Distributions

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Given the distribution

$$\Delta(x) = \text{span}\left\{\begin{pmatrix}1\\0\\x_1\end{pmatrix}, \begin{pmatrix}x_2x_3\\1\\0\end{pmatrix}\right\}$$ verify wether it's integrable

remarks → Frobenius theorem:

$\Delta(x)$ integrable $\Leftrightarrow$ involutive

$$\tau_1(x) = \begin{pmatrix}1\\0\\x_1\end{pmatrix}$$

$$\tau_2(x) = \begin{pmatrix}x_2x_3\\1\\0\end{pmatrix}$$

$\Delta(x)$ is involutive iff $[\tau_1(x), \tau_2(x)] \in \Delta(x)$

$$[\tau_1(x), \tau_2(x)] = \frac{\partial \tau_2}{\partial x}\tau_1(x) - \frac{\partial \tau_1}{\partial x}\tau_2(x)$$

$$\begin{pmatrix}0 & x_3 & x_2\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\x_1\end{pmatrix} - \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_2x_3\\1\\0\end{pmatrix} =$$

$$\begin{pmatrix}x_1x_2\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\x_2x_3\end{pmatrix} = \begin{pmatrix}x_1x_2\\0\\-x_2x_3\end{pmatrix}$$

$$T(x) = \left[\ \tau_1(x)\ \vdots\ \tau_2(x)\ \vdots\ [\tau_1(x), \tau_2(x)]\ \right]$$

$\det T(x) = 0$?

If this condition is verified, there is a rank drop. This means that $[\tau_1(x), \tau_2(x)]$ satisfy a new direction $\rightarrow [\tau_1(x), \tau_2(x)] \in \Delta(x)$

$$T(x) = \begin{pmatrix}1 & x_2x_3 & x_1x_2\\0 & 1 & 0\\x_1 & 0 & -x_2x_3\end{pmatrix} = -x_2x_3 - x_1^2x_2 \neq 0 \quad \text{rank}\{T(x)\} = 3$$

$$[\tau_1(x), \tau_2(x)] \notin \Delta(x)$$

$\Delta(x)$ not involutive $\Rightarrow$ $\Delta(x)$ not integrable

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$$\Delta(x) = \left\{\begin{pmatrix}x_1\\-x_2\\0\end{pmatrix}\begin{pmatrix}x_1\\0\\-x_3\end{pmatrix}\right\}$$ verify the integrability

$$[\tau_1(x), \tau_2(x)] = \frac{\partial \tau_2}{\partial x}\tau_1 - \frac{\partial \tau_1}{\partial x}\tau_2 =$$

$$= \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\-x_2\\0\end{pmatrix} - \begin{pmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\0\\-x_3\end{pmatrix} =$$

$$= \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = 0 \quad \Rightarrow \text{commutative}$$

$$[\tau_1, \tau_2] \in \Delta(x)$$

$\Rightarrow$ involutive and integrable

$$\Delta(x) = \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_{n-k}(x) \end{pmatrix} = 0 \qquad \begin{matrix} n = 3 \\ k = \text{rank}\{\Delta(x)\} = 2 \end{matrix}$$

$$\forall \tau_i \in \Delta(x) \Rightarrow \frac{\partial \Lambda(x)}{\partial x} \tau_i(x) = 0$$

we are looking for $n-k = 1$ functions

$$\lambda(x) : \begin{cases} \frac{\partial \lambda}{\partial x} \tau_1(x) = 0 \\ \frac{\partial \lambda}{\partial x} \tau_2(x) = 0 \end{cases}$$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_1 \\ -x_2 \\ 0 \end{pmatrix} = 0 \Rightarrow \frac{\partial \lambda}{\partial x_1} x_1 - \frac{\partial \lambda}{\partial x_2} x_2 = 0$$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ -x_3 \end{pmatrix} = 0 \Rightarrow \frac{\partial \lambda}{\partial x_1} x_1 - \frac{\partial \lambda}{\partial x_3} x_3 = 0$$

