12. Input-Output Decoupling problem (strict non-interacting control problem)

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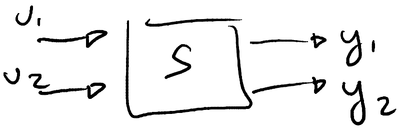
Consider a MIMO system with $p = q > 1$ ($= 2$ for simplicity)



with $\rho(B) = p = 2$
$\rho(C) = q = 2$ (square case)

We are looking for a static compensator C in the form

$$u = Fx + Gv = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^T x + (g_1 \; g_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

that makes y_1 independent from v_2 and
y_2 independent from v_1

Def: notion of vector relative degree

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \Rightarrow \text{each } r_i \text{ is the relative degree associated to the } y_i \text{ output}$$

$$r_i : \begin{cases} c_i A^k b_j = 0 & k = 0, \dots, r-2 \\ c_i A^k b_j \neq 0 & k = r-1 \end{cases}$$

where $B = (b_1 \; b_2)$ $C = (c_1 \; c_2)$

r_i is the differential delay between u and y

$$\begin{cases} y = Cx \\ \dot{y} = CAx + CBu \end{cases} \rightarrow r = 1$$

$$\begin{cases} y = Cx \\ \dot{y} = CAx \\ \ddot{y} = CAx + CABu \end{cases} \rightarrow r = 2$$

Def: Decoupling matrix

$$U = \begin{pmatrix} c_1 A^{r_1-1} b_1 & c_1 A^{r_1-1} b_2 \\ \hline c_2 A^{r_2-1} b_1 & c_2 A^{r_2-1} b_2 \end{pmatrix}$$

The system is said to have strong relative degree if $\det(U) \neq 0$, i.e. the decoupling matrix is invertible and non singular

Under the previous condition:

$$V^* = \ker \begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \\ \hdashline c_2 \\ \vdots \\ c_2 A^{r_2-1} \end{pmatrix} \begin{matrix} \}\, r_1 \\ \\ \}\, r_2 \end{matrix} \qquad u = U^{-1}\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} c_1 A^{r_1-1} \\ c_2 A^{r_2-1} \end{pmatrix} \right)$$

where the feedback is $F^* = -U^{-1}\begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix}$

and $G^* = U^{-1}$

Theorem: Let r be defined, the SNCP problem is solvable iff $\det(U) \neq 0$, i.e. the system has strong relative degree and

$$u = Fx + Gv = -U^{-1}\begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix} x + U^{-1} v$$

Proof (Nec) $\leftarrow$: suppose $\exists\, F, G$ which solve the IOS

$$w(t) = c e^{(A+BF)t} BG = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} c_1 (A+BF)^k BG = (* \ \ 0) \\ c_2 (A+BF)^s BG = (0 \ \ *) \end{cases} \quad \forall k, s$$

$$\Rightarrow \begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \end{pmatrix} G = UG = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

since UG is diagonal, $|U| \neq 0$

Proof (suff.) $\rightarrow$: Suppose $\exists\, \det(U) \neq 0$, (F, G) solve the SNCP

$\exists\, T$ coordinate transformation

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \\ \hline c_2 \\ \vdots \\ c_2 A^{r_2-1} \\ \hline T_3 \end{pmatrix} \qquad \det(T) \neq 0$$

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \\ \hline c_2 \\ \vdots \\ c_2 A^{r_2-1} \\ \hline T_3 \end{pmatrix} \qquad \det(T) \neq 0$$

and looking at the feedback system:

$$S_F : \begin{cases} \dot{x} = (A+BF)x + BGv \\ y = Cx \end{cases}$$

with $F = -U^{-1} \begin{pmatrix} c_1 A^{r_1-1} \\ c_2 A^{r_2-1} \end{pmatrix}$ $\quad G = U^{-1}$

$\tilde{A} = T(A+BF)T^{-1} \qquad \tilde{B} = TBG \qquad \tilde{C} = CT^{-1}$

which shows how the SNCP problem is solved

$$\tilde{C} = CT^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 - & 0 & 0 & - & 0 \\ 0 & - & 0 & 1 & - & 0 \end{array}\right)$$

$$\tilde{B} = TBG = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \cdot B \cdot \left(\begin{pmatrix} c_1 A^{r_1-1} \\ c_2 A^{r_2-1} \end{pmatrix} B \right)^{-1} =$$

$$= \begin{pmatrix} 0 \\ | \\ 0 \\ c_1 A^{r_1-1} B \\ \hline 0 \\ | \\ 0 \\ c_2 A^{r_2-1} B \\ \hline T_3 B \end{pmatrix} \begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ | & | \\ 1 & 0 \\ \hline 0 & 0 \\ | & | \\ 0 & 1 \\ \hline \multicolumn{2}{c}{\tilde{B}_3} \end{pmatrix} = \tilde{B}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} (A+BF) = \tilde{A} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1 & & 0 \\ | & & \ddots & 1 \\ 0 & - & & 0 \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & & \\ & & \ddots & 1 \\ 0 & - & & 0 \end{matrix} & 0 \\ \hline \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{array}\right) \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

