CALCULUS OF VARIATIONS

The calculus of variations is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals.
Functionals are often expressed as definite integrals:

$$J(z) = \int_{t_i}^{t_f} \underbrace{L(t, z(t), \dot{z}(t))}_{\text{Lagrangian}} \, dt$$

• The Lagrange Problem

Let D be an admissible set in $\bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$

$$D = \Big\{ (z, t_i, t_f) \in \bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} : \\ (z(t_i), t_i) \in D_i \subset \mathbb{R}^{\nu+1}, \ q(z, \dot{z}, t) \leq 0 \\ (z(t_f), t_f) \in D_f \subset \mathbb{R}^{\nu+1}, \ g(z(t), \dot{z}(t), t) = 0, \\ \int_{t_i}^{t_f} h(z, \dot{z}, t) \, dt = 0 \Big\}$$

$g \in \mathbb{R}^{\mu < \nu}$ of C^2 class
$h \in \mathbb{R}^{\sigma}$ of C^2 class
q of C^2 class, q_a of dimension βa

$$J(z, t_i, t_f) = \int_{t_i}^{t_f} \underbrace{L(z(t), \dot{z}(t), t)}_{\substack{\text{Lagrangian of } C^2 \text{ class} \\ L: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}}} \, dt$$

The aim is to minimize the evolution (integral) of the functional from an instant t_i to t_f

~ Lagrange theorem

Define the augmented lagrangian

$$l(z(t),\dot{z}(t),t,\lambda_0,\lambda(t),\rho) = \lambda_0 L(z(t),\dot{z}(t),t) + \eta^T(t) q(z,\dot{z},t) + \lambda^T(t) g(z(t),\dot{z}(t),t) + \rho^T h(z(t),\dot{z}(t),t)$$

If $\lambda_0 \neq 0 \to$ normal solution

Let $(z^*, t_i^*, t_f^*) \in D$ be such that

$$rk\left\{ \frac{\partial(g, q_a)}{\partial \dot{z}} \Big|^* \right\} = \nu + \beta_a \quad \forall [t_i^*, t_f^*]$$

If (z^*, t_i^*, t_f^*) is a local minimum for J over D, then there exist $\eta^* \in \bar{C}^0[t_i, t_f]$, $\lambda_0^* \in \mathbb{R}$, $\lambda^* \in \bar{C}^0[t_i^*, t_f^*]$, $\rho^* \in \mathbb{R}^6$ not simultaneously null in $[t_i^*, t_f^*]$

such that the following conditions hold:

- $\boxed{\text{Euler - Lagrange}}$ $\quad \dfrac{\partial l^*}{\partial z} - \dfrac{d}{dt}\dfrac{\partial l}{\partial \dot{z}}\Big|^* = 0^T \quad \forall t \in [t_i^*, t_f^*]$

- $\boxed{\text{Weierstress - Erdmann}}$ (for discontinuity points) $\quad \dfrac{\partial l}{\partial \dot{z}}\Big|^*_{\bar{t}^-} = \dfrac{\partial l}{\partial \dot{z}}\Big|^*_{\bar{t}^+}$ $\quad \bar{t}$ are corner points of z^*

$$\left(l - \frac{\partial l}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}^-} = \left(l - \frac{\partial l}{\partial \dot{z}}\dot{z}\right)^*_{\bar{t}^+}$$

- $\boxed{\text{Trasversality}}$ $\quad$ different cases: they depend on the nature of the boundary conditions.

~ Euler - Lagrange equation

Trajectories satisfying the E-L equation are called extremals

$$\boxed{\left.\frac{\partial \ell}{\partial z}\right|^{*} - \frac{d}{dt}\left.\frac{\partial \ell}{\partial \dot z}\right|^{*} = 0} \qquad \forall t \in [t_i, t_f]$$

Proof

Let's consider a curve C^1 $z : [a,b] \to \mathbb{R}$ with $z(a) = z_0$ and $z(b) = z_1$, and the functional

$$J(z) = \int_a^b \ell(z, \dot z, t)\, dt \qquad \ell \in C^2$$

The goal is to find the local minimum of J.

