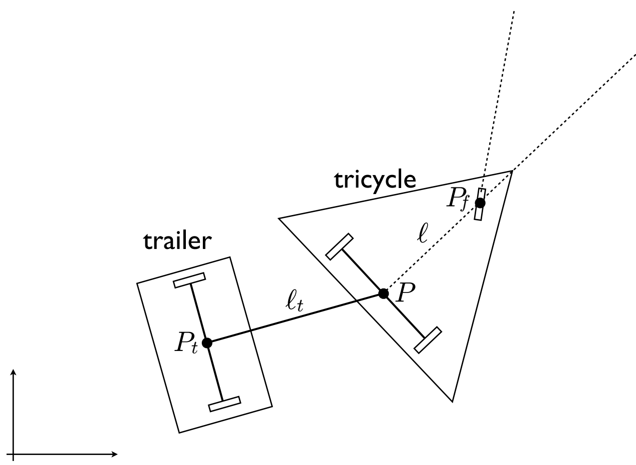
Autonomous and Mobile Robotics
Class Test no. 1

Problem 1

Consider the mobile robot shown in figure, obtained by attaching a trailer to a rear-wheel drive tricycle. The trailer is a rigid body with an axle carrying two fixed wheels, and is connected to the midpoint of the tricycle rear axle through a revolute joint. Both two-wheel axles can be assimilated to a single wheel located at the midpoint of the axle.



1. Find a set of generalized coordinates for the robot, and clearly indicate them on the drawing.
2. Write the Pfaffian kinematic constraints to which the robot is subject.
3. Derive a kinematic model of the system.

Problem 2

Consider the kinematic model of the rear-wheel drive car-like robot

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \omega$$

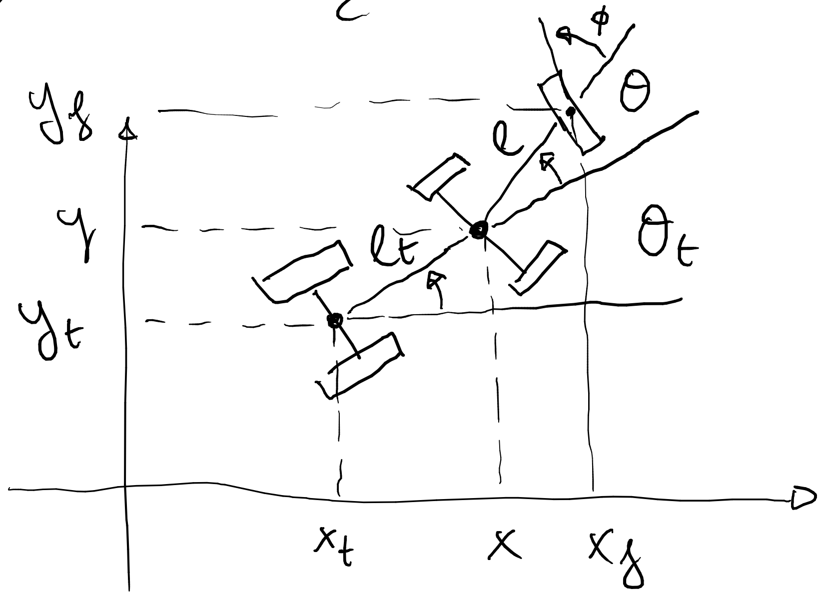
with the appropriate meaning for x, y, θ, ϕ, ℓ, v, ω.

In analogy to the case of the unicycle, identify a point B whose cartesian coordinates y_1, y_2, when taken as outputs of the system, allow to perform an input/output linearization via feedback [*Hint: you need to visualize a point, not necessarily on the robot body, that moves independently when the two control inputs v, ω are separately applied to the robot*]. Use this approach to derive a control law that achieves tracking of an arbitrary trajectory $(y_{1d}(t), y_{2d}(t))$.

[90 mins]

① Set of generalized coordinates

1.



2. 3 Rolling without slipping constraints

$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$ Rear

$\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$ front

$\dot{x}_t \sin(\theta_t) - \dot{y}_t \cos(\theta_t) = 0$ trailer

$$\begin{cases} x_f = x + l\cos\theta \\ y_f = y + l\sin\theta \end{cases} \qquad \begin{cases} x_t = x - l_t\cos\theta \\ y_t = y - l_t\sin\theta \end{cases}$$

Kinematic constraints:

$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta}\, l\cos\phi = 0$

$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$

$\dot{x}\sin\theta - \dot{y}\cos\theta + l_t\,\dot{\theta}_t$

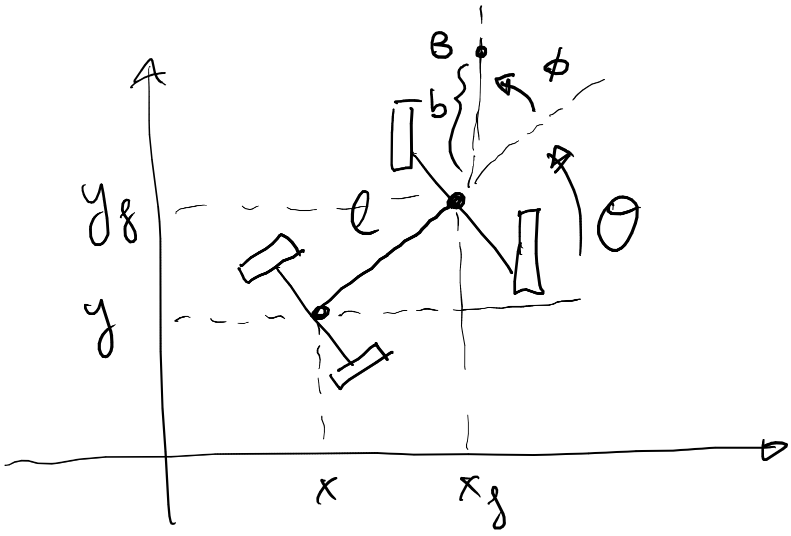
in Pfaffian form

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 0 & l_t \end{pmatrix} \dot{q} = 0$$

3. Kinematic model

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \frac{1}{\ell}\tan\phi \\ -\frac{1}{\ell_t}\sin(\theta_t - \theta) \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \omega$$

(2)



b: small offset

cartesian outputs:

$$y_1 = x + \ell\cos\theta + b\cos(\theta+\phi)$$
$$y_2 = y + \ell\sin\theta + b\sin(\theta+\phi)$$

$b \neq 0$

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{pmatrix}\cos\theta - \tan\phi\left(\sin\theta + b\sin(\theta+\phi)/\ell\right) & -b\sin(\theta+\phi)\\ \sin\theta + \tan\phi\left(\cos\theta + b\cos(\theta+\phi)/\ell\right) & b\cos(\theta+\phi)\end{pmatrix}\begin{pmatrix}v\\ \omega\end{pmatrix}$$

$$= T(\theta,\phi)\begin{pmatrix}v\\ \omega\end{pmatrix}$$

$$\begin{pmatrix}v\\ \omega\end{pmatrix} = T^{-1}(x)\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

→ new input

$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

Control simple integrator

$$e = \begin{pmatrix}y_1 - y_{1d}\\ y_2 - y_{2d}\end{pmatrix} \qquad \dot e = \begin{pmatrix}\dot y_1 - \dot y_{1d}\\ \dot y_2 - \dot y_{2d}\end{pmatrix} = \begin{pmatrix}u_1\\ u_2\end{pmatrix} - \begin{pmatrix}\dot y_{1d}\\ \dot y_{2d}\end{pmatrix} = -Ke$$

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}\dot y_{1d}\\ \dot y_{2d}\end{pmatrix} - \begin{pmatrix}K_1 & K_2\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

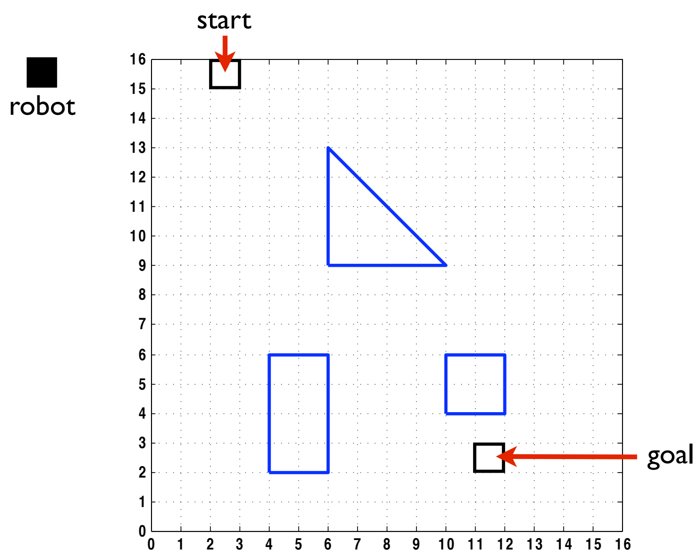
$K_1 > 0$
$K_2 > 0$

for exponential convergence to zero

Autonomous and Mobile Robotics
Class Test no. 2

Problem 1

Consider the planar motion planning problem shown in figure. The robot is a square of unit side and can translate freely in the plane without changing its orientation.



NO

Build the $\mathcal{C}$-obstacles and show all the steps of a solution obtained using the planning method based on approximate cell decomposition.

[*The steps should be graphically illustrated; in particular, for each step draw the current decomposition and the associated connectivity graph. Channels can be identified by visual path search. At the end, show a solution path extracted from the free channel.*]

Problem 2

Consider a rear-wheel drive car-like robot equipped with a sensor that can measure the distance between itself and a single landmark positioned at the origin of the Cartesian plane. The sensor is placed on the sagittal axis of the robot, at a distance $\ell/2$ from the midpoint of the rear wheel axis. The sensor can always see the landmark.

Build an Extended Kalman Filter for estimating the configuration of the robot.

[*The solution is obtained through the following steps (1) write a discrete-time model for the system and include noise (2) write the output equations and include noise (3) compute the linearization of the system and the output equations (4) write the EKF equations based on the obtained formulas.*]

[2 hrs]

② Kinematic model of the RWD car

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{\ell} \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \omega$$

Euler integration with process noise

$$x_{k+1} = x_k + v_k T_s \cos\theta_k + v_{1k}$$
$$y_{k+1} = y_k + v_k T_s \sin\theta_k + v_{2k}$$
$$\phi_{k+1} = \phi_k + v_k T_s \frac{\tan\phi}{\ell} + v_{3k}$$
$$\theta_{k+1} = \theta_k + \omega_k T_s + v_{4k}$$

$T_s = t_{k+1} - t_k$ $\qquad V_k = \begin{pmatrix} v_{1k} \\ \vdots \\ v_{4k} \end{pmatrix}$ white gaussian process noise with zero mean and covariance matrix V_k

$q_{k+1} = f(q_k, u_k) + v_k$ model in general form

the cartesian coordinates of the sensor are a function of the robot configuration q_k.

$$\begin{cases} x_{sk} = x_k + \frac{\ell}{2}\cos\theta_k \\ y_{sk} = y_k + \frac{\ell}{2}\sin\theta_k \end{cases} \longrightarrow p_s = (x_s, y_s)$$

Output (measurement) with noise

$$z_k = \sqrt{(x_{sk} - \underbrace{x_{ek}}_{\to 0})^2 + (y_{sk} - \underbrace{y_{ek}}_{\to 0})^2} + w_k = h(q_k) + w_k$$

white gaussian noise with zero mean and covariance matrix W_k

linearization of the process and output equations, evaluated at the previous estimate $\hat{q}_k$ and at the prediction $\hat{q}_{k+1|k}$:

$$F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k=\hat{q}_k} = \begin{pmatrix} 1 & 0 & -v_k T_s \sin\hat{\theta} & 0 \\ 0 & 1 & v_k T_s \cos\hat{\theta} & 0 \\ 0 & 0 & 1 & v_k T_s \frac{1}{\cos^2\phi\, l} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H_{k+1} = \left.\frac{\partial h}{\partial q_k}\right|_{q_k=\hat{q}_{k+1|k}} = \begin{pmatrix} x_{s\,k+1|k} - \overset{\circ}{x}_L & y_{s\,k+1|k} - \overset{\circ}{y}_L & & 0 \end{pmatrix}$$

$$\frac{\partial h}{\partial p_{sk}} \cdot \left.\frac{\partial p_{sk}}{\partial q_k}\right|_{q=\hat{q}_{k+1|k}} \qquad \frac{l}{2}\left(\hat{y}_{s,k+1|k}\cos\hat{\theta}_{k+1|k} - \hat{x}_{s,k+1|k}\sin\hat{\theta}_{k+1|k}\right)$$

EKF equations:

1. State and covariance prediction

$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$

$$P_{k+1|k} = F_k P_k F_k^T + V_k$$

→ noise covariance matrix

→ covariance of the estimate

2. Correction

$$\hat{q}_{k+1} = \hat{q}_{k+1|k} + R_{k+1} \nu_{k+1}$$

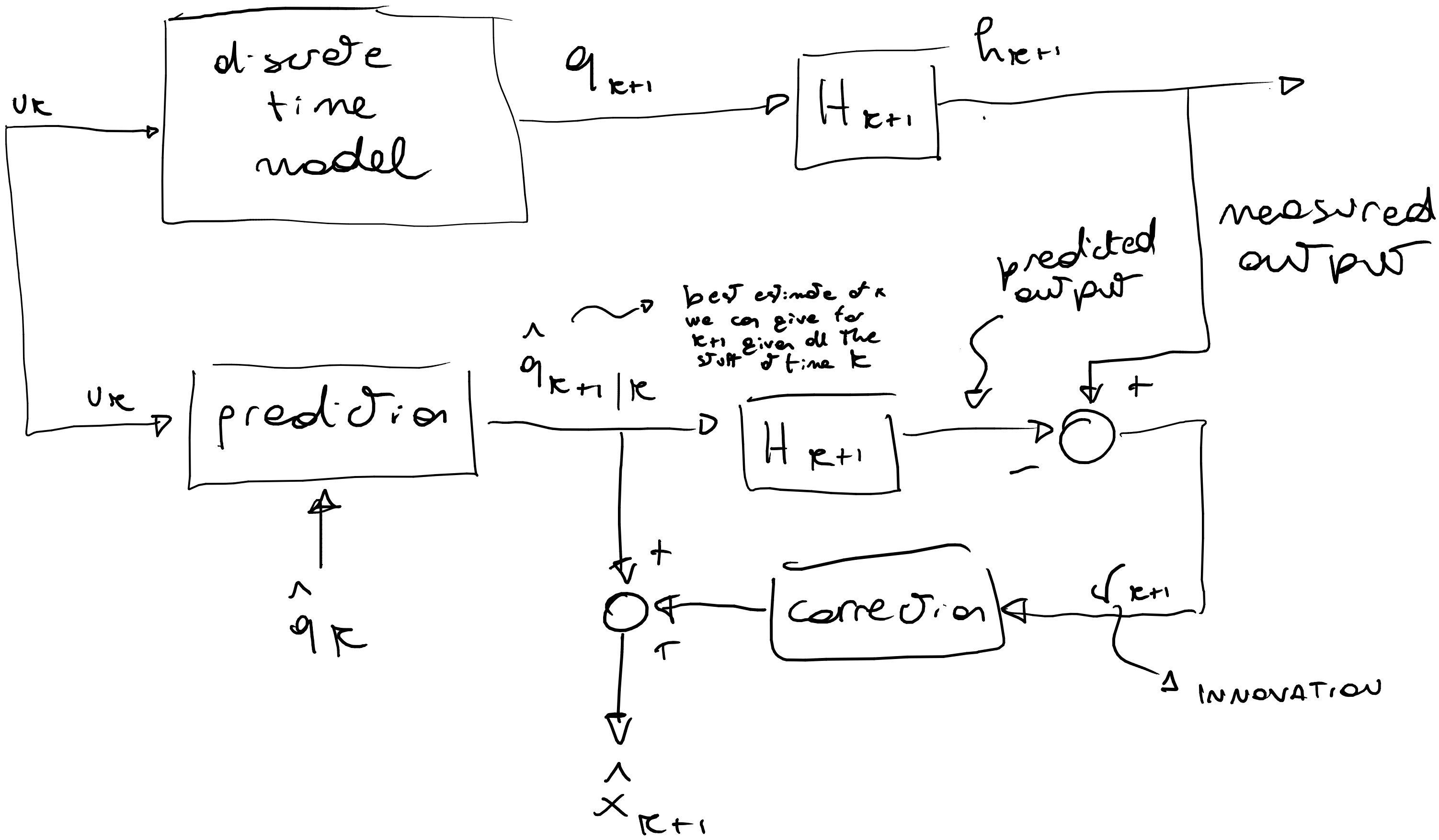
$$P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k}$$

where the innovation

$$\nu_{k+1} = z_{k+1} - h(\hat{q}_{k+1})$$

And the Kalman gain matrix

$$R_{k+1} = P_{k+1|k} H_{k+1}^T \left(H_{k+1} P_{k+1|k} H_{k+1}^T + W_{k+1}\right)^{-1}$$

