THE LINEAR QUADRATIC REGULATOR

~ Finite time interval

In calculus of variations and optimal control we had

$$\begin{cases} \dot{x} = f(x,u,t) \\ x(t_i) = x_i \text{ fixed} \end{cases}$$

$$J(x,u) = \int_{t_i}^{t_f} L(x,u,t)\,dt + G(x(t_f))$$

with the regulator problem we have now

$$\left.\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t) \\ y(t) &= C(t)\,x(t) \end{aligned}\right\} \text{linear system}$$

with $[t_i, t_f]$ fixed and $x(t_i) = x_i$ fixed.

$\underset{n\times n}{A(t)}, \underset{n\times m}{B(t)}, \underset{n\times p}{C(t)}$ matrices function of time

The linear control law is obtained minimizing the <u>Quadratic performance index</u>:

$$J(x,u) = \frac{1}{2}\int_{t_i}^{t_f}\left(x^T Q x + u^T R u\right)dt + \frac{1}{2}x^T(t_f)\,F\,x(t_f)$$

$\underset{\text{non neg}}{Q(t) \geq 0}$, $\underset{\text{pos def}}{R(t) > 0}$, $\underset{\text{non neg}}{F(t) \geq 0}$ all symmetric

To the regulator problem is associated the <u>Riccati equation</u>

$$\begin{cases} \dot{K}(t) = K B R^{-1} B^T K - KA - A^T K - Q \\ K(t_f) = F \end{cases}$$

The Riccati equation admits always a unique solution positive semidefinite in the control interval

<u>Theorem</u> : The optimal control for the regulator problem is given by the linear feedback law

$$u^\circ(t) = -R^{-1}B^T K \, x^\circ$$

where

$$\begin{aligned} \dot{x}^\circ(t) &= A x^\circ + B u^\circ \\ &= A x^\circ - B R^{-1} B^T K x^\circ \\ &= \left[A - B R^{-1} B^T K\right] x^\circ \end{aligned}$$

$$x^\circ(t_i) = x_i$$

and

$$J(x^\circ, u^\circ) = \frac{1}{2} x_i^T K(t_i) x_i$$

The solution of the Riccati equation is symmetric and can be found only numerically

Remark : matrix A is limited

1) The existence and uniqueness theorem may be applied only locally. Therefore $\exists \; (t, t_f]$ in which the Riccati equation admits a unique solution

2) It must be shown that $\nexists$ an element k_{ij} of the matrix K such that for any $t_1 < t_f$:

$$\lim_{t \to t_1^+} |k_{ij}(t)| = \infty$$

Proof

- Condition 1

We apply the results of the Pontryagin principle to our linear sys.

Hamiltonian: $H(x,u,\lambda,t) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + \lambda^T A x + \lambda^T B u$

normal case → λ_0 cannot be 0 because we don't have any condition on the final instant

e.g. If you don't have the Bolza term $\lambda(t_f) = 0$ so $\lambda_0 \neq 0$ itself, if there is the Bolza term we use the transversality conditions

The necessary & sufficient conditions are

$$\begin{cases} 1) \quad \dot{\lambda}^\circ = -\left.\frac{\partial H}{\partial x}\right|^{\circ T} = -Qx^\circ - A^T\lambda \\ 2) \quad \left.\frac{\partial H}{\partial u}\right|^{\circ} = Ru^\circ + B^T\lambda^\circ = 0 \\ 3) \quad \lambda^\circ(t_f) = \left.\frac{\partial G}{\partial x(t_f)}\right|^{\circ T} \end{cases}$$

Taking into account the RE (→ Riccati eq.) and defining the costate proportional to the state as:

$\boxed{\lambda^\circ = K x^\circ}$ <u>the result is proved!!</u>

In fact, checking:

$$1) \quad \dot{\lambda}^\circ = \dot{K}x^\circ + K\dot{x}^\circ = [KBR^{-1}B^TK - KA - A^TK - Q]x^\circ + K[Ax + B\overset{-R^{-1}B^T\lambda}{u}]$$

$$= KBR^{-1}B^TKx^\circ - KAx^\circ - A^TKx^\circ - Qx^\circ + KAx^\circ - KBR^{-1}B^TKx^\circ$$

$$= -A^TKx^\circ - Qx^\circ = -A^T\lambda^\circ - Qx^\circ \quad ✓$$

2) $R v^\circ + B^T \lambda^\circ = R(-R^{-1}B^T K x^\circ) + B^T \lambda^\circ =$

$= -B^T K x^\circ + B^T \lambda^\circ = 0$ ✓

3) $\lambda^\circ(t_f) = K(t_f^\circ)\, x^\circ(t_f^\circ) = F x^\circ(t_f^\circ)$ ✓

4) Value of the cost index in the minimum point:

$$J(x^\circ, v^\circ) = \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) + \frac{1}{2}\int_{t_i}^{t_f} x^{\circ T}(Q + K B R^{-1} B^T K) x^\circ dt$$

$$\frac{1}{2}\int_{t_i}^{t_f} x^T Q x + v^T R v$$

$$v^\circ = -R^{-1}B^T K x^\circ$$

I substitute Q from the Riccati eq.

