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Nonlinear Systems & Control
Part II
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Student: ____________________

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1. Given

$$\dot{x} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} x$$

state whether it is possible to achieve input/output noninteraction under feedback.

2. Given the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_3 + x_1^2 + \cos(x_2) d \\ \dot{x}_3 &= -x_3 + x_1^2 + (1 + x_1^2) u \\ y &= x_1 \end{aligned}$$

where d denotes a measurable and bounded disturbance $d \in \mathbb{R}$.

(a) Assuming at first $d \equiv 0$
   i. Compute the feedback that solves the input-output linearization problem;
   ii. Compute the zero dynamics;
   iii. Discuss on the stabilizability of the closed-loop system.

(b) When $d \neq 0$
   i. Compute, if any, the feedback solving the output-disturbance decoupling problem;
   ii. Qualitatively discuss on the boundedness of the internal dynamics of the closed-loop system.

3. Given a system $\dot{x} = f(x) + g(x)u$ discuss and provide condition for deducing the maximal input-to-state feedback linearizable component.

4. Compute, if any, a stabilizing high gain feedback for the origin of the dynamics

$$\begin{aligned} \dot{x}_1 &= -x_1 - \cos(x_1) x_2 \\ \dot{x}_2 &= x_1 - e^{-x_2} u \\ y &= x_2 \end{aligned}$$

5. Given a input-output feedback linearizable system in input-affine form, discuss about the design of an asymptotic tracking feedback over a bounded reference output $y_r(t)$.

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1 State wether it is possible to achieve I/O non interaction under feedback

$$\dot{x} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u \qquad p = q = 2$$

$$y = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} x$$

Rel. deg.:

$$r_1 : \; c_1 B_1 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \neq 0 \qquad r_1 = 1$$

$$r_2 : \; c_2 B_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \neq 0 \qquad r_2 = 1$$

[illegible]

$$U = \begin{pmatrix} C_1B_1 & C_1B_2 \\ C_2B_1 & C_2B_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\det U = 1 - 1 = 0 \rightarrow U$ not invertible

$u = Fx + Gv$

$G = U^{-1}$

$F = -U^{-1}\begin{pmatrix} C_1A \\ C_2A \end{pmatrix}$ $\qquad u = U^{-1}\left(v - \begin{pmatrix} C_1A^{r_1} \\ C_2A^{r_2} \end{pmatrix}x\right) = U^{-1}\left(v - \begin{pmatrix} C_1A \\ C_2A \end{pmatrix}x\right)$

Since U is not invertible I try to put it in $\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}$ form

$$U' = U \cdot M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ ok}$$

$a + c = *$

$b + d = 0$

$a + c = * \rightarrow a = * - c$ $\qquad * = 1$

$b + d = 0 \rightarrow b = -d$

$u = Mv$

$Bu = BMv$

$$BM = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\dot{x} = Ax + BMv = Ax + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \dot{v}_1 = \omega_1, \quad \dot{v}_2 = \omega_2$$

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{v}_1 \end{pmatrix} = \left(\begin{array}{ccc|c} & A & & \begin{matrix}1\\0\\0\end{matrix} \\ \hline 0 & 0 & 0 & 0 \end{array}\right) z + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{b_1}\omega_1 + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{b_2}\omega_2$$

$$y = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} z \qquad \tilde{A} = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rel. deg.

$r_1 \Rightarrow c_1 b_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$

$r_1 \Rightarrow c_1 b_1 = (1\ 0\ -1\ 0)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 0$

$c_1 b_2 = (1\ 0\ 0\ 0)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 0$

$c_1 \bar{A} b_1 = (-2\ 2\ 1\ 1)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 1 \neq 0$ $\quad r_1 = 2$

$r_2 \Rightarrow c_2 \bar{A} b_1 = (1\ 0\ 0\ 0)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1 \neq 0$ $\quad r_2 = 2$