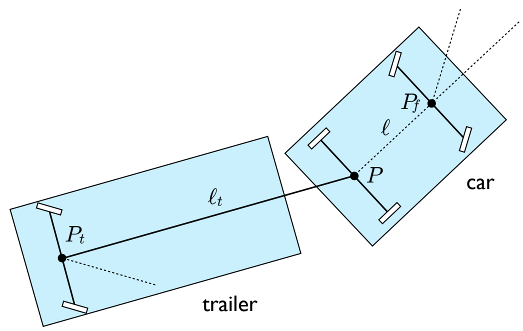


discrete time model
u_k
q_{k+1}
H_{k+1}
h_{k+1}
measured output
predicted output
prediction
$\hat{q}_{k+1|k}$
best estimate of x we can give for k+1 given all the stuff at time k
$\hat{q}_k$
correction
v_{k+1}
INNOVATION
$\hat{x}_{k+1}$

Autonomous and Mobile Robotics
Class Test no. 1, 2010/2011

Problem 1

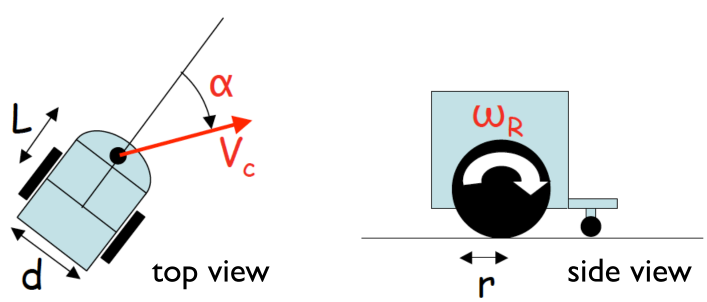
Consider the tractor-trailer system shown in figure, often referred to as the *firetruck* in the robotics literature. The tractor is a rear-wheel-drive car-like vehicle, while the trailer is a rigid body with an axle carrying two *steering* wheels, and is connected to the midpoint of the tractor rear axle through a revolute joint. The fact that the trailer wheels can be steered increases the maneuverability of the vehicle, which can thus negotiate sharp turn in spite of its size.



1. Find a set of generalized coordinates for the robot, and show them on the drawing.
2. Write the Pfaffian kinematic constraints to which the robot is subject (two-wheel axles can be assimilated to a single wheel located at the midpoint of the axle).
3. Derive a kinematic model of the system.

Problem 2

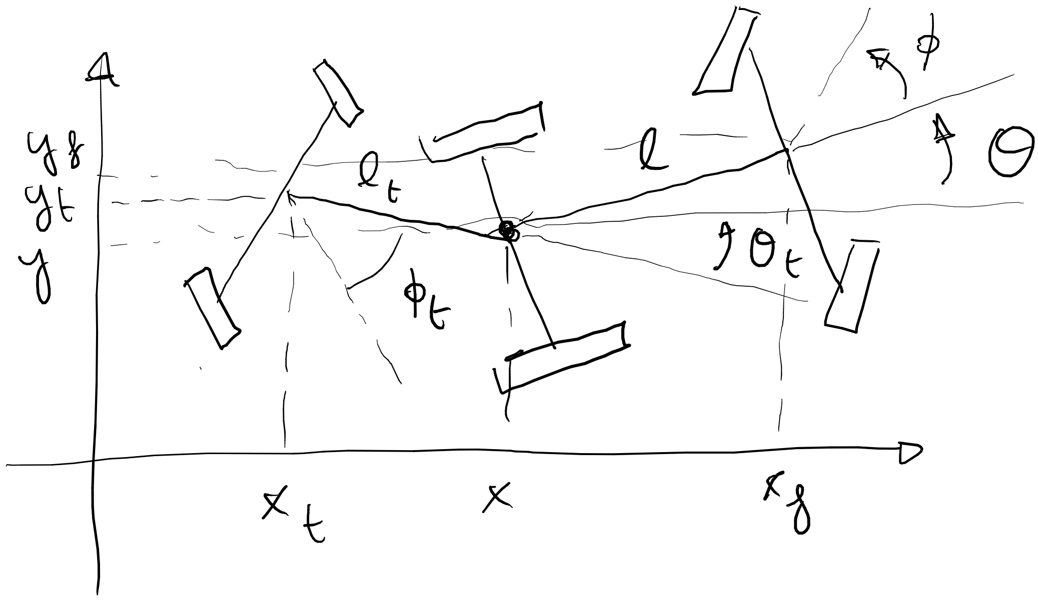
Consider the differential-drive robot shown below, where a passive sphere is used as a caster wheel.



Assume that we want to impose to the caster a velocity V_c directed as in figure. Compute the angular speeds ω_R and ω_L required to achieve this objective, using the following numerical data: $L = 0.3$ m, $d = 0.4$ m, $r = 0.15$ m, $\alpha = 45°$, $\|V_c\| = 0.1$ m/s [*Hint: you need a mapping between the velocity inputs ω_R, ω_L of the differential-drive robot and the velocity of a point located along the sagittal axis at a distance L from the midpoint between the wheels...*].

[1 h 45 mins]

1. Set of generalized coordinates



2.

3 RWS constraints

$$\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$$
$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0$$
$$\dot{x}_t \sin(\theta_t+\phi_t) - \dot{y}_t \cos(\theta_t+\phi_t) = 0$$

$$\begin{cases} x_f = x + l\cos\theta \\ y_f = y + l\sin\theta \end{cases} \qquad \begin{cases} x_t = x - l\cos\theta \\ y_t = y - l\sin\theta \end{cases}$$

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta} l\cos\phi = 0$$
$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
$$\dot{x}\sin(\theta_t+\phi_t) - \dot{y}\cos(\theta_t+\phi_t) + \dot{\theta}_t l_t \cos\phi_t = 0$$

In Pfaffian form

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 & 0 & 0 \\ \sin(\theta_t+\phi_t) & -\cos(\theta_t+\phi_t) & 0 & 0 & l_t\cos\phi_t & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \\ \dot{\phi}_t \end{pmatrix} = 0$$

$$
G(q) = \begin{array}{c} \begin{pmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ \tan\phi/\ell & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{\sin(\theta_t - \theta + \phi_t)}{\ell_t \cos\phi_t} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} g_1 & \qquad\qquad g_2 & \quad g_3 \end{array} \end{array}
$$

The kinematic control system is

$$
\dot{q} = g_1(q)\,v + g_2(q)\,\omega + g_3(q)\,\omega_t
$$

steering velocity of the trailer

② $P_c = (x_c, y_c)$ contact point of the caster

$$v = r\frac{\omega_R + \omega_L}{2} \qquad \omega = r\frac{\omega_R - \omega_L}{d}$$

$$\omega_R = \frac{2v + d\omega}{2r} \qquad \omega_L = \frac{2v - d\omega}{2r}$$

To write the velocity of P_c as a function of the velocity inputs ω_R, ω_L, one can first consider the robot as a unicycle and find the velocity inputs v, ω and then convert in ω_L, ω_R.

$$\begin{cases} x_c = x + L\cos\theta \\ y_c = y + L\sin\theta \end{cases}$$

cartesian outputs

$$V_c = \begin{pmatrix} \dot{x}_c \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} \cos\theta & -L\sin\theta \\ \sin\theta & L\cos\theta \end{pmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix} = T(\theta)\begin{pmatrix} v \\ \omega \end{pmatrix}$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = T^{-1}(\theta)\, V_c = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{L} & \frac{\cos\theta}{L} \end{pmatrix} \begin{pmatrix} \|V_c\|\cos(\theta - \alpha) \\ \|V_c\|\sin(\theta - \alpha) \end{pmatrix}$$

Autonomous and Mobile Robotics
Class Test no. 2, 2010/2011

Problem 1

Consider a 2R planar manipulator with the first joint at the origin of the plane and links of unit length. Regardless of the initial configuration, we want to bring the tip of the manipulator to the goal point $(1, -1)$ while avoiding collisions with a point obstacle located at $(1, 0)$. To solve this motion planning problem, one can take the following approach:

NO

1. define a suitable set of control points on the manipulator;
2. build an appropriate artificial force field for each control point;
3. control the robot by imposing to its joints the torques resulting from the combined action of the Cartesian force fields.

Compute the complete expression of the torques as a function of the robot configuration.

Problem 2

Consider a fixed-wing UAV flying at a constant altitude with zero pitch angle. In this particular condition, its configuration can be described as $(x \quad y \quad \psi \quad \phi)^T$, where (x, y) are the Cartesian coordinates of the center of gravity, ψ is the yaw angle, and ϕ is the roll angle. The UAV dynamic model is

$$\begin{aligned} \dot{x} &= v \cos \psi \\ \dot{y} &= v \sin \psi \\ \dot{\psi} &= -\frac{g}{v} \tan \phi \\ \dot{\phi} &= u_\phi \end{aligned}$$

where g is the gravity acceleration. The UAV speed v and roll rate u_ϕ are the available control inputs. Assume that the UAV is equipped with a radio sensor that can measure the bearing angle between the UAV main axis and two radio beacons, respectively placed at $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. For simplicity, assume that:

- the sensor is exacty located at the center of gravity;
- the beacons are located on the plane of flight;
- the sensor can distinguish between the two landmarks (e.g., by radio frequency);
- the sensor can always see both landmarks.

Build an Extended Kalman Filter for estimating the configuration of the UAV.

[2 hrs 30 mins]

② Euler integration + noise

$$\left.\begin{aligned} x_{k+1} &= x_k + v_k T_s \cos\psi_k + w_{p_1 k} \\ y_{k+1} &= y_k + v_k T_s \sin\psi_k + w_{p_2 k} \\ \psi_{k+1} &= \psi_k - \frac{g}{v_k} T_s \tan\phi_k + w_{p_3 k} \\ \phi_{k+1} &= \phi_k + u_{\phi,k} T_s + w_{p_4 k} \end{aligned}\right\} f(q_k, u_k) + w_{p,k}$$

Output measurement with noise (bearing of landmarks)

$$z_k = \begin{pmatrix} \mathrm{ATAN2}(y_k - y_1, x_k - x_1) - \psi_k \\ \mathrm{ATAN2}(y_k - y_2, x_k - x_2) - \psi_k \end{pmatrix} + \begin{pmatrix} w_{m_1,k} \\ w_{m_2 k} \end{pmatrix} = h(q_k) + w_{m,k}$$

Linearization of the process and output equations

$$F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k = \hat{q}_k} = \begin{pmatrix} 1 & 0 & -v_k T_s \sin\psi_k & 0 \\ 0 & 1 & v_k T_s \cos\psi_k & 0 \\ 0 & 0 & 1 & -\frac{g}{v_k}\frac{T_s}{\cos^2\phi_k} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H_{k+1} = \left.\frac{\partial h}{\partial q_k}\right|_{q_k = \hat{q}_{k+1|k}} =$$

$$= \begin{pmatrix} \frac{-(\hat{y}_{k+1|k} - y_1)}{(\hat{x}_{k+1|k} - x_1)^2 + (\hat{y}_{k+1|k} - y_1)^2} & \frac{\hat{x}_{k+1|k} - x_1}{(\hat{x}_{k+1|k} - x_1)^2 + (\hat{y}_{k+1|k} - y_1)^2} & -1 & 0 \\ \frac{-(\hat{y}_{k+1|k} - y)}{(\hat{x}_{k+1|k} - x_2)^2 + (\hat{y}_{k+1|k} - y_2)^2} & \frac{\hat{x}_{k+1|k} - x_2}{(\hat{x}_{k+1|k} - x_2)^2 + (\hat{y}_{k+1|k} - y_2)^2} & -1 & 0 \end{pmatrix}$$

EKF equations :

1. State and covariance prediction

$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$

$\longrightarrow$ noise covariance matrix

$$P_{k+1|k} = F_k P_k F_k^T + W_{pk}$$

$\longrightarrow$ covariance of the estimate

2. Correction

$$\hat{q}_{k+1} = \hat{q}_{k+1|k} + R_{k+1} V_{k+1}$$

$$P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k}$$

where the innovation

$$V_{k+1} = z_{k+1} - h(\hat{q}_{k+1})$$

And the Kalman gain matrix

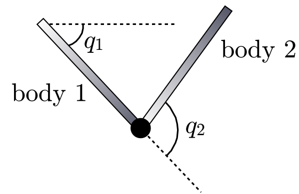
$$R_{k+1} = P_{k+1|k} H_{k+1}^T \left(H_{k+1} P_{k+1|k} H_{k+1}^T + W_{k+1} \right)^{-1}$$

The covariance of the estimate P_k will be initialized at a certain value P_0 reflecting the uncertainty of the initial estimate $\hat{q}_0$.

Autonomous and Mobile Robotics
Final Class Test, 2011/2012

Problem 1

Consider the two-body space robot shown in figure. The robot freely floats in the absence of gravity.



This system, whose configuration is described by the vector $(q_1 \;\; q_2)^T$, is subject to the conservation of the angular momentum. This can be expressed as a kinematic constraint:

$$a_1(q_2)\dot{q}_1 + a_2(q_2)\dot{q}_2 = 0$$

where a_1 and a_2 are functions of q_2 and of the dynamic parameters of the robot (masses, lengths, inertias).

1. Is this constraint Pfaffian?
2. Derive the corresponding kinematic model of the system.
3. Is the system controllable or not? Correspondingly, is the constraint holonomic or not?

Problem 2

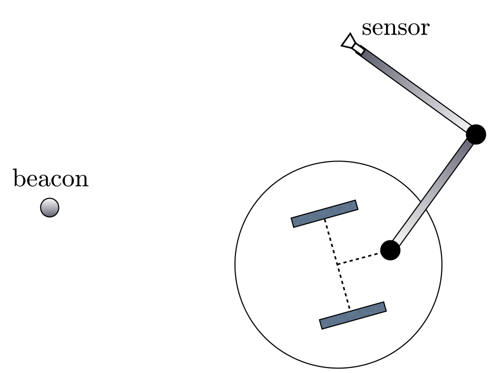
Consider a planar circular robot with differential-drive kinematics. Denote by $v_{\max}$, $\omega_{\max}$ the bounds on the absolute value of the robot velocity inputs, and by R the radius of the robot base. The robot must travel between cartesian points P_S (start) and P_g (goal) in a perfectly known planar environment that contains polygonal obstacles. Build a complete navigation system that integrates the following modules:

- a motion planner that generates a feasible collision-free path;
- a trajectory planner that computes admissible robot velocities along this path;
- a feedback controller that can track the reference trajectory;

Discuss in detail the possible options and the motivation behind your choices. Provide a block scheme of your system with a clear indication of the inputs and the outputs of each block. Points that deserve special attention are: (1) is your motion planner complete, and under which assumptions? (2) does your motion planner generate paths that the robot can follow? (3) will the reference trajectory belong to the class that your feedback controller can track?

Problem 3

Consider the mobile manipulator in figure, consisting of a differential-drive base carrying a 2R planar horizontal arm.



