20_02.pdf

venerdì 17 luglio 2020 16:05

20_02.pdf

Nonlinear Systems & Control
Part II
03/02/2020

Student: ______________________

Student ID: ______________________

1. Consider the linear system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} d + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & -1 \end{pmatrix} x.$$

Is it possible to compute a state-feedback such that that $y(t) \to 1$ as $t \to \infty$ for all disturbances d and with stability?

2. Given the nonlinear dynamics

$$\begin{aligned} \dot{x}_1 &= u_1 \cos(x_3) \\ \dot{x}_2 &= u_1 \sin(x_3) \\ \dot{x}_3 &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

compute, if any, a feedback for achiving

(a) input-output decoupling;
(b) asymptotic output tracking of a circular path (i.e., $x^2 + y^2 = 1$).

Finally, exhibit the zero-dynamics.

3. Compute a backstepping-based feedback making $x_\star = (0\ 1)^\top$ a globally asymptotically stable equilibrium for

$$\begin{aligned} \dot{x}_1 &= -x_1^3 x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 + u. \end{aligned}$$

4. The zero-dynamics and its role in local stabilization of nonlinear systems.
5. The control Lyapunov function and the Artstein-Sontag Theorem.

2 MIMO decoupling
Dynamic extension, AOTP
Zero dynamics

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \\ y_1 = x_1 \\ y_2 = x_2 \end{cases}$$

$$f = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad g_1 = \begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$dh_1 = (1\ 0\ 0) \qquad dh_2 = (0\ 1\ 0)$$

$$L_g h_1 = dh_1 g = (1\ 0\ 0)\begin{pmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{pmatrix} = (\underset{\neq 0}{\cos x_3}\ \ 0)$$

$$L_g h_2 = dh_2 g = (0\ 1\ 0)\begin{pmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{pmatrix} = (\underset{\neq 0}{\sin x_3}\ \ 0)$$

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 1 \end{aligned} \rightarrow r_1 + r_2 < n$$

decoupling matrix :

decoupling matrix:

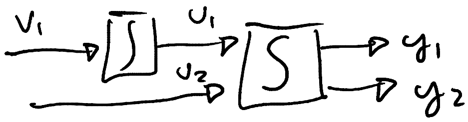
$$A(x) = \begin{pmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{pmatrix} \qquad \det(A(x)) = 0$$

the system does not have strong vector relative degree $\to$ NIC not solvable

- Dynamic extension:

add $n - r_1 - r_2 = 1$ state

$$\dot{x}_1 = \xi \cos(x_3)$$
$$\dot{x}_2 = \xi \sin(x_3)$$
$$\dot{x}_3 = u_2$$
$$\dot{\xi} = v_1$$
$$y_1 = x_1$$
$$y_2 = x_2$$



$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \xi \end{pmatrix}$$

the two inputs are now v_1 and u_2

$$\tilde{g}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \tilde{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \tilde{f} = \begin{pmatrix} \xi \cos x_3 \\ \xi \sin x_3 \\ 0 \\ 0 \end{pmatrix}$$

$$L_{\tilde{g}} h_1 = dh_1 \cdot \tilde{g} = (1\ 0\ 0\ 0) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (0 \quad 0)$$

$$L_{\tilde{f}} h_1 = dh_1 \cdot \tilde{f} = (1\ 0\ 0\ 0) \begin{pmatrix} \xi \cos x_3 \\ \xi \sin x_3 \\ 0 \\ 0 \end{pmatrix} = \xi \cos x_3$$

$$L_{\tilde{g}} L_{\tilde{f}} h_1 = \frac{\partial (\xi \cos x_3)}{\partial (x, \xi)} \cdot \tilde{g} = (0 \quad 0 \quad -\xi \sin x_3 \quad \cos x_3) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= (\cos x_3 \quad -\xi \sin x_3) \neq 0 \qquad r_1 = 2$$

$$L_{\tilde{g}} h_2 = dh_2 \cdot \tilde{g} = (0\ 1\ 0\ 0) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (0 \quad 0)$$

$$L_{\tilde{f}} h_2 = dh_2 \cdot \tilde{f} = (0\ 1\ 0\ 0) \begin{pmatrix} \xi \cos x_3 \\ \xi \sin x_3 \\ 0 \\ 0 \end{pmatrix} = \xi \sin x_3$$

$$L_{\tilde{g}} L_{\tilde{f}} h_2 = (0 \quad 0 \quad \xi \cos x_3 \quad \sin x_3) \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sin x_3 \quad \xi \cos x_3) \neq 0$$
$$r_2 = 2$$

$$\tilde{A}(x) = \begin{pmatrix} \cos x_3 & -\xi \sin x_3 \\ \sin x_3 & \xi \cos x_3 \end{pmatrix} \qquad \det(\tilde{A}(x)) = \xi \cos^2 x_3 + \xi \sin^2 x_3 =$$
$$= \xi \neq 0 \to \text{NIC solvable}$$

$$\begin{cases} r_1 = 2 \\ r_2 = 2 \end{cases} \qquad r_1 + r_2 = \tilde{n} = 4$$

