12.1. Describe the nature (including the dimension) of the configuration space for a mobile manipulator consisting of a unicycle-like vehicle carrying a six-DOF anthropomorphic arm, providing a choice of generalized coordinates for the system.

The configuration of the mobile manipulator is

$q = [x \;\; y \;\; \theta_0 \;\; \theta_1 \;\; \dots \;\; \theta_6]^T$

where (x, y) are the Cartesian coordinates of the contact point of the wheel with the ground

θ_0 is the orientation of the unicycle wrt x-axis.

$\theta_1 \dots \theta_6$ are the manipulator joint variables.

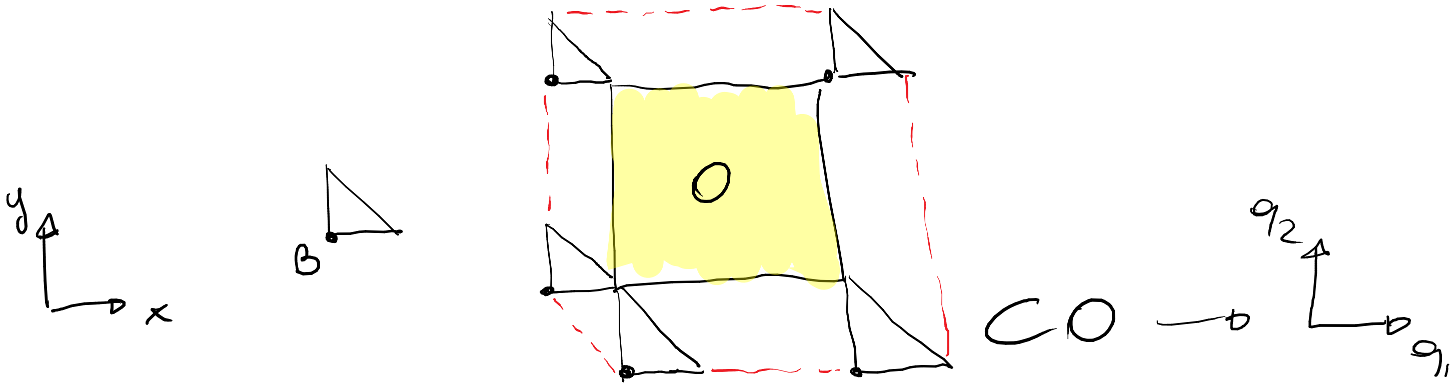
The configuration space is therefore

$C = \mathbb{R}^2 \times SO(2) \times \dots \times SO(2) \qquad \dim(C) = 9$

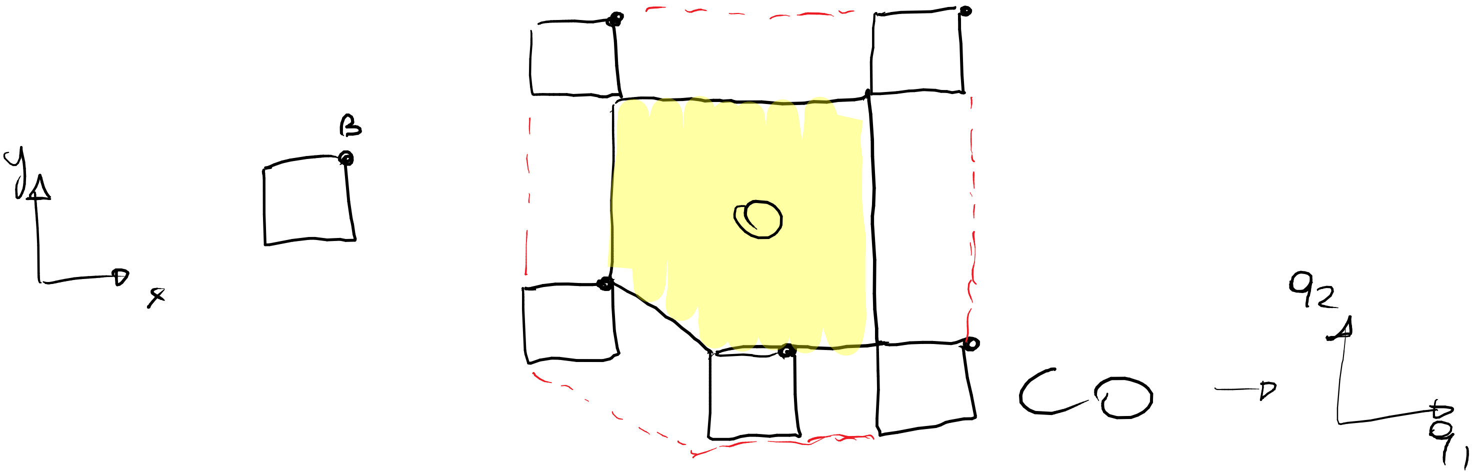
($SO(2)$ appearing 7 times)

12.3. Consider a polygonal robot translating at a fixed orientation in $\mathbb{R}^2$ among polygonal obstacles. Build an example showing that the same $\mathcal{C}$-obstacle region may correspond to robot and obstacles of different shapes.