The end-effector of the arm carries an exteroceptive sensor which can measure the distance to a beacon whose position in the environment is exactly known. For simplicity, assume that the sensor can always 'see' the beacon. The proprioceptive sensors are the wheel encoders for the base and the joint encoders for the arm. Derive the equations of an Extended Kalman Filter for estimating the configuration of the mobile manipulator, providing also a detailed block scheme of its structure. [*Hint: a kinematic model for the arm consists of simple integrators*]

[3 h]

Autonomous and Mobile Robotics
Solution of Final Class Test, 2011/2012

Note: only solutions to Problems 1 and 3 are provided. Problem 2 is actually a discussion and many different choices are possible.

Solution of Problem 1

The angular momentum conservation constraint is linear in the generalized velocities, and therefore Pfaffian. In particular, it can be rewritten as

$$\boldsymbol{a}^T(\boldsymbol{q})\,\dot{\boldsymbol{q}} = 0 \qquad \text{with} \quad \boldsymbol{a}^T(\boldsymbol{q}) = (a_1(q_2)\;\; a_2(q_2))$$

The corresponding kinematic model describes the admissible velocities as $\dot{\boldsymbol{q}} \in \mathcal{N}(\boldsymbol{a}^T(\boldsymbol{q}))$. The number of generalized coordinates is $n = 2$ while the number of constraints is $k = 1$; therefore, the dimension of the null space of $\boldsymbol{a}^T(\boldsymbol{q})$ is $n - k = 1$. One possible basis for this null space is clearly given by

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_2(q_2) \\ -a_1(q_2) \end{pmatrix}$$

Correspondingly, the kinematic model associated to the conservation of angular momentum is

$$\dot{\boldsymbol{q}} = v\,\boldsymbol{g}(\boldsymbol{q}) = v \begin{pmatrix} a_2(q_2) \\ -a_1(q_2) \end{pmatrix}$$

where $v \in I\!R$ is the velocity input for this model (no direct physical meaning).

The controllability of the above kinematic model is characterized by its accessibility distribution $\Delta_{\mathcal{A}}$, i.e., the involutive closure of $\Delta = \text{span}(\boldsymbol{g})$. Since Δ is generated by a single vector field, it is necessarily involutive, i.e., $\Delta_{\mathcal{A}} = \Delta$. Then we have $\dim \Delta_{\mathcal{A}} = \dim \Delta = 1 < 2$ and the system is *not controllable*. This means that the angular momentum conservation constraint is *holonomic*, i.e., it can be written as a geometric constraint $h(q_1, q_2) = 0$. It is therefore impossible to move the coordinates q_1 and q_2 to arbitrary values; once one of them is given, the other follows from the constraints.

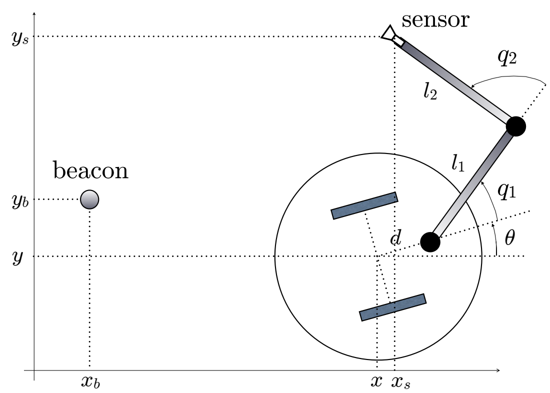
Note that the angular momentum is conserved only if no external forces act on the system. In the considered case, this means that the only available input is the torque at the rotational joint between the two bodies. We may then write a kinematic model whose velocity input is 'closer' to the actual input. To this end, consider the following alternative[1] basis for the null space of $\boldsymbol{a}^T(\boldsymbol{q})$

$$\boldsymbol{g}'(\boldsymbol{q}) = \begin{pmatrix} -\dfrac{a_2(q_2)}{a_1(q_2)} \\ 1 \end{pmatrix}$$

and the corresponding kinematic model

$$\dot{\boldsymbol{q}} = v_2 \begin{pmatrix} -\dfrac{a_2(q_2)}{a_1(q_2)} \\ 1 \end{pmatrix}$$

with the velocity input $v_2 = \dot{q}_2$, i.e., the velocity of the actuated rotational joint. Obviously, this model is also not controllable.



Solution of Problem 3

Refer to the figure for the definition of the relevant variables and quantities. The configuration vector of the mobile manipulator is $\boldsymbol{q} = (x \;\; y \;\; \theta \;\; q_1 \;\; q_2)^T$. The corresponding kinematic model is

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ u_1 \\ u_2 \end{pmatrix}$$

with obvious meaning for v, ω, u_1, u_2. Using Euler integration and including noise, a discrete-time nonlinear model of this system is obtained as

$$\boldsymbol{q}_{k+1} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \\ \theta_{k+1} \\ q_{1,k+1} \\ q_{2,k+1} \end{pmatrix} = \begin{pmatrix} x_k + v_k\, T_s \cos\theta_k \\ y_k + v_k\, T_s \sin\theta_k \\ \theta_k + \omega_k\, T_s \\ q_{1,k} + u_{1,k}\, T_s \\ q_{2,k} + u_{2,k}\, T_s \end{pmatrix} + \begin{pmatrix} w_{p1,k} \\ w_{p2,k} \\ w_{p3,k} \\ w_{p4,k} \\ w_{p5,k} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}_k, \boldsymbol{u}_k) + \boldsymbol{w}_{p,k}$$

where T_s is the sampling interval, $\boldsymbol{u}_k = (v_k \;\; \omega_k \;\; u_{1,k} \;\; u_{2,k})^T$ is the input vector in $[t_k, t_{k+1}]$, and $\boldsymbol{w}_{p,k} = (w_{p1,k} \;\; \ldots \;\; w_{p5,k})^T$ is a white gaussian process noise with zero mean and covariance matrix $\boldsymbol{W}_{p,k}$. As usual, v_k and ω_k will be reconstructed from wheel encoder readings (see, e.g., the formulas in the AMR slides "Odometric Localization"); whereas $u_{1,k}$, $u_{2,k}$ will be reconstructed from joint encoder readings, e.g., by numerical differentiation.

The Cartesian coordinates $\boldsymbol{p}_s = (x_s, y_s)$ of the sensor are expressed as

$$\begin{aligned} x_s &= x + d\cos\theta + l_1\cos(\theta + q_1) + l_2\cos(\theta + q_1 + q_2) \\ y_s &= y + d\sin\theta + l_1\sin(\theta + q_1) + l_2\sin(\theta + q_1 + q_2) \end{aligned}$$

The output equation is therefore

$$z_k = \sqrt{(x_{s,k} - x_b)^2 + (y_{s,k} - y_b)^2} + w_{m,k} = h(\boldsymbol{q}_k) + w_{m,k}$$

where $w_{m,k}$ is a white gaussian measurement noise with zero mean and (co)variance $W_{m,k}$. Note that z_k, $h(\boldsymbol{q}_k)$, $w_{m,k}$ and $W_{m,k}$ are all scalars.

The linearization of the process and output equations, respectively evaluated at the previous estimate $\hat{\boldsymbol{q}}_k$ and at the prediction $\hat{\boldsymbol{q}}_{k+1|k}$, gives

$$\boldsymbol{F}_k = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k = \hat{\boldsymbol{q}}_k} = \begin{pmatrix} 1 & 0 & -v_k T_s \sin\hat{\theta}_k & 0 & 0 \\ 0 & 1 & v_k T_s \cos\hat{\theta}_k & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[1]We are assuming that $a_1(q_2) \neq 0$, $\forall q_2$.

and

$$\begin{aligned}
\boldsymbol{H}_{k+1} &= \left.\frac{\partial h}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}_k=\hat{\boldsymbol{q}}_{k+1|k}} = \left.\frac{\partial h}{\partial \boldsymbol{p}_{s,k}}\frac{\partial \boldsymbol{p}_{s,k}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}} \\
&= \frac{1}{\sqrt{(\hat{x}_{s,k+1|k}-x_b)^2+(\hat{y}_{s,k+1|k}-y_b)^2}} \begin{pmatrix} \hat{x}_{s,k+1|k}-x_b & \hat{y}_{s,k+1|k}-y_b \end{pmatrix} \left.\frac{\partial \boldsymbol{p}_{s,k}}{\partial \boldsymbol{q}_k}\right|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}}
\end{aligned}$$

Here, $\hat{x}_{s,k+1|k}, \hat{y}_{s,k+1|k}$ are the sensor coordinates at the predicted configuration $\hat{\boldsymbol{q}}_{k+1|k}$. The 2×5 Jacobian matrix $\partial \boldsymbol{p}_{s,k}/\partial \boldsymbol{q}_k\big|_{\boldsymbol{q}=\hat{\boldsymbol{q}}_{k+1|k}}$ is easily computed from the above expressions for x_s and y_s.

The EKF equations are finally obtained as follows.

1. State and covariance prediction:

$$\begin{aligned}
\hat{\boldsymbol{q}}_{k+1|k} &= \boldsymbol{f}(\hat{\boldsymbol{q}}_k, \boldsymbol{u}_k) \\
\boldsymbol{P}_{k+1|k} &= \boldsymbol{F}_k \boldsymbol{P}_k \boldsymbol{F}_k^T + \boldsymbol{W}_{p,k}
\end{aligned}$$

2. Correction:

$$\begin{aligned}
\hat{\boldsymbol{q}}_{k+1} &= \hat{\boldsymbol{q}}_{k+1|k} + \boldsymbol{R}_{k+1}\nu_{k+1} \\
\boldsymbol{P}_{k+1} &= \boldsymbol{P}_{k+1|k} - \boldsymbol{R}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}
\end{aligned}$$

   where the innovation

$$\nu_{k+1} = z_{k+1} - \sqrt{(x_{s,k+1|k}-x_b)^2+(y_{s,k+1|k}-y_b)^2}$$

   is a scalar quantity and the Kalman gain matrix

$$\boldsymbol{R}_{k+1} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + W_{m,k+1})^{-1}$$

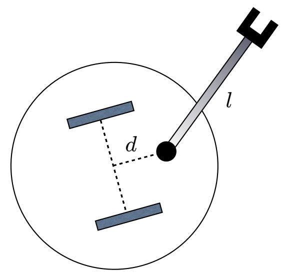
   is a 5×1 matrix (note that $\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + W_{m,k+1}$ is actually a scalar, so no matrix inverse computation is required).

In these equations, $\boldsymbol{P}_k$ obviously denotes the covariance of the estimate, which will be initialized at a certain value reflecting the uncertainty on the initial estimate $\hat{q}_0$.

Autonomous and Mobile Robotics
Final Class Test, 2012/2013

Problem 1

Consider the mobile manipulator in figure, consisting of a differential-drive base carrying a 1R planar horizontal arm of length l. The arm is hinged at a distance d from the center of the base, which is located in correspondence of the midpoint of the two wheels.



1. Define a configuration vector for the robot.
2. Write a kinematic model of the robot. [*Hint: a kinematic model for the arm is a simple integrator*]
3. Express the end effector position (x_{ee}, y_{ee}) as an output of the kinematic model. Is it a flat output?
4. Use the kinematic model to design a control law for tracking an assigned trajectory $(x^*_{ee}(t), y^*_{ee}(t))$ with the end-effector. [*Hint: use input-output linearization*]

Problem 2

Consider an RRT-based motion planner for a unicycle robot based on the use of the following motion primitives:

$$v = \bar{v} \qquad \omega = \{-\bar{\omega}, 0, \bar{\omega}\} \qquad t \in [t_k, t_{k+1}]$$

where v and w are, respectively, the driving and steering velocity inputs, $\bar{v}$ and $\bar{\omega}$ are positive constants, and t_k, t_{k+1} are two consecutive sampling instants. The unicycle body is a circle of radius r. Assume that the environment is an empty square room, so that the only obstacles are the room walls. The goal configuration is the center of the room.

NO

1. Which is the minimum necessary clearance (i.e., distance from the unicycle center to the obstacles) for the unicycle at the start to guarantee that the planner will find asymptotically a solution? [*Hint: if the unicycle is facing a wall...*]
2. How would you tune or modify the planner to reduce or possibly eliminate this clearance?

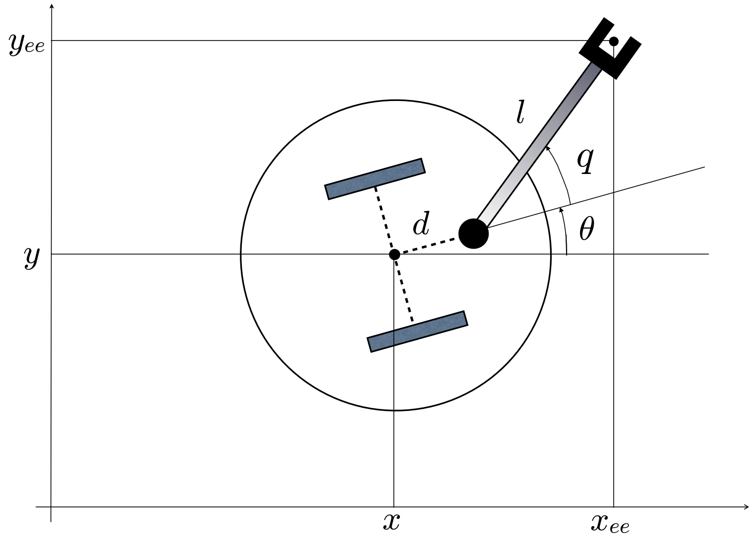
Problem 3

Consider two omnidirectional point robots navigating in an environment containing a single point landmark. Each robot is equipped with a sensor that can measure (1) the relative distance between itself and the other robot (2) the relative distance between itself and the landmark. For simplicity, assume that the sensor can distinguish between the 'other robot' and the landmark, and that it can always 'see' both. Build an EKF-based system for simultaneously localizing the robots and the landmark, given an initial estimate of their positions.

[3 h]

Autonomous and Mobile Robotics
Solution of Final Class Test, 2012/2013

Solution of Problem 1



Refer to the figure for the definition of the relevant variables. The configuration vector of the mobile manipulator is $\boldsymbol{q} = (x\ \ y\ \ \theta\ \ q)^T$. The corresponding kinematic model is

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ u \end{pmatrix}$$

with obvious meaning for v, ω, u. The coordinates of the end-effector are

$$\begin{aligned} x_{ee} &= x + d\cos\theta + l\cos(\theta+q) \\ y_{ee} &= y + d\sin\theta + l\sin(\theta+q) \end{aligned}$$

To design a trajectory tracking controller for this output, differentiate it w.r.t. time and use the kinematic model equation:

$$\begin{aligned} \dot{x}_{ee} &= v\cos\theta - d\,\omega\sin\theta - l\sin(\theta+q)(\omega+u) \\ \dot{y}_{ee} &= v\sin\theta + d\,\omega\cos\theta + l\cos(\theta+q)(\omega+u) \end{aligned}$$

which may be conveniently rewritten as

$$\begin{pmatrix} \dot{x}_{ee} \\ \dot{y}_{ee} \end{pmatrix} = \begin{pmatrix} \cos\theta & -d\,\sin\theta - l\sin(\theta+q) & -l\sin(\theta+q) \\ \sin\theta & d\,\cos\theta + l\cos(\theta+q) & l\cos(\theta+q) \end{pmatrix} \begin{pmatrix} v \\ \omega \\ u \end{pmatrix} = \boldsymbol{T}(\theta,q) \begin{pmatrix} v \\ \omega \\ u \end{pmatrix} \tag{1}$$

Matrix $\boldsymbol{T}(\theta,q)$ has always full row rank. In fact, its rank does not change if we replace the second column with the difference of the second and the third column, obtaining

$$\boldsymbol{T}'(\theta,q) = \boldsymbol{T}(\theta,q) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -d\,\sin\theta & -l\sin(\theta+q) \\ \sin\theta & d\,\cos\theta & l\cos(\theta+q) \end{pmatrix}$$

and this matrix has always rank 2 if $d \neq 0$ (compute the minor corresponding to the first two columns). We may therefore define the following input transformation

$$\begin{pmatrix} v \\ \omega \\ u \end{pmatrix} = \boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{2}$$

where $\boldsymbol{T}^{\dagger}(\theta, q) = \boldsymbol{T}^T(\boldsymbol{T}\boldsymbol{T}^T)^{-1}$ is the pseudoinverse of $\boldsymbol{T}(\theta, q)$. Using this in (1) we get

$$\begin{pmatrix} \dot{x}_{ee} \\ \dot{y}_{ee} \end{pmatrix} = \boldsymbol{T}(\theta, q)\boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The input-output channels (from the new inputs w_1, w_2 to the outputs x_{ee}, y_{ee}, respectively) are now simple integrators. We can then guarantee global exponential convergence of the output to the desired trajectory by letting

$$\begin{aligned} w_1 &= \dot{x}^*_{ee} + k_1(x^*_{ee} - x_{ee}) \\ w_2 &= \dot{y}^*_{ee} + k_2(y^*_{ee} - y_{ee}) \end{aligned}$$

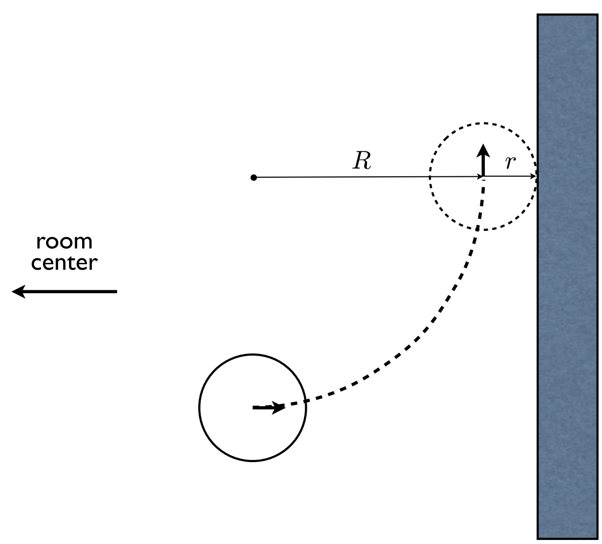
with k_1, k_2 positive gains. The actual control law to be implemented is readily obtained by substituting the latter expressions for w_1, w_2 in (2).

