LINEAR QUADRATIC GAUSSIAN PROBLEM (LQG)

We have a linear system with some noise and the cost index is very similar to the previous cases.

~ Optimal regulator with available state on finite time interval

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \quad , \; t \in [t_i, t_f]$$

w = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value
(x_i, w not correlated)

Covariance:

$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$ covariance (how much this this information is precise) → uncertainty

$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\,\delta(\tau)$

I want to minimize:

Bolza term

$$J(P) = \frac{1}{2} E \left\{ \int_{t_i}^{t_f} [x^T Q x + u^T R u]\, dt + x^T(t_f) F x(t_f) \right\}$$

→ expected value of the integral

In the previous cases our optimal control was

$$u^\circ = -R^{-1} B^T K x^\circ$$

In this case I have the noise but I can find a control as:

$$u^\circ(t) = P(t)x(t) \quad \text{where } P \in C^1[t_i, t_f]$$

$$Q \geq 0, \quad F \geq 0, \quad R > 0$$

Theorem: $\exists$ a unique solution

$$p^\circ(t) = -R^{-1} B^T(t) K(t)$$

with $K \geq 0$ solution of the Riccati equation:

$$\dot{K}(t) = -A^T(t) K(t) - K(t) A(t) +$$
$$+ K(t) B(t) R^{-1}(t) B^T(t) K(t) - Q(t)$$
$$K(t_f) = F$$

The optimal state found should be:

$$\dot{x}^\circ(t) = A(t) x^\circ(t) - B(t) R^{-1}(t) B^T(t) K(t) x^\circ(t) + w(t)$$
$$x^\circ(t_i) = x(t_i)$$

and the cost index has minimum value

$$J(p^\circ) = \frac{1}{2} x_i^T K(t_i) x_i +$$
$$+ Tr \left\{ \int_{t_i}^{t_f} K(t) \Psi_w(t) \, dt + \frac{1}{2} K(t_i) \Psi_{x_i} \right.$$

Trace (sum of the elements on the diagonal of the matrix)

The theorem gives only the best linear solution of the stochastic regulator problem

It can be proved that the linear feedback law is optimal when the white noise is gaussian

~ Optimal regulator with state available and noise with non null mean value

Consider the linear system

$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \bar{w}(t) \qquad t \in [t_i, t_f]$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $(A\,B)$ controllable

$x(t_i) = x_i$ gaussian vector

$\bar{w}$ is a noise $\rightsquigarrow$ $\bar{w} = \underset{\text{white gaussian noise}}{w} + \mu(t)$

Covariance:

$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$ uncertainty covariance (how much this this information is precise)

$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\delta(\tau)$

$E\{w(t)\} = \mu(t) \in C^0[t_i, t_f]$

↳ function (a systematic error always present)

x_i and w incorrelated.

In this case we cannot proceed with a control like $u = P(t)x(t)$ because it won't be probably a good choice because it was the best choice in the case of noise with 0 mean value, now we have to add something:

$u(t) = P(t)x(t) + q(t) \qquad P, q \in C^1[t_i, t_f]$

minimizing:

$J(P) = \frac{1}{2} E\left\{\int_{t_i}^{t_f} [x^T Q x + u^T R u]\,dt + x^T(t_f) F x(t_f)\right\}$

$Q \geq 0$, $F \geq 0$, $R > 0$ with elements of C^1 class

Theorem: $\exists$ a unique solution:

$$p^o(t) = -R^{-1}(t)\,B^T(t)\,K(t)$$
$$q^o(t) = -R^{-1}(t)\,B^T(t)\,g(t)$$

$K \geq 0$ solution of the Riccati equation

$$\dot{K}(t) = -A^T K - KA + KBR^{-1}B^T K - Q$$
$$K(t_f) = F$$

$$\dot{g}(t) = -\left[A^T - KBR^{-1}B\right]g - K(t)\,\mu(t)$$
$$g(t_f) = 0$$

The cost index has minimum value:

$$J(p^o, q^o) = \frac{1}{2}x_i^T K(t_i)\,x_i + Tr\left\{\int_{t_i}^{t_f} K(t) + \Psi_w(t)\,dt + \frac{1}{2}K(t_i)\,\Psi_{x_i}\right\} + x_i^T g(t_i) + h(t_i)$$

with h unique solution of the differential equation

$$\left.\begin{aligned} \dot{h}(t) &= \frac{1}{2}g^T(t)\,B(t)\,R^{-1}(t)\,B^T(t)\,g(t) - g^T\mu(t) \\ h(t_f) &= 0 \end{aligned}\right\} \text{similar to tracking problem}$$

