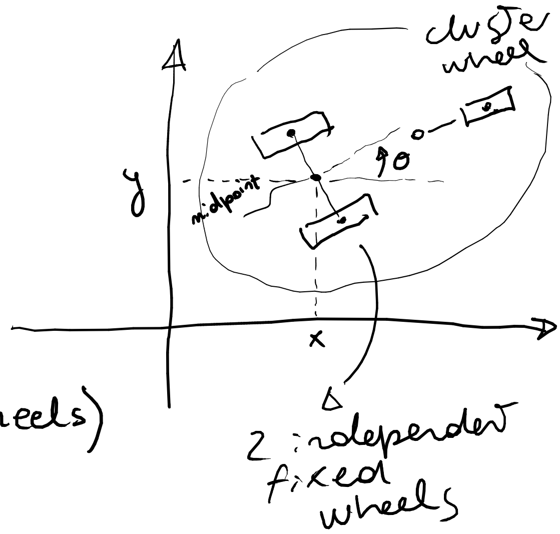
$$= \text{span}\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{\ell}{\cos^2\phi} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{\sin\theta}{\ell\cos^2\phi} \\ -\frac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{rank} = 4$$

• FWD $\qquad \dot{q} = \begin{pmatrix} \cos\theta \cos\phi \\ \sin\theta \cos\phi \\ (\sin\phi)/l \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} w$

exercise
- compute $g_3 = [g_1, g_2]$ $\quad g_4 = [g_1, g_3]$
- give a geometric interpretation
- prove that $\dim \Delta_A = \{g_1, g_2, g_3, g_4, \underbrace{\dots}_{\text{don't need}}\} = 4$

Balance

unicycle $\begin{cases} \text{differential-drive} \\ \text{synchro-drive} \end{cases}$



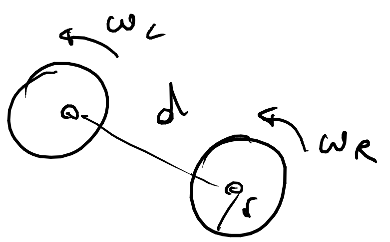
equivalent to unicycle
but statically balanced (3 wheels)

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

- kinematic model of the differential-drive

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega$$ as the unicycle

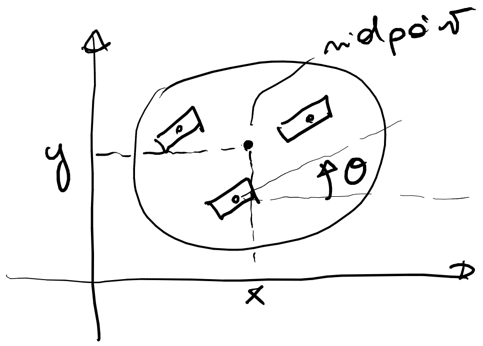
v and ω are different couse we have 2 wheels
then we consider the midpoint



$$v = r\,\frac{\omega_R + \omega_L}{2}$$

$$\omega = \frac{\omega_R - \omega_L}{d}$$

- kinematic model of the synchro-drive



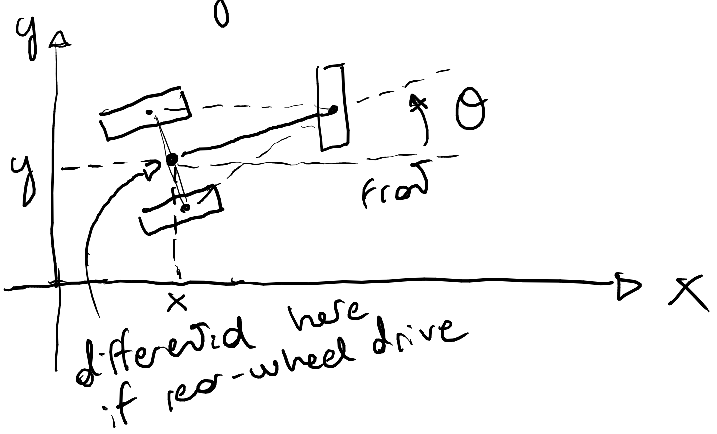
$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$$