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) + \frac{1}{2}\int_{t_i}^{t_f} x^{\circ T}(\underbrace{-\dot{K} + 2KBR^{-1}B^T K - KA - A^T K}_{x^{\circ T} K \dot{x}^\circ \qquad x^{\circ T}\dot{K} x^\circ}) x^\circ dt$$

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) - \frac{1}{2}\int_{t_i}^{t_f} (2x^{\circ T} K \dot{x}^\circ + x^{\circ T}\dot{K} x^\circ) dt$$

$$= \frac{1}{2} x^{\circ T}(t_f^\circ) F x^\circ(t_f^\circ) - \frac{1}{2}\int_{t_i}^{t_f} \frac{d}{dt}(x^{\circ T} K x^\circ)\, dt = \frac{1}{2} x^{iT} K(t_i) x^i$$ ✓

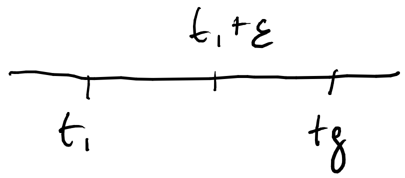
- Condition 2

We want to extend the solution of RE to the whole time interval $[t_i, t_f]$

By contradiction:

assume that $\lim_{t \to t_1^+} |K_{ij}(t)| = \infty$

Consider an $\varepsilon > 0$ such that $t_1 + \varepsilon < t_f$



In $[t_1+\varepsilon, t_f]$ RE admits a solution

So $\exists$ a solution of the regulator problem $\forall x(t_1+\varepsilon)$.

In particular

$J_{t_1+\varepsilon}(x^\circ, u^\circ) = \frac{1}{2} x^T(t_1+\varepsilon)\, K(t_1+\varepsilon)\, x(t_1+\varepsilon)$ and for the hypotheses on the matrices F, $Q(t)$, $R(t)$

- $J_{t_1+\varepsilon}(x^\circ, u^\circ) \geq 0$, $\forall x(t_1+\varepsilon) \in \mathbb{R}^n$
- $K(t_1+\varepsilon) \geq 0$, $\forall \varepsilon \in (0, t_f - t_1)$
- $K(t) \geq 0$, $\forall t \in (t_1, t_f]$

Trick: since K is not negative,

now consider the i-th column of the identity matrix e_i:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & 1 & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

e_i

Being $K(t)$ symmetric one has:

$(e_i \pm e_j)^T K (e_i \pm e_j) = k_{ii} + k_{jj} + 2k_{ij} \geq 0 \quad \forall t \in (t_1, t_f]$

example:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$e_i \quad e_j$

$k_{12} = k_{21}$ simmetry

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} k_{11} + k_{12} \\ k_{21} + k_{22} \end{pmatrix} = k_{11} + k_{12} + k_{21} + k_{22} =$$

$$= k_{11} + k_{22} + 2k_{12} \geq 0$$

$$K_{ii} + K_{jj} \geq 2 \underbrace{|K_{ij}|} \quad \forall t \in (t_1, t_f]$$

$$\lim_{t \to t_1^+} |K_{ij}| = \infty$$

One of the two elements K_{ii} or K_{jj} goes to ∞

Let's choose $K_{jj} \Rightarrow \lim_{t \to t_1^+} K_{jj} = \infty$

Let's consider then the optimal control problem in the interval $[t_1 + \varepsilon, t_f]$ with initial condition

$x(t_1 + \varepsilon) = e_j$

$\exists$ an optimal solution and one has:

$$(*) \quad \lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^\circ, u^\circ) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T K(t_1+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} K_{jj}(t_1+\varepsilon) = \infty$$

Now consider the <u>null control</u> over $[t_1 + \varepsilon, t_f]$ and the corresponding <u>free evolution</u> $\hat{x}$ on the state

Indicate with $\Phi(t, \tau)$ the transition matrix of the stat corresponding to A

$$\underset{\substack{\uparrow \\ \text{starting} \\ \text{instant}}}{J_{t_1+\varepsilon}} (\underset{\substack{\uparrow \\ \text{free} \\ \text{evolution}}}{\hat{x}}, \underset{\substack{\uparrow \\ \text{zero} \\ \text{control}}}{0}) = \underbrace{\frac{1}{2} \hat{x}^T(t_f) F \hat{x}(t_f)}_{\text{Bolze term}} + \frac{1}{2} \int_{t_i}^{t_f} \hat{x}^T(t) Q(t) \hat{x}(t) \, dt$$

$$\hat{x}(t_f) = \Phi(t_f, t_1 + \varepsilon) e_j$$

$$= \frac{1}{2} e_j^T \Phi^T(t_f, t_1 + \varepsilon) F \Phi(t_f, t_1 + \varepsilon) e_j$$

$$= \frac{1}{2} \int_{t_1+\varepsilon}^{t_f} e_j^T \Phi^T(t, t_1 + \varepsilon) Q(t) \Phi(t, t_1 + \varepsilon) e_j \, dt$$

