KINEMATIC MODELS:

UNICYCLE:

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \qquad \text{CONSTRAINTS: } 1.\ (\sin\theta \quad -\cos\theta \quad 0)\ \dot{q} = 0$$

$$\dot{q} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = u_2 \end{cases}$$

IF WE CONSIDER AS INPUT ω_R AND ω_L WE CAN APPLY AN INPUT TRANSFORMATION:

$$v = \frac{r(\omega_R + \omega_L)}{2} \qquad \omega = \frac{r(\omega_R - \omega_L)}{d}$$

r : RADIUS OF WHEELS
d : BASE LINE

BICYCLE:

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix}$$

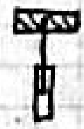
CONSTRAINTS:

RW. $\sin\theta\,\dot{x} - \cos\theta\,\dot{y} = 0$
FW $\sin(\theta+\phi)\,\dot{x} - \cos(\theta+\phi)\,\dot{y} - \ell\cos\phi\,\dot{\theta} = 0$

RWD:

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{\ell} \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

$\phi = \frac{\pi}{2} \rightarrow$ SINGULARITY



FWD:

$$\dot{q} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{\sin\phi}{\ell} \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

NO SINGULARITY

LIE BRACKET IS THE RESULT OF A PARTICULAR MANEUVER
CONSIDER

$$\dot{q} = g_1(q)\,u_1 + g_2(q)\,u_2$$

1. $u_1 = 1$ $\quad u_2 = 0$ $\quad t \in [0, \varepsilon) \rightarrow$ MOVING ALONG g_1
2. $u_1 = 0$ $\quad u_2 = 1$ $\quad t \in [\varepsilon, 2\varepsilon) \rightarrow$ MOVING ALONG g_2
3. $u_1 = -1$ $\quad u_2 = 0$ $\quad t \in [2\varepsilon, 3\varepsilon) \rightarrow$ MOVING ALONG $-g_1$
4. $u_1 = 0$ $\quad u_2 = -1$ $\quad t \in [3\varepsilon, 4\varepsilon] \rightarrow$ MOVING ALONG $-g_2$

$$q(4\varepsilon) = q_0 + \varepsilon^2 [g_1(q), g_2(q)] + o(\varepsilon^3)$$

1 APPUNTI
2 SCHEMI
3 COMPITI.

CONTROLLABLE $\rightarrow$ DIMENSION OF Δ_A EQUAL OF n

TRICYCLE = SAME MODEL OF THE BICYCLE
CAR-LIKE = SAME MODEL OF BICYCLE

CHAINED FORM:

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1 \end{aligned}$$

ALWAYS CONTROLLABLE

PUT A SYSTEM IN THIS FORM APPLING:

STATE TRANSFORMATION $z = \alpha(q)$

INPUT TRANSFORMATION $v = \beta(q) u$

UNICYCLE

$$\begin{cases} \dot{z}_1 = v_1 \\ \dot{z}_2 = v_2 \\ \dot{z}_3 = z_2 v_1 \end{cases}$$

TRANSFORMATIONS:

COORDINATES:

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta \end{aligned}$$

INPUT:

COORDINATES IN MOVING FRAME

$$\begin{aligned} v_1 &= \omega \\ v_2 &= v - z_3 \omega \end{aligned}$$

BICYCLE:

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = v\tan\phi / \ell \\ \dot{\phi} = \omega \end{cases} \longrightarrow \text{CHAIN FORM } (2,4) \quad \begin{cases} \dot{z}_1 = v_1 \\ \dot{z}_2 = v_2 \\ \dot{z}_3 = z_2 v_1 \\ \dot{z}_4 = z_3 v_1 \end{cases}$$

*

DIFFERENTIAL FLATNESS

IF THERE EXIST A SET OF OUTPUTS $w = h(q)$ CALLED FO SUCH THAT q AND u:

$$q = \alpha(w, \dot{w}, \ddot{w} \ldots)$$
$$u = \beta(w, \dot{w}, \ddot{w} \ldots)$$

UNICYCLE:

$$w = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{matrix} \dot{x} \\ \dot{y} \end{matrix}$$

$$\theta = \arctan\frac{\dot{y}}{\dot{x}} = \mathrm{ATAN2}(\dot{y}, \dot{x}) + k\pi$$

$$v = \pm\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\omega = \dot{\theta} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}$$

