Stability

martedì 16 giugno 2020 15:57

11/2019

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \dot{x}_2 = -x_2 - x_2^3 \end{cases}$$

stability

computation of equilibria

$$J_f(x_1, x_2)\Big|_{\binom{0}{0}} = \begin{pmatrix} -3x_1^2 & 1 \\ 0 & -1-3x_2^2 \end{pmatrix}\Big|_{\binom{0}{0}} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{matrix} A = 0 \\ B = -1 \end{matrix}$$

Center manifold analysis

try $x_2 = h(x_1) = 0$ satisfying $h(0) = 0$ and $\left.\frac{\partial f}{\partial x_1}\right|_0 = 0$

center manifold equation

$$\frac{\partial h}{\partial x_1}\left(Ax_1 + f(x_1, x_2)\right) - Bx_2 - g(x_1, x_2) = 0$$

$$\begin{cases} f(x_1, x_2) = -x_1^3 + x_2 \\ g(x_1, x_2) = -x_2^3 \end{cases}$$

The center manifold equation with $h(x) = 0$ becomes:

$$\frac{\partial h}{\partial x_1} \cancel{f(x_1, x_2)}^{0} - \cancel{Bx_2}^{0} + \cancel{x_2^3}^{0} = 0$$

$0 = 0 \longrightarrow h(x) = 0$ works

substituting $h(x) = 0$ in the reduced dynamics:

$\dot{x}_1 = -x_1^3 + \cancel{x_2}^{0} = -x_1^{3}$ AS

odd

$\boxed{\begin{array}{l} -ax^k \\ \text{stable if } a > 0 \\ \qquad k \text{ odd} \to \text{dispari} \end{array}}$

Lyapunov method

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\frac{\partial V}{\partial x_1 \partial x_2}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -x_1^3 + x_2 \\ -x_2 - x_2^3 \end{pmatrix} = -x_1^4 + x_1x_2 - x_2^4 - x_2^2 \leq 0?$$

nothing can be said with this Lyapunov f.

09/19

$$\begin{cases} \dot{x} = x(d - x^3) - \alpha xy = xd - x^4 - \alpha xy \\ \dot{y} = -y + \alpha xy \end{cases}$$

$$\begin{cases} \dot{x} = x(d - x^3) - \alpha x y = xd - x^4 - \alpha x y \\ \dot{y} = -y + \alpha x y \end{cases}$$

$$J_f(x,y)\Big|_{\binom{0}{0}} = \begin{pmatrix} d - 4x^3 - \alpha y & -\alpha x \\ +\alpha y & -1 + \alpha x \end{pmatrix}\Bigg|_{\binom{0}{0}} = \begin{pmatrix} d & 0 \\ 0 & -1 \end{pmatrix}$$

if $d > 0 \to$ unstable

if $d < 0 \to$ stable

if $d = 0 \to$ center manifold analysis

$$\begin{cases} \dot{x} = -x^4 - \alpha x y \\ \dot{y} = -y + \alpha x y \end{cases}$$

$y = h(x) = 0 \qquad h(0) = 0, \ \frac{\partial h}{\partial x}\Big|_0 = 0$

$A = 0 \qquad B = -1$

$$\begin{cases} f(x,y) = -x^4 - \alpha x y \\ g(x,y) = \alpha x y \end{cases}$$

$$\frac{\partial h}{\partial x}\big(\cancel{f(x,y)}\big) - B\,\cancel{h(x)} - \alpha x\, \cancel{h(x)} = 0 \qquad 0 = 0$$

reduced dynamics: $\dot{x} = -x^4 - \alpha x \cancel{h(x)} = -x^4 \to$ even unstable

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07/19

$$\begin{cases} \dot{x} = \dfrac{x^3 - 2}{1 - x^2} + 2 \\ \dot{y} = -2y + x^2 \end{cases}$$

note that $\dot{x}$ is an autonomous system (does not depend on y)

equilibrium: $(x \ y)_e = (0 \ 0)$

we can simply study the sign of the function $\dfrac{x^3 - 2}{1 - x^2} + 2$

$\varepsilon_1 > 0 = 10$

$$\frac{\varepsilon^3 - 2}{1 - \varepsilon^2} + 2 = -10$$

$\varepsilon_1 < 0 = -10$

$$\frac{(-\varepsilon)^3 - 2}{1 - (-\varepsilon)^2} + 2 = 10$$

$100 = \varepsilon_2 > \varepsilon_1$

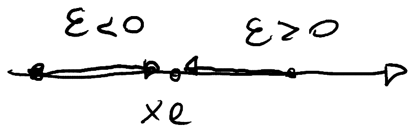
$$\frac{\varepsilon^3 - 2}{1 - \varepsilon^2} + 2 = -100$$

