DOUBLE INTEGRATOR

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = u(t)$

$x(t_i) = x_i$

$x(t_f) = 0$

$|u(t)| \leq 1$

$J = \int_{t_i}^{t_f} dt = t_f - t_i$

We can write the system in this way:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u$$

eigenvalues:

$p(\lambda) = |A - \lambda I| = \lambda^2 = 0 \longrightarrow \lambda_1 = 0$

$\lambda_2 = 0$

$$\det \begin{pmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix}$$

$$\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

controllability:

$\det(B \quad AB) = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$ ok!

$\Rightarrow \exists\ (x^\circ, u^\circ, t_f^\circ)$ unique, non singular, bang bang

number of switching points $\nu^\circ \leq n-1 = 1$

From any initial state I can reach the origin with zero switches or just one.

We know that $u(t) = \pm 1$, therefore

$$\dot{x}_1 = x_2 \qquad x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix}$$

$$\dot{x}_2 = \pm 1$$

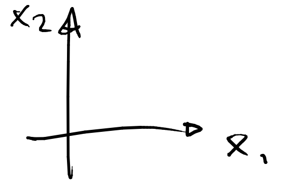
switch

integrating

$$x_2 = \pm (t - t_i) + x_{2i} \qquad (1)$$

$$x_1(t) = x_{1i} + x_{2i}(t - t_i) \pm \frac{1}{2}(t - t_i)^2 \qquad (2)$$

Instead of studying the evolution of x_1 and x_2 separately, it is useful to study them together in the state phase plane:



From (1) $\quad (t - t_i) = \pm [x_2(t) - x_{2i}]$

substituting in (2):

$$x_1(t) - x_{1i} = \pm x_{2i}[x_2(t) - x_{2i}] \pm \frac{1}{2}[x_2(t) - x_{2i}]^2$$

$$= \frac{\pm 2x_{2i}[x_2(t) - x_{2i}] \pm [x_2(t) - x_{2i}]^2}{2}$$

$$= \frac{\pm [x_2(t) - x_{2i}][2x_{2i} + [x_2(t) - x_{2i}]]}{2}$$

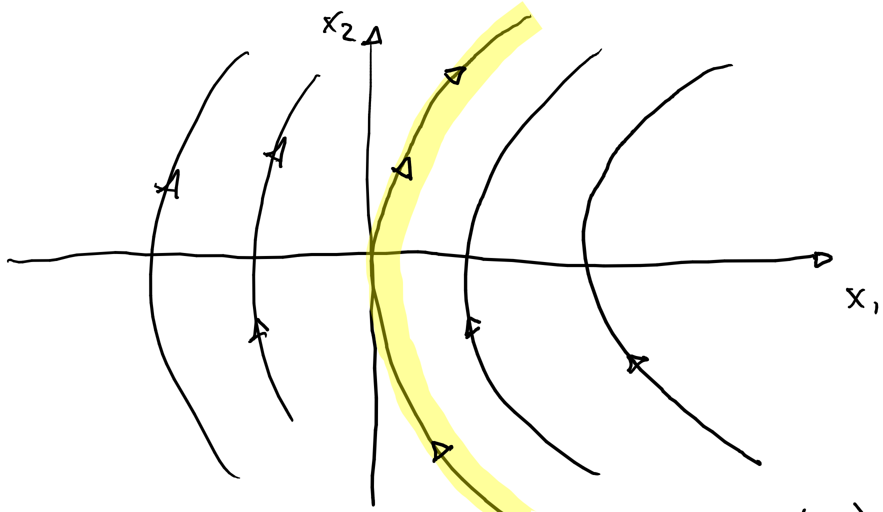
$$= \frac{\pm [x_2(t) - x_{2i}][x_2(t) + x_{2i}]}{2}$$

$$= \pm \frac{1}{2}[x_2^2(t) - x_{2i}^2] \quad \text{hyperbolic paraboloid}$$

The optimal trajectory is described by parabolic arcs:

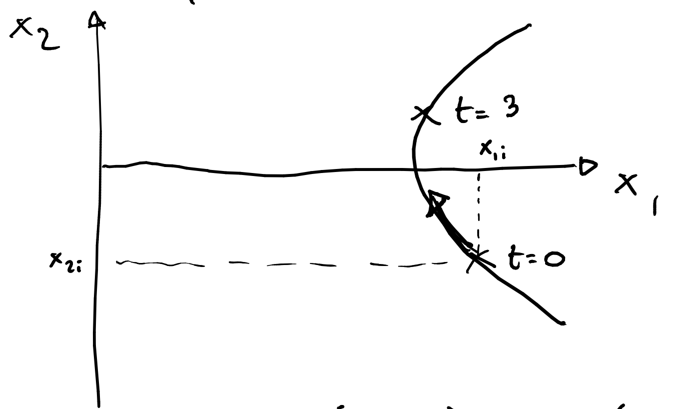
- consider $u(t) = +1$

$$x_1(t) - x_{1i} = +\frac{1}{2}\left[x_2^2(t) - x_{2i}^2\right] \rightarrow x_1(t) = x_{1i} + \dots$$



if $\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

What happen if time increases?