with $T_1 = \begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \end{pmatrix} \qquad T_2 = \begin{pmatrix} c_2 \\ \vdots \\ c_2 A^{r_2-1} \end{pmatrix} \qquad \rho(T_1) = r_1 \qquad \rho(T_2) = r_2$

and T_3 (with $\dim(T_3) = n - r_1 - r_2$) chosen s.t. $|T| \neq 0$ in the new coordinates the following results are obtained:

$$\begin{pmatrix} z_1 \\ z_2 \\ \eta \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} x$$

$$z_1 = T_1 x = \begin{cases} z_{11} = C_1 x = y_1 \\ z_{12} = C_1 A x = \dot{y}_1 \\ \dots \\ z_{1 r_1} = C_1 A^{r_1-1} x = y_1^{(r_1-1)} \end{cases}$$

$$\rightarrow \dot{z}_1 = \begin{cases} \dot{z}_{11} = C_1 x = z_{12} \\ \dot{z}_{12} = z_{13} \\ \dots \\ \dot{z}_{1,r_1-1} = z_{1,r_1} \\ \dot{z}_{1,r_1} = C_1 A^{r_1} x + C_1 A^{r_1-1} B u = v_1 \end{cases} \qquad \rightarrow \dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{b}_1 v_1$$

$$z_2 = T_2 x = \begin{cases} z_{21} = C_2 x = y_2 \\ z_{22} = C_2 A x = \dot{y}_2 \\ \dots \\ z_{2, r_2} = C_2 A^{r_2-1} x = y_2^{(r_2-1)} \end{cases}$$

$$\rightarrow \dot{z}_2 = \begin{cases} \dot{z}_{21} = C_2 A x = z_{22} \\ \dot{z}_{22} = C_2 A^2 x = z_{23} \\ \dots \\ \dot{z}_{2,r_2-1} = z_{2,r_2} \\ \dot{z}_{2,r_2} = C_2 A^{r_2} x + C_2 A^{r_2-1} B u = v_2 \end{cases} \qquad \rightarrow \dot{z}_2 = \tilde{A}_{22} z_2 + \tilde{b}_2 v_2$$

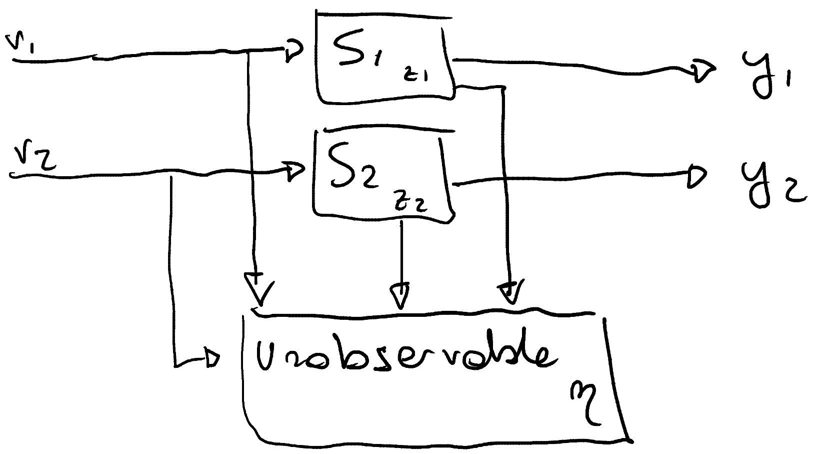
$$\eta = T_3 x$$

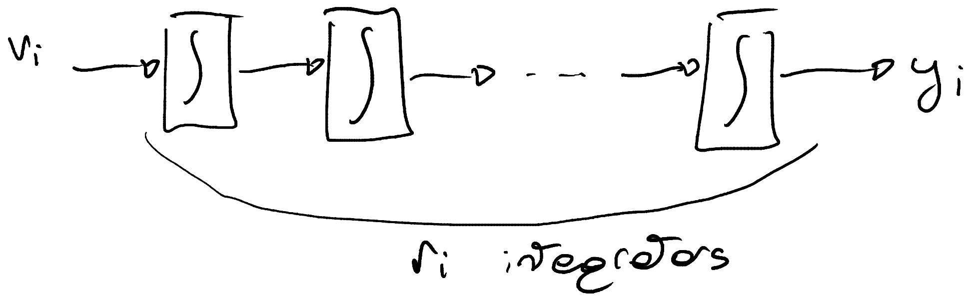
$$\rightarrow \dot{\eta} = T_3 \dot{x} = T_3 A x + T_3 B u = \tilde{A}_{31} z_1 + \tilde{A}_{32} z_2 + \tilde{A}_{33} \eta + \tilde{B}_3 v$$

$$y_1 = (1 0 \dots 0) z_1$$

$$y_2 = (1 0 \dots 0) z_2$$

the feedback system has some Brunowsky Canonical blocks, i.e. the system is a cascade of integrators





$$W(s) = \begin{pmatrix} \frac{1}{s^{r_1}} & 0 \\ 0 & \frac{1}{s^{r_2}} \end{pmatrix} \leftarrow \text{decoupling: only diagonal terms!}$$

