13. The realization problem

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When the starting point x_0 is an equilibrium point, separability property holds.

Since $f(x_0)=0$, follows

$$\tilde{\gamma}_m(t, \tau_1, \dots, \tau_m; x_0) = \gamma_m(t, \tau_1, \dots, \tau_m, t_0; x_0)$$

The first kernel can be written as:

$$\gamma_1(t, \tau; x_0) = \left.\frac{\partial \gamma_0(t, \tau; x_0)}{\partial x}\right|_{x_0} \cdot g(x_0) = \left.\frac{\partial \gamma_0(t; x)}{\partial x}\right|_{x_0} \cdot \left.\frac{\partial \gamma_0(-\tau_1; x)}{\partial x}\right|_{x_0} \cdot g(x_0)$$

and the same considerations are valid for kernels of every order.

Problem:

given a set of kernels $V_0(t)$, $V_1(t, \tau_1)$, $V_2(t, \tau_1, \tau_2)$, ..., $V_m(t, \tau_1, \dots, \tau_m)$ $\quad t \geq \tau_1 \geq \dots \geq \tau_m \geq t_0 = 0$

find the CAUSAL realization (n, x_0, f, g, h) s.t. the kernels of this quintuple, W_i, are the same of V_i

$$V_m(t, \tau_1, \dots, \tau_m) \equiv W_m(t, \tau_1, \dots, \tau_m) \quad \forall m$$

Theorem: CNES of the solvability of Non-Linear realization problem is:

a set of kernels V_m, $m \geq 0$, admits causal realization if and only if there exists an integer n and two functions

$P, Q \in C^1$: $\quad P \in \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \quad Q: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^q$

and $x_0 \in \mathbb{R}^n$ s.t.

$$\begin{cases} V_0(t) = Q(t, x_0) \\ V_m(t, \tau_1, \dots \tau_m) = \left. L_{P(\tau_m, \cdot)} \cdots L_{P(\tau_1, \cdot)} Q(t, \cdot)\right|_{x_0} \end{cases}$$

Theorem: in an equilibrium point:

i) kernels can be separated as

$$W_1(t,\tau_1,t_0;x_0) = W_{11}(t-\tau_1)$$
$$W_2(t,\tau_1,\tau_2,t_0;x_0) = W_{21}(t-\tau_1)\,W_2(\tau_1-\tau_2)$$
$$\vdots$$
$$W_n(t,\tau_1,\dots,\tau_n,t_0;x_0) = W_{n1}(t-\tau_1)\dots W_{nn}(\tau_{n-1}-\tau_n)$$

ii) every function $W_{nj}(\tau_{j-1}-\tau_j)$ is factorizable

$$W_{nj}(\tau_{j-1}-\tau_j) = Q_{nj}(\tau_{j-1})\cdot P_{nj}(\tau_j)$$
$$= C_{nj}\cdot e^{(\tau_{j-1}-\tau_j)A_{nj}}\cdot B_{nj}$$

the first kernel can be factorized as

$$W_1(t,\tau_1;x_e) = W_{11}(t-\tau_1) = Q_{11}(t)\,P_{11}(\tau_1) = C_{11}e^{(t-\tau_1)A_{11}}\cdot B_{11}$$

while kernel of order 2 assumes a bilinear form:

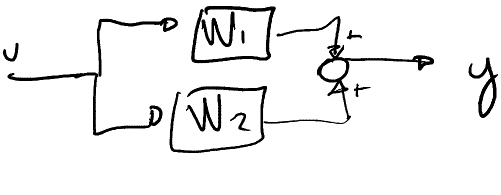
$$W_2(t,\tau_1,\tau_2;x_e) = W_{21}(t-\tau_1)\,W_{22}(\tau_1-\tau_2) = Q_{21}(t)\cdot P_{21}(\tau_1)\cdot Q_{22}(\tau_1)\cdot P_{22}(\tau_2)$$
$$= C_{21}e^{(t-\tau_1)A_{21}}B_{21}\cdot C_{22}e^{(\tau_1-\tau_2)A_{22}}B_{22}$$

