Given $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

$y_1 = h_1(x)$

...

$y_m = h_m(x)$

supposing $m = 2$, find a feedback law:

$$u_i = \alpha_i + \sum_{j=1}^{m} \beta_{ij}(x) v_j$$

defined on U of x_0, with $\alpha_i(0) = 0$, such that the closed loop system has vector relative degree well defined at equilibrium point $x = 0$ and for each $i \leq 1 \leq m$, the output y_i is affected by v_i.

Prop: N&S condition for NIC solvability is

$$\det(A(x)) \neq 0$$

i.e., the system has strong vector relative degree.

Proof: suppose r_i i-th relative degree of S

$\bar{r}_i$ i-th relative degree of S_F

and suppose first that $\bar{r}_i \geq r_i$.

From the definition of r_i

$$L_{(g\beta)_j} L_{f+g\alpha}^{k} h = 0 \quad \forall k < r_i - 1 \quad \forall j$$

if the problem is solved under $\alpha + \beta v$ the closed loop has for $\forall i$ a $\bar{r}_i$ defined and $\bar{r}_i \geq r_i$, therefore r_i is defined.

Suppose now that $\bar{r}_i > r_i$, then

$$\left(L_{(g\beta)_1} L_{f+g\alpha}^{r_i-1} h_i \; \dots \; L_{(g\beta)_m} L_{f+g\alpha}^{r_i-1} h_i \right) =$$

$$= \left(L_{g_1} L_f^{r_i-1} h_i \; \dots \; L_{g_m} L_f^{r_i-1} h_i \right) \beta = 0$$

$$= \left(L_{g_1} L_f^{r_i-1} h_i \;\cdots\; L_{g_m} L_f^{r_i-1} h_i\right)\beta = 0$$

this implies $\rho(\beta(0)) < m$.

In fact if $\rho(\beta(0)) = m$ then $\rho(\beta(x)) = m$ around $x_0 = 0$ and this contradicts $(L_{g_1} L_f^{r_i-1} h_i \ldots L_{g_m} L_f^{r_i-1} h_i) \neq 0$.

Thus $\rho(\beta(0)) < m \longrightarrow \rho(g(0)\beta(0)) < m$

Recall that, if the closed-loop system has strong vector relative degree, the matrix

$$\bar{A}(x) = \begin{pmatrix} L_{f+g\alpha}^{\bar{r}_1-1} h_1(x) \\ \cdots \\ L_{f+g\alpha}^{\bar{r}_m-1} h_m(x) \end{pmatrix} g(x)\,\beta(x)$$

is non singular at $x=0$. But this contradicts the fact that $\rho(g(0)\beta(0)) < m$, therefore $\bar{r}_i = r_i$ and it follows that

$\bar{A}(x) = A(x)\beta(x)$

from which it is deduced that $A(x)$ and $\beta(x)$ are nonsingular at $x=0$. ◀

This fact motivates the definition of the matrix $A(x)$ as the decoupling matrix.

All the previous concepts can be easily extended to the case of m inputs and q outputs with $m > q$. In the case of NIC the problem can be formulated in terms of existence of a partition of the inputs $(v_1 \ldots v_q)$ so that in the feedback the i-th element of the partition influences the i-th output.

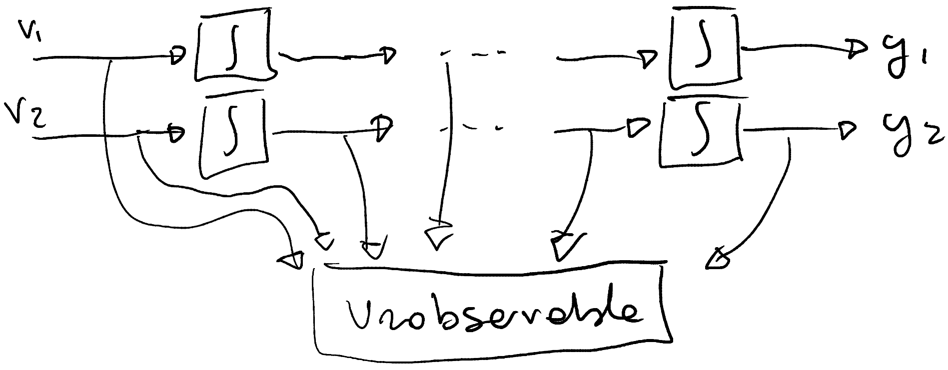
Remarks: Notice that, if $r_1 + r_2 < n$, then there exists a zero dynamics. The zero dynamics can be defined as the "residual dynamics" under which the system evolves when the input is:

$u = -A^{-1}(x)\, b(x)$

and $x_0 \in Z^* = \{x \in \mathbb{R}^n : h_1(x) = \ldots = L_f^{r_1-1} h_1(x) = h_2(x) = \ldots = L_f^{r_2-1} h_2(x) = 0\}$

The stability of the zero dynamics is a necessary condition

The stability of the zero dynamics is a necessary condition for the application of any feedback which renders it unobservable



$$W(s) = \begin{pmatrix} 1/s^{r_1} & 0 \\ 0 & \frac{1}{s^{r_2}} \end{pmatrix}$$