18. Frobenius theorem

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A distribution Δ (dim k) is completely integrable iff it is involutive ($\bar{\Delta} \equiv \Delta$)

Involutiveness means closure w.r.t. the brackets

procedure:

$$\Delta(x) = \begin{pmatrix} | & | \\ \tau_1(x) & \tau_2(x) \\ | & | \end{pmatrix}$$

$$\underbrace{\det \begin{pmatrix} | & | & | \\ \tau_1(x) & \tau_2(x) & [\tau_1, \tau_2] \\ | & | & | \end{pmatrix}}_{\bar{\Delta} = \text{ involutive closure}}$$

$\to = 0 \to \tau_1, \tau_2, [\tau_1, \tau_2]$ are dependent vectors therefore $\exists\, \alpha, \beta$ s.t. $[\tau_1, \tau_2] \in \Delta$

$\exists\, \lambda \ni \frac{\partial \lambda}{\partial x}(\tau_1, \tau_2) = 0$

$\to \neq 0 \to \tau_1, \tau_2, [\tau_1, \tau_2]$ are independent vectors so $\nexists\, \alpha, \beta$ s.t. $[\tau_1, \tau_2] \in \Delta$ and $\frac{\partial \lambda}{\partial x} \bar{\Delta} \equiv 0$

no solution

Proof:

① Necessity

$n = 3 \quad k = 2$

hyp: $\exists\, \lambda$ function, $\tau_1(x), \tau_2(x)$ vector fields s.t.

$L_{\tau_1}\lambda = L_{\tau_2}\lambda = 0$

I can compute: $L_{[\tau_1, \tau_2]}\lambda = L_{\tau_1}\underbrace{L_{\tau_2}\lambda}_{0} - L_{\tau_2}\underbrace{L_{\tau_1}\lambda}_{0} = 0$

and since, by assumption, $\frac{\partial \lambda}{\partial x} \perp \Delta$,

we deduce from $\langle \partial\lambda, [\tau_1, \tau_2] \rangle = 0$ that $[\tau_1, \tau_2] \in \Delta$

② Sufficiency

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Suppose $\Delta = \text{span}\{\tau_1(x) \dots \tau_k(x)\}$, involutive and $\tau_{k+1}(x) \dots \tau_n(x)$ be a complementary set of vector fields s.t.

$T_x\mathbb{R}^n = \text{span}\{\tau_1(x), \dots, \tau_k(x), \tau_{k+1}(x), \dots, \tau_n(x)\}$

Let $\Phi_t^\tau(x)$ the flow of the vector field τ with the property that $x(t) = \Phi_t^\tau(x^0)$ solves the ordinary differential equation:

$\dot{x} = \tau(x)$, $x(0) = x^0$

Therefore:

$$\frac{\partial}{\partial t}\Phi_t^\tau(x) = \tau(\Phi_t^\tau(x)), \quad \Phi_0^\tau(x) = x$$

From involutivity is proved that the mapping

$\psi: (z_1, \dots, z_n) \longmapsto \Phi_t^{\tau_1} \cdot \Phi_t^{\tau_2} \dots \Phi_t^{\tau_n}(x^0)$

has the following properties:

1. $\psi(z_1, \dots, z_n)$ is a local diffeomorphism
2. $\frac{\partial \psi}{\partial z}$, the Jacobian matrix, has the first k columns linearly independent vectors in $\Delta(\psi(z))$ (k columns are a basis for Δ)

By property ①

$$\psi^{-1} = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \qquad \text{cause } \psi \cdot \psi^{-1} = Id$$

observing by definition that

$$\partial \psi^{-1}(\psi(z)) = \left[\frac{\partial \psi^{-1}}{\partial x}\right]_{x = \psi(z)} \cdot \frac{\partial \psi}{\partial z} = Id$$

It follows that the first k columns of $\frac{\partial \psi}{\partial z}$ annihilate the differentials and

$\frac{\partial \phi_i}{\partial x}$ $i = k+1 \dots n$ are the $n-k$ functions representing $\mathcal{R}_i$