$\lambda(x) = f(x_1) x_2 x_3$ solution with f differentiable $\frac{\partial f(x_1)}{\partial x_1} \neq 0$

$$\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases}$$

$\Delta(x)$ is invariant wrt the dynamics f

$x_0 \in \Delta(x) \qquad x(t) \in \Delta(x) \qquad \forall t \geq 0$

$$f(x) = \begin{pmatrix} x_3 \\ x_2 \\ x_3 x_4 - x_1 x_2 x_3 \\ \sin x_3 + x_2^2 + x_1 + x_3 \end{pmatrix} \qquad \Delta(x) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ x_2 \end{pmatrix} \right\}$$

$$\text{rank}\{\Delta(x)\} = \dim(\Delta(x)) = 2 = k$$

(A) Check the integrability (Frobenius)

$$[\tau_1(x), \tau_2(x)] = \frac{\partial \tau_2}{\partial x} \tau_1 - \frac{\partial \tau_1}{\partial x} \tau_2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} - \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} = 0 \quad \text{commutative } [\tau_1(x), \tau_2(x)] \in \Delta(x)$$

$\Rightarrow$ Involutive & integrable

$\boxed{B}$ Check invariance

$$[f, \tau_1(x)] \in \Delta(x) \quad \& \quad [f, \tau_2(x)] \in \Delta(x)$$

$$[f, \tau_1(x)] = \frac{\partial \tau_1}{\partial x} f - \frac{\partial f}{\partial x} \tau_1$$

$$= \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix} \begin{pmatrix} * \\ * \\ x_3x_4 - x_1x_2x_3 \\ * \end{pmatrix} - \begin{pmatrix} 0 & * & * & 0\\ 0 & * & * & 0 \\ -x_2x_3 & * & * & x_3 \\ x_3 & * & * & 0 \end{pmatrix} \begin{pmatrix} 1\\0\\0\\x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0\\0\\0\\ x_3x_4 - x_1x_2x_3 \end{pmatrix} - \begin{pmatrix} 0\\0\\ -x_2x_3 + x_3^2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0\\0\\ +x_2x_3 - x_3^2 \\ x_3x_4 - x_1x_2x_3 - x_3 \end{pmatrix}$$

$$T(x) = \left[\tau_1(x), \tau_2(x), [f, \tau_1]\right]$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \\ x_3 & x_2 & * \end{pmatrix}$$

$$\text{rank}\{T(x)\} = 3 \neq 2 \Rightarrow [f, \tau_1(x)] \notin \Delta(x)$$

$\Delta(x)$ is not f-invariant, $[f, \tau_1]$ define a new direction.

find $n-k=2$ functions λ_1, λ_2 solving the PDEs:

$$\Lambda(x) = \begin{pmatrix} \lambda_1(x) \\ \vdots \\ \lambda_{n-k}(x) \end{pmatrix} = 0 \qquad \begin{matrix} n = 4 \\ k = \text{rank}\{\Delta(x)\} = 2 \end{matrix}$$

$$\forall \tau_i \in \Delta(x) \Rightarrow \frac{\partial \Lambda(x)}{\partial x} \tau_i(x) = 0$$

we are looking for $n-k=2$ functions

$$\lambda(x) : \begin{cases} \frac{\partial \lambda}{\partial x} \tau_1(x) = 0 \\ \frac{\partial \lambda}{\partial x} \tau_2(x) = 0 \end{cases}$$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial x_4} \end{pmatrix} \begin{pmatrix} 1\\0\\0\\x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial x_4} \end{pmatrix} \begin{pmatrix} 0\\1\\0\\x_2 \end{pmatrix} = 0$$

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$$\left(\frac{\partial \lambda}{\partial x_1} \quad \frac{\partial \lambda}{\partial x_2} \quad \frac{\partial \lambda}{\partial x_3} \quad \frac{\partial \lambda}{\partial x_4} \right) \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix} = 0$$

$$\begin{cases} \frac{\partial \lambda}{\partial x_1} + x_3 \frac{\partial \lambda}{\partial x_4} = 0 \Rightarrow \frac{\partial \lambda}{\partial x_1} = -x_3 \frac{\partial \lambda}{\partial x_4} \Rightarrow \frac{\cancel{\partial \lambda}}{\cancel{\partial \lambda}} \frac{\partial x_4}{\partial x_1}... \\ \frac{\partial \lambda}{\partial x_2} + x_2 \frac{\partial \lambda}{\partial x_4} = 0 \Rightarrow \dots \end{cases}$$

