THE MAXIMUM PRINCIPLE

~ Principle of optimality

Minimizing over $[t, t_f]$ is equivalent to minimize over $[t, t_1]$ and $[t_1, t_f]$

$$J(x(t), t) = \min_{u[t,t_1]} \left[\int_t^{t_1} \mathcal{L}(x,u,\tau)\,d\tau + \underbrace{\min_{u[t_1,t_f]} \int_{t_1}^{t_f} \mathcal{L}(x,u,\tau)\,d\tau + G(x(t_f))}_{\downarrow} \right]$$

or

$$J(x(t), t) = \min_{u[t,t_1]} \left[\int_t^{t_1} \mathcal{L}(x,u,\tau)\,d\tau + J(x(t_1), t_1) \right]$$

Define the function $C(x(t), t) = \min_{\substack{u(\tau) \in U \\ t_i \le \tau \le t}} \int_{t_i}^{t} \mathcal{L}(x(\tau), u(\tau), \tau)\,d\tau$

If $(x^\circ, u^\circ, t_f^\circ)$ is an optimal solution for the control problem

$$\Rightarrow \quad \boxed{C(x^\circ(t), t) = \int_{t_i}^{t} \mathcal{L}(x^\circ(\tau), u^\circ(\tau), \tau)\,d\tau \quad \forall t \in [t_i, t_f^\circ]} \quad (*)$$

that is if a solution is optimal, it is optimal in any subinterval.

Proof

Assume the relation $(*)$ is not true.

There exists a control u' and the state x' in $[t_i, t]$ s.t:

$$\int_{t_i}^{t} \mathcal{L}(x', u', \tau)\,d\tau < \int_{t_i}^{t} \mathcal{L}(x^\circ, u^\circ, \tau)\,d\tau$$

I can define a new solution of the optimal control:

$$\tilde{u} = \begin{cases} u' & \forall t \in [t_i, t] \\ u^\circ & \forall t \in]t, t_f^\circ] \end{cases} \qquad \tilde{x} = \begin{cases} x' & \forall t \in [t_i, t] \\ x^\circ & \forall t \in]t, t_f^\circ] \end{cases} \qquad \tilde{t}_f = t_f^\circ$$

$$J(\tilde{x}, \tilde{u}, \tilde{t}_f) = \int_{t_i}^{t_f} \mathcal{L}(\tilde{x}, \tilde{u}, t)\, dt = \int_{t_i}^{t} \mathcal{L}(x', u', \tau)\, d\tau + \int_{t}^{t_f^\circ} \mathcal{L}(x^\circ, u^\circ, \tau)\, d\tau$$

$$< \int_{t_i}^{t_f^\circ} \mathcal{L}(x^\circ, u^\circ, \tau)\, d\tau = \underbrace{J(x^\circ, u^\circ, t_f^\circ)}_{\text{contradiction!}}$$

End

$$C(x^\circ(t), t) = \int_{t_i}^{t} \mathcal{L}(x^\circ(\tau), u^\circ(\tau), \tau)\, d\tau \qquad \forall t \in [t_i, t_f^\circ]$$

$$\left.\frac{dC}{dt}\right|^\circ = \mathcal{L}(x^\circ, u^\circ, t)$$

$$\boxed{\left.\frac{\partial C}{\partial x}\right|^\circ \cdot \frac{dx}{dt} + \left.\frac{\partial C}{\partial t}\right|^\circ = \mathcal{L}(x^\circ, u^\circ, t) \qquad \frac{dx}{dt} \to \dot{x} = f(x^\circ, u^\circ, t)}$$

Hamilton - Jacobi equation

The variation of the C function wrt the state multiplied to the variation of the state plus the variation of C wrt time is equal to the Lagrangian evaluated in the optimal solution

~ Hamilton - Jacobi equation

It is an approach in some sense alternative to the minimum principle, combined with the Euler - Lagrange equation.

It is useful mainly for linear regulator problems.

The H-J equation is satisfied by the optimal performance index under suitable differentiability and continuity assumptions.

If a solution to the H-J equation has certain properties, this solution is the desired performance index.

The H-J represents only a <u>sufficient</u> condition on the optimal performance index

$$\left.\frac{\partial C}{\partial x}\right|^{\circ} f(x^{\circ}, u^{\circ}, t) + \left.\frac{\partial C}{\partial t}\right|^{\circ} + \mathcal{L}(x^{\circ}, u^{\circ}, t) = 0$$

$$\Downarrow$$

$$\boxed{\left.\frac{\partial C}{\partial t}\right|^{\circ} + H\left(x^{\circ}, u^{\circ}, \left.\frac{\partial C}{\partial x}\right|^{\circ}, t\right) = 0}$$

~ Pontryagin Principle

Consider the dynamical system $\dot{x} = f(x, u, t)$

f in the calculus of variations was C^2 class.

