18. Zero dynamics

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Concept similar to the concept of "zeros" of the transfer function in a linear system.

First note that, if $r = n$, there is no zero dynamics; in fact, in linear systems, if $r = n$ no zeros appear in the transfer function.

Consider now $r < n$, and denote

$$\dot{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_r \\ \hline z_{r+1} \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z \\ \hline \eta \end{pmatrix} \begin{matrix} \}\, r \\ \}\, n-r \end{matrix}$$

with this notation the normal form can be rewritten as

$$\dot{z}_1 = z_2$$
$$\dots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = b(z,\eta) + a(z,\eta)u$$
$$\dot{\eta} = q(z,\eta)$$
$$y = z_1$$

Problem of zeroing the output

Find, if any, pairs $(x_0, u_0(t))$ consisting of an initial state x_0 and of an input function $u_0(\cdot)$, defined $\forall t$ in a neighborhood of $t=0$, such that the output $y(t)$ is zero $\forall t$ near $t=0$.

$\rightarrow (x_0, u_0(t))$ such that $y(t) = 0$

In the normal form, having $y(t) = z_1(t)$, the constraint $y(t) = 0$ implies that:

$$\dot{z}_1(t) = \dot{z}_2(t) = \dots = \dot{z}_r(t) = 0$$

that is $z(t) = 0 \quad \forall t$

The input $u(t)$ must be the unique solution of the

Then $y(t) = 0 \ \forall t$

The input $u(t)$ must be the unique solution of the equation

$$\dot{z}_r = b(0,\eta) + a(0,\eta)u = 0$$

It is possible to deduce that if $y(t)$ has to be zero then it's necessary that the initial state of the system must be set to a value such that $z(0)=0$ while $\eta(0) = \eta_0 \neq 0$ can be chosen arbitrarily.

According to η_0

$$u(t) = \frac{-b(0,\eta(t))}{a(0,\eta(t))}$$

where $\eta(t)$ denotes the solution of the differential equation $\dot{\eta} = q(0,\eta(t))$, $\eta(0) = \eta_0$

This corresponds to the dynamics describing the internal behavior of the system when input and initial conditions have been chosen in such a way to constrain the output to be zero.

This dynamics is called "zero dynamics" of the system

Link with linear systems

Consider the minimal realization of $W(s)$

$$W(s) = k\,\frac{b_0 + b_1 s + \dots + b_{n-r-1}s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \dots + a_{n-1}s^{n-1} + s^n}$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_0 & -a_1 & -a_2 \ \cdots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k \end{pmatrix}$$

$$C = \begin{pmatrix} b_0 & b_1 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{pmatrix}$$

The normal form is the following:

$$z_1 = Cx = b_0 x_1 + b_1 x_2 + \dots + b_{n-r-1} x_{n-r} + x_{n-r+1}$$
$$z_2 = CAx = b_0 x_2 + b_1 x_3 + \dots + b_{n-r-1} x_{n-r+1} + x_{n-r+2}$$
$$\vdots$$
$$z_r = CA^{r-1}x = b_0 x_r + b_1 x_{r-1} + \dots + b_{n-r-1} x_{n-1} + x_n$$

while the last $n-r$ coordinates can be chosen as:

$$\eta_{r+1} = x_1$$
$$\eta_n = x_{n-r}$$

This is an admissible choice because $z = \phi(x)$ has a non singular Jacobian matrix.

$$\frac{\partial \phi}{\partial x} = \left(\begin{array}{c|c} \text{any} & \begin{matrix} 1 & & 0 \\ \# & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \end{array}\right)$$

In the new coordinates, because of linearity

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = Rz + S\eta + Ku$$
$$\dot{\eta} = Pz + Q\eta$$

Where R, S are row vectors and P, Q are matrices of suitable dimension.

The zero dynamics are $\dot{\eta} = Q\eta$ and the previous choice of the $n-r$ new coordinates involves a particular structure for Q:

$$\dot{\eta}_{r+1} = \dot{x}_1 = x_2 = \eta_2$$
$$\dot{\eta}_{r+2} = \dot{x}_2 = x_3 = \eta_3$$
$$\vdots$$

$$\dot{\eta}_{n-1} = \eta_n$$

$$\dot{\eta}_n = \dot{x}_{n-r} = x_{n-r+1} = -b_0 x_1 - b_1 x_2 - \dots - b_{n-r-1} x_{n-r} - z_1$$

$$Q = \begin{pmatrix} 0 & 1 & & 0 \\ | & & \ddots & \\ 0 & & & 1 \\ -b_0 & -b_1 & \dots & -b_{n-r-1} \end{pmatrix}$$

The eigenvalues of Q coincide with the zeros of $W(s)$.