Since $\Phi(t,\tau)$ is bounded (in norm), $\exists\, M>0$ matrix such that:

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(\hat{x},0) < M$$

so $J_{t_1+\varepsilon}(x^o,u^o) \leq J_{t_1+\varepsilon}(\hat{x},0) \quad \forall \varepsilon \in (0, t_f - t']$

by doing the limit of $\varepsilon \to 0^+$ also for $J_{t_1+\varepsilon}(x^o,u^o)$ we obtain

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^o,u^o) \leq \lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(\hat{x},0) < M$$

which is in contrast with $(*)$:

$$\lim_{\varepsilon \to 0^+} J_{t_1+\varepsilon}(x^o,u^o) = \lim_{\varepsilon \to 0^+} \frac{1}{2} e_j^T K(t_1+\varepsilon) e_j = \lim_{\varepsilon \to 0^+} \frac{1}{2} k_{jj}(t_1+\varepsilon) = \infty$$

So the condition $\lim_{t \to t_1^+} |k_{ij}(t)| = \infty$ is <u>not possible</u>

$$\boxed{\text{RE admits a unique solution} \geq 0 \text{ in } [t_i, t_f]}$$

semidefinite positive

\# End

<u>Theorem</u>: The regulator problem admits a unique solution

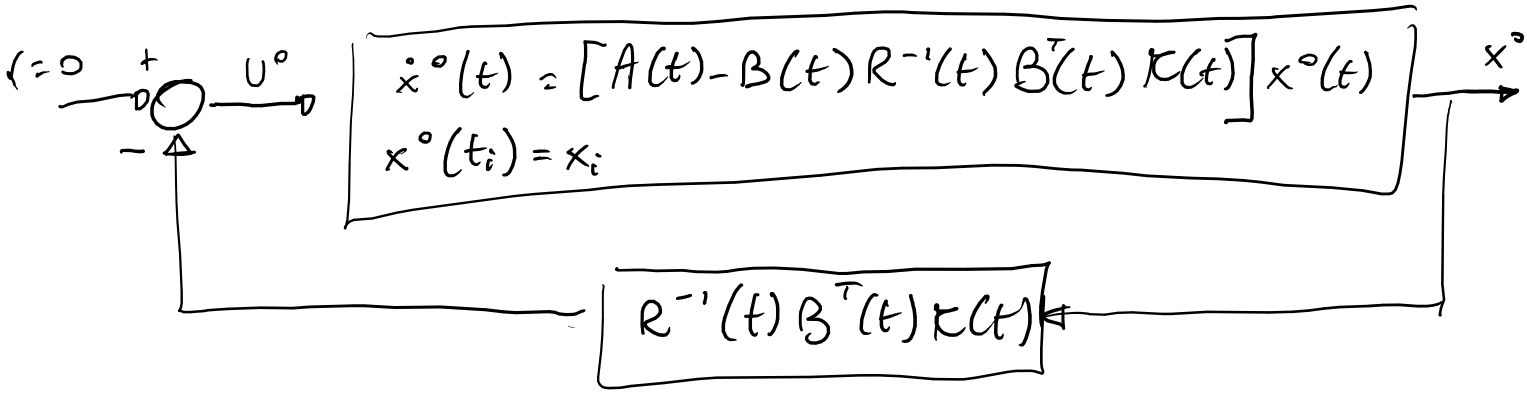
\# Remarks:

1) Q, R, F can be not simmetric $\xrightarrow{\text{trick}}$ $\frac{Q+Q^T}{2} \to$ (Q become symmetric)

2) The solution of RE does not depend on the initial state, therefore the solution can be found off line

3) The Riccati matrix K is a function of time, even if A, B, Q, R are constant

~ Regulator problem scheme



$$u^\circ = -R^{-1}(t)B^T(t)K(t)x^\circ(t)$$

~ Infinite time interval $[t_i, \infty)$

- Assume the matrices A, B bounded with elems of C^1 class
- Assume the dynamical system is completely controllable or exponentially stable
- Assume Q, R symmetric, $Q \geq 0$, $R > 0$, bounded with elems in C^1 class.