CHAINED FORM $\begin{pmatrix} z_1 \\ z_n \end{pmatrix}$

$$\begin{cases} \dot{z}_1 = v_1 \\ \dot{z}_2 = v_2 \\ \dot{z}_3 = z_2 v_1 \end{cases} \qquad \text{FO } \begin{pmatrix} z_1 \\ z_3 \end{pmatrix} \qquad z_2 = \frac{\dot{z}_3}{v_1} = \frac{\dot{z}_3}{\dot{z}_1}$$

$$v_1 = \dot{z}_1$$
$$v_2 = \dot{z}_2 = \frac{\ddot{z}_3\dot{z}_1 - \dot{z}_3\ddot{z}_1}{\dot{z}_1^2}$$

* TRANSFORMATION FOR BICYCLE:

$$\begin{aligned} z_1 &= x \\ z_2 &= \frac{1}{\ell}\sec^3\theta\tan\phi \\ z_3 &= \tan\theta \\ z_4 &= y \end{aligned}$$

$$v = \frac{v_1}{\cos\theta}$$

$$\omega = -\frac{3}{\ell} v_1 \sec\theta \sin^2\phi + \frac{1}{\ell} v_2 \cos^3\theta \cos^2\phi$$

PATH PLANNING

* USING DIFFERENTIAL FLATTENESS

1. COMPUTE

$w_i = h(q_i)$ VALUE OF FO AT BEGGINING AND

$w_f = h(q_f)$ AT END

2. GENERATE A PATH FROM w_i TO w_f RESPECTING THE BOUNDARY CONDITIONS.

3. RECOSTRUCT THE PATH FOR ALL STATE $q(s)$ AND FOR THE INPUTS

* PLANNING VIA THE CHAINED FORM

1. COMPUTE INITIAL AND FINALS VALUES z_i AND z_f THAT CORRESPOND TO q_i AND q_f BY USING THE CHANGE OF COORDINATES

2. INTERPOLATE z_1, z_3 WITH THE APPROPRIATE BOUNDARY CONDITIONS $z_2 = z_3' / z_1'$

UNDER THE ASSUMPTION $z_{1,i} \neq z_{1,f}$ CONSIDER THE FOLLOWING INTERPOLATION SCHEME:

$$z_1(s) = z_{1,f}\, s - (s-1) z_{1,i}$$

$$z_3(s) = s^3 z_{3,f} - (s-1)^3 z_{3,i} + \alpha_3 s^2 (s-1) + \beta_3 s (s-1)^2 \qquad s \in [0,1]$$

α_3, β_3 DETERMINED BY IMPOSING:

$$\frac{z_3'(0)}{z_1'(0)} = z_{2,i} \qquad \frac{z_3'(1)}{z_1'(1)} = z_{2f}$$

* PLANNING VIA PARAMETERIZED INPUTS

CONSIDER THE CHAINED FORM

$$z_1' = \tilde{v}_1$$
$$z_2' = \tilde{v}_2$$
$$z_3' = z_2 \tilde{v}_1$$
$$\vdots$$

LET THE GEOMETRIC INPUT BE CHOSEN AS:

$$\tilde{v}_1 = \operatorname{sgn}(\Delta)$$
$$\tilde{v}_2 = c_0 + c_1 s + \dots + c_{n-2} s^{n-2}$$

WITH $\Delta = z_{1,f} - z_{1,i}$ AND $s \in [s_i, s_f] = [0, |\Delta|]$

$c_0 \dots c_{n-2}$ MUST BE CHOSEN SO AS TO GIVE $z(s_f) = z_f$ $n-1$ CONDITIONS

KNOWING $\tilde{v}_1, \tilde{v}_2$ INTEGRATE $z_1, z_2 \dots z_n$

TRAJECTORY PLANNING

ONCE WE HAVE A PATH $q(s)$ $s \in [s_i, s_f]$ WE WANT TO ADD A TIMING LAW $s = s(t)$ SUCH THAT $s(t_i) = s_i$ $s(t_f) = s_f$

IF $v(t) \leq v_M$

REDEFINE $\tau = \frac{t}{T} = \frac{t}{t_f - t_i}$ $\longrightarrow$ $v(t) = v(s) \frac{ds}{d\tau} \frac{d\tau}{dt} = v(s) \frac{ds}{dt} \frac{1}{T}$

T INCREASE $\Rightarrow$ v DECREASE.

