11. Disturbance Decoupling Problem under State Feedback

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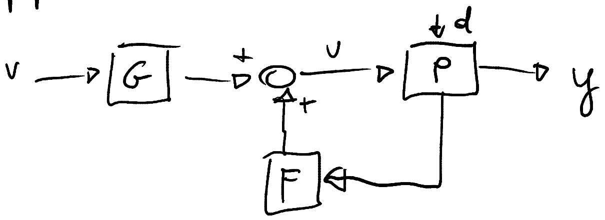
Suppose a disturbance d is acting on a process P:



The disturbance decoupling problem under state feedback regards the realization of a static state

$u = Fx + Gv$

Where G is the gain matrix associated to the input v, in such a way the output of the process y does not depend on the disturbance d.

Therefore the problem consists on finding (F, G) such that y does not depend on d.

Necessary & Sufficient condition for DDP solvability is:

$\exists\, V : Av \subset V,\ \forall v \in V$ such that $\mathrm{Im}(D) \subset V \subset \ker(C)$

i.e., the reachable states from d are contained in the unobservable set

$\mathcal{R}_D = \mathrm{Im}(D\ AD \cdots A^{n-1}D)$ if $x_0 \in \mathcal{R}_D \rightarrow Ax_0 \in \mathcal{R}_D$

$$\begin{pmatrix} C \\ \vdots \\ CA^{n-1} \end{pmatrix} (D\ AD \cdots A^{n-1}D) \equiv 0 \iff \begin{array}{ll} CA^kD = 0 & \forall k \\ Ce^{At}D = 0 & \forall t \end{array}$$

the system under state feedback becomes:

$$\begin{cases} \dot{x} = Ax + BFx + BGv + Dd \\ y = Cx \end{cases}$$

def.: V is (A,B)-invariant if $\exists\, F : (A+BF)v \subset V$

Prop: DDP is solvable iff. $\exists\, v$ (A,B)-invariant such that $\mathrm{Im}(D) \subset v \subset \ker(C)$

Proof (necessity $\rightarrow$): if $\exists\, F$ s.t. $(A+BF)v \subset V$, then $(A+BF)^k v \subset V\ \forall k$

since $v \subset \ker(C)$, then $C(A+BF)^k v = 0\ \forall k$

and since $\mathrm{Im}(D) \subset V$, then $C(A+BF)^k D = 0\ \forall D$

which assures the independence of y

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Proof (sufficiency) $\leftarrow$: Suppose $C(A+BF)^k D = 0 \ \forall k$
then $(A+BF)^k D \subset Ker(C) \rightarrow (A+BF)^k Im(D) \subset (C)$
$v \in V = Im(D \ \ (A+BF)D \cdots (A+BF)^k D)$
is (A,B)-invariant and by construction is $\supset D$.

Prop: V is (A,B) invariant iff $Av \subset V + Im(B) \ \ \forall v \in V$

Proof (Nec.) $\rightarrow$: if $\exists F : (A+BF)v \subset V \ \ \forall v \in V$ then
$(A+BF)v = w \quad \forall v, w \in V$
thus $Av + BFv = w \rightarrow Av = w - BFv$ which is
$AV \subset V + Im(B)$

Proof (Suff.) $\leftarrow$: Suppose $V = \{v_1, \dots, v_p\}$ where $v_1, \dots, v_p$ is a basis of V. According to the definition of invariance:
$A(v_1, \dots, v_p) = (w_1, \dots, w_p) + B(u_1, \dots, u_p)$.
Let F be a $p \times n$ matrix. If the identity
$F(v_1, \dots, v_p) = -(u_1, \dots, u_p)$ is satisfied by some F for the fixed $(v_1, \dots, v_p)$ and $(u_1, \dots, u_p)$ then the proof is completed.

Proposition: DDP is solvable iff $Im(D) \subset V^*$
with V^* maximal (A,B) invariant subspace contained in $Ker(C)$

$$Im(D) \subset V^* = ker \begin{pmatrix} C \\ C(A+BF^*) \\ \vdots \\ C(A+BF^*)^{n-1} \end{pmatrix} = 0$$

F^* is the feedback which maximizes the unobservability of the controlled system.
The maximal unobservability is the one obtained by deleting stable zeros and it can be generated in order to solve DDP and to guarantee the stability of the controlled system

Proposition: Let r be the relative degree of the system, and suppose $p = q = 1$

then $F^* = -\dfrac{CA^r}{CA^{r-1}B}$ cancel all zeros

and is called "friend" of $V^* = \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix}$

therefore $V^* \subset \ker(C)$ and it is invariant, in fact:

if $\bar{x} \in V^* \rightarrow CA^k \bar{x} = 0 \quad \forall k = 0, \dots, r-1$

$$(A + BF^*)\bar{x} = \left(A - \cancel{B} \frac{CA^r}{CA^{r-1}\cancel{B}}\right)\bar{x} \subset V^*$$

premultipling for:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} \left(A - \cancel{B}\frac{CA^r}{CA^{r-1}\cancel{B}}\right)\bar{x} = 0 \rightsquigarrow \begin{pmatrix} I \\ CA^r - CA^r \end{pmatrix}$$

because

$$\begin{cases} CA^k B = 0 & k = 0, \dots, r-2 \\ CA^k B \neq 0 & k = r-1 \end{cases}$$

Consider now $T = \begin{pmatrix} C \\ \vdots \\ CA^{r-1} \\ \hline \forall \end{pmatrix} \begin{matrix} \}\, r \\ \}\, n-r \end{matrix}$ and suppose that