Find the control $u^\circ \in \bar{C}^\circ [t_i, \infty)$ and the state $x^\circ \in \bar{C}^1 [t_i, \infty)$ satisfying the system, the initial condition and minimizing:

$$J(x, u) = \frac{1}{2} \int_{t_i}^{\infty} [x^T Q x + u^T R u] \, dt$$

Theorem: The unique optimal solution is

$$u^\circ(t) = -R^{-1} B^T \bar{K} x^\circ$$

$$\dot{x}^\circ(t) = [A - B R^{-1} B^T \bar{K}] x^\circ$$

$$x^\circ(t_i) = x_i$$

where $\bar{K}$ is the solution of the RE:

$$\dot{\bar{K}}(t) = \bar{K} B R^{-1} B^T \bar{K} - \bar{K} A - A^T \bar{K} - Q$$

with final condition

$$\lim_{t_f \to \infty} \bar{K}(t_f) = 0$$

and $J(x^\circ, u^\circ) = \frac{1}{2} x_i^T \bar{K}(t_i) x_i$

Proof

Let us consider an arbitrary finite instant t_f, such that the time interval is $[t_i, t_f]$, $t_f < \infty$

In this time interval we can write

$\dot{K} = KBR^{-1}B^TK - KA - A^TK - Q$

$K(t_f) = 0$

depends on t_f because it depends $K_{t_f}(t_i)$ and K depends on the interval $[t_i, t_f]$

Fixing t_f, $J_{t_f} = \frac{1}{2} x_i^T K_{t_f}(t_i) x_i$

Now I want to show that $J_{t_f}(x^\circ, u^\circ)$ is superiorly limited.

1) If the system is completely controllable, whatever the initial state is, $\exists$ a control u' able to transfer it to the origin in a finite time t_f' with an evolution x'

So:

x_i initial state

u' control to send the sys from x_i to 0

x' evolution of the sys

t_f' final instant once reached the origin

Chosen $t_f' < t_f$, I suppose $x' = u' = 0$ $\forall t \in [t_f', t_f]$

$$\underbrace{\frac{1}{2}\int_{t_i}^{t_f'} (x'^TQx' + u'^TRu')dt}_{J_{t_f}(x', u')} = \frac{1}{2}\int_{t_i}^{t_f'}(x'^TQx' + u'^TRu')dt + \frac{1}{2}\int_{t_f'}^{t_f}\cancel{(x'^TQx' + u'^TRu')dt} \to 0$$

$\geq J_{t_f}(x^\circ, u^\circ)$ $\rightsquigarrow$ therefore it is bounded

② If the system is <u>exponentially stable</u>, from any initial state with <u>zero input</u>, the corresponding <u>free evolution</u> of the state $\bar{x}$ satisfies this inequality:

$$\|\bar{x}(t)\| \le \alpha e^{-\beta t}, \quad \beta > 0$$

free evolution with 0 input

$$J_{t_f}(x^\circ, u^\circ) \le \underbrace{J_{t_f}(\bar{x}, 0)}_{\frac{1}{2}\int_{t_i}^{t_f} \bar{x}^T Q \bar{x}\, dt} \le \frac{1}{2}\int_{t_i}^{\infty} \bar{x}^T Q \bar{x}\, dt$$

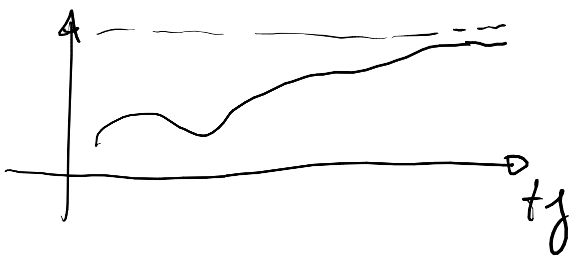
because

$$\int_{t_i}^{\infty} \dots dt = \int_{t_i}^{t_f} \dots dt + \int_{t_f}^{\infty} \underline{\text{quantity}}\, dt$$

$$\le \frac{1}{2}\int_{t_i}^{\infty} \|Q(t)\| \cdot \|\bar{x}(t)\|^2 dt \quad \text{property of inequalities}$$

$$\le c \int_{t_i}^{\infty} e^{-2\beta t} dt = \frac{c}{2\beta} e^{-2\beta t_i} \quad J_{t_f}(x^\circ, u^\circ) \text{ bounded}$$

$J_{t_f}(x^\circ, u^\circ)$ is <u>monotonically non decreasing</u> and <u>superiorly limited in t_f</u>



Consider now $(x^{\circ\prime}, u^{\circ\prime})$ for t_f' and $(x^{\circ\prime\prime}, u^{\circ\prime\prime})$ for t_f''

assume that $t_f' \le t_f''$

$$J_{t_f''}(x^{\circ\prime\prime}, u^{\circ\prime\prime}) = \frac{1}{2}\int_{t_i}^{t_f''} [x''^T Q x'' + u''^T R u'']\, dt \ge \frac{1}{2}\int_{t_i}^{t_f'} [x''^T Q x'' + u''^T R u'']\, dt$$

in the interval $[t_i, t_f']$ the solution is $(x^{o\prime}, u^{o\prime})$ and not $(x^{o\prime\prime}, u^{o\prime\prime})$, so:

$$J_{t_f''}(x^{o\prime\prime}, u^{o\prime\prime}) \geq J_{t_f'}(x^{o\prime}, u^{o\prime}) = \int_{t_i}^{t_f'} [x'^T Q x' + u'^T R u'] dt$$