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \underbrace{v}_{\text{driving vel of the 3 wheels}} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \underbrace{\omega}_{\text{steering vel of the 3 wheels}}$$ as the unicycle

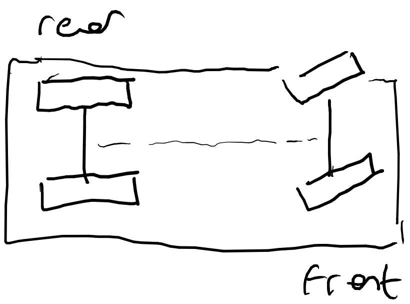
Statically balanced

robots equivalent to the bicycle

• tricycle

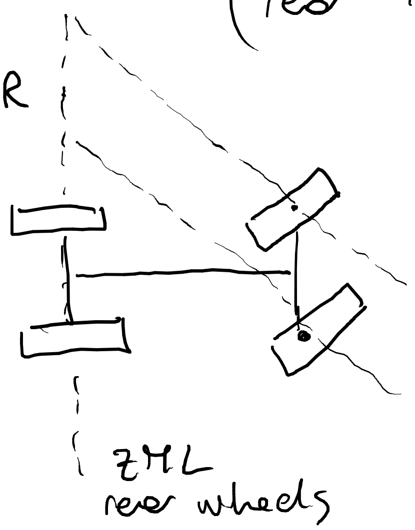


• Car-like robots

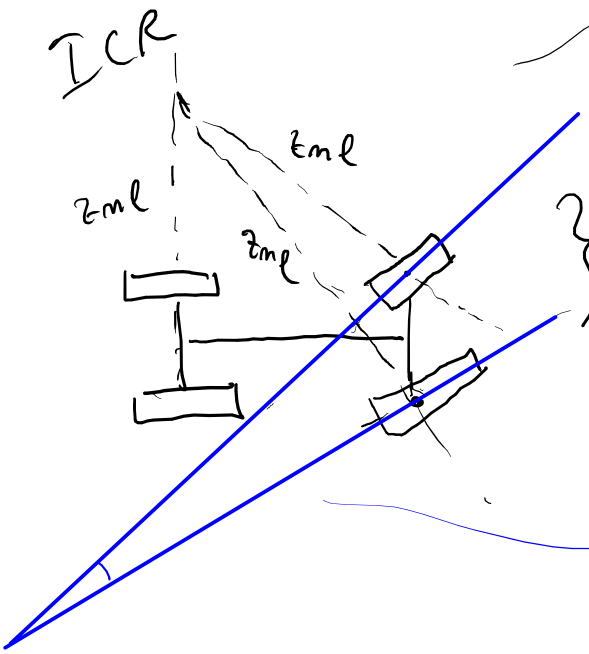


1 differential (front if FWD
rear if RWD)

NO
ICR
"slippage"



if the
front wheels
non parallel



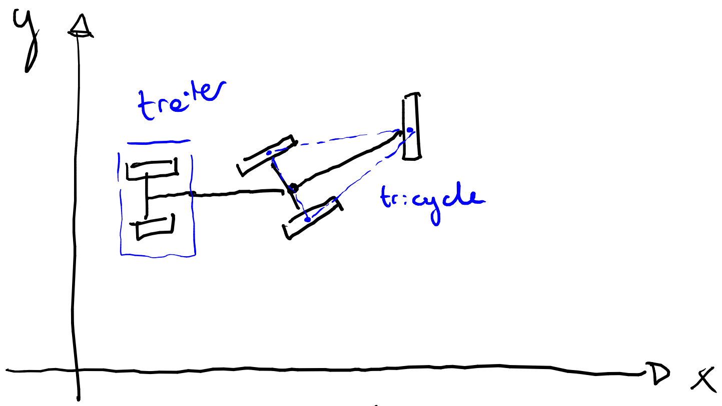
} not
parallel

thanks to
ACKERMANN
STEERING
(no slippage)

