Kernels

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# How to compute $e^{At}$

**Case 1:** real and distinguished eigenvalues

a) Compute eigenvectors $u_1, u_2$

b) compute $T^{-1} = [u_1 | u_2]^{-1}$ and get $v_1^T, v_2^T$

c) $e^{At} = e^{\lambda_1 t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T$

**Case 2:** real and coincident eigenvalues

a) cannot just compute 2 eigenvectors, so we need to compute the Jordan form

b) a fast alternative method is based on this fact

$$e^{At} = e^{-\lambda t + At + \lambda t} = e^{\lambda t} e^{At - \lambda t} =$$
$$= e^{+\lambda t} \sum \frac{1}{k!} \left((A - \lambda I)t\right)^k =$$
$$= e^{+\lambda t} \left(I + (A - \lambda I)t + \cancel{\frac{(A-\lambda I)^k t^k}{k!}}\right) \quad \to 0$$

in general for a $n \times n$ system all $k \geq n$ become zero.

**Case 3:** complex eigenvalues

$\lambda = \alpha + j\omega$

$$e^{At} = e^{\alpha t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

# How to compute kernels

$$\gamma_0(t; x_0) = e^{t L_f} x_0 = e^{At} x_0$$

$$W_0(t; x_0) = \frac{\partial h}{\partial x} e^{t L_f} x_0 = C e^{At} x_0$$

$$W_1(t, \tau_1; x_0) = e^{\tau_1 L_f} \circ L_g \circ e^{(t-\tau_1) L_f} h \Big|_{x_0} = \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1; x_0))$$
$$= \frac{\partial W_0(t - \tau_1; x_0)}{\partial x} g(x) \Big|_{x = \gamma_0(\tau_1; x_0)} = L_g W_0(t - \tau_1; x)$$

$$W_2(t, \tau_1, \tau_2; x_0) = e^{\tau_2 L_f} \circ L_g \circ e^{(\tau_1 - \tau_2) L_f} \circ L_g \circ e^{(t - \tau_1) L_f} h \Big|_{x_0} =$$
$$= e^{\tau_2 L_f} \tilde{W}_1(t, \tau_1, \tau_2; x_0) =$$
$$= e^{\tau_2 L_f} L_g \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1 - \tau_2; x_0))$$

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$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_2 - x_1 u \\ y = x_1 \end{cases}$$

Compute $w_0, w_1, w_2$

$$f(x) = Ax = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad N(x) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$h(x) = (1\ \ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$A$ diagonal $\rightarrow$ $\lambda_{12} = 0, 1$

eigenvectors:

$$(A - \lambda I)\, v = 0 \implies \begin{pmatrix} -\lambda & -1 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$

$$\begin{cases} -\lambda v_a - v_b = 0 \\ v_b - \lambda v_b = 0 \end{cases}$$

$\boxed{\lambda = 0} \rightarrow \begin{cases} -v_b = 0 \\ v_b = 0 \end{cases}$ $\qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\boxed{\lambda = 1} \rightarrow \begin{cases} -v_a - v_b = 0 \\ 0 = 0 \end{cases} \implies \begin{cases} v_a = -v_b \\ 0 = 0 \end{cases}$ $\qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$U = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad (v_1\ \ v_2)$$

$\det U = -1$

$$V = U^{-1} = -1\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{matrix} v_1^T \\ v_2^T \end{matrix}$$

$$\boxed{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^{-1} = \frac{1}{\det}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}$$

$$e^{At} = e^{\lambda_1 t} v_1 v_1^T + e^{\lambda_2 t} v_2 v_2^T$$

$$= e^{0t}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + e^{t}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

or

$$e^{At} = U e^{\Lambda t} U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^0 & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -e^t \end{pmatrix} = \begin{pmatrix} 1 & 1-e^t \\ 0 & e^t \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 1 & 1-e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} x_{01} + (1-e^t)x_{02} \\ e^t x_{02} \end{pmatrix}$$

$$W_0(t; x_0) = C e^{At} x_0 = (1\ \ 0)\,\gamma_0 = x_{01} + (1-e^t)x_{02}$$

$$W_1(t, \tau_1; x_0) = C e^{(t-\tau_1)A} N e^{\tau_1 A} x_0 =$$

$$= (1\ \ 0)\begin{pmatrix} 1 & 1-e^{(t-\tau_1)} \\ 0 & e^{(t-\tau_1)} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1-e^{\tau_1} \\ 0 & e^{\tau_1} \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-e^{(t-\tau_1)} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} =$$

$$= \begin{pmatrix} -1+e^{(t-\tau_1)} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1-e^{\tau_1} \\ 0 & e^{\tau_1} \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} =$$

$$= \begin{pmatrix} -1+e^{(t-\tau_1)} & (1-e^{\tau_1})(-1+e^{(t-\tau_1)}) \end{pmatrix}\begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} =$$

$$= \begin{pmatrix} -1+e^{(t-\tau_1)} & (1-e^{\tau_1})(-1+e^{(t-\tau_1)}) \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} =$$

$$\left(e^{(t-\tau_1)}-1\right) x_{01} + (1-e^{\tau_1})(-1+e^{(t-\tau_1)}) x_{02}$$

$$W_2(t,\tau_1,\tau_2;x_0) = C e^{(t-\tau_1)A} N e^{(\tau_1-\tau_2)A} N e^{\tau_2 A} \cdot x_0$$