I compute also $c_1 \bar{A} b_2 = (-2\ 2\ 1\ 1)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = 2$

and $c_2 \bar{A} b_2 = (1\ 0\ 0\ 0)\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = 1$

therefore

$$U = \begin{pmatrix} c_1 \bar{A} b_1 & c_1 \bar{A} b_2 \\ c_2 \bar{A} b_1 & c_2 \bar{A} b_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$\det U = -1 \neq 0 \rightarrow U$ invertible

$$U^{-1} = -1 \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$w = Fz + G(t) \rightsquigarrow$ new input

$$w = U^{-1}\left(t - \begin{pmatrix} c_1 A^2 \\ c_2 A^2 \end{pmatrix} z\right)$$

$$A^2 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -2 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$c_1 A^2 = (1\ 0\ -1\ 0) \cdot A^2 = (3\ \ -3\ \ -4\ \ -2)$

$c_2 \cdot A^2 = (1\ 0\ 0\ 0) \cdot A^2 = (2\ \ -2\ \ -2\ \ -1)$

the resulting sys is:

$$\begin{cases} \dot{\xi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t_1 \end{cases}$$

$$\begin{cases} \dot{\xi} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t_1 \\ \dot{\eta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t_2 \\ y_1 = \xi_1 \\ y_2 = \eta_1 \end{cases}$$

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② $$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_3 + x_1^2 + \cos(x_2) d \\ \dot{x}_3 = -x_3 + x_1^2 + (1+x_1^2) u \\ y = x_1 \end{cases}$$

d: measurable bounded $d \in \mathbb{R}$

2.1 Assuming $d \equiv 0$

- Compute the fb that solves u-y FL

rel. deg.

$r=1$ $$L_g h = \underbrace{(1 \quad 0 \quad 0)}_{\frac{\partial h}{\partial x}} \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix}}_{g} = 0$$

$r=2$ $$L_f h = \frac{\partial h}{\partial x} \cdot f = (1 \quad 0 \quad 0) \begin{pmatrix} x_3 \\ x_3 + x_1^2 \\ -x_3 + x_1^2 \end{pmatrix} = x_3$$

$$L_g L_f h = \frac{\partial L_f h}{\partial x} \cdot g$$
$$= (0 \quad 0 \quad 1) \begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix} = 1 + x_1^2 \neq 0$$

Coord. transf.

$$\begin{pmatrix} z \\ \hline \eta \end{pmatrix} \begin{matrix} \} r=2 \\ \} n-r=1 \end{matrix} \to \begin{pmatrix} z_1 \\ z_2 \\ \eta \end{pmatrix}$$

$z_1 = h = x_1$

$z_2 = L_f h = x_3$

η s.t. $\nabla \varphi \cdot g = 0$

η s.t. $\nabla\overset{0}{\varphi} \cdot \varepsilon = 0$

$$\begin{pmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix} = 0$$

$$\frac{\partial \varphi}{\partial x_3} + \frac{\partial \varphi}{\partial x_3} x_1^2 = 0$$

$\eta = x_2$

$$\begin{cases} \dot{z}_1 = \dot{x}_1 = x_3 = z_2 \\ \dot{z}_2 = \dot{x}_3 = -x_3 + x_1^2 + (1+x_1^2)u = -z_2 + z_1^2 + (1+z_1^2)u \\ \dot{\eta} = \dot{x}_2 = x_3 + x_1^2 = z_2 + z_1^2 \end{cases}$$

Having $u = \frac{(z_2 - z_1^2 + v)}{(1+z_1^2)}$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \\ \dot{\eta} = z_2 + z_1^2 \end{cases}$$

- Zero dynamics

$Z = \{ x \in \mathbb{R}^n \text{ s.t } y(t_0) = 0 \ \ u^*(x) \text{ s.t. } y(t) \equiv 0 \}$

$\quad = \{ x \in \mathbb{R}^n \text{ such that } x_1 = 0 \}$

$z = 0 \quad v = 0$

$\dot{\eta} = z_2 + z_1^2 = 0$

$Q = \left. \frac{\partial q(0,\eta)}{\partial \eta} \right|_0 = 0$ No zero dynamics

