35. Artstein-Sontag's Theorem

martedì 14 luglio 2020 16:50

It's a basic tool in the control Lyapunov Function (CLF) approach.

$\dot{x} = f(x) \quad f(0) = 0$

if $x_e = 0$ is AS then $\exists V$ positive definite such that $\dot{V} < 0$ around zero.

$\dot{x} = f(x) + g(x)u$

if $\exists$ $u = \alpha(x) \ni f + g\alpha$ is AS then $\exists V$ such that

$$\frac{\partial V}{\partial x}(f + g\alpha) < 0$$

CLF approach:

solve the previous inequality w.r.t. $\alpha(x)$ for a given V or (at the same time) compute V and $\alpha(x)$ such that the previous inequality holds true.

According to the inverse Lyapunov theorem, if a system is Globally Asymptotically stabilized by a smooth feedback $u = \alpha(x)$, then $\exists V$ positive and proper such that

$$\frac{\partial V}{\partial x}(f + g\alpha) = L_f V + \alpha L_g V < 0 \quad \forall x \neq 0$$

$\Rightarrow L_f V < 0$ at each $x \ni L_g V = 0$

then a necessary condition for a system to be globally asymptotically stabilizable (via smooth feedback) is the existence of a positive and proper V with the property that

$L_g V(x) = 0 \rightarrow L_f V(x) < 0 \quad \forall x \neq 0$

Such a function is called Control Lyapunov Function (CLF).

This condition is also sufficient for the existence of a stabilizer, $u = \alpha(x)$, $\alpha(0) = 0$, smooth in $x \in \mathbb{R}^n \setminus \{0\}$ and continuous at $x = 0$.

Such a function is called also "almost smooth function".

Theorem: (f, g) be smooth, $f(0)=0$

$\exists$ an almost smooth $\alpha(x)$ which solves the GAS problem iff $\exists$ V positive and proper such that:

(i) $L_g V(x) = 0 \Longrightarrow L_f V(x) < 0 \quad \forall x \neq 0$

i.e. V is a CLF

(ii) $\forall \varepsilon \; \exists \delta$: if $x \neq 0$, $\|x\| < \delta$ then $\exists$ some u, $|u| < \varepsilon \ni$

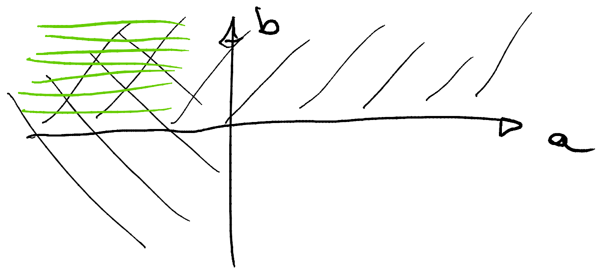
$L_f V + u L_g V < 0$

i.e., the control is small and continuous around $x=0$ (small control property)

Proof:

- Necessity: is obvious. For (ii) is continuity.
- Sufficiency:

$$S = \{(a,b) \in \mathbb{R}^2 : b \geq 0 \text{ or } a < 0\}$$



and let in S

$$\Phi(a,b) = \begin{cases} 0 & \text{if } b=0 \text{ and } a<0 \\ \dfrac{a+\sqrt{a^2+b^2}}{b} & \text{elsewhere} \end{cases}$$

it is needed to prove that Φ is real analytic in S

For this:

$$F(a,b,p) = bp^2 - 2ap - b$$

$$p_{12} = \frac{a \pm \sqrt{a^2+b^2}}{b}$$

note that $F(a,b,p) = 0 \quad \forall (a,b) \in S$ for which $p = \phi(a,b)$ (at any value of the parameter $p = \phi(a,b)$)

$$\Longrightarrow \left.\frac{\partial F}{\partial p}\right|_{p=\phi(a,b)} = (2pb - 2a)\big|_{\phi} = 2(pb-a)\big|_{\phi} =$$

$$= 2\left(\cancel{b}\,\frac{\cancel{a} \pm \sqrt{a^2+b^2}}{\cancel{b}} - \cancel{a}\right) \neq 0$$

it is never zero on S.

This definitely implies, by the implicit function theorem, that ϕ is real analytic; because $\phi(a,b)$ is the solution of $F(a,b,p)=0$ with F real analytic.

If V satisfy (i), $\forall$ pair $(a,b) = (L_f V, (L_g V)^2)$ is in S,

setting $\alpha(x) = \begin{cases} 0 & \text{if } x=0 \\ -L_g V \phi(L_f V, (L_g V)^2) & \text{elsewhere} \end{cases}$

this feedback, which is the composition of real analytic functions is real analytic itself (smooth).

Moreover (ii) $\Longrightarrow$ $\alpha(x)$ is continuous at $x=0$.

By construction $\alpha(x)$ is such that

$$\frac{\partial V}{\partial x}(f+g\alpha) = L_f V + L_g V\left(-L_g V\,\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{(L_g V)^2}\right)$$

$$= -\sqrt{(L_f V)^2 + (L_g V)^4} < 0$$

this implies GAS.

And the final form of the feedback is

$$\alpha(x) = \begin{cases} 0 & \text{if } L_g V = 0 \\ -\dfrac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{(L_g V)} & \text{elsewhere} \end{cases}$$