# 16_12.pdf

sabato 18 luglio 2020 13:19

16_12.pdf

ESAMI

**Nonlinear Systems & Control**
Part II
22/12/16

**Student:** ______________________

**Student ID:** ______________________

---

1. Discuss about the noninteracting control (with stability) for linear time invariant systems with two inputs and two outputs.
2. Given

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u + Dd \quad d \in \mathbb{R}$$
$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

characterise the column vector $D$ which specifies the action of disturbances which can be "decoupled" under feedback.

3. Given the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + x_1^2 \\ \dot{x}_3 &= -x_2 + x_1^2 + (1 + x_1^2)u \\ y &= -x_1 + x_3 + x_1^2 \end{aligned}$$

   **a.** Compute the feedback that solves the input-output linearization problem;
   **b.** Compute the zero dynamics; ?
   **c.** Discuss the stability of the closed-loop system. ??

4. The dynamic decoupling algorithm (the case $p = q = 2$).
5. Compute, a feedback yielding global asymptotic stability of the equilibrium of the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - \cos(x_1)x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

6. Given a nonlinear system in input-affine form, discuss about the asymptotic tracking problem via feedback linearization with respect to a bounded reference output $y_r(t)$.

① Characterize the vector D which specifies the action of disturbances which can be decoupled under FB

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u + Dd \qquad d \in \mathbb{R}$$

$$y = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x$$

<u>1. step</u>: compute the relative degree

It's a linear system so $r$ is the 1st integer s.t. $CA^{r-1}B \neq 0$

$r=1 \quad CB = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$

It's a linear system so $r$ is the 1st integer s.t. $CA^{r-1}B \neq 0$

$r=1 \quad CB = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$

$r=2 \quad CAB = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \neq 0$

2. step: compute $V^*$

$$V^* = \ker \begin{pmatrix} C \\ \cdots \\ CA^{r-1} \end{pmatrix} = \ker \begin{pmatrix} C \\ CA \end{pmatrix} = \ker \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{cases} x_3 = 0 \\ x_2 = 0 \\ x_1 = x_1 \end{cases} \to \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \quad \text{I set } x_1 = 1$$

$$V^* = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

For any $D$ s.t. $\operatorname{Im} D \subset \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ the DDP admits a solution, and from this I know that the DDP admits a solution $\forall D$ s.t.

$$D(s) = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} \quad s \in \mathbb{R}$$

---

## 3 Compute the FB that solves the I/O linearization problem

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 + x_1^2 \\ \dot{x}_3 = -x_2 + x_1^2 + (1+x_1^2)u \\ y = -x_1 + x_3 + x_1^2 \end{cases}$$

$$f = \begin{pmatrix} x_2 \\ x_3 + x_1^2 \\ -x_2 + x_1^2 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix}$$

$$dh = \begin{pmatrix} -1+2x_1 & 0 & 1 \end{pmatrix}$$

Compute the rel. degree

1-st integer s.t. $L_g L_f^{r-1} h \neq 0$

$$r=1 \quad L_g h = \frac{\partial h}{\partial x} g = \begin{pmatrix} -1+2x_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix} = 1+x_1^2 \neq 0$$

Calculate the derivative of the output:

$$\dot{y} = -\dot{x}_1 + \dot{x}_3 + \dot{x}_1^2 = -x_2 - x_2 + x_1^2 + (1+x_1^2)u + 2x_1 x_2$$

$(\dot{x_1^2})$ composite function $= 2x_1 \cdot \dot{x}_1 = 2x_1 x_2$

$$u = \frac{x_2 + x_2 - x_1^2 - 2x_1x_2}{1+x_1^2} = \frac{2x_2 - x_1^2 - 2x_1x_2}{1+x_1^2} \Rightarrow \left( = -\frac{L_f^r h}{L_g L_f^{r-1} h} = \frac{-L_f h}{L_g h} \right)$$

Now compute the normal form

$\begin{pmatrix} z \\ \eta \end{pmatrix} \begin{matrix} r=1 \\ n-r=2 \end{matrix}$ coord. transformation

$$z = \begin{pmatrix} h \\ \cdots \\ L_f^{r-1} h \end{pmatrix} = h = -x_1 + x_3 + x_1^2$$

I choose $\eta_1$ s.t. the non linearities are avoided $\Rightarrow \eta_1 = x_1$

I choose $\eta_2$ s.t. $\nabla\varphi_2\, g = 0$

$$\begin{pmatrix} \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1+x_1^2 \end{pmatrix} = 0$$

$$\frac{\partial \varphi_2}{\partial x_3}(1+x_1^2) = 0 \Rightarrow \frac{\partial \varphi_2}{\partial x_3} = -\frac{\partial \varphi_2}{\partial x_3} x_1^2 \quad \text{if } \varphi_2 = x_2 \Rightarrow 0 = 0$$