Therefore the function is <u>not decreasing</u> and

$J_{t_f}(x^o, u^o) = \frac{1}{2} x_i^T K(t_i) x_i$ has a limit as $t_f \to \infty$

* Being x_i arbitrary, it follows that any matrix $K(t_i)$ has a limit as $t_f \to \infty$
* As t_i is arbitrary, the solution K of the RE with final condition $K(t_f) = 0$ admits a limited solution $K'(t)$ $\forall t$ as $t_f \to \infty$
* <u>Existence</u>, <u>uniqueness</u> and the characterization <u>of the solution</u> can be justified as in the previous theorem considering <u>the limit for the solution</u> herein identified and recalling that

$$\lim_{t_f \to \infty} K(t_f) = K'(t)$$

End

Remark

If the matrix $Q(t)$ has eigenvalues $\geq \alpha$, with $\alpha > 0$ $\forall t \in [t_i, \infty)$ the optimal regulator is asymptotically stable; in fact for the optimal solution it results in:

$$\frac{\alpha}{2} \int_{t_i}^{\infty} \| x^o(t) \|^2 dt \leq \frac{1}{2} \int_{t_i}^{\infty} [x^{oT}(t) Q(t) x^o(t) + u^{oT}(t) R(t) u^o(t)] dt$$

$$= \frac{1}{2} x_i^T \bar{K}(t_i) x_i$$

Therefore

$$\lim_{t \to \infty} x^o(t) = 0 \quad \forall x_i \in \mathbb{R}^n$$

~ The steady state solution of the deterministic linear optimal regulator problem

Remark: in this case in place of a differential equation I have an algebraic equation

Theorem: from the previous problem

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$J(x,u) = \frac{1}{2}\int^{t_f}[x^T(t)\,Q(t)\,x(t) + u^T(t)\,R(t)\,u(t)]\,dt$$

and $$\dot{K}(t) = K^{t_i}(t)\,B(t)\,R^{-1}(t)\,B(t)\,K(t) - K(t)\,A(t) - A^T(t)\,K(t) - Q(t)$$

all the elements were time dependent

- If A, B, Q, R are constant
- $Q > 0$

$\rightarrow$ I can still write RE

$\exists$ a unique optimal solution:

$$u^\circ(t) = -R^{-1}B^T K_r\, x^\circ(t)$$

$$\dot{x}^\circ(t) = [A - BR^{-1}B^T K_r]\,x^\circ(t)$$

$$x^\circ(t_i) = x_i$$

where K_r constant matrix and unique solution of the algebraic Riccati equation:

$$K_r BR^{-1}B^T K_r - K_r A - A^T K_r - Q = 0$$

with

$$J(x^\circ, u^\circ) = \frac{1}{2}x_i^T K_r\, x_i$$

Previously I had $\dot{K}$ for $[t_i, t_f]$, $\bar{K}$ for $[t_i, \infty)$, now $\dot{\bar{K}} = 0$ for $[t_i, \infty)$

Proof

From the previous theorem we have that the value of the cost index for the considered problem is given by

$$J(x^o, u^o) = \frac{1}{2} x_i^T \bar{K}(t_i) x_i$$

Due to the stationarity of the problem defined on infinite time interval, it results that its value is independent from t_i.

Therefore the unique solution of the equation:

$$\dot{\bar{K}}(t) = \bar{K}(t) B(t) R^{-1}(t) B(t) \bar{K}(t) - \bar{K}(t) A(t) - A^T(t) K(t) - Q(t)$$

is constant $\bar{K}$

The existence and uniqueness of the optimal solution follow from the previous theorem

End

Remarks

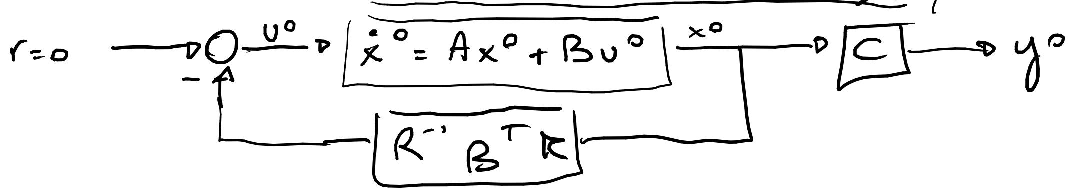
1) Consider the sys: $\begin{cases} \dot{x}(t) = A(t) x(t) + B(t) u(t) \\ y(t) = C(t) x(t) \end{cases}$