$$1 = -x_3 \frac{\partial x_1}{\partial x_4}$$

$$\int \partial x_4 = -x_3 \int \partial x_1$$

$$x_4 = -x_3 x_1 \Rightarrow -x_4 + x_3 x_1$$

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$$\begin{cases} \frac{\partial \lambda}{\partial x_1} + x_1 \frac{\partial \lambda}{\partial x_3} = 0 \\ x_2 \frac{\partial \lambda}{\partial x_1} - x_1 \frac{\partial \lambda}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{array}{l} \frac{\partial \lambda}{\partial x_1} = -x_1 \frac{\partial \lambda}{\partial x_3} \Rightarrow 1 = -x_1 \frac{\partial x_1}{\partial x_3} \\ \frac{\partial \lambda}{\partial x_1} = \frac{x_1}{x_2} \frac{\partial \lambda}{\partial x_2} \Rightarrow 1 = \frac{x_1}{x_2} \frac{\partial x_1}{\partial x_2} \end{array}$$

$$-\partial x_3 = x_1 \partial x_1 = x_2 \partial x_2$$

$$\begin{cases} \frac{\partial \lambda}{\partial x_1} = -x_1 \frac{\partial \lambda}{\partial x_3} \\ \frac{\partial \lambda}{\partial x_1} = \frac{x_1}{x_2} \frac{\partial \lambda}{\partial x_2} \end{cases} \Rightarrow \quad -\int \partial x_3 = \int x_2 \partial x_2$$

$$-x_3 = \frac{x_2^2}{2} \Rightarrow \lambda = -2x_3 + x_2^2 + x_1^2$$

play with the solution in order to verify the system

$$f(x) = \begin{pmatrix} x_3 \\ x_3 \cos(x_1) \\ -x_1 - x_3^3 + x_2^5 \end{pmatrix} \qquad \Delta(x) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ \cos x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(A) check integrability

$$[\tau_1(x), \tau_2(x)] = \frac{\partial \tau_2}{\partial x} \tau_1 - \frac{\partial \tau_1}{\partial x} \tau_2 =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \in \Delta(x)$$

involutive and integrable

$\Rightarrow n-k = 3-2 = 1$ ($\exists$ $n-k$ $\lambda(x)$ function)

$$\exists \lambda(x) \text{ s.t. } \frac{\partial \lambda}{\partial x} \left(\tau_1(x) \,\vdots\, \tau_2(x) \right) = 0$$

$$\begin{cases} \frac{\partial \lambda(x)}{\partial x_1} \cdot 1 + \frac{\partial \lambda(x)}{\partial x_2} \cos x_1 = 0 \\ \frac{\partial \lambda(x)}{\partial x_1} = 0 \end{cases} \Rightarrow \lambda(x) = \lambda(x_1, x_2) = x_2 - \sin(x_1)$$

$$\int 1 \, dx_2 + \int \cos x_1 \, dx_1$$

(B) check invariance $[f, \tau_i] \in \Delta(x) \quad i = 1,2$

$$[f, \tau_1] = \frac{\partial \tau_1}{\partial x} f - \frac{\partial f}{\partial x} \tau_1$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ * \\ * \end{pmatrix} - \begin{pmatrix} 0 & 0 & * \\ -x_3 \sin x_1 & 0 & * \\ -1 & 5x_2^4 & * \end{pmatrix} \begin{pmatrix} 1 \\ \cos x_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} * \end{pmatrix} \quad \begin{pmatrix} -1 & 5x_2^4 & * \end{pmatrix}\begin{pmatrix} 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -x_3 \sin x_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -x_3 \sin x_1 \\ -1 + 5x_2^4 \cos x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 + 5x_2^4 \cos x_1 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 1 & 0 & 0 \\ \cos x_1 & 0 & 0 \\ 0 & 1 & * \end{pmatrix} \qquad \text{rank}\{T(x)\} = 2 = k$$

$$\Rightarrow [f, \tau_1] \in \Delta(x)$$

$$[f, \tau_2] = \frac{\partial \tau_2}{\partial x} f - \frac{\partial f}{\partial x} \tau_2$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} * \\ * \\ * \end{pmatrix} - \begin{pmatrix} * & * & 1 \\ * & * & \cos x_1 \\ * & * & -3x_3^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ \cos x_1 \\ -3x_3^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -\cos x_1 \\ 3x_3^2 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 1 & 0 & -1 \\ \cos x_1 & 0 & -\cos x_1 \\ 0 & 1 & 3x_3^2 \end{pmatrix} \qquad \det(T(x)) = -\cos x_1 + \cos x_1 = 0$$

$$\text{Rank}(T(x)) = 2 = k$$

$$[f, \tau_2] \in \Delta(x)$$

$\Delta(x)$ is f-invariant

remarks: z has the following form: $z = \begin{pmatrix} z_1 \\ \hline 0 \end{pmatrix}$ with $z_2 = 0$