- NIC

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = Fx + Gv = -\tilde{A}^{-1}(x) \begin{pmatrix} L_{\tilde{f}}^{r_1} h_1 \\ L_{\tilde{f}}^{r_2} h_2 \end{pmatrix} x + \tilde{A}^{-1}(x)\, \varrho =$$

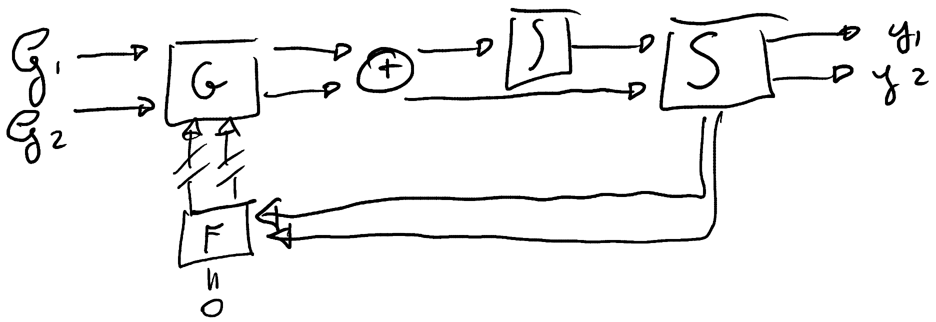
$$L_{\tilde{f}}^2 h_1 = L_{\tilde{f}} L_{\tilde{f}} h_1 = L_{\tilde{f}}\, \xi \cos x_3 = (0 \;\; 0 \;\; -\xi \sin x_3 \;\; \cos x_3) \begin{pmatrix} \xi \cos x_3 \\ \xi \sin x_3 \\ 0 \\ 0 \end{pmatrix} = 0$$

$\rightarrow$ no feedback needed

$$L_{\tilde{f}}^2 h_2 = L_{\tilde{f}} L_{\tilde{f}} h_2 = L_{\tilde{f}}\, \xi \sin x_3 = (0 \;\; 0 \;\; * \;\; *)\, \tilde{f} = 0$$

$\rightarrow$ no feedback needed

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\xi} \begin{pmatrix} \xi \cos x_3 & \xi \sin x_3 \\ -\sin x_3 & \cos x_3 \end{pmatrix} \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix} = \frac{1}{\xi} \begin{pmatrix} \varrho_1 \xi \cos x_3 + \varrho_2 \xi \sin x_3 \\ -\varrho_1 \sin x_3 + \varrho_2 \cos x_3 \end{pmatrix}$$



We can consider the decoupled system where the reference inputs are $(\varrho_1 \;\; \varrho_2)^T$.

- AOTP

can be approached considering 2 individual SISOs.

③ Back stepping

$$\begin{cases} \dot{x}_1 = -x_1^3 x_2 \\ \dot{x}_2 = -\sin x_1 - x_2 + \bar{u} \end{cases} \Rightarrow \begin{cases} \dot{z} = f(z) + g(z)\, \xi_1 \\ \dot{\xi}_r = b_r(z, \xi_1, \dots, \xi_{r-1}) + a_r(z, \xi_1, \dots, \xi_r)\, u \end{cases}$$

$$\gamma(x_1) = x_1^2 \Rightarrow \dot{\gamma}(x_1(t)) = \frac{\partial \gamma}{\partial x_1} \cdot \dot{x}_1 = 2 x_1 \dot{x}_1$$

$$\eta = x_2 - \gamma(x_1) = x_2 - x_1^2 \Rightarrow x_2 = \eta + \gamma(x_1) = \eta + x_1^2$$

$$\begin{cases} \dot{z} = f(z) + g(z)\gamma_1(z) + g(z)\eta_1 \\ \dot{\eta}_1 = b_1(z, \eta_1 + \gamma_1(z)) + a_1(z, \eta_1 + \gamma_1(z))\, \xi_2 - \dot{\gamma}_1(z) \end{cases}$$