$\varepsilon_2 < \varepsilon_1$

$$\frac{(-\varepsilon)^3 - 2}{1 - (-\varepsilon)^2} + 2 = 100$$

$\varepsilon < 0 \qquad \varepsilon > 0$

$1-\varepsilon$ $\qquad\qquad$ $1-(-\varepsilon)^2$



stable

6/19

$$\begin{cases} \dot{x} = \dfrac{x^5-1}{1-x} + 1 \\ \dot{y} = -3y + x^2 \end{cases}$$

Autonomous system

$\varepsilon_1 > 0 = 10$

$$\frac{\varepsilon^5-1}{1-\varepsilon} = -11111$$

$\varepsilon_2 = 100 > \varepsilon_1$

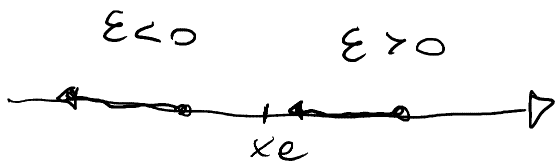
$$\frac{\varepsilon^5-1}{1-\varepsilon} = -10101010$$

$\varepsilon_1 < 0 = -10$

$$\frac{(-\varepsilon)^5-1}{1-(-\varepsilon)} = -9091$$

$\varepsilon_2 = -100 < \varepsilon_1$

$= -990099$



unstable

9/19

Given $\dot{x} = f(x)$ with $f(0) = 0$, prove that, if

$$\frac{\partial f}{\partial x}(0) + \left(\frac{\partial f}{\partial x}\right)^T < 0$$

then the origin is exponentially LES

$$\frac{\partial f}{\partial x}(0) = A \quad \text{suppose } A \text{ } 2\times 2 \text{ matrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Lyapunov direct theorem for LTI

$PA + A^TP < 0$

$A + A^T < 0$ $\qquad$ because $P = I$

$\{ \dot{x}_1 = -x_1^4 \sin(x_1) - x_2^2$

$$\begin{cases} \dot{x}_1 = -x_1^4 \sin(x_1) - x_2^2 \\ \dot{x}_2 = -x_2 - x_2^3 \end{cases} \qquad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$J_f(x_1, x_2)\Big|_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \left.\begin{pmatrix} -x_1^4 \cos(x_1) - 4x_1^3 \sin(x_1) & -2x_2 \\ 0 & -1 - 3x_2^2 \end{pmatrix}\right|_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} =$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{aligned} &A = 0 \leftarrow \text{center manifold} \\ &B = -1 \end{aligned}$$

$f(x_1, x_2) = -x_1^4 \sin(x_1) - x_2^2$

$g(x_1, x_2) = -x_2^3$

$h(x_1) = 0 = x_2$

$$\frac{\partial h}{\partial x_1} \left(Ax + f(x, h(x)) \right) - B h(x) - g(x, h(x))$$

$$\frac{\partial h}{\partial x_1} \left(\cancel{f(x_1, x_2)}^{0} \right) + \cancel{h(x_1)}^{0} + \cancel{h(x_1)^3}^{0} = 0 \qquad 0 = 0$$

substituing $\quad x_2 = h(x_1) = 0$

$\dot{x}_1 = -x_1^4 \sin(x_1)$

$$-\pi < \sin(x_1) < \pi$$

$\varepsilon_1 > 0 = \frac{\pi}{4}$
$= -5.21 \times 10^{-3}$

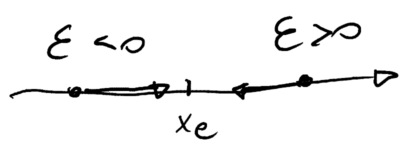
$\varepsilon_1 < 0 = -\frac{\pi}{4}$
$= 5.2157 \times 10^{-3}$

$\varepsilon_2 > \varepsilon_1 = \frac{\pi}{2}$
$= -0.1668$

$\varepsilon > 0$ decrease

$\varepsilon_2 < \varepsilon_1 = -\frac{\pi}{2}$
$= 0.1668$

$\varepsilon < 0$ increase



AS

$$\begin{cases} \dot{x} = x^2 y \\ \dot{y} = -e^{-x} y - x^3 \end{cases}$$

$$J_f(x,y)\Big|_0 = \begin{pmatrix} 2xy & x^2 \\ e^{-x}y - 3x^2 & -e^{-x} \end{pmatrix}\Bigg|_{\binom{0}{0}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$A = 0 \quad B = -1$

center manifold

$y = h(x) = 0$

$f(x,y) = x^2 y$

$g(x,y) = -e^{-x} y - x^3$

$$\frac{\partial h}{\partial x}\left(Ax + f(x,h(x))\right) - Bh(x) - g(x,h(x))$$

$$\frac{\partial h}{\partial x}\left(f(x,h(x))\right) + h(x) - g(x,h(x)) = 0$$

$$= 0 + 0 + x^3 = 0 \quad \neq 0 \quad h(x) = 0 \text{ does not work!}$$

$y = h(x) = ax^2 + O(|x|^3)$

$2ax(x^2 ax^2) + ax^2 - ax^2 e^{-x} + x^3$

if $a = 0 \Rightarrow x^3 \neq 0$ $\quad h(x) = ax^2$ does not work!

