34. Backstepping

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$$\dot{z} = f(z) + g(z)\xi_1$$
$$\dot{\xi}_1 = b_1(z, \xi_1) + a_1(z, \xi_1)\xi_2$$
$$\dot{\xi}_2 = b_2(z, \xi_1, \xi_2) + a_2(z, \xi_1, \xi_2)\xi_3$$
$$\vdots$$
$$\dot{\xi}_r = b_r(z, \xi_1, \dots, \xi_{r-1}) + a_r(z, \xi_1, \dots, \xi_r)u$$

Start from the knowledge of a feedback $\gamma(z)$ and a function $V(z)$ positive and proper such that z is GAS w.r.t ξ_1.
Thus

$$\exists\ \gamma_1(z),\ V(z) \ni \frac{\partial V}{\partial z}(f + g\gamma_1) < 0 \quad \forall z \neq 0$$

under the additional assumption that

$$a_i(z, \dots, \xi_i) \neq 0 \quad \forall (z, \xi_1, \dots, \xi_i) \neq 0$$

A static state feedback which globally asymptotically stabilizes the origin can be computed by means of the following procedure.

Set $\eta_1 = \xi_1 - \gamma_1(z)$, where the feedback $\gamma_1(z)$ stabilizes the dynamics in z, and perform a coordinate transformation:

$$\begin{pmatrix} z \\ \xi_1 \end{pmatrix} \to \begin{pmatrix} z \\ \eta_1 \end{pmatrix}$$

$$\begin{cases} \dot{z} = f(z) + g(z)\gamma_1(z) + g(z)\eta_1 \\ \dot{\eta}_1 = b_1(z, \eta_1 + \gamma_1(z)) + a_1(z, \eta_1 + \gamma_1(z))\xi_2 - \dot{\gamma}_1(z) \end{cases}$$

if $\xi_2 = u$, the following function V_1 positive and proper can be used to prove the Global asymptotic stabilizability of the dynamics (z, η_1), under a suitable $\xi_2 = \gamma_1(z, \eta_1) = \gamma_1(z, \xi_1 - \gamma(z)) = \bar{\gamma}(z, \xi_1)$

$$V_1(z, \eta_1) = V(z) + \frac{1}{2}\eta_1^2$$

choose $\xi_2 \ni \dot{V}_1(z, \eta_1) < \phi$

$$\Rightarrow \dot{V}_1 = \frac{\partial V}{\partial z}(f + g\gamma_1 + g\eta_1) + \eta_1(b_1 + a_1\xi_2 - \dot{\gamma}_1(z))$$

$$\Rightarrow \dot{V}_1 = \frac{\partial V}{\partial z}\left(f + g\gamma_1 + g\eta_1\right) + \eta_1\left(b_1 + a_1\xi_2 - \dot{\gamma}_1(z)\right)$$

with $\xi_2 = a_1^{-1}\left(-b_1 + \dot{\gamma}_1(z) - L_g V - k_1\eta_1\right)$

one obtains

$$\dot{V}_1(z_1,\eta_1) = \underbrace{\frac{\partial V}{\partial z}\left(f + g\gamma\right)}_{<0} - \underbrace{k_1\eta_1^2}_{<0}$$

i.e. GAS of $\begin{pmatrix} z \\ \eta_1 \end{pmatrix}$ equivalently of $\begin{pmatrix} z \\ \xi_1 \end{pmatrix}$.

Rewrite now:

$$\begin{aligned}
\dot{z} &= \overbrace{f(z) + g(z)\gamma_1(z) + g(z)\eta_1}^{f_1(z_1)} \overbrace{+\,0}^{g_1(z_1)} \\
\dot{\eta}_1 &= b_1(z,\eta_1+\gamma_1) - \dot{\gamma}_1(z) + a_1(z,\eta_1+\gamma_1)\,\xi_2 \\
\dot{\xi}_2 &= \dots
\end{aligned}$$