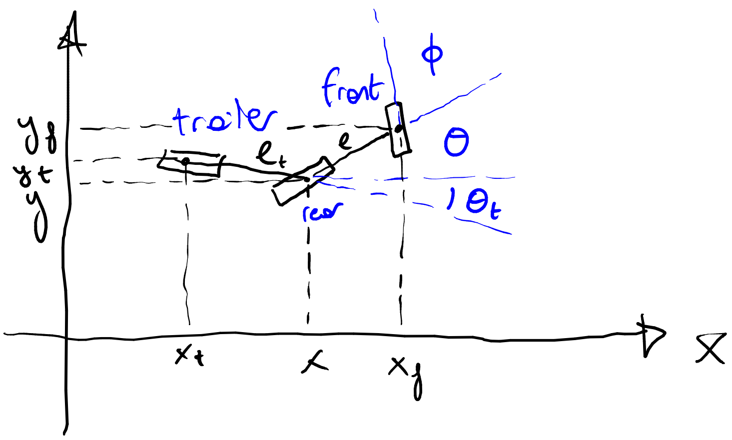
just an implementation
of mechanics
no effects on the
kinematics (no extra
angle)

Exercises

- tricycle with trailer $\Rightarrow$
  - write the kinematic constraints
  - derive kinematic model



$\Downarrow$ equivalent model : $\begin{cases} \text{bicycle : tricycle} \\ \text{unicycle : trailer} \end{cases}$



$n = 5$

- 3 wheels $\Rightarrow$ 3 RWS constraints

(2) $\dot x \sin\theta - \dot y \cos\theta = 0$ rear wheel

(1) $\dot x_f \sin(\theta + \phi) - \dot y_f \cos(\theta + \phi) = 0$ front wheel

(3) $\dot x_t \sin\theta_t - \dot y_t \cos\theta_t = 0$ trailer wheel

$$\begin{cases} x_f = x + l\cos\theta \\ y_f = y + l\sin\theta \end{cases} \qquad \begin{cases} x_t = x - l_t\cos\theta_t \\ y_t = y - l_t\sin\theta_t \end{cases}$$

$$\begin{cases} \dot x_f = \dot x - l\sin\theta\,\dot\theta \\ \dot y_f = \dot y + l\cos\theta\,\dot\theta \end{cases} \qquad \begin{cases} \dot x_t = \dot x + l_t\sin\theta_t\,\dot\theta_t \\ \dot y_t = \dot y - l_t\cos\theta_t\,\dot\theta_t \end{cases}$$

(1) $(\dot{x} - l\sin\theta\,\dot{\theta})\sin(\theta+\phi) - (\dot{y} + l\cos\theta\,\dot{\theta})\cos(\theta+\phi) = 0$

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\big(\underbrace{\sin\theta\sin(\theta+\phi) - \cos(\theta)\cos(\theta+\phi)}_{\cos\phi}\big) = 0 \quad \text{(front wheel)}$$

(3) $(\dot{x} + l_t\sin\theta_t\,\dot{\theta}_t)\sin\theta_t - (\dot{y} - l_t\cos\theta_t)\cos\theta_t = 0$

$$\dot{x}\sin\theta_t - \dot{y}\cos\theta_t + l_t\dot{\theta}_t = 0 \quad \text{(trailer wheel)}$$

$$A^T(q)\,\dot{q} = 0$$

bicycle

$$\begin{array}{r} \text{front wheel} \\ \text{rear wheel} \\ \text{trailer wheel} \end{array} \begin{pmatrix} s(\theta+\phi) & -c(\theta+\phi) & -l\cos\phi & 0 & 0 \\ s\theta & -c\theta & 0 & 0 & 0 \\ s\theta_t & -c\theta_t & 0 & 0 & l_t \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \end{pmatrix}$$

$$A^T(q) \in \mathbb{R}^{3\times 5}$$

$N(A^T(q))$

kinematic model: basis of the null space of $A^T(q)$ of dim 2 (5 variables − 3 constraints)

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

Basis for the bicycle: (RWD)

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/l \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

extending to the 5th variable ...