It can be defined an optimal regulation problem from the output y, considering the cost index:

$$\bar{J}(y, u) = \frac{1}{2} y^T(t_f) F y(t_f) + \frac{1}{2} \int^{t_f} [y^T(t) \bar{Q}(t) y(t) + u^T(t) R(t) u(t)] dt$$

$$= \frac{1}{2} x^T(t_f) C^T(t_f) \bar{F} C(t_f) x(t_f) + \frac{1}{2} \int_{t_i}^{t_i, t_f} [x^T C^T \bar{Q} C x + u^T R u] \, dt$$

Condition: the state must be accessible, for the feedback action



2) case $[t_i, t_f]$ → differential RE
case $[t_i, \infty)$ → same
case $[t_i, \infty)$ with steady state sys → algebraic RE

~ The optimal tracking problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_i) = x_i \end{cases}$$

$A(t), B(t)$ with elements functions of C^1 class

Consider the reference $r \in C^1[t_i, t_f]$

Determine the OC $u^\circ \in \bar{C}^0[t_i, t_f]$ and the state $x^\circ \in \bar{C}^1[t_i, t_f]$ minimizing:

$$J(x, u) = \frac{1}{2}\int_{t_i}^{t_f} \left\{ [r(t) - x(t)]^T Q(t)[r(t) - x(t)] + u^T(t)R(t)u(t) \right\} dt$$

where $Q(t) \geq 0$, $R(t) > 0$, symmetric

The RE is:

$$\begin{cases} \dot{K}(t) = K(t)B(t)R^{-1}(t)B^T(t)K(t) - K(t)A(t) - A^T(t)K(t) - Q(t) \\ K(t_f) = 0 \end{cases}$$

<u>Theorem</u>: the problem admits a <u>unique</u> optimal solution:

$$u^\circ(t) = R^{-1}(t)B^T(t)\left[g(t) - K(t)x^\circ(t)\right]$$

$$\dot{x}^\circ(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)K(t)\right]x^\circ(t) + B(t)R^{-1}(t)B^T(t)g(t)$$

\# Previously the OC was $u^\circ(t) = -R^{-1}B^T K x^\circ$

$g(t)$ is the solution of the differential equation

$$\begin{cases} \dot{g}(t) = \underbrace{\left[K(t)B(t)R^{-1}(t)B^T(t) - A^T(t)\right]}_{\text{not important}} g(t) - Q(t)\underbrace{r(t)}_{\text{important}} \\ g(t_f) = 0 \end{cases}$$

with:

$$J(x^\circ, u^\circ) = \frac{1}{2}x^{\circ T}(t_i)K(t_i)x^\circ(t_i) - x^{\circ T}(t_i)g(t_i) + v(t_i)$$

where v is the solution of the equation

$$\begin{cases} \dot{v}(t) = \frac{1}{2} g^T(t) B(t) R^{-1}(t) B^T(t) g(t) - \frac{1}{2} r(t)^T Q(t) r(t) \\ v(t_f) = 0 \end{cases}$$

important

$g(t)$ and $v(t)$ are functions that depends on the reference $r(t)$

Proof

It can be applied Pontryagin theorem (convex case).

Let us define the hamiltonian:

$$H(x, u, \lambda, t) = \frac{1}{2}(r-x)^T Q (r-x) + \frac{1}{2} u^T R u + \lambda^T A x + \lambda^T B u$$

The nec. & suff. conditions are:

$$\begin{cases} \dot{\lambda}^\circ = -Q x^\circ - A^T \lambda^\circ + Qr \rightsquigarrow \text{new term} \\ R u^\circ + B^T \lambda^\circ = 0 \\ \lambda^\circ(t_f) = 0 \end{cases}$$

With the following choice for the costate:

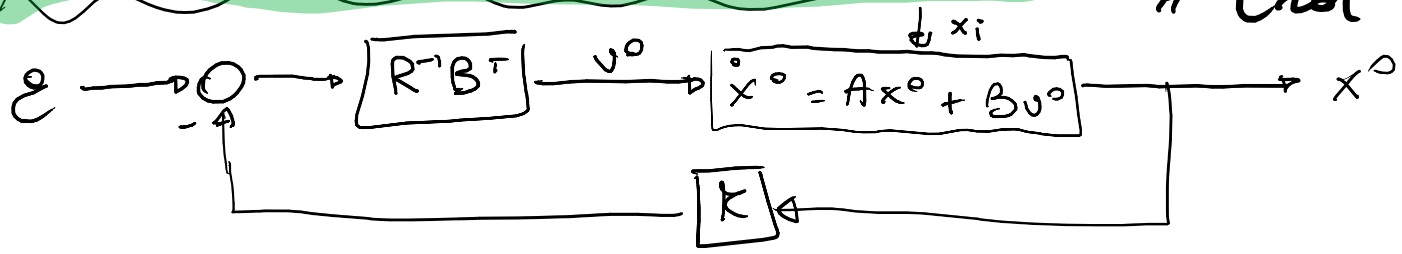
$$\lambda^\circ = K x^\circ - g$$

the theorem is proved.