Introduce now the perturbation $\eta : [a,b] \to \mathbb{R}$
$\eta(a) = 0$, $\eta(b) = 0$.
In fact, if $z(a) = z_0$, $z(b) = z_1$, then
$z(a) + \alpha\,\eta(a) = z_0 \Rightarrow \eta(a) = 0$
$z(b) + \alpha\,\eta(b) = z_1 \Rightarrow \eta(b) = 0$

$$J(z + \alpha\eta) = \int_a^b \ell(z + \alpha\eta, \dot z + \alpha\dot\eta, t)\, dt =$$

$$(1) = \underbrace{J(z)}_{J(z) = \int_a^b \ell(z, \dot z, t)\, dt} + \alpha \underbrace{\delta J|_{z^*}(\eta)}_{=0} + o(\alpha)$$

(first order necessary condition)

Taylor expansion with respect to α

$$J(z+\alpha\eta) = \int_a^b \ell(z+\alpha\eta, \dot{z}+\alpha\dot{\eta}, t)\,dt$$

2

$$J(z+\alpha\eta) = \int_a^b \left(\ell(z,\dot{z},t) + \frac{\partial \ell}{\partial z}(z,\dot{z},t)\alpha\eta + \frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\alpha\dot{\eta}\right)dt$$

From 1 and 2 (dividing both members by α):

$$\overset{2}{J}(z+\alpha\eta) = \cancel{\int \ell(z,\dot{z},t)dt} + \int \frac{\partial \ell}{\partial z}\cdot\cancel{\alpha}\,\eta\,dt + \int \frac{\partial \ell}{\partial \dot{z}}\cancel{\alpha}\,\dot{\eta}\,dt$$

1

$$J(z+\alpha\eta) = \cancel{\int \ell(z,\dot{z},t)dt} + \cancel{\alpha}\,\delta J|_{z^*}(\eta)$$

$$\Downarrow$$

$$\delta J|_{z^*}(\eta) = \int_a^b \frac{\partial \ell}{\partial z}(z,\dot{z},t)\,\eta\,dt + \int_a^b \frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\,\dot{\eta}\,dt$$

Integration by parts $\rightarrow \int_a^b \frac{\partial \ell}{\partial \dot{z}}\dot{\eta}\,dt = -\int_a^b \frac{d}{dt}\frac{\partial \ell}{\partial \dot{z}}\eta\,dt + \frac{\partial \ell}{\partial \dot{z}}\eta\Big|_a^b$

$$\downarrow$$

$$3 \quad \frac{\partial \ell}{\partial \dot{z}}\eta(b) - \frac{\partial \ell}{\partial \dot{z}}\eta(a)$$

For the boundary conditions, $\eta(a)=0$, $\eta(b)=0$ and $\delta J|_{z^*}\eta = 0$, therefore 3 $=0$ and

$$\boxed{\int_a^b \frac{\partial \ell}{\partial z}(z,\dot{z},t)\,\eta(t)\,dt - \int_a^b \frac{d}{dt}\frac{\partial \ell}{\partial \dot{z}}(z,\dot{z},t)\,\eta(t)\,dt = 0}$$

for all curves vanishing at the ends points

That is the same of:

$$\int_a^b \left[\frac{\partial \ell}{\partial z} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot z} \right] \eta(t)\, dt = 0$$

$$\eta(a) = 0 \quad , \quad \eta(b) = 0$$

Lemma : if a continuous function $g : [a,b] \to \mathbb{R}$ is such that $\int_a^b g(t)\,\eta(t)\,dt = 0 \quad \forall \eta : [a,b] \to \mathbb{R}$ with $\eta(a) = 0$ and $\eta(b) = 0$ then $g = 0$

Therefore

$$\boxed{\frac{\partial \ell}{\partial z} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot z} = 0}$$

Necessary condition for $z(\cdot)$ to be an extremum

End

~ Variable endpoint problem

$$\boxed{\left.\frac{\partial \mathcal{L}}{\partial \dot z}\right|_b^* = 0} \qquad \boxed{\left.\frac{\partial \mathcal{L}}{\partial \dot z}\right|_b^* + \left.\mathcal{L}\right|_b^* = 0}$$

Proof

Let us consider the C^1 curves $z : [a,b] \to \mathbb{R}$ s.t.
$z(a) = z_0 \quad , \quad z(b)$ free.