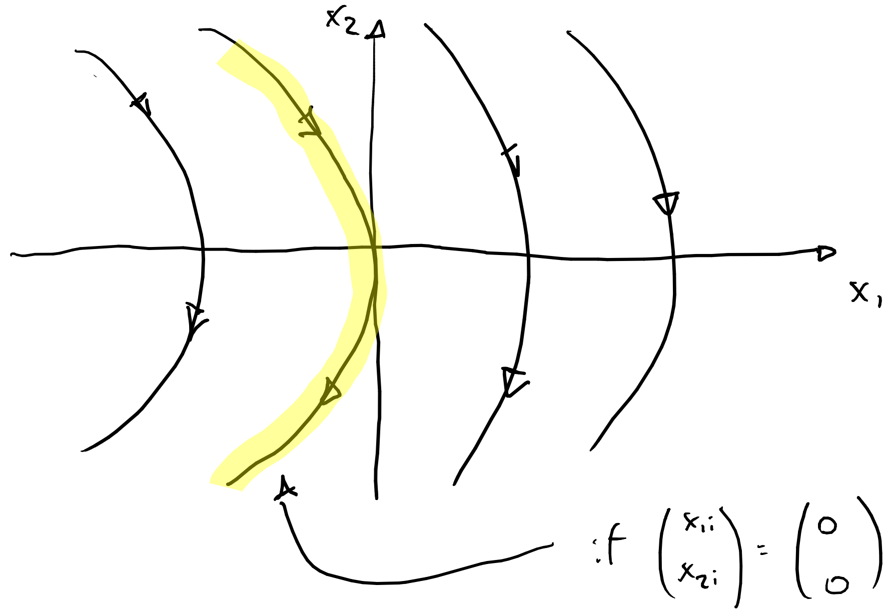
I go up

In fact in $(t - t_i) = + (x_2 - x_{2i})$ if time increases:

$$\begin{cases} t_i = 0 \\ x_{2i} = -2 \end{cases} \rightarrow \begin{array}{lll} \text{in } t = 0 & x_2 = -2 & \\ \text{in } t = 3 & x_2 = 1 & \rightarrow x_2 \text{ increases} \end{array}$$

- consider $u(t) = -1$

$$x_1(t) - x_{1i} = -\frac{1}{2}\left[x_2^2(t) - x_{2i}^2\right] \to x_1(t) = x_{1i} - \dots$$



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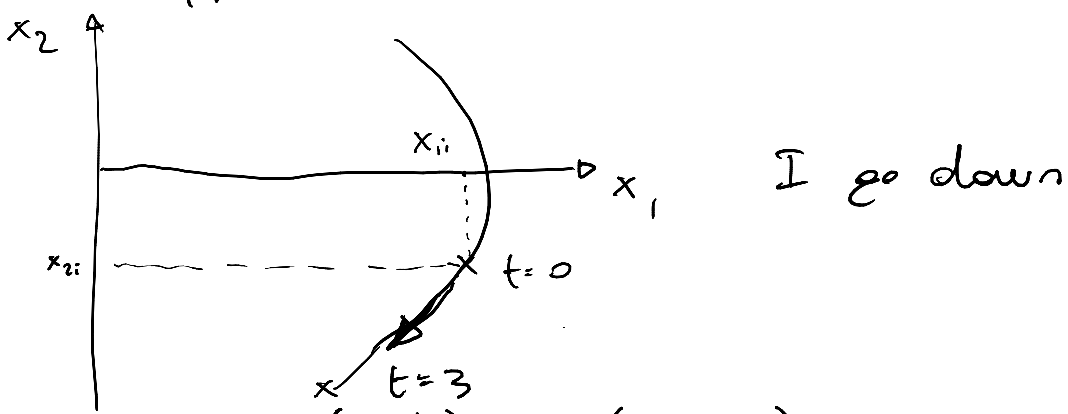
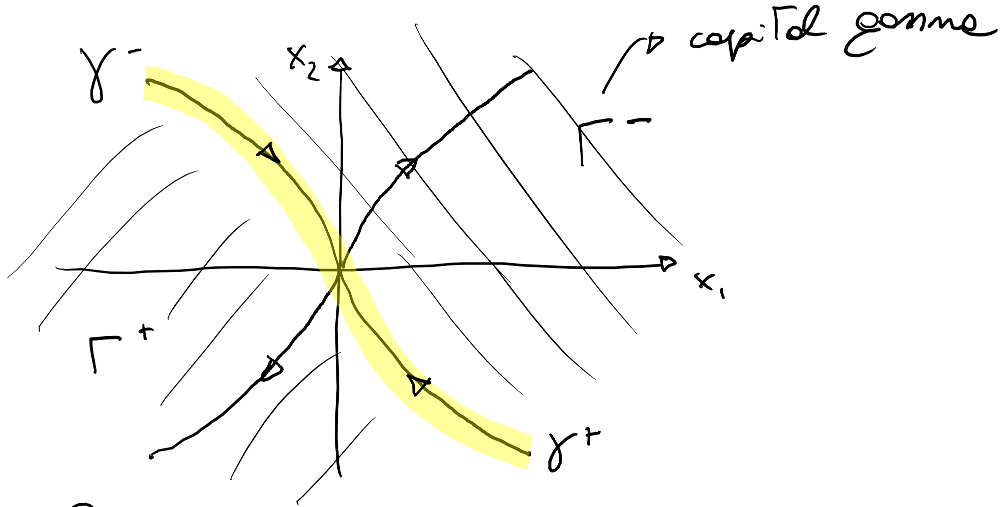


