HARMONIC OSCILLATOR

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = -\omega x_1(t) + u(t) \end{cases} \qquad \omega > 0$$

$x(t_i) = x_i \qquad x(t_f) = 0 \qquad |u(t)| \le 1$

$$J(t_f) = \int_{t_i}^{t_f} dt = t_f - t_i$$

$$\dot{x}(t) = \underbrace{\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(t)$$

Eigenvalues:

$$p(\lambda) = \begin{vmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{vmatrix} = \lambda^2 + \omega^2 = 0$$

$$\lambda^2 = -\omega^2 \longrightarrow \lambda_{1,2} = \pm\sqrt{-\omega}$$

$$\lambda = \begin{pmatrix} j\omega \\ -j\omega \end{pmatrix}$$

The natural modes are oscillatory

Controllability:

$$\det(B \;\; AB) = \begin{vmatrix} 0 & \omega \\ 1 & 0 \end{vmatrix} = -\omega \neq 0 \quad \text{ok!}$$

$\exists$ (x^o, u^o, t_f^o) unique, non singular, bang bang $\forall$ initial condition, even if $Re\{\lambda\} = 0$ since we are in the steady state case. Since the eigs are complex we can't apply the theorem about the maximum number of commutation points

To solve the problem we can apply the Pontryagin principle

$$H(x, v, 1, \lambda) = \mathcal{L} + \lambda^T f = 1 + \lambda_1(t)\omega x_2(t) - \lambda_2(t)\omega x_1(t) + \lambda_2(t) v(t)$$

Necessary conditions

$$\dot{\lambda}^\circ(t) = -\frac{\partial H}{\partial x}\bigg|^T = -A^T\lambda^\circ(t) \longrightarrow \begin{cases} \dot{\lambda}_1 = +\omega\lambda_2 \\ \dot{\lambda}_2 = -\omega\lambda_1 \end{cases}$$

$$1 + \cancel{\lambda_1^\circ(t)\,\omega x_2^\circ(t)} - \cancel{\lambda_2^\circ(t)\,\omega x_1^\circ(t)} + \lambda_2^\circ(t)\, v^\circ(t) \le$$
$$1 + \cancel{\lambda_1^\circ(t)\,\omega x_2^\circ(t)} - \cancel{\lambda_2(t)\,\omega x_1^\circ(t)} + \lambda_2^\circ(t)\, v(t)$$

$$\forall v : |v(t)| \le 1$$

$$\lambda_2^\circ(t)\, v(t) \le \lambda_2^\circ(t)\, v(t)$$

Deriving the second line for example:

$$\ddot{\lambda}_2^\circ = -\omega\dot{\lambda}_1^\circ(t) = -\omega^2\lambda_2^\circ(t)$$

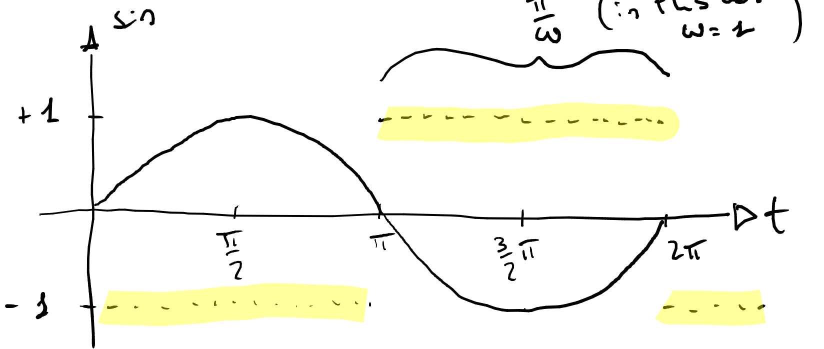
We know that:

$$\lambda_2^\circ(t) = K\sin(\omega(t - t_i) + \alpha)$$

From Pontryagin

$$v^\circ(t) = \begin{cases} -1 & \lambda_2(t) > 0 \\ 1 & \lambda_2(t) < 0 \end{cases} \Rightarrow v^\circ(t) \xrightarrow{-\text{sign}\{\lambda(t)B\}} = -\text{sign}\{\lambda_2(t)\}$$

$$= -\text{sign}\{K\sin(\omega(t - t_i) + \alpha)\}$$



The control is $-\text{sign}(\sin\ldots)$ therefore it is set as in the figure, with switches at every $\frac{\pi}{\omega}$, exception for the first and the last $\le \frac{\pi}{\omega}$

It is useful to describe the problem in the phase plane $x_1 - x_2$:

Integrating the system:

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = -\omega x_1(t) + \underbrace{u(t)}_{\to \pm 1} \end{cases}$$

$$x_1(t) = \left(x_{1i} \mp \frac{1}{\omega}\right) \cos\omega(t-t_i) + x_{2i} \sin\omega(t-t_i) \pm \frac{1}{\omega}$$