Moreover, at each channel can be assigned the dynamics through v. Since both channels are decoupled (independent), eigenvalues can be assigned:

Recalling $u = -U^{-1}\left(\begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix} x + v\right)$

it's possible to set:

$$v_1 = -\sum_{i=0}^{r_1} \alpha_{1,i} \cdot z_{1,i+1} + w_1 \quad \text{with } z_{1,i} = c_1 A^{i-1} x$$

$$\dot{z}_1 = \begin{pmatrix} 0 & 1 & & 0 \\ | & & \ddots & 1 \\ -\alpha_0^* & \cdots & & -\alpha^*_{r_1-1} \end{pmatrix} z_1 + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v_1$$

$$W_1(s) = \frac{1}{\alpha_0^* + \dots + \alpha^*_{r_1-1} s^{r_1-1} + s^{r_1}}$$

$$u = U^{-1}\left[\left(-\begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix} - \begin{pmatrix} \sum_{i=0}^{r_1-1} \alpha_{1,i}\, c_1 A^i \\ \sum_{i=0}^{r_2-1} \alpha_{2,i}\, c_2 A^i \end{pmatrix}\right) x + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right]$$

$$\left[\begin{pmatrix} \vdots \\ c_2 A^{\cdot} \end{pmatrix} \quad \begin{pmatrix} \vdots \\ \sum_{i=0}^{r_2-1} \alpha_{2,i}\, c_2 A^i \end{pmatrix} \quad \begin{pmatrix} \vdots \\ w_2 \end{pmatrix} \right]$$

$$= U^{-1}\left(- \begin{pmatrix} c_1 P_1^*(A) \\ c_2 P_2^*(A) \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right)$$

by using this feedback law both problems of input/output decoupling and of eigenvalues assignment can be solved.

Looking at the solution from a different point of view, computing the derivatives of the output:

$$y_1 = c_1 x \qquad\qquad y_2 = c_2 x$$

$$y_1^{(1)} = c_1 A x + \underset{\to 0}{c_1 B}\, u \qquad\qquad y_2^{(1)} = c_2 A x$$

$$\cdots \qquad\qquad \cdots$$

$$y^{(r_1)} = c_1 A^{r_1} x + c_1 A^{r_1-1} B u \qquad\qquad y^{(r_2)} = c_2 A^{r_2} x + c_2 A^{r_2-1} B u$$

and using

$$u = -U^{-1}\begin{pmatrix} c_1 A^{r_1-1} \\ c_2 A^{r_2-1} \end{pmatrix} x + B U^{-1} = Fx + Gv$$

one obtains:

$$\begin{pmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \end{pmatrix} = \begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix} x + \begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \end{pmatrix} u =$$

$$= \cancel{\begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \end{pmatrix} x} + \cancel{\underbrace{\begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \end{pmatrix}}_{U^{-1}}} \cancel{\underbrace{\begin{pmatrix} -c_1 A^{r_1-1} B \\ -c_2 A^{r_2-1} B \end{pmatrix}}_{-U^{-1}}} \cancel{\begin{pmatrix} c_1 A^{r_1-1} \\ c_2 A^{r_2-1} \end{pmatrix} x} + U B U^{-1} =$$

$$= U B U^{-1} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The feedback which achieve I-O decoupling (SUC) is nothing else than the feedback which maximizes the unobservability of S_F:

$$V^* = \ker \begin{pmatrix} c_1 \\ \vdots \\ c_1 A^{r_1-1} \\ c_2 \\ \vdots \\ c_2 A^{r_2-1} \end{pmatrix} \qquad F^* \text{ friend of } V^*$$

Like in the DDP case, the idea is to obtain unobservability under feedback.

F^* is the feedback which maximizes the unobservability of the controlled system.
The maximal unobservability is the one obtained by deleting stable zeros and it can be generated in order to solve DDP and to guarantee the stability of the controlled system

For what concerns the feasibility of the computed feedback (F^*, G), it must result that
$\sigma(\tilde{A}_{33}) = \sigma(A+BF)\big|_{V^*} \subset \mathbb{C}^-$, in fact feasibility depends on the "transmission zeros" of the square MIMO plant.