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07/11/19

$$\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = x_1 + e^{x_1+x_2} u \\ y = x_1 \end{cases}$$

$$A = \frac{\partial f(x)}{\partial x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} 1 \\ e^{x_1+x_2} \end{pmatrix}$$

$$h(x) = \underset{C}{\begin{pmatrix} 1 & 0 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda_{12} = \pm 1$$

eigenvectors

$$(A-\lambda I)v = 0 \qquad \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$

$$\begin{cases} -\lambda v_a + v_b = 0 \\ v_a - \lambda v_b = 0 \end{cases}$$

$$\boxed{\lambda = 1} \Rightarrow \begin{cases} -v_a + v_b = 0 \\ v_a - v_b = 0 \end{cases} \quad \begin{cases} v_a = v_b \\ 0 = 0 \end{cases} \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\lambda = -1} \Rightarrow \begin{cases} v_a + v_b = 0 \\ v_a + v_b = 0 \end{cases} \quad \begin{cases} v_a = -v_b \\ \text{———} \end{cases} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$U = \begin{array}{c} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ v_1 \;\; v_2 \end{array} \qquad U^{-1} = V = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{array}{c} v_1^T \\ v_2^T \end{array}$$

$$e^{At} = U e^{\Lambda t} V = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & e^t \\ e^{-t} & -e^{-t} \end{pmatrix} = \begin{pmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{pmatrix}$$

$$\gamma_0(t;x_0) = e^{At}x_0 = \frac{1}{2}\begin{pmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{pmatrix} x_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$W_0(t;x_0) = Ce^{At}x_0 = \frac{1}{2}(1 \;\; 0)\,\gamma_0(t;x_0) = \underbrace{\frac{1}{2}\left(e^t+e^{-t} \;\; e^t-e^{-t}\right)}_{a(t)} x_0$$

$$W_0(t;x_0) = Ce^{At}x_0 = \tfrac{1}{2}(1 \quad 0)\,\gamma_0(t;x_0) = \underbrace{\tfrac{1}{2}(e^{t}+e^{-t} \quad e^{t}-e^{-t})}_{a(t)}x_0$$

$$W_1(t,\tau_1;x) = \tilde{W}_0(t,\tau_1;\gamma_0(\tau_1,x_0))$$

$$\tilde{W}_0(t,\tau_1,x) = L_g\, W_0(t-\tau_1,x)$$

$$= L_g(a(t-\tau_1)x) = \frac{\partial a(t-\tau_1)x}{\partial x}\cdot\begin{pmatrix}1\\ e^{x_1+x_2}\end{pmatrix}$$

$$= a(t-\tau_1)\begin{pmatrix}1\\ e^{x_1+x_2}\end{pmatrix} =$$

$$= \tfrac{1}{2}\left(e^{(t-\tau_1)}+e^{-(t-\tau_1)} \;\vdots\; e^{(t-\tau_1)}-e^{-(t-\tau_1)}\right)\begin{pmatrix}1\\ e^{x_1+x_2}\end{pmatrix} =$$

$$= \tfrac{1}{2}\left[e^{(t-\tau_1)}+e^{-(t-\tau_1)}+\left(e^{(t-\tau_1)}-e^{-(t-\tau_1)}\right)\left(e^{x_1+x_2}\right)\right]$$

$$\gamma_0(\tau_1;x_0) = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}(e^{\tau_1}+e^{-\tau_1})x_{01}+(e^{\tau_1}-e^{-\tau_1})x_{02}\\ (e^{\tau_1}-e^{-\tau_1})x_{01}+(e^{\tau_1}+e^{-\tau_1})x_{02}\end{pmatrix}$$

$$x_1+x_2 = (e^{\tau_1}+e^{-\tau_1})x_{01}+(e^{\tau_1}-e^{-\tau_1})x_{02}+(e^{\tau_1}-e^{-\tau_1})x_{01}+(e^{\tau_1}+e^{-\tau_1})x_{02}$$

$$= 2e^{\tau_1}x_{01}+2e^{\tau_1}x_{02} = 2e^{\tau_1}(x_{01}+x_{02})$$

$$W_1(t,\tau_1;x_0) = \tfrac{1}{2}\left[e^{(t-\tau_1)}+e^{-(t-\tau_1)}+\left(e^{(t-\tau_1)}-e^{-(t-\tau_1)}\right)e^{2e^{\tau_1}(x_{01}+x_{02})}\right]$$

$$W_2(t,\tau_1,\tau_2,x_0) = e^{\tau_2 L_g}\,\tilde{W}_1(t,\tau_1,\tau_2;x) = \tilde{W}_1(t,\tau_1,\tau_2;\gamma_0(\tau_2;x_0))$$

$$\tilde{W}_1(t,\tau_1,\tau_2,x) = L_g\,\tilde{W}_0(t,\tau_1,\gamma_0(\tau_1-\tau_2,x)) \qquad \alpha = \tau_1-\tau_2$$

$$= L_g\, a_2(t-\tau_1)\begin{pmatrix}(e^{\alpha}+e^{-\alpha})x_1+(e^{\alpha}-e^{-\alpha})x_2\\ (e^{\alpha}-e^{-\alpha})x_1+(e^{\alpha}+e^{-\alpha})x_2\end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix}e^{\alpha}+e^{-\alpha} & e^{\alpha}-e^{-\alpha}\\ e^{\alpha}-e^{-\alpha} & e^{\alpha}+e^{-\alpha}\end{pmatrix}x = P\cdot x$$

$$= \frac{\partial a_2(t-\tau_1)P\cdot x}{\partial x}\begin{pmatrix}1\\ e^{x_1+x_2}\end{pmatrix}$$

$$= a_2(t-\tau_1)P\begin{pmatrix}1\\ e^{x_1+x_2}\end{pmatrix}$$

$$\gamma_0(\tau_2;x_0) = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}e^{\tau_2}+e^{-\tau_2} & e^{\tau_2}-e^{-\tau_2}\\ e^{\tau_2}-e^{-\tau_2} & e^{\tau_2}+e^{-\tau_2}\end{pmatrix}\begin{pmatrix}x_{01}\\ x_{02}\end{pmatrix}$$

$$x_1+x_2 = 2e^{\tau_2}(x_{01}+x_{02})$$

$$W_2(t,\tau_1,\tau_2;x_0) = a_2(t-\tau_1)\cdot P\cdot\begin{pmatrix}1\\ e^{2e^{\tau_2}(x_{01}+x_{02})}\end{pmatrix}$$