1. Triangular robot, single square obstacle



2. Square robot, pentagonal obstacle



Under the assumption that the robots can freely translate without changing their orientation, the C-obstacle region is exactly the same for the two scenes

12.4. With reference to the second workspace shown in Fig. 12.4, give the numerical value of three configurations of the manipulator that lie in the three connected components of $\mathcal{C}_{\text{free}}$. Moreover, sketch the manipulator posture for each of these configurations.

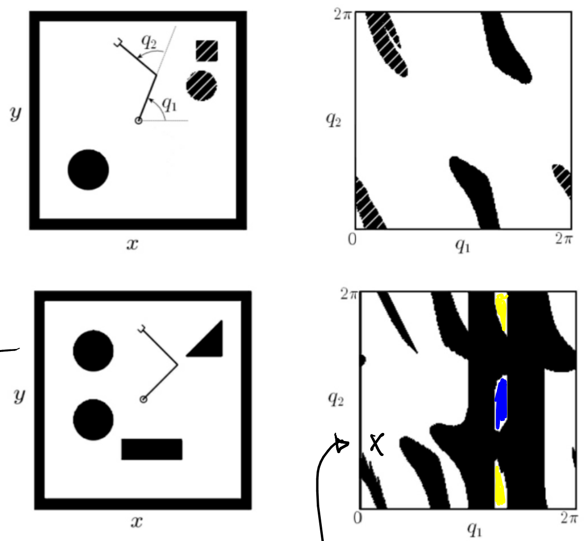


Fig. 12.4. $\mathcal{C}$-obstacles for a wire-frame 2R manipulator in two different cases; *left*: the robot and the obstacles in $\mathcal{W} = \mathbb{R}^2$, *right*: the configuration space $\mathcal{C}$ and the $\mathcal{C}$-obstacle region $\mathcal{CO}$

by visual inspection

$$q = \begin{pmatrix} \pi/4 \\ \pi/2 \end{pmatrix}$$

for the others the same procedure holds, the colored areas are the others connected components of $\mathcal{C}_{free}$

12.5. Discuss the basic steps of an algorithm for computing the generalized Voronoi diagram of a limited polygonal subset of $\mathbb{R}^2$. [*Hint*: a simple algorithm is obtained by considering all the possible side-side, side-vertex and vertex-vertex pairs, and generating the elementary arcs of the diagram on the basis of the intersections of the associated equidistance contours.]

Consider the lists $V = \{V_1, \dots, V_v\}$ and $S = \{S_1, \dots, S_s\}$ of all the vertices and all the sides of the given limited polygonal subset.

A rough sketch of a naive algorithm for computing the generalized Voronoi diagram is the following.

1. Build all the equidistance curves as follows:

   1a. For each vertex-vertex pair (V_i, V_j), derive the equation of the line $L_{V_i V_j}$ passing through the midpoint of the segment $V_i V_j$ and orthogonal to the segment itself.

   1b. For each vertex-side pair (V_i, S_j), denote by L_{S_j} the line containing S_j and derive the equation of the parabola $P_{V_i S_j}$ having V_i as focus and L_{S_j} as directrix.

   1c. For each side-side pair (S_i, S_j), denote by L_{S_i} and L_{S_j} the lines containing S_i and S_j, respectively, and derive the equation of the bisectrix $L_{S_i S_j}$ of the angle formed by L_{S_i} and L_{S_j} which contains S_i and S_j

2. For each possible pair of equidistance curves, compute the intersection points
   - at most 1 for line / line pair
   - at most 2 for line / parabola or parabola / parabola

   Each of this points is characterized by the two distances from the features (vertex / vertex, line / line or line / vertex) that generate the two intersecting equidistance curves.

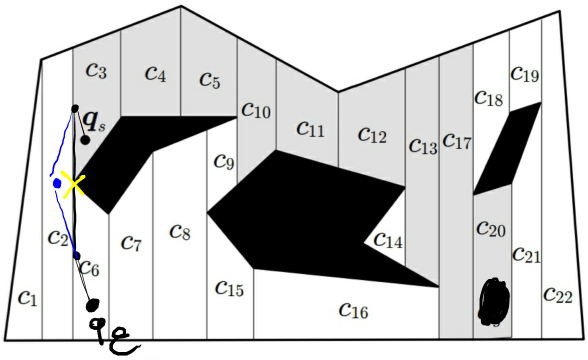
3. Discard all intersection points that do not belong to the subset of $\mathbb{R}^2$ in exam

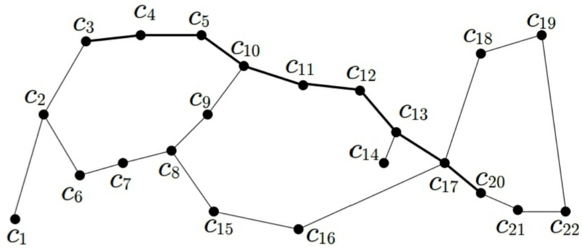
4. Discard all intersection points for which the two characteristic distances are not equal

5. For each remaining intersection point, compute it's clearance, and discard all intersection points for which the clearance is not equal to the characteristic distance.