$\eta_2 = \varphi_2$

And so, the coord. trasf:

$$z = -x_1 + x_3 + x_1^2 \quad \text{but } \eta_1 = x_1 \Rightarrow \begin{cases} z = -\eta_1 + x_3 + \eta_1^2 \\ \eta_1 = x_1 \\ \eta_2 = x_2 \end{cases}$$

$$u = -\frac{L_f^r h}{L_g L_f^{r-1} h} + \frac{v}{L_f^{r-1} L_g h} = \frac{-L_f h}{L_g h} + \frac{v}{L_g h} = \frac{2x_2 - x_1^2 - 2x_1x_2 + v}{1+x_1^2}$$

Applying this FB directly I obtain the normal form

$$\begin{cases} \dot{z} = -\dot{x}_1 + \dot{x}_3 + \dot{x}_1^2 = -\cancel{2x_2} + \cancel{x_1^2} + \cancel{(1+x_1^2)}\left(\frac{\cancel{2x_2} - \cancel{x_1^2} - \cancel{2x_1x_2} + v}{\cancel{(1+x_1^2)}}\right) + \cancel{2x_1x_2} = v \\ \dot{} \quad \dot{} \end{cases}$$

$$
\begin{cases}
\dot z = -\dot x_1 + \dot x_3 + \dot x_1^2 = -\cancel{(x_2 + x_1)} + \cancel{(1+x_1)}\left( - \dfrac{\cdots}{\cancel{(1+x_1^2)}} \right) + 2\cancel{x_1}x_2 = v \\
\dot\eta_1 = \dot x_1 = x_2 = \eta_2 \\
\dot\eta_2 = \dot x_2 = x_3 + x_1^2 = z + \eta_1 - \cancel{\eta_1^2} + \cancel{\eta_1^2} = z + \eta_1 \\
y = z
\end{cases}
$$

## Complete the zero dynamics

$Z = \{ x \in \mathbb{R}^n : y(t_0) \equiv 0 \quad u^*(x) \text{ s.t. } y(t) \equiv 0 \} =$

$= \{ x \in \mathbb{R}^n \text{ s.t. } -x_1 + x_3 + x_1^2 = 0 \}$ if $z = 0 \Rightarrow y = 0$

$\dot y = 0$ gives me the $u$ that assures that I'm in the zero dynamics, so from the normal form I obtain the zero dynamics $\dot\eta = q(0, \eta)$

$$
\begin{cases}
\dot\eta_1 = \eta_2 \\
\dot\eta_2 = \eta_1
\end{cases}
\quad \text{mantaining } z = 0
$$

## Discuss the stability of the closed loop sys.

LTM for $q(0,\eta) \Rightarrow Q = \left.\dfrac{\partial q(0,\eta)}{\partial \eta}\right|_{\eta=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{1}$

unstable zero dynamics

---

## 5 Compute a FB yelding GAS

$$
\begin{cases}
\dot x_1 = -x_1 - \cos(x_1) x_2 \\
\dot x_2 = u
\end{cases}
$$

1 $x_2 = \gamma(x_1)$ s.t. $\dot V(x_1) < 0$

$V(x_1) = \frac{1}{2} x_1^2 \Rightarrow \dot V(x_1) = \dot x_1 x_1 = -x_1^2 - \cos(x_1)\gamma(x_1)x_1$

$\gamma(x_1) = 0 \Rightarrow \dot V(x_1) = -x_1^2 < 0 \Rightarrow$ GAS

$V(x_1) = \frac{1}{2} x_1^2 \Rightarrow \dot{V}(x_1) = x_1 \dot{x}_1 = -x_1^2 - \cos(x_1)\,\gamma(x_1)\,x_1$

$\gamma(x_1) = 0 \Rightarrow \dot{V}(x_1) = -x_1^2 < 0 \Rightarrow \text{GAS}$

Given dynamics which are characterized by an affine structure

$$\begin{cases} \dot{z} = f(z) + g(z)\xi \\ \dot{\xi} = b(z,\xi) + a(z,\xi)u \end{cases}$$

suppose $\exists\, \gamma(z)$, $V$ positive and proper such that

$$\frac{\partial V}{\partial z}(f + g\gamma) < 0 \quad \forall z \neq 0$$

then $u = a^{-1}(z,\xi)\left(-b(z,\xi) - \frac{\partial V}{\partial z} g(z) + \dot{\gamma}(z) + v\right)$

with $\dot{\gamma} = \frac{\partial \gamma}{\partial z}(f + g\gamma) = \frac{\partial V}{\partial z}\dot{z}$

renders $x_e = 0$ GAS.

Since $x_2 = 0$, $x_2 = c$ so in change of coord.

$$\begin{cases} \dot{x}_1 = -x_1 - \cos(x_1)x_2 \\ \dot{x}_2 = u \end{cases}$$

$$V_2(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V}_2 = \dot{x}_1 x_1 + \dot{x}_2 x_2 = -x_1^2 - x_1\cos(x_1)x_2 + x_2 u$$

Picking by inspection $u = x_1\cos(x_1) - x_2$

one obtains

$$\dot{V}_2 = \underbrace{-x_1^2}_{<0} - \cancel{x_1\cos(x_1)x_2} + \cancel{x_1\cos(x_1)x_2} \underbrace{-x_2^2}_{<0}$$

Therefore $u$ achieves GAS