16. State-space exact linearization problem

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u-x feedback linearization

Consider $\dot{x} = f(x) + g(x)u$

Given a point x_0 find, if possible, a neighborhood U of x_0, a feedback $u = \alpha(x) + \beta(x)v$ defined on U, and a coordinate transformation $z = \Phi(x)$ on U such that

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v$$

in the z coordinates is linear and controllable, i.e.

$$\frac{\partial \phi}{\partial x}\left(f(x) + g(x)\alpha(x)\right)\Big|_{x=\phi^{-1}(z)} = Az$$

$$\frac{\partial \phi}{\partial x}\left(g(x)\beta(x)\right)\Big|_{x=\phi^{-1}(z)} = B$$

and $\rho(B \; AB \cdots A^{n-1}B) = n$

Lemma: the u-x feedback linearization problem is solvable iff there exists a neighborhood of x_0 and a real valued function $\lambda(x)$ on U, such that

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = \lambda(x) \end{cases}$$

has relative degree n at x_0

Proof (Necessity):

(1) r is invariant under coordinate transformations

$$\bar{f}(z) = \left(\frac{\partial \phi}{\partial x} f(x)\right)\Big|_{x=\phi^{-1}(z)} \qquad \bar{g}(z) = \left(\frac{\partial \phi}{\partial x} g(x)\right)\Big|_{x=\phi^{-1}(z)} \qquad \bar{h}(z) = h(\phi^{-1}(z))$$

$$L_{\bar{f}}\bar{h}(z) = \frac{\partial \bar{h}}{\partial z}\bar{f}(z) = \left(\frac{\partial h}{\partial x}\right)\Big|_{x=\phi^{-1}(z)} \left[\frac{\partial \phi^{-1}}{\partial z}\frac{\partial \phi}{\partial x} f(x)\right]\Big|_{x=\phi^{-1}(z)} =$$

$$= \left(\frac{\partial h}{\partial x} f(x)\right)\Big|_{x=\phi^{-1}(z)} = \left[L_f h(x)\right]\Big|_{x=\phi^{-1}(z)}$$

iterating... $L_{\bar{g}} L_{\bar{f}}^k \bar{h}(z) = [L_g L_f^k h(x)]_{x=\phi^{-1}(z)}$

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Therefore, the relative degree is invariant under coordinates transformation.

② One can similarly show that r is invariant under feedback:

$$L_{f+g\alpha}^{k} h(x) = L_f^k h(x) \quad \forall\, 0 \le k \le r-1$$

then, for sake of completeness:

$$L_{f+g\alpha}^{k+1} h(x) = L_{f+g\alpha} L_f^k h(x) = L_f^{k+1} h(x) + \cancel{L_g L_f^k h(x) \alpha(x)} =$$
$$= L_f^{k+1} h(x)$$

showing that the equality holds for $k+1$ too.

One deduces that

$$L_{g\beta} L_{f+g\alpha}^{k} h(x) = 0 \quad \forall\, 0 \le k < r-1$$

and if $\beta(x_0) \neq 0 \longrightarrow L_{g\beta} L_{f+g\alpha}^{r-1} h(x_0) \neq 0$

this shows that r is invariant under feedback.

Now, let (A, B) be a reachable pair. Then $\exists T$ and a vector K such that

$$T(A+BK)T^{-1} = \begin{pmatrix} 0 & 1 & & 0 \\ | & & \ddots & 1 \\ 0 & & — & 0 \end{pmatrix} \qquad TB = \begin{pmatrix} 0 \\ | \\ 0 \\ 1 \end{pmatrix} \qquad \begin{matrix} T \in \mathbb{R}^{n\times n} \\ K \in \mathbb{R}^{1\times n} \end{matrix}$$

set $\bar{z} = \bar{\Phi}(x) = T\Phi(x)$

$\bar{\alpha}(x) = \alpha(x) + \beta(x) K \phi(x)$

Then, it is easily seen that

$$\left[\frac{\partial\phi}{\partial x}\left(f(x) + g(x)\bar{\alpha}(x)\right)\right]\Big|_{x=\bar{\phi}^{-1}(z)} = \begin{pmatrix} 0 & 1 & & 0 \\ | & & \ddots & 1 \\ 0 & & — & 0 \end{pmatrix} \bar{z} = A_0 \bar{z}$$

$$\left[\frac{\partial\bar{\phi}}{\partial x}\left(g(x)\beta(x)\right)\right]\Big|_{x=\bar{\phi}^{-1}(z)} = \begin{pmatrix} 0 \\ | \\ 0 \\ 1 \end{pmatrix} = B_0$$

Define the output function as $y = (1\ 0 - 0)\bar{z}$.