6. All the remaining intersection points are nodes of the generalized Voronoi diagram.
   Its arcs are the portions of the equidistance curves that are enclosed by these nodes.

12.6. For the motion planning method via exact cell decomposition, give an example in which the path obtained as a broken line joining the midpoints of the common boundary of the channel cells goes through a vertex of the $\mathcal{C}$-obstacle region, and propose an alternative procedure for extracting a free path from the channel. [*Hint*: build a situation in which the channel from c_s to c_g contains a cell for which the entrance and the exit boundary lie on the same side.]





Consider for instance the previous cell decompos.
We have $c_s = c_3$ and $c_E = c_6$

The shortest path is clearly $\{c_3, c_2, c_6\}$.
But we can see that the midpoint of the common boundary of c_3 and c_2 is exactly the left vertex of the obstacle.

To solve the problem, we can include additional vertices in the path. For example, in addition to q_s, q_E and the midpoints of the common boundaries in the channel, one may include the centroids (or a generic internal point) of all the cells crossed by the channel, with the exception of c_s and c_E. In this case the vertices will be q_s, midpoint between c_3 and c_2, the centroid of c_2, midpoint of c_2 and c_6 and finally q_E

12.9. For the case $\mathcal{C} = \mathbb{R}^2$, build a continuously differentiable attractive potential having a paraboloidal profile inside the circle of radius ρ and a conical profile outside. [*Hint*: modify the expression (12.12) of the conical potential using a different constant k_b in place of k_a, which already characterizes the paraboloidal potential.]

Let

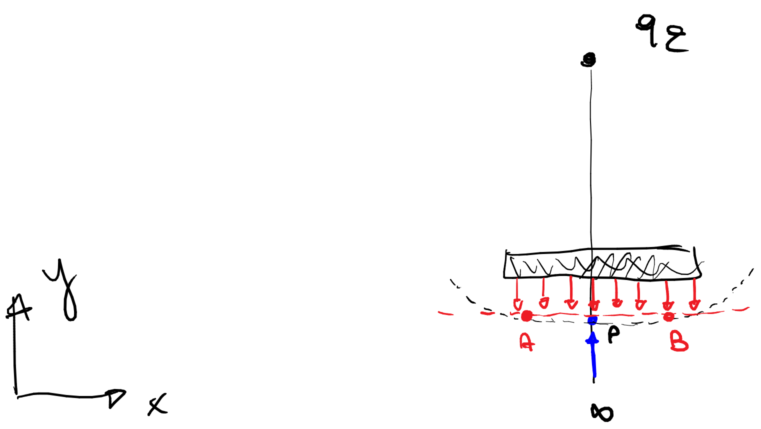
$$U_a(q) = \begin{cases} \frac{1}{2} k_a \|e(q)\|^2 & \text{if } \|e(q)\| \le \rho \\ k_b \|e(q)\| & \text{if } \|e(q)\| > \rho \end{cases}$$

Continuity of the attractive force at the transition radius ρ is guaranteed by imposing

$$k_a e(q) = k_b \frac{e(q)}{\|e(q)\|} \quad \text{for } \|e(q)\| = \rho$$

i.e. $k_b = \rho k_a$

12.10. For the case $\mathcal{C} = \mathbb{R}^2$, prove that the total potential U_t may exhibit a local minimum in areas where the equipotential contours of the repulsive potential U_r have lower curvature than those of the attractive potential U_a. [*Hint*: consider a polygonal $\mathcal{C}$-obstacle and use a geometric construction.]