~ Optimal regulator with available state on infinite time interval

$$\dot{x}(t) = A(t)\,\overset{\in \mathbb{R}^n}{x(t)} + B(t)\,\overset{\in \mathbb{R}^p}{u(t)} + w(t) \quad , \quad t \in [t_i, \infty]$$

$(A\ B)$ controllable

w = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value gaussian vector

(x_i, w not correlated)

Covariance:

$$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \Psi_{x_i}$$

uncertainty covariance (how much this information is precise)

$$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w \rightarrow \text{constant}$$

I want to minimize:

$$J(P) = \lim_{t_f \to \infty} \frac{1}{2(t_f - t_i)} E\left\{ \int_{t_i}^{t_f} [x^T Q x + u^T R u]\, dt \right\}$$

$Q > 0$, $R \geq 0$

the control has the form $u(t) = \overset{\text{constant matrix}}{P}\,x(t)$

Theorem: $\exists$ a unique solution:

$$P^o = -R^{-1} B^T K_r \rightsquigarrow \text{solution of the RE}$$

$$A^T K_r + K_r A - K_r B R^{-1} B^T K_r + Q = 0$$

Therefore:

$$u^o(t) = -R^{-1} B^T K_r x^o(t)$$

$$\dot{x}^o(t) = [A - BR^{-1}B^T K_r]\, x^o(t) + w(t)$$

$$x^o(t_i) = x(t_i)$$

And the minimum value is: $J(P^o) = Tr\{K_r \Psi_w\}$

~ Optimal gaussian linear tracking (state available)

$$\dot{x}(t) = A(t)\overbrace{x(t)}^{\in \mathbb{R}^n} + B(t)\overbrace{u(t)}^{\in \mathbb{R}^p} + w(t) \qquad \overset{A,B \in C^1}{,} \quad t \in [t_i, t_f]$$

w = white gaussian noise with 0 mean value

$E\{w(t)\} = 0$ and diagonal covariance matrix

$E\{x(t_i)\} = x_i$ initial mean value

(x_i, w not correlated)

Covariance:

$$E\{[x(t_i) - x_i][x(t_i) - x_i]^T\} = \overbrace{\Psi_{x_i}}^{\text{uncertainty}}$$ covariance (how much this this information is precise)

$$E\{w(t)w^T(t)(t+\tau)\} = \Psi_w(t)\,\delta(\tau)$$

Now we have a <u>reference</u> whose dynamics is:

$$\dot{r}(t) = A_r(t)r(t) + \theta(t) \qquad t \in [t_i, t_f]$$

r, θ, w, $x(t_i)$ incorrelated

Moreover:

$$E\{r(t_i)\} = r_i \qquad E\{[r(t_i) - r_i][r(t_i) - r_i]^T\} = \Psi_{ri}$$

$$E\{\theta(t)\} = m(t)$$

$$E\{[\theta(t) - m(t)][\theta(\tau) - m(\tau)]^T\} = \Psi_\theta(t)\,\delta(t-\tau)$$

We are looking for a control like

$$u(t) = P(t)x(t) + P_r(t)r(t) + q(t) \qquad P, P_r, q \in C^1[t_i, t_f]$$

that minimizes

$$J(P, P_r, q) = \frac{1}{2} E\left\{\int_{t_i}^{t_f} [(r(t) - x(t))^T Q(t)(r(t) - x(t_i)) + u^T(t)R(t)u(t)]\,dt\right.$$

$Q > 0$, $R > 0$ elements of C^1 class

Theorem: ∃ solution unique:

$$P^{\circ}(t) = -R^{-1}(t)\, B^{T}(t)\, K_{11}(t)$$
$$P_r^{\circ}(t) = -R^{-1}(t)\, B^{T}(t)\, K_{12}(t)$$
$$q^{\circ}(t) = -R^{-1}(t)\, B^{T}(t)\, g_1(t)$$

where

$$\left.\begin{aligned} \dot{K}_{11}(t) &= -A^T K_{11} - K_{11} A + K_{11} B R^{-1} B^T K_{11} - Q \\ K_{11}(t_f) &= 0 \end{aligned}\right\}$$ RE concerning A, B, Q and $F = 0$

$$\left.\begin{aligned} \dot{K}_{12}(t) &= -A^T K_{12} - K_{12} A + K_{11} B R^{-1} B^T K_{12} + Q \\ K_{12}(t_f) &= 0 \end{aligned}\right\}$$ Not an RE

$$\dot{g}_1(t) = -\left[A^T - K_{11} B R^{-1} B^T\right] g_1 - K_{12}\, m$$
$$g_1(t_f) = 0$$

The problem admits a unique optimal solution:

$$u^{\circ} = P_x^{\circ} x^{\circ} + P_r^{\circ} r + q^{\circ} = -R^{-1} B^T \left(K_{11} x^{\circ} + K_{12} r + g_1\right)$$