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A straightforward calculation shows that a linear system with A and B in the form of A_0, B_0 and with this output function has relative degree equal to n.
Thus, since r is invariant under coordinates transformation and feedback, the proof is completed.

Finding $\lambda(x)$ such that $r = n$, more precisely
$L_g \lambda(x) = L_g L_f \lambda(x) = \dots = L_g L_f^{n-2} \lambda(x) = 0 \ \forall x$
$L_g L_f^{n-1} \lambda(x_0) \neq 0$
is a problem involving the solution of a system of PDE in which $\lambda(x)$ is differentiated $n-1$ times.
The existence of a function satisfying these relations is a consequence of the Frobenius Theorem.

Lemma: $\exists$ a real valued function $\lambda(x)$ defined on U of x_0 solving the PDEs and satisfying:

① $L_g \lambda(x) = L_g L_f \lambda(x) = \dots = L_g L_f^{n-2} \lambda(x) = L_{ad_f g} \lambda(x) = \dots$
$= L_{ad_f^{n-2} g} \lambda(x) = 0$

② $L_g L_f^{n-1} \lambda(x_0) = L_{ad_f^{n-1} g} \lambda(x_0) \neq 0$
if and only if:

(i) $\rho \left[g(x_0) \ \ ad_f g(x_0) \dots ad_f^{n-2} g(x_0) \ \ ad_f^{n-1} g(x_0) \right] = n$

(ii) the distribution $\Delta = \text{span}\{ g, ad_f g, \dots, ad_f^{n-2} g \}$ is involutive in a neighborhood of x_0.

$\rightarrow$ U-X FL problem solvable iff (i) and (ii) holds true.

- Basically, the U-X FL procedure is the following:

step 1 - from $f(x)$ and $g(x)$ construct the vector fields

Basically, the U-X FL procedure is the following:

step 1 - from $f(x)$ and $g(x)$ construct the vector fields $g(x),\ ad_f\, g(x), \dots, ad_f^{\,n-2} g(x),\ ad_f^{\,n-1} g(x)$ and check conditions (i) and (ii)

step 2 - if both are satisfied, solve for $\lambda(x)$ the partial differential equation (1)

step 3 - set $\alpha(x) = \dfrac{-L_f^n \lambda(x)}{L_g L_f \lambda(x)}$ $\qquad \beta(x) = \dfrac{1}{L_g L_f^{n-1} \lambda(x)}$

step 4 - set $\Phi(x) = \begin{pmatrix} \lambda(x) \\ L_f \lambda(x) \\ \dots \\ L_f^{n-1} \lambda(x) \end{pmatrix}$ "linearizing coordinates"

• Feedback linearization schematic resume

$U\text{-}Y \xrightarrow{FL} r \underset{(\text{from } x)}{(\alpha, \beta)}$ $\qquad W(s) = \dfrac{1}{s^r}$ + pole placement

$U\text{-}X \xrightarrow{FL} \exists \lambda : r_\lambda = n$ $\qquad W_\lambda(s) = \dfrac{1}{s^n}$ + pole placement

$\dot{z} = A_0 z + B_0 v$ + eigenvalues assignment

• Additional results on U-X FL

As already pointed out, the U-X FL problem is solvable if and only if

(1) $\rho\left(g(x_0)\ \ ad_f\, g(x_0)\ \dots\ ad_f^{\,n-1} g(x_0)\right) = n$ and

(2) $\Delta = \text{span}\{g(x_0), ad_f\, g(x_0), \dots, ad_f^{\,n-2} g(x_0)\}$ involutive

If these 2 conditions are not satisfied there is no way of achieving a linear and controllable system by mean of feedback and coordinate transformation.

However, even if it is not possible, one can always decompose the system in two subsystems: a linear

However, even if it is not possible, one can always decompose the system in two subsystems: a linear one and a non linear one.