In a local minima the robot is subjected to a force $f=0$ and then it stops.
This is couse by the fact that near the obstacles the repulsive force dominates (it goes to ∞), while, sufficiently far from them, the attractive force dominates (as the repulsive one is smaller or null).
Considering the figure with the point P:

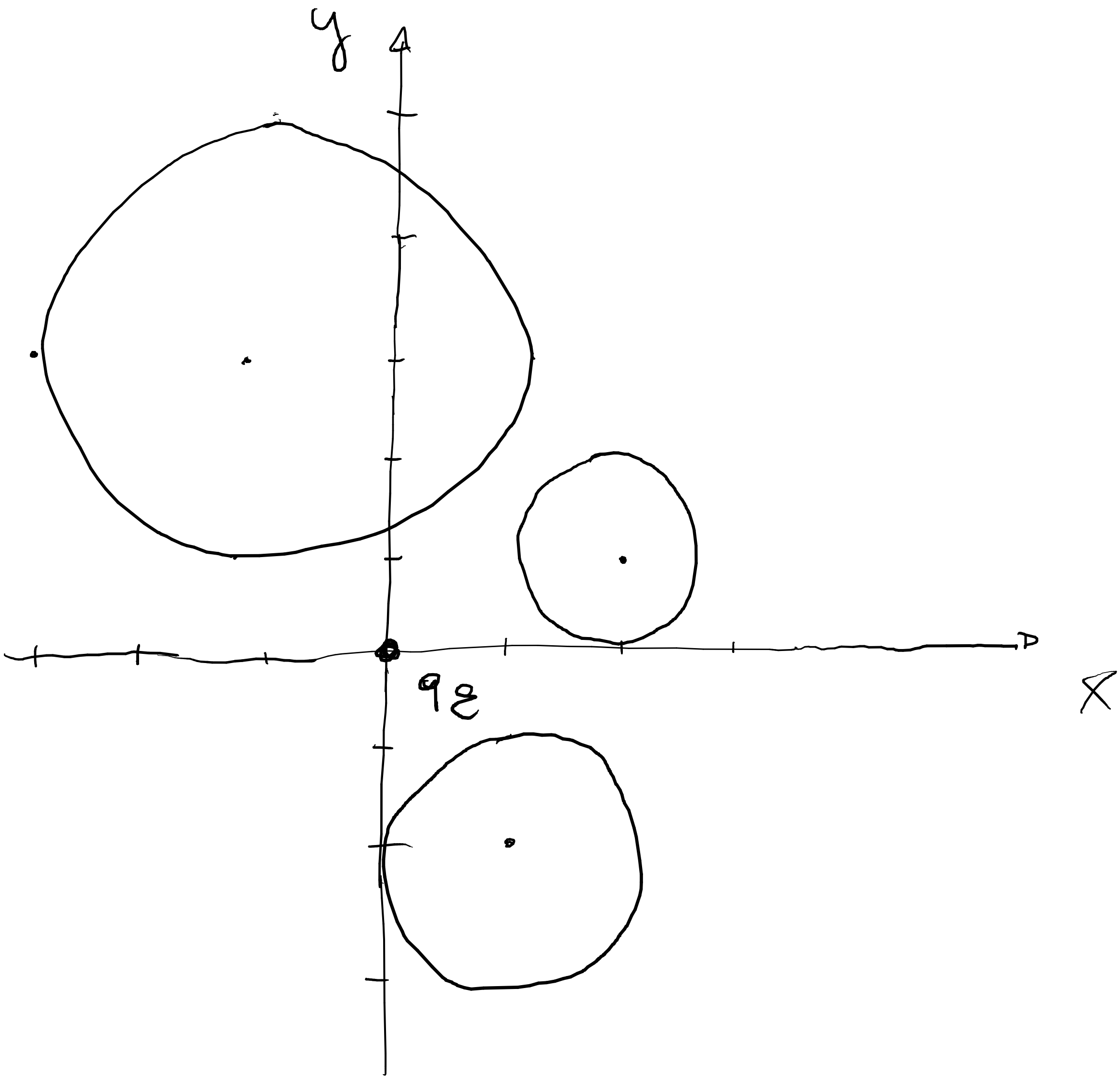
$$\frac{\partial U}{\partial x}\Big|_P = 0 \qquad \frac{\partial U}{\partial y}\Big|_P = 0$$

which implies that P is a stationary point for U, i.e., the gradient of U is zero at P.
To show that P is a local minimum we can consider the Hessian matrix of U

$$\begin{bmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial U}{\partial x \partial y} \\ \frac{\partial U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} \end{bmatrix}$$

and prove that it is positive definite in P. This can be verified by using the analytic expression of U. Note that the elements of the diagonal are certainly positive in P.

12.11. Consider a point robot moving in a planar workspace containing three circular obstacles, respectively of radius 1, 2 and 1 and centre in $(2, 1)$, $(-1, 3)$ and $(1, -2)$. The goal is the origin of the workspace reference frame. Build the total potential U_t resulting from the superposition of the attractive potential U_a to the goal and the repulsive potential U_r from the obstacles, and derive the corresponding artificial force field. Moreover, compute the coordinates of the saddle points of U_t.



Denote by (x, y) the cartesian coordinates of the robot, by (x_i, y_i) the cartesian coordinates of the centre C_i of the i-th circular obstacle, and by ρ_i its radius.