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Class: 16/10/19

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = x_1+x_2^2 u\\ y = x_1\end{cases}$$

input affine

$$A = \frac{\partial f(x)}{\partial x} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad g(x) = \begin{pmatrix}0\\ x_2^2\end{pmatrix}$$

$$h(x) = (1 \quad 0)\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

eigenvalues:

$$(A-\lambda I) = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \qquad \det(A-\lambda I) = \lambda^2 - 1 = 0$$
$$\lambda_{12} = \pm 1$$

eigenvectors:

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \implies \begin{cases} -\lambda v_a + v_b = 0 \\ v_a - \lambda v_b = 0 \end{cases}$$

$$\lambda = 1 \implies \begin{cases} v_a = v_b \\ \text{---} \end{cases} \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \implies \begin{cases} v_a = -v_b \\ \text{---} \end{cases} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$e^{At} = U e^{\Lambda t} U^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^t & e^t \\ e^{-t} & -e^{-t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{pmatrix}$$

$$\gamma_0(t;x_0) = e^{At}x_0 = \frac{1}{2}\begin{pmatrix} (e^t+e^{-t})x_{01} + (e^t-e^{-t})x_{02} \\ (e^t-e^{-t})x_{01} + (e^t+e^{-t})x_{02} \end{pmatrix}$$

$$W_0(t;x_0) = Ce^{At}x_0 = (1\ 0)\gamma_0(t;x_0) = \frac{1}{2}\left((e^t+e^{-t})x_{01} + (e^t-e^{-t})x_{02}\right)$$
$$= \underbrace{\frac{1}{2}\left(e^t+e^{-t} \mid e^t-e^{-t}\right)}_{a(t)} x_0$$

$$W_1(t,\tau_1;x) = \widetilde{W}_0(t,\tau_1;\gamma_0(\tau_1;x_0))$$

$$\widetilde{W}_0(t,\tau_1;x) = L_g W_0(t-\tau_1;x)$$
$$= \frac{\partial a(t-\tau_1)x}{\partial x} \circ \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}$$
$$= a(t-\tau_1) \cdot \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$$
$$= \frac{1}{2}\left(e^{t-\tau_1} - e^{-t-\tau_1}\right)x^2$$

$$\gamma_0(\tau_1;x_0) = \frac{1}{2}\begin{pmatrix} (e^{\tau_1}+e^{-\tau_1})x_{01} + (e^{\tau_1}-e^{-\tau_1})x_{02} \\ (e^{\tau_1}-e^{-\tau_1})x_{01} + (e^{\tau_1}+e^{-\tau_1})x_{02} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_2^2 = \frac{1}{2}\left((e^{\tau_1}-e^{-\tau_1})x_{01} + (e^{\tau_1}+e^{-\tau_1})x_{02}\right)^2$$

$$W_1(t,\tau_1;x) = \frac{1}{4}\left(e^{t-\tau_1} - e^{-(t-\tau_1)}\right)\left((e^{\tau_1}-e^{-\tau_1})x_{01} + (e^{\tau_1}+e^{-\tau_1})x_{02}\right)^2$$

$$W_2(t,\tau_1,\tau_2;x) = e^{\tau_2 L_g}\tilde{W}_1(t,\tau_1,\tau_2;x) = \tilde{W}_1(t,\tau_1,\tau_2;\gamma_0(\tau_2;x_0))$$

$$\tilde{W}_1(t,\tau_1,\tau_2;x) = L_g \tilde{W}_0(t,\tau_1;\gamma_0(\tau_1-\tau_2;x)) = \qquad \tau_1-\tau_2=\alpha$$

$$= L_g \cdot a(t-\tau_1)\cdot\frac{1}{2}\underbrace{\begin{pmatrix}(e^{\alpha}+e^{-\alpha}) & +(e^{\alpha}-e^{-\alpha})\\ (e^{\alpha}-e^{-\alpha}) & +(e^{\alpha}+e^{-\alpha})\end{pmatrix}}_{P}x$$

$$= \frac{\partial a(t-\tau_1)\cdot P\cdot x}{\partial x}\cdot\begin{pmatrix}0\\ x_2^2\end{pmatrix} =$$

$$= \frac{1}{2}a(t-\tau_1)\cdot\begin{pmatrix}(e^{\alpha}-e^{-\alpha})x_2^2\\ (e^{\alpha}+e^{-\alpha})x_2^2\end{pmatrix}$$

$$x_2^2 = \frac{1}{2}\left((e^{\tau_1-\tau_2}-e^{-(\tau_1-\tau_2)})x_{01} + (e^{\tau_1-\tau_2}+e^{-(\tau_1-\tau_2)})x_{02}\right)^2$$

$$W_2(t,\tau_1,\tau_2;x) = \frac{1}{4}a(t-\tau_1)\cdot P(\tau_1-\tau_2)\cdot\left[(e^{\tau_1-\tau_2}-e^{-(\tau_1-\tau_2)})x_{01}+\dots\right]^2$$

