# 6. Linear finite-dimensional state space representations

venerdì 5 giugno 2020 12:48

Combining the hypotheses of linearity and stationariety

$$\begin{cases} x(t) = \varphi_L(t, t_0, x(t_0)) + \varphi_F(t, t_0, u|_{[t_0,t)}) \\ y(t) = \eta_L(t, x(t)) + \eta_F(t, u(t)) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x(t) = \varphi_L(t-t_0, x(t_0)) + \varphi_F(t-t_0, u_{-t_0}|_{[t_0,t)}) \\ y(t) = \eta_L(x(t)) + \eta_F(u(t)) \end{cases}$$

assuming $y \in \mathbb{R}^q$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

**Output:** $y_L(t) = \eta_L(x(t)) = C\,x(t)$ $\qquad C \in \mathbb{R}^{q\times n}$

$y_F(t) = \eta_F(u(t)) = D\,u(t)$ $\qquad D \in \mathbb{R}^{q\times p}$

**State:** $x_L(t) = \varphi_L(t-t_0, x(t_0)) = \underbrace{\Phi(t-t_0)}_{n\times n \text{ time dependent transition matrix}} x(t_0)$

$x_F(t) = \varphi_F(t-t_0, u_{-t_0}|_{[t_0,t)})$

pieces of function

We assume <u>discrete time</u> $T \equiv \mathbb{Z}$

$u|_{[t_0,t)}$ = sequence of discrete values
$= (u(t_0), u(t_0+1), \dots, u(t-1), \cancel{u(t)})$

not considered due to causality

sum of contributions $= (u(t_0), 0, \dots, 0) + (0, u(t_0+1), 0, \dots, 0) + \dots + (0, 0, \dots, u(t-1))$

$$\to x_F(t) = \varphi_{F_1}(t-t_0, (u(t_0), 0, \dots, 0)) + \varphi_{F_2}(t-t_0, (0, u(t_0+1), 0, \dots, 0)) + $$
$$+ \varphi_{F_N}(t-t_0, (0, 0, \dots, u(t-1)))$$
$$= H(t-t_0)u(t_0) + H(\underbrace{t-t_0-1}_{t-(t_0+1)})u(t_0+1) + \underbrace{H(1)}_{t-(t-1)}u(t-1) =$$
$$= \sum_{\tau=t_0}^{t-1} H(t-\tau)\,u(\tau)$$

$H(t,\tau)$ impulse response matrix

assuming now <u>continuous time</u> $T \equiv \mathbb{R}$

$$x_F = \varphi_F(t-t_0, u|_{[t_0,t)}) = \int^{t} H(t-\tau)\,u(\tau)\,d\tau$$

$$x_F = \varphi_F(t-t_0, u|_{[t_0,t)}) = \int_{t_0}^{t} H(t-\tau)\,u(\tau)\,d\tau$$

**Output:** $y(t) = Cx(t) + Du(t)$

$\boxed{\text{in } T \equiv \mathbb{Z}}$

$$= C\left[\Phi(t-t_0)\,x(t_0) + \sum_{\tau=t_0}^{t-1} H(t-\tau)u(\tau)\right] + Du(t)$$

$$= \underbrace{C\Phi(t-t_0)}_{\Psi}\,x(t_0) + \sum_{\tau=t_0}^{t-1} \underbrace{CH(t-\tau)u(\tau) + Du(t)}$$

$$W(t,\tau) = \begin{cases} CH & \tau < t \\ D & \tau = t \end{cases}$$

$$y(t) = \Psi(t-t_0)\,x(t_0) + \sum_{\tau=t_0}^{t} W(t-\tau)\,u(\tau)$$

$\boxed{\text{in } T \equiv \mathbb{R}}$

$$y(t) = Cx(t) + Du(t) =$$

$$= \Psi(t-t_0)\,x_0 + \int_{t_0}^{t} CH(t-\tau)u(\tau) + Du(t)$$

$$y(t) = \Psi(t-t_0)\,x_0 + \int_{t_0}^{t} CH(t-\tau)\,u(\tau)\,d\tau + \int_{t_0}^{t} Du(\tau)\,\delta(t-\tau)\,d\tau =$$

impulse properties

$$\int_a^b f(\tau)\,\delta(t-\tau)\,dt = f(t)$$

$$\rightarrow \int_a^b f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$

$$= \Psi(t-t_0)\,x_0 + \int_{t_0}^{t} [\underbrace{CH(t-\tau) + D\delta(t-\tau)}_{W(t-\tau)}]\,u(\tau)\,d\tau =$$

$$= \Psi(t-t_0)\,x_0 + \int_{t_0}^{t} W(t-\tau)\,u(\tau)\,d\tau$$

$W(t,\tau)$ impulse response matrix

# Properties

- **Causality:** it is respected because the input is not considered beyond $t$
- **Consistency** verified if $x(t_0) = \underbrace{\Phi(t_0-t_0)}\,x(t_0) \quad \forall x(t_0)$