In this way $\begin{pmatrix} z \\ \eta_1 \end{pmatrix}^{n}_{1} = z_1$ of dimension $n+1$ and the dynamics takes the form

$$\dot{z}_1 = f_1(z_1) + g_1(z_1)\,\xi_2$$

By adding the dynamics of ξ_2 one obtains:

$$\begin{cases} \dot{z}_1 = f_1(z_1) + g_1(z_1)\,\xi_2 \\ \dot{\xi}_2 = b_2(z,\eta_1+\gamma_1,\xi_2) + a_2(z,\eta_1+\gamma_1,\xi_2)\,\xi_3 \end{cases}$$

That is:

$$\dot{\xi}_2 = \bar{b}_2(z_1,\xi_2) + \underbrace{\bar{a}_2(z_1,\xi_2)}_{\neq 0}\,\xi_3$$

i.e. a dynamics of the same form as the one at the beginning of the procedure.

It is known that $\exists\ \xi_2 = \gamma_1(z_1)$ and $V_1(z_1)$ such that

$$\frac{\partial V_1}{\partial z_1}\left(f_1 + g_1\gamma_1\right)(z_1) < 0 \qquad \forall z_1 \neq 0$$

then it is possible to repeat the entire procedure from the then it is possible to repeat the entire procedure from the start.

Set $\eta_2 = \xi_2 - \gamma_1(z_1)$ and rewrite the dynamics in the new coordinates

$\begin{pmatrix} z_1 \\ \eta_2 \end{pmatrix}$ in order to get:

$$\begin{cases} \dot{z}_1 = f_1(z_1) + g_1(z_1)\gamma_1(z_1) + g_1(z_1)\eta_2 \\ \dot{\eta}_2 = \bar{b}_2(z_1, \eta_2 + \gamma_1(z_1)) - \dot{\gamma}_1(z_1) + \bar{a}_2(z_1, \eta_2 + \gamma_1(z_1))\xi_3 \end{cases}$$

which has the same structure of the previous computations with z_1, η_2, ξ_3 in place of z, η_1, ξ_2.

The same computations in fact can be repeated: the following function $V_2(z_1, \eta_2)$ positive and proper

$$V_2(z_1, \eta_2) = V_1(z_1) + \frac{1}{2}\eta_2^2 \Rightarrow \bar{V}_2(z, \xi_1, \xi_2) = \bar{V}_1(z, \xi_1) + \frac{1}{2}(\xi_2 - \bar{\gamma}_1(z, \xi_1))$$

can be used to compute

$$\xi_3 = \gamma_2(z_1, \eta_2)$$

in order to render $\dot{V}_2 < 0 \quad \forall (z_1, \eta_2) \neq 0$.

It can be verified that:

$$\gamma_2(z_1, \eta_2) = -\bar{a}_2^{-1}(\bar{b}_2 + \dot{\gamma}_1(z_1) - L_{g_1}V_1 - k_2\eta_2)$$

and by setting

$$\begin{array}{r} {\scriptstyle n+1} \\ {\scriptstyle 1} \end{array}\begin{pmatrix} z_1 \\ \eta_2 \end{pmatrix} = z_2 \quad \text{of dimension } n+2 \quad \text{one obtains:}$$

$$\begin{cases} \dot{z}_2 = f_2(z_2) + g_2(z_2)\xi_3 \\ \dot{\xi}_3 = \bar{b}_3(z_2, \xi_3) + \bar{a}_3(z, \xi_3)\xi_4 \end{cases}$$

always a structure like the previous ones with the property that the function $V_2(z_2)$ together with the computed $\gamma_2(z_2)$ are such that

$\dot{V}_2 < 0$

the computed $\gamma_2(\xi_2)$ be such that

$$\dot{V}_2 < 0$$

assuring GAS of the dynamics of ξ_2, and equivalently of z, ξ_1, ξ_2.

In this way it is possible to proceed with $\xi_4 = \gamma_3(\xi_2, \eta_3)$ which has the same form of γ_1 and γ_2, and repeating the same iterative computations.

In order to apply profitably the backstepping procedure a suitable control Lyapunov function for the dynamics

$$\dot{z} = f(z) + g(z)\,\xi \rightarrow (u)$$

must be computed.

A control Lyapunov function (CLF) is a positive and proper V which can be rendered a strict Lyapunov function under a control law like $u = \gamma(x)$ i.e.

$$\frac{\partial V}{\partial x}\left(f + g\alpha\right) < 0$$