FEEDBACK CONTROL

* TRAJECTORY PLANNING:

- APPROXIMATE LINEARIZATION:

EXPRESS THE ERROR IN THE MOVING FRAME:

$$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{pmatrix}$$

INPUT TRANSFORMATION:

$$v = v_d \cos e_3 - \dots$$
$$\omega = \omega_d - u_2$$

DIFFERENTIATING AND SUBSTITUTE $\dot{q}_d$ AND $\dot{q}$

$$\dot{e} = \begin{pmatrix} 0 & \omega_d & 0 \\ -\omega_d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 \\ \sin e_3 \\ 0 \end{pmatrix} v_d + \begin{pmatrix} 1 & -e_2 \\ 0 & e_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

AT EQUILIBRIUM WE APPROXIMATE $e = 0$

$$\dot{e} = \begin{pmatrix} 0 & \omega_d & 0 \\ -\omega_d & 0 & v_d \\ 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u_1 = -K_1 e_1$$
$$u_2 = -K_2 e_2 - K_3 e_3$$

IN CLOSED-LOOP WE HAVE:

$$\dot{e} = A(t) e = \begin{pmatrix} -K_1 & \omega_d & 0 \\ -\omega_d & 0 & v_d \\ 0 & -K_2 & -K_3 \end{pmatrix} e$$

→ CHARACTERISTIC POLYNOMIAL OF MATRIX A IS

$$p(\lambda) = \lambda(\lambda + K_1)(\lambda + K_3) + \omega_d^2(\lambda + K_3) + v_d K_2 (\lambda + K_1)$$

WE WANT EIGENVALUE FIXED AND REAL PART < 0

$$K_1 = K_3 = 2\xi a \qquad K_2 = \frac{a^2 - \omega_d^2}{a v_d} \qquad \xi \in (0,1), \quad a > 0$$

$$\Downarrow$$

$$p(\lambda) = (\lambda + 2\xi a)(\lambda^2 + 2\xi a \lambda + a^2)$$

- INPUT/OUTPUT LINEARIZATION

$$\dot{y} = \dot{h}(x) = \frac{\partial h}{\partial x}\dot{x} = \underbrace{\boxed{\frac{\partial h}{\partial x} G(x)}}_{T(x)} u \qquad u = T^{-1}(x) \underbrace{v}_{\text{NEW INPUT}}$$

$$\dot{y} = T(x) u = T(x) T^{-1}(x) v \quad \rightarrow \quad \dot{y} = v$$

CONTROL SIMPLE INTEGRATOR

$$\dot{y} = v \qquad e = y - y_d$$

$$\dot{e} = \dot{y} - \dot{y}_d = v - \dot{y}_d = -Ke \quad \Rightarrow \quad v = \underbrace{\dot{y}_d}_{\text{FEED FORWARD}} - \underbrace{Ke}_{\text{PROPORTIONAL ERROR}}$$

DOUBLE INTEGRATOR

$$\ddot{y} = v \qquad e = y - y_d$$

$$\ddot{e} = \ddot{y} - \ddot{y}_d = v - \ddot{y}_d = -K_p e - K_d \dot{e}$$

$$\Rightarrow v = \ddot{y}_d - K_p e - K_d \dot{e}$$

* POSTURE REGULATION

- PARTIAL REGULATION $(x, y) \to (x_d, y_d)$

THEN ROTATE ON THE SPOT TO RECOVER θ.

$$e_p = \begin{pmatrix} 0 - x \\ 0 - y \end{pmatrix} \qquad n = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \longrightarrow \quad v = K_1 e_p^T n \qquad w = K_2 (\mathrm{ATAN2}(y, x) - \theta + \pi)$$

- FULL REGULATION

EXPRESS IN POLAR COORDINATES

$$\rho = \sqrt{x^2 + y^2}$$

$$\gamma = \mathrm{ATAN2}(y, x) + \pi - \theta$$

$$\delta = \gamma + \theta$$

THE KINEMATIC MODEL BECOMES:

$$\begin{cases} \dot{\rho} = -v \cos(\gamma) \\ \dot{\gamma} = \dfrac{\sin\gamma}{\rho} v - w \\ \dot{\delta} = \dfrac{\sin\gamma}{\rho} v \end{cases}$$