Finally, equation (1) shows that the end-effector position is *not* a flat output. In fact, once $\dot{x}^*_{ee}$, $\dot{y}^*_{ee}$ are specified by the assigned trajectory, there exist an infinity of velocity inputs that realize them:

$$\begin{pmatrix} v^* \\ \omega^* \\ u^* \end{pmatrix} = \boldsymbol{T}^{\dagger}(\theta, q) \begin{pmatrix} \dot{x}^*_{ee} \\ \dot{y}^*_{ee} \end{pmatrix} + (\boldsymbol{I} - \boldsymbol{T}^{\dagger}(\theta, q)\boldsymbol{T}(\theta, q))\boldsymbol{z}$$

where $\boldsymbol{z}$ is an arbitrary 3-vector.

Solution of Problem 2



Consider the initial arrangement shown in figure. Clearly, the only way for the unicycle to avoid collision with the wall is to turn left (or right) as much as possible, by choosing $\omega = \bar{\omega}$ ($\omega = -\bar{\omega}$) for as many consecutive intervals as needed. This obviously corresponds to the robot moving along an arc of circle of radius $R = \bar{v}/\bar{\omega}$. The desired clearance is therefore $r + R = r + \bar{v}/\bar{\omega}$.

To reduce the clearance, one may either decrease $\bar{v}$ or increase $\bar{\omega}$. To eliminate it, extend the primitive set by including $v = 0$ (this will allow rotation on the spot) or $v = -\bar{v}$ (this will allow backward motions).

Solution of Problem 3

Define the extended state vector to be estimated as $\boldsymbol{\chi} = (x_1\ y_1\ x_2\ y_2\ x_l\ y_l)^T$, where (x_i, y_i) are the cartesian coordinates of robot i $(i = 1, 2)$ and (x_l, y_l) are the cartesian coordinates of the landmark. The discrete-time model describing the motion of the extended robots+landmark system is

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{x1,k} \\ u_{y1,k} \\ u_{x2,k} \\ u_{y2,k} \end{pmatrix} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ v_{4,k} \\ 0 \\ 0 \end{pmatrix}$$

where $(u_{x1,k}\ \ u_{y1,k}\ \ u_{x2,k}\ \ u_{y2,k})^T$ collects the robots' velocity inputs and $v_{i,k}$ is a white gaussian noise with zero mean and covariance $V_{i,k}$ $(i = 1, \ldots, 4)$. Note how the landmark being fixed reflects on the last two rows of the above equation. This is clearly a linear model of the form

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{A}\boldsymbol{\chi}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{v}_k$$

where $\boldsymbol{A} = \boldsymbol{I}$, the 6×6 identity matrix, $\boldsymbol{u}_k$ is the input vector, and $\boldsymbol{v}_k$ is the process noise vector, whose covariance is

$$\boldsymbol{V}_k = \begin{pmatrix} V_{1,k} & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{2,k} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{3,k} & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{4,k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Coming to the measurement model, we have four measurements coming from the two robot sensors:

$$\boldsymbol{h}_k = \begin{pmatrix} h_1(\boldsymbol{\chi}_K) \\ h_2(\boldsymbol{\chi}_K) \\ h_3(\boldsymbol{\chi}_K) \\ h_4(\boldsymbol{\chi}_K) \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \\ w_{3,k} \\ w_{4,k} \end{pmatrix} = \begin{pmatrix} \sqrt{(x_{1,k} - x_{2,k})^2 + (y_{1,k} - y_{2,k})^2} \\ \sqrt{(x_{1,k} - x_l)^2 + (y_{1,k} - y_l)^2} \\ \sqrt{(x_{2,k} - x_{1,k})^2 + (y_{2,k} - y_{1,k})^2} \\ \sqrt{(x_{2,k} - x_l)^2 + (y_{2,k} - y_l)^2} \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \\ w_{3,k} \\ w_{4,k} \end{pmatrix}$$

where w_i, k is a white gaussian noise with zero mean and covariance $W_{i,k}$ $(i = 1, \ldots, 4)$. Ideally, the first and third components should be identical; however, since they are measured by two different sensors, their measured values will in general be different and therefore the entries must be duplicated in the model.

We are now ready to derive the EKF equations. Note that, since the process dynamics is linear, there is no need to linearize it. As for the linearization of the output equations, we have

$$\boldsymbol{H}_{k+1} = \begin{pmatrix} \left.\dfrac{\partial h_1}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_2}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_3}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \\ \left.\dfrac{\partial h_4}{\partial \boldsymbol{\chi}}\right|_{\boldsymbol{\chi}=\hat{\boldsymbol{\chi}}_{k+1|k}} \end{pmatrix}$$

where

$$
\begin{aligned}
\frac{\partial \boldsymbol{h}_1}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_2}{\delta(\boldsymbol{\chi})} & \frac{y_1 - y_2}{\delta(\boldsymbol{\chi})} & \frac{x_2 - x_1}{\delta(\boldsymbol{\chi})} & \frac{y_2 - y_1}{\delta(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_2}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_l}{\eta_1(\boldsymbol{\chi})} & \frac{y_1 - y_l}{\eta_1(\boldsymbol{\chi})} & 0 & 0 & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_3}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} \frac{x_1 - x_2}{\delta(\boldsymbol{\chi})} & \frac{y_1 - y_2}{\delta(\boldsymbol{\chi})} & \frac{x_2 - x_1}{\delta(\boldsymbol{\chi})} & \frac{y_2 - y_1}{\delta(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix} \\
\frac{\partial \boldsymbol{h}_4}{\partial \boldsymbol{\chi}} &= \begin{pmatrix} 0 & 0 & \frac{x_2 - x_l}{\eta_2(\boldsymbol{\chi})} & \frac{y_2 - y_l}{\eta_2(\boldsymbol{\chi})} & 0 & 0 \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta(\boldsymbol{\chi}) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
\eta_1(\boldsymbol{\chi}) &= \sqrt{(x_1 - x_l)^2 + (y_1 - y_l)^2} \\
\eta_2(\boldsymbol{\chi}) &= \sqrt{(x_2 - x_l)^2 + (y_2 - y_l)^2}
\end{aligned}
$$

The EKF equations are finally obtained as follows.

1. State and covariance prediction:

$$
\begin{aligned}
\hat{\boldsymbol{\chi}}_{k+1|k} &= \hat{\boldsymbol{\chi}}_k + \boldsymbol{B}\boldsymbol{u}_k \\
\boldsymbol{P}_{k+1|k} &= \boldsymbol{P}_k + \boldsymbol{V}_k
\end{aligned}
$$

2. Correction:

$$
\begin{aligned}
\hat{\boldsymbol{\chi}}_{k+1} &= \hat{\boldsymbol{\chi}}_{k+1|k} + \boldsymbol{R}_{k+1}\boldsymbol{\nu}_{k+1} \\
\boldsymbol{P}_{k+1} &= \boldsymbol{P}_{k+1|k} - \boldsymbol{R}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}
\end{aligned}
$$

where the innovation is

$$\boldsymbol{\nu}_{k+1} = \boldsymbol{h}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\chi}}_{k+1|k})$$

and the Kalman gain matrix is computed as

$$\boldsymbol{R}_{k+1} = \boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T(\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^T + \boldsymbol{W}_{k+1})^{-1}$$

where $\boldsymbol{W}_{k+1} = \text{diag}\{W_{1,k+1}, \ldots, W_{4,k+1}\}$.

In these equations, $\boldsymbol{P}_k$ obviously denotes the covariance of the estimate, which will be initialized at a certain value reflecting the uncertainty on the initial estimate $\hat{\boldsymbol{\chi}}_0$.

Autonomous and Mobile Robotics
Final Class Test, 2013/2014

Problem 1

Consider the following nonlinear system that arises in quantum mechanics

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1\end{aligned}$$

This is called a *nonholonomic integrator* or *Heisenberg control system.*

1. Write the differential constraint underlying the above model.
2. Prove that such constraint is nonholonomic.
3. Prove that the system is controllable.
4. Compute the final state displacement after a Lie Bracket control maneuver.

Problem 2

Discuss the nature and the dimension of the configuration space for the following mechanical systems.

1. A team of two mobile robots that can translate and rotate in the plane.
2. A team of two mobile robots that can translate and rotate in the plane and are connected by a rope.
3. A team of two mobile robots that can translate and rotate in the plane and are connected by a rigid bar.
4. A spacecraft with a 6R robot arm.
5. The last link of 6R robot arm mounted on a spacecraft.

Problem 3

Consider a fixed-wing Unmanned Aerial Vehicle (UAV) flying at a constant altitude $z = \bar{z}$ with zero pitch angle. In this particular condition, the UAV configuration can be described as $(x \quad y \quad \psi \quad \phi)^T$, where (x, y) are the Cartesian coordinates of its center of gravity, ψ is the yaw angle, and ϕ is the roll angle. The UAV dynamic model is

$$\begin{aligned}\dot{x} &= v\cos\psi\\ \dot{y} &= v\sin\psi\\ \dot{\psi} &= -\frac{g}{v}\tan\phi\\ \dot{\phi} &= u_\phi\end{aligned}$$

where g is the gravity acceleration. The UAV speed v and roll rate u_ϕ are the available control inputs. The UAV is equipped with a sensor, located at the center of gravity, that can measure the distance between itself and two radio beacons, located on the ground ($z = 0$) at unknown positions. Under the assumption that the sensor can distinguish between the two beacons, build an Extended Kalman Filter for estimating simultaneously the configuration of the UAV and the position of the beacons.

[3 h]

Autonomous and Mobile Robotics
Solution of Final Class Test, 2013/2014

Solution of Problem 1

Substituting the first and second model equations into the third, one immediately obtains the underlying differential constraint

$$x_2 \dot{x}_1 - x_1 \dot{x}_2 + \dot{x}_3 = 0 \qquad \text{or} \qquad \begin{pmatrix} x_2 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = 0$$

As for system controllability, calling $\boldsymbol{g}_1 = (1 \;\; 0 \;\; -x_2)^T$ and $\boldsymbol{g}_2 = (0 \;\; 1 \;\; x_1)^T$ the two input vector fields, their Lie Bracket is easily obtained as

$$[\boldsymbol{g}_1, \boldsymbol{g}_2] = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

Since $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is always linearly independent on $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, the accessibility rank condition is satisfied and the system is controllable. This also means that the above differential constraint is nonholonomic.

Coming to the last question, a Lie Bracket control maneuver consists in the following control sequence

$$u(t) = \begin{cases} u_1(t) = +1, u_2(t) = 0 & t \in [0, \epsilon) \\ u_1(t) = 0, u_2(t) = +1 & t \in [\epsilon, 2\epsilon) \\ u_1(t) = -1, u_2(t) = 0 & t \in [2\epsilon, 3\epsilon) \\ u_1(t) = 0, u_2(t) = -1 & t \in [3\epsilon, 4\epsilon), \end{cases}$$

where ϵ is a small time interval. Denoting by $(x_{10} \;\; x_{20} \;\; x_{30})^T$ the initial system state, integration of the system equations over the first time interval $[0, \epsilon]$ easily gives

$$\begin{aligned} x_1(\epsilon) &= x_{10} + \epsilon \\ x_2(\epsilon) &= x_{20} \\ x_3(\epsilon) &= x_{30} - \epsilon\, x_{20}. \end{aligned}$$

Similarly, integrating over the second time interval $[\epsilon, 2\epsilon]$ we get

$$\begin{aligned} x_1(2\epsilon) &= x_{10} + \epsilon \\ x_2(2\epsilon) &= x_{20} + \epsilon \\ x_3(2\epsilon) &= x_{30} - \epsilon\, x_{20} + (x_{10} + \epsilon)\epsilon. \end{aligned}$$

After similar computations for the third and fourth time intervals, one obtains

$$\begin{aligned} x_1(4\epsilon) &= x_{10} \\ x_2(4\epsilon) &= x_{20} \\ x_3(4\epsilon) &= x_{30} + 2\,\epsilon^2. \end{aligned}$$

so that the final state displacement is *exactly equal* to $\epsilon^2\, [\boldsymbol{g}_1, \boldsymbol{g}_2]$. Note that this result is stronger than the theoretical result provided by Taylor expansion for general drifless systems: in fact, it shows that for the considered system the $O(\epsilon^3)$ terms are identically zero.

Solution of Problem 2

- $\mathcal{C} = SE(2) \times SE(2)$, $\dim \mathcal{C} = 6$.
- $\mathcal{C}$ is a subset of $SE(2) \times SE(2)$ (for each position of the first robot, the position of the second robot must be within a circle centered at the first robot), $\dim \mathcal{C} = 6$ (the constraint entailed by the rope is an inequality constraint and therefore it does not decrease the dimension of $\mathcal{C}$).
- $\mathcal{C} = SE(2) \times SO(2) \times SO(2)$ (one possible choice of configuration is: position and orientation of the first robot, orientation of the bar, orientation of the second robot), $\dim \mathcal{C} = 5$.
- $\mathcal{C} = SE(3) \times (SO(2))^6$ (need 3D position and orientation for the spacecraft, plus six joint angles for the robot), $\dim \mathcal{C} = 12$.
- $\mathcal{C} = SE(3)$ (the last link of the robot is a rigid body which can be arbitrarily positioned and oriented in space by moving the spacecraft and/or the robot arm), $\dim \mathcal{C} = 6$.

Solution of Problem 3

Since the position of the beacons is unknown, this is a SLAM problem. Define the extended state vector to be estimated as $\boldsymbol{\chi} = (x \;\; y \;\; \psi \;\; \phi \;\; x_1 \;\; y_1 \;\; x_2 \;\; y_2)^T$, where (x_i, y_i), for $i = 1, 2$, are the cartesian coordinates of the two beacons (the z coordinate of the beacon is known to be zero). The nonlinear discrete-time model describing the motion of the extended UAV+beacons system is then

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} v_k \cos\psi_k \, T_s \\ v_k \sin\psi_k \, T_s \\ -g/v_k \tan\phi_k \, T_s \\ u_{\phi,k} \, T_s \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ v_{4,k} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\boldsymbol{v}_k = (v_{1,k} \;\, v_{2,k} \;\, v_{3,k} \;\, v_{4,k})^T$ is a white gaussian noise with zero mean and covariance V_k. Note how the beacons being fixed reflects on the last four rows of the above equation.