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class: 16/10/19

$$\begin{cases}\dot{x}_1 = x_2+\cos(x_1)u\\ \dot{x}_2 = -x_1+u\\ y = x_2\end{cases}\qquad A=\frac{\partial f(x)}{\partial x}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\qquad g(x)=\begin{pmatrix}\cos(x_1)\\1\end{pmatrix}$$

$$h(x) = (0\ \ 1)\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

eigenvalues

$$(A-\lambda I)=\begin{pmatrix}-\lambda&1\\-1&-\lambda\end{pmatrix} \Rightarrow \det(A-\lambda I)=\lambda^2+1=0$$

$$\lambda^2=-1$$

$$\lambda_{12}=\sqrt{1}\cdot\sqrt{-1}=\pm j$$

eigenvectors:

$$(A-\lambda I)u=0 \qquad \begin{pmatrix}-\lambda&1\\-1&-\lambda\end{pmatrix}\begin{pmatrix}u_{11}\\u_{12}\end{pmatrix}=0$$

$\lambda=j$

$$\begin{cases}-ju_{11}+u_{12}=0\\ -u_{11}-ju_{12}=0\end{cases}\qquad\begin{cases}u_{12}=ju_{11}\\ \text{———}\end{cases}\qquad u_1=\begin{pmatrix}1\\j\end{pmatrix}$$

$$u_2=u_1^*=\begin{pmatrix}1\\-j\end{pmatrix}$$

$$U=\begin{pmatrix}1&1\\j&-j\end{pmatrix} \Rightarrow u_a=\frac{u_1+u_1^*}{2}=\frac{1}{2}\left[\begin{pmatrix}1\\j\end{pmatrix}+\begin{pmatrix}1\\-j\end{pmatrix}\right]=\begin{pmatrix}1\\0\end{pmatrix}$$

$$u_b=\frac{u_1-u_1^*}{2j}=\frac{1}{2j}\left[\begin{pmatrix}1\\j\end{pmatrix}-\begin{pmatrix}1\\-j\end{pmatrix}\right]=\frac{1}{2j}\begin{pmatrix}0\\2j\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$z_j \qquad z_j \left[ (j)^{\cdot} (-j) \right] = \bar{z}_j |z_j|^{\cdot} (1)$$

$$\bar{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \bar{U}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} Re[\,] & Im[\,] \\ -Im[\,] & Re[\,] \end{pmatrix} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{matrix} \alpha = 0 \\ \omega = 1 \end{matrix}$$

$$e^{At} = \bar{U} e^{\Lambda t} \bar{U}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha t}\cos\omega t & e^{\alpha t}\sin\omega t \\ -e^{\alpha t}\sin\omega t & e^{\alpha t}\cos\omega t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\alpha t}\cos\omega t & e^{\alpha t}\sin\omega t \\ -e^{\alpha t}\sin\omega t & e^{\alpha t}\cos\omega t \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\gamma_0(t; x_0) = e^{At} x_0 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x_0 = \begin{pmatrix} \cos t\, x_{01} + \sin t\, x_{02} \\ -\sin t\, x_{01} + \cos t\, x_{02} \end{pmatrix}$$

$$W_0(t; x_0) = \frac{\partial h(x)}{\partial x} \gamma_0(t; x_0) = (0 \;\; 1) \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x_0 =$$

$$= \underbrace{\begin{pmatrix} -\sin t & \cos t \end{pmatrix}}_{a(t)} x_0$$

$$W_1(t, \tau_1; x) = \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1; x_0))$$

$$\tilde{W}_0(t, \tau_1; x) = L_g \tilde{W}_0(t - \tau_1; x) = \frac{\partial a(t-\tau_1) x}{x} \cdot \begin{pmatrix} \cos(x_1) \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin(t-\tau_1) & \cos(t-\tau_1) \end{pmatrix} \begin{pmatrix} \cos(x_1) \\ 1 \end{pmatrix}$$

$$= -\sin(t-\tau_1)\cos(x_1) + \cos(t-\tau_1)$$

$$\gamma_0(\tau_1; x_0) = \begin{pmatrix} \cos\tau_1 & \sin\tau_1 \\ -\sin\tau_1 & \cos\tau_1 \end{pmatrix} x_0 \Rightarrow x_1 = \cos\tau_1\, x_{01} + \sin\tau_1\, x_{02}$$

$$W_1(t, \tau_1; x) = \cos(t-\tau_1) - \sin(t-\tau_1)\cos(\cos(\tau_1) x_{01} + \sin(\tau_1) x_{02})$$

$$W_2 = e^{\tau_2 L_g} \tilde{W}_1(t, \tau_1, \tau_2; x) = \tilde{W}_1(t, \tau_1, \tau_2; \gamma_0(\tau_2; x_0))$$

$$\tilde{W}_1(t, \tau_1, \tau_2; x) = L_g \cdot \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1 - \tau_2; x)) \qquad \alpha = \tau_1 - \tau_2$$

$$= L_g\, a(t-\tau_1) \cdot \underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{P} x$$

$$= \frac{\partial a(t-\tau_1) \cdot P \cdot x}{\partial x} \cdot \begin{pmatrix} \cos(x_1) \\ 1 \end{pmatrix}$$