$\Phi_2(x) = 0$ when $x \in \Delta(x)$

this means: $z = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = \begin{array}{c|l} \begin{pmatrix} \text{complement} \\ \hline \lambda(x) \end{pmatrix} & \begin{matrix} k \\ n-k \end{matrix} \end{array}$

$\left| \dfrac{\partial \Phi}{\partial x} \right| \neq 0$ at least in a given neighborhood

$$d\lambda = \begin{pmatrix} -\cos x_1 & 1 & 0 \end{pmatrix}$$

$$\frac{\partial \Phi}{\partial x} = \begin{array}{c|l} \begin{pmatrix} \text{complement} \\ \hline d\lambda(x) \end{pmatrix} & \begin{matrix} k \\ n-k \end{matrix} \end{array} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline -\cos x_1 & 1 & 0 \end{pmatrix}$$

$$\left.\frac{\partial \Phi}{\partial x}\right|_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow \Phi_1(x) = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} z_1^1 \\ z_1^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \Phi_1(x) \qquad z_2 = x_2 - \sin x_1 = \Phi_2(x)$$

$$x_2 = z_2 + \sin z_1^1$$

$$\dot{z} = \frac{\partial \Phi}{\partial x} \circ f(x)\Big|_{x = \Phi^{-1}(z)} = \begin{pmatrix} x_3 \\ -x_1 - x_3^3 + x_2^5 \\ 0 \end{pmatrix}$$

$$\dot{z} = \begin{cases} \dot{z}_1^1 = z_1^2 \\ \dot{z}_1^2 = -z_1^1 - (z_1^2)^3 + (z_2 + \sin z_1^1)^5 \end{cases} \Big\} \text{ unobservable subsystem}$$

$$\dot{z}_2 = 0$$

$$y = h(x) = x_2 - \sin(x_1) = z_2$$

supposing a $g(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow z_2$

$$\begin{cases} \dot{z}_2 = u \\ y = z_2 \end{cases}$$ ← this component evolves linearly

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From Isidori

$$f = \begin{pmatrix} x_2 \\ x_3 \\ x_3 x_4 - x_1 x_2 x_3 \\ \sin x_3 + x_2^2 + x_1 x_3 \end{pmatrix} \qquad \Delta(x) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ x_1 \end{pmatrix} \right\}$$

(A) Integrability test

$$[\tau_1(x), \tau_2(x)] = \frac{\partial \tau_2}{\partial x}\tau_1 - \frac{\partial \tau_1}{\partial x}\tau_2$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \in \Delta(x)$$ involutive & integrable

$\rightarrow n - k = 4 - 2 = 2$ functions λ such that $\frac{\partial \lambda}{\partial x}\Delta = 0$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial x_4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial x_4} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ x_1 \end{pmatrix} = 0$$

$$\begin{cases} \frac{\partial \lambda}{\partial x_1} + x_2 \frac{\partial \lambda}{\partial x_4} = 0 \\ \frac{\partial \lambda}{\partial x_2} + x_1 \frac{\partial \lambda}{\partial x_4} = 0 \end{cases}$$

by inspection this must be (-) to satisfy both conditions

$$\int \partial x_4 + \int x_2 \, \partial x_1 = 0 \qquad x_4 + x_1 x_2 = 0$$

satisfy both conditions

$$\int \partial x_4 + \int x_2 \, \partial x_1 = 0 \qquad\qquad x_4 + x_1 x_2 = 0$$

$$\Rightarrow$$

$$\int \partial x_4 + \int x_1 \, \partial x_2 = 0 \qquad\qquad x_4 + x_1 x_2 = 0$$

$$\lambda_1(x) = x_4 - x_1 x_2$$

$\lambda_2(x)$ = chosen by inspection: there's no derivatives in x_3

$\Rightarrow \frac{\partial x_3}{\partial x_i} = 0$ $i \neq 3$ $\rightarrow$ verifies all the conditions