As for the measurement model, we have two distance readings coming from the robot sensor. An elementary geometric construction provides

$$\boldsymbol{h}_k = \begin{pmatrix} \sqrt{(x_k - x_{1,k})^2 + (y_k - y_{1,k})^2 + \bar{z}^2} \\ \sqrt{(x_k - x_{2,k})^2 + (y_k - y_{2,k})^2 + \bar{z}^2} \end{pmatrix} + \begin{pmatrix} w_{1,k} \\ w_{2,k} \end{pmatrix}$$

where $\boldsymbol{w}_k = (w_{1,k} \;\; w_{2,k})^T$ is a white gaussian noise with zero mean and covariance W_k. Note that the sensor-beacon distance depends also on $\bar{z}$, since the beacons are on the ground.

The rest of the problem is trivial: linearize the process and measurement models and then derive the EKF equations.

Autonomous and Mobile Robotics
Final Class Test, 2014/2015

Problem 1

Consider a differential-drive robot with velocity inputs ω_R, ω_L (respectively, angular speed of the right and the left wheel).

1. Write the kinematic model of the system in the above inputs.
2. Prove that this kinematic model is controllable.
3. Draw a sketch of a Lie Bracket maneuver.

Problem 2

NO

Consider a circular robot with unicycle kinematics moving in a known environment containing circular obstacles. Denote by $\boldsymbol{q} = (\boldsymbol{p}, \theta)$ the robot configuration, with $\boldsymbol{p} = (x, y)$, and by v, ω its velocity inputs. The robot must reach a certain destination $\boldsymbol{p}_{\text{goal}}$ (final orientation is not assigned). Build a navigation system that integrates the following components:

1. A robot-independent module that uses the environment geometry and the assigned goal to build an artificial force field $\boldsymbol{f}(\boldsymbol{p})$.
2. A module that transforms the artificial command $\boldsymbol{f}(\boldsymbol{p})$ into actual velocity inputs v, ω.
3. A module that computes the robot state needed by the first two.

Discuss in detail the possible options and the motivation behind your choices. Provide a block scheme of your system with a clear indication of the inputs and the outputs of each block. Points that deserve special attention are: (1) which is the main difficulty in building the transformation module? (2) with your navigation system, is the robot guaranteed to converge to the destination, and under which assumptions?

Problem 3

Consider a differential-drive robot equipped with a sensor that can measure relative range and bearing between itself and certain landmarks distributed in the environment. The number of landmarks is L but their position is unknown. For simplicity, assume that (1) the sensor is located above the midpoint of the wheel axis (2) the sensor can detect and identify all landmarks, irrespective of their position. Build a localization system for estimating simultaneously the configuration of the robot and the position of the landmarks. How should the system be modified if the sensor cannot reconstruct the identity of the landmarks?

[3 h]

Autonomous and Mobile Robotics
Solution of Final Class Test, 2014/2015

Solution of Problem 1

Letting $\boldsymbol{q} = (x, y, \theta)$, the kinematic model is readily expressed as

$$\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta \\ \frac{r}{d} \end{pmatrix} \omega_R + \begin{pmatrix} \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta \\ -\frac{r}{d} \end{pmatrix} \omega_L = \boldsymbol{g}_1(\boldsymbol{q})\omega_R + \boldsymbol{g}_2(\boldsymbol{q})\omega_L,$$

where r is the radius of the wheels and d is the distance between their centers.

As for system controllability, the Lie Bracket of the two input vector fields is easily obtained as

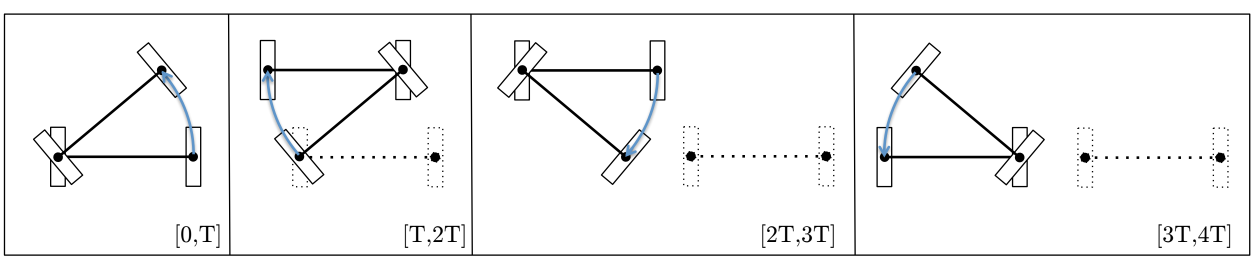
$$\boldsymbol{g}_3 = [\boldsymbol{g}_1, \boldsymbol{g}_2] = \frac{r^2}{d} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix},$$

i.e., it is directed along the Zero Motion Line. A simple computation shows that

$$\det(\boldsymbol{g}_1\, \boldsymbol{g}_2\, \boldsymbol{g}_3) = r^4/d^2.$$

Hence, the accessibility rank condition is satisfied and the system is controllable.

The result of a Lie Bracket control maneuver is easily drawn as follows (the initial configuration is shown dashed):



Note that the achieved displacement is exactly in the direction of the Zero Motion Line; this means that for the considered system the $O(\epsilon^3)$ terms are identically zero.

Solution of Problem 2

(just the main ideas are sketched, other options are possible)
The potential field module is trivial (see slides on Motion Planning 3). The problem, however, is that a unicycle robot is not *free-flying* in its configuration space due to its nonholonomy; in particular, its representative point P (with coordinates $\boldsymbol{p}$) can only move instantaneously in the direction of the sagittal axis, whereas the artificial force field $\boldsymbol{f}$ at $\boldsymbol{p}$, which depends only on the obstacle and goal placement, may be oriented in any direction. As a consequence, setting $\dot{\boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{p})$ is impossible in this case.

To transform the artificial force field $\boldsymbol{f}(\boldsymbol{p})$ in actual velocity inputs v, ω, one possibility is to assume that $\boldsymbol{f}$ acts on a point B (with coordinates $\boldsymbol{p}_B$) which is displaced from P of a certain distance b along the sagittal axis, as in input-output linearization control (see slides on Motion Control of WMRs: Trajectory Tracking). In fact, B can move in any direction, and the velocity inputs v, ω that realize a certain Cartesian velocity for B are easily computed by inverting the input-output map. The idea is then to set $\dot{\boldsymbol{p}}_B = \boldsymbol{f}(\boldsymbol{p}_B)$, and then compute v, ω that realize $\dot{\boldsymbol{p}}_B$.

A localization module will also be necessary for making (an estimate of) $\boldsymbol{q}$ available to the transformation module (the input-output matrix depends on $\boldsymbol{q}$, in particular on θ). From $\boldsymbol{q}$, it is straightforward to compute (an estimate of) $\boldsymbol{p}_B$ to be passed to the potential field module.

As for the effectiveness of the above navigation strategy, the potential field itself will be free of local minima because the robot and the obstacles are circular (*world of spheres)*. Isolated saddle points will be present; the total field is zero there but the robot can easily escape with a small perturbation. This means that a point robot subject to such field (including isolated perturbations) would always converge to the destination, provided that the latter is outside the range of infuence of all obstacles. For our unicycle robot, however, it will be point B that converges to the destination, while point P will actually lie on a circle of radius b centered at the destination. If b can be chosen small (this depends on the bounds on the input velocities), the final navigation error will be acceptable.

Solution of Problem 3

Since the position of the landmarks is unknown, this is a SLAM problem. Define the extended state vector to be estimated as $\boldsymbol{\chi} = (x \;\; y \;\; \theta \;\; x_{l,1} \;\; y_{l,1} \;\; \ldots \;\; x_{l,L} \;\; y_{l,L})^T$, where (x, y, θ) are the robot generalized coordinates and $(x_{l,i}, y_{l,i})$, for $i = 1, \ldots, L$, are the Cartesian coordinates of the landmarks. The nonlinear discrete-time model describing the motion of the extended robot+beacons system is then

$$\boldsymbol{\chi}_{k+1} = \boldsymbol{\chi}_k + \begin{pmatrix} v_k \cos\theta_k \\ v_k \sin\theta_k \\ \omega_k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

where $v_{i,k}$ is a white gaussian noise with zero mean and covariance $V_{i,k}$ $(i = 1, \ldots, 3)$. Note how the landmarks being fixed reflects on the last $2L$ rows of the above equation.

As for the model of the measurement $\boldsymbol{y}$, we have L pairs of readings (range+bearing) coming from the robot sensor:

$$\boldsymbol{y}_k = \begin{pmatrix} \boldsymbol{h}_1(\boldsymbol{q}_k) \\ \vdots \\ \boldsymbol{h}_L(\boldsymbol{q}_k) \end{pmatrix} + \begin{pmatrix} \boldsymbol{w}_{1,k} \\ \vdots \\ \boldsymbol{w}_{L,k} \end{pmatrix}$$

with

$$\boldsymbol{h}_i(\boldsymbol{q}_k, i) = \begin{pmatrix} \sqrt{(x_k - x_{l,i})^2 + (y_k - y_{l,i})^2} \\ \text{atan2}(y_{l,i} - y_k, x_{l,i} - x_k) - \theta_k \end{pmatrix}$$

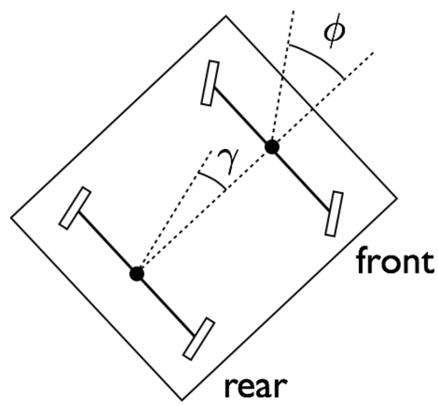
where $\boldsymbol{w}_i, k$ is a white gaussian noise with zero mean and covariance $\boldsymbol{W}_{i,k}$ $(i = 1, \ldots, L)$. The rest of the problem is trivial: linearize the process and measurement models and then derive the EKF equations.

If the sensor cannot identify the landmark, an association map should be used, to be estimated on the basis of the innovation norm weighted by matrix S_{ij}^{-1} (Mahalanobis distance, see slides on Localization 2).

Autonomous and Mobile Robotics
Midterm Class Test, 2015/2016

Problem 1

The Cycab is an autonomous electrical vehicle developed in France by INRIA.



From a kinematic point of view, it is a robot with four orientable wheels, arranged in pairs on a front axle and a rear axle. Each pair of wheels can be steered independently. This results in a higher maneuverability with respect to a car-like robot.

1. Derive a kinematic model of the Cycab for the case of rear-wheel drive.
2. *Optional*: Prove that this kinematic model is controllable.
   (hint: one may exploit the fact that the car-like robot is controllable...)

Problem 2

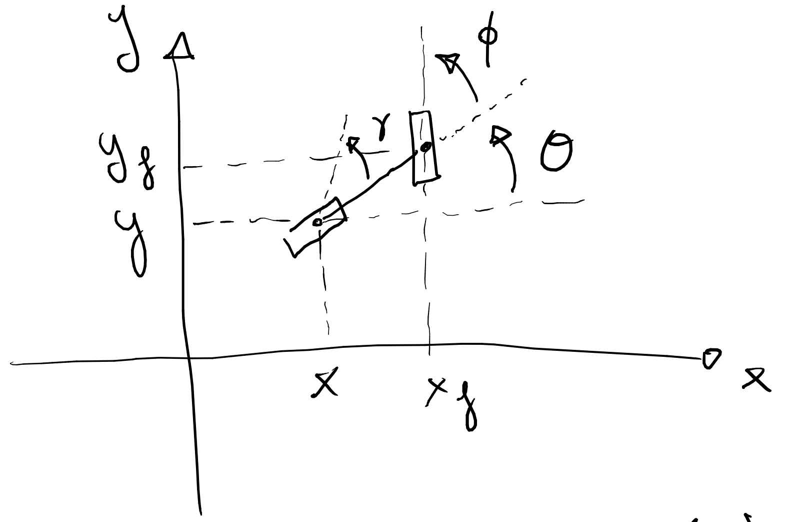
Using the obtained kinematic model of the Cycab, design a feedback control law such that the robot orientation θ asymptotically tracks a time-varying reference value $\theta_d(t)$. Discuss the conditions under which the controller achieves its objective.
(hint: identify the output variable to be controlled and try input-output linearization...)

Problem 3

Assume that the Cycab is equipped with wheel encoders and a laser range finder that can measure the distance to a charging station. For simplicity, suppose that (1) the sensor is located above the midpoint of the rear wheel axis (2) the sensor can always see the charging station. Build a localization system for estimating the configuration of the robot. Provide both its equations and a block scheme to clarify where sensor data are used in the localization process.

[3 h]

1.2 RWD car-like model (equivalent to bicycle)



Configuration space $q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \\ \gamma \end{pmatrix}$ $\quad C = \mathbb{R}^2 \times SO(2)^3$

Constraints : Roll without slipping

R) $\dot{x}\sin(\theta+\gamma) - \dot{y}\cos(\theta+\gamma) = 0$

F) $\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$

$$\begin{cases} x_f = x + \ell\cos\theta \\ y_f = y + \ell\sin\theta \end{cases} \longrightarrow \begin{cases} \dot{x}_f = \dot{x} - \ell\sin\theta\,\dot{\theta} \\ \dot{y}_f = \dot{y} + \ell\cos\theta\,\dot{\theta} \end{cases}$$

F) $\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) \underset{-}{+} \ell\dot{\theta}\Big(\underbrace{\underset{+}{-}\sin\theta(\sin(\theta+\phi)) \underset{+}{-} c\theta(c(\theta+\phi))}_{\cos\phi}\Big)$

$$A^T(q) = \begin{pmatrix} \sin(\theta+\gamma) & -\cos(\theta+\gamma) & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 & 0 \end{pmatrix}$$

$A^T(q)\dot{q} = 0 \qquad A^T$ 2×5 matrix

a basis for $N(A^T(q)) \longrightarrow \underset{g_1(q)}{\begin{pmatrix} \cos(\theta+\gamma) \\ \sin(\theta+\gamma) \\ \frac{\sin(\phi-\gamma)}{\ell\cos\phi} \\ 0 \\ 0 \end{pmatrix}}, \underset{g_2(q)}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{g_3(q)}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$

$m = 5 - 2 = 3$

$\dot{q} = g_1 u_1 + g_2 u_2 + g_3 u_3 \longrightarrow u_1^2 = v^2 = \dot{x}^2 + \dot{y}^2 \qquad u_2 = \omega = \dot{\phi}$

$u_3 = \omega_r = \dot{\gamma}$

1.2 Controllable iff $\operatorname{rank}(g_1, g_2, [g_1,g_2], [g_1,[g_1,g_2]], \dots) = 4$

$$[g_1,g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = -\frac{\partial g_1}{\partial q} g_2$$

$$-\begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{\ell\cos^2\phi} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{pmatrix} \quad \text{wriggle } g_3$$

$$[g_1,g_3] = g_4 = \frac{\partial g_3}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_3$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{\tan\phi}{\ell} \\ 0 \end{pmatrix} - \frac{\partial g_1}{\partial q} \cdot \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sin\theta}{\ell\cos^2\phi} \\ \frac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{pmatrix} \quad \text{slide}$$

$$\operatorname{rank}\begin{pmatrix} | & | & | & | \\ g_1 & g_2 & g_3 & g_4 \\ | & | & | & | \end{pmatrix} = 4 \qquad \det = \frac{1}{\ell^2\cos^4\phi}$$

② Output variable to be controlled: θ

Try an input-output linearization

$$\dot{\theta} = \underbrace{\frac{\sin(\phi-\gamma)}{\ell\cos\phi}}_{T(x)} u_1 = \frac{\partial h}{\partial x} G(x)\, u$$

$$u_1 = T(x)^{-1} \underbrace{v}_{\text{new input}} = \frac{\ell\cos\phi}{\sin(\phi-\gamma)}\, v$$

We obtain a linear map between the derivative of the output and the new input v

$$\dot{\theta} = T(x)\, u_1 = \cancel{T(x)} \cdot \cancel{T(x)^{-1}}\, v$$

Control law simple integrator

$$\dot{\theta} = v \qquad e = \theta - \theta_d$$

$$\dot{e} = \dot{\theta} - \dot{\theta}_d = v - \dot{\theta}_d = -ke \Rightarrow v = \overbrace{\dot{\theta}_d}^{\text{feed forward}} - \underbrace{ke}_{\text{proportional error}}$$

(3) A straightforward approach for designing the required localization system is the EKF.

