27. Local coordinates transformations

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Given the state space form

$$\begin{cases} \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \\ y_1 = h_1(x) \\ y_2 = h_2(x) \end{cases}$$

m input
q output

Note first that input and output are independent, i.e.

$\text{rank}\left(g_1(x), \dots, g_m(x)\right) = m$, $\text{rank}\begin{pmatrix} dh_1(x) \\ \dots \\ dh_q(x) \end{pmatrix} = q$

The system has a (MIMO) vector Relative degree $r = \begin{pmatrix} r_1 \\ \dots \\ r_q \end{pmatrix}$ and it is defined if each r_i is defined ($r_i < \infty$), that is:

$L_{g_j} L_f^k h_i = 0 \quad k = 0, \dots, r_j - 2 \quad \forall i, j$

$L_{g_j} L_f^{r_i - 1} h_i \neq 0$ for at least a value of i

Supposing $m = q = 2$ (square case), it is possible to design the matrix $A(x)$ such that:

$$A(x) = \begin{pmatrix} L_{g_1} L_f^{r_1 - 1} h_1 & L_{g_2} L_f^{r_1 - 1} h_1 \\ L_{g_1} L_f^{r_2 - 1} h_2 & L_{g_2} L_f^{r_2 - 1} h_2 \end{pmatrix}$$

if $f(x) = Ax \quad g_1(x) = b_1 \quad g_2(x) = b_2$
$h_1(x) = c_1 x \quad h_2(x) = c_2 x$

then $A(x)$ is the Decoupling Matrix

where $a_{ij}(x) = \dfrac{\partial}{\partial u_j} y_i^{(r_i)}(t)$ are the elements.

The system has strong relative degree if $r_i < \infty$ and $\det(A(x)) \neq 0$.

If the system has strong relative degree then

$\Big/ \, da_1$

If the system has strong relative degree then

$$\rho \begin{pmatrix} dh_1 \\ \cdots \\ dL_g^{r_1-1} h_1 \\ dh_2 \\ \cdots \\ dL_g^{r_2-1} h_2 \end{pmatrix} = r_1 + r_2$$

Prop: Suppose the system has a vector relative degree at x_0. Then

$r_1 + r_2 \leq n$

and

$$\exists \begin{pmatrix} z_1 \\ z_2 \\ \eta \end{pmatrix} = \begin{pmatrix} \phi_1(x) \\ \hline \phi_2(x) \\ \hline \phi_3(x) \end{pmatrix} = \begin{pmatrix} h_1 \\ \cdots \\ L_g^{r_1-1} h_1 \\ \hline h_2 \\ \cdots \\ L_g^{r_2-1} h_2 \\ \hline \phi_3 \end{pmatrix}, \text{ in particular,}$$

if $r = r_1 + r_2 < n$, it is always possible to find $n-r$ more functions $\phi_{r+1}(x), \dots, \phi_n(x)$ such that the mapping

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_{r+1} \\ \cdots \\ \phi_n \end{pmatrix}$$ has a Jacobian matrix non singular at x_0

then it is qualified as a local coordinates transformation.
Moreover, if $G = \mathrm{span}\{g_1, g_2\}$ is involutive near x_0, it's always possible to find $\phi_{r+1}(x), \dots, \phi_n(x)$ such that

$L_{g_j} \phi_i(x) = 0 \quad j = 1,2 \quad \forall r+1 \leq i \leq n$

It is possible to express now, as for SISO systems, the system in the new coordinates:

$$\begin{cases} z_1^1 = h_1 \\ z_2^1 = L_g h_1 \\ \cdots \\ z^1 = L_g^{r_1-1} h_1 \end{cases}$$

$$
\begin{cases}
\quad \cdots \\
z_{r_1}^1 = L_f^{r_1-1} h_1 \\
z_1^2 = h_2 \\
\quad \cdots \\
z_{r_2}^2 = L_f^{r_2-1} h_2 \\
\eta = \phi_3(x)
\end{cases}
$$

therefore

$$\dot{z}_1^1 = z_2^1$$

$$\dot{z}_2^1 = z_3^1$$

$$\cdots$$

$$\dot{z}_{r_1-1}^1 = z_{r_1}^1$$

$$\dot{z}_{r_1}^1 = L_f^{r_1} h_1 + u_1 L_{g_1} L_f^{r_1-1} h_1 + u_2 L_{g_2} L_f^{r_2-1} h_2 =$$

$$= L_f^{r_1} h_1 + \begin{pmatrix} a_{11}(x) & a_{12}(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\dot{z}_1^2 = z_2^2$$