- Stabilizability of the closed loop sys

$\dot{\eta} = 0$ is stable since η is a constant

(2.2) Assuming $d \neq 0$

Compute if and the output solving the

(2.c) Assuming $d \neq 0$

Compute if any the output solving the output DDP problem

It's assumed that d is a bounded disturbance.
Remember that u found in case $d=0$ is the fb that decouples the output from the disturbance if $r_u < r_w$ and

$$L_p h = 0$$
$$\vdots$$
$$L_p^{r-1} h = 0$$

$y = x_1$, $\dot{y} = x_3$, $\ddot{y} = x_3 + x_1^2 + (1+x_1^2)u$, so $\boxed{r_u = 2}$

since there's no w in the last derivative it's obvious that $r_u < r_w$

The DDP problem admits a solution

From $\ddot{y}$ I get $u = \frac{v - x_1^2 - x_3}{1+x_1^2}$

with this fb you just decouple the effect of the disturbance on the output.

So I set $v = -k_1 h - k_2 L_f h = -k_1 x_1 - k_2 x_3$

Discuss on the boundedness of the internal dynamics of the closed loop system

A necessary condition for boundedness is asymptotic stability.

Since $\dot{\eta} = \cos(\eta) d$ $\left(= \dot{x}_2 = z_1^2 + z_2 + \cos(\eta) d \big|_{z=0} = \cos(\eta) d\right)$ is the zero dynamics, which is unstable, nothing can be concluded.

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(4) Stabilize if possible via high gain fb

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 \cos(x_1) \\ \dot{x}_2 = x_1 - e^{-x_2} u \\ y = x_2 \end{cases}$$

$$f = \begin{pmatrix} -x_1 - x_2 \cos(x_1) \\ x_1 \end{pmatrix}$$

$g = \Big(0 \quad \Big)$

$y = x_2$

$$g = \begin{pmatrix} 0 \\ -e^{-x_2} \end{pmatrix}$$

$h = x_2 \quad \frac{\partial h}{\partial x} = (0 \;\; 1)$

equilibrium at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Relative deg.

$r = 1 \quad L_g h = \frac{\partial h}{\partial x} \cdot g = (0 \;\; 1) \begin{pmatrix} 0 \\ -e^{-x_2} \end{pmatrix} = -e^{-x_2} \neq 0$

then

$u = -k(c_0 h + \dots + L_g^{r-1} h) \quad r > 1$ where $(c_0, \dots, c_{r-2})$ Hurwitz

$u = -k h \,\text{sign}(L_g h) \Rightarrow$ a fb of this kind if $r = 1$ stabilizes the sys:

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

In this case $\text{sign}(-\overbrace{e^{-x_2}}^{pos}) = \text{sign}\left(-\frac{1}{e^{x_2}}\right) = -1$

So $u = +kh$ with k big enough

$$\bar{f} = \begin{pmatrix} -x_1 - x_2 \cos(x_1) \\ x_1 - k e^{-x_2} x_2 \end{pmatrix}$$

LTM:

$$\bar{A} = \frac{\partial \bar{f}}{\partial x}\Big|_{x=0} = \begin{pmatrix} -1 + x_2 \sin(x_1) & -\cos(x_1) \\ 1 & +k e^{-x_2} x_2 - k e^{-x_2} \end{pmatrix}\Big|_{x=0} = \begin{pmatrix} -1 & -1 \\ 1 & -k \end{pmatrix}$$

$$|\bar{A} - \lambda I| = \begin{pmatrix} -1-\lambda & -1 \\ 1 & -k-\lambda \end{pmatrix} = (-1-\lambda)(-k-\lambda) + 1 = 0$$

$$= +k + \lambda + \lambda k + \lambda^2 + 1 = 0$$

$$\lambda^2 + \lambda(k+1) + k + 1 = 0$$

$$\lambda_{1,2} = \frac{-(k+1) \pm \sqrt{(k+1)^2 - 4(k+1)}}{2}$$

$$= \frac{-k-1}{2} \pm \frac{\sqrt{k^2+1+2k-4k-4}}{2}$$

with $k = 10$ (for instance)

$\lambda_1 = -1.11$

$\lambda_2 = -9.88$

So non linear sys is AS under high gain fb