The discrete-time model of the robot is:

<u>Euler integration</u> $\quad T_s$ = sampling interval

$$x_{k+1} = x_k + u_{1k} T_s \cos(\theta_k + \gamma_k)$$
$$y_{k+1} = y_k + u_{1k} T_s \cos(\theta_k + \gamma_k)$$
$$\theta_{k+1} = \theta_k + u_{1k} T_s \frac{\sin(\phi_k - \gamma_k)}{\ell \cos\phi_k} \qquad \alpha$$
$$\phi_{k+1} = \phi_k + u_{2k} T_s$$
$$\gamma_{k+1} = \gamma_k + u_{3k} T_s$$

x_c, y_c = cartesian coordinates of the charging station (landmark)

x_k, y_k = actual cartesian coordinates of the sensor at time t_k

h_k = measurement model = $\sqrt{(x_k - x_c)^2 + (y_k - y_c)^2} + w_{m,k}$

$w_{m,k}$ = white gaussian measurement noise with zero mean and covariance $W_{m,k}$

The linearization of the process and output equations, respectively evaluated at the previous estimate $\hat{q}_k$ and at the prediction $\hat{q}_{k+1|k}$, gives:

$$F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k = \hat{q}_k} = \begin{pmatrix} 1 & 0 & -u_{1k} T_s\, s(\theta_k+\gamma_k) & 0 & -u_{1k} T_s\, s(\theta_k+\gamma_k) \\ 0 & 1 & u_{1k} T_s\, c(\theta_k+\gamma_k) & 0 & u_{1k} T_s\, c(\theta_k+\gamma_k) \\ 0 & 0 & 1 & u_{1k} T_s \left.\frac{\partial \alpha}{\partial \phi}\right|_{\phi_k} & u_{1k} T_s \left.\frac{\partial \alpha}{\partial \gamma}\right|_{\gamma_k} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H_{k+1} = \frac{\partial h}{\partial q_k}\bigg|_{q_k = \hat{q}_{k+1|k}} = \frac{\left(\hat{x}_{k+1|k} - x_c \quad \hat{y}_{k+1|k} - y_c\right)}{\sqrt{(\hat{x}_{k+1|k} - x_c)^2 + (\hat{y}_{k+1|k} - y_c)^2}}$$

$\hat{q}_{k+1|k}$ → sensor coordinates

The EKF equations are:

1. state & covariance prediction

$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$

$$P_{k+1|k} = F_k P_k F_k^T + W_{p,k}$$

2. Correction

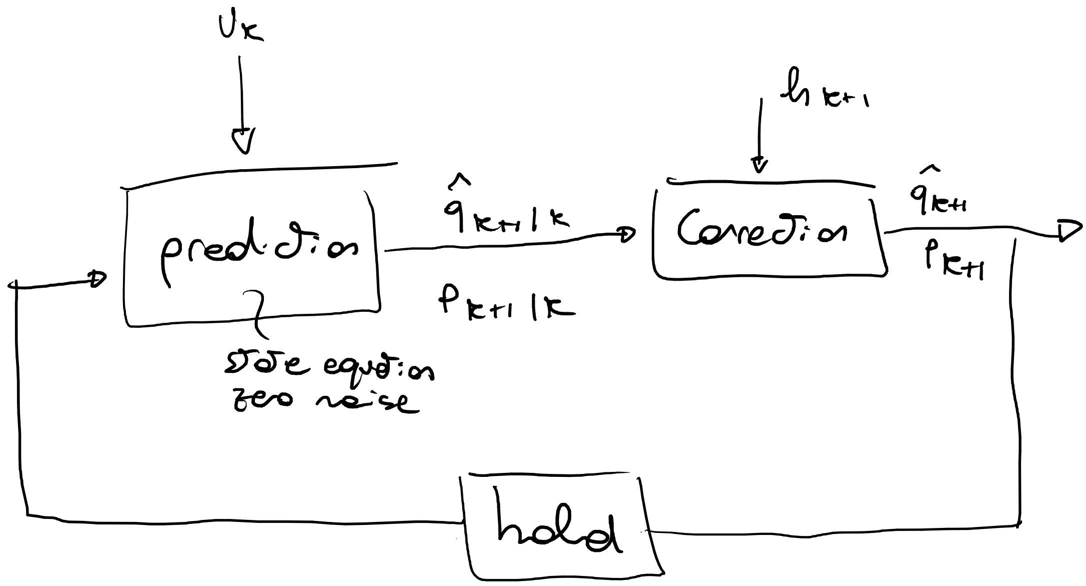
$$\hat{q}_{k+1} = q_{k+1|k} + R_{k+1} \nu_{k+1}$$

$$P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k}$$

with the innovation $\nu_{k+1} = h_{k+1} - h_{k+1|k}$

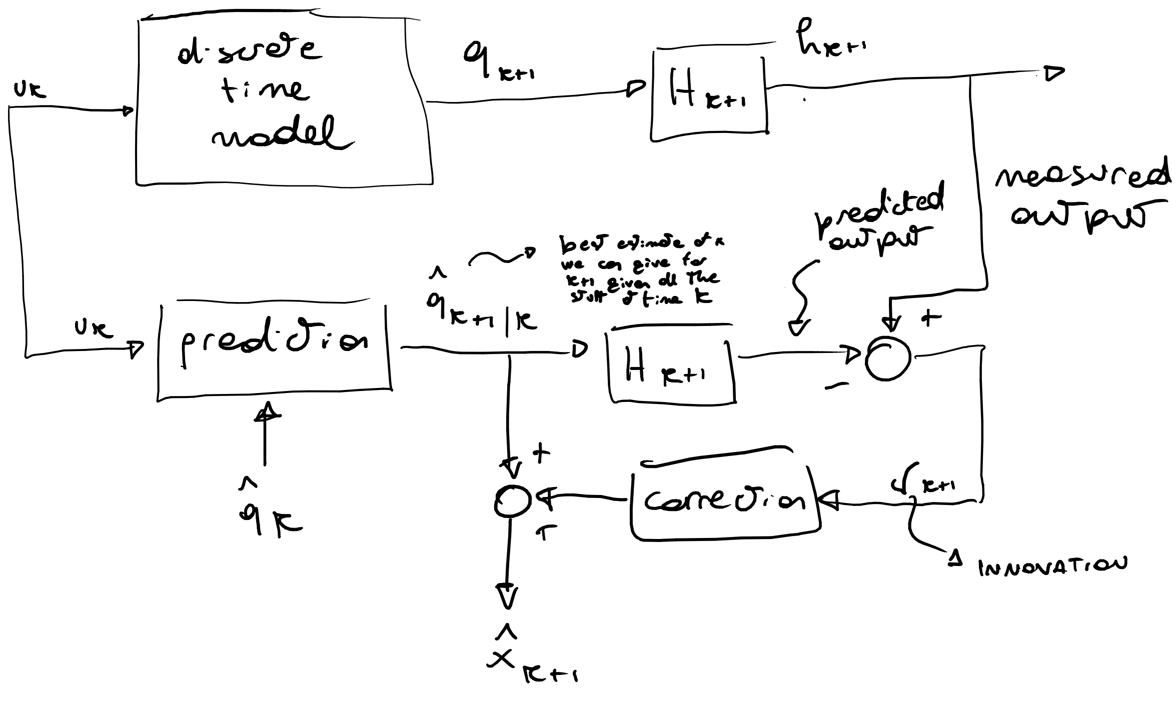
and the Kalman gain matrix

$$R_{k+1} = P_{k+1|k} H^T_{k+1} \left(H_{k+1} P_{k+1|k} H^T_{k+1} + W_{m,k+1}\right)^{-1}$$



u_k
h_{k+1}
prediction
q̂_{k+1|k}
P_{k+1|k}
Correction
q̂_{k+1}
P_{k+1}
state equation
zero noise
hold

in detail

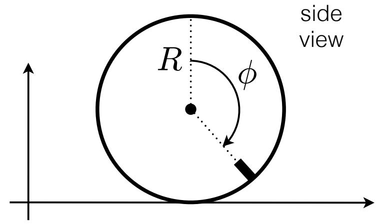


discrete
time
model
q_{k+1}
H_{k+1}
h_{k+1}
u_k
measured
output
predicted
output
best estimate of x
we can give for
k+1 given all the
stuff at time k
q̂_{k+1|k}
prediction
H_{k+1}
q̂_k
Correction
ν_{k+1}
INNOVATION
x̂_{k+1}

Autonomous and Mobile Robotics
Midterm Class Test, 2016/2017

Problem 1

Consider a unicycle moving on a plane and augment its configuration vector $\boldsymbol{q}$ by including the wheel angle ϕ.



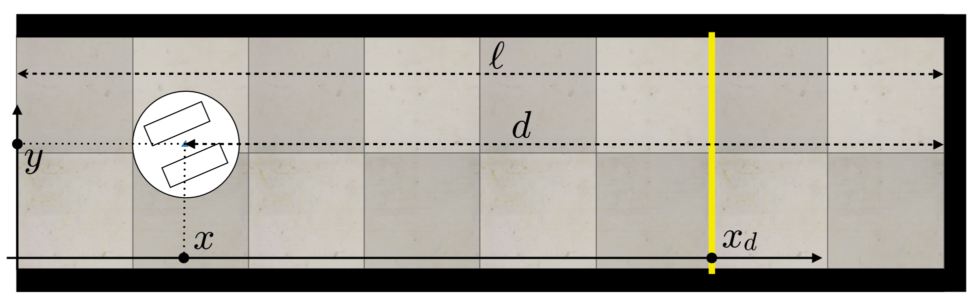
1. Define the augmented configuration space and its dimension.
2. Write the augmented kinematic model of the robot in the appropriate velocity inputs.
3. Show that the augmented kinematic model is controllable.
4. Describe a sequential maneuver for moving the robot between two augmented configurations $\boldsymbol{q}_s$ and $\boldsymbol{q}_g$.

Problem 2

Consider a (2,3) chained form system. Plan a feasible geometric path that connects the origin of the configuration space to point $(1, 1, 1)$. Be sure to provide the parametric expression of all the configuration variables.

Problem 3

A unicycle robot moves in a corridor whose end is closed by a wall. The length of the corridor is ℓ. Note the world frame.



1. Design a feedback control law for driving the robot coordinate x to a desired value x_d. The y coordinate is not of interest; however, the robot should not collide with the lateral walls.
2. Assume that the robot is equipped with a compass that measures its orientation θ and a range finder that measures the distance d to the end wall. For simplicity, assume that the range finder is located at the center of the robot. Design a localization system for estimating in real time the state variables needed by your controller. Provide equations (be sure to define all symbols) and a block scheme.

[210 min; *turn page for some hints. . .*]

Hints (depending on your chosen solution approach, these may be useful or not)

Problem 1

2. *As an alternative to the classical procedure, one may also write the required kinematic model by direct augmentation of the classical unicycle model.*

4. *The sequential maneuver may be found by adding one step to the basic maneuver for reconfiguring a rolling coin.*

Problem 2

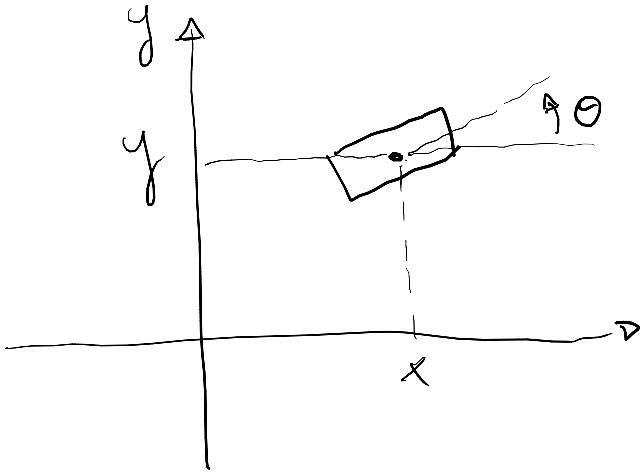
The flat outputs of a (2,3) chained form are the first and the third coordinate (z_1 and z_3).

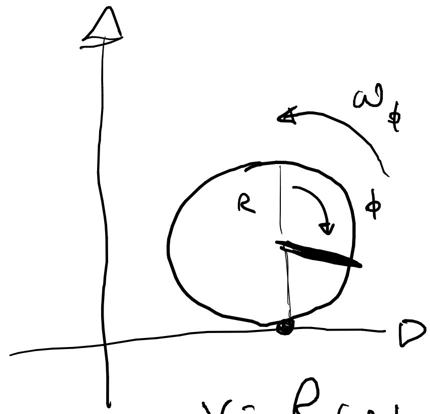
Problem 3

1. *Identify the output variable to be controlled and try input-output linearization. You can use the remaining input to keep the robot parallel to the corridor…*

(1)

1.





$v = R\omega_\phi$

$$C = \mathbb{R}^2 \times SO(2) \times SO(2) \qquad n = 4$$

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \qquad v = \dot{x}^2 + \dot{y}^2$$

2. 2 constraints

RWS : $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$

Wheel : $v - R\omega_\phi = 0 \to \dot{\phi}$

$$v(\cos^2\theta + \sin^2\theta) = v\cos\theta\cos\theta + v\sin\theta\sin\theta$$
$$= \dot{x}\cos\theta + \dot{y}\sin\theta$$

$$\dot{x}\cos\theta + \dot{y}\sin\theta - R\dot{\phi} = 0$$

$$A^T(q) = \begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -R \end{pmatrix}$$ Admissible velocities

$m = n - k = 4 - 2 = 2$

$$A^T(q)\dot{q} = 0$$

Inputs:
$\omega_\phi \quad \omega_\theta$

basis of the null space $N(A^T(q))$

$$v = R\omega_\phi$$

$$g_1 = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix}\omega_\phi \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\omega_\theta$$

3. Controllability

$$\text{rank}\left(g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]] \dots\right) = 4$$

$$g_3 = [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

$$= -\begin{pmatrix} 0 & 0 & -R\sin\theta & 0 \\ 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R\sin\theta \\ -R\cos\theta \\ 0 \\ 0 \end{pmatrix}$$

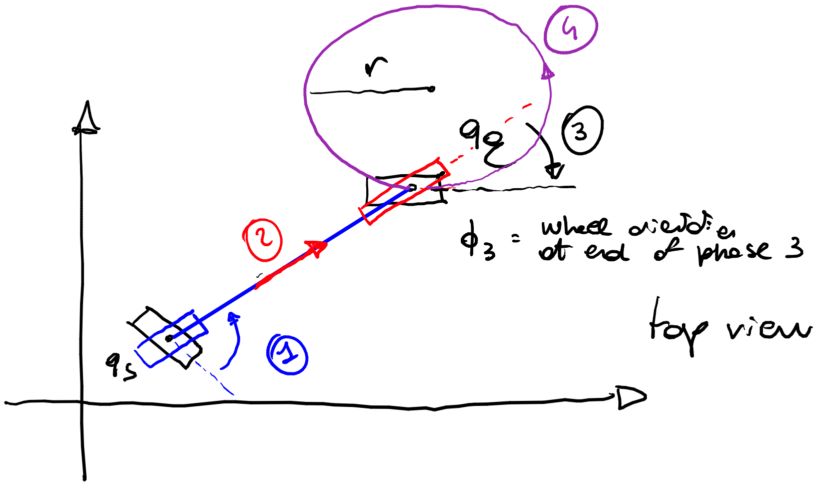
$$g_4 = [g_1, g_3] = 0 \qquad \text{NO}$$

$$g_5 = [g_2, g_3] =$$

$$\begin{pmatrix} 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & R\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 0 \end{pmatrix}$$

$$\text{rank}\begin{pmatrix} R\cos\theta & 0 & R\sin\theta & R\cos\theta \\ R\sin\theta & 0 & -R\cos\theta & R\sin\theta \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4$$

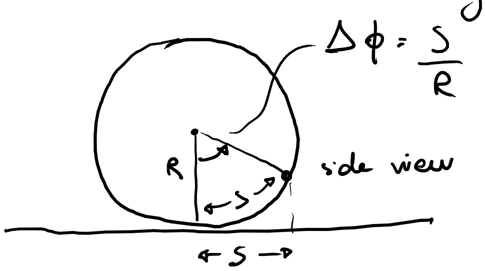
4. Sequential Manouver



1 - align starting orientation with goal orientation
2 - roll to e position

3 - align with final orientation

4 - a circle to adjust the orientation



$$\underbrace{2\pi r}_{\text{travel space}} = R\,\Delta\phi = R\left(\phi_e - \phi_3\right)$$

$$r = \frac{\left(\phi_e - \phi_3\right)}{2\pi}$$

2

Geometric version of the chained form (2,3)

$z_1' = \tilde{v}_1$
$z_2' = \tilde{v}_2$
$z_3' = z_2 \tilde{v}_1$

First we have to convert to the chained form the initial and desired configurations.
Therefore we need to transform the coordinates from q_i, q_g to z_i, z_g.
The flat outputs are z_1, z_3.