Find the local minima of $J(z) = \int_a^b \mathcal{L}(z, \dot z, t)\,dt$
$\hookrightarrow C^2$ class

The perturbations η must satisfy $\eta(a) = 0$ but $\eta(b)$ arbitrary

In this case the first variation is given by (*)

$$\delta J|_z(\eta) = \boxed{\int_a^b \frac{\partial \mathcal{L}}{\partial z}(t,z,\dot z)\,\eta(t)\,dt - \int_a^b \frac{d}{dt}\frac{\partial}{\partial \dot z}(z,\dot z,t)\,\eta(t)\,dt} +$$

$$+ \frac{\partial \mathcal{L}}{\partial \dot z}(b, z(b), \dot z(b))\,\eta(b) = 0$$

if z is an extremum

Perturbations $\eta(b)=0$ are still allowed, in that case I obtain the previous E-L condition, which is still a necessary condition for optimality.

After this consideration we know that (*) $=0$ for all admissible perturbations η because

$$\frac{\partial \mathcal{L}}{\partial z}(z,\dot z,t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot z}(z,\dot z,t) = 0 \quad \text{(E-L condition)}$$

Therefore the extra condition is found:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot z}(b, z(b), \dot z(b)) = 0}$$

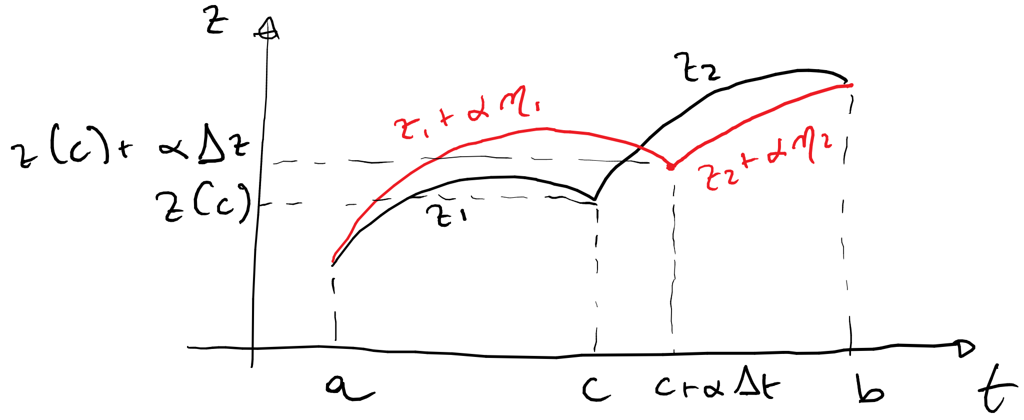
~ Weierstrass - Erdmann corner condition

Additional conditions at corner points, in order for z to be a strong extremum

Proof

Consider $z \in \bar{C}^1$ solution (continuous first derivative almost everywhere), we can have some points in which the derivative is not continuous.

Assume $c \in [a,b]$ es a corner point of z



Split z in $z_1 : [a,c] \to \mathbb{R}$ and $z_2 : [c,b] \to \mathbb{R}$

The perturbed versions are $z_1 + \alpha \eta_1$ and $z_2 + \alpha \eta_2$ with $\eta_1(a) = \eta_2(b) = 0$

The location of the corner point is not fixed, so the corner point could deviate from the point c

Therefore the domain of $z_1 + \alpha \eta_1$ should be extended to $[a, c + \Delta t]$.

