33. Global Asymptotic Stabilization

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Recalling that a smooth $V: \mathbb{R}^n \to \mathbb{R}$ is
- Positive definite if $V(0)=0$ and $V(x)>0$ for $x \neq 0$
- Proper if, $\forall a>0$, $V^{-1}([0,a]) = \{x \in \mathbb{R}^n;\ 0 \leq V(x) \leq a\}$ is compact

Prop: Given $\begin{cases} \dot z = f(z,\xi) \\ \dot \xi = u \end{cases}$ $\quad z \in \mathbb{R}^n \quad \xi \in \mathbb{R} \quad f(0,0)=0$

Suppose $\exists V$ positive definite and proper such that

$$\frac{\partial V}{\partial z} f(z,0) < 0 \quad , \quad \forall z \neq 0$$

then there exists a static feedback law $u(z,\xi)$, $u(0,0)=0$ and a function $W(z,\xi)$ positive definite and proper such that

$$\begin{pmatrix} \frac{\partial W}{\partial z} & \frac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u(z,\xi) \end{pmatrix} < 0 \qquad \forall (z,\xi) \neq (0,0)$$

i.e. $\exists u(z,\xi)$ which renders GAS $x_e = 0$

Proof: Observe that $f(z,\xi) = f(z,0) + p(z,\xi)\xi$ where $p(z,\xi)$ smooth function.

Observe that $\bar f(z,\xi) = f(z,\xi) - f(z,0)$ is a smooth function vanishing at $\xi = 0$

$$\bar f(z,\xi) = \int_0^1 \frac{\partial \bar f(z, s\xi)}{\partial s}\, ds = \underbrace{\int_0^1 \left.\frac{\partial \bar f(z,\zeta)}{\partial \zeta}\right|_{\zeta = s\xi} \xi \, ds}_{p(z,\xi)\cdot \xi}$$

$$f(z,\xi) = f(z,0) + p(z,\xi)\xi$$

Consider now $W(z,\xi) = V(z) + \frac{1}{2}\xi^2$ and note that

$$\begin{pmatrix} \frac{\partial W}{\partial z} & \frac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u \end{pmatrix} = \frac{\partial V}{\partial z} f(z,0) + \frac{\partial V}{\partial z} p(z,\xi)\xi + \xi u$$

$$\underbrace{\frac{\partial V}{\partial z} \cdot f(z,0)}_{<0} + \left(\frac{\partial V}{\partial z} p(z,\xi) + u \right)\xi$$

choosing $u = u(z,\xi) = -\xi - \frac{\partial V}{\partial z} p(z,\xi)$
yelds the result

$$\frac{\partial V}{\partial z} f(z,0) - \xi^2 < 0$$

therefore $\dot{W} < 0$

Prop: given $\begin{cases} \dot{z} = f(z,\xi) \\ \dot{\xi} = u \end{cases}$

suppose $\exists\ \xi = \gamma(z)$, $\gamma(0) = 0$
and V positive definite and proper such that

$$\frac{\partial V}{\partial z} f(z,\gamma(z)) < 0 \quad \forall z \neq 0$$

then $\exists\ u(z,\xi)$, $u(0,0) = 0$, and $W(z,\xi) > 0$ and proper
such that

$$\begin{pmatrix} \frac{\partial W}{\partial z} & \frac{\partial W}{\partial \xi} \end{pmatrix} \begin{pmatrix} f(z,\xi) \\ u(z,\xi) \end{pmatrix} < 0 \quad \forall (z,\xi) \neq (0,0)$$

note that $\gamma(z)$ is the feedback which renders V a strict Lyapunov function for $\dot{z} = f(z,\gamma(x))$ and note that from V and γ one can compute $u(z,\xi)$ such that $W = V + \frac{1}{2}\xi^2$ is a strict Lyapunov function itself.

Proof: define $y = -\xi - \gamma(z) \longrightarrow \xi = \gamma(z) + y$

$$\begin{pmatrix} z \\ \xi \end{pmatrix} \to \begin{pmatrix} z \\ y \end{pmatrix} \quad \Rightarrow \begin{cases} \dot{z} = f(z, y+\gamma(z)) \\ \dot{y} = \dot{\xi} - \dot{\gamma}(z) = u - \frac{\partial \gamma}{\partial z} f(z, y+\gamma(z)) = u' \end{cases}$$

$$\to \begin{cases} \dot{z} = \tilde{f}(z,y) \\ \dot{y} = u' \end{cases} \qquad \begin{aligned} u' &= u - \dot{\gamma}(z) \\ u &= u' + \dot{\gamma}(z) \end{aligned}$$

$$W(z,y) = V(z) + \frac{1}{2}y^2 = V(z) + \frac{1}{2}(\xi - \gamma(z))^2 = W(z,\xi)$$

therefore the feedback law u yelds GAS.

Prop: Given dynamics which are characterized by an affine structure

$$\begin{cases} \dot{z} = f(z) + g(z)\xi \\ \dot{\xi} = b(z, \xi) + a(z, \xi) u \end{cases}$$

suppose $\exists\, \gamma(z)$, V positive and proper such that

$$\frac{\partial V}{\partial z}(f + g\gamma) < 0 \quad \forall z \neq 0$$

then $u = a^{-1}(z, \xi)\left(-b(z, \xi) - \frac{\partial V}{\partial z} g(z) + \dot{\gamma}(z) + v\right)$

with $\dot{\gamma} = \frac{\partial \gamma}{\partial z}(f + g\gamma) = \frac{\partial V}{\partial z}\dot{z}$

renders $x_e = 0$ GAS.

Proof: $y = \xi - \gamma(z) \qquad \xi = \gamma(z) + y$

$$\begin{cases} \dot{z} = f(z) + g(z)\gamma(z) + g(z) y \\ \dot{y} = \dot{\xi} - \frac{\partial \gamma}{\partial z}\dot{z} = b + a u - \frac{\partial \gamma}{\partial z}\left(f + g\gamma + g(z) y\right) = -\frac{\partial V}{\partial z} g(z) - k y \end{cases}$$

$$W(z, \xi) = V(z) + \frac{1}{2}(\xi - \gamma(z))^2 = V(z) + \frac{1}{2} y^2$$

$$\dot{W}(z, \xi) = \frac{\partial V(z)}{\partial z}(f + g\gamma + \cancel{g y}) - y\left(\cancel{\frac{\partial V}{\partial z} g} - k y\right) =$$

$$= \underset{\substack{<0 \\ \text{by hypothesis}}}{\frac{\partial V}{\partial z}(f + g\gamma)} \underbrace{- k y^2}_{<0} < 0 \quad \forall (z, y) \rightarrow \forall (z, \xi)$$