System dynamics can be also represented as:

$$\dot{z} = Az + \left[f(z) - \frac{\partial f}{\partial z}(0)\, z\right] = Az + \tilde{f}(z)$$

$$\tilde{f}(0) = 0 \quad , \quad \frac{\partial \tilde{f}}{\partial z}(0) = 0$$

Suppose A has k eigenvalues with 0 real part
and $m = n-k$ eigenvalues with negative real part
we can always find T s.t. $T^{-1} = \left(\begin{array}{c|c} \text{basis in } E^c & \text{basis in } E^s \end{array} \right)$

$$TAT^{-1} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \qquad A\ (k \times k) \quad B\ (m \times m)$$

$$\sigma(A) \subset \mathrm{Im} \qquad \sigma(B) \subset \mathbb{C}^-$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = Tz \qquad x \in \mathbb{R}^k \quad y \in \mathbb{R}^m$$

$$\Rightarrow \begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = Bx + g(x,y) \end{cases} \qquad f(0,0) = g(0,0) = 0$$

$$\left(\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \right)\Big|_0 = \left(\frac{\partial g}{\partial x} \ \frac{\partial g}{\partial y} \right)\Big|_0 = (0,0)$$

For a given vector field around the equilibrium

$$W^c(0) = \left\{ (x,y) \in \mathbb{R}^c \times \mathbb{R}^s : y = h(x) \ ,\ |x| < \delta ,\ h(0) = 0 ,\ \frac{\partial h}{\partial x}\Big|_0 = 0 \right\}$$

An invariant manifold is said to be the "central manifold" for $W_c(0)$.
In fact with $h(0) = 0$ and $\frac{\partial h}{\partial x}\Big|_0 = 0 \Rightarrow E_c$ is tangent to $W^c(0)$

Theorem 1

The center manifold exists and the evolutions on it for small δ are characterized by

$$\dot{u} = Au + f(u, h(u)) \qquad u \in \mathbb{R}^c$$

Theorem 2

If $u=0$ is stable / AS / unstable, $(x,y) = (0,0)$ in the system $(\dot{x}, \dot{y})$ is stable / AS / unstable

Moreover, if $(x(t), y(t))$ solution of $(\dot{x}, \dot{y})$ with (x_0, y_0) small:
$\exists\ u(t)$ s.t. for $t \to \infty$. $\gamma > 0$

Moreover, if $(x(t), y(t))$ solution of $(\dot{x}, \dot{y})$ with (x_0, y_0) small:
$\exists\, u(t)$ s.t. for $t \to \infty$, $\gamma > 0$

$$x(t) = u(t) + \mathcal{O}(e^{-\gamma t})$$
$$y(t) = h(u(t)) + \mathcal{O}(e^{-\gamma t})$$

The equations that characterizes the central manifold are:

$$y = h(x) \Rightarrow \dot{y} = \frac{\partial h}{\partial x}\dot{x} \quad \text{where} \quad \begin{aligned} \dot{x} &= Ax + f(x, h(x)) \\ \dot{y} &= Bh(x) + g(x, h(x)) \end{aligned}$$

$$Bh(x) + g(x, h(x)) = \frac{\partial h}{\partial x}\left(Ax + f(x, h(x))\right)$$

expressed also as:

$$N(h(x)) = \frac{\partial h}{\partial x}\left(Ax + f(x, h(x))\right) - Bh(x) - g(x, h(x)) = 0$$

$\hookrightarrow$ Non linear differential operator

With the approximation of $h(x)$ (Taylor series in x) we can find an approximated solution:

Theorem 3

Let $\phi : \mathbb{R}^c \to \mathbb{R}^s$ be a C^1 map with $\phi(0) = 0$ and $\frac{\partial \phi}{\partial x}(0) = 0$.
$N(\phi(x)) = \mathcal{O}(|x|^p)$, $x \to 0$ for $p > 1$.
Then for sufficiently small $|x|$

$$|h(x) - \phi(x)| = \mathcal{O}(|x|^p)$$
$$\longrightarrow \dot{x} = Ax + f(x, \phi(x)) + \mathcal{O}(|x|^{p+1})$$