This can be done by linear continuation:

$z_1(t) = z(c) + \dot{z}(c^-)(t-c)$

$z_1 \in C^1$ in c with $z_1(c) = z(c)$ and $\dot{z}_1(c) = \dot{z}(c^-)$

The same assumptions hold for z_2

New cost functional

$$J(z) = \int_a^b L(z,\dot z,t)\,dt = \int_a^c L(z_1,\dot z_1,t)\,dt + \int_c^b L(z_2,\dot z_2,t)\,dt$$

$$= J_1(z_1) + J_2(z_2)$$

- $J_1(z_1)$:

Perturbed $J_1(z_1+\alpha\eta_1) = \int_a^{c+\alpha\Delta t} L(z_1+\alpha\eta_1, \dot z_1+\alpha\dot\eta_1, t)\,dt$

$$\delta J_1\big|_{z_1}(\eta_1) = \frac{d}{d\alpha}\Big|_{\alpha=0} J_1(z_1+\alpha\eta_1) =$$

$$= L(z_1(c),\dot z_1(c),c)\,\Delta t + \int_a^c \left[\frac{\partial L}{\partial z}(z_1,\dot z_1,t)\,\eta_1 + \frac{\partial L}{\partial \dot z}(z_1,\dot z_1,t)\,\dot\eta_1\right]dt$$

$\Big|$ $z_1(c)=z(c)$, $\dot z_1(c) = \dot z(c^-)$, $\eta_1(a)=0$

integration by parts

$$= L(z(c),\dot z(c^-),c)\,\Delta t + \frac{\partial L}{\partial \dot z}(z(c),\dot z(c^-),c)\,\eta_1(c) +$$

$$+ \cancel{\int_a^c \left[\frac{\partial L}{\partial z}(z_1,\dot z_1,t) - \frac{d}{dt}\frac{\partial L}{\partial \dot z}(z_1,\dot z_1,t)\right]\eta_1(t)\,dt}$$

- $J_2(z_2)$:

Perturbed: $J_2(z_2+\alpha\eta_2) = \int_{c+\alpha\Delta t}^{b} L(z_2+\alpha\eta_2, \dot z_2+\alpha\dot\eta_2, t)\,dt$

$$\delta J_2\big|_{z_2} = -L(z(c),\dot z(c^+),c)\,\Delta t - \frac{\partial L}{\partial \dot z}(z(c),\dot z(c^+),c)\,\eta_2(c) +$$

$$+ \cancel{\int_c^b \left[\frac{\partial L}{\partial z}(z_2,\dot z_2,t) - \frac{d}{dt}\frac{\partial L}{\partial \dot z}(z_2,\dot z_2,t)\right]\eta_2(t)\,dt}$$

For $\alpha \to 0$ the perturbed curve is close to the original curve z.

The cost index J as function of α must have a minimum at $\alpha=0$

$$0 = \frac{d}{d\alpha}\Big|_{\alpha=0} J(z,\alpha) = \frac{d}{d\alpha}\Big|_{\alpha=0} \left[J_1(z_1+\alpha\eta_1) + J_2(z_2+\alpha\eta_2) \right]$$
$$= \delta J_1\big|_{z_1}(\eta_1) + \delta J_2\big|_{z_2}(\eta_2)$$

The two portions z_1, z_2 of z must be extremals of the correspondings J_1, J_2

Therefore, since the E-L condition holds, it holds also for the subintervals $[a,c]$ and $[c,b]$.

$$L(z(c),\dot z(c^-),c)\Delta t + \frac{\partial L}{\partial \dot z}(z(c),\dot z(c^-),c)\,\eta_1(c) +$$
$$- L(z(c),\dot z(c^+),c)\Delta t - \frac{\partial L}{\partial \dot z}(z(c),\dot z(c^+),c)\,\eta_2(c) = 0 \qquad (*)$$

with this new condition I'm evaluating the function in c^- and c^+ together

The perturbed curve it is known it's continuous at $t = c+\alpha\Delta t$ ($\bar C^1$ class), therefore η_1 and η_2 are not independent

$$z_1(c+\alpha\Delta t) + \alpha\eta_1(c+\alpha\Delta t) = z_2(c+\alpha\Delta t) + \alpha\eta_2(c+\alpha\Delta t)$$
$$=: z(c) + \alpha\,\Delta z + o(\alpha) \quad \text{(first approximation)}$$

first order (in α) vertical displacement

Remark: $\dot z_1(c) = \dot z(c^-)$, $\dot z_2(c) = \dot z(c^+)$

Obtaining:

$$\Delta z \simeq \frac{1}{\alpha}\left[z_1(c+\alpha\Delta t) - z(c) + \alpha\eta_1(c+\alpha\Delta t)\right] \simeq \dot z(c^-)\Delta t + \eta_1(c)$$
$$\Delta z \simeq \frac{1}{\alpha}\left[z_2(c+\alpha\Delta t) - z(c) + \alpha\eta_2(c+\alpha\Delta t)\right] \simeq \dot z(c^+)\Delta t + \eta_2(c)$$