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/l \\ 0 \\ (*) \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

we can find $(*)$ multiplying $A^T(q)\cdot g_1 \triangleq 0$

3rd row $(A^T(q)) \cdot g_1 =$

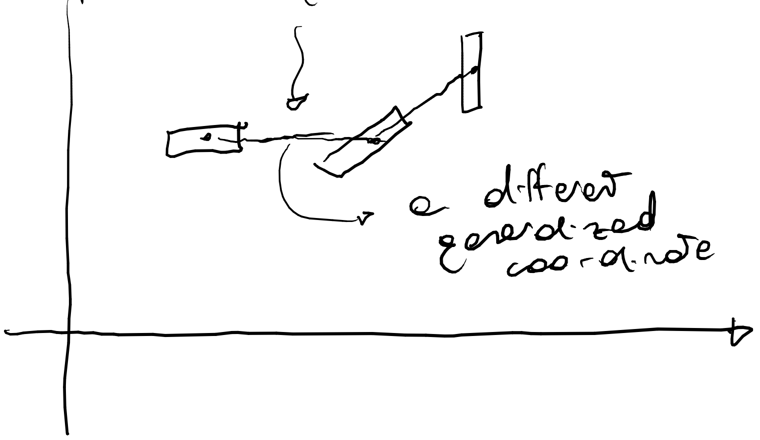
$$\underbrace{\sin\theta_t\cos\theta - \cos\theta_t\sin\theta}_{\sin(\theta_t-\theta)} + l_t(*) = 0$$

$$(*) = -\frac{\sin(\theta_t - \theta)}{l_t} = \frac{\sin(\theta - \theta_t)}{e_t}$$

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi / l \\ 0 \\ \frac{\sin(\theta - \theta_t)}{e_t} \end{pmatrix}$$

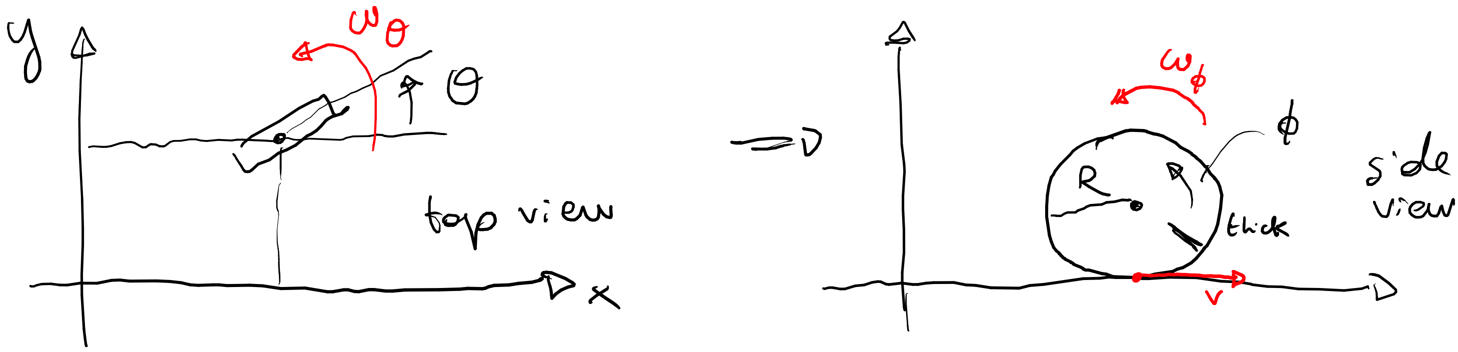
$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/l \\ 0 \\ \frac{\sin(\theta - \theta_t)}{e_t} \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \omega$$

$$(\theta - \theta_t) = \delta$$



$\Rightarrow$ exercise: KM model

Exercise: unicycle with wheel orientation



We want to know also the thick orientation with the kin model.

- kin model
- prove controllability
- build manouver for going from q_s to q_g, $\forall q_s, q_g$

(q_s: start, q_g: goal)

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix}$$

Kin Model: augmentation approach

$$\dot q = \begin{pmatrix} \dot x \\ \dot y \\ \dot\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega \qquad \text{unicycle}$$

$v \to$ driving velocity

$v = R\,\omega_\phi \to \dot\phi$

$$\Rightarrow \begin{pmatrix} \dot x \\ \dot y \\ \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \omega_\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \omega_\theta$$

Kin model : <u>canonical approach</u>

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix}$$

2 constraints

(1) $\dot{x} \sin\theta - \dot{y} \cos\theta = 0$

(2) $v - R\dot{\phi} = 0 \Rightarrow \overbrace{\pm \sqrt{\dot{x}^2 + \dot{y}^2}}^{v} - R\dot{\phi} = 0$ a kinematic constraint

