# INTRODUCTION

## Notations :

$x(t) \in \mathbb{R}^n$ state variable
$u(t) \in \mathbb{R}^p$ control variable

$$f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} : \to \mathbb{R}$$

$\bar{C}^k$ function with a derivative up to the k-th order continuous almost everywhere

example:

$\bar{C}^0 \rightsquigarrow$ continuous almost everywhere
$\bar{C}^1 \rightsquigarrow$ continuous ($C^0$) and $C^1$ almost everywhere
$\vdots$
$\bar{C}^6 \rightsquigarrow$ continuous $C^0, C^1, \ldots, C^5$ and $C^6$ almost everywhere

Optimal control sets out to provide analytical designs of a special appealing type.

The final system is suppose to be the best possible system of a particular type

There is a COST INDEX because the system has to behave in the best way
$\downarrow$
best depends on the specific criteria chosen

- **Linear Optimal Control (LOC)**

special sort of optimal control

↳ linear plant

↳ controller constrained to be linear

Linear controllers are achieved thanks to <u>quadratic cost indices</u>

Pros:

1) LOC may be applied to nonlinear systems
2) LOC have mainly computational solutions
3) If a nonlinear system has not strong nonlinearities it is possible to model it approximating as a linear system

Birth of Optimal Control (1696)

↓

Bernoulli - Brachistochrone problem

Control the path of the behaviour of a dynamical system:

Particle of mass M moves along a wire from A to B under gravity. Find the shape of the wire in order to reach B in minimum time.

- **Feedback**

The actual operation of the control system is compared to the desired operation and the input to the plant is adjusted on the basis of this comparison

Feedback control systems are able to operate satisfactorily despite adverse conditions, such as disturbances and variations in plant properties

- **Optimal control problem**

$$\left.\begin{aligned} \dot{x} &= f(x,u,t) \\ x(t_0) &= x_0 \end{aligned}\right\} \text{ control system}$$

$$J(u) = \int_{t_0}^{t_f} L(x(t), u(t), t)\,dt + K(x(t_f), t_f)$$

↳ the cost index depends on the goal chosen

# Remark : the optimal control $u(t)$ is a function of time.
$J$ is a functional (a function of a function) which returns a number (cost)

The problem is to choose the best path among all paths feasible for the system, with respect to the given cost function.

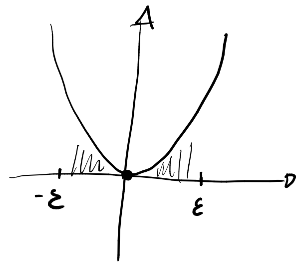
This is an infinite dimensional problem because the space of paths is an infinite dimensional function space

The problem is a dynamic optimization because it involves a dynamical system and time

# • local / strict / global minimum (maximum)

Find a minimum of $f : \mathbb{R}^n \to \mathbb{R}$

example:




$y = x^2$

$y' = 2x \rightsquigarrow y' = 0 \to 2x = 0 \to x = 0$

$y'' = 2 > 0$

$f : \mathbb{R}^n \to \mathbb{R}$ $\qquad x = (x_1 \dots x_n)^T$

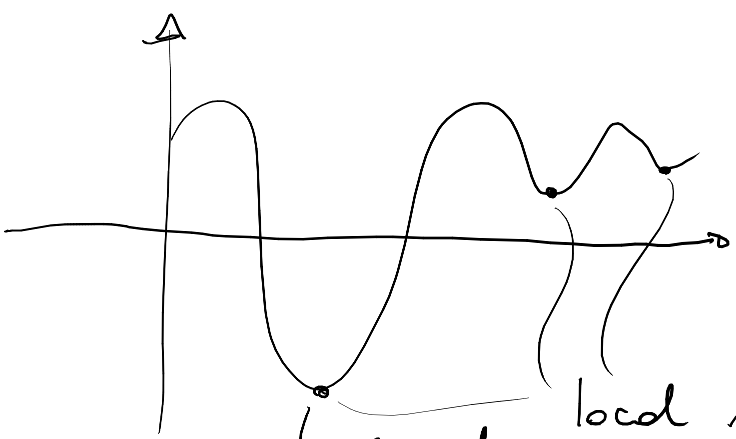
$D \subseteq \mathbb{R}^n$

$|\cdot|$ = Euclidean norm

A point $x^* \in D$ is a LOCAL MINIMUM of $f$ over $D \subseteq \mathbb{R}^n$ if

$\exists\, \varepsilon > 0$ s.t. $\forall x \in D$ satisfying $|x - x^*| < \varepsilon$

$\Rightarrow \boxed{f(x^*) \le f(x)} \rightsquigarrow$ if $f(x^*) < f(x)$ there's a STRICT LOCAL MINIMUM $\forall x \neq x^*$



local minima in their subintervals

Global minimum

A point $x^* \in D$ is a GLOBAL MINIMUM of $f$ over $D \subseteq \mathbb{R}^n$ if

for all $x \in D \Rightarrow f(x^*) \leq f(x)$

If a point is either a maximum or a minimum is called EXTREMUM

- **Unconstrained optimization** (no limitations)

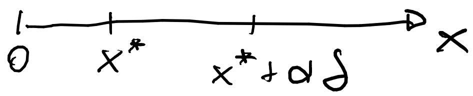
  - first order necessary conditions

All points $x$ sufficiently near $x^*$ in $\mathbb{R}^n$ are in $D$

Assume $f \in C^1$ and $x^*$ is its local minimum $(f(x^*) \le f(x))$
Let $\delta \in \mathbb{R}^n$ (arbitrary vector)

Being in the unconstrained case:

$x^* + \alpha\delta \in D \quad \forall \alpha \in \mathbb{R}$ close enough to $0$




Let's define a function $k(\alpha) := f(x^* + \alpha\delta)$

So $\boxed{0 \text{ is a minimum of } k}$

First order Taylor expansion of $k$ around $\alpha = 0$

$$k(\alpha) = k(0) + k'(0)\,\alpha + o(\alpha)$$

$$\lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0$$

$\Rightarrow \boxed{k'(0) = 0}$

GOAL

# Proof:

I want to prove that demonstrating by contraddiction

assume $\varphi'(0) \neq 0 \Rightarrow \exists\ \varepsilon > 0$ small enough s.t.
for $|\alpha| < \varepsilon$ $\quad |o(\alpha)| < |\varphi'(0)\alpha|$

For these values of $\alpha$

$$\varphi(\alpha) - \varphi(0) = \varphi'(0)\alpha + o(\alpha) < \varphi'(0)\alpha + |\varphi'(0)\alpha|$$

If we restrict $\alpha$ to have the opposite sign of $\varphi'(0)$

$\varphi(\alpha) - \varphi(0) < 0 \Rightarrow$ Contraddiction $(\varphi'(0) = 0)$

# End

$\varphi'(\alpha) = \nabla f(x^* + \alpha d)$

$\hookrightarrow \nabla f = (f_{x_1} \dots f_{x_n})^T$ gradient of $f$

$\varphi'(0) = 0 = \nabla f(x^*) \cdot d$

Being $d$ arbitrary:

$\boxed{\nabla f(x^*) = 0}$ First order necessary condition for optimality

A point $x^*$ satisfying this condition is a stationary point. The result is valid when $f \in C^1$ and $x^*$ is an interior point (inside D).

Therefore when $\alpha = 0$ (not $x^* = 0$), $\varphi(\alpha)$ has a minimum (we are not saying that $f(x^*) = 0$) so $\varphi'(0) = 0$.

- Second order conditions

(Like before but second order)

Assume $f \in C^2$ and $x^*$ its local minimum

Let $\delta \in \mathbb{R}^n$ be an arbitrary vector.

Second order Taylor expansion of $k$ around $\alpha = 0$

$$k(\alpha) = k(0) + k'(0)\alpha + \frac{1}{2}k''(0)\alpha^2 + o(\alpha^2),$$

$$\lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0$$

Since $k'(0) = 0 \Rightarrow k''(0) \geq 0$ (not negative)

# Proof :

Suppose $k''(0) < 0 \Rightarrow \exists\, \varepsilon > 0$ small enough so that for $|\alpha| < \varepsilon$ $|o(\alpha)^2| < \frac{1}{2}|k''(0)|\alpha^2$

For these values of $\alpha$, $k(\alpha) - k(0) < 0 \rightarrow$ <u>Contraddiction</u>

We already know that $k'(0) = 0$ from the previous case, therefore $k'' \geq 0$ (non negative)

# End

it means that

$$k(\alpha) = f(x^* + \alpha\delta)$$

$$k'(\alpha) = \nabla f(x^* + \alpha\delta)\delta = \sum_{i=1}^{n} f_{x_i}(x^* + \alpha\delta)\delta_i$$

By differentiating both sides of

$$k'(\alpha) = \sum_{i=1}^{n} f_{x_i}(x^* + \alpha\delta)\,\delta_i$$ with respect to $\alpha$

$$k''(\alpha) = \sum_{i,j=1}^{n} f_{x_i x_j}(x^* + \alpha\delta)\,\delta_i\,\delta_j$$

$$\Rightarrow k''(0) = \sum_{i,j=1}^{n} f_{x_i x_j}(x^*)\,\delta_i\,\delta_j = \delta^T \underbrace{\nabla^2 f(x^*)}\,\delta$$

Hessian matrix

$$\nabla^2 f = \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}$$

$x^*$ is a strict local minimum of $f$

Second order necessary condition for optimality

$$\boxed{\nabla^2 f(x^*) \geq 0}$$

The result is valid when $f \in C^2$ and $x^*$ is an interior point

# Remark :

Being $\delta$ arbitrary :

$f \in C^1$ $\quad \nabla f(x^*) = 0$

$f \in C^2$ $\quad \nabla^2 f(x^*) \geq 0$

The second order condition distinguishes minima from maxima

- At a local maximum $\nabla^2 f(x^*) \leq 0$ (negative semidefinite)
- At a local minimum $\nabla^2 f(x^*) \geq 0$ (positive semidefinite)

- Weierstrass Theorem (Existence result)

Determining points of global minimum (optimal) of $f$ in $D$.

$$D = \{ z \in \mathbb{R}^n : h(z) = 0 \ , \ g(z) \leq 0 \}$$

means finding every point $z^\circ \in D$ s.t.:

$$f(z^\circ) \leq f(z) \ , \ \forall z \in D$$

For the existence of the optimal solutions, Weierstrass theorem gives the sufficient conditions:

$C^0(D)$

$$\boxed{\text{If } D \text{ is a compact set and } f \text{ is continuous on } D, \text{ there exist global minimum (optimal point)}}$$

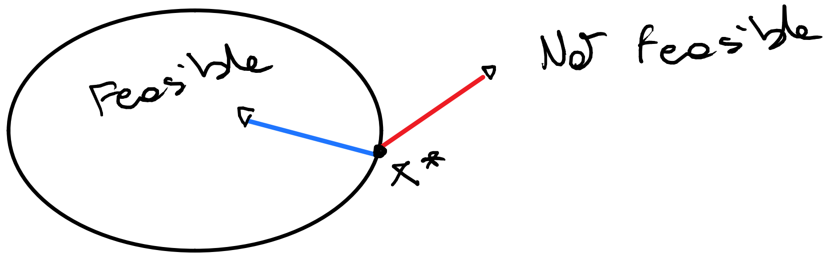
# Compact: property of closure and limitation of the domain

- Closure: given by the structure of $h$ and $g$ (they need to be continuous).

- Domain Limitation: it's not guaranteed by the structure of the domain, it depends only on the constraints: it holds only if $\lim_{z \to \infty} h \neq 0$ and/or some components of $g$ are $> 0$ $\forall z \to \infty$

## ~ Feasible directions

A vector $\delta \in \mathbb{R}^n$ is a feasible direction at $x^*$ if $x^* + \alpha\delta \in \Delta$ for small enough $\alpha > 0$




> If not all directions $\delta$ are feasible, then the condition $\nabla f(x^*) = 0$ is no longer necessary for optimality

Previously $\nabla f(x^*) = 0$.
If $x^*$ is a local minimum $\Rightarrow$ $\nabla f(x^*) \cdot \delta \geq 0$
$\forall$ feasible direction $\delta$

## ~ Procedure for finding a global minimum

1. Find all interior points of $\Delta$ satisfying $\nabla f(x^*) = 0$ (stationary points)
2. If $f$ is not differentiable everywhere, include also points where $\nabla f(x^*) = 0$ does not exist (critical points)
3. Find all boundary points satisfying $\nabla f(x^*)\delta \geq 0$ for all feasible $\delta$
4. Compare values at all these candidate points and choose the smallest one.

## ~ Convexity

If $f$ is a convex function and $D \subset \mathbb{R}^n$ is a convex set, a local minimum is automatically a global one and the first order necessary condition (for $f \in C^1$) is also a sufficient condition.

- # Constrained optimization

~ **Lagrangian**

Let $D \subset \mathbb{R}^n$ and $f \in C^1$

$h_1(x) = \dots = h_p(x) = 0$, $h \in C^1$ $\Rightarrow$ Equality constraints

$g_1(x) \leq 0, \dots, g_q(x) \leq 0$, $g \in C^1$ $\Rightarrow$ Inequality constraints

$\text{Rank}\left\{ \left. \dfrac{\partial(h, g_a)}{\partial x} \right|_{x^*} \right\} = p + q_a$ $\Rightarrow$ Regularity condition

Useful to avoid strange conditions like redundant constraints or strange situations

the gradients of $h$ and $g_a$ should be linearly independent

$g_a$ = active constraint of $g$ with dimension $q_a$ (the constraint $g$ when it is equal to 0)

$$\mathcal{L}(x, \lambda_0, \lambda, \eta) = \lambda_0 f(x) + \underbrace{\lambda^T}_{\text{Lagrange multipliers}} g(x) + \underbrace{\eta^T}_{\text{Kuhn-Tucker multipliers}} g(x)$$

↳ the Lagrangian is something like a perturbation

> If $\lambda_0 \neq 0$ (usually $\lambda_0 = 1$) the stationary point $x^*$ is called <u>normal</u>

## ~ First order necessary conditions

$f: \mathbb{R}^n \to \mathbb{R}$ with $f, h, g \in C^1$ with $x^*$ its local minimum

$$\boxed{\left.\frac{\partial \mathcal{L}}{\partial x}\right|_{x^*} = 0^T \qquad \mu_i g_i(x^*) = 0, \ \forall i \qquad \mu_i \geq 0 \ \forall i}$$

If $f$ and $g$ convex and $h$ linear $\Rightarrow$ necessary & sufficient conditions

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# Proof (only for equality constraints)

Assume that $x^*$ is a local minimum and a regular point
$\rightarrow \nabla h_i, \ i = 1 \ldots m$ are linearly independent in $x^*$

Let $x(\alpha) \in D$ such that $x(0) = x^*$ with $x(\alpha)$ a family of curves passing through $x^*$ $(\alpha \in \mathbb{R})$

Consider the function

$K(\alpha) = f(x(\alpha))$, $K(0) = f(x^*)$ is a minimum of $K$

$$K'(\alpha) = \nabla f(x(\alpha))x'(\alpha), \quad K'(0) = \nabla f(x^*)\, x'(0) = \nabla f(x^*)\, \delta = 0$$

$\forall$ tangent vector $\delta$

Consider the Taylor expansion (1-st order)

$$x(\alpha) = \underbrace{x'(0)}_{\text{tangent vector to } D \text{ at } x^*} \alpha + o(\alpha)$$

The tangent space to $D$ at $x^*$ is characterized by

$h_i(x(\alpha)) = 0 \quad \forall \alpha$ with $i = 1, \ldots, m$

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} h_i(x(\alpha)) = \nabla h_i(x^*)\, x'(0) = \underbrace{\nabla h_i(x^*)\, \delta = 0}_{} \quad \forall \delta \in \mathbb{R}^n$$

The tangent vectors to $D$ at $x^*$ are exactly $\delta$ for which the condition holds

$$\Rightarrow \boxed{\nabla f(x^*) \in \text{span}\{\nabla h_i(x^*), \ i = 1 \ldots m\}}$$

There exist real numbers $\lambda_1^*, \ldots, \lambda_m^*$ such that:

$$\nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \ldots + \lambda_m^* \nabla h_m(x^*) = 0$$

# End

## ~ Second order sufficient condition (not always applicable)

Let $x^* \in D$ and $f, h, g \in C^2$ and assume the conditions
$\hookrightarrow$ very regular

$$\frac{\partial \mathcal{L}}{\partial x}\bigg|_{x^*} = 0^T \qquad \eta_i g_i(x^*) = 0 \;, \quad \eta_i \geq 0 \quad \forall i$$

$x^*$ is a strict local minimum if

$$\delta^T \frac{\partial^2 \mathcal{L}}{\partial x^2}\bigg|_{x^*} \delta > 0 \quad \forall \delta \text{ such that } \frac{d h_i(x)}{dx}\bigg|_{x^*} \delta = 0 \quad i = 1 \dots p$$

- Bordered Hessian (Not used in this course)

$$H = \begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{\partial h_j}{\partial x_i} \\ \left(\dfrac{\partial h_j}{\partial x_i}\right)^T & 0 \end{pmatrix} \hookrightarrow \text{borders}$$

A point $x^*$ in which $\nabla \mathcal{L} = 0$ and $\det(H) \neq 0$ is called a non-degenerate critical point of the constrained problem

• **Function spaces**

Functional (function of functions): $J: V \to R$ with $A \subseteq V$
$\downarrow$
vector space

## ~ Norm $\|\cdot\|$

It is a real valued function on $V$

- Positive definite $\|y\| > 0$ if $y \neq 0$
- Homogeneous $\|\lambda y\| = |\lambda| \cdot \|y\|$ $\forall \lambda \in R, y \in V$
- Satisfies the triangle inequality $\|y+z\| < \|y\| + \|z\|$

## ~ Distance or metric

$$d(y,z) = \|y-z\|$$

## ~ Strong extreme (0-Norm)

$\rightsquigarrow$ A strong minimum is also a weak one (not the inverse)

On the space $C^0([a,b], R^n)$

$\|y\|_0 = \max_{a \le x \le b} |y(x)|$ $\Rightarrow$ Euclidean norm

$$\boxed{\text{Extreme of } J \text{ wrt the 0-Norm are strong extreme}}$$

## ~ Weak extreme (1-Norm)

On the space $C^1([a,b], R^n)$

$\|y\|_1 = \max_{a \le x \le b} |y(x)| + \max_{a \le x \le b} |y'(x)|$

$$\boxed{\text{Extreme of } J \text{ wrt the 1-Norm are weak extreme}}$$

## ~ k-th Norm

On the space $C^\ell([a,b], R^n)$, $\ell \ge k$ $\Rightarrow$ $\|y\|_p = \left(\int_a^b |y(x)|^p dx\right)^{1/p}$

~ **Definition of extreme**

$z^* \in A$ is a local minimum of $J$ over $A$ if
$\exists \varepsilon > 0 : \forall z \in A$ such that $\|z - z^*\| < \varepsilon \Rightarrow J(z^*) \leq J(z)$

~ **Variation**

Notation: $\delta$ = variation (like the incremental ratio)
$\downarrow$
derivative

Let $z + \alpha\eta$, $\eta \in V$, $\alpha \in \mathbb{R}$ a function in $V$.

It is an admissible perturbation wrt a subset $A$
if $z + \alpha\eta \in A \ \forall \alpha \to 0$

The first variation of $J$ at $z$ is the linear function
$\delta J|_z : V \to \mathbb{R}$ such that $J(z+\alpha\eta) = J(z) + \delta J|_z(\eta)\alpha + o(\alpha)$

or defined as:

$$\boxed{\delta J|_z(\eta) = \lim_{\alpha \to 0} \frac{J(z+\alpha\eta) - J(z)}{\alpha}}$$

- First order necessary condition for optimality

$$\boxed{\delta J|_{z^*}(\eta) = 0}$$

- Second order necessary condition

$\boxed{\delta^2 J|_{z^*}(\eta) \geq 0}$ with a second variation of $J$ at $z$ if
$J(z+\alpha\eta) = J(z) + \delta J|_z(\eta)\alpha + \delta^2 J|_z(\eta)\alpha^2 + o(\alpha^2)$

## ~ Weierstrass theorem

> If $A$ is a compact set and $J$ is continuous on $A$, there exist global minimum

## ~ Convexity

$A \subset V$ convex with $J: A \to \mathbb{R}$

$J$ is a convex functional on $A$ if

$$J(\alpha z' + (1-\alpha) z'') \leq \underbrace{\alpha J(z') + (1-\alpha) J(z'')}^{\text{linear combination}}$$

$z'$ and $z''$ other points

> If $J$ is a convex function and $A \subset V$ is a convex set, a local minimum is automatically a global one and the first order conditions are necessary and sufficient conditions for a minimum

If $A = \mathbb{R}^n$ and $J \in C^1(A)$: convexity $\Leftrightarrow$ $J(z) \geq J(z') + \left.\frac{dJ}{dz}\right|_{z'} \overbrace{(z-z')}^{\text{increment}}$

If $A = \mathbb{R}^n$ and $J \in C^2(A)$: convexity $\Leftrightarrow$ $(z-z')^T \left.\frac{d^2J}{dz^2}\right|_{z'} (z-z') \geq 0$

# CALCULUS OF VARIATIONS

The calculus of variations is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals.
Functionals are often expressed as definite integrals:

$$J(z) = \int_{t_i}^{t_f} \underbrace{L(t, z(t), \dot{z}(t))}_{\text{Lagrangian}} \, dt$$

- **The Lagrange Problem**

Let $D$ be an admissible set in $\bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$

$$D = \Big\{ (z, t_i, t_f) \in \bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} :$$
$$(z(t_i), t_i) \in D_i \subset \mathbb{R}^{v+1}, \quad q(z, \dot{z}, t) \leq 0$$
$$(z(t_f), t_f) \in D_f \subset \mathbb{R}^{v+1}, \quad g(z(t), \dot{z}(t), t) = 0,$$
$$\int_{t_i}^{t_f} h(z, \dot{z}, t)\, dt = 0 \Big\}$$

$g \in \mathbb{R}^{\mu < v}$ of $C^2$ class
$h \in \mathbb{R}^{\sigma}$ of $C^2$ class
$q$ of $C^2$ class, $q_a$ of dimension $\beta_a$

$$J(z, t_i, t_f) = \int_{t_i}^{t_f} \underbrace{L(z(t), \dot{z}(t), t)}_{\substack{\text{Lagrangian of } C^2 \text{ class} \\ L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}}} \, dt$$

The aim is to minimize the evolution (integral) of the functional from an instant $t_i$ to $t_f$

## ~ Lagrange theorem

Define the augmented lagrangian

$$\ell(z(t),\dot z(t),t,\lambda_0,\lambda(t),\rho) = \lambda_0 L(z(t),\dot z(t),t) + \eta^T(t)\, q(z,\dot z,t) + \lambda^T(t)\, g(z(t),\dot z(t),t) + \rho^T h(z(t),\dot z(t),t)$$

If $\lambda_0 \neq 0 \rightarrow$ normal solution

Let $(z^*, t_i^*, t_f^*) \in D$ be such that

$$\mathrm{rk}\left\{ \frac{\partial(g, q_a)}{\partial \dot z}\Big|^* \right\} = \nu + \beta_a \quad \forall [t_i^*, t_f^*]$$

If $(z^*, t_i^*, t_f^*)$ is a local minimum for $J$ over $D$, then there exist $\eta^* \in \bar C^0[t_i, t_f]$, $\lambda_0^* \in \mathbb{R}$, $\lambda^* \in \bar C^0[t_i^*, t_f^*]$, $\rho^* \in \mathbb{R}^6$ not simultaneously null in $[t_i^*, t_f^*]$

such that the following conditions hold:

- $\boxed{\text{Euler - Lagrange}}$ $\quad \dfrac{\partial \ell^*}{\partial z} - \dfrac{d}{dt}\dfrac{\partial \ell}{\partial \dot z}\Big|^* = 0^T \quad \forall t \in [t_i^*, t_f^*]$

- $\boxed{\text{Weierstress - Erdmann}}$ $\quad \dfrac{\partial \ell}{\partial \dot z}\Big|^*_{\bar t^-} = \dfrac{\partial \ell}{\partial \dot z}\Big|^*_{\bar t^+}$ $\quad \bar t$ are corner points of $z^*$

(for discontinuity points)

$$\left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^*_{\bar t^-} = \left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^*_{\bar t^+}$$

- $\boxed{\text{Trasversality}}$ $\quad$ different cases: they depend on the nature of the boundary conditions.