Observe that:

$$J(x^\circ, u^\circ) = \frac{1}{2} x^{\circ T}(t_i) K(t_i) x^\circ(t_i) - x^{\circ T}(t_i) g(t_i) + v(t_i) = \int_{t_i}^{t_f} -\frac{d}{dt}\left(\frac{1}{2} x^{\circ T} K x^\circ - x^{\circ T} g + v\right) dt$$

End



Remark

- Note the quadratic error on the final instant
  $r(t_f) - x(t_f)$ is not present

- The realization of the optimal control for the tracking problem needs the solution of RE K, and the solution of the diff. equation in g

- To solve the tracking problem the reference variable <u>r must be known in advance</u> in the interval

- The <u>steady state</u> tracking problem over an infinite time interval requires a steady state solution for the differential equation in the g function.

  eg. with $\bar{r}$ constant reference:

$$g_r = [K_r B R^{-1} B^T - A^T] Q \bar{r}$$

~ The optimal regulator problem with null final error

<u>Problem</u>: Let us consider the linear system:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$x(t_i) = x_i \qquad x(t_f) = 0$$

If I pick $x(t_f) = 0$ I don't need the Bolza term in the functional

Determine the control $u^\circ \in \bar{C}^0[t_i, t_f]$ and the state $x^\circ \in \bar{C}^1[t_i, t_f]$ minimizing

$$J(x,u) = \frac{1}{2}\int_0^{t_f} \left[x^T Q x + u^T R u\right] dt$$

$Q(t) \geq 0$, $R(t) > 0$, symmetric with C^1 class elements

<u>Theorem</u>: Let us introduce the matrix of dim $2n \times 2n$

$$\underset{\text{doublu}}{\Omega(t)} = \begin{pmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{pmatrix}$$

and indicate with $\Phi(t,\tau) = \begin{pmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{pmatrix}$ $\quad n \times n$

its transition matrix partitioned in submatrices of dimension $n \times n$

Assume that the submatrix $\phi_{12}(t_f, t_i)$ is <u>not singular</u>
↳ $\det \neq 0$

The optimal regulator problem with null final error admits a <u>unique optimal solution</u>:

$$u^\circ(t) = -R^{-1}(t)B^T(t)\left[\phi_{21}(t,t_i) - \phi_{22}(t,t_i)\,\phi_{12}^{-1}(t_f,t_i)\,\phi_{11}(t_f,t_i)\right] x_i$$

$$x^\circ(t) = \left[\phi_{11}(t,t_i) - \phi_{12}(t,t_i)\,\phi_{12}^{-1}(t_f,t_i)\,\phi_{11}(t_f,t_i)\right] x_i \longrightarrow$$

initial state (not the current)

Proof

We can apply the Pontryagin theorem, convex case.

Hamiltonian: $H(x,u,\lambda,t) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T A x + \lambda^T B u$

Nec. & Suff. conditions:

$$\dot{\lambda} = - \frac{\partial H}{\partial x}\Big|^T = -A^T \lambda - Q x$$

$$0 = \frac{\partial H}{\partial u}\Big|^T = R u + B^T \lambda \Rightarrow u = -R^{-1} B^T \lambda$$

I have two equations:

$$\dot{x} = A x + B u$$

$$\dot{\lambda} = -A^T \lambda - Q x$$

Trick: let's put them together

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -B R^{-1} B^T \\ -Q & -A^T \end{pmatrix}}_{\Omega} \begin{pmatrix} x \\ \lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \overbrace{\Omega}^{2n \times 2n} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

indicating with $\underbrace{\Phi(t,\tau)}_{\text{transition matrix}} = \begin{pmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{pmatrix}$

The solution of the differential equation, taking into account the initial condition is:

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t,t_i) & \phi_{12}(t,t_i) \\ \phi_{12}(t,t_i) & \phi_{22}(t,t_i) \end{pmatrix} \begin{pmatrix} x(t_i) \\ \lambda(t_i) \end{pmatrix}$$

free evolution of the state

I don't know $\lambda(t_i)$ but I know $x(t_f) = 0$.

So multiplying the first row of the free evolution for the first column in the case of $t = t_f$

$$x(t_f) = \phi_{11}(t_f, t_i)\, x(t_i) + \phi_{12}(t_f, t_i)\, \lambda(t_i)$$

$\hookrightarrow 0$

Taking into account the non singularity of submatrix

$\phi_{12}(t_f, t_i) \neq 0 \rightarrow$ non singular (hypothesis) $\rightarrow$ If it is singular no solution exists or ∞ solutions depending on

it results in

$$0 = \phi_{11}(t_f, t_i)\, x(t_i) + \phi_{12}(t_f, t_i)\, \lambda(t_i)$$

$$\lambda(t_i) = -\frac{\phi_{11}(t_f, t_i)}{\phi_{12}(t_f, t_i)}\, x(t_i) = -\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)$$