$$= a(t-\tau_1) \cdot P \cdot \begin{pmatrix} \cos(x_1) \\ 1 \end{pmatrix}$$

$$\gamma_0(\tau_2; x_0) = \begin{pmatrix} \cos(\tau_2) & \sin(\tau_2) \\ -\sin(\tau_2) & \cos(\tau_2) \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$$

$$\gamma_0(\tau_2; x_0) = \begin{pmatrix} \cos(\tau_2) & \sin(\tau_2) \\ -\sin(\tau_2) & \cos(\tau_2) \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}$$

$$x_1 = \cos(\tau_2)\, x_{01} + \sin(\tau_2)\, x_{02}$$

$$W_2(t, \tau_1, \tau_2; x) = \begin{pmatrix} -\sin(t-\tau_1) & \cos(t-\tau_1) \end{pmatrix} \cdot \begin{pmatrix} \cos(\tau_1-\tau_2) & \sin(\tau_1-\tau_2) \\ -\sin(\tau_1-\tau_2) & \cos(\tau_1-\tau_2) \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \cos(\cos(\tau_2)x_{01} + \sin(\tau_2)x_{02} \\ 1 \end{pmatrix}$$

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3/9/19

$$\begin{cases} \dot{x}_1 = x_2 u \\ \dot{x}_2 = x_2 - x_2 u \\ y = x_2 + x_1 \end{cases} \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

eigenvalues

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \qquad \det(A - \lambda I) = -\lambda(1-\lambda) = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$

eigenvectors

$$(A - \lambda I)v = 0 \implies \begin{pmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$

$$\lambda = 0 \implies \begin{cases} 0 = 0 \\ v_b = 0 \end{cases} \implies v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = 1 \implies \begin{cases} -v_a = 0 \\ 0 = 0 \end{cases} \implies v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = U e^{\Lambda t} U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$$

$$\gamma_0(t; x_0) = e^{At} x_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} x_0$$

$$W_0(t; x_0) = C e^{At} x_0 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} x_0$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_{01} \\ e^t x_{02} \end{pmatrix} = x_{01} + e^t x_{02}$$

$$= \begin{pmatrix} 1 & e^t \end{pmatrix} x_0$$

$$W_1(t,\tau_1;x_0) = Ce^{A(t-\tau_1)} N e^{A\tau_1} x_0 =$$

$$= (1\ 1)\begin{pmatrix}1 & 0\\ 0 & e^{(t-\tau_1)}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{\tau_1}\end{pmatrix} x_0$$

$$W_2(t,\tau_1,\tau_2;x_0) = Ce^{A(t-\tau_1)} N e^{A(\tau_1-\tau_2)} N e^{A\tau_2} x_0 =$$

$$= (1\ \ 1)\begin{pmatrix}1 & 0\\ 0 & e^{(t-\tau_1)}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{(\tau_1-\tau_2)}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{\tau_2}\end{pmatrix} x_0$$

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04/07/19

$$\begin{cases}\dot{x}_1 = u\\ \dot{x}_2 = x_1 + \cos(x_1)u\\ y = x_2 - \sin(x_1)\end{cases}$$

$$A = \frac{\partial f(x)}{\partial x} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad g(x) = \begin{pmatrix}1\\ \cos(x_1)\end{pmatrix}$$

$$h(x) = -\sin(x_1) + x_2$$

$$\det(A-\lambda I) = \begin{vmatrix}-\lambda & 0\\ 1 & -\lambda\end{vmatrix} = \lambda^2 = 0 \Rightarrow \lambda_{12} = 0 \qquad m = 2$$ algebraic mult.

$$(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix}-\lambda & 0\\ 1 & -\lambda\end{pmatrix}\begin{pmatrix}v_a\\ v_b\end{pmatrix} = 0$$

$$\lambda = 0 \Rightarrow \begin{cases}\text{—}\\ v_a = 0\end{cases} \Rightarrow v_1 = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$v_2$: $(A-\lambda I)v = v_1$

$$\begin{pmatrix}-\lambda & 0\\ 1 & -\lambda\end{pmatrix}\begin{pmatrix}v_a\\ v_b\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\lambda = 0 \Rightarrow \begin{cases}\text{—}\\ v_a = 1\end{cases} \Rightarrow v_2 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$U = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad U^{-1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad J = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$e^{At} = Ue^{Jt}U^{-1} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} =$$

$e^{0t} = 1$

$$= \begin{pmatrix}0 & 1\\ 1 & t\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

$$\boxed{J = \begin{bmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{bmatrix}}$$

$$J = \begin{bmatrix} 0 & \lambda & 1 & \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma_0(t;x_0) = e^{At}x_0 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} x_0 = \begin{pmatrix} x_{01} \\ t\,x_{01} + x_{02} \end{pmatrix}$$

$$W_0(t;x_0) = \frac{\partial h(x)}{\partial x}\gamma_0(t;x_0) = (-\cos(x_1) \,\vdots\, 1)\begin{pmatrix} x_{01} \\ t x_{01} + x_{02} \end{pmatrix}$$

$$= -\cos(x_1)\,x_{01} + t\,x_{01} + x_{02}$$

$$= (t - \cos(x_1))\,x_{01} + x_{02} =$$

$$= (\underbrace{t - \cos(x_1) \,\vdots\, 1}_{a(t)})\,x_0$$

$$W_1(t,\tau_1,x) = \tilde{W}_0(t,\tau_1;\gamma_0(\tau_1;x_0))$$

$$\tilde{W}_0(t,\tau_1,x) = L_g \cdot W_0(t-\tau_1;x) =$$

$$= \frac{\partial a(t-\tau_1)x}{\partial x} \circ \begin{pmatrix} 1 \\ \cos(x_1) \end{pmatrix} =$$