~ Euler - Lagrange equation

Trajectories satisfying the E-L equation are called extremals

$$\boxed{\left.\frac{\partial \ell}{\partial z}\right|^{*} - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{z}}\right|^{*} = 0} \qquad \forall t \in [t_i, t_f]$$

# Proof

Let's consider a curve $C^1$ $z : [a,b] \to \mathbb{R}$ with $z(a) = z_0$ and $z(b) = z_1$, and the functional

$$J(z) = \int_a^b \ell(z, \dot{z}, t)\, dt \qquad \ell \in C^2$$

The goal is to find the local minimum of $J$.

Introduce now the perturbation $\eta : [a,b] \to \mathbb{R}$
$\eta(a) = 0$, $\eta(b) = 0$.
In fact, if $z(a) = z_0$, $z(b) = z_1$, then
$z(a) + \alpha\,\eta(a) = z_0 \Rightarrow \eta(a) = 0$
$z(b) + \alpha\,\eta(b) = z_1 \Rightarrow \eta(b) = 0$

$$J(z + \alpha\eta) = \int_a^b \ell(z + \alpha\eta, \dot{z} + \alpha\dot{\eta}, t)\, dt =$$

$$1 \quad = \underbrace{J(z)}_{} + \alpha \underbrace{\delta J|_{z^*}(\eta)}_{=0} + o(\alpha)$$

(first order necessary condition)

$$J(z) = \int_a^b \ell(z, \dot{z}, t)\, dt$$

Taylor expansion with respect to $\alpha$

$$J(z+\alpha\eta) = \int_a^b \ell(z+\alpha\eta, \dot{z}+\alpha\dot{\eta}, t)\,dt$$

②

$$J(z+\alpha\eta) = \int_a^b \left(\ell(z,\dot{z},t) + \frac{\partial \ell}{\partial z}(z,\dot{z},t)\alpha\eta + \frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\alpha\dot{\eta}\right)dt$$

From ① and ② (dividing both members by $\alpha$):

$$\overset{②}{J}(z+\alpha\eta) = \cancel{\int \ell(z,\dot{z},t)dt} + \int \frac{\partial \ell}{\partial z}\cdot\cancel{\alpha}\,\eta\,dt + \int \frac{\partial \ell}{\partial \dot{z}}\cancel{\alpha}\,\dot{\eta}\,dt$$

$$\overset{①}{J}(z+\alpha\eta) = \cancel{\int \ell(z,\dot{z},t)dt} + \cancel{\alpha}\,\delta J|_{z^*}(\eta)$$

$$\Downarrow$$

$$\delta J|_{z^*}(\eta) = \int_a^b \frac{\partial \ell}{\partial z}(z,\dot{z},t)\,\eta\,dt + \int_a^b \frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\,\dot{\eta}\,dt$$

Integration by parts $\rightarrow$ $\displaystyle\int_a^b \frac{\partial \ell}{\partial \dot{z}}\dot{\eta}\,dt = -\int_a^b \frac{d}{dt}\frac{\partial \ell}{\partial \dot{z}}\eta\,dt + \frac{\partial \ell}{\partial \dot{z}}\eta\Big|_a^b$

$$③ \quad \frac{\partial \ell}{\partial \dot{z}}\eta(b) - \frac{\partial \ell}{\partial \dot{z}}\eta(a)$$

For the boundary conditions, $\eta(a)=0$, $\eta(b)=0$ and $\delta J|_{z^*}\eta = 0$, therefore ③ $=0$ and

$$\boxed{\int_a^b \frac{\partial \ell}{\partial z}(z,\dot{z},t)\,\eta(t)\,dt - \int_a^b \frac{d}{dt}\frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\,\eta(t)\,dt = 0}$$

for all curves vanishing at the ends points

That is the same of:

$$\int_a^b \left[ \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \right] \eta(t)\, dt = 0$$

$$\eta(a) = 0 \quad , \quad \eta(b) = 0$$

$\boxed{\text{Lemma: if a continuous function } g:[a,b]\to\mathbb{R} \text{ is such that } \int_a^b g(t)\,\eta(t)\,dt = 0 \ \ \forall \eta:[a,b]\to\mathbb{R} \text{ with } \eta(a)=0 \text{ and } \eta(b)=0 \text{ then } g = 0}$

Therefore

$$\boxed{\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} = 0}$$

Necessary condition for $z(\cdot)$ to be an extremum

# End

~ Variable endpoint problem

$$\boxed{\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_b^* = 0} \qquad \boxed{\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_b^* + \mathcal{L}\Big|_b^* = 0}$$

# Proof

Let us consider the $C^1$ curves $z:[a,b]\to\mathbb{R}$ s.t.
$z(a) = z_0$ , $z(b)$ free.

Find the local minima of $J(z) = \int_a^b \mathcal{L}(z,\dot{z},t)\,dt$
$\hookrightarrow$ $C^2$ class

The perturbations $\eta$ must satisfy $\eta(a)=0$ but $\eta(b)$ arbitrary

In this case the first variation is given by (*)

$$\delta J|_z(\eta) = \boxed{\int_a^b \frac{\partial \mathcal{L}}{\partial z}(t,z,\dot z)\eta(t)\,dt - \int_a^b \frac{d}{dt}\frac{\partial}{\partial \dot z}(z,\dot z,t)\eta(t)\,dt} + \frac{\partial \mathcal{L}}{\partial \dot z}(b, z(b), \dot z(b))\,\eta(b) = 0$$

if $z$ is an extremum

Perturbations $\eta(b)=0$ are still allowed, in that case I obtain the previous E-L condition, which is still a necessary condition for optimality.

After this consideration we know that (*) $=0$ for all admissible perturbations $\eta$ because

$$\frac{\partial \mathcal{L}}{\partial z}(z,\dot z,t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot z}(z,\dot z,t) = 0 \quad (\text{E-L condition})$$

Therefore the extra condition is found:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot z}(b, z(b), \dot z(b)) = 0}$$

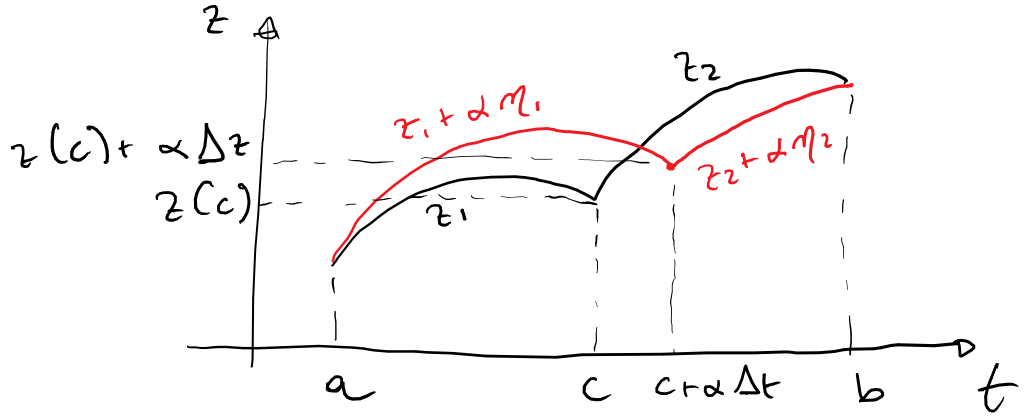
# ~ Weierstrass – Erdmann corner condition

Additional conditions at corner points, in order for $z$ to be a strong extremum

# Proof

Consider $z \in \bar{C}^1$ solution (continuous first derivative almost everywhere), we can have some points in which the derivative is not continuous.

Assume $c \in [a,b]$ is a corner point of $z$




Split $z$ in $z_1 : [a,c] \to \mathbb{R}$ and $z_2 : [c,b] \to \mathbb{R}$

The perturbed versions are $z_1 + \alpha\eta_1$ and $z_2 + \alpha\eta_2$ with $\eta_1(a) = \eta_2(b) = 0$

The location of the corner point is not fixed, so the corner point could deviate from the point $c$

Therefore the domain of $z_1 + \alpha\eta_1$ should be extended to $[a, c + \Delta t]$.

This can be done by linear continuation:

$z_1(t) = z(c) + \dot{z}(c^-)(t-c)$

$z_1 \in C^1$ in $c$ with $z_1(c) = z(c)$ and $\dot{z}_1(c) = \dot{z}(c^-)$

The same assumptions hold for $z_2$

New cost functional

$$J(z) = \int_a^b \mathcal{L}(z,\dot z,t)\,dt = \int_a^c \mathcal{L}(z_1,\dot z_1,t)\,dt + \int_c^b \mathcal{L}(z_2,\dot z_2,t)\,dt$$

$$= J_1(z_1) + J_2(z_2)$$

- $J_1(z_1)$:

Perturbed $J_1(z_1+\alpha\eta_1) = \int_a^{c+\alpha\Delta t} \mathcal{L}(z_1+\alpha\eta_1, \dot z_1+\alpha\dot\eta_1, t)\,dt$

$$\delta J_1\big|_{z_1}(\eta_1) = \frac{d}{d\alpha}\Big|_{\alpha=0} J_1(z_1+\alpha\eta_1) =$$

$$= \mathcal{L}(z_1(c),\dot z_1(c),c)\Delta t + \int_a^c \left[\frac{\partial\mathcal{L}}{\partial z}(z_1,\dot z_1,t)\eta_1 + \frac{\partial\mathcal{L}}{\partial\dot z}(z_1,\dot z_1,t)\dot\eta_1\right]dt$$

$$\Big|\ z_1(c) = z(c),\ \dot z_1(c) = \dot z_1(c^-),\ \eta_1(a) = 0$$

integration by parts

$$= \mathcal{L}(z(c),\dot z(c^-),c)\Delta t + \frac{\partial\mathcal{L}}{\partial\dot z}(z(c),\dot z(c^-),c)\,\eta_1(c) +$$

$$+ \cancel{\int_a^c \left[\frac{\partial\mathcal{L}}{\partial z}(z_1,\dot z_1,t) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot z}(z_1,\dot z_1,t)\right]\eta_1(t)\,dt}$$

- $J_2(z_2)$:

Perturbed: $J_2(z_2+\alpha\eta_2) = \int_{c+\alpha\Delta t}^b \mathcal{L}(z_2+\alpha\eta_2, \dot z_2+\alpha\dot\eta_2, t)\,dt$

$$\delta J_2\big|_{z_2} = -\mathcal{L}(z(c),\dot z(c^+),c)\Delta t - \frac{\partial\mathcal{L}}{\partial\dot z}(z(c),\dot z(c^+),c)\,\eta_2(c) +$$

$$+ \cancel{\int_c^b \left[\frac{\partial\mathcal{L}}{\partial z}(z_2,\dot z_2,t) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot z}(z_2,\dot z_2,t)\right]\eta_2(t)\,dt}$$

For $\alpha\to 0$ the perturbed curve is close to the original curve $z$.

The cost index $J$ as function of $\alpha$ must have a minimum at $\alpha = 0$

$$0 = \frac{d}{d\alpha}\Big|_{\alpha=0} J(z, \alpha) = \frac{d}{d\alpha}\Big|_{\alpha=0} [J_1(z_1 + \alpha \eta_1) + J_2(z_2 + \alpha \eta_2)]$$
$$= \delta J_1|_{z_1}(\eta_1) + \delta J_2|_{z_2}(\eta_2)$$

The two portions $z_1, z_2$ of $z$ must be extremals of the correspondings $J_1, J_2$

Therefore, since the E-L condition holds, it holds also for the subintervals $[a, c]$ and $[c, b]$.

$$L(z(c), \dot{z}(c^-), c)\Delta t + \frac{\partial L}{\partial \dot{z}}(z(c), \dot{z}(c^-), c)\eta_1(c) +$$
$$- L(z(c), \dot{z}(c^+), c)\Delta t - \frac{\partial L}{\partial \dot{z}}(z(c), \dot{z}(c^+), c)\eta_2(c) = 0 \qquad (*)$$

with this new condition I'm evaluating the function in $c^-$ and $c^+$ together

The perturbed curve it is known it's continuous at $t = c + \alpha\Delta t$ ($\bar{C}^1$ class), therefore $\eta_1$ and $\eta_2$ are not independent

$$z_1(c + \alpha\Delta t) + \alpha\eta_1(c + \alpha\Delta t) = z_2(c + \alpha\Delta t) + \alpha\eta_2(c + \alpha\Delta t)$$
$$=: z(c) + \alpha\Delta z + o(\alpha) \quad \left(\begin{array}{c}\text{first}\\ \text{approximation}\end{array}\right)$$

Remark: $\dot{z}_1(c) = \dot{z}(c^-)$, $\dot{z}_2(c) = \dot{z}(c^+)$

first order (in $\alpha$) vertical displacement

Obtaining:

$$\Delta z \simeq \frac{1}{\alpha}[z_1(c + \alpha\Delta t) - z(c) + \alpha\eta_1(c + \alpha\Delta t)] \simeq \dot{z}(c^-)\Delta t + \eta_1(c)$$

$$\Delta z \simeq \frac{1}{\alpha}[z_2(c + \alpha\Delta t) - z(c) + \alpha\eta_2(c + \alpha\Delta t)] \simeq \dot{z}(c^+)\Delta t + \eta_2(c)$$

So $\dot{z}(c^-)\Delta t + \eta_1(c) = \dot{z}(c^+)\Delta t + \eta_2(c) = \Delta z$ (second approximation)
We obtain $\eta_1(c) = \Delta z - \dot{z}(c^-)\Delta t$, $\eta_2(c) = \Delta z - \dot{z}(c^+)\Delta t$
to use in $(*)$

The result is :

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c),\dot{z}(c^-),c) - \frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c),\dot{z}(c^+),c)\right]\Delta z \; +$$

$$-\left[\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c),\dot{z}(c^-),c) - \mathcal{L}(z(c),\dot{z}(c^-),c)\right) + \right.$$

$$\left. -\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c),\dot{z}(c^+),c) + \mathcal{L}(z(c),\dot{z}(c^+),c)\right)\right]\Delta t$$

$$\boxed{= -\frac{\partial \mathcal{L}}{\partial \dot{z}}(z,\dot{z},t)\Big|_{c^-}^{c^+}\Delta z + \left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z,\dot{z},t)\,\dot{z}(t) - \mathcal{L}(z,\dot{z},t)\right)\Big|_{c^-}^{c^+}\Delta t = 0}$$

$\Delta z$, $\Delta t$ are arbitrary and independent therefore

$$\frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|_{c^-}^{c^+} = 0 \quad \text{and} \quad \left[\frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z} - \mathcal{L}\right]_{c^-}^{c^+} = 0$$

so these quantities are continuous.

# End

$\frac{\partial \mathcal{L}}{\partial \dot{z}}$ and $\frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z} - \mathcal{L}$ are continuous in $t = c$

$\rightarrow$ Weierstrass - Erdmann corner condition

~ **4 Case studies of Lagrange Problem**

$$J = \int l(z, \dot{z}, t)\, dt \qquad z \in C^1 \text{ or } z \in \bar{C}^1$$

Case 1-2 : $z(a) = z_0$ , $z(b) = z_1$

- Euler - Lagrange equation
- Weierstrass - Erdmann corner condition (if $\exists \bar{t}$ discontinuity point of $\dot{z}^*$)

Case 3-4 : $z(a) = z_0$ , $z(b)$ free

- Euler - Lagrange equation
- Weierstrass - Erdmann condition (if $\exists \bar{t}$ discontinuity point of $z^*$)
- Extra condition

~ Lagrange Problem

$z : \mathbb{R}^{\nu} \to \mathbb{R} \in \bar{C}^1$ with

$$D : \left\{ (z, t_i, t_f) \in \bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} : \begin{array}{l} (z(t_i), t_i) \in \mathcal{D}_i \subseteq \mathbb{R}^{\nu+1} \\ (z(t_f), t_f) \in \mathcal{S}_f \subseteq \mathbb{R}^{\nu+1} \end{array} \right\}$$

$J(z, t_i, t_f) = \int_{t_i}^{t_f} \mathcal{L}(z, \dot{z}, t)\,dt \leftarrow$ cost function with $\mathcal{L} \in C^2$

Find $(z^\circ, t_i^\circ, t_f^\circ)$ that minimizes the cost function over $D$

$$\boxed{J(z^\circ, t_i^\circ, t_f^\circ) \leq J(z, t_i, t_f) \quad \forall (z, t_i, t_f) \in D}$$

If $(z^\circ, t_i^\circ, t_f^\circ)$ is a local minimum then:
Euler - Lagrange equation, W-E condition and transversality conditions are satisfied:

- If $(z^*, t_i^*, t_f^*) \in D$ is a local minimum then

$$\left.\frac{\partial \mathcal{L}}{\partial z}\right|^* - \frac{d}{dt}\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in [t_i, t_f] \qquad \text{Euler equation}$$

- In any discontinuity point $t$ of $\dot{z}^*$: W-E condition

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^*_{t^-} = \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^*_{t^+}, \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)^*_{t^-} = \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)^*_{t^+}$$

- Trasversality conditions

1) $\Delta_i, \Delta_f$ open subsets

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = 0^T, \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = 0^T, \quad \left.\mathcal{L}\right|_{t_i^*}^{*} = 0, \quad \left.\mathcal{L}\right|_{t_f^*}^{*} = 0$$

2) $\Delta_i, \Delta_f$ closed subsets

$(z(t_i), t_i)$ initial point satisfy $\gamma(z(t_i), t_i) = 0$
$(z(t_f), t_f)$ final point satisfy $\chi(z(t_f), t_f) = 0$

these conditions must be regular

$\sigma_i, \sigma_f < n+1$

$$rk\left\{\left.\frac{\partial \gamma}{\partial (z(t_i), t_i)}\right|^{*}\right\} = \sigma_i$$

$$rk\left\{\left.\frac{\partial \chi}{\partial (z(t_f), t_f)}\right|^{*}\right\} = \sigma_f$$

Given two vectors $\overset{xi}{\xi} \in \mathbb{R}^{\sigma_i}$ and $\overset{zeta}{\zeta} \in \mathbb{R}^{\sigma_f}$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = \xi^T \left.\frac{\partial \gamma}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = \zeta^T \left.\frac{\partial \chi}{\partial z(t_f)}\right|^{*}$$

$$\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)_{t_i^*} = \xi^T \left.\frac{\partial \gamma}{\partial t_i}\right|^{*}, \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)_{t_f^*} = \zeta^T \left.\frac{\partial \chi}{\partial t_f}\right|^{*}$$

3) $D_i, D_f$ defined by $\omega(z(t_i), t_i, z(t_f), t_f) = 0$ of 6 components of $C^1$ class

regularity: $\quad rk \left\{ \frac{\partial \omega}{\partial (z(t_i), t_i, z(t_f), t_f} \Big|^* \right\} = 6$

$$\frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|^*_{t_i^*} = \theta^T \frac{\partial \omega}{\partial z(t_i)}\Big|^* , \quad \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|^*_{t_f^*} = -\theta^T \frac{\partial \chi}{\partial z(t_f)}\Big|^* , \quad \theta \in R^6$$

$$\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)_{t_i^*} = \theta^T \frac{\partial \omega}{\partial t_i}\Big|^* , \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)_{t_f^*} = -\theta^T \frac{\partial \omega}{\partial t_f}\Big|^*$$

## ~ Extremum and non-singularity

An extremum is a candidate to be the minimum and is any admissible point satisfying the Euler equation, the W-E conditions and the transversality ones.

An extremum is NON singular if $\frac{\partial^2 \mathcal{L}}{\partial \dot{z}^2}\Big|^*$ is non singular $(\neq 0)$ in $[t_i^*, t_f^*]$

A non singular extremum is a $C^2$ function

## ~ LP with fixed time instants

A standard LP Problem +

- $t_i$ & $t_f$ fixed
- $\mathcal{L}$ convex wrt $z, \dot{z}$ → Global minimum

### – Transversality conditions

1) $D_i, D_f$ open subsets of $\mathbb{R}^\nu$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i}^{\circ} = 0 \qquad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f}^{\circ} = 0$$

2) $D_i, D_f$ are closed subsets

$$\left.\begin{aligned} \gamma(z(t_i)) &= 0 \\ \chi(z(t_f)) &= 0 \end{aligned}\right\}$$

these conditions must be regular

$$\operatorname{rk}\left\{ \left.\frac{\partial \gamma}{\partial (z(t_i))}\right|^{*} \right\} = 6_i < \nu+1$$

$$\operatorname{rk}\left\{ \left.\frac{\partial \chi}{\partial (z(t_f))}\right|^{*} \right\} = 6_f < \nu+1$$

3) $D_i, D_f$ defined by $\omega(z(t_i), z(t_f)) = 0$ of 6 components of $C^1$ class

regularity: $\operatorname{rk}\left\{ \left.\frac{\partial \omega}{\partial (z(t_i), z(t_f))}\right|^{*} \right\} = 6$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = \theta^T \left.\frac{\partial \omega}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = -\theta^T \left.\frac{\partial \chi}{\partial z(t_f)}\right|^{*}, \quad \theta \in \mathbb{R}^6$$

If $\mathcal{L}$ strictly convex, if $\exists$ a solution, it is unique

~ **Lagrange problem with constraints**

- Integral constraint

$z : [a,b] \to \mathbb{R}$, $C^1$ curve, $z(a) = z_0$ $z(b) = z_1$

Find the local minima of the cost index

$$J(z) = \int_a^b \mathcal{L}(t, z, \dot{z})\, dt \qquad \mathcal{L} \ C^2 \text{ class}$$

with $\boxed{C(z) = \int_a^b h(t, z, \dot{z})\, dt = k}$

# Proof

Given the curve $z$, the perturbed family is given by

$C(z + \alpha\eta) = k \quad \forall \alpha \approx 0 \quad \to \quad \delta C|_z(\eta) = 0$

from the computations of the basic variation problem

$$\int_a^b \left[ \frac{\partial h}{\partial z}(t, z, \dot{z}) - \frac{d}{dt} \frac{\partial h}{\partial \dot{z}}(t, z, \dot{z}) \right] \eta(t)\, dt = 0$$

In fact, since in $\delta J|_z(\eta) = 0$

$$\int_a^b \left[ \frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} \right] \eta(t)\, dt = 0 \quad \forall \eta$$

then also $\int_a^b \left[ \frac{\partial h}{\partial z} - \frac{d}{dt} \frac{\partial h}{\partial \dot{z}} \right] \eta(t)\, dt = 0$

How can we put these conditions together?