$$\lambda_2(x) = x_3$$

(B) invariance test

$$[f, \tau_1] = \frac{\partial \tau_1}{\partial x} f - \frac{\partial f}{\partial x} \tau_1 =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ x_3 \\ * \\ * \end{pmatrix} - \begin{pmatrix} 0 & * & * & 0 \\ 0 & * & * & 0 \\ -x_2 x_3 & * & * & x_3 \\ x_3 & * & * & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3 \end{pmatrix} = 0 \in \Delta(x)$$

$$[f, \tau_2] = \frac{\partial \tau_2}{\partial x} f - \frac{\partial f}{\partial x} \tau_2 =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ * \\ * \\ * \end{pmatrix} - \begin{pmatrix} * & 1 & * & 0 \\ * & 0 & * & 0 \\ * & -x_1 x_3 & * & x_3 \\ * & 2x_2 & * & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ x_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -x_2 \end{pmatrix} = -\tau_1(x) \in \Delta(x)$$

$\Delta(x)$ is f-invariant

$$d\lambda_1 = \begin{bmatrix} -x_2 & -x_1 & 0 & 1 \end{bmatrix}$$

$$d\lambda_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

new coordinates:

$$z_i = \Phi_i(x) \qquad i = 1, 2, 3, 4 \qquad z = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$\partial \Phi |$ (complement) (1 0 0 0)

$$\left.\frac{\partial \Phi}{\partial x}\right|_{x=0} = \left.\begin{pmatrix} \text{complement} \\ \hline \lambda_1(x) \\ \hline \lambda_2(x) \end{pmatrix}\right|_{x=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Phi_1 = x_1 \quad \Phi_2 = x_2 \quad \Phi_3 = \lambda_1 \quad \Phi_4 = \lambda_2$

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= x_4 - x_1 x_2 \\ z_4 &= x_3 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_4 &= z_3 + z_1 z_2 \\ x_3 &= z_4 \end{aligned}$$

$$\bar{f}(z) = \left.\frac{\partial \Phi(x)}{\partial x} \cdot f(x)\right|_{x=\Phi^{-1}(z)} =$$

$$= \left.\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -x_2 & -x_1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot f(x)\right|_{x=\Phi^{-1}(z)} =$$

$$= \left.\begin{pmatrix} x_2 \\ x_3 \\ \cancel{-x_2^2} - \cancel{x_1 x_3} + \cancel{x_2^2} + \cancel{x_1 x_3} + \sin x_3 \\ x_3 x_4 - x_1 x_2 x_3 \end{pmatrix}\right|_{x=\Phi^{-1}(z)}$$

$$= \begin{pmatrix} z_2 \\ z_4 \\ \sin z_4 \\ z_4(z_3 + z_1 z_2) - z_1 z_2 z_4 \end{pmatrix}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_4 \\ \dot{z}_3 = \sin z_4 \\ \dot{z}_4 = z_3 z_4 \end{cases}$$

Local reachability decomposition

$$f = \begin{bmatrix} x_1 x_3 + x_2 e^{x_2} \\ x_3 \\ x_4 - x_2 x_3 \\ x_3^2 + x_2 x_4 - x_2^2 x_3 \end{bmatrix} \qquad g = \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix}$$

compute
$\langle f, g \mid \mathrm{span}\{g\} \rangle = \Delta_R$

$\Delta_0 = \mathrm{span}\{g\}$

$= \Delta_R$

$\Delta_0 = \text{span}\{g\}$

$\Delta_1 = \Delta_0 + [f, g] \longrightarrow \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix} e^{x_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Delta_1 = \text{span}\left\{ \underbrace{\begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3 \end{bmatrix}}_{g}, \underbrace{\begin{bmatrix} e^{x_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{[f,g]} \right\} \Rightarrow [g, [f,g]] = 0 \quad \text{involutive}$$

$\rho(\Delta_0) = 1 \quad \rho(\Delta_1) = 2$

$$\Delta_2 = \Delta_1 + [f, \Delta_1] + [g, \Delta_1] = \Delta_1 + [f, [f,g]] + [g, [f,g]]$$

$$[f, \Delta_1] = [f, [f,g]] = \begin{bmatrix} e^{x_2} x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow \text{involutive}$$