$h(x) = ax^2 + bx^3 + \ldots = y$

$$\underbrace{(2ax + 3bx^2 + \ldots)}^{\partial h/\partial x}(ax^4 + bx^5 \ldots) + ax^2 + bx^3 + e^{-x}(ax^2 + bx^3 \ldots) + x^3 =$$

$b = -1 \Rightarrow h(x) \cong bx^3 = -x^3 = y$

reduced dynamics

$\dot{x} = -x^{5}$ → odd $\quad$ AS

Lyapunov

$V(x,y) = \frac{1}{2}(x^2 + y^2)$

$$V(x,y) = \frac{1}{2}(x^2+y^2)$$

$$\frac{\partial V}{\partial x \partial y}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = (x \;\; y)\begin{pmatrix} x^2 y \\ -e^{-x}y - x^3\end{pmatrix} = \cancel{x^3 y} - e^{-x}y^2 - \cancel{x^3 y} =$$
$$= -e^{-x}y^2 < 0 \; ?$$

in $y=0$ $\dot{V}=0$ $\Rightarrow$ $\dot{V}$ neg. semidefinite

Using the Krasowsky - La Salle principle, the system is also AS because $y=0$ makes $\dot{V} \leq 0$

$$E = \{ x \in \Omega : \dot{V} = 0 \}$$

$\dot{V}$ is NSD in $S(0,\infty)$ and no other solution can stay in E $\Rightarrow$ the solution $x_e = 0$ is GAS

01/17

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_1 - x_2 - x_3 - x_1 x_3 \\ \dot{x}_3 = x_2 + x_1 x_2 \end{cases} \qquad V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

$$\frac{\partial V}{\partial x} f(x) = (x_1 \;\; x_2 \;\; x_3)\begin{pmatrix} -x_1 \\ -x_1 - x_2 - x_3 - x_1 x_3 \\ x_2 + x_1 x_2 \end{pmatrix} = -x_1^2 - x_1 x_2 - x_2^2 - \cancel{x_3 x_2} - \cancel{x_1 x_2 x_3} + \cancel{x_2 x_3} + \cancel{x_1 x_2 x_3} =$$
$$= -x_1^2 - x_2^2 - x_1 x_2 \leq 0$$

Using La Salle:

$$E = \{ x \in \Omega : \dot{V}(x) = 0 \}$$

$\dot{V}$ is NSD and no other solution can stay in E, then $x_e = 0$ is GAS

$$A = J_f(x_1 \;\; x_2 \;\; x_3)\Big|_{\begin{pmatrix}0\\0\\0\end{pmatrix}} = \begin{pmatrix} -1 & 0 & 0 \\ -1-x_3 & -1 & -1-x_1 \\ x_2 & x_1+1 & 0 \end{pmatrix}_{\begin{pmatrix}0\\0\end{pmatrix}} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

eigenvalues:

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 0 & 0 \\ -1 & -1-\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} \qquad \det(A - \lambda I) = (-1-\lambda)\left[-\lambda(-1-\lambda) + 1\right] = 0$$

$$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1} = \sqrt{3}\, i$$

$\lambda_1 = -1$ $\qquad \lambda + \lambda^2 + 1 = 0$

$-1 \pm \sqrt{1-4}$ $\Rightarrow$ $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \lambda_2$

$\lambda_1 = -1$ $\quad \lambda + \lambda^2 + 1 = 0$

$\frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \lambda_{1,2}$

$Re(\lambda_i) \subset \mathbb{C}^- \Rightarrow AS$

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01 / 20

$$\begin{cases} \dot{x}_1 = x_1 x_2^3 \\ \dot{x}_2 = -x_2 + x_1^2 \end{cases} \qquad J_f(x)\Big|_{\binom{0}{0}} = \begin{pmatrix} x_2^3 & 3x_1x_2^2 \\ 2x_1 & -1 \end{pmatrix}_{\binom{0}{0}}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$A = 0 \quad B = -1$

$\dot{x} = \dot{x}_1 = x_1 x_2^3 = f(x,y) = xy^3$

$\dot{y} = \dot{x}_2 = -x_2 + x_1^2 = -y + g(x,y) = -y + x^2$

$f(x,y) = x_1 x_2^3 \qquad g(x,y) = x_1^2$

Center manifold analysis

$y = h(x) = 0 \qquad h(0) = 0 \quad \frac{\partial h}{\partial x}\Big|_0 = 0$

$\frac{\partial h}{\partial x}\left(Ax + f(x, h(x))\right) - B\,h(x) - g(x, h(x)) =$