In this case $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$

$x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^p$.

$x(t_i) = x^i$

For the final values assume $\chi(x(t_f), t_f) = 0$

$\chi \in C^1(\mathbb{R}^{6f \leq n+1})$

Constraints:

$$\int_{t_i}^{t_f} h(x, u, \tau)\, d\tau = k \quad \text{with } h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$$

Cost index:

$$J(x, u, t_f) = \int_{t_i}^{t_f} \mathcal{L}(x, u, \tau)\, d\tau \quad \text{with } \mathcal{L}, \frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial t} \in C^0(\mathbb{R}^n \times U \times \mathbb{R})$$

<u>Goal</u>:

- $t_f \in [t_i, \infty)$
- $u^0 \in \bar{C}^0(\mathbb{R})$
- $x^0 \in \bar{C}^1(\mathbb{R})$

satisfying:

- dynamical system
- control's constraint
- initial and final conditions
- minimize the cost

- Hamiltonian

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t) f(x,u) + \rho^T h(x,u,t)$$

$$J = \int \mathcal{L} \qquad \dot{x} = f \qquad \int h = \chi$$

Theorem:

Consider an admissible solution (x^*, u^*, t_f^*) s.t.

$$\text{rank}\left\{ \frac{\partial \chi}{\partial (x(t_f), t_f)} \Big|^* \right\} = 6f$$

If it is a local minimum

$\exists$ $\lambda_0 \geq 0$, $\rho^* \in \mathbb{R}^6$, $\lambda^* \in \bar{C}^1[t_i, t_f^*]$ not simultaneously null such that:

- $$\dot{\lambda}^* = -\frac{\partial H}{\partial x}\Big|^{*T} \qquad \text{Co-state equation}$$

- $$\boxed{H(x^*, w, \lambda_0^*, \lambda^*) \geq H(x^*, u^*, \lambda_0^*, \lambda^*), \quad \forall w \in U}$$
  Pontryagin inequality — $\hookrightarrow$ any admissible control

And $\exists$ a vector $\xi \in \mathbb{R}^{6f}$ such that:

- $$\dot{\lambda}(T) = \frac{\partial \chi}{\partial x(t_f)}\Big|^{*T} \xi$$

- $$H\Big|^*_{t_f} = -\frac{\partial \chi}{\partial t_f}\Big|^{*T} \xi$$

- discontinuity of $\dot{\lambda}^*$ may occur in t_k where u^* has a discontinuity and
  $$H\Big|^*_{t_k^-} = H\Big|^*_{t_k^+}$$

- If $U = \mathbb{R}^p$ the minimum condition reduces to $\boxed{\frac{\partial H}{\partial u} = 0}$ $\forall t \in [t_i, t_f^o]$ (only in this case the condition holds in place of the inequality of Pontryagin

Proof

We want to prove the Pontryagin inequality.

Let's assume $(x^\circ, u^\circ, t_f^\circ)$ optimal normal solution and $\varepsilon > 0$.

optimal → global minimum
normal → $\lambda_0 = 1$

C over two subintervals:

$$C(x(t),t) = \min_{\substack{u(\tau)\in U \\ t_i \le \tau \le t-\varepsilon}} \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[\int_{t_i}^{t-\varepsilon} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau + \int_{t-\varepsilon}^{t} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau \right]$$

$$= \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[\min_{\substack{u(\tau)\in U \\ t_i \le \tau \le t-\varepsilon}} \int_{t_i}^{t-\varepsilon} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau + \int_{t-\varepsilon}^{t} \mathcal{L}(x(\tau),u(\tau),\tau)\, d\tau \right]$$

$$= \min_{\substack{u(\tau)\in U \\ t-\varepsilon \le \tau \le t}} \left[C(x(t-\varepsilon), t-\varepsilon) + \int_{t-\varepsilon}^{t} \mathcal{L}(x,u,\tau)\, d\tau \right]$$

$$\boxed{x(t-\varepsilon) \cong x(t) - \varepsilon \dot{x} = x(t) - \varepsilon f(x,u,t)}$$

Taylor expansion first order

$$= \min_{u(\tau)\in U} \left[\underbrace{C(x(t) - \varepsilon f(x,u,t), t-\varepsilon)} + \int_{t-\varepsilon}^{t} \mathcal{L}(x,u,\tau)\, d\tau \right]$$