$\nexists\, \gamma : \gamma T = 0$

We are looking for a representation of the feedback system under coordinate transformation

$$T(A + BF^*)T^{-1} = \tilde{A} \rightarrow T(A + BF^*) = \tilde{A}T$$

$$TBG^* = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline \forall \end{pmatrix} \quad \text{with } G^* = \frac{1}{CA^{r-1}B}$$

the system $S(F^*, G^*)$ in the $z = Tx$ coordinates is:

$$\tilde{A} = \left(\begin{array}{cccc|c} 0 & 1 & & 0 & \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & & 0 & \\ \hline & \tilde{A}_{21} & & & \tilde{A}_{22} \end{array}\right) \qquad \tilde{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \hline \tilde{B}_2 \end{pmatrix} \qquad CT^{-1} = \tilde{C} = (1\ 0 - 0)$$

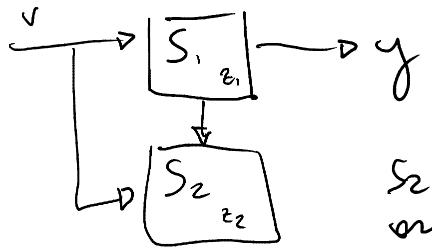
$$\begin{cases} \dot{z}_1 = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} v \\ \dot{z}_2 = \tilde{A}_{21} z_1 + \tilde{A}_{22} z_2 + \tilde{B}_2 v \end{cases}$$

Brunowsky Canonical form

$$\begin{cases} \dot{z}_2 = \tilde{A}_{21} z_1 + \tilde{A}_{22} z_2 + \tilde{B}_2 v \\ y = (1\ 0 \dots 0) z_1 \end{cases}$$

Brunowsky Canonical form

The decomposition wrt observability can be represented as:

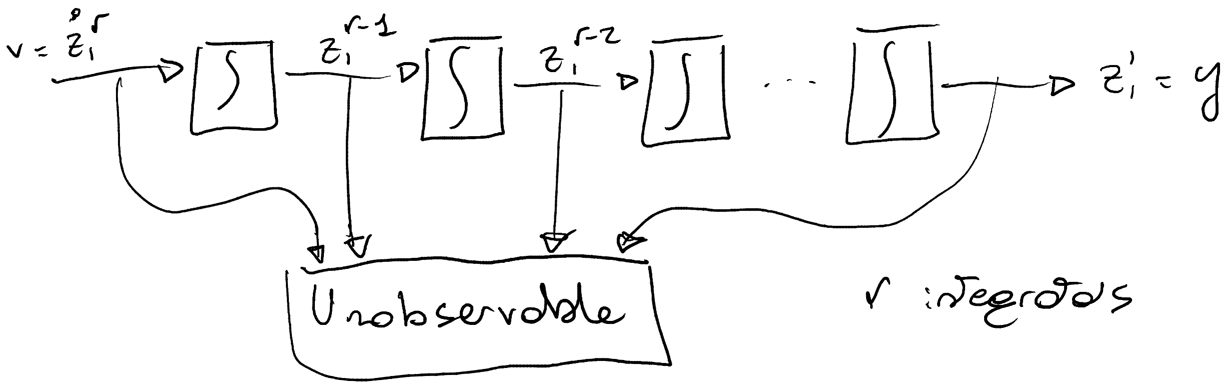


S_2 is the unobservable subsystem and the disturbance acts here.

Therefore the feedback renders the disturbance unobservable, $r-1$ is the dimension of the maximal unobservability.

$$\begin{cases} \dot{z}_1^1 = z_1^2 \\ \dot{z}_1^2 = z_1^3 \\ \vdots \\ \dot{z}_1^r = v \\ y = z_1^1 \end{cases}$$

the new system is a cascade of integrators



r integrators

and, under F^*

$$C(sI - A - BF^*)B = \begin{pmatrix} 1/s^{r_1} & & \\ & \ddots & \\ & & \frac{1}{s^{r_m}} \end{pmatrix}$$

DDP with stability

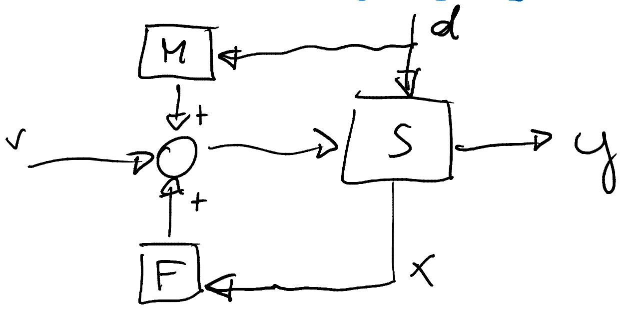
Stability can be reached by a solution which maximizes unobservability iff

$$\sigma\left((A+BF^*)|_{V^*}\right) \subset \mathbb{C}^-$$

i.e. the eigenvalues which characterize the restriction of the dynamics to V^*, and specify the unobservable dynamics are stable (they are the zeros of A, B, C)

Therefore a stable solution exists iff the disturbances matrix D can be included into the maximal unobservability subspace which can be generated without cancelling unstable zeros.

DDP with measure of the perturbation



d: measurable

DDP with measure is solvable iff
$\mathrm{Im}(BM + D) \subset V^*$ for some M.

M is used to cancell the effect of the perturbation which matches with the input.

$\exists M : \mathrm{Im}(BM+D) \subset V \iff \mathrm{Im}\, D \subset V + \mathrm{Im}\, B$

Prop: DDP with measure of d is solvable iff

$\mathrm{Im}\, D \subset V^* + \mathrm{Im}\, B$

the procedure is:
- compute K: $BK = D$, $D = (D_1, D_2)$, $D_1 \subset \mathrm{Im}\, B$
- set $M = -K$
- compute F which solves DDP with D_2