$z_2 = \frac{z_3'}{z_1'}$ (remaining state variable)

We must choose $z_1(s)$ and $z_3(s)$ $s \in [s_i, s_g]$ so as to satisfy their endpoint conditions

$z_1(s_i) = z_3(s_i) = 0 \qquad z_1(s_g) = z_3(s_g) = 1$

and the boundary conditions

$z_2(s_i) = 0 \qquad z_2(s_g) = 1$

Letting $s \in [0, 1]$, we can use a 1-st order polynomial for z_1 and 3-rd order polynomial for z_3

$z_1(s) = a_1 s + b_1$
$z_3(s) = a_2 s^3 + b_2 s^2 + c_2 s + d_2$

whose derivatives wrt s

$z_1'(s) = a_1$
$z_3'(s) = 3a_2 s^2 + 2b_2 s + c_2$ $\Rightarrow$ by imposing the previous conditions it is possible to to reconstruct $z_1(s), z_2(s), z_3(s)$

(3) The output variable is x.

Kinematic model $\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$

$\dot{x} = u_1 \cos\theta$

Input - Output linearization

$\dot{x} = \underbrace{\cos\theta}_{T(x)} u_1 \Rightarrow u_1 = T(x)^{-1} v$ → new input

$\dot{x} = T(x)\, T(x)^{-1} v \Rightarrow \dot{x} = v$

Control law simple integrator

$e = x - x_d \Rightarrow \dot{e} = \dot{x} - \dot{x}_d = v - \dot{x}_d = -ke$

$v = -k(x - x_d)$

- The robot should not collide with the lateral walls.

u_2 is free (ω), we can use it to keep the robot parallel to the corridor.

$\dot{\theta} = u_2$

$u_2 = -k_\theta \theta$ → if $k_\theta > 0$, $\theta \to 0$ exponentially

2. Localization system with real time estimates of θ and d.

We need the state variables $\underbrace{x \text{ and } \theta}_{\text{2-dim filter}}$.

Euler integration (T_s sampling time)

$$x_{k+1} = x_k + u_{1k} T_s \cos\theta_k$$

$$y_{k+1} = y_k + u_{1k} T_s \sin\theta_k \longrightarrow \text{not used}$$

$$\theta_{k+1} = \theta_k + u_{2k} T_s$$

$T_s = t_{k+1} - t_k$

noise free model

Noise-free output equation (measurement model)

$$h_k = \begin{pmatrix} \theta \\ \ell - x_k \end{pmatrix}$$

$\longrightarrow$ It is already linear
no need to linearize

The linearization of the process equations respectively evaluated at the previous estimate $\hat{q}_k$ and at the prediction $\hat{q}_{k+1|k}$, gives:

$$F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k=\hat{q}_k} = \begin{pmatrix} 1 & -u_{1k} T_s \cos\hat{\theta}_k \\ 0 & 1 \end{pmatrix}$$

$$H_k \Big|_{q_k=\hat{q}_{k+1|k}} = \begin{pmatrix} \hat{\theta}_{k+1|k} \\ \ell - \hat{x}_{k+1|k} \end{pmatrix}$$

The EKF equations are:

1. state & covariance prediction

$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$

$$P_{k+1|k} = F_k P_k F_k^T + W_{p,k}$$

process noise covariance

2. Correction

$$\hat{q}_{k+1} = q_{k+1|k} + K_{k+1} \nu_{k+1}$$

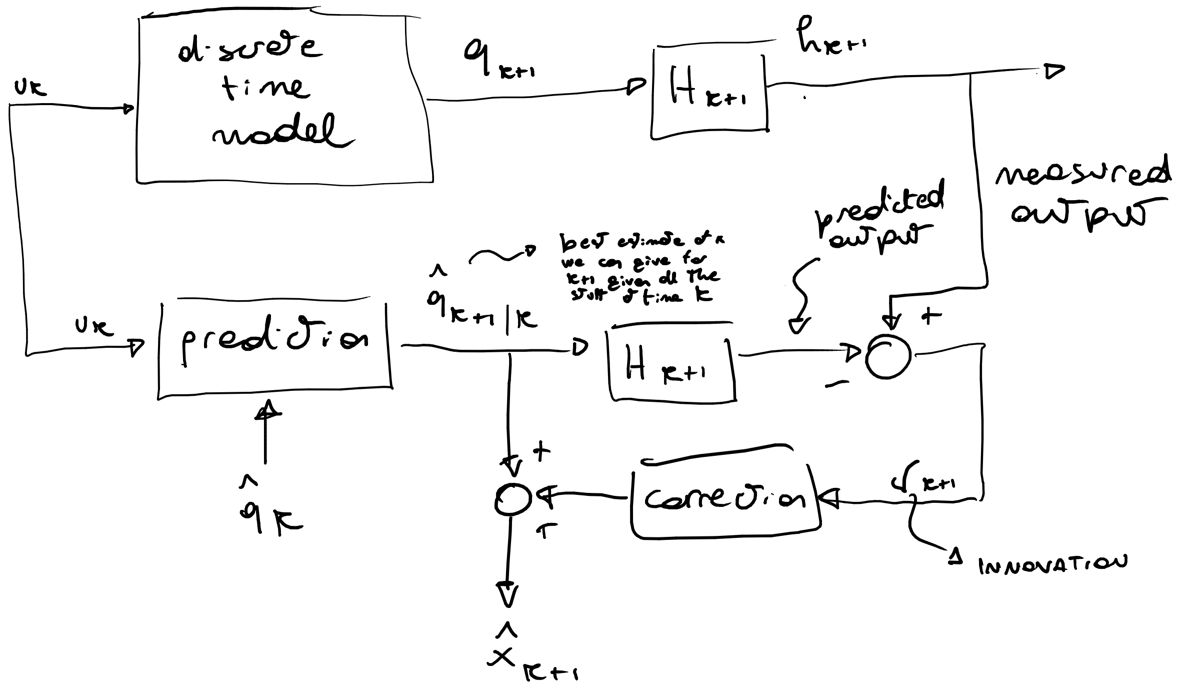
$$P_{k+1} = P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}$$

with the innovation $\nu_{k+1} = h_{k+1} - h_{k+1|k}$

and the Kalman gain matrix

$$K_{k+1} = P_{k+1|k} H^T_{k+1} \left(H_{k+1} P_{k+1|k} H^T_{k+1} + W_{m,k+1} \right)^{-1}$$

measurement noise covariance



Autonomous and Mobile Robotics
Midterm Class Test, 2017/2018

Problem 1

Consider the following kinematic model

$$
\begin{aligned}
\dot{q}_1 &= q_2 u_1 + q_3 u_2 \\
\dot{q}_2 &= -q_1 u_1 \\
\dot{q}_3 &= -q_1 u_2
\end{aligned}
$$

1. Study the controllability of the system.
2. Write the differential constraint underlying the model, and indicate whether it is holonomic or nonholonomic based on the previous controllability study.
3. Give a geometric description of the local and global mobility of the system in configuration space.

Problem 2

Consider a unicycle robot whose driving and steering velocities are subject to the following bounds:

$$|v| \leq 1 \text{ m/s} \qquad |\omega| \leq 1 \text{ rad/sec}$$

1. Plan a geometric path that brings the robot from the origin of the configuration space to point $(1, 1, \pi/2)$ [m,m,rad].
2. Describe in detail the steps of a computational procedure for associating a feasible timing law to the planned path.

Problem 3

Consider a differential-drive robot whose control inputs are the left and right wheel angular accelerations, respectively a_L and a_R. The robot is equipped with *(i)* an accelerometer which measures (by integration) the linear and angular velocity of the robot, and *(ii)* a sensor which measures the relative bearing between the robot and a single landmark whose position is unknown. For simplicity, assume that all measurements are updated every T_s seconds, where T_s is the sampling interval of the robot control loop.

1. Write the kinematic model of the system (with a_L and a_R as control inputs).
2. Derive a discrete-time model of the system that can be used for odometric localization under the assumption that a_L and a_R are known.
3. Build an EKF for estimating simultaneously the configuration of the robot and the position of the landmark. Be sure to provide all the filter equations and a block scheme showing all the signals involved in the process.

[3 h]

(1) This is a (2,3) chained form

1. $g_1 = \begin{pmatrix} q_2 \\ -q_1 \\ 0 \end{pmatrix}$ $g_2 = \begin{pmatrix} q_3 \\ 0 \\ -q_1 \end{pmatrix}$

$$g_3 = [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

$$= -\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_3 \\ 0 \\ -q_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_2 \\ -q_1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ -q_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -q_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ q_3 \\ -q_2 \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} q_2 & q_3 & 0 \\ -q_1 & 0 & q_3 \\ 0 & -q_1 & -q_2 \end{pmatrix} = 0 - \left(-q_1 q_3 q_2 + q_2 q_1 q_3\right) = 0$$

Also all the higher-order brackets will lead to the same result.
The accessibility rank condition is then violated and the system is not controllable

2. multiplying the first equation by q_1

$$q_1 \dot{q}_1 - q_1 q_2 v_1 - q_1 q_3 v_2 = 0$$

substituting the other conditions

$$q_1 \dot{q}_1 + q_2 \dot{q}_2 + q_3 \dot{q}_3 = 0$$

$$(q_1 \; q_2 \; q_3)\, \dot{q} = 0$$

$A^T(q)$

Writing it as a geometric constraint it is integrable

$$q_1^2 + q_2^2 + q_3^2 = c \xrightarrow{\text{derivative}} \cancel{2} q_1 \dot{q}_1 + \cancel{2} q_2 \dot{q}_2 + \cancel{2} q_3 \dot{q}_3 = 0$$

Both global and local mobility are restricted: $q \in C_1$ must satisfy the integral of $A(q)\dot{q} = 0$.
Therefore the model is holonomic

3. From a geometric point of view, for the local mobility limitation $\dot{q}$ must be contained in the tangent plane to the surface defined at q by the global mobility limitation: in this case the motion of the system in Configuration Space is constrained to take place on the sphere centered at the origin and passing through q_0.

② Since unicycle's system is flat I can describe the whole evolution of the state and the inputs from the FOs

Geometric model

$$q_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \tilde{v} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tilde{\omega} \qquad q_g = \begin{pmatrix} 1 \\ 1 \\ \frac{\pi}{2} \end{pmatrix}$$

flat outputs are $W = \begin{pmatrix} x \\ y \end{pmatrix}$

Boundary conditions: $s \in [0,1]$

$$w_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad w_g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x'(0) = \tilde{v}(0) \cos\theta_i$

$y'(0) = \tilde{v}(0) \sin\theta_i$

$x'(1) = \tilde{v}(1) \cos\theta_g$

$y'(1) = \tilde{v}(1) \sin\theta_g$

Generate a path for x and y:

$x(s) = a_x s^3 + b_x s^2 + c_x s + d_x$

$y(s) = a_y s^3 + b_y s^2 + c_y s + d_y$

We get

$x(s) = s^3 x_g - (s-1)^3 x_i + \alpha_x s^2 (s-1) + \beta_x s(s-1)$

$y(s) = s^3 y_g - (s-1)^3 y_i + \alpha_y s^2 (s-1) + \beta_y s(s-1)$

$$\begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} k\cos\theta_g - 3x_g \\ k\sin\theta_g - 3y_g \end{pmatrix} \qquad \begin{pmatrix} \beta_x \\ \beta_y \end{pmatrix} = \begin{pmatrix} k\cos\theta_i - 3x_i \\ k\sin\theta_i - 3y_i \end{pmatrix}$$

$v(0) = v(1) = k \rightsquigarrow$ free choice

Reconstruct the path

$\theta(s) = \mathrm{ATAN2}\,(y'(s), x'(s)) + k\pi$ → $k=0$ moving forward

$\tilde{v}(s) = \pm\sqrt{x'^2 + y'^2}$

$$\tilde{\omega}(s) = \frac{(y''x' - y'x'')}{x'^2 + y'^2}$$

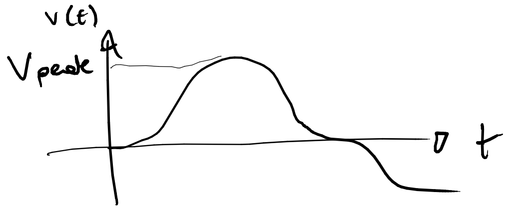
2. Timing low

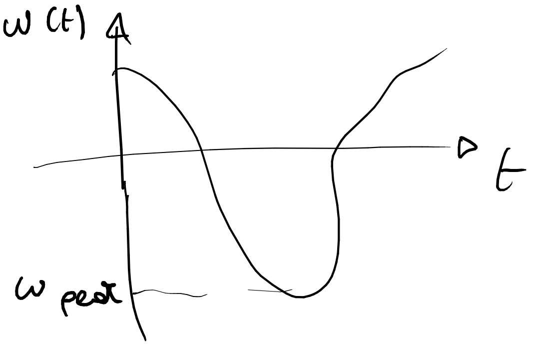
$s = s(t) \qquad t \in [t_i, t_f] \qquad \begin{array}{l} s(t_i) = s_i \\ s(t_f) = s_f \end{array}$

$v = \tilde{v} \cdot \dot{s} \qquad \omega = \tilde{\omega} \cdot \dot{s}$

let $t = \alpha s$

1. Pick $\alpha = 1$ and compute $v(t)$, $\omega(t)$





2. If $|v_{peak}| \leq v_{max}$
$|\omega_{peak}| \leq \omega_{max}$

then $\alpha = 1$ is ok ⇒ $s = t$

3. else, $\eta = \max\left(\dfrac{|v_{peak}|}{v_{max}}, \dfrac{|\omega_{peak}|}{\omega_{max}}\right)$

and let $\alpha = \dfrac{1}{\eta}$

At the end the bounds are respected with at least one velocity saturating the bound at peak value

③ Kinematic model of the differential drive

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega \qquad \begin{cases} v_R = \omega_R \cdot r \\ v_L = \omega_L \cdot r \end{cases}$$

$$v = r\,\frac{\omega_R + \omega_L}{2} \qquad \omega = r\,\frac{\omega_R - \omega_L}{d}$$

r = wheel radius
d = wheel base

$$\dot{\omega}_R = u_1 \qquad u_1 = a_R \qquad u_2 = a_L$$

$$\dot{\omega}_L = u_2$$

2. Euler integration $\quad T_s = t_{k+1} - t_k$

$$x_{k+1} = x_k + T_s \cos\theta_k \, r \, \frac{\omega_{Rk} + \omega_{Lk}}{2}$$

$$y_{k+1} = y_k + T_s \sin\theta_k \, r \, \frac{\omega_{Rk} + \omega_{Lk}}{2}$$

$$\theta_{k+1} = \theta_k + T_s \, r \, \frac{\omega_{Rk} - \omega_{Lk}}{d}$$

$$\omega_{R\,k+1} = \omega_{Rk} + T_s \, u_{1k}$$

$$\omega_{L\,k+1} = \omega_{Lk} + T_s \, u_{2k}$$

3. Since the position of the landmark is unknown we are dealing with a SLAM problem.

Denote the extended state vector to be estimate as $X = \begin{pmatrix} x \\ y \\ \theta \\ \omega_R \\ \omega_L \\ x_L \\ y_L \end{pmatrix}$