- **Consistency** verified if $\check{x}(t_0) = \underbrace{\Phi(t_0 - t_0)}_{\Phi(0) = I} x(t_0) \quad \forall x(t_0)$
- **Separation:** verified if $\forall t > t_1 > t_0$

$$\Phi(t-t_0)x(t_0) + \int_{t_0}^{t} H(t-\tau)u(\tau)d\tau = \Phi(t-t_1)x(t_1) + \int_{t_1}^{t} H(t-\tau)u(\tau)d\tau$$

$$x(t_1) = \Phi(t_1 - t_0)x(t_0) + \int_{t_0}^{t_1} H(t_1-\tau)u(\tau)d\tau$$

if the property is verified:

- Free evolution:

$$\Phi(t-t_0)\cancel{x(t_0)} = \Phi(t-t_1)\Phi(t_1-t_0)\cancel{x(t_0)}$$

- Forced evolution:

$$\int_{t_0}^{t} H(t-\tau)\,d\tau = \int_{t_0}^{t_1} \Phi(t-t_1)H(t_1-\tau)u(\tau)d\tau + \int_{t_1}^{t} H(t-\tau)u(\tau)d\tau$$

In order to verify the equality of the convolutional integral

$$\Phi(t_1-t_0)H(t_1-\tau) = H(t-\tau)$$

$$\boxed{\begin{cases} x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau \\ y(t) = \Psi(t,t_0)x_0 + \int_{t_0}^{t} W(t,\tau)u(\tau)d\tau \end{cases}} \quad \text{Explicit Representation}$$

$\boxed{\text{in } T \equiv \mathbb{Z}}$

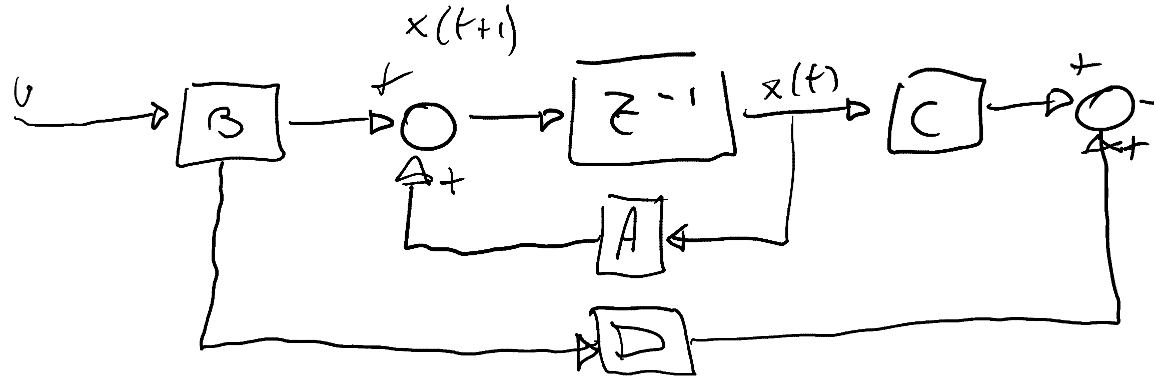
$$\begin{cases} x(t+1) = \varphi(t+1, t, x(t), u(t)) = f(t, x(t), u(t)) \\ y(t) = \eta(t, x(t), u(t)) \end{cases}$$

$A(t) = \Phi(t+1, t) \qquad B(t) = H(t+1, t)$

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) + D_u(t) \end{cases}$$ Implicit representation discrete-time

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Implicit representation discrete-time




Real time!

$\boxed{\text{in } T \equiv \mathbb{R}}$

under the regularity assumption $\varphi$ is solution of

$$\frac{\partial \varphi}{\partial t} = \dot{x}(t) = f(t, \varphi, u(t))$$

if $\phi$ and $H$ continuous functions on $(T \times T)^*$ and $\Phi$ derivable w.r.t. $t$

then

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

$$\frac{\partial \Phi(t,\tau)}{\partial t} = \lim_{\varepsilon \to 0} \frac{\Phi(t+\varepsilon,\tau) - \Phi(t,\tau)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left( \frac{\Phi(t+\varepsilon,t) - \Phi(t,t)}{\varepsilon} \right) \Phi(t,\tau) = A(t)\Phi(t,\tau)$$

$$\frac{\partial H(t,\tau)}{\partial t} = \frac{\partial}{\partial t} \left( \Phi(t,\tau) H(t,t) \right) = A(t) H(t,\tau)$$

therefore

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$ has a unique solution with

$$B(t) = H(t,t) \qquad A(t) = \left. \frac{\partial \Phi(t,\tau)}{\partial t} \right|_{\tau = t}$$

- $\Phi(t,t_0) \cdot X(t_0) \rightsquigarrow I$ is the solution of

$$\dot{X}(t) = A(t)X(t)$$

Therefore $x(t) = \Phi(t, t_0)\, x(t_0)$ is the solution of $\dot{x}(t) = A(t) x(t)$

$\Phi(t, t_0)$ admits the Neumann series expansion

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)\, d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)\, d\tau_2\, d\tau_1 + \dots$$

$\to$ in the stationary case $\Phi(t - t_0) = e^{A(t - t_0)}$

$$\boxed{\begin{cases} \dot{x}(t) = A(t) x(t) + B(t) u(t) \\ y(t) = C(t) x(t) + D(t) u(t) \end{cases}}$$

Implicit representation continuous-time