There exists a constant $p$ (Lagrange multiplier) s.t. :

$$\left[\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}\right] + p\left[\frac{\partial h}{\partial z} - \frac{d}{dt}\frac{\partial h}{\partial \dot{z}}\right] = 0$$

$$\Downarrow$$

$$\frac{\partial(\mathcal{L}+ph)}{\partial z} = \frac{d}{dt}\frac{\partial(\mathcal{L}+ph)}{\partial \dot{z}}$$

$\boxed{\mathcal{L}+ph}$ is the augmented Lagrangian for which the Euler equation holds

\# End

It means that $z$ is an extremal of the augmented cost functional

$$(J+pC)(z) = \int_a^b \left[\mathcal{L}(t,z,\dot{z}) + p\,h(t,z,\dot{z})\right] dt$$

\# Remark :

It's a global constraint $\rightarrow$ it applies to the entire curve

- Equality constraints (non-integral)

$z : [a,b] \to \mathbb{R} \quad z(a) = z_0 \quad z(b) = z_1$

Find the local minima of the cost index

$$J(z) = \int_a^b L(t, z, \dot z)\, dt \qquad L \text{ of } C^2 \text{ class}$$

with the constraint $\boxed{g(t, z, \dot z) = 0}$

The EL equation holds for the augmented lagrange

$\boxed{L + \lambda(t) g}$

↳ here we consider the minimization wrt $z$ and $\lambda$ of:

$$\int_a^b L\, dt + \int_a^b \lambda(t) g\, dt$$

$\lambda$ no longer needs to be constant since $g$ is identically null

# Remark: the constraint is local and there is no difference locally around each curve

~ Augmented Lagrangian

$$l(t, z, \dot z, \lambda_0, \lambda(t), p) = \lambda_0 L(t, z, \dot z) + \lambda^T(t) g(t, z, \dot z) + p^T h(t, z, \dot z)$$

if $\lambda_0 \neq 0 \to$ normal solution

# CALCULUS OF VARIATIONS & OPTIMAL CONTROL

Let's consider a dynamical system

$\dot{x} = f(x,u,t)$

$\hookrightarrow C^2$ class

$u(t) \in \mathbb{R}^p$ control vector

$x(t) \in \mathbb{R}^n$ state vector

$x(t_i) = x_i$ known

Constraints:

$\chi(x(t_f), t_f) = 0$ class $C^1$ of dim $6 \delta \leq n+1$

$q(x,u,t) \leq 0$ class $C^2$ of dim $\beta$

$q_a = 0$ (active constraint) dim $\beta_a$

Norm: $\|(x,u,t_f)\| = \sup_t \|x(t)\| + \sup_t \|\dot{x}(t)\| + \sup_t \left\| \int_{t_i}^{t_f} u(\tau) d\tau \right\| + \sup_t \|u(t)\| + |t_f|$

Cost index: $J(x,u,t_f) = \int_{t_i}^{t_f} \mathcal{L}(x,u,t)dt$

$\hookrightarrow C^2$ class

Goal: find $t_f^\circ$, $u^\circ \in \bar{C}^0(\mathbb{R})$, $x^\circ \in \bar{C}^1(\mathbb{R})$

that satisfy the constraints and minimize $J$

Hamiltonian function (scalar)

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 \mathcal{L}(x,u,t) + \lambda^T(t) \underbrace{f(x,u,t)}_{\text{dynamical part of } x}$$

Theorem:

Let $(x^*, u^*, t_f^*)$ be an admissible solution s.t.

$$\operatorname{rk}\left\{ \frac{\partial \chi}{\partial(x(t_f), t_f)}\Bigg|^* \right\} = \sigma_f \quad \operatorname{rk}\left\{ \frac{\partial q_{active}}{\partial u}\Bigg|^* \right\} = \beta_a(t) \ \forall t \in [t_i, t_f^*]$$

regularity

if $(x^*, u^*, t_f^*)$ is a local minimum

$\exists\ \lambda_0^* \geq 0$, $\lambda^* \in \bar{C}^1[t_i, t_f^*]$, $\eta^* \in \bar{C}^0[t_i, t_f^*]$

not simultaneously null in $[t_i, t_f^*]$ such that:

- $$\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Bigg|^{*T} - \frac{\partial q}{\partial x}\Bigg|^{*T} \eta^*$$ costate equation

- $$0 = \frac{\partial H}{\partial u}\Bigg|^{*T} + \frac{\partial q}{\partial u}\Bigg|^{*T} \eta^*$$ control equation

- $\eta_j^*(t)\, q_j(x^*, u^*, t) = 0$, $\quad \eta_j^*(t) \geq 0$, $j = 1, 2, \dots \beta$

- $$\lambda^*(t_f^*) = -\frac{\partial \chi}{\partial(x(t_f))}\Bigg|_{t_f^*}^{*T} \xi, \quad \xi \in \mathbb{R}^{\sigma_f}$$

- $$H\Big|_{t_f^*}^{*} = \frac{\partial \chi}{\partial t_f}\Bigg|_{t_f^*}^{*T} \xi$$

transversality conditions

The discontinuity of $\dot{\lambda}^*$ and $\eta^*$ may occur only in the points $\bar{t}$ where $u^*$ has a discontinuity and

$$H\Big|_{\bar{t}^-}^{*} = H\Big|_{\bar{t}^+}^{*}$$

# Proof:

Rewrite the optimal control problem as a Lagrange problem and find the solution:

Introduce the new function $v$ and the new variable $z$

$$v(t) = \int_{t_i}^{t_f} u(\tau)\,d\tau \qquad \dot v(t) = u(t) \qquad v(t_i) = 0$$

$\hookrightarrow \hat C^0$

$$z = \begin{pmatrix} x \\ v \end{pmatrix} \in \hat C^1(\mathbb{R}) \qquad \begin{cases} \text{if } v \text{ has a cusp } \int u \\ \quad u \text{ has a jump} \end{cases}$$

admissible set (the same of before with $\dot v$ instead of $u$):

$$D = \Big\{ (z, t_f) \in \hat C^1(\mathbb{R}) \times \mathbb{R},\ z(t_i) = \begin{pmatrix} x_i \\ 0 \end{pmatrix},\ \chi(x(t_f), t_f) = 0,$$
$$v(t_f) \in \mathbb{R}^p,\ f(x(t), \dot v(t), t) - \dot x(t) = 0,\ q(x(t), \dot v(t), t) \le 0 \Big\}$$

$\hookrightarrow \dot x = f(x, v, t) \rightarrow \varepsilon$ constraint

Cost function $J(z, t_f) = \int_{t_i}^{t_f} \mathcal{L}(x(t), v(t), t)\,dt$

Applying the necessary condition of the LP

$$\ell(x, \dot x, \dot v, \lambda_0, \lambda, \eta, t) = \lambda_0 \mathcal{L}(x, \dot v, t) + \lambda^T(t)\big[\overbrace{f(x, \dot v, t) - \dot x(t)}^{\varepsilon}\big]$$
$$+ \eta^T(t)\, q(x, \dot v, t) =$$
$$= H(x, \dot v, \lambda_0, \lambda, t) - \lambda^T \dot x + \eta^T q(x, \dot v, t)$$

Rank condition:

$$rk\left\{ \frac{\partial(\varepsilon, q_a)}{\partial \dot z}\Big|^* \right\} = rk\left\{ \begin{pmatrix} \frac{\partial \varepsilon}{\partial \dot x} & \frac{\partial \varepsilon}{\partial \dot v} \\ \frac{\partial q_a}{\partial \dot x} & \frac{\partial q_a}{\partial \dot v} \end{pmatrix}\Big|^* \right\} = rk\left\{ \begin{pmatrix} -I & \frac{\partial \varepsilon}{\partial u}\Big|^* \\ 0 & \frac{\partial q_a}{\partial u}\Big|^* \end{pmatrix} \right\} = n + \beta_a(t)$$

- Euler - Lagrange

$$\begin{cases} \left.\frac{\partial \ell}{\partial x}\right|^{\circ} - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{x}}\right|^{\circ} = 0^T & (1) \\ \left.\frac{\partial \ell}{\partial v}\right|^{\circ} - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot{v}}\right|^{\circ} = 0^T & (2) \end{cases}$$

$$\boxed{\frac{d}{dx} f(g(\dots(n(x)))) = \frac{df}{dg}\cdot\frac{dg}{dh}\cdot\dots\cdot\frac{dn}{dx}}$$

Recall: $\ell = H - \lambda^T \dot{x} + \eta^T q$

(1) $\frac{\partial H}{\partial x} + \frac{\partial \eta^T q}{\partial x} - \frac{d}{dt}(-\lambda^T) = 0^T$

$$\frac{\partial H}{\partial x} + \eta^T \frac{\partial q}{\partial x} = -\dot{\lambda}^T \Rightarrow \boxed{\dot{\lambda} = -\left.\frac{\partial H}{\partial x}\right|^T - \left.\frac{\partial q}{\partial x}\right|^T \eta}$$

costate equation

$\frac{\partial \ell}{\partial v} = 0$

(2) $0 - \frac{d}{dt}\frac{\partial \ell}{\partial \dot{v}} = 0^T$

since $\frac{d}{dt}(\dots) = 0$ means that $(\dots)$ is constant

then $\frac{\partial \ell}{\partial \dot{v}} = c^T$ $\rightarrow$ We know from the transversality conditions that

$$\left.\frac{\partial \ell}{\partial \dot{z}}\right|_{t_f} = \xi^T \frac{\partial \chi}{\partial(z(t_f), t_f)}$$

$$\left.\frac{\partial \ell}{\partial \dot{v}}\right|_{t_f} = \xi^T \frac{\partial \chi}{\partial(v(t_f, t_f))} = 0$$

$\boxed{c^T = 0}$

$$\boxed{0 = \left.\frac{\partial H}{\partial u}\right|^{*T} + \left.\frac{\partial q}{\partial u}\right|^{*T} \eta^*}$$

Control equation

rewriting the E-L equation:

$$\frac{\partial \ell}{\partial \dot{z}} - \int_{t_i}^{t_f} \frac{\partial \ell}{\partial z}\, d\tau = c^T$$

$$-\lambda^T = \int_{t_i}^{t_f} \frac{\partial \ell}{\partial z}\, d\tau + c^T$$

$\rightarrow \bar{C}^1$ regularity of $\lambda$

3) From the necessary condition of the LP

$$\eta_j^*(t)\, q_j(x^*, v^*, t) = 0 \qquad \text{(directly)}$$

4) The second W-E condition of the LP

$$\underbrace{\left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^*} = \ell - \left(\frac{\partial \ell}{\partial \dot z}\ \ \frac{\partial \ell}{\partial \dot v}\right)\begin{pmatrix}\dot x \\ \dot v\end{pmatrix} = H\Big|^* \quad \text{continuous}$$

$$H = \cancel{\lambda^T \dot x} + \cancel{\eta^T q} - \underbrace{\frac{\partial \ell}{\partial \dot x}\dot x}_{\lambda^T \dot x} - \cancel{\frac{\partial \ell}{\partial \dot v}\dot v}$$

it is continuous

The transversality conditions of the LP imply:

$$\left(\ell - \frac{\partial \ell}{\partial \dot z}\dot z\right)^{*T}_{t_f^*} = H^*_{t_f} = \frac{\partial \chi}{\partial t_f}\Big|^{*T} \xi$$

$$\left(\frac{\partial \ell}{\partial \dot x}\right)^{*T}_{t_f^*} = -\lambda^{*T}(t_f^*) = \xi^T \frac{\partial \chi}{\partial x(t_f)}\Big|^*_{t_f^*} \quad \text{does not depend on } v$$

# End

If $\mathcal{L}, q, f$ do not depend on $t$ (stationary problem)

$$\boxed{H\big|^* = c, \quad \forall t \in [t_i, t_f^*]}$$

## ~ Control Problem

$$\begin{cases} \dot{x} = f(x,u,t) \\ x(t_i) = x_i \end{cases} \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^p,\ f \in C^2$$

Constraints

$$\chi(x(t_f), t_f) = 0 \qquad \chi \in C^1(\mathbb{R}^{6f}) \qquad \mathrm{rk}\left\{ \left( \frac{\partial \chi}{\partial (x(t_f), t_f)} \right) \Big|^{*} \right\} = 6f \le n+1$$

$$\int_{t_i}^{t_f} h(x(t), u(t), t)\, dt = k \qquad h \in C^2(\mathbb{R}^6)$$

Cost functional : $J(x,u,t_f) = \int_{t_i}^{t_f} L(x,u,t)\, dt$

$L \to C^2$ class

Hamiltonian:

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 L(x,u,t) + \lambda^T(t) f(x,u,t) + \rho^T h(x,u,t)$$

If $(x^*, u^*, t_f^*)$ is a local minimum

$\exists\ \lambda_0^* \in \mathbb{R},\ \lambda^* \in \bar{C}^1[t_i, t_f^*],\ \rho^* \in \mathbb{R}$ not simultaneously null in $[t_i, t_f^*]$ s.t :

- $\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T}$ <u>costate equation</u>
- $0 = \frac{\partial H}{\partial u}\Big|^{*T}$ <u>control equation</u>
- $\lambda^*(t_f) = -\frac{\partial \chi}{\partial (x(t_f))}\Big|^{*T}_{t_f^*} \xi \ ,\ \xi \in \mathbb{R}^{6f}$
- $H\Big|^*_{t_f^*} = \frac{\partial \chi}{\partial t_f}\Big|_{t_f^*} \xi$

$\Big\}$ <u>transversality conditions</u>

- <u>discontinuity</u> of $\dot{\lambda}^*$ may occur in $t_k$ where $u^*$ has a discontinuity and $H\Big|^*_{t_k^-} = H\Big|^*_{t_k^+}$

~ Control problem with linear system and convexity

$$\begin{cases} \dot{x} = A(t)x + B(t)u & x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^p,\ A, B \in C^2 \\ x(t_i) = x_i \\ x(t_f) \in D_f \text{ fixed point of } \mathbb{R}^n \end{cases}$$

$q(x,u,t) \leq 0 \quad q \in C^2 \quad \text{rank}\left(\frac{\partial q_a}{\partial u}\Big|^*\right) = \beta_a(t) \ \forall t \in [t_i, t_f]$

$$J(x,u) = \underbrace{G(x(t_f))}_{\text{Boltza term}} + \int_{t_i}^{t_f} \mathcal{L}(x,u)\,dt \qquad \mathcal{L} \in C^2,\ G \in C^3$$

$$H(x,u,\lambda_0,\lambda,t) = \lambda_0 \mathcal{L}(x,u,t) + \lambda^T(t) f(x,u,t)$$

$\rightarrow (\lambda_0 = 1)$

$(x^*, u^*)$ is a <u>normal</u> optimal solution if

▷ $\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T} - \frac{\partial q}{\partial x}\Big|^{*T} \eta^*$

▷ $0 = \frac{\partial H}{\partial u}\Big|^{*T} + \frac{\partial q}{\partial u}\Big|^{*T} \eta^*$

▷ $\eta_j^*(t)\, q_j(x^*, u^*, t) = 0 \qquad j = 1, 2, \dots, \beta$

▷ $\eta^*(t) \geq 0$

and <u>if</u> $D_f = \mathbb{R}^n \quad \lambda^*(t_f) = \frac{dG}{dx(t_f)}\Big|^{*T}$

# THE MAXIMUM PRINCIPLE

## ~ Principle of optimality

Minimizing over $[t, t_f]$ is equivalent to minimize over $[t, t_1]$ and $[t_1, t_f]$

$$J(x(t),t) = \min_{u[t,t_1]} \left[ \int_t^{t_1} \mathcal{L}(x,u,\tau)\,d\tau + \min_{u[t_1,t_f]} \int_{t_1}^{t_f} \mathcal{L}(x,u,\tau)\,d\tau + G(x(t_f)) \right]$$

or

$$J(x(t),t) = \min_{u[t,t_1]} \left[ \int_t^{t_1} \mathcal{L}(x,u,\tau)\,d\tau + J(x(t_1),t_1) \right]$$

Define the function $C(x(t),t) = \min_{\substack{u(\tau)\in U \\ t_i \le \tau \le t}} \int_{t_i}^{t} \mathcal{L}(x(\tau),u(\tau),\tau)\,d\tau$

If $(x^\circ, u^\circ, t_f^\circ)$ is an optimal solution for the control problem

$$\Rightarrow \quad (*) \quad \boxed{C(x^\circ(t),t) = \int_{t_i}^{t} \mathcal{L}(x^\circ(\tau), u^\circ(\tau), \tau)\,d\tau \quad \forall t \in [t_i, t_f^\circ]}$$

that is if a solution is optimal, it is optimal in <u>any</u> subinterval.

---

# Proof

Assume the relation $(*)$ is not true.

There exists a control $u'$ and the state $x'$ in $[t_i, t]$ s.t.:

$$\int_{t_i}^{t} \mathcal{L}(x', u', \tau)\,d\tau < \int_{t_i}^{t} \mathcal{L}(x^\circ, u^\circ, \tau)\,d\tau$$

I can define a new solution of the optimal control:

$$\tilde{u} = \begin{cases} u' & \forall t \in [t_i, t] \\ u^\circ & \forall t \in ]t, t_f^\circ] \end{cases} \qquad \tilde{x} = \begin{cases} x' & \forall t \in [t_i, t] \\ x^\circ & \forall t \in ]t, t_f^\circ] \end{cases} \qquad \tilde{t}_f = t_f^\circ$$

$$J(\tilde{x}, \tilde{u}, \tilde{t}_f) = \int_{t_i}^{t_f} \mathcal{L}(\tilde{x}, \tilde{u}, t)\, dt = \int_{t_i}^{t} \mathcal{L}(x', u', \tau)\, d\tau + \int_{t}^{t_f^\circ} \mathcal{L}(x^\circ, u^\circ, \tau)\, d\tau$$

$$< \int_{t_i}^{t_f^\circ} \mathcal{L}(x^\circ, u^\circ, \tau)\, d\tau = \underbrace{J(x^\circ, u^\circ, t_f^\circ)}_{\text{contradiction!}}$$

# End

$$C(x^\circ(t), t) = \int_{t_i}^{t} \mathcal{L}(x^\circ(\tau), u^\circ(\tau), \tau)\, d\tau \qquad \forall t \in [t_i, t_f^\circ]$$

$$\left.\frac{dC}{dt}\right|^\circ = \mathcal{L}(x^\circ, u^\circ, t)$$

$$\boxed{\left.\frac{\partial C}{\partial x}\right|^\circ \cdot \frac{dx}{dt} + \left.\frac{\partial C}{\partial t}\right|^\circ = \mathcal{L}(x^\circ, u^\circ, t)}$$

$$\rightarrow \dot{x} = f(x^\circ, u^\circ, t)$$

Hamilton-Jacobi equation

The variation of the C function wrt the state multiplied to the variation of the state plus the variation of C wrt time is equal to the Lagrangian evaluated in the optimal solution

## ~ Hamilton - Jacobi equation

It is an approach in some sense alternative to the minimum principle, combined with the Euler - Lagrange equation.

It is useful mainly for linear regulator problems.

The H-J equation is satisfied by the optimal performance index under suitable differentiability and continuity assumptions.

If a solution to the H-J equation has certain properties, this solution is the desired performance index.

The H-J represents only a <u>sufficient</u> condition on the optimal performance index

$$\left.\frac{\partial C}{\partial x}\right|^{\circ} f(x^{\circ}, u^{\circ}, t) + \left.\frac{\partial C}{\partial t}\right|^{\circ} + \mathcal{L}(x^{\circ}, u^{\circ}, t) = 0$$

$$\Downarrow$$

$$\boxed{\left.\frac{\partial C}{\partial t}\right|^{\circ} + H\left(x^{\circ}, u^{\circ}, \left.\frac{\partial C}{\partial x}\right|^{\circ}, t\right) = 0}$$

~ Pontryagin Principle

Consider the dynamical system $\dot{x} = f(x,u,t)$

$f$ in the calculus of variations was $C^2$ class.

In this case $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$

$x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^p$.

$x(t_i) = x^i$

For the final values assume $\chi(x(t_f), t_f) = 0$

$\chi \in C^1(\mathbb{R}^{6f \leq n+1})$

Constraints:

$$\int_{t_i}^{t_f} h(x,u,\tau)\,d\tau = k \quad \text{with } h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$$

Cost index:

$$J(x,u,t_f) = \int_{t_i}^{t_f} \mathcal{L}(x,u,\tau)\,d\tau \quad \text{with } \mathcal{L}, \frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$$

<u>Goal</u>:

- $t_f \in [t_i, \infty)$
- $u^0 \in \bar{C}^0(\mathbb{R})$
- $x^0 \in \bar{C}^1(\mathbb{R})$

satisfying:

- dynamical system
- control's constraint
- initial and final conditions
- minimize the cost

- Hamiltonian

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t)\, f(x,u) + \rho^T h(x,u,t)$$

$$J = \int \mathcal{L} \qquad \dot{x} = f \qquad \int h = \kappa$$

Theorem:

Consider an admissible solution $(x^*, u^*, t_f^*)$ s.t.

$$\operatorname{rank}\left\{ \frac{\partial \chi}{\partial(x(t_f), t_f)} \Big|^* \right\} = 6f$$

If it is a local minimum

$\exists\ \lambda_0 \geq 0,\ \rho^* \in \mathbb{R}^6,\ \lambda^* \in \bar{C}^1[t_i, t_f^*]$ not simultaneously null such that:

- $$\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T} \qquad \text{Co-state equation}$$

- $$\boxed{H(x^*, w, \lambda_0^*, \lambda^*) \geq H(x^*, u^*, \lambda_0^*, \lambda^*), \quad \forall w \in U}$$
  Pontryagin inequality — $\hookrightarrow$ any admissible control

And $\exists$ a vector $\xi \in \mathbb{R}^{6f}$ such that:

- $$\dot{\lambda}(T) = \frac{\partial \chi}{\partial x(t_f)}\Big|^{*T} \xi$$

- $$H\Big|^*_{t_f} = -\frac{\partial \chi}{\partial t_f}\Big|^{*T} \xi$$

- discontinuity of $\dot{\lambda}^*$ may occur in $t_k$ where $u^*$ has a discontinuity and
  $$H\Big|^*_{t_k^-} = H\Big|^*_{t_k^+}$$

- If $U = \mathbb{R}^p$ the minimum condition reduces to $\boxed{\frac{\partial H}{\partial u} = 0}$ $\forall t \in [t_i, t_f^0]$ (only in this case the condition holds in place of the inequality of Pontryagin)

# Proof

We want to prove the Pontryagin inequality.

