III Conditioning and orthogonal projections

- Given: $Y \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$
- Consider:

Set of $\mathcal{F}^Y$-measurable random var. (linear space $\mathcal{M}^Y$)

(Set of functions $Z = f(Y)$ for some $\mathcal{B}(\mathbb{R})$-measurable function $f: \mathbb{R} \to \mathbb{R}$

- Definition: the orthogonal projection of X on $\mathcal{M}^Y$ $(\Pi(X|\mathcal{M}^Y))$ is $E\{X|\mathcal{F}^Y\}$

Proof: Since $X = X_{/\!/} + X_\perp$

$X_{/\!/} \in \mathcal{M}^Y$, $X_\perp \in (\mathcal{M}^Y)^\perp$ unique random var.

It is sufficient to prove:

i) $X - E\{X|\mathcal{F}^Y\} \in (\mathcal{M}^Y)^\perp$

ii) $E\{X|\mathcal{F}^Y\} \in \mathcal{M}^Y$

ii) True because $E\{X|\mathcal{F}^Y\}$ is $\mathcal{F}^Y$-measurable

i) Prove that the scalar product $\langle X - E\{X|\mathcal{F}^Y\}, Z \rangle = 0 \quad \forall Z \in \mathcal{M}^Y$

If $Z \in \mathcal{M}^Y$:

$$\langle X - E\{X|\mathcal{F}^Y\}, Z \rangle = E\{(X - E\{X|\mathcal{F}^Y\})Z\} =$$

$$= E\{XZ - E\{XZ|\mathcal{F}^Y\}\} =$$

$$= E\{XZ\} - E\{E\{XZ|\mathcal{F}^Y\}\} =$$

$$= E\{XZ\} - E\{XZ\} = 0$$

- **Projection Theorem** V

Given H a linear space with $\langle \cdot, \cdot \rangle_H$,
$M \in H$ closed subspace

$$\begin{cases} \forall v \in H \;\; \exists! \; m_0 \in M : \\ \| v - m_0 \|_H \leq \| v - m \|_H , \;\; \forall m \in M \quad (*) \end{cases}$$

$\| \cdot \|_H = \langle \cdot, \cdot \rangle$

Nec. & Suff. cond. for $(*)$ to be true is:

$\langle v - m_0 , m \rangle_H = 0 \quad \forall m \in M$

→ Alternative formulation:

$\forall v \in H , \; \underset{m \in M}{\operatorname{argmin}} \| v - m \| = \Pi (v \mid M)$