Taylor expansion with $(x(t),t)$ as initial point:

$$\left\{ g(x) = g(x_0) + \frac{dg}{dx}(x - x_0) \right\}$$

↳ increment → ε in our case

$$= \min_{u(\tau)\in U} \left[C(x(t),t) + \frac{\partial C}{\partial x(t)}\left[-\varepsilon f(x,u,t)\right] + \frac{\partial C}{\partial t}(-\varepsilon) + \underbrace{\varepsilon \mathcal{L}(x,u,\tau)}_{\text{approximation because } \varepsilon \text{ is very little}} \right]$$

Since many quantities don't depend on $u(\tau)$:

$$\cancel{C(x(t),t)} = \cancel{C(x(t),t)} - \cancel{\varepsilon}\frac{\partial C}{\partial t} + \min_{u(\tau)\in U}\left[-\frac{\partial C}{\partial x(t)} \cancel{\varepsilon} f(x,u,t) + \cancel{\varepsilon}\mathcal{L} \right]$$

$$\frac{\partial C}{\partial t} = \min_{u(t)\in U}\left[-\frac{\partial C}{\partial x(t)} f(x,u,t) + L(x,u,t)\right] \quad \forall (x(t),t) \in \mathbb{R}^{n+1}$$

Let us choose any $t \in [t_i, t_f^\circ]$ and $x(t) = x^\circ(t)$

$$\left.\frac{\partial C}{\partial t}\right|^\circ = \min_{u(t)\in U}\left[-\left.\frac{\partial C}{\partial x(t)}\right|^\circ f(x^\circ(t), u, t) + L(x^\circ, u, t)\right]$$

Given the Hamilton-Jacobi equation:

$$\left.\frac{\partial C}{\partial x}\right|^\circ f(x^\circ, u^\circ, t) + \left.\frac{\partial C}{\partial t}\right|^\circ = L(x^\circ, u^\circ, t)$$

$$\left.\frac{\partial C}{\partial t}\right|^\circ = -\frac{\partial C}{\partial x(t)} f(x^\circ, u^\circ, t) + L(x^\circ, u^\circ, t) =$$

$$= \min_{u(t)\in U}\left[-\left.\frac{\partial C}{\partial x(t)}\right|^\circ f(x^\circ(t), u, t) + L(x^\circ, u, t)\right]$$

$\to$ similar to the Hamiltonian with $\lambda_0 = 1$

$$H = L + \lambda^T f$$

Let us define $-\lambda^{\circ T}(t) = \left.\frac{\partial C}{\partial x(t)}\right|^\circ$

$$L(x^\circ, u^\circ, t) + \lambda^{\circ T}(t) f(x^\circ, u^\circ, t) = \min_{u(t)\in U}\left[L(x^\circ, u^\circ, t) + \lambda^{\circ T}(t) f(x^\circ(t), u, t)\right]$$

$$H(x^\circ, u^\circ, \lambda^\circ, t) = \min_{u(t)\in U} H(x^\circ(t), u(t), \lambda^\circ(t), t)$$

My Hamiltonian evaluated in the optimal solution is less or equal to the Hamiltonian evaluated in any other point, in particular in the optimal state but with another admissible control

End

~ Convex case

$$\begin{cases} \dot{x} = A(t) + B(t)u \\ x(t_i) = x_i \end{cases} \quad \text{with } x(t) \in \mathbb{R}^n, \; u(t) \in U \subset \mathbb{R}^p$$

$x(t_f) = x_f$, $A, B \in C^1 \quad \forall t \in [t_i, t_f]$

U is a <u>convex set</u>

- Performance index

$$J(x,u) = \int_{t_i}^{t_f} \underbrace{\mathcal{L}(x,u,\tau)}_{\text{convex}} + \overbrace{\underbrace{G(x(t_f))}_{\text{convex } C^2 \text{ class}}}^{\text{Bolza term}}$$

$$\mathcal{L}, \frac{\partial \mathcal{L}}{\partial x_i}, \frac{\partial \mathcal{L}}{\partial t} \in C^0(\mathbb{R}^n \times U \times [t_i, t_f])$$

We want to find if exist ($u^\circ \in \bar{C}^0[t_i,t_f]$, $x^\circ \in \bar{C}^1[t_i,t_f]$) that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t) f(x,u)$$

The necessary and sufficient condition for $(x^\circ, u^\circ, t_f^\circ)$ to be a local minimum:

$\exists \; \lambda^\circ \in \bar{C}^1[t_i,t_f]$ n-dimensional vector such that:

1) $\dot{\lambda}^\circ = -\left.\frac{\partial H}{\partial x}\right|^{\circ T}$

2) $H(x^\circ, \omega, \lambda_0^\circ, \lambda^\circ, t) \geq H(x^\circ, u^\circ, \lambda_0^\circ, \lambda^\circ, t) \quad \forall \omega \in U$