$\Delta_1 \equiv \Delta_2 \equiv \Delta_R$

$R_{x_0}(t) = \mathcal{Y}_{\Delta R}(e^{tf} x_0) \qquad n = 4 \qquad k = 2$

Δ_R is integrable if $\exists\ \Lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-k} \end{bmatrix}$ s.t.

$$\frac{\partial \Lambda}{\partial x} \Delta_R(x) = 0$$

$$\begin{bmatrix} \frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial x_4} \end{bmatrix} \left[\begin{array}{c|c} x_1 & e^{x_2} \\ 1 & 0 \\ 0 & 0 \\ x_3 & 0 \end{array}\right] = [0 \mid 0]$$

$$\begin{cases} x_1 \frac{\partial \lambda}{\partial x_1} + \frac{\partial \lambda}{\partial x_2} + x_3 \frac{\partial \lambda}{\partial x_4} = 0 \\ e^{x_2} \frac{\partial \lambda_1}{\partial x_1} = 0 \end{cases} \Rightarrow \begin{cases} 0 + \int \partial x_4 + \int x_3 \partial x_2 = 0 \\ \frac{\partial \lambda_1}{\partial x_1} = 0 \end{cases}$$

$\longrightarrow \lambda_1$ does not depend on x_1

$\lambda_1 = x_3$

$\lambda_2 = x_4 - x_2 x_3$

$\Phi_1 = x_1 \qquad \Phi_2 = x_2$

$\Phi_3 = \lambda_1 \qquad \Phi_4 = \lambda_2$

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= x_3 \\ z_4 &= x_4 - x_2 x_3 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= z_3 \\ x_4 &= z_4 + z_2 z_3 \end{aligned}$$

$$\tilde{f}(z) = \frac{\partial \Phi(x)}{\partial x} \cdot f(x) \Big|_{x=\Phi^{-1}(z)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -x_3 & -x_2 & 1 \end{pmatrix} \cdot f(x) \Big|_{x=\Phi^{-1}(z)}$$

$$= \begin{pmatrix} x_1 x_3 + x_2 e^{x_2} \\ x_3 \\ x_4 - x_2 x_3 \\ \cancel{-x_3^2} - \cancel{x_2 x_4} + \cancel{x_2^2 x_3} + \cancel{x_3^2} + \cancel{x_2 x_4} - \cancel{x_2^2 x_3} \end{pmatrix}_{x=\Phi^{-1}(z)}$$

$$= \begin{pmatrix} z_1 z_3 + z_2 e^{z_2} \\ z_3 \\ z_4 + \underbrace{\cancel{z_2 z_3} - \cancel{z_2 z_3}}_{0} \end{pmatrix} \begin{matrix} \}\, k \\ \}\, n-k \end{matrix}$$

$$e(z) = \begin{pmatrix} z_1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} k \\ n-k \end{matrix} \rightarrow \text{are always zero because } z_3 \text{ and } z_4 \text{ are unreachable}$$

$$J_{\tilde{f}} = \frac{\partial \tilde{f}(z)}{\partial z} = \begin{pmatrix} \frac{\partial \tilde{f}_{12}}{\partial \xi_1} & \frac{\partial \tilde{f}_{12}}{\partial \xi_2} \\ 0 & \frac{\partial \tilde{f}_{34}}{\partial \xi_2} \end{pmatrix} \qquad \xi_1 = (z_1, z_2) \qquad \xi_2 = (z_3, z_4)$$

$$y = h(\Phi^{-1}(z))$$

$$\begin{cases} x_3 \frac{\partial \lambda}{\partial x_3} = 0 \\ x_1 \frac{\partial \lambda}{\partial x_1} + x_1 x_2 \frac{\partial \lambda}{\partial x_2} = 0 \end{cases}$$

Does this system have a solution?

$$\begin{pmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_3} \end{pmatrix} \underbrace{\left(\begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 x_2 \\ 0 \end{pmatrix} \right)}_{\Delta(x)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$\exists$ a solution iff define a solution which is involutive

Frobenius:

$$\frac{\partial \varphi}{\partial x} \cdot \underbrace{\Delta}_{k} = 0 \qquad \exists \ n-k \text{ solutions to } \frac{\partial \varphi}{\partial x} \Delta = 0$$

The distribution must be integrable