Let $e = \underbrace{q_g}_{(0,0)} - q = (-x, -y)$

The attractive potential is

$$U_a = \frac{1}{2} \underset{\substack{\downarrow \\ >0}}{k_a} \| e(q) \|^2 = \frac{1}{2} k_a (x^2 + y^2)$$

While the repulsive potentials $(\gamma = 2)$ are

$$U_{r,i}(x,y) = \begin{cases} \frac{k_{r,i}}{2} \left(\frac{1}{\eta_i(x,y)} - \frac{1}{\eta_{0,i}} \right)^2 & \text{if } \eta_i(x,y) \le \eta_{0,i} \\ 0 & \text{if } \eta_i(x,y) > \eta_{0,i} \end{cases}$$

$i = 1, \dots, 3$

with $\eta_i(x,y) = \text{clearance} = \sqrt{(x-x_i)^2 + (y-y_i)^2} - \rho_i$.

The total potential is

$$U_t(x,y) = U_a(x,y) + \sum_{i=1}^{3} U_{r,i}(x,y)$$

The total force is

$$f(x,y) = f_a(x,y) + \sum_{i=1}^{3} f_{r,i}(x,y)$$

$$f_a(x,y) = -k_a \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f_{r,i}(x,y) = \begin{cases} \frac{k_{r,i}}{\eta_i^2(x,y)} \left(\frac{1}{\eta_i(x,y)} - \frac{1}{\eta_{0,i}} \right)^{\gamma-1} \nabla \eta_i(x,y) & \text{if } \eta_i(x,y) \le \eta_{0,i} \\ 0 & \text{if } \eta_i(x,y) > \eta_{0,i} \end{cases}$$

$$\nabla \eta_i(x,y) = \begin{bmatrix} \frac{\partial \eta_i}{\partial x} \\ \frac{\partial \eta_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x - x_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2}} \\ \frac{y - y_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2}} \end{bmatrix}$$

Assuming that the range of influence of the obstacles do not overlap, the saddle points of the total potential field will be located along the lines that join the origin of the reference frame (the goal) with the centre of the circular obstacles

In particular, the saddle points are found by solving the equations

$$-k_a \begin{bmatrix} x \\ y \end{bmatrix} = \frac{k_{r,i}}{\eta_i^2(x,y)} \left(\frac{1}{\eta_i(x,y)} - \frac{1}{\eta_{0,i}} \right) \nabla \eta_i(x,y) \quad i = 1,2,3$$

each representing the balance between the attractive force and the i-th repulsive force.

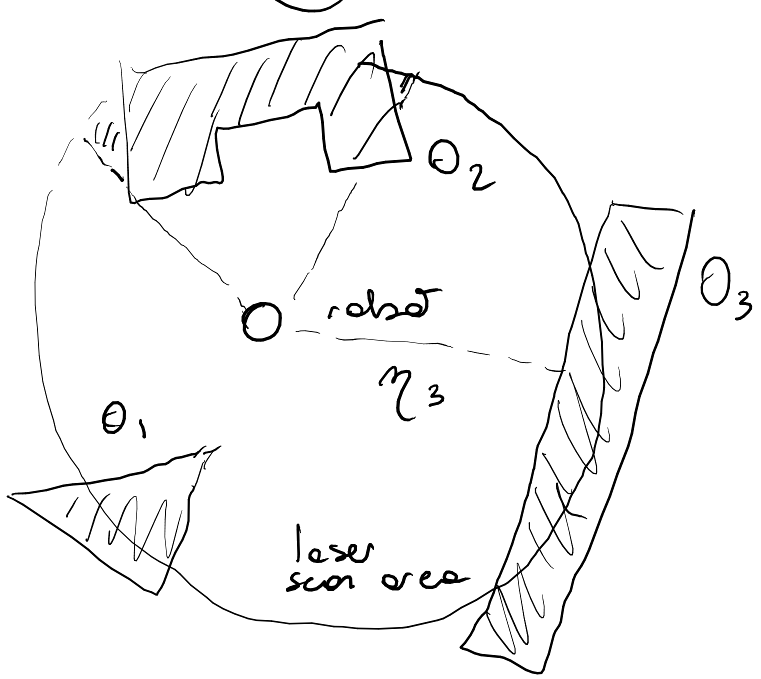
12.12. Discuss the main issues arising from the application of the artificial potential technique for planning on-line the motion of an omnidirectional circular robot. Assume that the robot is equipped with a rotating laser range finder placed at its centre that measures the distance between the sensor and the closest obstacles along each direction. If the distance is larger than the maximum measurable range R, the sensor returns R as a reading. Sketch a possible extension of the method to a unicycle-like mobile robot.

There are 2 main approaches for on-line motion planning based on sensor data:

1. Store the data coming from sensor in a map and use it to plan in an incremental fashion (Deliberative planning)

2. React only to the obstacles that are currently perceived by the robot, without any form of memory (reactive planning).

