34. Jordan canonical form

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$\dot{x} = Ax \quad A \in \mathbb{R}^{n \times n}$

$\exists \; T : \; TAT^{-1} = J$

$A = T^{-1} J T = T^{-1} e^{Jt} T$

$$e^{At} = e^{(T^{-1}JT)t} = I + T^{-1}JTt + \frac{t^2}{2!} T^{-1}J^2T + \dots$$
$$= T^{-1}\left(I + tJ + \frac{t^2}{2!}J^2 + \dots\right)T$$

A regular (it can be diagonalized)

$|\lambda I - A| = 0 \Rightarrow d(\lambda) = a_0 + a_1\lambda + \dots + a_{n-1}\lambda^{n-1} + \lambda^n = 0$

$\rightarrow \lambda_1, \lambda_2, \dots, \lambda_i$ with corresponding algebraic multiplicity m_{a_i}

$m_{a_1}, m_{a_2}, \dots, m_{a_i}, \quad i = 1 \dots v$

$$\Downarrow$$

$$\sum_{i=1}^{v} m_{a_i} = n$$

To be diagonalizable A must satisfy $\forall i = 1 \dots v$:
$\text{rank}\{(A - \lambda_i I)\} = n - m_g(\lambda_i)$

or in other words: $\dim\{\ker(A - \lambda_i I)\} = m_g(\lambda_i)$ that is the geometric multiplicity.

If this condition holds I can find n independent eigenvectors v_i s.t., applying T, A becomes diagonal

$$T^{-1} = [v_1, \dots, v_n] \rightarrow TAT^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

complex case

$$T^{-1} = \left(v \;\middle|\; v_a \; v_b \right)$$

$$TAT^{-1} = \left(\begin{array}{c|cc} \lambda & \multicolumn{2}{c}{0} \\ \hline 0 & \alpha & \omega \\ & -\omega & \alpha \end{array}\right) = \Lambda_R \rightsquigarrow \text{real diagonal form of the operator}$$

$A = T^{-1}\Lambda_R T$

$$e^{At} = T^{-1}e^{\Lambda_R t}T \rightarrow e^{\Lambda_R t} = \left(\begin{array}{c|c} e^{\lambda t} & 0 \\ \hline 0 & e^{\begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} t} \end{array}\right)$$

$$e^{At} = T^{-1} e^{\Lambda_R t} T \longrightarrow e^{\cdots} = \left(\begin{array}{c|c} 0 & e^{\begin{pmatrix}\alpha & \omega \\ -\omega & \alpha\end{pmatrix}t} \end{array} \right)$$

$$e^{\begin{pmatrix}\alpha & \omega \\ -\omega & \alpha\end{pmatrix}t} = e^{\alpha t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

A non-regular

Let $\{\lambda_1, \dots, \lambda_v\}$ the set of distinct eigenvalues of A

m_i = algebraic multiplicity of λ_i

μ_i = geometric "

suppose that at least for one λ_i is $m_i \neq \mu_i$

$\exists$ a squared non-singular matrix $T \in \mathbb{C}^{n \times n}$ s.t.

(*) $TAT^{-1} = J = \mathrm{diag}\{J_1, \dots, J_v\}$

$\mathrm{dim}\{J_i\} = m_i \quad i = 1, \dots, v \qquad J_i$ = Jordan blocks of A

$\quad \hookrightarrow$ square

(**) $J_i = \mathrm{diag}\{J(\lambda_i, d_{i1}), \dots, J(\lambda_i, d_{i\mu_i})\}$

λ_i = eigenvalue $\qquad i = 1, \dots, v$

d_{ij} = dimension $\qquad j = 1, \dots, \mu_i$

$$J(\lambda_i, d_{ij}) = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

for a given matrix A the Jordan form is unique.

Property

μ_i = number of Jordan blocks of A associated to λ_i

Generalized eigenvectors of A

$TAT^{-1} = J \Rightarrow AT^{-1} = T^{-1}J = T^{-1}\,\mathrm{diag}\{J_1, \dots, J_v\}$

partitioning T^{-1}:

$$A[W_1 | \dots | W_v] = [W_1 | \dots | W_v] \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_v \end{bmatrix}$$

$A W_i = W_i J_i \qquad i = 1, \dots, v$

Partitioning W_i :

$$A[W_{i1}, \dots, W_{i\mu_i}] = [W_{i1}, \dots, W_{i\mu_i}] \begin{bmatrix} J(\lambda_i, d_{i1}) & & 0 \\ & \ddots & \\ 0 & & J(\lambda_i, d_{i\mu_i}) \end{bmatrix}$$

$$A W_{ij} = W_{ij} J(\lambda_i, d_{ij}) \quad \begin{matrix} i = 1 \dots v \\ j = 1, \dots, \mu_i \end{matrix}$$

Let w_{ijh}, $h = 1, \dots, d_{ij}$ be the colums of W_{ij} :

$$A[w_{ij1}, \dots, w_{ijd_{ij}}] = [w_{ij1}, \dots, w_{ijd_{ij}}] \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

after some computations we obtain the relation

$$\begin{cases} (A - \lambda_i I)^h w_{ijh} = 0 & h = 1, \dots, d_{ij} \\ (A - \lambda_i I)^{h-1} w_{ijh} = w_{ij1} \neq 0 & h = 2, \dots, d_{ij} \end{cases}$$