I can now solve for $x(t)$:

$$x(t) = \phi_{11}(t, t_i)\, x(t_i) + \phi_{12}(t, t_i)\left[-\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)\right]$$

$$\lambda(t) = \phi_{21}(t, t_i)\, x(t_i) + \phi_{22}(t, t_i)\left[-\phi_{12}^{-1}(t_f, t_i)\, \phi_{11}(t_f, t_i)\, x(t_i)\right]$$

That substituted into the expression of control

$u^o = -R^{-1} B^T \lambda$ yelds the optimal solution.

end

~ The optimal regulator problem with limited control

Problem: Optimal regulator problem on finite time interval with additional constraint

$$|u_j(t)| \le 1 \quad , j = 1, 2, \cdots, p \qquad \forall t \in [t_i, t_f]$$

and the hypothesis that the weighting matrix R is diagonal

$$\dot{x} = Ax + Bu$$

Bolza Term (there's no final state condition)

$$J = \frac{1}{2} \int x^T Q x + u^T R u + \frac{1}{2} x^T(t_f) F x(t_f)$$

I want that each control is limited.

Theorem: All the normal solutions (no more than one) can be found solving the differential system:

$$\dot{x}^\circ(t) = A(t) x^\circ(t) - B(t) \operatorname{sat}\{R^{-1}(t) B^T(t) \lambda^\circ(t)\}$$

$$\dot{\lambda}^\circ(t) = -Q(t) x^\circ(t) - A^T(t) \lambda^\circ(t) \qquad \cancel{\ast}$$

$$x^\circ(t_i) = x_i \qquad \lambda^\circ(T) = F x^\circ(t_f)$$

Now with the condition of limited control the optimal control is:

$$u^\circ(t) = -\underbrace{\operatorname{sat}}_{\text{saturation function}}\{R^{-1}(t) B^T(t) \lambda^\circ(t)\}$$

Proof

Let's apply Pontryagin theorem (convex case);

It's not possible to apply, like before, $\frac{\partial H}{\partial u} = 0$, because u is bounded.

Let's consider the hamiltonian

$$H = \cancel{\tfrac{1}{2} x^T Q x} + \tfrac{1}{2} u^{\circ T} R u^\circ + \cancel{\lambda^T A x} + \lambda^T B u^\circ \leq$$ Pontryagin

$$\leq \cancel{\tfrac{1}{2} x^T Q x} + \tfrac{1}{2} \omega^{\circ T} R \omega^\circ + \cancel{\lambda^T A x} + \lambda^T B \omega^\circ$$

$$\forall \omega : |\omega_j| \leq 1 , \quad j = 1, 2, \dots, p$$

And the nec & suff. conditions

$$\dot{\lambda}^\circ = -Q x^\circ - A^T \lambda^\circ = -\frac{\partial H}{\partial u}\Big|^T$$

$$\lambda^\circ(t_f) = F x^\circ(t_f) \longrightarrow \text{Bolza term (boundary condition)}$$

By adding the quantity $\frac{1}{2} \lambda^{\circ T} B R^{-1} B^T \lambda^\circ$ to both members of the inequality, we can write it as follows:

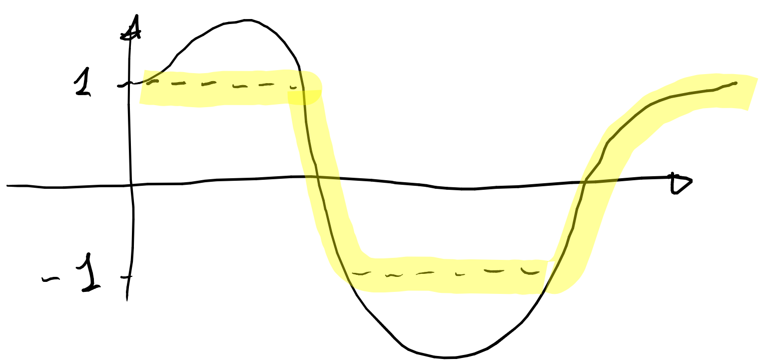
$$(u^\circ + \psi)^T R (u^\circ + \psi) \leq (\omega + \psi)^T R (\omega + \psi)$$

where $\psi = R^{-1} B^T \lambda$

Taking into account that the matrix R is diagonal and positive definite, in order to satisfy the Pontryagin theorem by making the first term $\leq$ of the second one, $\underline{u^\circ = -\psi}$ in order to obtain 0 (since we have quadratic forms we cannot have less than 0). Since $|u| \leq 1$ we have to saturate ψ.

$$\text{if} \begin{cases} \psi \leq -1 \\ \psi \geq 1 \end{cases} \Rightarrow -1 \leq \psi \leq 1$$

I can put $\boxed{u^\circ = -(R^{-1} B^T \lambda)}$ satisfying the equation



End

Remarks

- With the new constraint state-costate is non-linear (*)
- The existence of the solution is not guaranteed
- It is possible to consider the same problem with the final condition fixed

$$x(t_f) = x_f$$