$$\begin{cases} \dot{x}_1 = -x_1^3 x_2 = -x_1^3 \eta - x_1^5 \\ \dot{\eta} = \dot{x}_2 - \dot{x}_1^2 = -\sin x_1 - \eta - x_1^2 - \dot{\gamma}(x_1(t)) \end{cases}$$
$$= -\sin x_1 - \eta - x_1^2 - 2x_1(\dot{x}_1)$$
$$= -\sin x_1 - \eta - x_1^2 + 2x_1^4 \eta + 2x_1^6 + \bar{u}$$

$$V = \frac{1}{2}(x_1^2 + \eta^2)$$
$$\dot{V} = x_1\dot{x}_1 + \eta\dot{\eta} = -x_1^4\eta - x_1^6 - \eta\sin x_1 - \eta^2 - \eta x_1^2 + 2x_1^4\eta^2 + 2x_1^6\eta + \bar{u}\eta$$
$$= \underset{<0}{-x_1^6} \underset{<0}{-\eta^2} + \eta(-x_1^4 - \sin x_1 - x_1^2 + 2x_1^4\eta + 2x_1^6\eta + \bar{u})$$

by choosing $\bar{u} = -(-x_1^4 - \sin x_1 - x_1^2 + 2x_1^4\eta + 2x_1^6)$

$\rightarrow \dot{V} = -x_1^6 - \eta^2 < 0$ GAS $\forall x_1, \eta$

$\bar{u} = x_1^4 + \sin x_1 + x_1^2 - 2x_1^4 x_2 - \cancel{2x_1^6} + \cancel{2x_1^6}$

$\boxed{u = \bar{u} + 1}$

$$\begin{cases} \dot{x}_1 = -x_1^3 \overset{f}{x_2} \\ \dot{x}_2 = \underset{f}{-x_2} + \underbrace{x_1^4 - 2x_1^4 x_2 + x_1^2}_{g} + 1 \end{cases} \quad x_e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_e} = \left.\begin{pmatrix} -3x_1^2 x_2 & -x_1^3 \\ 3x_1 - 8x_1^3 x_2 & -1 - 2x_1^4 \end{pmatrix}\right|_{x_e} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -1 \end{array}\right)$$

→ Center manifold analysis

$A = 0$

$B = -1$

Center manifold analysis

try $x_2 = h(x_1) = 0$ satisfying $h(0) = 0$ and $\left.\frac{\partial h}{\partial x_1}\right|_0 = 0$

center manifold equation

$$\frac{\partial h}{\partial x_1}(Ax_1 + f(x_1, h(x))) - Bh(x) - g(x_1, h(x)) = 0$$

$$\begin{cases} f(x_1, x_2) = -x_1^3 x_2 \\ g(x_1, x_2) = x_1^4 + x_1^2 - 2x_1^4 x_2 \end{cases}$$

The center manifold equation with $h(x) = 0$ becomes:

The center manifold equation with $h(x)=0$ becomes:

$$\frac{\partial h}{\partial x_1} \cancel{f(x_1, x_2)}^{\,0} - \cancel{\beta h(x)}^{\,0} - x_1^4 + x_1^2 \neq 0$$

$h(x)=0$ does not work

$$h(x) = a x_1^2 + b x_1^3 + c x_1^4 \dots$$

$$\frac{\partial h(x)}{\partial x_1} = 2a x_1 + 3b x_1^2 + 4c x_1^3 \dots$$

$-a x^k$
stable if $a > 0$
k even $\rightarrow$ pari

$$\frac{\partial h}{\partial x_1}\left(\cancel{A}^{\,0} x + f(x, h(x))\right) - \beta h(x) - g(x, h(x)) = 0$$

$$= (2a x_1 + 3b x_1^2 + 4c x_1^3 \dots)(-a x_1^5 - b x_1^6 - c x_1^7 \dots) +$$

$$+ a x_1^2 + b x_1^3 + c x_1^4 \dots - x_1^4 - x_1^2 - 2x_1^4 (h(x) \dots)$$

$$c x_1^4 - x_1^4 + a x_1^2 - x_1^2 = 0$$

$$c = 1 \quad a = 1 \implies 0 = 0$$

$$h(x) \simeq x_1^2 + x_1^4$$

reduced dynamics:

$$\dot{x}_1 = -x_1^3 (x_1^2 + x_1^4) = -\overbrace{x_1^5}^{\text{odd}} - \overbrace{x_1^7}^{\text{odd}} \rightarrow \text{unstable!}$$