$\hookrightarrow$ not Pfaffian (non linear)

$$v = v(\cos^2\theta + \sin^2\theta) = \underbrace{v\cos\theta}_{\dot{x}}\cos\theta + \underbrace{v\sin\theta}_{\dot{y}}\sin\theta$$

(2) $= \dot{x}\cos\theta + \dot{y}\sin\theta - R\dot{\phi} = 0 \rightarrow$ Pfaffian form

$$A^T(q)\,\dot{q} = 0$$

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -R \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = 0$$

a null-space basis

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \omega_\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \omega_\theta$$

Controllability

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

$$= - \begin{pmatrix} 0 & 0 & -R\sin\theta & 0 \\ 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R\sin\theta \\ -R\cos\theta \\ 0 \\ 0 \end{pmatrix} = g_3$$

linearly independent from g_1, g_2

We need 4 vectors

g_4 will be a 2nd order vector

$[g_1, g_3] = 0$ no good (no direction)

$$[g_2, g_3] = \begin{pmatrix} 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & R\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 0 \end{pmatrix}$$

g_4

Accessibility distribution:

$$\Delta_A = \text{span}\{g_1, g_2, g_3, g_4, \dots\}$$

don't need

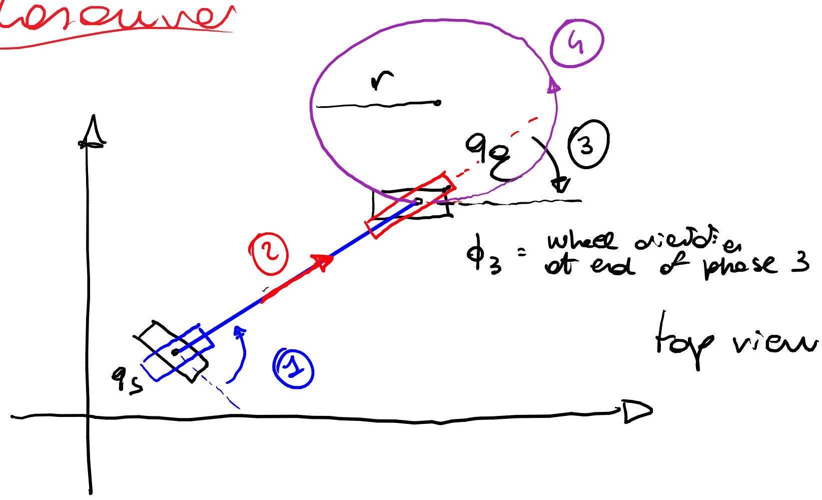
$$\text{rank}\begin{pmatrix} | & | & | & | \\ g_1 & g_2 & g_3 & g_4 \\ | & | & | & | \end{pmatrix} = \text{rank}\begin{pmatrix} | & | & | & | \\ g_3 & g_4 & g_1 & g_2 \\ | & | & | & | \end{pmatrix} =$$

$$\text{rank}\begin{pmatrix} R\sin\theta & R\cos\theta & R\cos\theta & 0 \\ -R\cos\theta & R\sin\theta & R\sin\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 4 \quad (\det = -R^2)$$

block triangular matrix

Therefore we have proven controllability

Maneuver

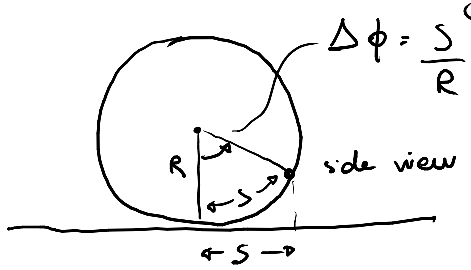


1 - align starting orientation with goal orientation

2 - roll to e position

3 - align with final orientation

4 - a circle to adjust the orientation



$$\underbrace{2\pi r}_{\text{travel space}} = R\,\Delta\phi = R\left(\phi_e - \phi_3\right)$$

$$r = \frac{\left(\phi_e - \phi_3\right)}{2\pi}$$