In fact in $(t - t_i) = -(x_2 - x_{2i})$ if time increases:

$$\begin{cases} t_i = 0 \\ x_{2i} = -2 \end{cases} \to \begin{array}{lll} \text{in } t = 0 & x_2 = -2 & \\ \text{in } t = 3 & x_2 = -5 & \to x_2 \text{ decreases} \end{array}$$

This curves represent the path on which I can move switching zero or maximum 1 occ.

~ Initial point



We define 2 curves and 2 regions:

Curves:

$$\gamma^+ = \left\{ x \in \mathbb{R}^2 : \; x_1 = \frac{1}{2} x_2^2 \; , \; x_2 < 0 \right\}$$

$$\gamma^- = \left\{ x \in \mathbb{R}^2 : \; x_1 = -\frac{1}{2} x_2^2 \; , \; x_2 > 0 \right\}$$

$$\Downarrow \quad \gamma = \gamma^+ \cup \gamma^+$$

$$\gamma = \left\{ x \in \mathbb{R}^2 : \; x_1 = -\frac{1}{2} x_2 |x_2| \; , \; x_2 \neq 0 \right\}$$

Regions:

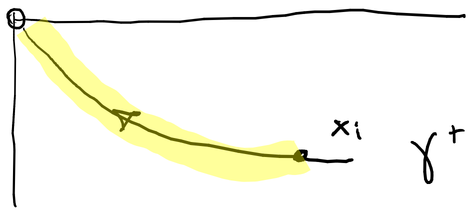
$$\Gamma^+ = \left\{ x \in \mathbb{R}^2 : \; x_1 < -\frac{1}{2} x_2 |x_2| \right\}$$

$$\Gamma^- = \left\{ x \in \mathbb{R}^2 : \; x_1 > -\frac{1}{2} x_2 |x_2| \right\}$$

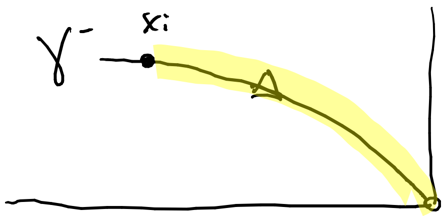
$$\Gamma^+ \cup \Gamma^- \cup \gamma = \mathbb{R}^2 \setminus \{0\}$$

~ 4 cases

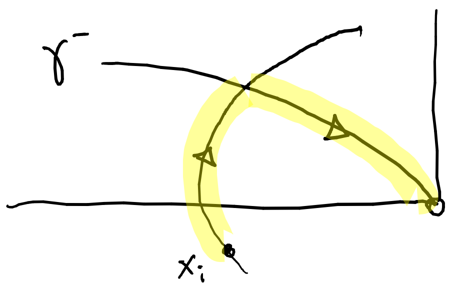
1) $x_i \in \gamma^+$ $\rightarrow$ with control $u = +1$ and zero switches