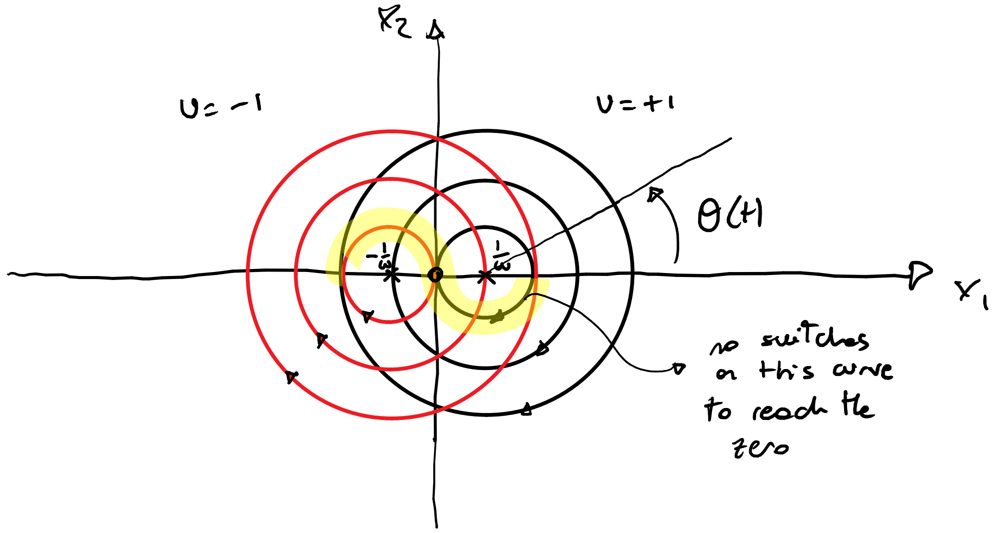
$$x_2(t) = \left(x_{1i} \mp \frac{1}{\omega}\right) \sin\omega(t-t_i) + x_{2i} \cos\omega(t-t_i)$$

$$\Downarrow \quad \cos^2\alpha + \sin^2\alpha = 1$$

$$\left(x_1(t) \mp \frac{1}{\omega}\right) + x_2^2(t) = \left(x_{1i} \mp \frac{1}{\omega}\right)^2 + (x_{2i})^2$$

The optimal trajectory is described by circumferences

$\to$ centered in $\left(\pm\frac{1}{\omega}, 0\right)$ passing through x_i

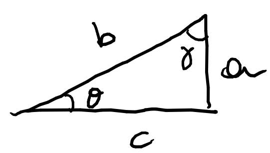


As time increase I go in the clockwise direction:

$\dot{x}_1(t) = \omega x_2(t) \to$ if $x_2 > 0$, $\dot{x}_1 > 0$ (x_1 increasing)
if $x_2 < 0$, $\dot{x}_1 < 0$ (x_1 decreasing)

The angle $\theta(t)$ can be measured with trigonometry:

$$\theta(t) = tg^{-1}\left(\frac{x_2(t)}{x_1(t) \mp \frac{1}{\omega}}\right)$$



$\sin(\theta) = \frac{a}{b}$

$\cos(\theta) = \frac{c}{b}$

$\tan(\theta) = \frac{a}{c}$

$$\dot{\theta}(t) = \dots \text{lot of calculations} \dots = -\omega \to \text{constant}$$

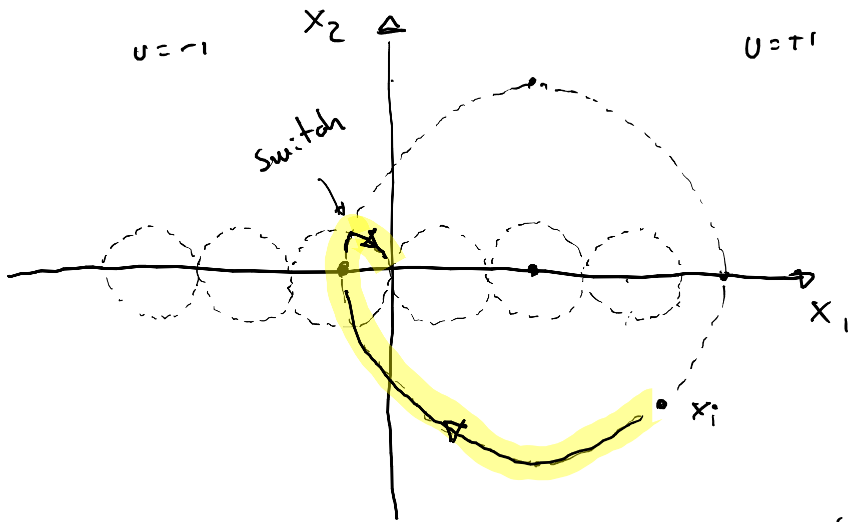
The trajectories are traversed with constant angular velocity