So $\dot{z}(c^-)\Delta t + \eta_1(c) = \dot{z}(c^+)\Delta t + \eta_2(c) = \Delta z$ (second approximation)
We obtain $\eta_1(c) = \Delta z - \dot{z}(c^-)\Delta t$, $\eta_2(c) = \Delta z - \dot{z}(c^+)\Delta t$
to use in $\circledast$

The result is :

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c), \dot{z}(c^-), c) - \frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c), \dot{z}(c^+), c)\right] \Delta z \; +$$

$$- \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c), \dot{z}(c^-), c) - \mathcal{L}(z(c), \dot{z}(c^-), c)\right) + \right.$$

$$\left. - \left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z(c), \dot{z}(c^+), c) + \mathcal{L}(z(c), \dot{z}(c^+), c)\right)\right] \Delta t$$

$$\boxed{= - \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}(z, \dot{z}, t)\right|_{c^-}^{c^+} \Delta z + \left.\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}(z, \dot{z}, t)\,\dot{z}(t) - \mathcal{L}(z, \dot{z}, t)\right|_{c^-}^{c^+} \Delta t = 0}$$

Δz, Δt are arbitrary and independent therefore
$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{c^-}^{c^+} = 0$ and $\left[\frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z} - \mathcal{L}\right]_{c^-}^{c^+} = 0$

so these quantities are continuous.

End

$\frac{\partial \mathcal{L}}{\partial \dot{z}}$ and $\frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z} - \mathcal{L}$ are continuous in $t = c$

$\rightarrow$ Weierstress - Erdmann corner condition

~ 4 Case studies of Lagrange Problem

$J = \int l(z, \dot{z}, t)\, dt \quad z \in C^1 \text{ or } z \in \bar{C}^1$

Case 1-2 : $z(a) = z_0$, $z(b) = z_1$

- Euler - Lagrange equation
- Weierstrass - Erdmann corner condition (if $\exists \bar{t}$ discontinuity point of $\dot{z}^*$)

Case 3-4 : $z(a) = z_0$, $z(b)$ free

- Euler - Lagrange equation
- Weierstrass - Erdmann condition (if $\exists \bar{t}$ discontinuity point of z^*)
- Extra condition

~ Lagrange Problem

$z : \mathbb{R}^{\nu} \to \mathbb{R} \in \bar{C}^1$ with

$$D : \left\{ (z, t_i, t_f) \in \bar{C}^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} : \begin{array}{l} (z(t_i), t_i) \in \mathcal{S}_i \subseteq \mathbb{R}^{\nu+1} \\ (z(t_f), t_f) \in \mathcal{S}_f \subseteq \mathbb{R}^{\nu+1} \end{array} \right\}$$

$J(z, t_i, t_f) = \int_{t_i}^{t_f} \mathcal{L}(z, \dot{z}, t)\, dt \leftarrow$ cost function with $\mathcal{L} \in C^2$

Find $(z^\circ, t_i^\circ, t_f^\circ)$ that minimizes the cost function over D

$$\boxed{J(z^\circ, t_i^\circ, t_f^\circ) \leq J(z, t_i, t_f) \quad \forall (z, t_i, t_f) \in D}$$

If $(z^\circ, t_i^\circ, t_f^\circ)$ is a local minimum then:
Euler - Lagrange equation, W-E condition and transversality conditions are satisfied:

- If $(z^*, t_i^*, t_f^*) \in D$ is a local minimum then

$$\left.\frac{\partial \mathcal{L}}{\partial z}\right|^* - \frac{d}{dt} \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^* = 0^T \quad \forall t \in [t_i, t_f] \qquad \underline{\text{Euler equation}}$$