$$= (t-\tau_1) - \cancel{\cos(x_1)} + \cancel{\cos(x_1)}$$

$$W_2(t,\tau_1,\tau_2;x) = e^{\tau_2 L_g}\tilde{W}_1(t,\tau_1,\tau_2;x) = \tilde{W}_1(t,\tau_1,\tau_2;\gamma_0(\tau_2;x_0))$$

$$\tilde{W}_1(t,\tau_1,\tau_2;x) = L_g\,W_0(t,\tau_1;\gamma_0(\tau_1-\tau_2;x))$$

$$= L_g \cdot a(t-\tau_1)\begin{pmatrix} 1 & 0 \\ \tau_1-\tau_2 & 1 \end{pmatrix} x$$

$$= \frac{\partial a(t-\tau_1)\,P\,x}{x} \circ \begin{pmatrix} 1 \\ \cos(x_1) \end{pmatrix}$$

$$= a(t-\tau_1)\,P\begin{pmatrix} 1 \\ \cos(x_1) \end{pmatrix} =$$

$$= (t-\tau_1-\cos(x_1) \,\vdots\, 1)\begin{pmatrix} 1 & 0 \\ \tau_1-\tau_2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \cos(x_1) \end{pmatrix} =$$

$$= (t-\tau_1-\cos(x_1) \,\vdots\, 1)\begin{pmatrix} 1 \\ \tau_1-\tau_2+\cos(x_1) \end{pmatrix}$$

$$= t - \cancel{\tau_1} - \cancel{\cos(x_1)} + \cancel{\tau_1} - \tau_2 + \cancel{\cos(x_1)} = t - \tau_2$$

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10/01/19

$$\begin{cases} \dot{x}_1 = -x_1 + x_2^2 \\ \dot{x}_2 = u \\ y = x_1 \end{cases}$$

$$f(x) = \begin{pmatrix} -x_1 + x_2^2 \\ 0 \end{pmatrix}$$

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = \left.\begin{pmatrix} -1 & 2x_2 \\ 0 & 0 \end{pmatrix}\right|_{x=0} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad h(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(-1-\lambda) = 0 \qquad \lambda_1 = 0, \quad \lambda_2 = -1$$

eigenvectors

$$(A - \lambda I)u = 0 \Rightarrow \begin{pmatrix} -1-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} u_a \\ u_b \end{pmatrix} = 0$$

$$\lambda = 0 \Rightarrow \begin{cases} -u_a = 0 \\ \text{—} \end{cases} \Rightarrow u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = -1 \Rightarrow \begin{cases} \text{—} \\ u_b = 0 \end{cases} \Rightarrow u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ e^{-t} & 0 \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_0(t; x_0) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} x_0$$

$$W_0(t; x_0) = \frac{\partial h}{\partial x} \gamma_0(t; x_0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} x_0$$

$$= \underbrace{\begin{pmatrix} e^{-t} & 0 \end{pmatrix}}_{a(t)} x_0$$

$$W_1(t, \tau_1; x) = \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1; x_0))$$

$$\tilde{W}_0(t, \tau_1; x) = L_g W_0(t - \tau_1; x)$$

$$\tilde{W}_0(t,\tau_1;x) = L_g \tilde{W}_0(t-\tau_1;x)$$
$$= \frac{\partial a(t-\tau_1)x}{\partial x} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$
$$= (e^{-t} \quad 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

The realization problem: find $A, N, B, C$

$W_0(t;x_0) = Q(t;x_0) \qquad W_k = 0 \quad \forall k>0$

$$\gamma_1(t,\tau_1;x_0) = \tilde{\gamma}_0(t,\tau_1;\gamma_0(\tau_1;x_0))$$
$$\tilde{\gamma}_0(t,\tau_1;x) = L_g \tilde{\gamma}_0(t-\tau_1;x)$$
$$= \begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\gamma_1(t,\tau_1;x_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} x(t) = \gamma_0(t,t_0;x_0) + \int_{t_0}^{t} \gamma_1(t,\tau_1,t_0;x_0)u(\tau_1)d\tau_1 + \dots \\ y(t) = W_0(t,t_0;x_0) + \int_{t_0}^{t} \underbrace{W_1(t,t_0,\tau_1;x_0)}_{\to 0} u(\tau_1)d\tau_1 + \int_{t_0}^{t}\int_{t_0}^{\tau_1} \underbrace{W_2 \dots}_{\to 0} \end{cases}$$

$$\gamma_2(t,\tau_1,\tau_2;x) = \tilde{\gamma}_1(t,\tau_1,\tau_2;\gamma_0(\tau_2;x_0))$$
$$\tilde{\gamma}_1(t,\tau_1,\tau_2,x) = L_g \cdot \tilde{\gamma}_0(t,\tau_1;\gamma_0(\tau_1-\tau_2;x_0))$$
$$= \begin{pmatrix} e^{-(t-\tau_1)} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\gamma_2(t,\tau_1,\tau_2;x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} x + \begin{pmatrix} 0 & B_{21}C_{22} \\ 0 & 0 \end{pmatrix} xu + \begin{pmatrix} 0 \\ B_{22} \end{pmatrix} u$$
$$y = (C_{21} \quad 0)x$$