This is the case 2



Since there is a visibility based sensor the O_2 obstacle is seen bigger due to perspective

Consider first the case of a point robot.

Since the goal is known, the attractive potential can be built as in the off-line case, since we have a localization module.

For the repulsive field, we choose:

η_{0i} = range of influence of the obstacles $= R \quad \forall i$

If the robot doesn't see the obstacle it doesn't react

The perceived obstacle region must be decomposed in convex components to guarantee continuity of the repulsive potential field; in the present case only O_2 needs to undergo this procedure.

At this point, the repulsive force produced by each convex component is computed with

$$f_{r,i}(q) = -\nabla U_{r,i}(q) = \begin{cases} \dfrac{k_{ri}}{\eta_i^2(q)} \left(\dfrac{1}{\eta_i(q)} - \dfrac{1}{\eta_{0,i}} \right)^{\gamma-1} \nabla \eta_i(q) & \text{if } \eta_i(q) \leq \eta_{0,i} \\ 0 & \text{if } \eta_i(q) > \eta_{0,i} \end{cases}$$

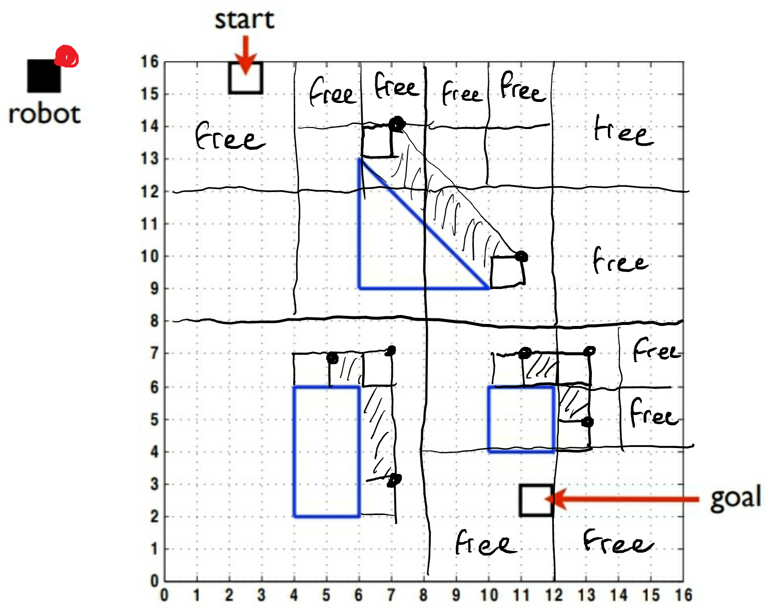
with η_i, $\nabla \eta_i$ directly obtained from the range scan profile.

If the robot has a finite nonzero radius ρ, one simply uses $\eta_i - \rho$ in place of η_i in $f_{r,i}(q)$, and obtains $\nabla \eta_i$ as in the point robot case.

It is easy to realize that this online reactive artificial potential method behaves exactly as the off-line case, in the sense that the resulting motion is the same. This is true because the environment is continuously scanned during the motion.

Test 2009

Consider the planar motion planning problem shown in figure. The robot is a square of unit side and can translate freely in the plane without changing its orientation.



Build the $\mathcal{C}$-obstacles and show all the steps of a solution obtained using the planning method based on approximate cell decomposition.

[*The steps should be graphically illustrated; in particular, for each step draw the current decomposition and the associated connectivity graph. Channels can be identified by visual path search. At the end, show a solution path extracted from the free channel.*]

To build the C-obstacles, it is necessary to choose a representative point for the robot.

For example the upper right vertex.

Slide the square robot along the obstacle boundaries and keep track of the associated motion of the representative point.

Approximate Cell decomposition

- Divide C in 4 cells and classify each cell as
  - free : if its interior is completely in C_{free}
  - occupied : if it is completely in CO
  - mixed : if neither free nor occupied

- Remove occupied cells
- Build a connectivity graph associated to the current level of decomposition, with free and mixed cells as nodes and arcs between adjacent nodes
- Search the graph for a path (a channel of cells) from the cell containing the start configuration to the cell containing the goal configuration.
  - → If such a channel exists:
    mixed cells are further partitioned and the cycle repeated until a free channel is found (if possible)

Test 2010

Consider a 2R planar manipulator with the first joint at the origin of the plane and links of unit length. Regardless of the initial configuration, we want to bring the tip of the manipulator to the goal point $(1, -1)$ while avoiding collisions with a point obstacle located at $(1, 0)$. To solve this motion planning problem, one can take the following approach:

1. define a suitable set of control points on the manipulator;
2. build an appropriate artificial force field for each control point;
3. control the robot by imposing to its joints the torques resulting from the combined action of the Cartesian force fields.

Compute the complete expression of the torques as a function of the robot configuration.