- In any discontinuity point t of $\dot{z}^*$: $\qquad$ W-E condition

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^*_{t^-} = \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|^*_{t^+}, \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)^*_{t^-} = \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)^*_{t^+}$$

- Trasversality conditions

1) D_i, D_f open subsets

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = 0^T , \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = 0^T , \quad \left.\mathcal{L}\right|_{t_i^*}^{*} = 0 , \quad \left.\mathcal{L}\right|_{t_f^*}^{*} = 0$$

2) D_i, D_f closed subsets

$(z(t_i), t_i)$ initial point satisfy $\gamma(z(t_i), t_i) = 0$
$(z(t_f), t_f)$ final point satisfy $\chi(z(t_f), t_f) = 0$

these conditions must be regular

$$\boxed{\sigma_i , \sigma_f < n+1}$$

$$rk\left\{ \left.\frac{\partial \gamma}{\partial (z(t_i), t_i)}\right|^{*} \right\} = \sigma_i$$

$$rk\left\{ \left.\frac{\partial \chi}{\partial (z(t_f), t_f)}\right|^{*} \right\} = \sigma_f$$

Given two vectors $\overset{\text{xi}}{\xi} \in \mathbb{R}^{\sigma_i}$ and $\overset{\text{zeta}}{\zeta} \in \mathbb{R}^{\sigma_f}$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = \xi^T \left.\frac{\partial \gamma}{\partial z(t_i)}\right|^{*} , \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = \zeta^T \left.\frac{\partial \chi}{\partial z(t_f)}\right|^{*}$$

$$\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)_{t_i^*} = \xi^T \left.\frac{\partial \gamma}{\partial t_i}\right|^{*} , \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}}\dot{z}\right)_{t_f^*} = \zeta^T \left.\frac{\partial \chi}{\partial t_f}\right|^{*}$$

3) D_i, D_f defined by $\omega(z(t_i), t_i, z(t_f), t_f) = 0$ of 6 components of C^1 class

regularity: $rk \left\{ \frac{\partial \omega}{\partial(z(t_i), t_i, z(t_f), t_f} \Big|^* \right\} = 6$

$$\frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|^*_{t_i^*} = \theta^T \frac{\partial \omega}{\partial z(t_i)}\Big|^* , \quad \frac{\partial \mathcal{L}}{\partial \dot{z}}\Big|^*_{t_f^*} = -\theta^T \frac{\partial \chi}{\partial z(t_f)}\Big|^* , \quad \theta \in \mathbb{R}^6$$

$$\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)_{t_i^*} = \theta^T \frac{\partial \omega}{\partial t_i}\Big|^* , \quad \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{z}\right)_{t_f^*} = -\theta^T \frac{\partial \omega}{\partial t_f}\Big|^*$$

~ Extremum and non-singularity

An extremum is a candidate to be the minimum and is any admissible point satisfying the Euler equation, the W-E conditions and the trasversality ones.

An extremum is NON singular if $\frac{\partial^2 \mathcal{L}}{\partial \dot{z}^2}\Big|^*$ is non singular ($\neq 0$) in $[t_i^*, t_f^*]$

A non singular extremum is a C^2 function

~ LP with fixed time instants

A standard LP Problem +

- ▷ t_i & t_f fixed
- ▷ $\mathcal{L}$ convex wrt $z, \dot{z}$ → Global minimum

- Transversality conditions

1) D_i, D_f open subsets of $\mathbb{R}^{\nu}$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i}^{\circ} = 0 \qquad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f}^{\circ} = 0$$

2) D_i, D_f are closed subsets

$$\left.\begin{aligned} \gamma(z(t_i)) &= 0 \\ \chi(z(t_f)) &= 0 \end{aligned}\right\}$$

these conditions must be regular

$$rk\left\{\left.\frac{\partial \gamma}{\partial(z(t_i))}\right|^{*}\right\} = 6_i < \nu+1$$

$$rk\left\{\left.\frac{\partial \chi}{\partial(z(t_f))}\right|^{*}\right\} = 6_f < \nu+1$$