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Exam (21/12/18)

$$\begin{cases} \dot{x}_1 = -x_1 + e^{x_2}u \\ \dot{x}_2 = -x_1 - x_2 + u \\ y = \end{cases}$$

$$f(x) = \begin{pmatrix} -x_1 \\ -x_1 - x_2 \end{pmatrix} \qquad A = \left.\frac{\partial f(x)}{\partial x}\right|_0 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} e^{x_2} \\ \end{pmatrix}$$

$$\begin{cases} \dot{x_i} = -x_1 - x_2 + u \\ y = x_2 \end{cases}$$

$$0 \quad \cdot \quad (-x_1 - x_2) \qquad \partial x \,|_0 \, (-1 \;\; -1)$$

$$g(x) = \begin{pmatrix} e^{x_2} \\ 1 \end{pmatrix} \qquad C = \frac{\partial h(x)}{\partial x} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$h(x) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

eigenvalues

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 = 0$$

$$\lambda_{12} = -1 \qquad m = 2$$

eigenvectors

$$(A - \lambda I) v = 0 \qquad \begin{pmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$

$$\lambda = -1 \Rightarrow \begin{cases} 0 = 0 \\ -v_a = 0 \end{cases} \qquad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$v_2$:

$$(A - \lambda I) v = v_1 \Rightarrow \overset{\lambda = -1}{\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ -v_a = 1 \end{cases} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad U^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow e^{Jt} = e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$$e^{At} = U e^{Jt} U^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -te^{-t} & e^{-t} \\ -e^{-t} & 0 \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{pmatrix}$$

$$\gamma_0(t; x_0) = e^{At} x_0 = \begin{pmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{pmatrix} x_0$$

$$W_0(t; x_0) = C \gamma_0(t; x_0) = \underbrace{(-te^{-t} \mid e^{-t})}_{a(t)} x_0$$

$$W_1(t,\tau_1;x) = \widetilde{W}_0(t,\tau_1;\gamma_0(\tau_1;x_0))$$

$$\widetilde{W}_0(t,\tau_1;x) = L_{\xi}\cdot \widetilde{W}_0(t-\tau_1;x)$$

$$= \frac{\partial a(t-\tau_1)x}{\partial x} \cdot \begin{pmatrix} e^{x_2} \\ 1 \end{pmatrix}$$

$$= \left(-(t-\tau_1)e^{-(t-\tau_1)} \;\vdots\; e^{-(t-\tau_1)}\right)\begin{pmatrix} e^{x_2} \\ 1 \end{pmatrix}$$

$$\gamma_0(\tau_1;x_0) = \begin{pmatrix} e^{-\tau_1} & 0 \\ -\tau_1 e^{-\tau_1} & e^{-\tau_1} \end{pmatrix} x_0 = \begin{pmatrix} e^{-\tau_1}x_{01} \\ -\tau_1 e^{-\tau_1}x_{01} + e^{-\tau_1}x_{02} \end{pmatrix}$$

$$x_2 = -\tau_1 e^{-\tau_1}x_{01} + e^{-\tau_1}x_{02}$$

$$W_1(t,\tau_1;x) = \left(-(t-\tau_1)e^{-(t-\tau_1)} \;\vdots\; e^{-(t-\tau_1)}\right)\begin{pmatrix} e^{(-\tau_1 e^{-\tau_1}x_{01} + e^{-\tau_1}x_{02})} \\ 1 \end{pmatrix}$$

$$W_2(t,\tau_1,\tau_2;x) = \widetilde{W}_1(t,\tau_1,\tau_2;\gamma_0(\tau_2;x_0))$$

$$\widetilde{W}_1(t,\tau_1,\tau_2;x) = L_{\xi}\,\widetilde{W}_0(t-\tau_1;\gamma_0(\tau_1-\tau_2;x))$$

$$= L_{\xi}\cdot a(t-\tau_1)\begin{pmatrix} e^{-(\tau_1-\tau_2)} & 0 \\ -(\tau_1-\tau_2)e^{-(\tau_1-\tau_2)} & e^{-(\tau_1-\tau_2)} \end{pmatrix} x$$

$$= \frac{\partial a(t-\tau_1)\cdot P(\tau_1-\tau_2)x}{\partial x} \cdot \begin{pmatrix} e^{x_2} \\ 1 \end{pmatrix}$$

with $P$ written above the matrix factor.

$$= a(t-\tau_1)\cdot P(\tau_1-\tau_2)\cdot\begin{pmatrix} e^{x_2} \\ 1 \end{pmatrix}$$

$$\gamma_0(\tau_2;x_0) = \begin{pmatrix} e^{-\tau_2} & 0 \\ -\tau_2 e^{-\tau_2} & e^{-\tau_2} \end{pmatrix} x_0$$

$$x_2 = -\tau_2 e^{-\tau_2}x_{01} + e^{-\tau_2}x_{02}$$

$$W_2(t,\tau_1,\tau_2;x) = \left(-(t-\tau_1)e^{-(t-\tau_1)} \;\vdots\; e^{-(t-\tau_1)}\right)\begin{pmatrix} e^{-(\tau_1-\tau_2)} & 0 \\ -(\tau_1-\tau_2)e^{-(\tau_1-\tau_2)} & e^{-(\tau_1-\tau_2)} \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} e^{(-\tau_2 e^{-\tau_2}x_{01} + e^{-\tau_2}x_{02})} \\ 1 \end{pmatrix}$$

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07/06/17

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_2 u \\ \dot{x}_2 = -x_2 + u \end{cases}$$

