# 19. Invariant distributions

martedì 9 giugno 2020 16:28

$\Delta$ invariant under the vectorfield $f$ if:

$\forall \tau \in \Delta \Rightarrow [f, \tau] \in \Delta$ or $[f, \Delta] \subset \Delta$

if $\tau_i$ $i = 1 \dots d$ are vector fields that locally span $\Delta$ we can say that $\Delta$ is invariant under $f$ if:

$[f, \tau_i] \in \Delta$ for $i = 1, \dots, d$

making a coordinate transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} \begin{matrix} \}d \\ \}n-d \end{matrix} \qquad \text{with } \phi_2(x) : \frac{\partial \phi_2}{\partial x} \cdot \Delta = 0$$

I obtain a sys of this form

$$\dot x = f(x) \xrightarrow{\phi(x)} \begin{cases} \dot z_1 = f_1(z_1, z_2) & \}d \\ \dot z_2 = f_2(z_2) & \}n-d \end{cases}$$

$z = \phi(x)$ $x = \phi^{-1}(z)$

## Importance of invariance

In linear systems, if the invariance property holds, an LTI system can be represented in different ways using a linear transformation which links the old state vector $x$ with the new $z$

$z = Tx$ $T$: square, non-sing

$\tilde A = TAT^{-1}$ $\tilde B = T^{-1}B$ $\tilde C = CT$

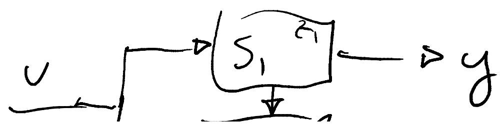
$\tilde A, \tilde B, \tilde C$ maintain the same geometrical properties of $A, B, C$ in terms of stability, observability, reachability

Also the eigenvalues of $\tilde A$ remain the same.

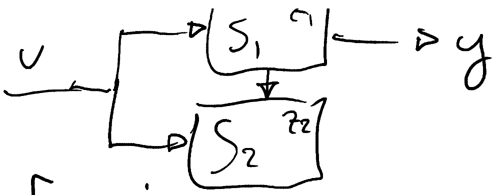
- OBSERVABILITY decomposition

$$\tilde A = \begin{bmatrix} \tilde A_{11} & 0 \\ \tilde A_{21} & \tilde A_{22} \end{bmatrix} \qquad \tilde B = \begin{bmatrix} \tilde B_1 \\ \tilde B_2 \end{bmatrix} \qquad \tilde C = [\tilde C_1 \ 0]$$

$\dot z_1 = \tilde A_{11} z_1 + \tilde B_1 u$



$$\begin{cases} \dot z_1 = \tilde A_{11} z_1 + \tilde B_1 u \\ \dot z_2 = \tilde A_{21} z_1 + \tilde A_{22} z_2 + \tilde B_2 u \\ y = \tilde C_1 z_1 \end{cases}$$

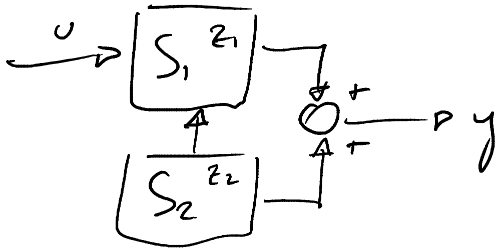



choosing $T = \left[\, \forall \;\vdots\; \text{base of } \mathcal{V} \,\right]$

- REACHABILITY decomposition

$$\hat A = \begin{bmatrix} \tilde A_{11} & \tilde A_{12} \\ 0 & \tilde A_{22} \end{bmatrix} \qquad \tilde B = \begin{bmatrix} \tilde B_1 \\ 0 \end{bmatrix} \qquad \tilde C = \begin{bmatrix} \tilde C_1 & \tilde C_2 \end{bmatrix}$$

$$\begin{cases} \dot z_1 = \tilde A_{11} z_1 + \tilde A_{12} z_2 + \tilde B_1 u \\ \dot z_2 = \tilde A_{22} z_2 \\ y = \tilde C_1 z_1 + \tilde C_2 z_2 \end{cases}$$




choosing $T = \left[\, \text{base of } \mathcal{V} \;\vdots\; \forall \,\right]$

The same reasoning can be extended to the non linear case
In fact if:

<u>a</u> $\Delta$ non singular distribution of dim = d

<u>b</u> $\Delta$ involutive

<u>c</u> $\Delta$ invariant under $f$

Then, at each point $x^\circ$ exists a coordinate transformation

$$z = \Phi(x)$$

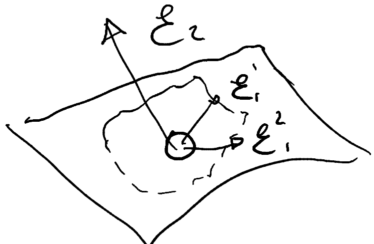
defined on a neighborhood of $x^\circ$, in which $f$ is represented by:

$$\bar f(z) = \begin{bmatrix} \bar f_1(z_1, \dots, z_d, z_{d+1}, \dots, z_n) \\ \dots \\ \bar f_d(z_1, \dots, z_d, z_{d+1}, \dots, z_n) \\ \bar f_{d+1}(z_{d+1}, \dots, z_n) \\ \dots \\ \bar f_n(z_{d+1}, \dots, z_n) \end{bmatrix}$$

defining: $\mathcal{Z}_1 = (z_1, \dots, z_d)$

$$\xi_2 = (z_{d+1}, \dots, z_n)$$

It is possible to obtain the same triangular decomposition as in the linear case




$$\begin{cases} \dot{\xi}_1 = f_1(\xi_1, \xi_2) \\ \dot{\xi}_2 = f_2(\xi_2) \end{cases}$$

With this decomposition the evolution of the system on the $\xi_2$ direction does not depend on the control input

$$\begin{cases} \dot{\xi}_1 = f_1(\xi_1, \xi_2) + g(\xi_1, \xi_2)u \\ \dot{\xi}_2 = f_2(\xi_2) \\ y = h(\xi_1, \xi_2) \end{cases}$$

and the Jacobian has an upper triangular structure