Also in this case the dimension of the linear subsystem can be maximized by appropriate $\phi(x)$ and v

Lemma: Consider Δ and suppose λ real valued function such that $\partial\lambda(x_0) \neq 0$ and $\partial\lambda \in \Delta^\perp$.
Then, in a neighborhood of x_0, $\partial\lambda \in \bar{\Delta}^\perp$

Proof: Consider the distribution $\Gamma = (\mathrm{span}\{\partial h\})^\perp$ $n-1$ dimensional in a neighborhood of x_0, and involutive by Frobenius theorem.
By construction $\Delta \subset \Gamma$, and since $\bar{\Delta}$ is the smallest involutive distribution containing Δ, then $\bar{\Delta} \subset \Gamma$, that is $\mathrm{span}\{\partial h\} \subset \bar{\Delta}^\perp$

· **Max u-x FL**

Theorem: Consider $f(x)$ and $g(x)$ vector fields
Suppose, for some integer γ

overbar denotes involutive closure

(i) $\rho(\bar{\Delta}_{\gamma-2}) = \rho(\overline{\mathrm{span}\{g, ad_f g, \dots, ad_f^{\gamma-2} g\}}) = k < n$
$\forall x$ around x_0 and

(ii) $\rho(\bar{\Delta}_{\gamma-1}) = \rho(\overline{\mathrm{span}\{g, ad_f g, \dots, ad_f^{\gamma-1} g\}}) = n$
at each $x = x_0$

then, in a neighborhood U_0 of x_0, there exists a function $\lambda(x)$ such that

$L_g \lambda(x) = L_g L_f \lambda(x) = \dots = L_g L_f^{\gamma-2} \lambda(x) = 0$ for all $x \in U_0$ and
$L_g L_f^{\gamma-1} \lambda(x) \neq 0$.

$\exists\, \lambda : r_\lambda = \gamma$ and $\forall\, \bar{\lambda} \neq \lambda$, $r_{\bar{\lambda}} \leq \gamma$

Proof: The distribution $\bar{\Delta}_{\gamma-2} = \overline{\mathrm{span}\{g, ad_f g, \dots, ad_f^{\gamma-2} g\}}$ is involutive by construction and has dimension $k < n$.

involutive by construction and has dimension $k<n$.
By Frobenius theorem: $\exists$ $n-k$ functions $\lambda_1(x), \dots, \lambda_{n-k}(x)$ whose differentials locally span the annihilator of $\bar{\Delta}_{r-2}$.
$\exists\ \lambda_1, \dots, \lambda_{n-k} : L_g \lambda, \dots, L_{ad_g g^{\gamma-2}} \lambda \equiv 0 \quad \forall \lambda = \{\lambda_1, \dots, \lambda_{n-k}\}$
while $L_{ad_g^{\gamma-1} g} \lambda(x) \neq 0$
If this is not true, the non zero covector $\partial\lambda(x)$ would be an element of span $\{g, ad_g g, \dots, ad_g^{\gamma-1} g\}^\perp$ and, by previous lemma, also an element of $\bar{\Delta}_{\gamma-1}^\perp$.
But, since $\bar{\Delta}_{\gamma-1}$ has dimension n, $\bar{\Delta}_{\gamma-1}^\perp$ has dimension $\emptyset$.
Therefore $\lambda(x)$ has the required properties.
Consider now $\bar{\lambda}(x)$ any other function such that
$$\partial\bar{\lambda} \in \text{span}\{g, ad_g g, \dots, ad_g^{\gamma-2} g\}^\perp = \Delta_{\gamma-2}^\perp$$
by previous lemma
$$\partial\bar{\lambda} \in \bar{\Delta}_{\gamma-2}^\perp$$
$$\exists\ \bar{\lambda} : r_{\bar{\lambda}} = \bar{r}$$
from (ii) follows that $\bar{r} \leq \gamma$, and the proof is completed.

Note that:

- If 1 and 2 are satisfied, then $\gamma = n$
- If 1 and 2 not satisfied, in order to find an output map $\lambda(x)$ one has to solve PDE of the form
$$\partial\lambda(x)\ (\tau_1, \dots, \tau_k) = 0$$
where $\tau_1, \dots, \tau_k$ are such that
$$\text{span}\{\tau_1, \dots, \tau_k\} = \bar{\Delta}_{\gamma-2}$$
then the feedback $u = \dfrac{1}{L_g L_f^{r-1} h(x)} \left(-L_f^r h(x) + v\right)$

will transform the system into one which, in the suitable coordinates, contains a linear subsystem of maximal dimension.