3) D_i, D_f defined by $\omega(z(t_i), z(t_f)) = 0$ of 6 components of C^1 class

regularity: $rk\left\{\left.\frac{\partial \omega}{\partial(z(t_i), z(t_f))}\right|^{*}\right\} = 6$

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_i^*}^{*} = \theta^T \left.\frac{\partial \omega}{\partial z(t_i)}\right|^{*}, \quad \left.\frac{\partial \mathcal{L}}{\partial \dot{z}}\right|_{t_f^*}^{*} = -\theta^T \left.\frac{\partial \chi}{\partial z(t_f)}\right|^{*}, \quad \theta \in \mathbb{R}^6$$

If $\mathcal{L}$ strictly convex, if $\exists$ a solution, it is unique

~ Lagrange problem with constraints

- Integral constraint

$z : [a,b] \to \mathbb{R}$, C^1 curve, $z(a) = z_0$ $z(b) = z_1$

Find the local minima of the cost index

$$J(z) = \int_a^b \mathcal{L}(t, z, \dot{z})\, dt \qquad \mathcal{L} \ C^2 \text{ class}$$

with $\boxed{C(z) = \int_a^b h(t, z, \dot{z})\, dt = k}$

Proof

Given the curve z, the perturbed family is given by

$$C(z + \alpha\eta) = k \quad \forall \alpha \approx 0 \quad \longrightarrow \quad \delta C|_z(\eta) = 0$$

from the computations of the basic variation problem

$$\int_a^b \left[\frac{\partial h}{\partial z}(t, z, \dot{z}) - \frac{d}{dt} \frac{\partial h}{\partial \dot{z}}(t, z, \dot{z}) \right] \eta(t)\, dt = 0$$

In fact, since in $\delta J|_z(\eta) = 0$

$$\int_a^b \left[\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} \right] \eta(t)\, dt = 0 \quad \forall \eta$$

then also $\int_a^b \left[\frac{\partial h}{\partial z} - \frac{d}{dt} \frac{\partial h}{\partial \dot{z}} \right] \eta(t)\, dt = 0$

How can we put these conditions together?

There exists a constant p (Lagrange multiplier) s.t. :

$$\left[\frac{\partial \mathcal{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}}\right] + p\left[\frac{\partial h}{\partial z} - \frac{d}{dt}\frac{\partial h}{\partial \dot{z}}\right] = 0$$

$$\Downarrow$$

$$\frac{\partial(\mathcal{L}+ph)}{\partial z} = \frac{d}{dt}\frac{\partial(\mathcal{L}+ph)}{\partial \dot{z}}$$

$\boxed{\mathcal{L}+ph}$ is the augmented Lagrangian for which the Euler equation holds

\# End

It means that z is an extremal of the augmented cost functional

$$(J+pC)(z) = \int_a^b \left[\mathcal{L}(t,z,\dot{z}) + p\,h(t,z,\dot{z})\right] dt$$

\# Remark :

It's a global constraint $\rightarrow$ it applies to the entire curve

- Equality constraints (non-integral)

$z : [a,b] \to \mathbb{R} \quad z(a) = z_0 \quad z(b) = z_1$

Find the local minima of the cost index

$$J(z) = \int_a^b \mathcal{L}(t, z, \dot{z})\,dt \qquad \mathcal{L} \text{ of } C^2 \text{ class}$$

with the constraint $\boxed{g(t, z, \dot{z}) = 0}$

The EL equation holds for the augmented lagrange

$\boxed{\mathcal{L} + \lambda(t) g}$

↳ here we consider the minimization wrt z and λ of:

$$\int_a^b \mathcal{L}\,dt + \int_a^b \lambda(t) g\,dt$$

λ no longer needs to be constant since g is identically null

Remark: the constraint is local and there is no difference locally around each curve

~ Augmented Lagrangian

$$\ell(t, z, \dot{z}, \lambda_0, \lambda(t), p) = \lambda_0 \mathcal{L}(t, z, \dot{z}) + \lambda^T(t) g(t, z, \dot{z}) + p^T h(t, z, \dot{z})$$

if $\lambda_0 \neq 0 \to$ normal solution