→ cartesian coordinates of the landmark

The discrete-time model + landmark system system is

$$
\chi_{k+1} = \chi_k + \begin{pmatrix} '' \\ '' \\ '' \\ '' \\ '' \\ 0 \\ 0 \end{pmatrix} \left.\vphantom{\begin{matrix} '' \\ '' \\ '' \\ '' \\ '' \end{matrix}}\right\} \text{Same as previous model} + \begin{pmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \\ v_{4,k} \\ v_{5,k} \\ 0 \\ 0 \end{pmatrix}
$$

v_{ik} is a white Gaussian noise with zero mean and covariance $V_{i,k}$ $(i = 1, \dots, 5)$

For the output we have 3 measurements: the linear and angular velocity of the robot and the relative bearing of the landmark

$$
y_k = \begin{pmatrix} r \cdot \dfrac{\omega_{Rk} + \omega_{Lk}}{2} \\ r \cdot \dfrac{\omega_{Rk} - \omega_{Lk}}{d} \\ \mathrm{ATAN2}\,(y_\ell - y_k,\ x_\ell - x_k) - \theta_k \end{pmatrix} + \begin{pmatrix} \omega_{1k} \\ \omega_{2k} \\ \omega_{3k} \end{pmatrix}
$$

$W_{m,k}$ gaussian noise

The linearization of the process and measurement equations respectively evaluated at the previous estimate $\hat{q}_k$ and at the prediction $\hat{q}_{k+1|k}$, gives:

$$
F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k = \hat{q}_k} =
$$

$$
= \begin{pmatrix}
1 & 0 & -T_s \sin\hat{\theta}_k\, r\, \frac{\hat{\omega}_{Rk} + \hat{\omega}_{Lk}}{2} & T_s \cos\hat{\theta}_k \frac{r}{2} & T_s \cos\hat{\theta}_k \frac{r}{2} & 0 & 0 \\
0 & 1 & T_s \cos\hat{\theta}_k\, r\, \frac{\hat{\omega}_{Rk} + \hat{\omega}_{Lk}}{2} & T_s \sin\hat{\theta}_k \frac{r}{2} & T_s \sin\hat{\theta}_k \frac{r}{2} & 0 & 0 \\
0 & 0 & 1 & T_s \frac{r}{d} & -T_s \frac{r}{d} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$H_{k+1} = \begin{pmatrix} \left.\dfrac{\partial h}{\partial q_k}\right|_{q_k = \underset{\substack{\downarrow \\ \text{sensor coordinates}}}{\hat{q}_{k+1|k}}} \\ \vdots \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{r}{2} & \frac{r}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{r}{d} & -\frac{r}{d} & 0 & 0 \\ \dfrac{-(\hat{y}_{k+1|k} - y_e)}{(x_{k+1|k} - x_e)^2 + (y_{k+1|k} - y_e)^2} & \dfrac{\hat{x}_{k+1|k} - x_e}{(x_{k+1|k} - x_e)^2 + (y_{k+1|k} - y_e)^2} & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The EKF equations are:

1. state & covariance prediction

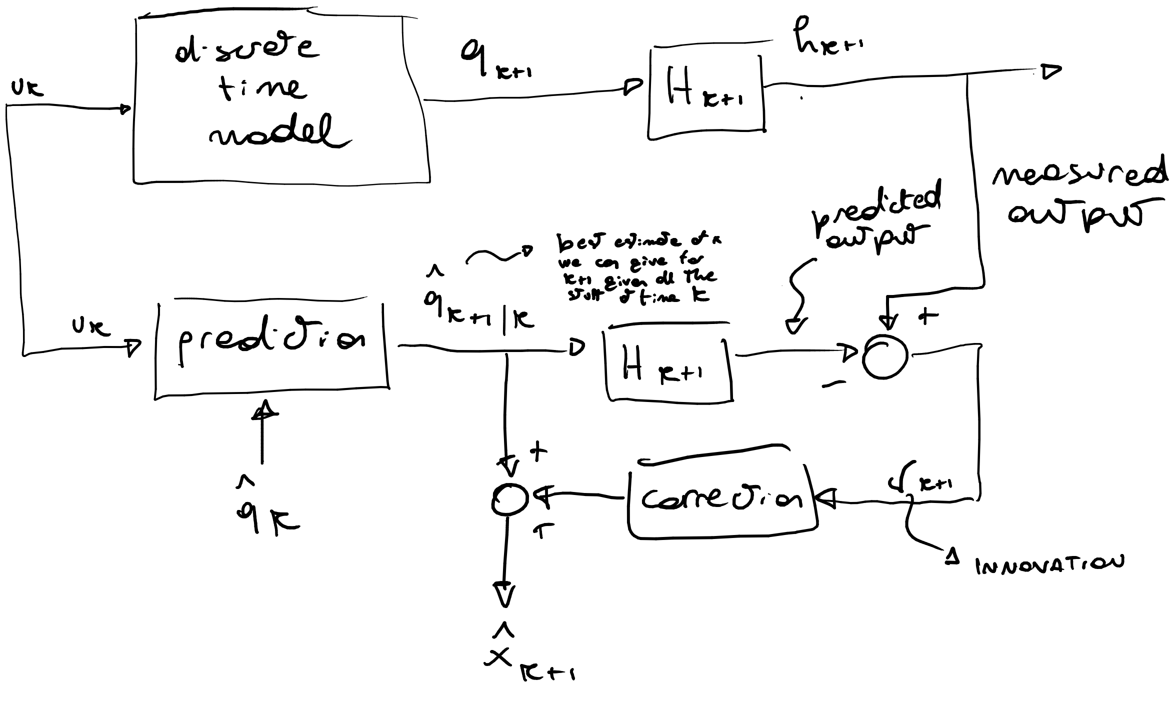
$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$
$$P_{k+1|k} = F_k P_k F_k^T + W_{p,k}$$

2. Correction

$$\hat{q}_{k+1} = q_{k+1|k} + R_{k+1} \nu_{k+1}$$
$$P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k}$$

with the innovation $\nu_{k+1} = h_{k+1} - h_{k+1|k}$
and the Kalman gain matrix

$$R_{k+1} = P_{k+1|k} H^T_{k+1} \left(H_{k+1} P_{k+1|k} H^T_{k+1} + W_{m,k+1} \right)^{-1}$$



discrete time model
u k
q k+1
H k+1
h k+1
measured output
predicted output
best estimate of x we can give for k+1 given all the stuff of time k
q k+1|k
prediction
H k+1
q k
correction
v k+1
INNOVATION
x k+1

Autonomous and Mobile Robotics
Midterm Class Test, 2019/2020

Problem 1

The kinematic model of a unicycle in polar coordinates is expressed as

$$\begin{aligned}\dot{\rho} &= -v\cos\gamma \\ \dot{\gamma} &= \frac{\sin\gamma}{\rho}v - \omega \\ \dot{\delta} &= \frac{\sin\gamma}{\rho}v\end{aligned}$$

where ρ, γ and δ are defined as in Fig. 1 and v, ω are the driving and steering velocity inputs.

1. Prove that the above kinematic model is controllable.
2. Assume that $v = \bar{v}$, with $\bar{v}$ constant and positive. Design a feedback control law for ω that will bring γ asymptotically to $\pi/2$ (*look at the second equation...*). What kind of Cartesian motion will the unicycle perform at steady state?

Problem 2

Using the bicycle equations, prove analytically that in this robot the velocity of the rear wheel (i.e., the velocity of the contact point of the rear wheel) is never larger than the velocity of the front wheel (i.e., the velocity of the contact point of the front wheel), and give a geometric interpretation of this fact.

Problem 3

Consider the unicycle robot of Fig. 2 moving in a corridor of width a (note the world frame). The robot uses a digital control scheme where the inputs are the driving and steering *accelerations*, respectively a_v and a_ω. The sensing equipment includes (1) a range finder located at the center of the robot that measures the distance d to the upper wall (2) an IMU that measures the orientation of the robot and its velocity along the x axis.

1. Write the kinematic model of the system with a_v and a_ω as control inputs (*it should be a 5-dimensional system...*).
2. Derive a discrete-time model of the system that can be used for odometric localization under the assumption that a_v and a_ω are known.
3. Build an EKF for estimating the complete *state* of the robot. Provide the filter equations and a block scheme showing all the signals involved in the process.

[2 h 50 min; see back for Figures 1-2]

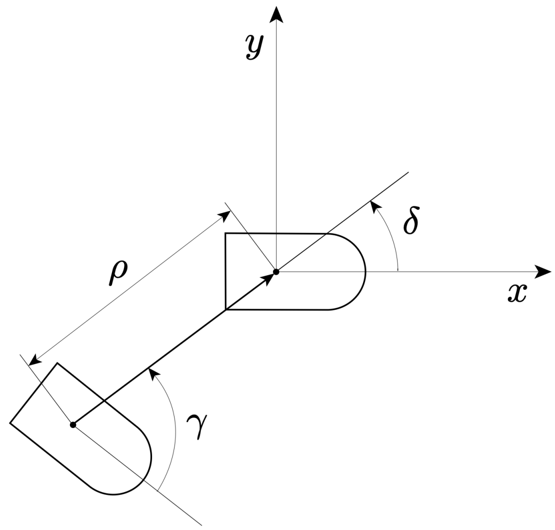


Figure 1: Polar coordinates for Problem 2

Figure 2: The geometric setting for Problem 3

(1)

1. Controllability

$$\begin{pmatrix}\dot\rho\\ \dot\gamma\\ \dot\delta\end{pmatrix} = \underbrace{\begin{pmatrix}-\cos\gamma\\ \frac{\sin\gamma}{\rho}\\ \frac{\sin\gamma}{\rho}\end{pmatrix}}_{g_1} v + \underbrace{\begin{pmatrix}0\\ -1\\ 0\end{pmatrix}}_{g_2} \omega \qquad n = 3$$

$x^{-1} \Rightarrow -1 \cdot x^{-2}$

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = -\begin{pmatrix}0 & \sin\gamma & 0\\ -\frac{\sin\gamma}{\rho^2} & \frac{\cos\gamma}{\rho} & 0\\ -\frac{\sin\gamma}{\rho^2} & \frac{\cos\gamma}{\rho} & 0\end{pmatrix}\begin{pmatrix}0\\ -1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}\sin\gamma\\ \frac{\cos\gamma}{\rho}\\ \frac{\cos\gamma}{\rho}\end{pmatrix}$$

$$\text{rank}\begin{pmatrix}-\cos\gamma & \sin\gamma & 0\\ \frac{\sin\gamma}{\rho} & \frac{\cos\gamma}{\rho} & -1\\ \frac{\sin\gamma}{\rho} & \frac{\cos\gamma}{\rho} & 0\end{pmatrix} = -\left(-\frac{\cos^2\gamma}{\rho} - \frac{\sin^2\gamma}{\rho}\right) = \frac{1}{\rho}$$

the system is controllable

2. Control law

the output variable is $\dot\gamma = \frac{\sin\gamma}{\rho}\bar{v} - \omega$ ($\bar{v}$ → known)

input-output linearization

$\omega = \frac{\sin\gamma}{\rho}\bar{v} - u$ → new input

in fact

$\dot\gamma = \frac{\sin\gamma}{\rho}\bar{v} - \frac{\sin\gamma}{\rho}\bar{v} + u \Rightarrow \dot\gamma = u$

The desired set-point $\gamma_d = \frac{\pi}{2}$ can be made globally exponentially stable by a proportional feedback

$e = \gamma - \gamma_d \Rightarrow \dot e = \dot\gamma - \dot\gamma_d = u - \dot\gamma_d = -ke$

$\Rightarrow u = \underbrace{\dot\gamma_d}_{\text{feedforward}} - ke$ → proportional error

At steady state $\gamma = \frac{\pi}{2}$ and $\dot\rho = 0$

The unicycle is moving along a circle centered in (0,0)

② Consider the kinematic model of the FWD:

$$\dot{q} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{\sin\phi}{\ell} \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_2$$

with $v_1^2 = v^2 = \dot{x}_f^2 + \dot{y}_f^2$ $\quad v =$ rear wheel velocity

$v_2 = \omega = \dot{\phi}$

Note that the instantaneous point of rotation C is the contact point of the rear wheel

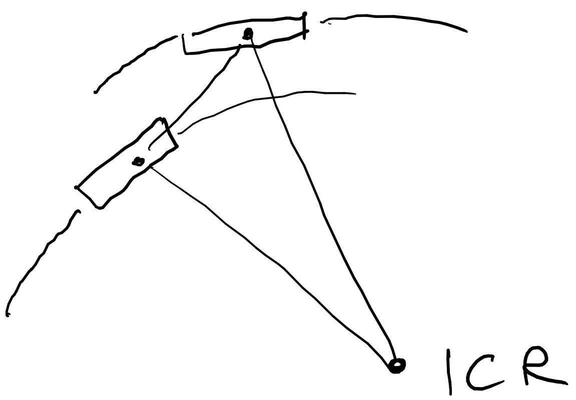
Since $v^2 = v_f^2 \cos^2\theta \cos^2\phi + v_f^2 \sin^2\theta \cos^2\phi$

$= v_f^2 \cos^2\phi$

it is evident that the velocity of the rear wheel is never larger of front wheel's one.

It is equal only when $\phi = 0$ (robot moving on a straight line)

Geometrically:



radius of the rear wheel

<

radius of the front wheel

Same results would have been obtained in the case of RWD

(3) Kinematic model of the unicycle

1. $\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega$

+

$\dot{q}_4 = \dot{v} = a_v$

$\dot{q}_5 = \dot{\omega} = a_\omega$

2. Euler integration $\quad T_s = t_{k+1} - t_k$

$$x_{k+1} = x_k + v_k T_s \cos\theta_k$$
$$y_{k+1} = y_k + v_k T_s \sin\theta_k$$
$$\theta_{k+1} = \theta_k + \omega_k T_s$$
$$v_{k+1} = v_k + a_{vk} T_s$$
$$\omega_{k+1} = \omega_k + a_{\omega k} T_s$$

3. EKF

measurement model $\quad h_k = \begin{pmatrix} \overbrace{e - y_k}^{d} \\ \theta_k \\ v_k \cos\theta_k \end{pmatrix}$

Linearization

$$F_k = \left.\frac{\partial f}{\partial q_k}\right|_{q_k = \hat{q}_k} = \begin{pmatrix} 1 & 0 & -\hat{v}_k T_s \sin\hat{\theta}_k & T_s \cos\hat{\theta}_k & 0 \\ 0 & 1 & \hat{v}_k T_s \cos\hat{\theta}_k & T_s \sin\hat{\theta}_k & 0 \\ 0 & 0 & 1 & 0 & T_s \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H_{k+1} = \frac{\partial h}{\partial q_k}\bigg|_{q_k = \hat{q}_{k+1|k}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\hat{V}_{k+1|k} \sin\hat{\theta}_{k+1|k} & \cos\hat{\theta}_{k+1|k} & 0 \end{pmatrix}$$

The EKF equations are:

1. state & covariance prediction

$$\hat{q}_{k+1|k} = f(\hat{q}_k, u_k)$$

$$P_{k+1|k} = F_k P_k F_k^T + \underbrace{\cancel{W_{p,k}}}_{\text{no noise}}$$

2. Correction

$$\hat{q}_{k+1} = q_{k+1|k} + R_{k+1} \gamma_{k+1}$$

$$P_{k+1} = P_{k+1|k} - R_{k+1} H_{k+1} P_{k+1|k}$$

with the innovation $\gamma_{k+1} = h_{k+1} - h_{k+1|k}$

and the Kalman gain matrix

$$R_{k+1} = P_{k+1|k} H^T_{k+1} (H_{k+1} P_{k+1|k} H^T_{k+1} + \underbrace{\cancel{W_{m,k+1}}}_{\text{no noise}})^{-1}$$