Let's assume $(x^0, u^0, t_f^0)$ optimal normal solution and $\varepsilon > 0$.

optimal $\rightarrow$ global minimum

normal $\rightarrow$ $\lambda_0 = 1$

C over two subintervals:

$$C(x(t),t) = \min_{\substack{u(\tau)\in U \\ t_i \le \tau \le t-\varepsilon}} \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[ \int_{t_i}^{t-\varepsilon} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau + \int_{t-\varepsilon}^{t} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau \right]$$

$$= \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[ \min_{\substack{u(\tau)\in U \\ t_i \le \tau \le t-\varepsilon}} \int_{t_i}^{t-\varepsilon} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau + \int_{t-\varepsilon}^{t} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau \right]$$

$$= \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[ C(x(t-\varepsilon), t-\varepsilon) + \int_{t-\varepsilon}^{t} \mathcal{L}(x,u,\tau)\, d\tau \right]$$

$$\boxed{x(t-\varepsilon) \cong x(t) - \varepsilon \dot{x} = x(t) - \varepsilon f(x,u,t)}$$

Taylor expansion first order

$$= \min_{u(\tau)\in U} \left[ C(x(t) - \varepsilon f(x,u,t), t-\varepsilon) + \int_{t-\varepsilon}^{t} \mathcal{L}(x,u,\tau)\, d\tau \right]$$

Taylor expansion with $(x(t),t)$ as initial point:

$$\left\{ g(x) = g(x_0) + \frac{dg}{dx}(x - x_0) \right\}$$

$\hookrightarrow$ increment $\rightarrow \varepsilon$ in our case

$$= \min_{u(\tau)\in U} \left[ C(x(t),t) + \frac{\partial C}{\partial x(t)}\left[-\varepsilon f(x,u,t)\right] + \frac{\partial C}{\partial t}(-\varepsilon) + \underbrace{\varepsilon \mathcal{L}(x,u,\tau)}_{\text{approximation because } \varepsilon \text{ is very little}} \right]$$

Since many quantities don't depend on $u(\tau)$:

$$\cancel{C(x(t),t)} = \cancel{C(x(t),t)} - \cancel{\varepsilon}\frac{\partial C}{\partial t} + \min_{u(\tau)\in U} \left[ -\frac{\partial C}{\partial x(t)} \cancel{\varepsilon} f(x,u,t) + \cancel{\varepsilon} \mathcal{L} \right]$$

$$\frac{\partial C}{\partial t} = \min_{u(\tau)\in U}\left[-\frac{\partial C}{\partial x(t)} f(x,u,t) + L(x,u,\tau)\right] \quad \forall (x(t),t) \in \mathbb{R}^{n+1}$$

Let us choose any $t \in [t_i, t_f^\circ]$ and $x(t) = x^\circ(t)$

$$\left.\frac{\partial C}{\partial t}\right|^\circ = \min_{u(\tau)\in U}\left[-\left.\frac{\partial C}{\partial x(t)}\right|^\circ f(x^\circ(t), u, t) + L(x^\circ, u, \tau)\right]$$

Given the Hamilton-Jacobi equation:

$$\left.\frac{\partial C}{\partial x}\right|^\circ f(x^\circ, u^\circ, t) + \left.\frac{\partial C}{\partial t}\right|^\circ = L(x^\circ, u^\circ, t)$$

$$\left.\frac{\partial C}{\partial t}\right|^\circ = \left(-\frac{\partial C}{\partial x(t)} f(x^\circ, u^\circ, t) + L(x^\circ, u^\circ, t)\right) =$$

$$= \min_{u(\tau)\in U}\left[-\left.\frac{\partial C}{\partial x(t)}\right|^\circ f(x^\circ(t), u, t) + L(x^\circ, u, \tau)\right]$$

$\to$ similar to the Hamiltonian with $\lambda_0 = 1$

$$H = L + \lambda^T f$$

Let us define $-\lambda^{\circ T}(t) = \left.\frac{\partial C}{\partial x(t)}\right|^\circ$

$$L(x^\circ, u^\circ, t) + \lambda^{\circ T}(t) f(x^\circ, u^\circ, t) = \min_{u(t)\in U}\left[L(x^\circ, u^\circ, t) + \lambda^{\circ T}(t) f(x^\circ(t), u, t)\right]$$

$$H(x^\circ, u^\circ, \lambda^\circ, t) = \min_{u(t)\in U} H(x^\circ(t), u(t), \lambda^\circ(t), t)$$

My Hamiltonian evaluated in the optimal solution is less or equal to the Hamiltonian evaluated in any other point, in particular in the optimal state but with another admissible control

# End

~ Convex case

$$\begin{cases} \dot{x} = A(t) + B(t)u \\ x(t_i) = x_i \end{cases}$$ with $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^p$

$x(t_f) = x_f$, $A, B \in C^1$ $\forall t \in [t_i, t_f]$

$U$ is a <u>convex set</u>

- Performance index

Bolza term

$$J(x,u) = \int_{t_i}^{t_f} \mathcal{L}(x,u,\tau) + G(x(t_f))$$

$\mathcal{L}$ ↳ convex, $G$ ↳ convex $C^2$ class

$$\mathcal{L}, \frac{\partial \mathcal{L}}{\partial x_i}, \frac{\partial \mathcal{L}}{\partial t} \in C^0(\mathbb{R}^n \times U \times [t_i, t_f])$$

We want to find if exist $(u^\circ \in \bar{C}^0[t_i,t_f], x^\circ \in \bar{C}^1[t_i,t_f])$ that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t) f(x,u)$$

the necessary and sufficient condition for $(x^\circ, u^\circ, t_f^\circ)$ to be a local minimum:

$\exists\ \lambda^\circ \in \bar{C}^1[t_i, t_f]$ n-dimensional vector such that:

1) $\dot{\lambda}^\circ = -\left.\dfrac{\partial H}{\partial x}\right|^{\circ T}$

2) $H(x^\circ, \omega, \lambda_0^\circ, \lambda^\circ, t) \geq H(x^\circ, u^\circ, \lambda_0^\circ, \lambda^\circ, t)$ $\forall \omega \in U$

3) if $x(t_f) \in \mathbb{R}^n \rightarrow \lambda^\circ(t_f) = \left.\dfrac{\partial G}{\partial x(t_f)}\right|^{\circ T}$

if $\mathcal{L}$ is strictly convex wrt $x$ and $u$ and $G$ is strictly convex wrt $x(t_f)$ the solution is <u>unique</u>

~ Stationary problem

$$\begin{cases} \dot{x} = f(x,u) \\ x(t_i) = x_i \end{cases}$$ with $x(t) \in \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^p$, $f, \frac{\partial f}{\partial x} \in C^0(\mathbb{R}^n \times U)$

$x(t_f) = x_f$

$$J(x,u,t_f) = \int_{t_i}^{t_f} \mathcal{L}(x,u)\,dt \qquad \mathcal{L}, \frac{\partial \mathcal{L}}{\partial x} \in C^0(\mathbb{R}^n \times U)$$

We want to find if exist $(t_f^\circ \in (t_i, \infty), u^\circ \in \bar{C}^0[t_i, t_f^\circ],$
$x^\circ \in \bar{C}^1[t_i, t_f^\circ])$ that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t) f(x,u)$$

The necessary condition for $(x^\circ, u^\circ, t_f^\circ)$ to be a local minimum:

$\exists\ \lambda_0^\circ \in \mathbb{R}$, $\lambda^\circ \in \bar{C}^1[t_i, t_f^\circ]$ not simultaneously null

1) $\dot{\lambda}^\circ = -\frac{\partial H}{\partial x}\Big|^{\circ T}$

2) $H(x^\circ, w, \lambda_0^\circ, \lambda^\circ) \geq H(x^\circ, u^\circ, \lambda_0^\circ, \lambda^\circ) \quad \forall w \in U$

3) $H\big|^\circ = \begin{cases} 0 & (t_f \text{ not fixed}) \\ k & (t_f \text{ fixed}) \end{cases}$

~ Unstationary Problem

$$\begin{cases} \dot{x} = f(x,u,t) & x(t) \in \mathbb{R}^n,\ u(t) \in U \subseteq \mathbb{R}^p \\ x(t_i) = x_i \end{cases}$$

$x(t_f) = x_f$

$f, \frac{\partial f}{\partial x}, \frac{\partial L}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$

$J(x,u,t_f) = \int_{t_i}^{t_f} L(x,u,t)\,dt \qquad L, \frac{\partial L}{\partial x}, \frac{\partial L}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$

We want to find if exist $(t_f^* \in (t_i, \infty),\ u^\circ \in \bar{C}^0[t_i, t_f^\circ]$
$x^\circ \in \bar{C}^1[t_i, t_f^\circ]$ that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 L(x,u) + \lambda^T(t) f(x,u)$$

The condition for $(x^\circ, u^\circ, t_f^\circ)$ to be a local minimum
$\exists\ \lambda_0^\circ \in \mathbb{R},\ \lambda^\circ \in \bar{C}^1[t_i, t_f^\circ]$ not simultaneously null

1) $\dot{\lambda}^\circ = -\frac{\partial H}{\partial x}\Big|^{\circ T}$

2) $H(x^\circ, \omega, \lambda_0^\circ, \lambda^\circ, t) \geq H(x^\circ, u^\circ, \lambda_0^\circ, \lambda^\circ) \quad \forall \omega \in U$

3) $$H\Big|^\circ + \int_{t_i}^{t_f^\circ} \frac{\partial H}{\partial \tau}\,d\tau = \begin{cases} 0 & (t_f \text{ not fixed}) \\ k & (t_f \text{ fixed}) \end{cases}$$

If $J = \int L\,d\tau + G(x(t_f), t_f)$ (the Bolza term is present):
The necessary conditions are:

$\lambda^\circ(t_f) = \frac{\partial \chi}{\partial x(t_f)}\Big|^{*T} \xi + \lambda_0^* \frac{\partial G}{\partial x(t_f)}\Big|^{*T}$

$H\Big|^\circ + \int_{t_i}^{t_f} \frac{\partial H}{\partial \tau}\,d\tau + \lambda_0^\circ \frac{\partial G}{\partial t}\Big|^\circ + \int_{t_i}^{t_f^\circ} \lambda_0^\circ \frac{\partial^2 G}{\partial \tau^2}\,d\tau$

## THE LINEAR QUADRATIC REGULATOR

### ~ Finite time interval

In calculus of variations and optimal control we had

$$\begin{cases} \dot{x} = f(x,u,t) \\ x(t_i) = x_i \text{ fixed} \end{cases}$$

$$J(x,u) = \int_{t_i}^{t_f} L(x,u,t)\,dt + G(x(t_f))$$

with the regulator problem we have now

$$\left.\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) \end{aligned}\right\} \text{ linear system}$$

with $[t_i, t_f]$ fixed and $x(t_i) = x_i$ fixed.

$\underset{n\times n}{A(t)}, \underset{n\times m}{B(t)}, \underset{n\times p}{C(t)}$ matrices function of time

The linear control law is obtained minimizing the <u>Quadratic performance index</u>:

$$J(x,u) = \frac{1}{2}\int_{t_i}^{t_f} \left(x^T Q x + u^T R u\right) dt + \frac{1}{2} x^T(t_f)\, F\, x(t_f)$$

$\underset{\text{non neg}}{Q(t) \geq 0}$, $\underset{\text{pos def}}{R(t) > 0}$, $\underset{\text{non neg}}{F(t) \geq 0}$ all symmetric

To the regulator problem is associated the <u>Riccati equation</u>

$$\begin{cases} \dot{K}(t) = K B R^{-1} B^T K - KA - A^T K - Q \\ K(t_f) = F \end{cases}$$

The Riccati equation admits always a <u>unique solution</u> positive semidefinite in the control interval

<u>Theorem</u> : The optimal control for the regulator problem is given by the linear feedback law

$$u^\circ(t) = -R^{-1}B^T K x^\circ$$

where

$$\dot{x}^\circ(t) = A x^\circ + B u^\circ$$
$$= A x^\circ - B R^{-1} B^T K x^\circ$$
$$= \left[A - B R^{-1} B^T K\right] x^\circ$$

$$x^\circ(t_i) = x_i$$

and

$$J(x^\circ, u^\circ) = \frac{1}{2} x_i^T K(t_i) x_i$$

The solution of the Riccati equation is symmetric and can be found only numerically

# Remark : matrix A is limited

1) The <u>existence and uniqueness</u> theorem may be applied <u>only locally</u>. Therefore $\exists$ $(t, t_f]$ in which the Riccati equation admits a unique solution

2) It must be shown that $\nexists$ an element $k_{ij}$ of the matrix $K$ such that for any $t_1 < t_f$ :

$$\lim_{t \to t_1^+} |k_{ij}(t)| = \infty$$

# Proof

- Condition 1

We apply the results of the Pontryagin principle to our linear sys.

Hamiltonian: $H(x,u,\lambda,t) = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu + \lambda^TAx + \lambda^TBu$

normal case $\rightarrow$ $\lambda_0$ cannot be 0 because we don't have any condition on the final instant

e.g. If you don't have the Bolza term $\lambda(t_f)=0$ so $\lambda_0 \neq 0$ itself, if there is the Bolza term we use the trasversality conditions

The necessary & sufficient conditions are

$$
\begin{cases}
1)\ \dot{\lambda}^\circ = -\left.\dfrac{\partial H}{\partial x}\right|^{\circ T} = -Qx^\circ - A^T\lambda \\
2)\ \left.\dfrac{\partial H}{\partial u}\right|^\circ = Ru^\circ + B^T\lambda^\circ = 0 \\
3)\ \lambda^\circ(t_f) = \left.\dfrac{\partial G}{\partial x(t_f)}\right|^{\circ T}
\end{cases}
$$

Taking into account the $\overset{\text{Riccati eq.}}{RE}$ and defining the costate proportional to the state as:

$\boxed{\lambda^\circ = K x^\circ}$ <u>the result is proved</u> !!

In fact, checking:

$$1)\ \dot{\lambda}^\circ = \dot{K}x^\circ + K\dot{x}^\circ = [KBR^{-1}B^TK - KA - A^TK - Q]x^\circ + K[Ax + B\overset{-R^{-1}B^T\lambda}{u}]$$

$$= KBR^{-1}B^TKx^\circ - KAx^\circ - A^TKx^\circ - Qx^\circ + KAx^\circ - KBR^{-1}B^TKx^\circ$$

$$= -A^TKx^\circ - Qx^\circ = -A^T\lambda^\circ - Qx^\circ$$

✓

2) $R u^\circ + B^T \lambda^\circ = R(-R^{-1}B^T K x^\circ) + B^T\lambda^\circ =$

$= -B^T K x^\circ + B^T \lambda^\circ = 0$

3) $\lambda^\circ(t_f) = K(t_f^\circ)\, x^\circ(t_f^\circ) = F x^\circ(t_f^\circ)$

4) Value of the cost index in the minimum point:

$$J(x^\circ,u^\circ) = \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) + \frac{1}{2}\int_{t_i}^{t_f} x^{\circ T}(Q + KBR^{-1}B^TK)x^\circ dt$$

I substitute Q from the Riccati eq.

$$\frac{1}{2}\int_{t_i}^{t_f} x^TQx + u^TRu$$

$$u^\circ = -R^{-1}B^TKx^\circ$$

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) + \frac{1}{2}\int_{t_i}^{t_f} x^{\circ T}(\underbrace{-\dot K + 2KBR^{-1}B^TK - KA - A^TK}_{x^{\circ T}K\dot x^\circ \qquad x^{\circ T}\dot K x^\circ})x^\circ dt$$

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) - \frac{1}{2}\int_{t_i}^{t_f}(2x^{\circ T}K\dot x^\circ + x^{\circ T}\dot K x^\circ)dt$$

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) - \frac{1}{2}\int_{t_i}^{t_f}\frac{d}{dt}(x^{\circ T}Kx^\circ)\,dt = \frac{1}{2}x^{iT}K(t_i)x^i$$

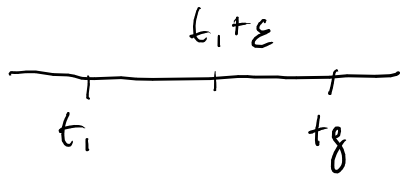
- Condition 2

We want to extend the solution of RE to the whole time interval $[t_i, t_f]$

By contradiction:

assume that $\lim_{t \to t_1^+} |K_{ij}(t)| = \infty$

Consider an $\varepsilon > 0$ such that $t_1 + \varepsilon < t_f$




In $[t_1+\varepsilon, t_f]$ RE admits a solution

So $\exists$ a solution of the regulator problem $\forall x(t_1+\varepsilon)$.

In particular

$J_{t_1+\varepsilon}(x^\circ, u^\circ) = \frac{1}{2} x^T(t_1+\varepsilon)\, K(t_1+\varepsilon)\, x(t_1+\varepsilon)$ and for the hypotheses on the matrices $F$, $Q(t)$, $R(t)$

- $J_{t_1+\varepsilon}(x^\circ, u^\circ) \geq 0$ , $\forall x(t_1+\varepsilon) \in \mathbb{R}^n$
- $K(t_1+\varepsilon) \geq 0$ , $\forall \varepsilon \in (0, t_f - t_1)$
- $K(t) \geq 0$ , $\forall t \in (t_1, t_f]$

Trick: since $K$ is not negative,

now consider the i-th column of the identity matrix $e_i$:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & | & | \\ | & & 1 & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

$e_i$

Being $K(t)$ symmetric one has:

$(e_i \pm e_j)^T K (e_i \pm e_j) = k_{ii} + k_{jj} \pm 2 k_{ij} \geq 0 \quad \forall t \in (t_1, t_f]$

example:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$e_i \quad e_j$

$k_{12} = k_{21}$ simmetry

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} k_{11} + k_{12} \\ k_{21} + k_{22} \end{pmatrix} = k_{11} + k_{12} + k_{21} + k_{22} =$$

$$= k_{11} + k_{22} + 2 k_{12} \geq 0$$

$$\kappa_{ii} + \kappa_{jj} \geq 2\,\underbrace{|\kappa_{ij}|}\quad \forall t \in (t_1, t_f]$$

$$\lim_{t \to t_1^+} |\kappa_{ij}| = \infty$$

One of the two elements $\kappa_{ii}$ or $\kappa_{ij}$ goes to $\infty$

Let's choose $\kappa_{jj} \Rightarrow \lim_{t \to t_1^+} k_{jj} = \infty$

Let's consider then the optimal control problem in the interval $[t_1 + \varepsilon, t_f]$ with initial condition

$x(t_1 + \varepsilon) = e_j$

$\exists$ an optimal solution and one has:

(*) $$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^\circ, u^\circ) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T \kappa(t_1+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} \kappa_{jj}(t_1+\varepsilon) = \infty$$

Now consider the <u>null control</u> over $[t_1 + \varepsilon, t_f]$ and the corresponding <u>free evolution</u> $\hat{x}$ on the state

Indicate with $\Phi(t, \tau)$ the transition matrix of the stat corresponding to $A$

$$\underset{\substack{\uparrow\\ \text{starting instant}}}{J_{t_1+\varepsilon}}(\underset{\substack{\uparrow\\ \text{free evolution}}}{\hat{x}}, \underset{\substack{\uparrow\\ \text{zero control}}}{0}) = \underbrace{\frac{1}{2}\hat{x}^T(t_f) F \hat{x}(t_f)}_{\text{Bolze term}} + \frac{1}{2}\int_{t_i}^{t_f} \hat{x}^T(t) Q(t) \hat{x}(t)\, dt$$

$$\hat{x}(t_f) = \Phi(t_f, t_1+\varepsilon) e_j$$

$$= \frac{1}{2} e_j^T \Phi^T(t_f, t_1+\varepsilon) F \Phi(t_f, t_1+\varepsilon) e_j$$

$$= \frac{1}{2}\int_{t_1+\varepsilon}^{t_f} e_j^T \Phi^T(t, t_1+\varepsilon) Q(t) \Phi(t, t_1+\varepsilon) e_j\, dt$$

Since $\Phi(t,\tau)$ is bounded (in norm), $\exists\ M>0$ matrix such that:

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(\hat{x},0) < M$$

so $J_{t_1+\varepsilon}(x^o,u^o) \leq J_{t_1+\varepsilon}(\hat{x},0) \quad \forall \varepsilon \in (0, t_f - t']$

by doing the limit of $\varepsilon \to 0^+$ also for $J_{t_1+\varepsilon}(x^o,u^o)$ we obtain

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^o,u^o) \leq \lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(\hat{x},0) < M$$

which is in contrast with $(*)$:

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^o,u^o) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T K(t_1+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} k_{jj}(t_1+\varepsilon) = \infty$$

So the condition $\lim_{t \to t_1^+} |k_{ij}(t)| = \infty$ is not possible

$$\boxed{\text{RE admits a unique solution } \geq 0 \text{ in } [t_i, t_f]}$$

semidefinite positive

\# End

Theorem: The regulator problem admits a unique solution

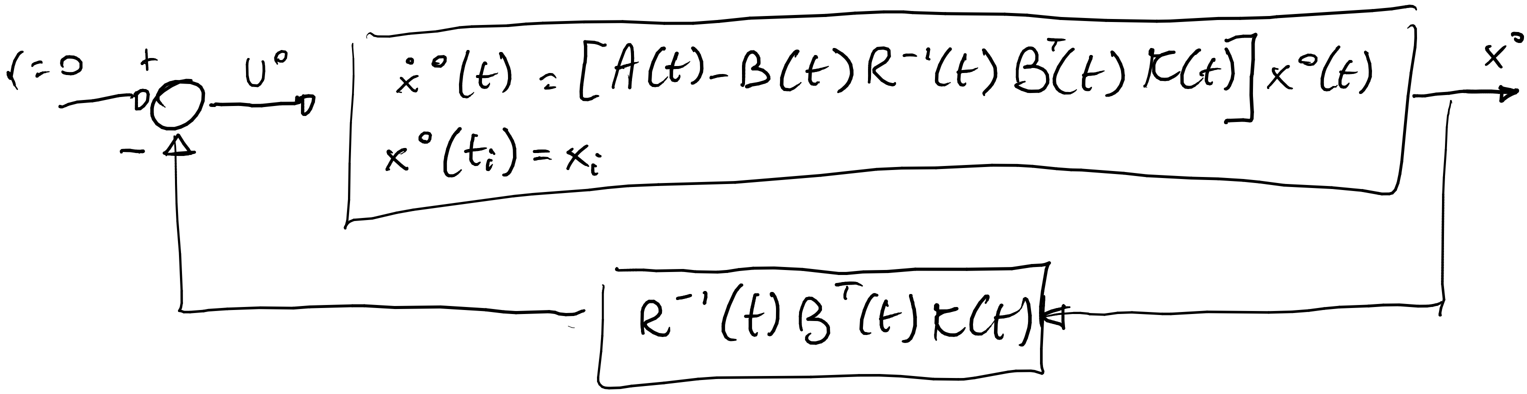
\# Remarks:

1) $Q, R, F$ can be not simmetric $\xrightarrow{\text{trick}}$ $\frac{Q+Q^T}{2}$ $\to$ ($Q$ become symmetric)

2) The solution of RE does not depend on the initial state, therefore the solution can be found offline

3) The Riccati matrix $K$ is a function of time, even if $A, B, Q, R$ are constant

# ~ Regulator problem scheme




$$u^\circ = -R^{-1}(t)B^T(t)K(t)x^\circ(t)$$

~ Infinite time interval $[t_i, \infty)$

- Assume the matrices $A, B$ bounded with elems of $C^1$ class
- Assume the dynamical system is completely controllable or exponentially stable
- Assume $Q, R$ symmetric, $Q \geq 0$, $R > 0$, bounded with elems in $C^1$ class.

Find the control $u^\circ \in \bar{C}^\circ [t_i, \infty)$ and the state $x^\circ \in \bar{C}^1 [t_i, \infty)$ satisfying the system, the initial condition and minimizing:

$$J(x,u) = \frac{1}{2} \int_{t_i}^{\infty} [x^T Q x + u^T R u] \, dt$$

Theorem: The unique optimal solution is

$$u^\circ (t) = -R^{-1} B^T \bar{K} x^\circ$$

$$\dot{x}^\circ (t) = [A - B R^{-1} B^T \bar{K}] x^\circ$$

$$x^\circ (t_i) = x_i$$

where $\bar{K}$ is the solution of the RE:

$$\dot{\bar{K}}(t) = \bar{K} B R^{-1} B^T \bar{K} - \bar{K} A - A^T \bar{K} - Q$$

with final condition

$$\lim_{t_f \to \infty} \bar{K}(t_f) = 0$$

and $J(x^\circ, u^\circ) = \frac{1}{2} x_i^T \bar{K}(t_i) x_i$

# Proof

Let us consider an arbitrary finite instant $t_f$, such that the time interval is $[t_i, t_f]$, $t_f < \infty$

In this time interval we can write

$\dot{K} = KBR^{-1}B^TK - KA - A^TK - Q$

$K(t_f) = 0$

depends on $t_f$ because it depends $K_{t_f}(t_i)$ and $K$ depends on the interval $[t_i, t_f]$

Fixing $t_f$, $\quad J_{t_f} = \frac{1}{2} x_i^T K_{t_f}(t_i)\, x_i$

Now I want to show that $J_{t_f}(x^\circ, u^\circ)$ is <u>superiorly limited</u>.

