(at least local)

Zero dynamics can be helpful in dealing with the problem of asymptotically stabilizing a nonlinear system at a given equilibrium point x_e.

Suppose as usual

$$\dot{x} = f(x) + g(x)u \qquad f(0) = 0 \qquad x_e = 0$$

the problem consists on finding $u = \alpha(x)$ such that

$\dot{x} = f(x) + g(x)\alpha(x) = \tilde{f}(x)$ has $x_e = 0$ as an AS equilibrium

If LTM is stabilizable then the problem can be solved by the linear feedback $\alpha(x) = Fx$, the same which stabilizes the LTM:

$$A = \left.\frac{\partial f}{\partial x}\right|_0 \qquad B = g(0) \qquad (A, B) \text{ stabilizable}$$

$$\Rightarrow \exists F : \sigma(A+BF) \subset \mathbb{C}^-$$

therefore $\dot{x} = f + g\underbrace{Fx}_{\alpha(x)} = \tilde{f}$

because of the indirect Lyapunov method:

$$\tilde{A} = \left.\frac{\partial \tilde{f}}{\partial x}\right|_0 = \left.\frac{\partial f}{\partial x}\right|_0 + g(0)F = A + BF$$

$\tilde{f}$ is locally asymptotically stable.

Prop: If $\exists$ uncontrollable modes of the LTM which are unstable ($\operatorname{Re}(\lambda) > 0$) a solution does not exist, both in linear and non linear problem.

Proof: by contradiction, suppose $\exists \alpha$:

$f + g\alpha$ <u>AS</u>

then $\left.\frac{\partial f}{\partial x}\right|_0 + g(0)\frac{\partial \alpha}{\partial x} = (A + BF)x$

Unrespectively from the feedback α

$\sigma(A+BF) \cap \mathbb{C}^+ \neq 0$, therefore α cannot stabilize.

Eigenvalues with 0 real part

This is a critical problem and zero dynamics is useful.

Consider

$$\dot z_1 = z_2$$
$$\dot z_2 = z_3$$
$$\vdots$$
$$\dot z_{r-1} = z_r$$
$$\dot z_r = b(z,\eta) + a(z,\eta)u$$
$$\dot\eta = q(z,\eta)$$

And suppose $(z,\eta)=(0,0)$ is an equilibrium point.
With a feedback in the form

$$u = \frac{1}{a(z,\eta)}\left(-b(z,\eta) - c_0 z_1 - c_1 z_1 - \dots - c_{r-1} z_r\right)$$

where $c_0, c_1, \dots, c_{r-1}$ are real numbers, the CL system is

$$\dot z = Az$$
$$\dot\eta = q(z,\eta)$$

with

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ | & & \ddots & \\ 0 & & & 1 \\ -c_0 & -c_1 & \dots & -c_{r-1} \end{pmatrix} \qquad \sigma(A) \subset \mathbb{C}^-$$

The corresponding LTM is:

$$\begin{cases} \dot z = Az \\ \dot\eta = Pz + Q\eta \end{cases} \qquad P = \frac{\partial q}{\partial z}\Big|_0 \qquad Q = \frac{\partial q}{\partial \eta}\Big|_0$$

analyzing the stability of the zero dynamics:

(1) $\sigma(Q) \subset \mathbb{C}^- \implies Q = \dfrac{\partial q(0,\eta)}{\partial\eta}\Big|_0$
and $\dot\eta = Q\eta$ is the LTM of the zero dynamics, which is stable in its first approximation from the indirect Lyapunov method, and the stability of the equilibrium of the NL dynamics is exponentially stable.
In fact the LTM is LAS:

$$\underset{\substack{\downarrow \\ \text{LTM}}}{\bar A} = \begin{pmatrix} A & 0 \\ P & Q \end{pmatrix} \qquad \sigma(\bar A) \subset \mathbb{C}^-$$

If the zero dynamics is GAS and locally exponentially stable, the feedback $\alpha(x)$ which linearizes and stabilizes the linear input-output link, also globally asymptotically stabilizes the origin of the and stabilizes the linear input-output link, also globally asymptotically stabilizes the origin of the state space.

Zero Dynamics is AS

Suppose the equilibrium $\eta = 0$ of the zero dynamics of the system is LAS and all roots of the characteristic polynomial have negative real part.

Theorem: U-Y FL can be used to get (L)AS of the equilibrium $x=0$ if the system has an AS ZD.

$$\begin{cases} \dot{z} = Az + p(z,\eta) \\ \dot{\eta} = q(z,\eta) \end{cases}$$

$$\begin{cases} p(z,0)=0 \quad \text{around } (0,0) \text{ for } z \text{ small} \\ \left.\frac{\partial p}{\partial z}\right|_{(0,0)} = 0 \\ \sigma(A) \subset \mathbb{C}^- \\ \dot{\eta} = q(0,\eta) \quad \eta_e = 0 \end{cases}$$

if the above conditions hold, $(0,0)$ is an AS equilibrium for the system.

proof makes use of the Center Manifold theorem.

$$q(z,\eta) = Q\eta + Pz + g(z,\eta)$$

and perform a coordinate transformation such that

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = T\eta + Kz \quad \text{and}$$

$$\begin{cases} \dot{\eta}_1 = Q_1\eta_1 + g_1(\eta_1, \eta_2, z) \\ \dot{\eta}_2 = Q_2\eta_2 + P_2 z + g_2(\eta_1,\eta_2,z) \\ \dot{z} = Az + p(\eta_1,\eta_2,z) \end{cases}$$

$\sigma(Q_2) \subset \mathbb{C}^-$ $\quad\sigma(Q_1) \in$ Im axis

g_1 and g_2 vanishing at $(0,0)$ with their first order partial derivatives.