3) if $x(t_f) \in \mathbb{R}^n \rightarrow \lambda^\circ(t_f) = \left.\frac{\partial G}{\partial x(t_f)}\right|^{\circ T}$

if $\mathcal{L}$ is strictly convex wrt x and u and G is strictly convex wrt $x(t_f)$ the solution is <u>unique</u>

~ Stationary problem

$$\begin{cases} \dot{x} = f(x,u) \\ x(t_i) = x_i \end{cases}$$ with $x(t) \in \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^p$, $f, \frac{\partial f}{\partial x} \in C^0(\mathbb{R}^n \times U)$

$x(t_f) = x_f$

$$J(x,u,t_f) = \int_{t_i}^{t_f} \mathcal{L}(x,u)\,dt \qquad \mathcal{L}, \frac{\partial \mathcal{L}}{\partial x} \in C^0(\mathbb{R}^n \times U)$$

We want to find if exist ($t_f^\circ \in (t_i, \infty)$, $u^\circ \in \bar{C}^0[t_i, t_f^\circ]$, $x^\circ \in \bar{C}^1[t_i, t_f^\circ]$) that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \mathcal{L}(x,u) + \lambda^T(t) f(x,u)$$

The necessary condition for $(x^\circ, u^\circ, t_f^\circ)$ to be a local minimum:

$\exists\ \lambda_0^\circ \in \mathbb{R}$, $\lambda^\circ \in \bar{C}^1[t_i, t_f^\circ]$ not simultaneously null

1) $\dot{\lambda}^\circ = -\left.\frac{\partial H}{\partial x}\right|^{\circ T}$

2) $H(x^\circ, \omega, \lambda_0^\circ, \lambda^\circ) \geq H(x^\circ, u^\circ, \lambda_0^\circ, \lambda^\circ) \quad \forall \omega \in U$

3) $H|^\circ = \begin{cases} 0 & (t_f \text{ not fixed}) \\ k & (t_f \text{ fixed}) \end{cases}$

~ Unstationary Problem

$$\begin{cases}\dot{x} = f(x,u,t) & x(t)\in\mathbb{R}^n,\ u(t)\in U\subseteq\mathbb{R}^p\\ x(t_i)=x_i\end{cases}$$

$x(t_f)=x_f$

$f, \frac{\partial f}{\partial x}, \frac{\partial L}{\partial t} \in C^0(\mathbb{R}^n\times U\times\mathbb{R})$

$J(x,u,t_f) = \int_{t_i}^{t_f} L(x,u,t)\,dt \qquad L, \frac{\partial L}{\partial x}, \frac{\partial L}{\partial t}\in C^0(\mathbb{R}^n\times U\times\mathbb{R})$

We want to find if exist $(t_f^* \in (t_i,\infty)$, $u^\circ\in\bar{C}^0[t_i,t_f^\circ]$, $x^\circ\in\bar{C}^1[t_i,t_f^\circ]$ that minimize the cost function

$$H(x,u,\lambda_0,\lambda) = \lambda_0 L(x,u) + \lambda^T(t) f(x,u)$$

The condition for $(x^\circ,u^\circ,t_f^\circ)$ to be a local minimum

$\exists\ \lambda_0^\circ\in\mathbb{R},\ \lambda^\circ\in\bar{C}^1[t_i,t_f^\circ]$ not simultaneously null

1) $\dot{\lambda}^\circ = -\frac{\partial H}{\partial x}\Big|^{\circ T}$

2) $H(x^\circ,\omega,\lambda_0^\circ,\lambda^\circ,t) \geq H(x^\circ,u^\circ,\lambda_0^\circ,\lambda^\circ) \quad \forall\omega\in U$

3) $H\Big|^\circ + \int_{t_i}^{t_f^\circ}\frac{\partial H}{\partial\tau}\,d\tau = \begin{cases}0 & (t_f \text{ not fixed})\\ k & (t_f \text{ fixed})\end{cases}$

If $J=\int L\,d\tau + G(x(t_f),t_f)$ (the Bolza term is present):

The necessary conditions are:

$\lambda^\circ(t_f) = \frac{\partial\chi}{\partial x(t_f)}\Big|^{*T}\xi + \lambda_0^*\frac{\partial G}{\partial x(t_f)}\Big|^{*T}$

$H\Big|^\circ + \int_{t_i}^{t_f^\circ}\frac{\partial H}{\partial\tau}\,d\tau + \lambda_0^\circ\frac{\partial G}{\partial t}\Big|^\circ + \int_{t_i}^{t_f^\circ}\lambda_0^\circ\frac{\partial^2 G}{\partial\tau^2}\,d\tau$