1) If the system is <u>completely controllable</u>, whatever the initial state is, $\exists$ a <u>control $u'$</u> able to transfer it to the origin in a <u>finite time $t_f'$</u> with an <u>evolution $x'$</u>

So:

$x_i$ initial state

$u'$ control to send the sys from $x_i$ to $0$

$x'$ evolution of the sys

$t_f'$ final instant once reached the origin

Chosen $t_f' < t_f$, I suppose $x' = u' = 0 \quad \forall t \in [t_f', t_f]$

$$\underbrace{\frac{1}{2}\int_{t_i}^{t_f'} \left(x'^TQx' + u'^TRu'\right)dt}_{\geq J_{t_f}(x^\circ,\, u^\circ)} = \frac{1}{2}\int_{t_i}^{t_f'}\left(x'^TQx' + u'^TRu'\right)dt + \frac{1}{2}\underbrace{\int_{t_f'}^{t_f}\left(x'^TQx' + u'^TRu'\right)dt}_{\to 0}$$

$J_{t_f}(x', u')$

$\geq J_{t_f}(x^\circ, u^\circ)$ $\rightsquigarrow$ therefore it is <u>bounded</u>

② If the system is exponentially stable, from any initial state with zero input, the corresponding free evolution of the state $\bar{x}$ satisfies this inequality:

$$\|\bar{x}(t)\| \le \alpha e^{-\beta t}, \quad \beta > 0$$

free evolution with 0 input

$$J_{t_f}(x^\circ, u^\circ) \le \underbrace{J_{t_f}(\bar{x}, 0)}_{\frac{1}{2}\int_{t_i}^{t_f} \bar{x}^T Q \bar{x}\, dt} \le \frac{1}{2}\int_{t_i}^{\infty} \bar{x}^T Q \bar{x}\, dt$$

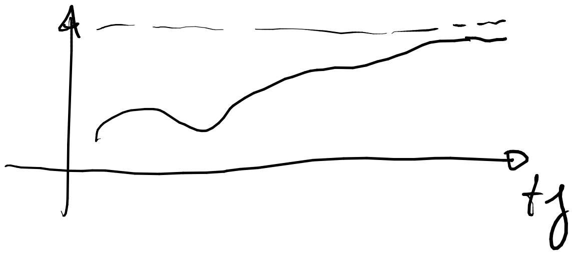
because

$$\int_{t_i}^{\infty} \dots dt = \int_{t_i}^{t_f} \dots dt + \int_{t_f}^{\infty} \underline{\text{quantity}}\, dt$$

$$\le \frac{1}{2}\int_{t_i}^{\infty} \|Q(t)\| \cdot \|\bar{x}(t)\|^2 dt \quad \text{property of inequalities}$$

$$\le c\int_{t_i}^{\infty} e^{-2\beta t} dt = \frac{c}{2\beta} e^{-2\beta t_i} \quad J_{t_f}(x^\circ, u^\circ) \text{ bounded}$$

$J_{t_f}(x^\circ, u^\circ)$ is monotonically non decreasing and superiorly limited in $t_f$




Consider now $(x^{\circ\prime}, u^{\circ\prime})$ for $t_f'$ and $(x^{\circ\prime\prime}, u^{\circ\prime\prime})$ for $t_f''$

assume that $t_f' \le t_f''$

$$J_{t_f''}(x^{\circ\prime\prime}, u^{\circ\prime\prime}) = \frac{1}{2}\int_{t_i}^{t_f''}[x''^T Q x'' + u''^T R u'']dt \ge \frac{1}{2}\int_{t_i}^{t_f'}[x''^T Q x'' + u''^T R u'']dt$$

in the interval $[t_i, t_f']$ the solution is $(x^{o\prime}, u^{o\prime})$ and not $(x^{o\prime\prime}, u^{o\prime\prime})$, so:

$$J_{t_f''}(x^{o''}, u^{o''}) \geq J_{t_f'}(x^{o'}, u^{o'}) = \int_{t_i}^{t_f'} [x'^T Q x' + u'^T R u'] dt$$

Therefore the function is <u>not decreasing</u> and $J_{t_f}(x^o, u^o) = \frac{1}{2} x_i^T K(t_i) x_i$ has a limit as $t_f \to \infty$

* Being $x_i$ arbitrary, it follows that any matrix $K(t_i)$ has a limit as $t_f \to \infty$
* As $t_i$ is arbitrary, the solution $K$ of the RE with final condition $K(t_f) = 0$ admits a limited solution $K'(t)$ $\forall t$ as $t_f \to \infty$
* <u>Existence</u>, <u>uniqueness</u> and the characterization <u>of the solution</u> can be justified as in the previous theorem considering <u>the limit for the solution</u> herein identified and recalling that

$$\lim_{t_f \to \infty} K(t_f) = K'(t)$$

# End

# Remark

If the matrix $Q(t)$ has eigenvalues $\geq \alpha$, with $\alpha > 0$ $\forall t \in [t_i, \infty)$ the optimal regulator is asymptotically stable; in fact for the optimal solution it results in:

$$\frac{\alpha}{2} \int_{t_i}^{\infty} \| x^o(t) \|^2 dt \leq \frac{1}{2} \int_{t_i}^{\infty} [x^{oT}(t) Q(t) x^o(t) + u^{oT}(t) R(t) u^o(t)] dt$$

$$= \frac{1}{2} x_i^T \bar{K}(t_i) x_i$$

Therefore

$$\lim_{t \to \infty} x^o(t) = 0 \quad \forall x_i \in \mathbb{R}^n$$

~ The steady state solution of the deterministic linear optimal regulator problem

# Remark: in this case in place of a differential equation I have an algebraic equation

<u>Theorem</u>: from the previous problem

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$J(x,u) = \frac{1}{2}\int^{t_f}[x^T(t)\,Q(t)\,x(t) + u^T(t)\,R(t)\,u(t)]\,dt$$

and
$$\dot{K}(t) = K^{t_i}(t)\,B(t)\,R^{-1}(t)\,B(t)\,K(t) - K(t)\,A(t) - A^T(t)\,K(t) - Q(t)$$

all the elements were time dependent

- If $A, B, Q, R$ are <u>constant</u>
- $Q > 0$

$\rightarrow$ I can still write RE

$\exists$ a unique optimal solution:

$$u^\circ(t) = -R^{-1}B^T K_r\, x^\circ(t)$$

$$\dot{x}^\circ(t) = [A - BR^{-1}B^T K_r]\,x^\circ(t)$$

$$x^\circ(t_i) = x_i$$

where $K_r$ <u>constant matrix</u> and <u>unique</u> solution of the <u>algebraic Riccati equation</u>:

$$K_r B R^{-1} B^T K_r - K_r A - A^T K_r - Q = 0$$

with

$$J(x^\circ, u^\circ) = \frac{1}{2} x_i^T K_r\, x_i$$

Previously I had $\dot{K}$ for $[t_i, t_f]$, $\bar{K}$ for $[t_i, \infty)$, now $\dot{\bar{K}} = 0$ for $[t_i, \infty)$

# Proof

From the previous theorem we have that the value of the cost index for the considered problem is given by

$$J(x^o, u^o) = \frac{1}{2} x_i^T \bar{K}(t_i)\, x_i$$

Due to the stationarity of the problem defined on infinite time interval, it results that its value is independent from $t_i$.

Therefore the unique solution of the equation:

$$\dot{\bar{K}}(t) = \bar{K}(t) B(t) R^{-1}(t) B(t) \bar{K}(t) - \bar{K}(t) A(t) - A^T(t) K(t) - Q(t)$$

is constant $K_r$

The existance and uniqueness of the optimal solution follow from the previous theorem

# End

# Remarks

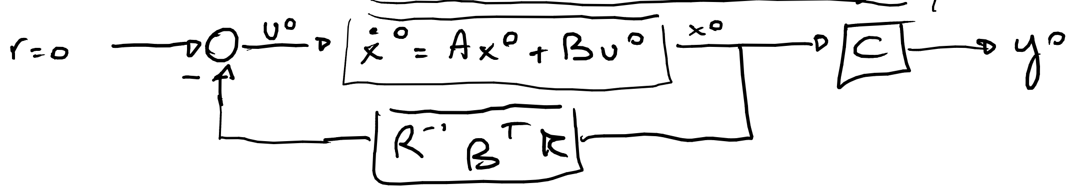
1) Consider the sys: $\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases}$

It can be defined an optimal regulation problem from the output $y$, considering the cost index:

$$\bar{J}(y,u) = \frac{1}{2} y^T(t_f) \bar{F} y(t_f) + \frac{1}{2}\int^{t_f} [y^T(t)\bar{Q}(t) y(t) + u^T(t) R(t) u(t)]\, dt$$

$$= \frac{1}{2} x^T(t_f) C^T(t_f) \bar{F} C(t_f) x(t_f) + \frac{1}{2}\int_{t_i}^{t_i, t_f} [x^T C^T \bar{Q} C x + u^T R u]\, dt$$

Condition: the state must be accessible, for the feedback action




2) case $[t_i, t_f]$ → differential RE
case $[t_i, \infty)$ → same
case $[t_i, \infty)$ with steady state sys → algebraic RE

~ The optimal tracking problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_i) = x_i \end{cases}$$

$A(t), B(t)$ with elements functions of $C^1$ class

Consider the reference $r \in C^1[t_i, t_f]$

Determine the OC $u^\circ \in \bar{C}^0[t_i, t_f]$ and the state $x^\circ \in \bar{C}^1[t_i, t_f]$ minimizing:

$$J(x,u) = \frac{1}{2}\int_{t_i}^{t_f} \left\{ [r(t) - x(t)]^T Q(t)[r(t) - x(t)] + u^T(t)R(t)u(t) \right\} dt$$

where $Q(t) \geq 0$, $R(t) > 0$, symmetric

The RE is:

$$\begin{cases} \dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t) \\ K(t_f) = 0 \end{cases}$$

Theorem: the problem admits a unique optimal solution:

$$u^\circ(t) = R^{-1}(t)B^T(t)[g(t) - K(t)x^\circ(t)]$$

$$\dot{x}^\circ(t) = [A(t) - B(t)R^{-1}(t)B^T(t)K(t)]x^\circ(t) + B(t)R^{-1}(t)B^T(t)g(t)$$

\# Previously the OC was $u^\circ(t) = -R^{-1}B^TKx^\circ$

$g(t)$ is the solution of the differential equation

$$\begin{cases} \dot{g}(t) = \underbrace{[K(t)B(t)R^{-1}(t)B^T(t) - A^T(t)]}_{\text{not important}}g(t) - Q(t)\underbrace{r(t)}_{\text{important}} \\ g(t_f) = 0 \end{cases}$$

with:

$$J(x^\circ, u^\circ) = \frac{1}{2}x^{\circ T}(t_i)K(t_i)x^\circ(t_i) - x^{\circ T}(t_i)g(t_i) + v(t_i)$$

where $v$ is the solution of the equation

important

$$\begin{cases} \dot v(t) = \frac{1}{2} g^T(t) B(t) R^{-1}(t) B^T(t) g(t) - \frac{1}{2} r(t)^T Q(t) r(t) \\ v(t_f) = 0 \end{cases}$$

$g(t)$ and $v(t)$ are functions that depends on the reference $r(t)$

# Proof

It can be applied Pontryagin theorem (convex case).

Let us define the hamiltonian:

$$H(x, u, \lambda, t) = \frac{1}{2}(r-x)^T Q (r-x) + \frac{1}{2} u^T R u + \lambda^T A x + \lambda^T B u$$

The nec. & suff. conditions are:

$$\begin{cases} \dot\lambda^\circ = -Q x^\circ - A^T \lambda^\circ + Qr \rightsquigarrow \text{new term} \\ R u^\circ + B^T \lambda^\circ = 0 \\ \lambda^\circ(t_f) = 0 \end{cases}$$

With the following choice for the costate:

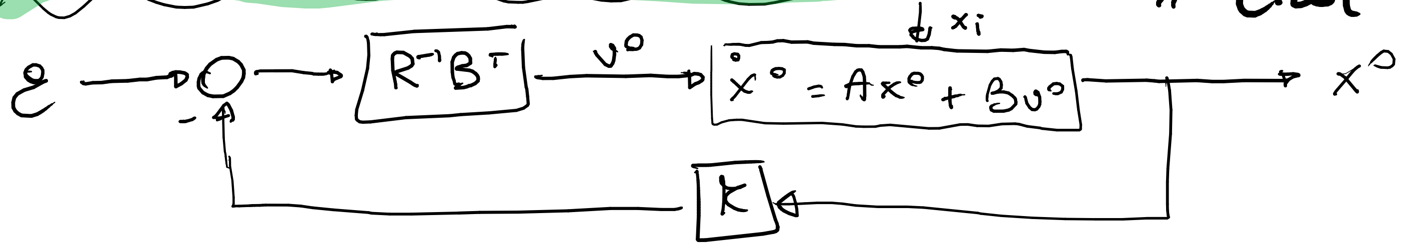
$$\lambda^\circ = K x^\circ - g$$

the theorem is proved.

Observe that:

$$J(x^\circ, u^\circ) = \frac{1}{2} x^{\circ T}(t_i) K(t_i) x^\circ(t_i) - x^{\circ T}(t_i) g(t_i) + v(t_i) =$$
$$= \int_{t_i}^{t_f} -\frac{d}{dt}\left(\frac{1}{2} x^{\circ T} K x^\circ - x^{\circ T} g + v\right) dt$$

# End



# Remark

- Note the quadratic error on the final instant $r(t_f) - x(t_f)$ is not present

- The realization of the optimal control for the tracking problem needs the solution of RE $K$, and the solution of the diff. equation in $g$

- To solve the tracking problem the reference variable <u>$r$ must be known in advance</u> in the interval

- The <u>steady state</u> tracking problem over an infinite time interval requires a steady state solution for the differential equation in the $g$ function.

eg. with $\bar{r}$ constant reference:

$$g_r = \left[ K_r B R^{-1} B^T - A^T \right] Q \bar{r}$$

## ~ The optimal regulator problem with null final error

**Problem**: Let us consider the linear system:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$x(t_i) = x_i \qquad x(t_f) = 0$$

If I pick $x(t_f)=0$ I don't need the Bolza term in the functional

Determine the control $u^\circ \in \bar{C}^0[t_i,t_f]$ and the state $x^\circ \in \bar{C}^1[t_i,t_f]$ minimizing

$$J(x,u) = \frac{1}{2}\int_0^{t_f}\left[x^T Q x + u^T R u\right] dt$$

$Q(t) \geq 0$, $R(t) > 0$, symmetric with $C^1$ class elements

**Theorem**: Let us introduce the matrix of dim $2n \times 2n$

$$\underset{\text{deltaU}}{\Omega(t)} = \begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{pmatrix}$$

and indicate with $\Phi(t,\tau) = \begin{pmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{pmatrix}$ ($n \times n$)

its transition matrix partitioned in submatrices of dimension $n \times n$

Assume that the submatrix $\phi_{12}(t_f,t_i)$ is <u>not singular</u>
↳ $\det \neq 0$

The optimal regulator problem with null final error admits a <u>unique optimal solution</u>:

$$u^\circ(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\,\phi_{12}^{-1}(t_f,t_i)\,\phi_{11}(t_f,t_i)\right] x_i$$

$$x^\circ(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\,\phi_{12}^{-1}(t_f,t_i)\,\phi_{11}(t_f,t_i)\right] x_i$$

$\longrightarrow$ initial state (not the current)

# Proof

We can apply the Pontryagin theorem, convex case.

Hamiltonian: $H(x,u,\lambda,t) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \lambda^T A x + \lambda^T B u$

Nec. & Suff. conditions:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T = -A^T\lambda - Qx$$

$$0 = \frac{\partial H}{\partial u}\Big|^T = Ru + B^T\lambda \implies u = -R^{-1}B^T\lambda$$

I have two equations:

$$\dot{x} = Ax + Bu$$

$$\dot{\lambda} = -A^T\lambda - Qx$$

Trick: let's put them together

$$\begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix} = \underbrace{\begin{pmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{pmatrix}}_{\Omega}\begin{pmatrix}x\\ \lambda\end{pmatrix} \longrightarrow \begin{pmatrix}\dot{x}\\ \dot{\lambda}\end{pmatrix} = \Omega\begin{pmatrix}x\\ \lambda\end{pmatrix}$$

$\Omega \to 2n \times 2n$

indicating with $\underset{\text{transition matrix}}{\Phi(t,\tau)} = \begin{pmatrix}\phi_{11}(t,\tau) & \phi_{12}(t,\tau)\\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau)\end{pmatrix}$

The solution of the differential equation, taking into account the initial condition is:

$$\begin{pmatrix}x(t)\\ \lambda(t)\end{pmatrix} = \begin{pmatrix}\phi_{11}(t,t_i) & \phi_{12}(t,t_i)\\ \phi_{12}(t,t_i) & \phi_{22}(t,t_i)\end{pmatrix}\begin{pmatrix}x(t_i)\\ \lambda(t_i)\end{pmatrix}$$

free evolution of the state

I don't know $\lambda(t_i)$ but I know $x(t_f) = 0$.

So multiplying the first row of the free evolution for the first column in the case of $t = t_f$

$$x(t_f) = \phi_{11}(t_f, t_i)\, x(t_i) + \phi_{12}(t_f, t_i)\, \lambda(t_i)$$

$\hookrightarrow 0$

Taking into account the non singularity of submatrix

$\phi_{12}(t_f, t_i) \neq 0 \longrightarrow$ non singular (hypothesis) $\longrightarrow$ If it is singular no solution exists or $\infty$ solutions depending on

it results in

$$0 = \phi_{11}(t_f, t_i)\, x(t_i) + \phi_{12}(t_f, t_i)\, \lambda(t_i)$$

$$\lambda(t_i) = -\frac{\phi_{11}(t_f, t_i)}{\phi_{12}(t_f, t_i)}\, x(t_i) = -\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)$$

I can now solve for $x(t)$:

$$x(t) = \phi_{11}(t, t_i)\, x(t_i) + \phi_{12}(t, t_i)\left[-\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)\right]$$

$$\lambda(t) = \phi_{21}(t, t_i)\, x(t_i) + \phi_{22}(t, t_i)\left[-\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)\right]$$

That substitued into the expression of control

$u^o = -R^{-1} B^T \lambda$ yelds the optimal solution.

# end

## ~ The optimal regulator problem with limited control

Problem: Optimal regulator problem on finite time interval with additional constraint

$$|u_j(t)| \leq 1 \quad , j=1,2,\dots,p \quad \forall t \in [t_i, t_f]$$

and the hypothesis that the weighting matrix R is diagonal

$$\dot{x} = Ax + Bu$$

Bolza Term (there's no final state condition)

$$J = \frac{1}{2} \int x^T Q x + u^T R u + \frac{1}{2} x^T(t_f) F x(t_f)$$

I want that each control is limited.

Theorem: All the normal solutions (no more than one) can be found solving the differential system:

$$\dot{x}^\circ(t) = A(t) x^\circ(t) - B(t) \text{sat}\{R^{-1}(t) B^T(t) \lambda^\circ(t)\}$$

$$\dot{\lambda}^\circ(t) = -Q(t) x^\circ(t) - A^T(t) \lambda^\circ(t)$$

$$x^\circ(t_i) = x_i \qquad \lambda^\circ(T) = F x^\circ(t_f)$$

Now with the condition of limited control the optimal control is:

$$u^\circ(t) = -\underbrace{\text{sat}}_{\text{saturation function}}\{R^{-1}(t) B^T(t) \lambda^\circ(t)\}$$

### Proof

Let's apply Pontryagin theorem (convex case);

It's not possible to apply, like before, $\frac{\partial H}{\partial u} = 0$, because $u$ is bounded.

Let's consider the hamiltonian

$$H = \cancel{\tfrac{1}{2} x^T Q x} + \tfrac{1}{2} u^{\circ T} R u^\circ + \cancel{\lambda^T A x} + \lambda^T B u^\circ \leq$$

Pontryagin

$$\leq \cancel{\tfrac{1}{2} x^T Q x} + \tfrac{1}{2} \omega^{\circ T} R \omega^\circ + \cancel{\lambda^T A x} + \lambda^T B \omega^\circ$$

$$\forall \omega : |\omega_j| \leq 1 \quad , \quad j = 1, 2, \dots, p$$

And the nec & suff. conditions

$$\dot{\lambda}^\circ = -Q x^\circ - A^T \lambda^\circ = -\left.\frac{\partial H}{\partial u}\right|^T$$

$\lambda^\circ(t_f) = F x^\circ(t_f) \longrightarrow$ Bolza term (boundary condition)

By adding the quantity $\frac{1}{2} \lambda^{\circ T} B R^{-1} B^T \lambda^\circ$ to both members of the inequality, we can write it as follows:

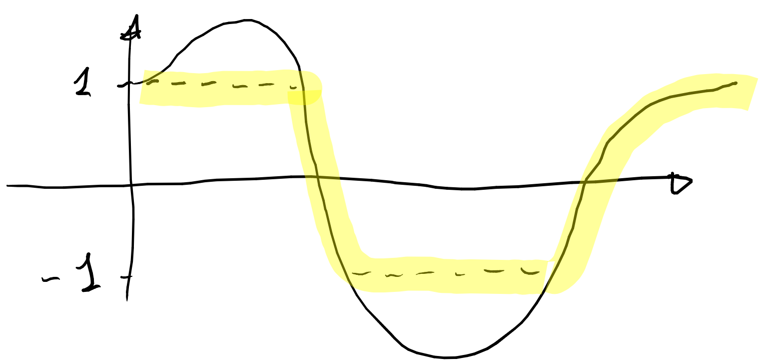
$$(u^\circ + \psi)^T R (u^\circ + \psi) \leq (\omega + \psi)^T R (\omega + \psi)$$

where $\psi = R^{-1} B^T \lambda$

Taking into account that the matrix $R$ is diagonal and positive definite, in order to satisfy the Pontryagin theorem by making the first term $\leq$ of the second one, $u^\circ = -\psi$ in order to obtain 0 (since we have quadratic forms we cannot have less than 0). Since $|u| \leq 1$ we have to saturate $\psi$.

$$\text{if} \begin{cases} \psi \leq -1 \\ \psi \geq 1 \end{cases} \Rightarrow -1 \leq \psi \leq 1$$

I can put $\boxed{u^\circ = -(R^{-1} B^T \lambda)}$ satisfying the equation




# End

# Remarks

- With the new constraint state-costate is non-linear
- The existence of the solution is not guaranteed
- It is possible to consider the same problem with the final condition fixed

$$x(t_f) = x_f$$

# SINGULAR SOLUTIONS

$$\dot{x} = f(x, u, t)$$

$$x(t_i) = x_i$$

$$J = \int_{t_i}^{t_f} L(x, u, t)\, dt$$

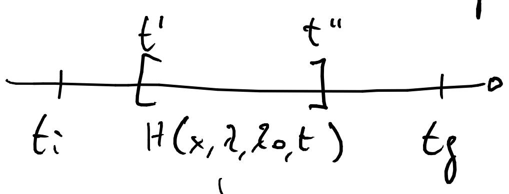
We write the Hamiltonian

$$H = \lambda_0 L + \lambda^T f = \lambda_0 L(x, u, t) + \lambda^T(t) f(x, u, t)$$

<u>Definition</u>: Let $(x^\circ, u^\circ, t_f^\circ)$ be an optimal solution of the above problem, and $\lambda_0^\circ, \lambda^\circ$ the corresponding multipliers.