$$|\lambda I - Q| = b_0 + b_1 \lambda + \dots + b_{n-r-1} \lambda^{n-r-1} + \lambda^{n-r}$$

Relation between NL zero dynamics and zero dynamics of the LTM (linear Tangent model)

$$\begin{tikzcd} S_{NL} \arrow[r, bend left, "ZD"] \arrow[d, "L"'] & \dot{\eta} = q(0,\eta) \arrow[d, "L"] \\ LTM \arrow[r, bend right, "ZD"'] & \dot{\eta} = Q\eta \end{tikzcd}$$

And it is True that $Q = \dfrac{\partial q(0,\eta)}{\partial \eta}$

To understand this fact consider the relative degree of LTM, assuming $x_0 = 0$

$$f(x) = Ax + f_2(x) \qquad f_2(0) = 0$$
$$h(x) = Cx + h_2(x) \qquad h_2(0) = 0$$
$$g(x) = B + g_2(x) \qquad g_2(0) = 0$$

One deduce that

$$CA^k B = L_g L_f^k h(x)|_{x_0} = 0 \qquad k = 0, \dots, r-2$$
$$CA^{k-1} B = L_g L_f^{r-1} h(x)|_{x_0} \neq 0$$

the relative degree does not change under linearization at equilibrium the relative degree does not change under linearization at equilibrium

$$b(z,\eta) = Rz + S\eta + b_2(z,\eta)$$
$$a(z,\eta) = K + a_2(z,\eta)$$
$$q(z,\eta) = Pz + Q\eta + q_2(z,\eta)$$

i.e., taking the linear approximation of the normal form yelds a linear system in normal form. Thus, the Jacobian $Q = \left(\frac{\partial q}{\partial \eta}\right)_{(z,\eta)=0}$

which describes the linear approximation at $\eta = 0$ of the zero dynamics of the nonlinear system has eigenvalues which coincides with the zeros of the transfer function of the linear approximation of the system at $x=0$

The state feedback $u^*(x) = \dfrac{-L_f^r h(x)}{L_g L_f^{r-1} h(x)}$

is such that $\begin{pmatrix} \partial h(x) \\ \partial L_f h(x) \\ \cdots \\ \partial L_f^{r-1} h(x) \end{pmatrix} (f(x) + g(x) u^*) =$

$$= \begin{pmatrix} L_f h(x) + L_g h(x) u^*(x) \\ \cdots \\ L_f^r h(x) + L_g L_f^{r-1} h(x) u^*(x) \end{pmatrix} = 0 \quad \forall x \in \Delta^*$$

$\Delta^* = \ker \begin{pmatrix} \partial h(x) \\ \cdots \\ \partial L_f^{r-1} h(x) \end{pmatrix}$ is a distribution of dimension $n-r$

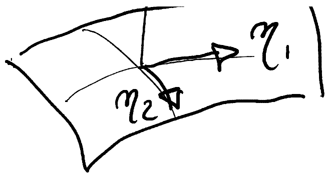
and therefore $f^*(x) = f(x) + g(x) u^*(x)$ is tangent to Δ^*.

Z is an integral variety of Δ^* through x_0

$$Z = \{x \in \mathbb{R}^n : h(x) = L_f h(x) = \dots = L_f^{r-1} h(x) = 0\}$$

$n = 3$

$r = 2$



$\dot{r} = \dot{z}$

$Z = \ker\{\partial h\}$

Consider again the system in normal form, with input
$$u = \frac{1}{a(z)}\left(-b(z) + v\right)$$
under which the system becomes fully linearized, i.e., the input-output behavior acts as a chain of r integrators.
The closed-loop system can be rewritten as:

$$\dot{z} = Az + Bv \qquad A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ & & & 1 \\ 0 & & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$
$$\dot{\eta} = q(z, \eta)$$
$$y = Cz \qquad C = (1 \; 0 \; \cdots \; 0)$$

If no input is applied, then $y(t) = 0$, and the internal dynamics of the closed loop system coincides with the zero dynamics of the open loop system.

Zero Dynamics of non-strict normal forms

Working on equations:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = b(z, \eta) + a(z, \eta)u$$
$$\dot{\eta} = q(z, \eta) + p(z, \eta)u$$

The same conditions are imposed:

$z(t) = 0$ and $0 = b(0, \eta(t)) + a(0, \eta(t))u$

Replacing $u(t)$ from this equation into the last one of $\dot{\eta}$ the result is:

$$\dot{\eta} = q(0, \eta(t)) - p(0, \eta(t))\frac{b(0, \eta(t))}{a(0, \eta(t))}$$

$$\eta = q(0, \eta(t)) - p(0, \eta(t)) \frac{b(0, \eta(t))}{a(0, \eta(t))}$$