The time interval to move on an arc of length β with $u = \pm 1$ is given by:

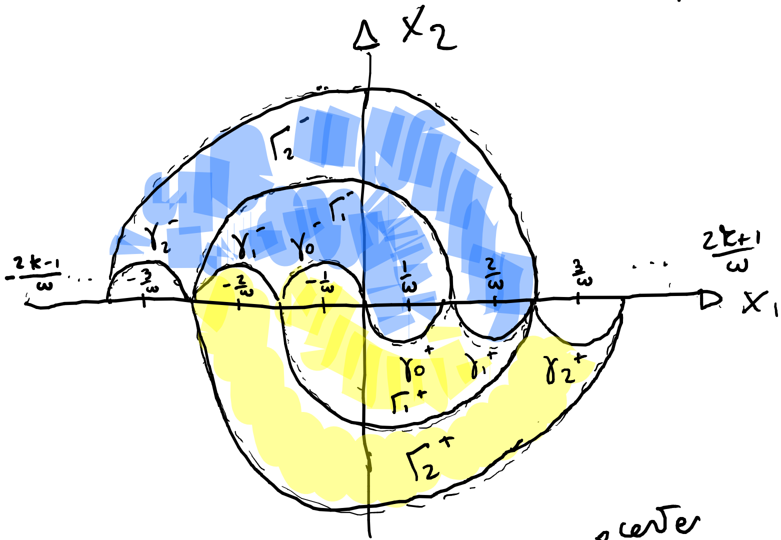
$$\boxed{\Delta t = \frac{\beta}{\omega}} \leq \frac{\pi}{\omega} \Rightarrow \boxed{\beta \leq \pi}$$

Therefore we must reach the origin along arcs with amplitude $\leq \pi$

$\Rightarrow \exists$ a <u>unique</u> way to do it



arc lenght must be $\beta \leq \pi$



center

radius

$$\gamma_k^+ = \left\{ x \in \mathbb{R}^2 : \left(x_1 - \frac{2k+1}{\omega}\right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \leq 0 \right\} \quad k = 0, \dots, n$$

$$\gamma_k^- = \left\{ x \in \mathbb{R}^2 : \left(x_1 + \frac{2k+1}{\omega}\right)^2 + x_2^2 = \frac{1}{\omega^2},\ x_2 \geq 0 \right\} \quad k = 0, \dots, n$$

Γ_k^+, Γ_k^- are the colored sets obtained rotating γ_k^+ and γ_k^- around $\left(\pm\frac{1}{\omega}, 0\right)$

~ Initial point

1) $x_i \in \gamma_0^+$ and $x_i \in \gamma_0^-$

Control $u = \pm 1$ without switches

2) $x_i \in \Gamma_1^+ \setminus (\gamma_0^+ \cup \gamma_0^-)$

First $u=+1$ to reach the curve γ_0^- with $\beta \le \pi$
then switch to $u=-1$ to reach the origin

3) $x_i \in \Gamma_1^- \setminus (\gamma_0^+ \cup \gamma_0^-)$

First $u=-1$ to reach the curve γ_0^+ with $\beta \le \pi$
then switch to $u=+1$ to reach the origin

4) $x_i \in \Gamma_k^+ \setminus \gamma^-$ or $x_i \in \Gamma_k^- \setminus \gamma^+$

$$u^\circ(x^\circ(t)) = \begin{cases} 1 & \text{if } x^\circ(t) \in \Gamma^+ \setminus \gamma^- \\ -1 & \text{if } x^\circ(t) \in \Gamma^- \setminus \gamma^+ \end{cases}$$

The number of switches is given by the minimum index k among the ones characterizing the sets Γ_k^+ and Γ_k^-

Example: $P \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^+$

1 → 1 switch

~ Minimum time

$$\begin{cases} \beta' = tg^{-1}\left(\dfrac{\omega x_{2i}}{1-\omega x_{1i}}\right) & \text{if } x_i \in \gamma_0^+ \\ \beta' = tg^{-1}\left(\dfrac{\omega x_{2i}}{1+\omega x_{1i}}\right) & \text{if } x_i \in \gamma_0^- \end{cases}$$

ν° → number of commutations

$$(t_f^\circ - t_i) = \frac{\beta'}{\omega} + (\nu^\circ - 1)\frac{\pi}{\omega} + \frac{\beta''}{\omega} \Rightarrow (t_f^\circ - t_i) = \frac{\beta'}{\omega} \text{ if } \nu^\circ = 0$$

β' rotation of the optimal trajectory from the initial point to the first point of commutation

β'' rotation to go from the final point to the origin