The solution is <u>singular</u> if $\exists$ a subinterval $[t', t'']$, $t'' > t'$ in which the Hamiltonian

$H(x^\circ(t), \omega, \lambda_0^\circ, \lambda^\circ(t), t)$ is <u>independent</u> from at least one component of $\omega$ in $[t'; t'']$




In this interval the Hamiltonian does not depend on the control (<u>Bad thing</u>)

<u>Idea</u>: $\exists$ a subset in which the Hamiltonian does not depend on the control

$$\frac{\partial H}{\partial u} = 0^T \implies \underline{\text{Singularity depends on } f \text{ and } L}$$

The idea is to find a cost index $J$ that avoid

To do that we split the Hamiltonian in 2 parts, one depending on the state and one on the control.

Theorem: Assume the Hamiltonian of the form:

$$H(x,u,\lambda_0,\lambda,t) = H_1(x,\lambda_0,\lambda,t) + H_2(x,\lambda_0,\lambda,t)\,N(x,u,\lambda_0,\lambda,t)$$

Let $(x^*, u^*, t_f^*)$ be an extremum and $\lambda_0^*, \lambda^*$ the multipliers such that $N(x^*, \omega, \lambda_0^*, \lambda^*, t)$ is <u>dependent</u> on any component $\omega$ in any subinterval $[t_i, t_f]$

A nec. & suff. condition for $(x^*, u^*, t_f^*)$ to be a singular extremum is that $\exists$ a subinterval $[t', t''] \subset [t_i, t_f]$, $t'' > t'$ such that

$$H_2(x, \lambda_0, \lambda, t) = 0 \quad , \quad \forall t \in [t', t'']$$

\# example

$$\dot{x} = Ax + Bu \qquad |u| \leq 1$$

$$x(t_i) = x_i$$

$$J = \int_{t_i}^{t_f} 1 \cdot dt$$ ⇝ minimum time problem

Hamiltonian: $H = \underbrace{\lambda_0 1 + \lambda^T A x}_{H_1} + \underbrace{\lambda^T B}_{H_2} \underbrace{u}_{N}$

$\lambda d$

$H_1$ → does not depend on $u$

If in a subinterval this quantity $= 0$, we have a singular solution

↓

this never occurs with a <u>special hypothesis</u>

~ The linear minimum time optimal control

OC of a linear sys with:

- fixed initial & final state
- constrained control
- cost index equal to the lenght of the time interval

Problem: $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $|u_j(t)| \leq 1$, $j = 1, 2, \dots, p$

$$\underbrace{A(t) \in C^{n-2}, \; B(t) \in C^{n-1}}_{\text{at least } C^1 \text{ class}} \quad \forall t \in \mathbb{R}$$

$$x(t_i) = x_i \;, \quad x(t_f) = 0$$

The aim is to determine the final instant $t_f \in \mathbb{R}$ and the control $u^\circ \in \bar{C}^\circ(\mathbb{R})$ → continuous almost everywhere and the state $x^\circ \in \bar{C}^1(\mathbb{R})$, minimizing:

$$J(t_f) = \int_{t_i}^{t_f} \underbrace{\mathcal{L}}_{\mathcal{L} = 1} \, dt = t_f - t_i$$

Theorem: Nec conditions for $(x^*, u^*, t_f^*)$ to be an optimal solution are that $\exists$ a constant $\lambda_0^* \geq 0$ and a n-dimensional function $\lambda^* \in \bar{C}^1[t_i, t_f]$ not simultaneously null and such that:

$$\dot{\lambda}^* = -A^T \lambda^* = -\frac{\partial H}{\partial x}^T$$

$$\underbrace{\lambda^{*T} B w \geq \lambda^{*T} B u^*}_{\text{Pontryagin principle}} \quad \forall w \in \mathbb{R}^p : |w_j| \leq 1, \; j = 1, 2 \dots p$$

Possible discontinuities in $\lambda^*$ can appear only in the points in which $u^*$ has a discontinuity.

Moreover we have the fixed final state $x(t_f)=0$ but we don't have $t_f$, therefore we have the trasversality condition

$$H\Big|_{t_f^*}^{*} = 0$$

# Proof

The Hamiltonian associated to the problem is:

$$H = \lambda_0 + \lambda^T A x + \lambda^T B u$$

Applying the minimum principle

$$\dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T$$

$$H(\dots w \dots) \geq H(\dots u \dots) \quad \forall |w| \leq 1$$

and applying the trasversality conditions, the theorem is proved

# End

~ Strong controllability

It's a new hypothesis.

When we had $\dot{x}(t) = Ax(t) + Bu(t)$, a system in the steady state case, with $A$ and $B$ constant matrices, for the controllability we had

$$\text{rank}\{(B \;\; AB \; \dots \; A^{n-1}B)\}$$

Now we have $A(t)$ and $B(t)$ so it's not possible anymore.

<u>Strong controllability</u> corresponds to the controllability in any instant $\underline{t_i}$, in any <u>time interval</u> and by any component of the <u>control vector</u>

Let us indicate with $b_j(t)$, the j-th column of $B(t)$

$$B(t) = \begin{pmatrix} | & | & \\ b_1(t) & b_2(t) & \dots \\ | & | & \end{pmatrix}$$

j-th column $\rightarrow b_2(t) = b_j^{(1)}(t)$

$$G_j(t) = \begin{pmatrix} | & | & \\ b_j^{(1)}(t) & b_j^{(2)}(t) & \dots \\ | & | & \end{pmatrix} \rightsquigarrow$$

If the det $\neq 0$ $\forall t \geq t_i$ the strong controllability is <u>guaranteed</u>

$$\begin{cases} b_j^{(1)}(t) = b_j(t) \\ \vdots \\ b_j^{(k)}(t) = \dot{b}_j^{(k-1)}(t) - A(t)b_j^{(k-1)}(t) \\ \quad k = 2, 3, \dots, n \end{cases}$$

- In the <u>NON steady state</u> case is a <u>suff</u> condition for strong controllability
- In the <u>steady state</u> case is <u>nec & suff</u> condition and may be written as usual:

$$\det\{(b_j \;\; Ab_j \; \dots \; A^{n-1}b_j)\} \neq 0 \quad \forall j = 1, \dots, p$$

~ Characterization of the optimal solution

Theorem: Consider the minimum time optimal problem

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$x(t_i) = x_i \qquad x(t_f) = 0 \qquad |u_j(t)| \leq 1$

$J = \int_{t_i}^{t_f} dt = t_f - t_i$

If the strong controllability condition is satisfied ($\det\{G\} \neq 0$)

If $\exists$ the solution:

1) It's non singular
2) $U_j$ is a bang-bang solution: every component of the optimal solution is <u>piecewise constant</u> assuming only the extreme values $\pm 1$
3) The number of switches (discontinuity instants) is limited ($t_j < \infty$)

---

# Proof

1) By <u>contradiction</u>

If the solution were singular, from the Pontryagin principle

$$H = \underbrace{\lambda_0 1 + \lambda^T A x}_{H_1} + \underbrace{\lambda^T B}_{H_2} \underbrace{u}_{N}$$

$\rightarrow \lambda^T B w \geq \lambda^T B u \quad \forall w : |w_j| \leq 1$

The singularity exists when $\lambda^T B = H_2 = 0$, so when

$\lambda^T(t)\, b_j(t) = 0$ in $[t', t''] \subset [t_i, t_f^\circ]$

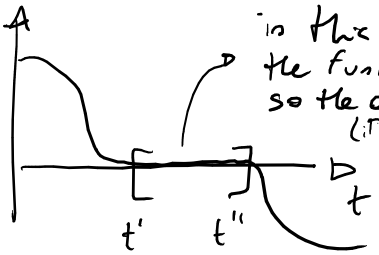
<u>Trick</u>:

I derive many times $\lambda^T(t)\, b_j(t)$ and using $\dot{\lambda} = -A^T\lambda$:

$\lambda^T(t)\, b_j(t) = 0$

$\frac{d}{dt}$

$\dot{\lambda}\, b_j + \lambda \dot{b}_j = 0 \quad \forall t \in [t', t'']$




$\rightarrow -A\lambda\, b_j + \lambda \dot{b}_j \rightarrow$ # Remember the matrix $G_j(t)$ of strong controllability

$\vdots$

$$\frac{d(\lambda^T(t)\, b_j(t))}{dt^i} = \lambda^{\circ T}(t)\, b_j^{(i+1)}(t) = 0 \qquad i = 1 \dots n-1$$

$$\rightarrow \lambda^{\circ T}(t)\, G_j(t) = 0 \qquad \forall t \in [t', t'']$$

On the other hand $\lambda^\circ(t)$ <u>must be $\neq 0$</u> $\forall t \in [t', t'']$ otherwise, from the <u>nec conditions</u> $\dot{\lambda} = -A^T\lambda$ it should be null in the whole interval and in particular $\lambda^\circ(t_f) = 0$

and if $\lambda(t_f) = 0$ we have the
$\rightarrow$ impossible because $x(t_f) = 0$

transversality condition $H|^*_{t_f^*} = 0$ because $t_f$ is not fixed, and this leads to $\lambda_0^\circ = 0$ in fact:

$H = \lambda_0 1 + \underbrace{\lambda^T A x + \lambda^T B u}_{0 \text{ in this hypothesis}} \rightarrow H = \lambda_0 1 \rightarrow \lambda_0$ cannot be 0 !!

<u>absurd</u>

Therefore, from the condition

$\lambda^{\circ T}(t)\, G_j(t) = 0$, $\forall t \in [t', t'']$ it results that $\det\{G_j(t)\} = 0$, in fact from Cramer, if $\det\{G_j\} = 0$ we should have that $\lambda(t) = 0$ which is <u>not possible</u> from <u>strong controllability</u>

$\rightarrow$ The solution is non singular

2) From the nonsingularity of the optimal solution, it results that the quantity

$\lambda^{\circ T}(t)\, b_j(t)$ can be null only on isolated points

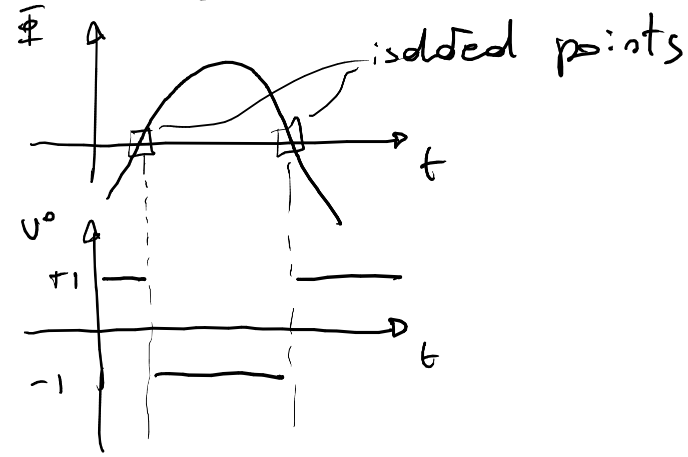
From the nec. condition : $\lambda^{\circ T} B w \geq \lambda^{\circ T} B v^{\circ}$

$v^{\circ}(t) = \begin{cases} +1 \\ 0 \longrightarrow \text{only on isolated points} \\ -1 \end{cases}$

$\lambda^{\circ T} B = \Phi$ = switching function $\rightarrow$ if $\Phi = 0$, $0 \cdot w \geq 0 \cdot v$ (I cannot say anything about the control)

The switching function can be 0 only on isolated points

Therefore $v_j^{\circ}(t) = -\operatorname{sign}\{\lambda^{\circ T}(t)\, b_j(t)\}$ $j = 1, 2, \dots, p$ $\forall t \in [t_i, t_f^{\circ}]$



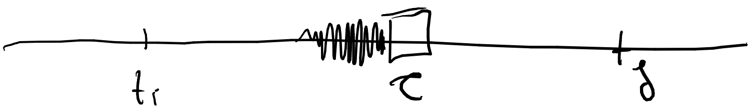

3) To demonstrate that the number of discontinuity instants is finite, we can proceed by contradiction:

Assume $\det\{b_j(t)\} = 0 \quad \forall t \in [t', t'']$

which contradicts the hypothesis of strong controllability

In fact let us assume that at least for one component of the control $v_j^{\circ}(t)$ it is not true

Then an accumulation point $\tau \in [t_i, t_f^{\circ}]$ of switching instants $\bar{t}_k^{(j)}$ exists




$\rightsquigarrow$ accumulation point (infinite $t_k$ in a finite interval)

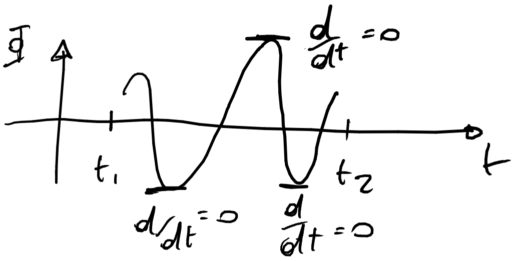
By continuity, given $u_j^\circ(t) = -\text{sign}\{\lambda^{\circ T}(t)\, b_j(t)\}$

$$\lambda^T(\bar{t}_k^{(j)})\, b_j(\bar{t}_k^{(j)}) = 0$$

$$\lambda^{\circ T}(\tau)\, b_j(\tau) = 0$$

<u>Trick:</u> for the continuity of $\lambda^{\circ T}(t)\, b_j(t) = 0$ between $\bar{t}_k^{(j)}$ and $\bar{t}_{k+1}^{(j)}$,

$\exists$ an instant $\tilde{t}_k^{(j)}$ in which $\frac{d}{dt}\left(\lambda^{\circ T}(t)\, b_j(t)\right) = 0$




the derivative $= 0$ when for example we have maxima and minima

Therefore, by continuity also in $\tau$: $\left.\frac{d}{dt}\left(\lambda^{\circ T}(t)\, b_j(t)\right)\right|_\tau = 0$

and analogously,

$$\left.\frac{d^i}{dt^i}\left(\lambda^{\circ T}(t)\, b_j(t)\right)\right|_\tau = 0 \qquad i = 0, 1, \dots (n-1)$$

$$\Rightarrow \det\{G_j(t)\} = 0 \quad \forall t \in [t', t'']$$

<u>contradiction with SC hypothesis</u>

\# End

A control function that assumes only the limit values is called <u>bang-bang control</u> and the instants of discontinuity are called <u>commutation instants</u>

## ~ Uniqueness

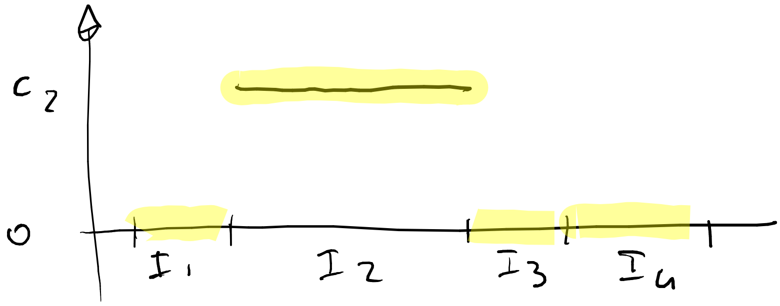
If the hypothesis of strong controllability is satisfied, if an optimal solution exists, it is unique

## ~ Measurable function

Let $S(t_i, t_f)$ be the space of piecewise constant function defined as follows:

$$s(t) = \begin{cases} c_j & \forall t \in I_j, \ j = 1, 2, \dots, m \\ 0 & \forall t \notin \bigcup_{j=1}^{m} I_j \end{cases}$$

$$\{I_j\} \subset [t_i, t_f], \quad I_j \cap I_k = \emptyset$$



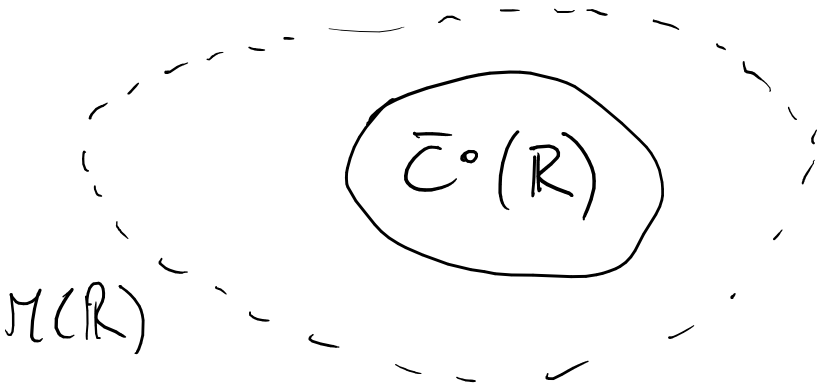

Let $z(t)$ be a function s.t. $\exists$ a sequence $\{s^{(k)}\} \subset S(t_i, t_f)$ such that $\forall t \in [t_i, t_f]$, with the exception of isolated points, one has: $\lim_{k \to \infty} s^{(k)}(t) = z(t)$

$z(t)$ is a measurable function, the limit of a piecewise constant function

The space $M(t_i, t_f)$ of measurable function is linear.

# Remark

These results hold also to the case in which the control is in the space of measurable function defined on the space of real number $\mathbb{R}$




~ Existence of the Optimal Solution

Theorem: If the condition of strong controllability is satisfied and if an admissible solution exists:
- $\exists$ an optimal solution
- The solution is unique and non singular
- The control is bang - bang

# Proof

The existence theorem is proved on the basis of the following results:

Theorem: Let assume that the control functions belong to the space of measurable functions. If $\exists$ an admissible solution, then the optimal solution exists

Trivial case: finite number of admissible solutions

We have for example $(t_1^\circ, u_1^\circ, x_1^\circ), (t_2^\circ, u_2^\circ, x_2^\circ), (t_3^\circ, u_3^\circ, x_3^\circ)$

Supposing $t_2^\circ$ is the lower bound of the sequence $t_1^\circ, t_2^\circ, t_3^\circ$, then $(t_2^\circ, u_2^\circ, x_2^\circ)$ is the solution of the minimum time problem

Non trivial case: $\infty$ admissible solutions

$t_f^\circ$ is the minimum extremum of the instants $t_f$ corresponding to the admissible solutions $\rightarrow t_f^\circ = \inf\{t_f\}$

Can we find a minimum?

We build a sequence of admissible solutions

$\{(x^{(k)}, u^{(k)}, t_f^{(k)})\}$, $t_f^{(k)} \geq t_f^\circ$

$\hookrightarrow$ larger because $t_f^\circ$ is the minimum extremum of the sequence

$\lim_{k \to \infty} t_f^{(k)} = t_f^\circ$

Now consider the transition matrix of the system $\Phi(t, \tau)$, from the linear sys:

$\dot{x} = A(t)\,x(t) + B(t)\,u(t)$

$\dot{x}^{(k)} = A(t)\,x^{k}(t) + B(t)\,u^{k}(t)$

$\downarrow$ solution of the equation

$$x^{(k)}(t_f^{(k)}) = \phi(t_f^{(k)}, t_i)\,x_i + \int_{t_i}^{t_f^{(k)}} \phi(t_f^{(k)}, t)\,B(t)\,u^{(k)}(t)\,dt$$

$$x^{(k)}(t_f^\circ) = \phi(t_f^\circ, t_i)\,x_i + \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\,B(t)\,u^{(k)}(t)\,dt$$

$$\lim_{k\to\infty}\left[x^{(k)}(t_f^{(k)}) - x^{(k)}(t_f^{\circ})\right] = \lim_{k\to\infty}\underbrace{\left[\phi(t_f^{(k)},t_i) - \phi(t_f^{\circ},t_i)\right]}_{\to 0} x_i \; +$$

$$+ \lim_{k\to\infty}\int_{t_i}^{t_f^{\circ}}\underbrace{\left[\phi(t_f^{(k)},t) - \phi(t_f^{\circ},t)\right]}_{\to 0} B(t)\, u^{(k)}(t)\, dt \; +$$

$$+ \lim_{k\to\infty}\underbrace{\int_{t_f^{\circ}}^{t_f^{(k)}}\phi(t_f^{(k)},t)\, B(t)\, u^{(k)}(t)\, dt}_{\to 0} = 0$$

The first 2 limits goes to 0 cause $\phi$ is continuous and $t_f^{(k)} \to t_f^{\circ}$

The last one goes to 0 cause independently to the quantity in the integral, the extrema are from $t_f^{\circ}$ to $t_f^{(k)}$, but $t_f^{(k)} \to t_f^{\circ}$ therefore due to the small interval the integral is 0.

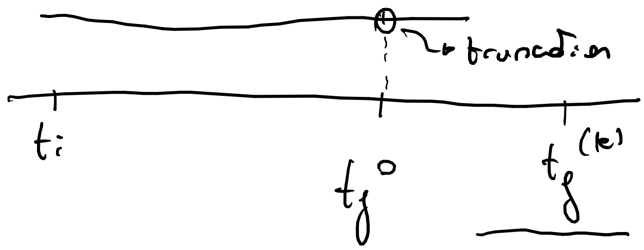
So we know that each $\boxed{x^{(k)}(t_f^{(k)}) = 0}$ because it is an admissible solution therefore:

$$\lim_{k\to\infty}\left[\underbrace{x^{(k)}(t_f^{(k)})}_{\to 0} - x^{(k)}(t_f^{\circ})\right] = 0$$

$$\Downarrow$$

$$\boxed{\lim_{k\to\infty} x^{(k)}(t_f^{\circ}) = 0}$$

Only the control now is missing.




Given the admissible solution $\{x^{(k)}, u^{(k)}, t_f^{(k)}\}$, we consider $\bar{u}^{(k)}$ as the truncation of $u^{(k)}$ in $t_f^{\circ}$

This function is limited and $L_2$, this means that if we do the integral $\int |f|^2 < \infty \rightarrow$ it is bounded, since $|u_j^{(k)}(t)| \leq 1$.

The sequence of functions $\{\bar{u}^{(k)}\} \in M$ of measurable function such that $|u_j(t)| \leq 1$, $\forall t \in [t_i, t_f^\circ]$. Therefore this sequence admits a subsequence, indicated still with $\{\bar{u}^{(k)}\}$, <u>weakly</u> converging to a function $u^\circ \in M$.

In synthesis:

$$\begin{tikzcd}
\text{from } \{u^{(k)}\} \arrow[r, "\text{truncation in } t_f^\circ"] & \{\bar{u}^{(k)}\} \text{ limited } L_2 \text{ fun } \int |f|^2 < \infty \arrow[d, "\text{find a subsequence}"] & \\
& \{\bar{u}^{(k)}\} \arrow[r, "\text{weakly}"] & u^\circ
\end{tikzcd}$$

- Weak convergence of $\forall$ measurable function $h$

$$\lim_{k \to \infty} \int_{t_i}^{t_f^\circ} h^T(t)\, \bar{u}^{(k)}(t)\, dt = \int_{t_i}^{t_f^\circ} h^T(t)\, u^\circ(t)\, dt$$

The limit of the integral of $h$ times the sequence, is equal to the integral of $h$ times $u^\circ$

Given $x^\circ$, the evolution of the state corresponding to $u^\circ$:

$$\lim_{k \to \infty} x^{(k)}(t_f^\circ) = \phi(t_f^\circ, t_i)\, x_i + \lim_{k \to \infty} \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\, B(t)\, \bar{u}^{(k)}(t)\, dt$$

$$= \phi(t_f^\circ, t_i)\, x_i + \int_{t_i}^{t_f^\circ} \phi(t_f^\circ, t)\, B(t)\, u^\circ(t)\, dt = \boxed{x^\circ(t_f^\circ)}$$

Therefore, given $\lim_{k \to \infty} x^{(k)}(t_f^\circ) = 0$, we deduce $x^\circ(t_f^\circ) = 0$

So we have found the optimal solution $(x^\circ, v^\circ, t_f^\circ)$ with a control able to transfer the state to the origin.

\# End

# Remark:

The existence of the optimal solution is guaranteed only for the couples $(t_i, x_i)$ for which the admissible solution exists.