Robot configuration $q = (q_1, q_2)$

Control points:

- p_1, located on the first link at a distance $\alpha < 1$ from the first joint
- p_2, located on the second link at a distance $\beta < 1$ from the second joint
- p_3, the manipulator tip

The required torque is

$$\tau = -J_1^T(q)\,\nabla U_r(p_1(q)) - J_2^T(q)\,\nabla U_r(p_2(q)) + \\ - J_3^T(q)\,\nabla U_t(p_3(q))$$

where

$J_i(q)$, $i = 1,2,3$ is the Jacobian matrix of the forward kinematics associated to the control point p_i

$U_r(p_i)$, $i = 1,2,3$ is the repulsive potential acting on the control point p_i due to the point obstacle.

$U_t = U_r + U_a$

$\hookrightarrow U_a(p_3)$ is the attractive potential acting on the control point p_3 due to the goal

$$p_1(q) = (\alpha \cos q_1, \alpha \sin q_1)$$
$$p_2(q) = (\cos q_1 + \beta \cos(q_1+q_2), \sin q_1 + \beta \sin(q_1+q_2))$$
$$p_3(q) = (\cos q_1 + \cos(q_1+q_2), \sin q_1 + \sin(q_1+q_2))$$

$$J_1(q) = \begin{pmatrix} -\alpha \sin q_1 & 0 \\ \alpha \cos q_1 & 0 \end{pmatrix}$$

$$J_2(q) = \begin{pmatrix} -\sin q_1 - \beta \cos(q_1+q_2) & -\beta \sin(q_1+q_2) \\ \cos q_1 + \beta \cos(q_1+q_2) & \beta \cos(q_1+q_2) \end{pmatrix}$$

$$J_3(q) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}$$

Artificial force field at the generic Cartesian point $p = (x,y)$:

Attractive field with the goal at $(1,-1)$

$$U_a(p) = \frac{1}{2} k_a \| e(p) \|^2 \qquad e = \begin{pmatrix} 1-x \\ -1-y \end{pmatrix}$$
$$f_a(p) = -\nabla U_a(p) = \underbrace{k_a}_{>0} e(p)$$

Repulsive force field at the generic Cartesian point $p = (x,y)$ by the obstacle in $(-1,0)$ $(\gamma = 2)$

$$f_{r,i}(p) = -\nabla U_r(p) = \begin{cases} \frac{k_r}{\eta^2(p)} \left(\frac{1}{\eta(p)} - \frac{1}{\eta_0} \right)^{\gamma-1} \nabla \eta(p) & \eta(p) \le \eta_0 \\ 0 & \text{if } \eta(p) > \eta_0 \end{cases}$$

$k_r > 0$

η_0 is the obstacle range of influence, chosen to be smaller than 1 to preserve a global minimum

$\eta(p) = \sqrt{(x-1)^2 + y^2}$ = distance between p and the obstacle

$$\nabla \eta(p) = \begin{pmatrix} \frac{x-1}{\sqrt{(x-1)^2 + y^2}} \\ \frac{y}{\sqrt{x-1+y^2}} \end{pmatrix}$$

Test 2011

Problem 2

Consider a planar circular robot with differential-drive kinematics. Denote by $v_{\max}$, $\omega_{\max}$ the bounds on the absolute value of the robot velocity inputs, and by R the radius of the robot base. The robot must travel between cartesian points P_S (start) and P_g (goal) in a perfectly known planar environment that contains polygonal obstacles. Build a complete navigation system that integrates the following modules:

- a motion planner that generates a feasible collision-free path;
- a trajectory planner that computes admissible robot velocities along this path;
- a feedback controller that can track the reference trajectory;

Discuss in detail the possible options and the motivation behind your choices. Provide a block scheme of your system with a clear indication of the inputs and the outputs of each block. Points that deserve special attention are: (1) is your motion planner complete, and under which assumptions? (2) does your motion planner generate paths that the robot can follow? (3) will the reference trajectory belong to the class that your feedback controller can track?

- Motion planner: cell decomposition
- Polynomial interpolation between the centers of the cells computed in the motion planner
- Input-output linearization to track the trajectory

exploration of pros and cons of each point.

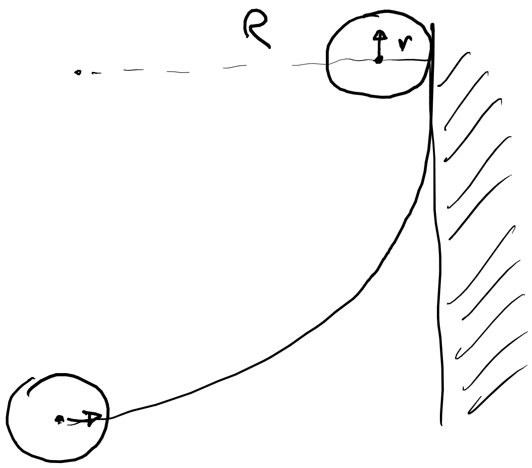
Test 2012

Consider an RRT-based motion planner for a unicycle robot based on the use of the following motion primitives:

$$v = \bar{v} \qquad \omega = \{-\bar{\omega}, 0, \bar{\omega}\} \qquad t \in [t_k, t_{k+1}]$$

where v and w are, respectively, the driving and steering velocity inputs, $\bar{v}$ and $\bar{\omega}$ are positive constants, and t_k, t_{k+1} are two consecutive sampling instants. The unicycle body is a circle of radius r. Assume that the environment is an empty square room, so that the only obstacles are the room walls. The goal configuration is the center of the room.