CONTROL LAW:

$$v = K_1 \rho \cos\gamma$$

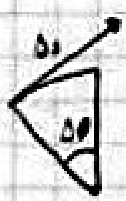
$$w = K_2 \gamma + \underbrace{K_1 \frac{\sin\gamma \cos\gamma}{\gamma} (\gamma + K_3 \delta)}$$ IT USED TO REGULATE θ

LOCALIZATION

$$\begin{cases} \dot{x} = v_k \cos\theta \\ \dot{y} = v_k \sin\theta \\ \dot{\theta} = w_k \end{cases}$$

INTEGRATION:

1. EULER



$$\theta_{k+1} = \theta_k + w_k T_s$$
$$x_{k+1} = x_k + v_k T_s \cos\theta_k$$
$$y_{k+1} = y_k + v_k T_s \sin\theta_k$$

2. RUNGE-KUTTA

$$x_{k+1} = x_k + v_k T_s \cos\left(\theta_k + \frac{w_k T_s}{2}\right)$$
$$y_{k+1} = y_k + v_k T_s \sin\left(\theta_k + \frac{w_k T_s}{2}\right)$$
$$\theta_{k+1} = \theta_k + w_k T_s$$

3. EXACT INTEGRATION

$$x_{k+1} = x_k + \frac{v_k}{w_k}\left(\sin\theta_{k+1} - \sin\theta_k\right)$$
$$y_{k+1} = y_k + \frac{v_k}{w_k}\left(\cos\theta_{k+1} - \cos\theta_k\right)$$
$$\theta_{k+1} = \theta_k + w_k T_s$$

FROM ENCODERS TO ROB INPUT $\rightarrow$ $\Delta s = \frac{r}{2}\left(\Delta\phi_R + \Delta\phi_L\right)$

$$\Delta\theta = \frac{r}{d}\left(\Delta\phi_R - \Delta\phi_L\right)$$

KALMAN

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + v_k \\ y_k = C_k x_k + w_k \end{cases}$$

PREDICTION: $$\begin{cases} x_{k+1|k} = A_k \hat{x}_k + B_k u_k \\ P_{k+1|k} = A_k P_k A_k^T + V_k \end{cases}$$

CORRECTION: $$\begin{cases} \hat{x}_{k+1} = \hat{x}_{k+1|k} + R_{k+1} v_{k+1} \\ P_{k+1} = P_{k+1|k} - R_{k+1} C_{k+1} P_{k+1|k} \end{cases}$$

$$R_{k+1} = P_{k+1|k} C_{k+1}^T \left(C_{k+1} P_{k+1|k} C_{k+1}^T + W_{k+1}\right)^{-1}$$

EKF

$$\begin{cases} x_{k+1} = f_k(x_k, u_k) + v_k \\ y_k = h_k(x_k) + w_k \end{cases}$$

~~CORR~~ PREDICTION
$$\begin{cases} \hat{x}_{k+1|k} = f_k(\hat{x}_k, u_k) \\ P_{k+1|k} = F_k P_k F_k^T + V_k \end{cases}$$

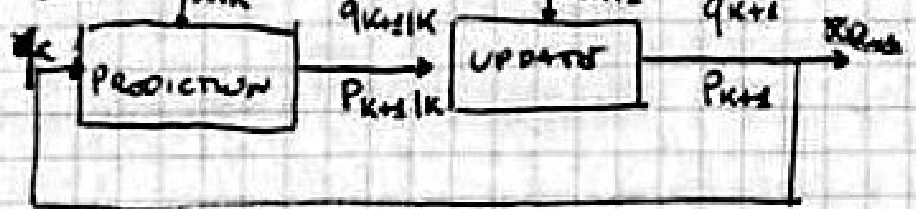
CORRECTION
$$\begin{cases} \hat{x}_{k+1} = \hat{x}_{k+1|k} + R_{k+1} v_{k+1} \\ P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k} \end{cases}$$

WITH
$$F_k = \frac{\partial f_k}{\partial x}\Big|_{x=\hat{x}_k} \qquad H_{k+1} = \frac{\partial h_{k+1}}{\partial x}\Big|_{x=\hat{x}_{k+1|k}}$$

$$R_{k+1} = P_{k+1|k} H^T_{k+1} \left(H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1} \right)^{-1}$$

$$v_{k+1} = h_{k+1} - h(q x_{k+1|k})$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$
$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$



$$\frac{1}{1+x^2}$$