# ~ Minimum time problem for steady state system

$$\dot{x}(t) = \underbrace{A}_{\text{constant}} x(t) + \underbrace{B}_{\text{constant}} u(t)$$

In this case there's a result on the number of <u>commutation points</u> ⇝ I can say the maximum number of switching points

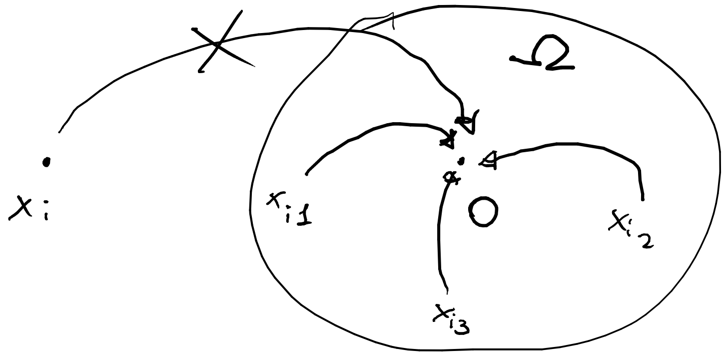
Given $\quad x(t_i) = x_i \quad x(t_f) = 0 \quad |u| \leq 1$

$$J = \int_{t_i}^{t_f} dt = t_f - t_i$$

<u>Theorem</u>: consider the control function belonging to the space of measurable function.

If the system is <u>controllable</u>

⇒ ∃ a neighbor $\Omega$ of the origin such that $\forall x_i \in \Omega$ there exists an optimal solution




# Proof

If I can show in this case that ∃ an admissible solution, I have solved it due to the previous results.

Let us fix any $t_f > 0$. The fact that the system is controllable can be described by saying that we can find a control able to transfer the initial state $x_i \in \Omega$ to the origin at the instant $t_f$:

$$\dot{x} = Ax + Bu \qquad\qquad x(t_i) = x_i$$

$$x(t) = e^{A(t-t_i)}x_i + \int_{t_i}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

I want that $\underline{x(t_f) = 0}$, so $t = t_f$

must be 0

$$x(t_f) = e^{A(t_f-t_i)}x_i + \int_{t_i}^{t_f} e^{A(t_f-\tau)}Bu(\tau)\,d\tau = 0$$

Since $A$, $B$ and $t_f$ are fixed, we miss only $u$.

Assume:

$u(\tau) = B^T e^{-A^T\tau} v$ with $v \in \mathbb{R}^p$

substituting in the last integral:

$$-e^{-At_i}x_i = \underbrace{\left(\int_{t_i}^{t_f} e^{-A\tau}BB^T e^{-A^T\tau}\,d\tau\right)}_{\text{Gramian matrix}} v$$

In order to show the controllability, it is equivalent to say that the Gramian matrix is <u>non singular</u>.

From the previous considerations:

$$v = -(\text{Gramian})^{-1} e^{-At_i} x_i$$

substituting in $u(\tau)$:

$$u(\tau) = -B^T e^{-A^T\tau} (\text{Gramian})^{-1} e^{-At_i} x_i$$

This is an admissible control, and it exists if the initial state

is sufficiently near the origin.
The control $u(t)$ to be admissible, must be $|u| \le 1$ as known, and to respect this condition, since everything is given with exception of $x_i$, therefore choosing it very small, of course $u(t)$ could be $\le 1$

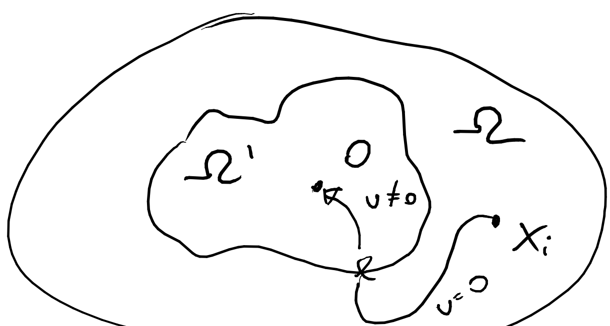
# End

How can we avoid the constraint of being near the origin?
If the eigenvalues of $A$ have negative real part ($Re\{\lambda\} < 0$), $\exists$ an optimal solution whatever the initial state is.

# Proof

- Let $\Omega$ be a neighbour of the origin including the original state $x_i$ for which an admissible solution exists.
- Let $\Omega'$ be a closed subset in $\Omega$

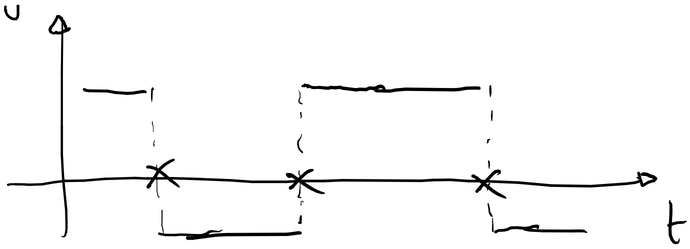
$\rightarrow$ From the asymptotic stability of the system it is possible to reach $\Omega'$ in a finite time with null input (free evolution), from any initial state




This probably isn't the optimal solution but I don't care because I need only an admissible one

# End

- How many <u>commutation instants</u>?




If we have the controllability condition, if $Re\{\lambda\} \leq 0$, the number of commutation instants for any component of the control is $\leq n-1$ whatever the initial state is.

$\hookrightarrow$ dimension of A

# Proof

Given the optimal control

$u_j^\circ(t) = -\text{sign}\{\lambda^{\circ T}(t)\, b_j\}$, $j = 1, 2, \dots, p$ $\quad \forall t \in [t_i, t_f^\circ]$

where the costate $\lambda^\circ$ in the steady state case is:

$H = \lambda_0 \cdot 1 + \lambda^T A x + \lambda^T B u \quad \rightarrow \dot{\lambda} = -\frac{\partial H}{\partial x}\Big|^T = -A^T \lambda$

$\downarrow$

$\lambda^\circ(t) = e^{-A^T(t-t_i)} \lambda_i$

Now:

$\dot{x}(t) = Ax(t) + Bu(t) \qquad A \in \mathbb{R}^{n \times n} \rightarrow n$ eigenvalues

denoting by $\alpha_k$ the eigenvalues and by $m_k$ their multiplicity, the expression of the r-th component of $\lambda^\circ$ with $r = 1, \dots, n$ is:

$\lambda_r^\circ(t) = \sum_{s=1}^{k} P_{rs}(t)\, e^{-\alpha_s(t-t_i)}$ $\rightarrow$ polynomial function of degree $< m_s$

Now substituting $\lambda_r^\circ(t)$ in the expression of the bang-bang control we obtain:

$$u_j^\circ(t) = -\,\text{sign}\left\{\sum_{s=1}^{k}\underbrace{\left(\sum_{r=1}^{n} b_{jr}\, p_{rs}(t)\right)} e^{-\alpha_s(t-t_i)}\right\} \qquad j = 1, \dots, p$$

$$= -\,\text{sign}\left\{\sum_{s=1}^{k} \underbrace{p_{js}(t)}_{\text{polynomial function of degree } < m_s} e^{-\alpha_s(t-t_i)}\right\} \qquad \forall t \in [t_i, t_f^\circ]$$

The optimal control is given by the sum of polynomial functions in which each of them has degree $< m_s$, so the argument of the sign function has at most $m_1 + m_2 + \dots + m_{k-1} = n-1$ real solutions

Therefore the control $u_j^\circ$ has at most $n-1$ roots

$$\begin{array}{cc} \alpha_1 & m_1 \\ \alpha_2 & m_2 \\ \vdots & \vdots \\ \alpha_k & m_k \end{array} \longrightarrow \text{the sum of these is } n$$

#End

# Remark:

In the minimum time problem it is not possible to relate directly the control and the state esplicitly

$u^\circ(t) = -\,\text{sign}\left\{\lambda^T(t) B\right\}$ and not

$\cancel{u^\circ(t) = f(x^\circ(t))}$

# DOUBLE INTEGRATOR

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$x(t_i) = x_i$$
$$x(t_f) = 0$$
$$|u(t)| \leq 1$$
$$J = \int_{t_i}^{t_f} dt = t_f - t_i$$

We can write the system in this way:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

eigenvalues:

$$p(\lambda) = |A - \lambda I| = \lambda^2 = 0 \longrightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \end{matrix}$$

$$\det \begin{pmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix}$$

$$\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

controllability:

$$\det (B \;\; AB) = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0 \quad \text{ok!}$$

$\Rightarrow \exists\ (x^\circ, u^\circ, t_f^\circ)$ unique, non singular, bang bang
number of switching points $\nu^\circ \leq n-1 = 1$

From any initial state I can reach the origin with zero switches or just one.

We know that $u(t) = \pm 1$, therefore

$\dot{x}_1 = x_2$

$\dot{x}_2 = \pm 1$

$x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix}$

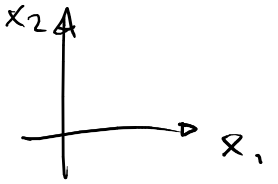
switch

$\hookrightarrow$ integrating

$$x_2 = \pm (t - t_i) + x_{2i} \quad (1)$$

$$x_1(t) = x_{1i} + x_{2i}(t - t_i) \pm \frac{1}{2}(t - t_i)^2 \quad (2)$$

Instead of studying the evolution of $x_1$ and $x_2$ separately, it is useful to study them together in the state phase plane:




From (1) $(t - t_i) = \pm [x_2(t) - x_{2i}]$

substituting in (2):

$$x_1(t) - x_{1i} = \pm x_{2i}[x_2(t) - x_{2i}] \pm \frac{1}{2}[x_2(t) - x_{2i}]^2$$

$$= \frac{\pm 2x_{2i}[x_2(t) - x_{2i}] \pm [x_2(t) - x_{2i}]^2}{2}$$

$$= \frac{\pm [x_2(t) - x_{2i}][2x_{2i} + [x_2(t) - x_{2i}]]}{2}$$

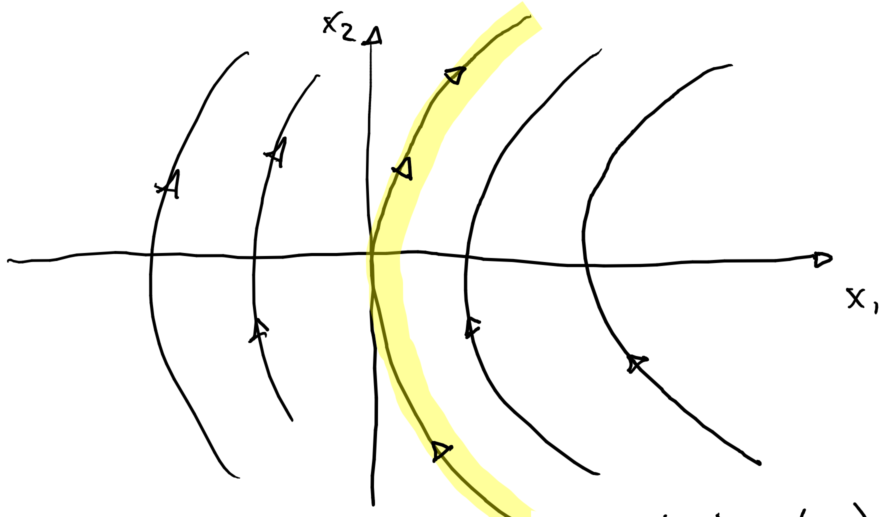
$$= \pm \frac{[x_2(t) - x_{2i}][x_2(t) + x_{2i}]}{2}$$

$$= \pm \frac{1}{2}[x_2^2(t) - x_{2i}^2]$$ hyperbolic paraboloid

The optimal trajectory is described by parabolic arcs:

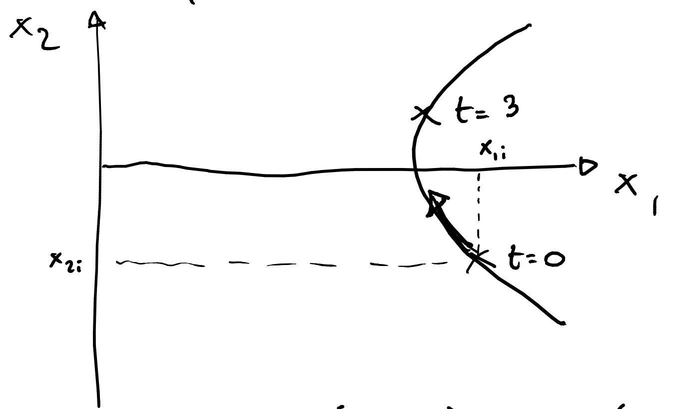
- consider $u(t) = +1$

$$x_1(t) - x_{1i} = +\frac{1}{2}\left[x_2^2(t) - x_{2i}^2\right] \to x_1(t) = x_{1i} + \dots$$




if $\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

What happen if time increases?



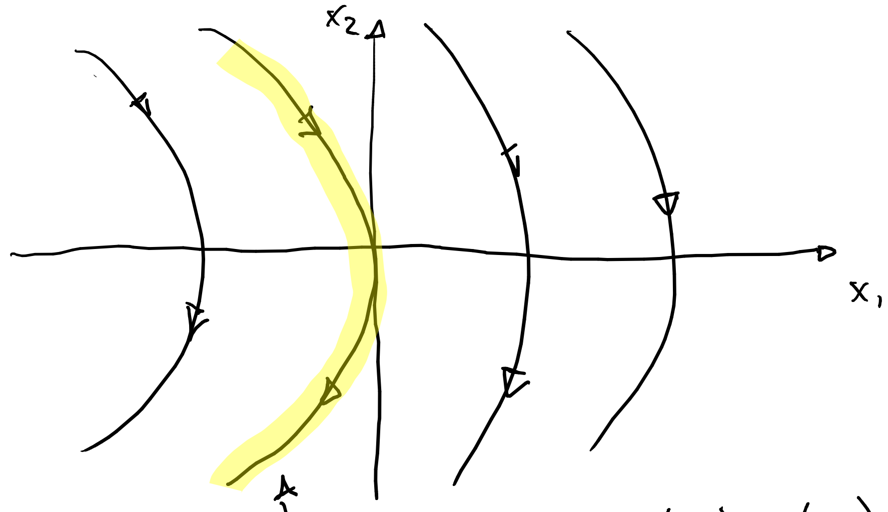

I go up

In fact in $(t - t_i) = + (x_2 - x_{2i})$ if time increases:

$$\begin{cases} t_i = 0 \\ x_{2i} = -2 \end{cases} \to \begin{array}{lll} \text{in } t = 0 & x_2 = -2 & \\ \text{in } t = 3 & x_2 = 1 & \to x_2 \text{ increases} \end{array}$$

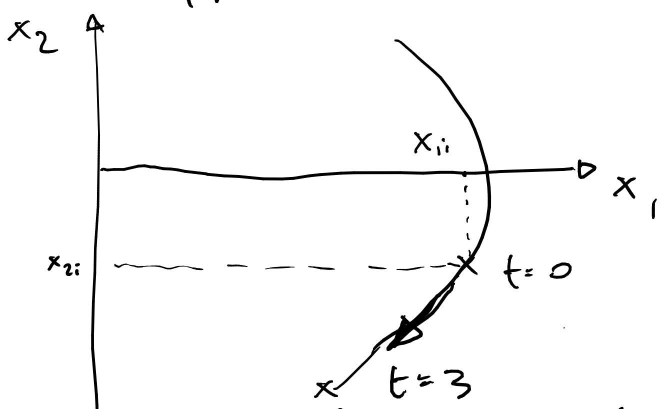
- consider $u(t) = -1$

$$x_1(t) - x_{1i} = -\frac{1}{2}\left[x_2^2(t) - x_{2i}^2\right] \rightarrow x_1(t) = x_{1i} - \dots$$




if $\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

What happen if time increases?



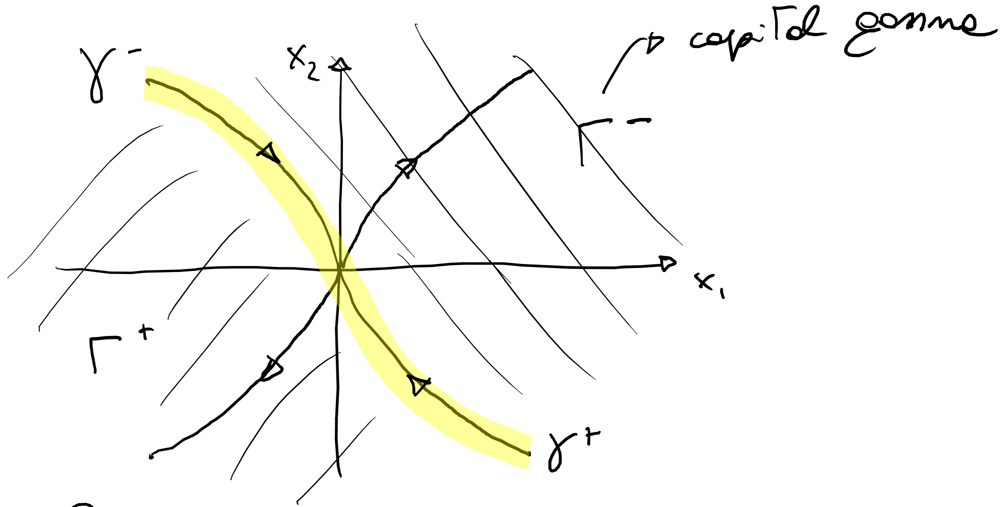

I go down

In fact in $(t - t_i) = -(x_2 - x_{2i})$ if time increases:

$$\begin{cases} t_i = 0 \\ x_{2i} = -2 \end{cases} \rightarrow \begin{array}{lll} \text{in } t = 0 & x_2 = -2 & \\ \text{in } t = 3 & x_2 = -5 & \rightarrow x_2 \text{ decreases} \end{array}$$

This curves represent the path on which I can move switching zero or maximum 1 arc.

## ~ Initial point




We define 2 curves and 2 regions:

Curves:

$$\gamma^+ = \left\{ x \in \mathbb{R}^2 : \ x_1 = \frac{1}{2} x_2^2 , \ x_2 < 0 \right\}$$

$$\gamma^- = \left\{ x \in \mathbb{R}^2 : \ x_1 = -\frac{1}{2} x_2^2 , \ x_2 > 0 \right\}$$

$$\Downarrow \quad \gamma = \gamma^+ \cup \gamma^+$$

$$\gamma = \left\{ x \in \mathbb{R}^2 : \ x_1 = -\frac{1}{2} x_2 |x_2| , \ x_2 \neq 0 \right\}$$

Regions:

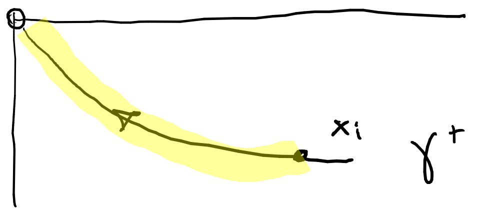
$$\Gamma^+ = \left\{ x \in \mathbb{R}^2 : \ x_1 < -\frac{1}{2} x_2 |x_2| \right\}$$

$$\Gamma^- = \left\{ x \in \mathbb{R}^2 : \ x_1 > -\frac{1}{2} x_2 |x_2| \right\}$$

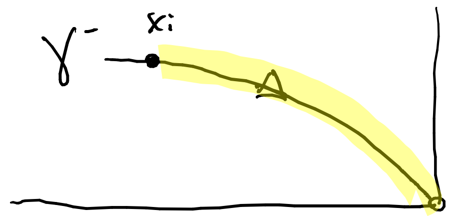
$$\Gamma^+ \cup \Gamma^- \cup \gamma = \mathbb{R}^2 \setminus \{0\}$$

~ 4 cases

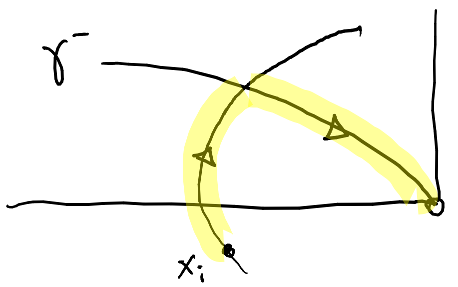
1) $x_i \in \gamma^+$ → with control $u=+1$ and zero switches



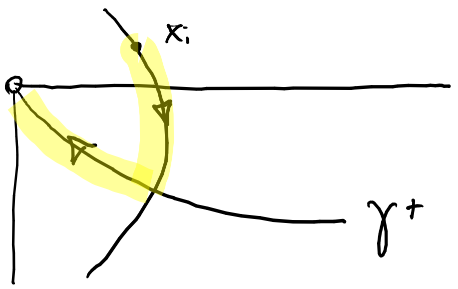

2) $x_i \in \gamma^-$ → with control $u=-1$ and zero switches




3) $x_i \in \Gamma^+$ → first $u=+1$ to reach the curve $\gamma^-$
then $u=-1$ to reach the origin (1 switch)




4) $x_i \in \Gamma^-$ → first $u=-1$ to reach the curve $\gamma^+$
then $u=+1$ to reach the origin (1 switch)




~ Optimal control

$$u^\circ(x^\circ(t)) = \begin{cases} 1 & \text{if } x^\circ(t) \in \Gamma^+ \cup \gamma^+ \\ -1 & \text{if } x^\circ(t) \in \Gamma^- \cup \gamma^- \end{cases}$$

## ~ Minimum Time

It depends on the location of the initial point $x_i$

A) $x_i \in \gamma = \gamma^+ \cup \gamma^-$

No switches

from $(t - t_i) = \pm [x_2(t) - x_{2i}]$

$$\underbrace{t_f}_{\text{unknown}} - t_i = + [\underbrace{x_2(t_f)}_{\to 0 \text{ (the origin}} - x_{2i}] \quad \text{if } u = +1$$

$$\boxed{t_f^\circ = t_i - x_{2i} \quad \text{if } u = +1}$$

$$t_f - t_i = - [\underbrace{x_2(t_f)}_{\to 0} - x_{2i}] \quad \text{if } u = -1$$

$$\boxed{t_f^\circ = t_i + x_{2i} \quad \text{if } u = -1}$$

B) $x_i \in \Gamma^+$

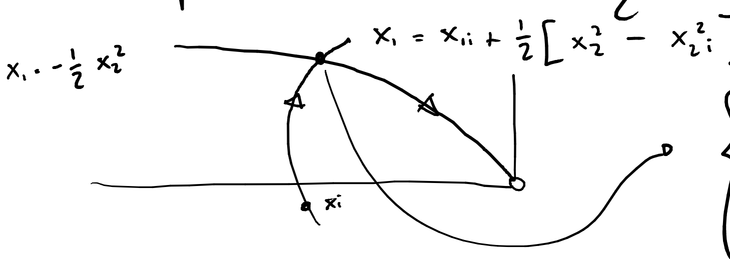
The control switches at the instant $\bar{t}$ at the position $\bar{x}_2$

$$(t_f^\circ - t_i) = \underbrace{(t_f^\circ - \bar{t})}_{\substack{u=-1 \\ \text{(after switching)}}} + \underbrace{(\bar{t} - t_i)}_{u=+1 \text{ (before switching)}}$$

$$(t_f^\circ - t_i) = \pm [x_2(t) - x_{2i}] \Rightarrow \begin{cases} \bar{t} - t_i = \bar{x}_2 - x_{2i} \\ t_f^\circ - \bar{t} = \bar{x}_2 \end{cases}$$

$$(t_f^\circ - t_i) = 2\bar{x}_2 - x_{2i}$$

The position $\bar{x}_2$ belongs to the two parabolic arcs:




$$\begin{cases} \bar{x}_1 = x_{1i} + \frac{1}{2}[\bar{x}_2^2 - x_{2i}^2] \\ \bar{x}_1 = -\frac{1}{2}\bar{x}_2^2 \end{cases}$$