1. Which is the minimum necessary clearance (i.e., distance from the unicycle center to the obstacles) for the unicycle at the start to guarantee that the planner will find asymptotically a solution? [*Hint: if the unicycle is facing a wall...*]
2. How would you tune or modify the planner to reduce or possibly eliminate this clearance?



The only way for the unicycle to avoid collision with the wall is to turn as much as possible choosing $\omega = \pm\bar{\omega}$ fixed for more consecutive intervals. The robot in this way moves along an arc of circle of radius $R = \frac{\bar{v}}{\bar{\omega}}$

The clearance is therefore $R + r = \frac{\bar{v}}{\bar{\omega}} + r$

To reduce the clearance one may decrease the value of $\bar{v}$ or increase $\bar{\omega}$.

To eliminate it rotation at $v = 0$ are needed or $v = -\bar{v}$ to allow backward motion.

Test 2014

Problem 2

Consider a circular robot with unicycle kinematics moving in a known environment containing circular obstacles. Denote by $\boldsymbol{q} = (\boldsymbol{p}, \theta)$ the robot configuration, with $\boldsymbol{p} = (x, y)$, and by v, ω its velocity inputs. The robot must reach a certain destination $\boldsymbol{p}_{\text{goal}}$ (final orientation is not assigned). Build a navigation system that integrates the following components:

1. A robot-independent module that uses the environment geometry and the assigned goal to build an artificial force field $\boldsymbol{f}(\boldsymbol{p})$.

2. A module that transforms the artificial command $\boldsymbol{f}(\boldsymbol{p})$ into actual velocity inputs v, ω.

3. A module that computes the robot state needed by the first two.

Discuss in detail the possible options and the motivation behind your choices. Provide a block scheme of your system with a clear indication of the inputs and the outputs of each block. Points that deserve special attention are: (1) which is the main difficulty in building the transformation module? (2) with your navigation system, is the robot guaranteed to converge to the destination, and under which assumptions?

Solution of Problem 2

(just the main ideas are sketched, other options are possible)
The potential field module is trivial (see slides on Motion Planning 3). The problem, however, is that a unicycle robot is not *free-flying* in its configuration space due to its nonholonomy; in particular, its representative point P (with coordinates $\mathbf{p}$) can only move instantaneously in the direction of the sagittal axis, whereas the artificial force field $\boldsymbol{f}$ at $\mathbf{p}$, which depends only on the obstacle and goal placement, may be oriented in any direction. As a consequence, setting $\dot{\boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{p})$ is impossible in this case.

To transform the artificial force field $\boldsymbol{f}(\boldsymbol{p})$ in actual velocity inputs v, ω, one possibility is to assume that $\boldsymbol{f}$ acts on a point B (with coordinates $\boldsymbol{p}_B$) which is displaced from P of a certain distance b along the sagittal axis, as in input-output linearization control (see slides on Motion Control of WMRs: Trajectory Tracking). In fact, B can move in any direction, and the velocity inputs v, ω that realize a certain Cartesian velocity for B are easily computed by inverting the input-output map. The idea is then to set $\dot{\boldsymbol{p}}_B = \boldsymbol{f}(\boldsymbol{p}_B)$, and then compute v, ω that realize $\dot{\boldsymbol{p}}_B$.

A localization module will also be necessary for making (an estimate of) $\boldsymbol{q}$ available to the transformation module (the input-output matrix depends on $\boldsymbol{q}$, in particular on θ). From $\boldsymbol{q}$, it is straightforward to compute (an estimate of) $\boldsymbol{p}_B$ to be passed to the potential field module.

As for the effectiveness of the above navigation strategy, the potential field itself will be free of local minima because the robot and the obstacles are circular (*world of spheres*). Isolated saddle points will be present; the total field is zero there but the robot can easily escape with a small perturbation. This means that a point robot subject to such field (including isolated perturbations) would always converge to the destination, provided that the latter is outside the range of infuence of all obstacles. For our unicycle robot, however, it will be point B that converges to the destination, while point P will actually lie on a circle of radius b centered at the destination. If b can be chosen small (this depends on the bounds on the input velocities), the final navigation error will be acceptable.