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$B = ($ … $\quad C = ($ …

$$\begin{cases} \dot{x}_2 = -x_2 + u \\ y = x_1 \end{cases} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

eigenvalues

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 = 0 \implies \lambda_{1,2} = -1 \qquad M = 2$$

eigenvectors

$$(A - \lambda I)\,U = 0 \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$$

$$\begin{cases} v_b = 0 \\ 0 = 0 \end{cases} \implies v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$v_2$:

$$(A - \lambda I)\,v = v_1 \implies \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} v_b = 1 \\ 0 = 0 \end{cases} \implies v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad e^{Jt} = e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$$e^{At} = U e^{Jt} U^{-1} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

$$\gamma_0(t; x_0) = e^{At} x_0$$

$$W_0(t; x_0) = C e^{At} x_0$$

$$W_1(t, \tau_1; x_0) = C e^{A(t-\tau_1)} N e^{A\tau_1} x_0$$

$$W_2(t, \tau_1, \tau_2; x_0) = C e^{A(t-\tau_1)} N e^{A(\tau_1 - \tau_2)} N e^{A\tau_2} x_0$$

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02/2020

$$\begin{cases} \dot{x}_1 = x_1 + x_2 x_1 u \\ \dot{} \end{cases} \qquad f(x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \qquad A = \frac{\partial f(x)}{\partial x}\Big|_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \dot{x}_1 = x_1 + x_2 x_1 u \\ \dot{x}_2 = x_1 u \\ y = x_1 - \frac{1}{2} x_2^2 \end{cases}$$

$$f(x) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \qquad A = \frac{\partial f(x)}{\partial x}\Big|_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$$

$$h(x) = x_1 - \frac{1}{2} x_2^2 \qquad C = \frac{\partial h(x)}{\partial x}\Big|_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = 0$$

$$(A - \lambda I) v = 0$$

$$\lambda = 1 \qquad \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \Rightarrow \begin{cases} 0 = 0 \\ -v_b = 0 \end{cases} \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = 0 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0 \Rightarrow \begin{cases} v_a = 0 \\ 0 = 0 \end{cases} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = U e^{\Lambda t} U^{-1} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_0(t; x_0) = e^{At} x_0 = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} x_0$$

$$W_0(t; x_0) = \begin{pmatrix} 1 & 0 \end{pmatrix} \gamma_0(t; x_0) = \underbrace{\begin{pmatrix} e^t & 0 \end{pmatrix}}_{a(t)} x_0$$

$$W_1(t, \tau_1; x) = \tilde{W}_0(t, \tau_1; \gamma_0(\tau_1; x_0))$$

$$\tilde{W}_0(t, \tau_1, x) = L_g \cdot \tilde{W}_0(t - \tau_1; x)$$

$$= \frac{\partial a(t - \tau_1) x}{\partial x} \cdot \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-\tau_1)} & 0 \end{pmatrix} \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$$

$$= e^{(t-\tau_1)} x_2 x_1$$

$$\gamma_0(\tau_1; x_0) = \begin{pmatrix} e^{\tau_1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} e^{\tau_1} x_{01} \\ x_{02} \end{pmatrix}$$

$$x_2 x_1 = e^{\tau_1} x_{01} x_{02}$$

$$W_1(t, \tau_1; x) = e^{(t-\tau_1)} e^{\tau_1} x_{01} x_{02}$$

$$W_2(t,\tau_1,\tau_2;x) = \tilde{W}_1(t,\tau_1,\tau_2,\gamma_0(\tau_2;x_0))$$

$$\tilde{W}_1(t,\tau_1,\tau_2;x) = L_g \cdot \tilde{W}_0(t-\tau_1;\gamma_0(\tau_1-\tau_2;x))$$

$$= L_g \cdot a(t-\tau_1)\begin{pmatrix} e^{\tau_1-\tau_2} & 0 \\ 0 & 1 \end{pmatrix} x$$

$$= \frac{\partial a(t-\tau_1) P x}{\partial x} \cdot \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$$

$$= a(t-\tau_1) P \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$$

(with $P$ written above the $\begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}$ in the fourth line)

$$\gamma_0(\tau_2;x_0) = \begin{pmatrix} e^{\tau_2} x_{01} \\ x_{02} \end{pmatrix}$$

$$x_2 x_1 = e^{\tau_2} x_{01} x_{02}$$

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**synthesis:**

$$\begin{cases} \dot{x}_1 \\ x_2 \\ y \end{cases} \qquad A = \left.\frac{\partial f(x)}{\partial x}\right|_0 \qquad C = \left.\frac{\partial h(x)}{\partial x}\right|_0$$

$$\gamma_0(t;x_0) = e^{At} x_0$$

$$W_0(t;x_0) = C e^{At} x_0 = \underbrace{(\,-\,-\,-\,-\,-\,)}_{a(t)} x_0$$

$$W_1(t,\tau_1;x_0) = \tilde{W}_0(t,\tau_1;\gamma_0(\tau_1;x_0))$$

$$\tilde{W}_0(t,\tau_1;x) = L_g \cdot \tilde{W}_0(t-\tau_1;x)$$

$$= \frac{\partial a(t-\tau_1) x}{\partial x} \cdot g(x)$$

$$\gamma_0(\tau_1;x_0)$$

$$W_2(t,\tau_1,\tau_2;x_0) = \hat{W}_1(t,\tau_1,\tau_2;\gamma(\tau_2;x_0))$$

$$\hat{W}_1(t,\tau_1,\tau_2;x) = L_g \cdot \hat{W}_0(t-\tau_1;\gamma_0(\tau_1-\tau_2;x))$$

$$= \frac{\partial a(t-\tau_1) \cdot \gamma_0(\tau_1-\tau_2) x}{\partial x} \cdot g(x)$$

$$\gamma(\tau_2;x_0)$$