Result of the system:

$$\bar{x}_2^2 = -x_{1i} + \frac{1}{2} x_{2i}^2, \quad x_2 > 0$$

$$\bar{x}_2 = \sqrt{-x_{1i} + \frac{1}{2} x_{2i}^2}$$

$$\boxed{(t_f^o - t_i) = \sqrt{-4 x_{1i} + 2 x_{2i}^2} - x_{2i}}$$

c) $x_i \in \Gamma^-$

The same calculations yeld:

$$\boxed{(t_f^o - t_i) = \sqrt{4 x_{1i} + 2 x_{2i}^2} + x_{2i}}$$

## ~ Commutation curve

$\gamma$ is the commutation curve so:

$$\varphi(x) = x_1 + \frac{1}{2} x_2 |x_2|$$

$$u^o(t) = -\operatorname{sign}\{\varphi(x)\} = -\operatorname{sign}\left\{x_1 + \frac{1}{2} x_2 |x_2|\right\}$$

# HARMONIC OSCILLATOR

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = -\omega x_1(t) + u(t) \end{cases} \qquad \omega > 0$$

$x(t_i) = x_i \qquad x(t_f) = 0 \qquad |u(t)| \leq 1$

$$J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i$$

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(t)$$

Eigenvalues:

$$p(\lambda) = \begin{vmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0$$

$$\lambda^2 = -\omega^2 \rightarrow \lambda_{1,2} = \pm\sqrt{-\omega}$$

$$\lambda = \begin{pmatrix} j\omega \\ -j\omega \end{pmatrix}$$

The natural modes are oscillatory

Controllability:

$$\det(B \;\; AB) = \begin{vmatrix} 0 & \omega \\ 1 & 0 \end{vmatrix} = -\omega \neq 0 \quad \text{ok!}$$

$\exists \; (x^\circ, u^\circ, t_f^\circ)$ unique, non singular, bang bang $\forall$ initial condition, even if $Re\{\lambda\} = 0$ since we are in the steady state case. Since the eigs are complex we can't apply the theorem about the maximum number of commutation points

To solve the problem we can apply the Pontryagin principle

$$H(x, v, 1, \lambda) = \mathcal{L} + \lambda^T f = 1 + \lambda_1(t)\,\omega\, x_2(t) - \lambda_2(t)\,\omega\, x_1(t) + \lambda_2(t)\, v(t)$$

Necessary conditions

$$\dot{\lambda}^\circ(t) = -\frac{\partial H}{\partial x}\bigg|^T = -A^T \lambda^\circ(t) \longrightarrow \begin{cases} \dot{\lambda}_1 = +\omega \lambda_2 \\ \dot{\lambda}_2 = -\omega \lambda_1 \end{cases}$$

$$\cancel{1 + \lambda_1^\circ(t)\,\omega\, x_2^\circ(t)} - \cancel{\lambda_2^\circ(t)\,\omega\, x_1^\circ(t)} + \lambda_2^\circ(t)\, v^\circ(t) \le$$
$$\cancel{1 + \lambda_1^\circ(t)\,\omega\, x_2^\circ(t)} - \cancel{\lambda_2(t)\,\omega\, x_1^\circ(t)} + \lambda_2^\circ(t)\, v(t)$$

$$\forall v : |v(t)| \le 1$$

$$\lambda_2^\circ(t)\, v^\circ(t) \le \lambda_2^\circ(t)\, v(t)$$

Deriving the second line for example:

$$\ddot{\lambda}_2^\circ = -\omega\, \dot{\lambda}_1^\circ(t) = -\omega^2 \lambda_2^\circ(t)$$

We know that:

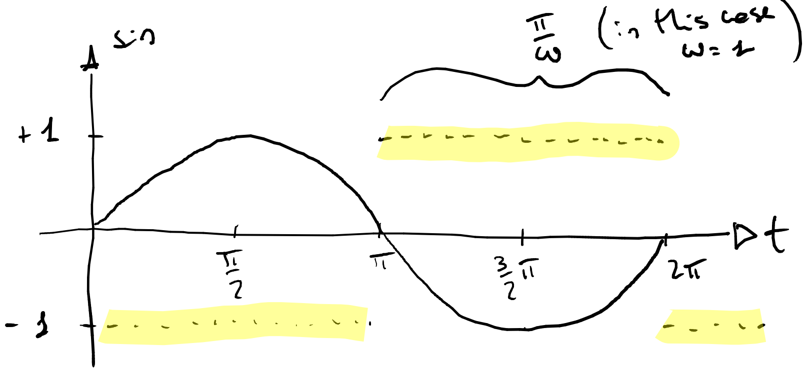
$$\lambda_2^\circ(t) = K \sin(\omega(t - t_i) + \alpha)$$

From Pontryagin

$-\text{sign}\{\lambda(t) B\}$

$$v^\circ(t) = \begin{cases} -1 & \lambda_2(t) > 0 \\ 1 & \lambda_2(t) < 0 \end{cases} \Rightarrow v^\circ(t) = -\text{sign}\{\lambda_2(t)\}$$

$$= -\text{sign}\{K \sin(\omega(t - t_i) + \alpha)\}$$




The control is $-\text{sign}(\sin\ldots)$ therefore it is set as in the figure, with switches at every $\frac{\pi}{\omega}$, exception for the first and the last $\le \frac{\pi}{\omega}$

It is useful to describe the problem in the phase plane $x_1 - x_2$:

Integrating the system:

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = -\omega x_1(t) + \underbrace{u(t)}_{\pm 1} \end{cases}$$

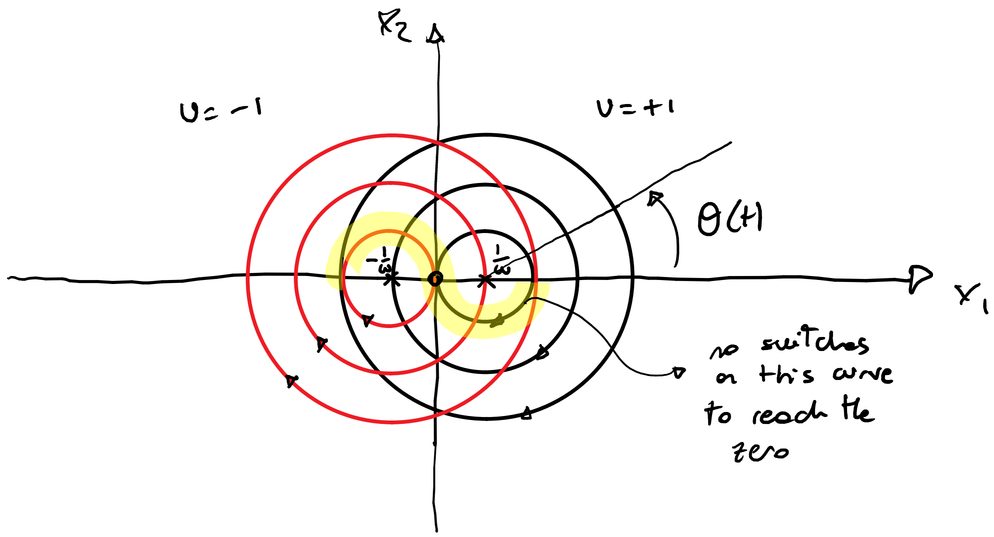
$$x_1(t) = \left(x_{1i} \mp \frac{1}{\omega}\right) \cos\omega(t-t_i) + x_{2i} \sin\omega(t-t_i) \pm \frac{1}{\omega}$$

$$x_2(t) = \left(x_{1i} \mp \frac{1}{\omega}\right) \sin\omega(t-t_i) + x_{2i} \cos\omega(t-t_i)$$

$$\Downarrow \quad \cos^2\alpha + \sin^2\alpha = 1$$

$$\left(x_1(t) \mp \frac{1}{\omega}\right) + x_2^2(t) = \left(x_{1i} \mp \frac{1}{\omega}\right)^2 + (x_{2i})^2$$

The optimal trajectory is described by circumferences → centered in $\left(\pm\frac{1}{\omega}, 0\right)$ passing through $x_i$

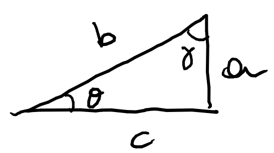



As time increase I go in the clockwise direction:

$\dot{x}_1(t) = \omega x_2(t)$ → if $x_2 > 0$, $\dot{x}_1 > 0$ ($x_1$ increasing)
if $x_2 < 0$, $\dot{x}_1 < 0$ ($x_1$ decreasing)

The angle $\theta(t)$ can be measured with trigonometry:

$$\theta(t) = tg^{-1}\left(\frac{x_2(t)}{x_1(t) \mp \frac{1}{\omega}}\right)$$




$\sin(\theta) = \frac{a}{b}$

$\cos(\theta) = \frac{c}{b}$

$\tan(\theta) = \frac{a}{c}$

$\dot{\theta}(t) = \ldots$ lot of calculations $\ldots = -\omega$ → constant

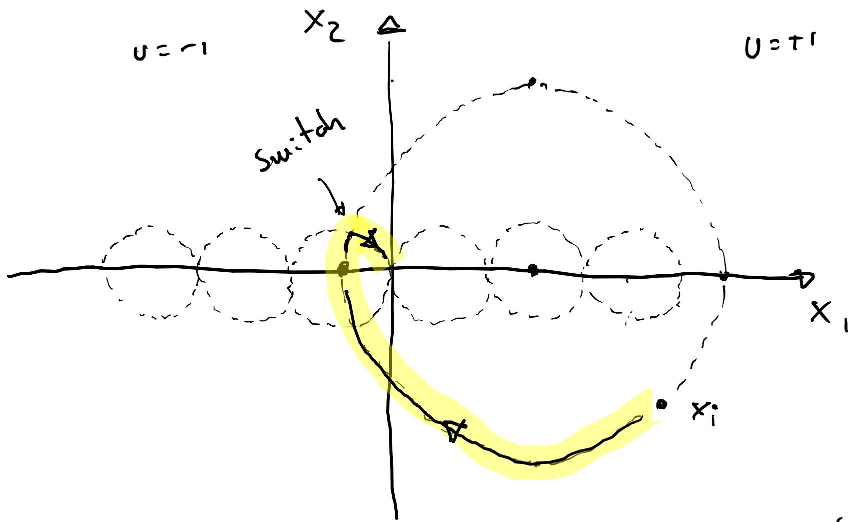
The trajectories are traversed with constant angular velocity

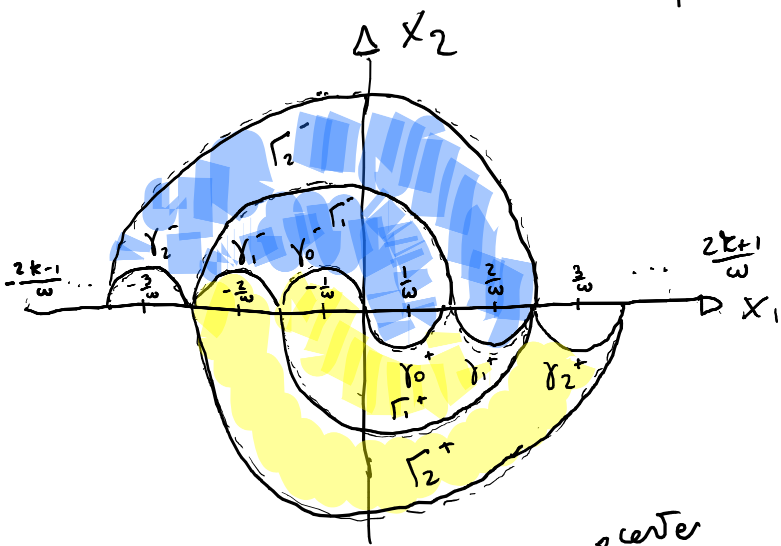
The time interval to move on an arc of length $\beta$ with $u = \pm 1$ is given by:

$$\boxed{\Delta t = \frac{\beta}{\omega}} \le \frac{\pi}{\omega} \Rightarrow \boxed{\beta \le \pi}$$

Therefore we must reach the origin along arcs with amplitude $\le \pi$

$\Rightarrow$ $\exists$ a unique way to do it




arc lenght must be $\beta \le \pi$

center    radius

$$\gamma_k^+ = \left\{ x \in \mathbb{R}^2 : \left(x_1 - \frac{2k+1}{\omega}\right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \le 0 \right\} \quad k = 0, \dots, n$$

$$\gamma_k^- = \left\{ x \in \mathbb{R}^2 : \left(x_1 + \frac{2k+1}{\omega}\right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \ge 0 \right\} \quad k = 0, \dots, n$$

$\Gamma_k^+$, $\Gamma_k^-$ are the colored sets obtained rotating $\gamma_k^+$ and $\gamma_k^-$ around $\left(\pm\frac{1}{\omega}, 0\right)$

## ~ Initial point

1) $x_i \in \gamma_0^+$ and $x_i \in \gamma_0^-$

Control $u = \pm 1$ without switches

2) $x_i \in \Gamma_1^+ \setminus (\gamma_0^+ \cup \gamma_0^-)$

First $u = +1$ to reach the curve $\gamma_0^-$ with $\beta \leq \pi$
then switch to $u = -1$ to reach the origin

3) $x_i \in \Gamma_1^- \setminus (\gamma_0^+ \cup \gamma_0^-)$

First $u = -1$ to reach the curve $\gamma_0^+$ with $\beta \leq \pi$
then switch to $u = +1$ to reach the origin

4) $x_i \in \Gamma_k^+ \setminus \gamma^-$ or $x_i \in \Gamma_k^- \setminus \gamma^+$

$$u^\circ(x^\circ(t)) = \begin{cases} 1 & \text{if } x^\circ(t) \in \Gamma^+ \setminus \gamma^- \\ -1 & \text{if } x^\circ(t) \in \Gamma^- \setminus \gamma^+ \end{cases}$$

The number of switches is given by the minimum index $k$ among the ones characterizing the sets $\Gamma_k^+$ and $\Gamma_k^-$

Example: $P \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^+$ → 1 switch

## ~ Minimum time

$$\begin{cases} \beta' = tg^{-1}\left(\dfrac{\omega x_{2i}}{1-\omega x_{1i}}\right) & \text{if } x_i \in \gamma_0^+ \\ \beta' = tg^{-1}\left(\dfrac{\omega x_{2i}}{1+\omega x_{1i}}\right) & \text{if } x_i \in \gamma_0^- \end{cases}$$

$\nu^\circ$ → number of commutations

$$(t_f^\circ - t_i) = \frac{\beta'}{\omega} + (\nu^\circ - 1)\frac{\pi}{\omega} + \frac{\beta''}{\omega} \Rightarrow (t_f^\circ - t_i) = \frac{\beta'}{\omega} \text{ if } \nu^\circ = 0$$

$\beta'$ rotation of the optimal trajectory from the initial point to the first point of commutation

$\beta''$ rotation to go from the final point to the origin

# LINEAR QUADRATIC GAUSSIAN PROBLEM (LQG)

We have a linear system with some noise and the cost index is very similar to the previous cases.

## ~ Optimal regulator with available state on finite time interval

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \quad , \quad t \in [t_i, t_f]$$

$w$ = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value

($x_i$, $w$ not correlated)

Covariance :

uncertainty

$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$ covariance (how much this this information is precise)

$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\,\delta(\tau)$

I want to minimize:

Bolza term

$$J(P) = \frac{1}{2} E\left\{\int_{t_i}^{t_f} [x^T Q x + u^T R u]\,dt + x^T(t_f) F x(t_f)\right\}$$

$\hookrightarrow$ expected value of the integral

In the previous cases our optimal control was

$u^\circ = -R^{-1} B^T K x^\circ$

In this case I have the noise but I can find a control as:

$u^\circ(t) = P(t)x(t)$ where $P \in C^1[t_i, t_f]$

$Q \geq 0$ , $F \geq 0$ , $R > 0$

Theorem: $\exists$ a unique solution

$$p^\circ(t) = -R^{-1} B^T(t) K(t)$$

with $K \geq 0$ solution of the Riccati equation:

$$\dot{K}(t) = -A^T(t) K(t) - K(t) A(t) + $$
$$+ K(t) B(t) R^{-1}(t) B^T(t) K(t) - Q(t)$$
$$K(t_f) = F$$

The optimal state found should be:

$$\dot{x}^\circ(t) = A(t) x^\circ(t) - B(t) R^{-1}(t) B^T(t) K(t) x^\circ(t) + w(t)$$
$$x^\circ(t_i) = x(t_i)$$

and the cost index has minimum value

$$J(p^\circ) = \frac{1}{2} x_i^T K(t_i) x_i +$$
$$+ \mathrm{Tr}\left\{ \int_{t_i}^{t_f} K(t) \Psi_w(t)\, dt + \frac{1}{2} K(t_i) \Psi_{x_i} \right.$$

Trace (sum of the elements on the diagonal of the matrix)

The theorem gives only the best linear solution of the stochastic regulator problem

It can be proved that the linear feedback law is optimal when the white noise is gaussian

~ Optimal regulator with state available and noise with non null mean value

Consider the linear system

$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \bar{w}(t) \qquad t \in [t_i, t_f]$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $(A\ B)$ controllable

$x(t_i) = x_i$ gaussian vector

$\bar{w}$ is a noise $\leadsto$ $\bar{w} = \overset{\text{white gaussian noise}}{w} + \mu(t)$

Covariance:

$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$ $\leftarrow$ uncertainty covariance (how much this this information is precise)

$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\,\delta(\tau)$

$E\{w(t)\} = \mu(t) \in C^0[t_i, t_f]$

↳ function (a systematic error always present)

$x_i$ and $w$ incorrelated.

In this case we cannot proceed with a control like $u = P(t)x(t)$ because it won't be probably a good choice because it was the best choice in the case of noise with 0 mean value, now we have to add something:

$u(t) = P(t)x(t) + q(t) \qquad P, q \in C^1[t_i, t_f]$

minimizing:

$$J(P) = \frac{1}{2} E\left\{ \int_{t_i}^{t_f} [x^T Q x + u^T R u]\, dt + x^T(t_f) F x(t_f) \right\}$$

$Q \geq 0$, $F \geq 0$, $R > 0$ with elements of $C^1$ class

Theorem: $\exists$ a unique solution:

$$p^\circ(t) = -R^{-1}(t)\, B^T(t)\, K(t)$$
$$q^\circ(t) = -R^{-1}(t)\, B^T(t)\, g(t)$$

$K \geq 0$ solution of the Riccati equation

$$\dot{K}(t) = -A^T K - KA + KBR^{-1}B^T K - Q$$
$$K(t_f) = F$$

$$\dot{g}(t) = -[A^T - KBR^{-1}B]\, g - K(t)\, \mu(t)$$
$$g(t_f) = 0$$

The cost index has minimum value:

$$J(p^\circ, q^\circ) = \frac{1}{2} x_i^T K(t_i)\, x_i + Tr\left\{ \int_{t_i}^{t_f} K(t) + \Psi_w(t)\, dt + \frac{1}{2} K(t_i)\, \Psi_{x_i} \right\}$$
$$+ x_i^T g(t_i) + h(t_i)$$

with $h$ unique solution of the differential equation

$$\dot{h}(t) = \frac{1}{2} g^T(t)\, B(t)\, R^{-1}(t)\, B^T(t)\, g(t) - g^T \mu(t)$$
$$h(t_f) = 0$$

} similar to tracking problem

~ Optimal regulator with available state on infinite time interval

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \quad , \; t \in [t_i, \infty]$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $(A\ B)$ controllable

$w$ = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value gaussian vector
($x_i$, $w$ not correlated)

Covariance:

$$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$$

$\Psi_{x_i}$ → uncertainty covariance (how much this information is precise)

$$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w \to \text{constant}$$

I want to minimize:

$$J(P) = \lim_{t_f \to \infty} \frac{1}{2(t_f - t_i)} E\left\{ \int_{t_i}^{t_f} [x^T Q x + u^T R u] \, dt \right\}$$

$Q > 0$, $R \geq 0$

The control has the form $u(t) = Px(t)$ → constant matrix

Theorem: $\exists$ a unique solution:

$$P^o = -R^{-1}B^T K_r \rightsquigarrow \text{solution of the RE}$$

$$A^T K_r + K_r A - K_r B R^{-1} B^T K_r + Q = 0$$

Therefore:

$$u^o(t) = -R^{-1}B^T K_r x^o(t)$$

$$\dot{x}^o(t) = [A - BR^{-1}B^T K_r] x^o(t) + w(t)$$

$$x^o(t_i) = x(t_i)$$

And the minimum value is: $J(P^o) = \mathrm{Tr}\{K_r \Psi_w\}$

~ Optimal gaussian linear tracking (state available)

$$\dot{x}(t) = A(t)\underbrace{x(t)}_{\in\mathbb{R}^n} + B(t)\underbrace{u(t)}_{\in\mathbb{R}^p} + w(t) \qquad A, B \in C^1 \qquad , t \in [t_i, t_f]$$

$w$ = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value

($x_i$, $w$ not correlated)

Covariance:

$$E\{[x(t_i)-x_i][x(t_i)-x_i]^T\} = \underbrace{\Psi_{x_i}}_{\text{uncertainty}} \quad \text{covariance}$$

(how much this this information is precise)

$$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\,\delta(\tau)$$

Now we have a <u>reference</u> whose dynamics is:

$$\dot{r}(t) = A_r(t)\,r(t) + \theta(t) \qquad t \in [t_i, t_f]$$

$r$, $\theta$, $w$, $x(t_i)$ incorrelated

Moreover:

$$E\{r(t_i)\} = r_i \qquad E\{[r(t_i)-r_i][r(t_i)-r_i]^T\} = \Psi_{ri}$$

$$E\{\theta(t)\} = m(t)$$

$$E\{[\theta(t)-m(t)][\theta(\tau)-m(\tau)]^T\} = \Psi_\theta(t)\,\delta(t-\tau)$$

We are looking for a control like

$$u(t) = P(t)\,x(t) + P_r(t)\,r(t) + q(t) \qquad P, P_r, q \in C^1[t_i, t_f]$$

that minimizes

$$J(P, P_r, q) = \frac{1}{2} E\left\{\int_{t_i}^{t_f} \left[(r(t)-x(t))^T Q(t)\,(r(t)-x(t_i)) + u^T(t)R(t)u(t)\right] dt\right.$$

$Q > 0$, $R > 0$ elements of $C^1$ class

Theorem: ∃ solution unique:

$$p^o(t) = -R^{-1}(t)\, B^T(t)\, K_{11}(t)$$
$$p_r^o(t) = -R^{-1}(t)\, B^T(t)\, K_{12}(t)$$
$$q^o(t) = -R^{-1}(t)\, B^T(t)\, g_1(t)$$

where

$$\left.\begin{aligned} \dot{K}_{11}(t) &= -A^T K_{11} - K_{11} A + K_{11} B R^{-1} B^T K_{11} - Q \\ K_{11}(t_f) &= 0 \end{aligned}\right\}$$ RE concerning $A, B, Q$ and $F = 0$

$$\left.\begin{aligned} \dot{K}_{12}(t) &= -A^T K_{12} - K_{12} A + K_{11} B R^{-1} B^T K_{12} + Q \\ K_{12}(t_f) &= 0 \end{aligned}\right\}$$ Not an RE

$$\dot{g}_1(t) = -\left[A^T - K_{11} B R^{-1} B^T\right] g_1 - K_{12}\, m$$
$$g_1(t_f) = 0$$

The problem admits a unique optimal solution:

$$u^o = P_x^o x^o + P_r^o r + q^o = -R^{-1} B^T \left(K_{11} x^o + K_{12} r + g_1\right)$$