in fact in the state space we have

$$W_1: \quad \dot{x}_1 = A_{11}x_1 + B_{11}u$$

$$W_2: \quad \begin{pmatrix}\dot{x}_{21}\\ \dot{x}_{22}\end{pmatrix} = \begin{pmatrix}A_{21} & 0\\ 0 & A_{22}\end{pmatrix}\begin{pmatrix}x_{21}\\ x_{22}\end{pmatrix} + \begin{pmatrix}0 & B_{21}C_{22}\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_{21}\\ x_{22}\end{pmatrix}u + \begin{pmatrix}0\\ B_{22}\end{pmatrix}u$$

this structure represents a dynamics of order 2



Theorem: there exists a bilinear representation of a finite number of input-output kernels related to an input-affine representation (starting from an equilibrium)

$n=1$ $\quad W_1(t,\tau_1,t_0;x_0) = Q_{11}(t)\,P_{11}(\tau) = C_{11}e^{(t-\tau_1)A_{11}}B_{11}$

$$\begin{cases}\dot{z} = A_{11}z + B_{11}u & z_0 = 0\\ y = C_{11}z + Du\end{cases}$$

$n=2$ $\quad W_2(t,\tau_1,\tau_2,t_0;x_0) = W_{21}(t-\tau_1)W_{21}(\tau_1-\tau_2)$
$= C_{21} e^{(t-\tau_1)} B_{21} C_{22} e^{(\tau_1-\tau_2)A_{22}} B_{22}$

$$\begin{cases} \dot{z}_2 = \begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} z_2 + \begin{pmatrix} 0 & B_{21}C_{22} \\ 0 & 0 \end{pmatrix} z_2 u + \begin{pmatrix} 0 \\ B_{22} \end{pmatrix} u \\ y = \begin{pmatrix} C_{21} & 0 \end{pmatrix} z_2 \qquad\qquad z_2(0)=0 \end{cases}$$

the only realization of W_2 is, in fact:

$$W_2 = \begin{pmatrix} C_{21} \mid 0 \end{pmatrix} e^{(t-\tau_1)\begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}} \cdot \begin{pmatrix} 0 & B_{21}C_{22} \\ 0 & 0 \end{pmatrix} z_2 u +$$
$$+ \begin{pmatrix} 0 \\ B_{12} \end{pmatrix} e^{(\tau_1-\tau_2)\begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}} \begin{pmatrix} 0 \\ B_{22} \end{pmatrix}$$

• Conditions for realizability of NL systems

① $\exists\ \underset{q\times n}{P(\tau)},\ \underset{n\times p}{Q(t)}$ s.t. $x(t) = Q(t)P(\tau)$

② $rk(H) = n$ (finite) $\longrightarrow$ H = Hankel Matrix = $\begin{bmatrix} x(0) & \frac{\partial x}{\partial t}\big|_0 & \ddot{x}(0) & \\ \frac{\partial x}{\partial t}\big|_0 & \ddot{x}(0) & \ddots & \ddots \\ \ddot{x}(0) & \cdots & & \end{bmatrix}$

③ $\mathcal{L}(x(t)) = \dfrac{N(s)}{D(s)}$

can be written in the Laplace domain as a rational function

Linear case (realization problem)

Given $K(t)$, Kernel $q\times p$, find
$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{n\times q}$, $D \in {}^{q\times p}$
with

$$K(t) = Ce^{At}B + D\,\delta(t)$$

and

$$y_c(t) = \int_0^t K(t-\tau)u(\tau)d\tau$$

setting

$$P(\tau,x) = e^{\tau L_g} L_g e^{-\tau L_g} Id\big|_x$$
$$Q(t,x) = W_0(t,x) = e^{tL_g} h$$

$$k(t-\tau) = Q(t)\cdot P(-\tau)$$

$k(s)$ is rational and proper function $\Rightarrow \mathcal{L}(k(t)) = \frac{N(s)}{D(s)}$

and $\exists\ P(\tau, x) = P(\tau) \quad Q(t,x) = Q(t)x$ s.t.

$$k(t-\tau) = L_{P(\tau)} Q(t,x) = L_{e^{-\tau A}B}\left(Ce^{At}x\right)$$

$$= Ce^{A(t-\tau)}B = Ce^{At}\cdot e^{-A\tau}B$$