$x^2 = 0$ NOT possible

$h(x) = ax^2 + O(|x|^3)$

$2ax(ax^7) + ax^2 - x^2 = 0$

$a = 1 \Rightarrow h(x) \cong x^2 = y$

reduced dynamics

$\dot{x} = x(x^2)^3 = x^7 \Rightarrow$ UNSTABLE

$$\begin{cases} \dot{x} = ax^3 + xy - xy^2 \\ \dot{y} = -y + bx^2 + x^2 y \end{cases} \qquad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$J_f(x,y)\Big|_{\binom{0}{0}} = \begin{pmatrix} 3ax^2 + y - y^2 & x - 2xy \\ 2bx + 2xy & -1 + x^2 \end{pmatrix}_{\binom{0}{0}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Center manifold analysis

$A = 0 \qquad f(x,y) = ax^3 + xy - xy^2$

$B = -1 \qquad g(x,y) = bx^2 + x^2 y$

$h(x) = 0 \qquad \frac{\partial h}{\partial x}(0) = 0 \qquad h(0) = 0$

$$\frac{\partial h}{\partial x}\left(Ax + f(x, h(x))\right) - B h(x) - g(x, h(x)) = 0$$

$$0 + 0 - bx^2 \neq 0$$

$h(x) = cx^2 + dx^3 \dots$

$$(2cx + 3dx^2)\left(ax^3 + (cx^3 + dx^4 \dots) - (cx^4 + dx^5 \dots)\right) +$$

$$+ cx^2 + dx^3 \dots - bx^2 + \left(x^2(cx^2 + dx^3 \dots)\right) = 0$$

if $c = b \implies h(x) = bx^2 = y$

reduced dynamics

$$\dot{x} = ax^3 + bx^3 - x b^2 x^4 = ax^3 + bx^3 - b^2 x^5$$

$$= ax^3 + b(x^3 \underbrace{- bx^5}_{\text{stable}}) \simeq ax^3 + bx^3 = x^3(a+b)$$

AS if $(a+b) < 0$

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05 / 2020

$$\begin{cases} \dot{x} = -x + ax^2 y^2 \\ \dot{y} = -y^3 + y^5 \end{cases} \implies \begin{cases} \dot{x}_1 = -x_1^3 + x_1^5 \\ \dot{x}_2 = -x_2 + a x_2^2 x_1^2 \end{cases}$$

in order to have the form

$$\begin{cases} \dot{x}_1 = Ax_1 + f(x_1, x_2) \\ \dot{x}_2 = Bx_2 + g(x_1, x_2) \end{cases}$$

$J_f\Big| = \begin{pmatrix} -3x_1^2 + 5x_1^4 & 0 \\ \dots & \dots \end{pmatrix}\Big|$

$$\begin{cases} \dot{x}_1 = Ax_1 + f(x_1, x_2) \\ \dot{x}_2 = Bx_2 + g(x_1, x_2) \end{cases}$$

$$J_f\Big|_0 = \begin{pmatrix} -3x_1^2 + 5x_1^4 & 0 \\ 2a x_2^2 x_1 & -1 + 2a x_2 x_1^2 \end{pmatrix}\Bigg|_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$A = 0 \quad B = -1$

$f(x_1, x_2) = -x_1^3 + x_1^5$

$g(x_1, x_2) = a x_2^2 x_1^2$

$h(x_1) = 0 = x_2$

$$\frac{\partial h}{\partial x_1}\left(Ax_1 + f(x_1, h(x_1))\right) - B h(x_1) - g(x_1, h(x_1)) = 0$$

$$0 - 0 - 0 = 0$$

$x_2 = h(x) = 0$

reduced dynamics:

$\dot{x}_1 = -x_1^3 + x_1^5$

$\varepsilon_1 > 0 = 10$

$-(10)^3 + (10)^5 = 99000$

$\varepsilon_2 > \varepsilon_1 = 100$

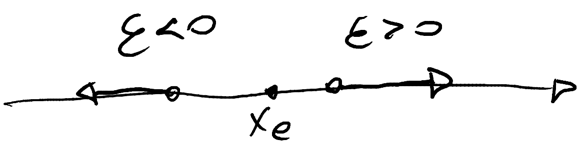
$-(100)^3 + (100)^5 = 9999000..$

$\varepsilon_1 < 0 = -10$

$-(-10)^3 + (-10)^5 = -99000$

$\varepsilon_2 < \varepsilon_1 = -100$

$-(-100)^3 + (-100)^5 = -99000...$



UNSTABLE