14. Local coordinates transformation

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$$S: \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad f, g \text{ smooth vector field}$$

r is the relative degree of S at point x_0 if

$L_g L_f^k h(x) = 0 \quad \forall x$ in a neighborhood of x_0, $k < r-1$

$L_g L_f^{r-1} h(x) \neq 0$

There may be points where a relative degree is not defined. In fact, remembering that r is exactly the number of times one has to differentiate the output at time $t = t_0$ in order to have $u(t_0)$ explicitly appearing:

$$y(t_0) = h(x(t_0)) = h(x_0)$$

$$y^{(1)}(t) = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}\left(f(x(t)) + g(x(t))u(t)\right) =$$

$$= L_f h(x(t)) + \cancel{L_g h(x(t)) u(t)} = L_f h(x(t))$$

$$y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial L_f h}{\partial x}\left(f(x(t)) + g(x(t))u(t)\right) =$$

$$= L_f^2 h(x(t)) + \cancel{L_g L_f h(x(t)) u(t)} = L_f^2 h(x(t))$$

$$y^{(k)}(t) = L_f^k h(x(t)) \quad \forall k < r, \ t \text{ near } t_0$$

$$y^{(r)}(t) = L_f^r h(x_0(t)) + L_g L_f^{r-1} h(x_0) u(t)$$

if $r = \infty$, the Taylor expansion of $y(t)$ at the point $t = t_0$ has the form:

$$y(t) = \sum_{k=0}^{\infty} L_f^k h(x_0) \frac{(t-t_0)^k}{k!}$$

and it has a not defined relative degree, then $y(t)$ does not depend on u.

Proposition: Suppose that the system has relative degree $r \leq n$, and set

$\phi_1(x) = h(x)$

$\phi_2(x) = L_f h(x)$

$\vdots$

$\phi_r(x) = L_g^{r-1} h(x)$

if $r < n$ (strict) it is possible to find $n-r$ more functions $\phi_{r+1}(x) \dots \phi_n(x)$ such that the mapping

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \dots \\ \phi_n(x) \end{pmatrix}$$

has Jacobian non singular at x_0 and therefore qualifies as a local coordinates transformation in a neighborhood of x_0.

It is always possible to choose $\phi_{r+1}(x) \dots \phi_n(x)$ in such a way that $L_g \phi_i(x) = 0 \quad \forall r+1 < i < n$ and all x around x_0.

$\phi_1(x), \dots, \phi_r(x)$ are linearly independent.

Defining the new coordinates $\begin{pmatrix} z \\ \eta \end{pmatrix}$ such that:

$$\begin{pmatrix} z \\ \eta \end{pmatrix} = \begin{pmatrix} \left.\begin{matrix} h(x) \\ \vdots \\ L_g^{r-1} h(x) \end{matrix}\right\} r \\ \hline \left.\begin{matrix} \phi_{r+1}(x) \\ \vdots \\ \phi_n(x) \end{matrix}\right\} n-r \end{pmatrix}$$

Lemma: Given S, r defined, then:

$$\begin{pmatrix} h \\ L_g h \\ \vdots \\ L_g^{r-1} h \end{pmatrix} \text{ are independent } \Longleftrightarrow \rho \begin{pmatrix} dh \\ dL_g h \\ \vdots \\ dL_g^{r-1} h \end{pmatrix} = r$$

The system in the new coordinates:

$$\dot z_1 = \frac{\partial \phi_1}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} = L_g h(x(t)) = \phi_2(x) = z_2(t) = \dot y$$

$$\dots \qquad L_g^2 h(x(t)) = \phi_3(x) = z_3(t) = \ddot y$$

$$\dot z_{r-1} = \frac{\partial \phi_{r-1}}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial L_g^{r-2} h}{\partial x}\frac{\partial x}{\partial t} = L_g^{r-1} h(x(t)) = \phi_r(x(t)) = z_r(t)$$

$$\frac{\partial z_r}{\partial t} = L_g^r h(x(t)) + L_g L_g^{r-1} h(x(t))\, u(t)$$

$$\dot z_r = L_g^r h(x(t)) + L_g L_g^{r-1} h(x(t))\, u(t)$$

$$a(z) = L_g L_g^{r-1} h(\phi^{-1}(z))$$

$$b(z) = L_g^r h(\phi^{-1}(z))$$

$$b(z) = L_f^r h(\phi^{-1}(z))$$

$$\dot z_r = b(z(t)) + a(z(t))\, u(t)$$

where $a(z)$ is non zero for all z near z_0

if $\phi_{r+1}(x) \dots, \phi_n(x)$ have been chosen in such a way that
$L_g \phi_i(x) = 0$
then

$$\dot\eta_i = \frac{\partial \phi_i}{\partial x}\left(f(x(t)) + g(x(t))\, u(t)\right) = L_f \phi_i(x(t)) + L_g \phi_i(x(t))\, u(t) =$$
$$= L_f \phi_i(x(t)) \quad \forall\, r+1 \le i \le n$$

setting $q_i(z) = L_f \phi_i(\phi^{-1}(z))$ it is possible to rewrite

$$\dot z_i = q_i(z(t)) \quad r+1 \le i \le n$$

and the system in the new coordinates becomes

$$\dot z_1 = z_2$$
$$\dot z_2 = z_3$$
$$\dots$$
$$\dot z_{r+1} = z_r$$
$$\dot z_r = b(z) + a(z) u$$
$$\dot z_{r+1} = q_{r+1}(z)$$
$$\dots$$
$$\dot z_n = q_n(z)$$
$$y = z_1$$

This is the strict normal form.

It is not always easy to find $n-r$ functions such that $L_g \phi_1(x) = 0$. It is much simpler to find $\phi_{r+1}, \dots, \phi_n$ satisfying $\det(J_\phi(x_0)) \neq 0$.

So the last $n-r$ appear

$$\dot z_{r-1} = q_{r+1}(z) + p_{r+1}(z) u$$
$$\dots$$
$$\dot z_n = q_n(z) + p_n(z) u$$

Note a particular structure:

$\begin{pmatrix} dh \\ dL_f h \end{pmatrix} \dots L_f^{r-1} \dots \quad \begin{bmatrix} 0 & \dots & 0 & L_g L_f^{r-1} h \end{bmatrix}$

$$\begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{r-1} h \end{pmatrix} \begin{pmatrix} g & ad_f g & \cdots & ad_f^{r-1} g \end{pmatrix} = \begin{bmatrix} 0 & \cdots & 0 & L_g L_f^{r-1} h \\ \vdots & & \iddots & \\ 0 & \iddots & & \\ L_g L_f^{r-1} h & & & \# \end{bmatrix}$$

where $ad_f g = [f, g]$, $ad_f^2 g = ad_f\, ad_f g = [f, ad_f g] = [f, [f, g]]$

$ad_f^k g = [f, [f, [f, \cdots [f, g] \cdots]$