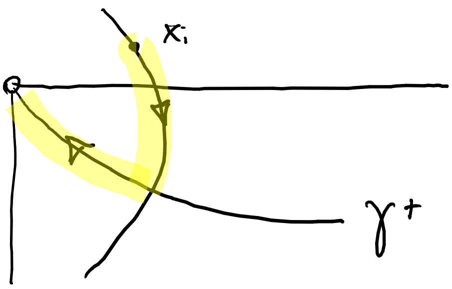
2) $x_i \in \gamma^-$ $\rightarrow$ with control $u = -1$ and zero switches



3) $x_i \in \Gamma^+$ $\rightarrow$ first $u = +1$ to reach the curve γ^-
then $u = -1$ to reach the origin (1 switch)



4) $x_i \in \Gamma^-$ $\rightarrow$ first $u = -1$ to reach the curve γ^+
then $u = +1$ to reach the origin (1 switch)



~ Optimal control

$$u^\circ(x^\circ(t)) = \begin{cases} 1 & \text{if } x^\circ(t) \in \Gamma^+ \cup \gamma^+ \\ -1 & \text{if } x^\circ(t) \in \Gamma^- \cup \gamma^- \end{cases}$$

~ Minimum Time

It depends on the location of the initial point x_i

A) $x_i \in \gamma = \gamma^+ \cup \gamma^-$

No switches

from $(t - t_i) = \pm [x_2(t) - x_{2i}]$

$t_f - t_i = + [\cancel{x_2(t_f)} - x_{2i}]$ if $u = +1$

$\underset{\text{unknown}}{\uparrow}$ $\qquad x_2(t_f) \to 0$ (the origin

$$\boxed{t_f^\circ = t_i - x_{2i} \quad \text{if} \quad u = +1}$$

$t_f - t_i = - [\cancel{x_2(t_f)} - x_{2i}]$ if $u = -1$

$x_2(t_f) \to 0$

$$\boxed{t_f^\circ = t_i + x_{2i} \quad \text{if} \quad u = -1}$$

B) $x_i \in \Gamma^+$

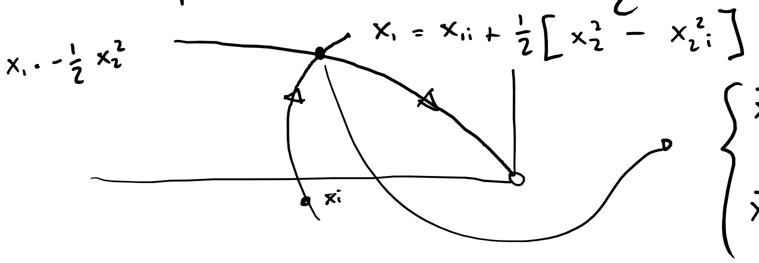
The control switches at the instant $\bar{t}$ at the position $\bar{x}_2$

$$(t_f^\circ - t_i) = \underbrace{(t_f^\circ - \bar{t})}_{\substack{\uparrow \\ u = -1 \\ \text{(after switching)}}} + \underbrace{(\bar{t} - t_i)}_{\substack{\uparrow \\ u = +1 \\ \text{(before switching)}}}$$

$$(t_f^\circ - t_i) = \pm [x_2(t) - x_{2i}] \Rightarrow \begin{cases} \bar{t} - t_i = \bar{x}_2 - x_{2i} \\ t_f^\circ - \bar{t} = \bar{x}_2 \end{cases}$$

$$(t_f^\circ - t_i) = 2\bar{x}_2 - x_{2i}$$

The position $\bar{x}_2$ belongs to the two parabolic arcs:



$$\begin{cases} \bar{x}_1 = x_{1i} + \frac{1}{2} [\bar{x}_2^2 - x_{2i}^2] \\ \bar{x}_1 = -\frac{1}{2} \bar{x}_2^2 \end{cases}$$

Result of the system:

$$\bar{x}_2^{\,2} = -x_{1i} + \frac{1}{2} x_{2i}^2 \, , \quad x_2 > 0$$

$$\bar{x}_2 = \sqrt{-x_{1i} + \frac{1}{2} x_{2i}^2}$$

$$\boxed{(t_f^o - t_i) = \sqrt{-4 x_{1i} + 2 x_{2i}^2} - x_{2i}}$$

c) $x_i \in \Gamma^-$

The same calculations yeld:

$$\boxed{(t_f^o - t_i) = \sqrt{4 x_{1i} + 2 x_{2i}^2} + x_{2i}}$$

~ Commutation curve

γ is the commutation curve so:

$$\varphi(x) = x_1 + \frac{1}{2} x_2 |x_2|$$

$$u^o(t) = -\text{sign}\{\varphi(x)\} = -\text{sign}\left\{x_1 + \frac{1}{2} x_2 |x_2|\right\}$$