By assumption $(\eta_1, \eta_2) = (0,0)$ is AS

$$\begin{cases} \dot{\eta}_1 = Q_1\eta_1 + g_1(\eta_1,\eta_2,z) \end{cases}$$

$$\begin{cases} \dot{\eta}_1 = Q_1 \eta_1 + g_1(\eta_1, \eta_2, z) \\ \dot{\eta}_2 = Q_2 \eta_2 + g_2(\eta_1, \eta_2, z) \end{cases}$$

- Let $h_2(\eta_1) = \eta_2$ be a Center Manifold at $(0,0)$.

$$\frac{\partial h_2}{\partial \eta_1}\left(Q\eta_1 + g_1(\eta_1, \eta_2, 0)\right) = Q_2 h_2(\eta_1) + g_2(\eta_1, h_2(\eta_1), 0)$$

and for the reduction principle
$\dot{x} = Q_1 + g_1(x, h_2(x), 0)$ has necessarily an AS equilibrium at $x=0$.

- Now consider the full system and compute a Center Manifold for it.

$\eta_2 = k_2(\eta_1) \qquad z = k_1(\eta_1)$

$$\frac{\partial k_2}{\partial \eta_1}\left(Q_1\eta_1 + g_1(\eta_1, k_2(\eta_1), k_1(\eta_1))\right) = \\ = Q_2 k_2(\eta_1) + P_2 k_1(\eta_1) + g_2(\eta_1, k_2(\eta_1), k_1(\eta_1))$$

$$\frac{\partial k_1}{\partial \eta_1}\left(Q_1\eta_1 + g_1(\eta_1, k_2(\eta_1), k_1(\eta_1))\right) = \\ = A k_1(\eta_1) + p(\eta_1, k_2(\eta_1), k_1(\eta_1))$$

such expressions are solved by
$k_2(\eta_1) = h_2(\eta_1)$ and $k_1(\eta_1) = 0$

the second of the previous equations annihilates and by using the reduction principle the stability of $(0,0)$ in the system reduces to the stability of $x=0$ in
$\dot{x} = Q_1 x + g_1(x, h_2(x), 0)$, i.e. the reduced dynamics of $\dot{\eta} = q(0, \eta)$ which is A.S.

Stabilization through FL can be used also to overcome the obstructions linked to the presence of not controllable modes with zero real part.

The feedback law

$$v = \frac{1}{a(z,\eta)}\left(-b(z,\eta) - c_0 z_1 - c_1 z_2 - \dots - c_{r-1} z_r\right)$$

Locally asymptotically stabilizes the equilibrium $(z,\eta) = (0,0)$

Locally asymptotically stabilizes the equilibrium $(z,\eta)=(0,0)$
In the original coordinates

$$u=\frac{1}{L_g L_f^{r-1} h(x)}\left(-L_f^r h(x) - c_0 h(x) - c_1 L_f h(x) - \dots - c_{r-1} L_f^{r-1} h(x)\right)$$

Prop: the eigenvalues associated to the not controllable modes of a LTM, if any, belong to $\sigma(Q)$

Proof: look at the LTM of the Normal form

$$\begin{cases} \dot z_1 = z_2 \\ \vdots \\ \dot z_r = Rz + S\eta + Ku \\ \dot\eta = Pz + Q\eta \end{cases} \qquad R=\left.\frac{\partial b}{\partial z}\right|_{0,0} \quad S=\left.\frac{\partial a}{\partial \eta}\right|_{0,0} \quad K=a(0,0) \qquad P=\left.\frac{\partial q}{\partial z}\right|_{0,0} \quad Q=\left.\frac{\partial q}{\partial \eta}\right|_{0,0}$$

From the Hautus criterion:
the mode associated to λ^* is not controllable iff:
$\rho(\lambda^* I - A \,\vdots\, B) < n$

defining the Hautus matrix

$$\begin{array}{c} r\left\{ \begin{array}{c} \\ \\ \\ \end{array}\right. \\ n-r\left\{ \begin{array}{c} \\ \\ \end{array}\right. \end{array}
\left(\begin{array}{ccc|c|c}
\lambda^* & -1 & 0 & 0 & 0 \\
0 & \lambda^* & -1 & 0 & 0 \\
-r_1 & \cdots & -r_r & -S & K\neq 0 \\ \hline
 & P & & \lambda I - Q & 0 \\
 & & & & 0
\end{array}\right)$$

$$\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{r}\ \underbrace{\qquad}_{n-r}$$

the only possibility for the Hautus matrix to have the full rank is that λ^* is an eigenvalue of Q

Remark: if any output function is not defined ZD is not defined too. One may be able to design a dummy output whose associated ZD has AS equilibrium