$$\dot{z}_2^2 = z_3^2$$

$$\cdots$$

$$\dot{z}_{r_2-1}^2 = z_{r_2}^2$$

$$\dot{z}_{r_2}^2 = L_f^{r_2} h_2 + \begin{pmatrix} a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\dot{\eta} = \frac{\partial \phi_3}{\partial x} \left(f + u_1 g_1 + u_2 g_2 \right) \Big|_{\phi^{-1}(z^1, z^2, \eta)} =$$

being $\phi_3 \ni p_i(z,\eta) = \frac{\partial \phi_3}{\partial x} \cdot g_i \Big|_{\phi^{-1}(z,\eta)} = L_{g_i} \phi_3 \Big|_{\phi^{-1}}$

$$= q(z,\eta) + \begin{pmatrix} p_1(z,\eta) & p_2(z,\eta) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$u = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}^{-1} \left(- \begin{pmatrix} L_f^{r_1} h_1 \\ L_f^{r_2} h_2 \end{pmatrix} + v \right) = A^{-1}(x) \left(-b(x) + v \right)$$

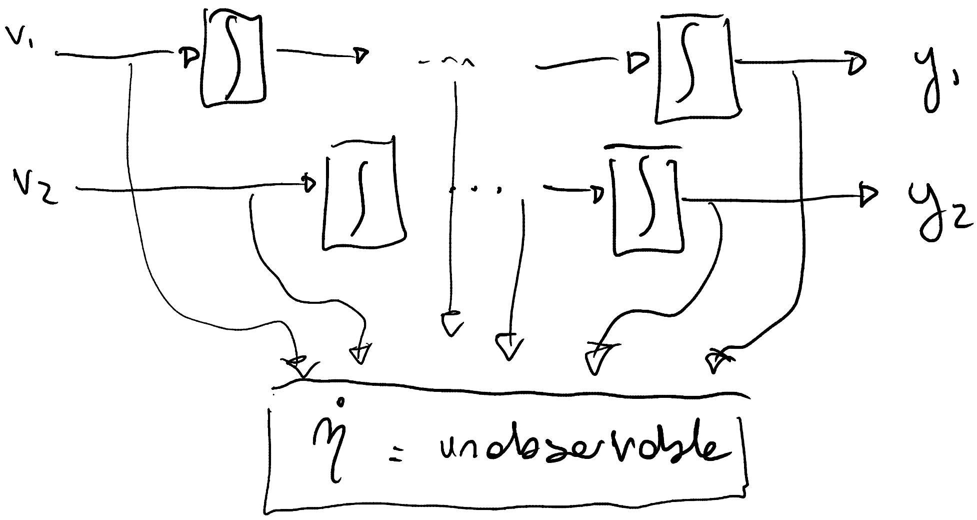
In this way the final system obtained has the following structure

$$\begin{cases} \dot{z}^1 = A^1 z^1 + B^1 v \end{cases}$$

structure

$$\begin{cases} \dot z^1 = A_0^1 z^1 + B_0^1 v_1 \\ \dot z^2 = A_0^2 z^2 + B_0^2 v_2 \\ \dot \eta = \tilde q(z,\eta) + \tilde p(z,\eta) v \\ y_1 = z_1^1 \\ y_2 = z_1^2 \end{cases}$$

where $A_0^{r_1, r_2} = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & \cdots & 0 \end{pmatrix}$ $\quad B_0^{r_1, r_2} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$



Prop:

U-Y feedback linearization

feedback linearization can be achieved around any point at which the system has strong relative degree.

Normal form

Around any point at which the strong relative degree is defined there exists $\Phi_3(x) \ni L_{g_i} \phi_3 \equiv 0 \quad i = 1, \dots, m$

if and only if $G_0 = \operatorname{span}\{g_1 \cdots g_m\} = \bar G$ involutive

Proof:

$$A(x) = \begin{pmatrix} dL_f^{r_1-1} h_1 \\ dL_f^{r_2-1} h_2 \end{pmatrix} (g_1 \; g_2) \quad \text{is non singular}$$

$\to g(g_1, g_2) = ?$ if it is involutive

$\hookrightarrow \rho(g_1, g_2) = 2$ if it is involutive

then $\exists\ \lambda_1, \dots, \lambda_{n-m}$ functions such that

$$\ker d\begin{pmatrix}\lambda_1\\ \cdots \\ \lambda_{n-m}\end{pmatrix} = G^{\perp}$$

Consider now

$$\begin{pmatrix} dH_1 \\ \hline dH_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ \cdots \\ L_f^{r_1-1} h_1 \\ \hline h_2 \\ \cdots \\ L_f^{r_2-1} h_2 \end{pmatrix}$$

and note that

$$\ker\begin{pmatrix} dH_1 \\ dH_2 \end{pmatrix} \cap G = \{0\}$$

in fact, if it is not true, there exists

$g = \alpha_1(x)\, g_1 + \alpha_2(x)\, g_2$ such that

$$\begin{pmatrix} dL_f^{r_1-1} h_1 \\ dL_f^{r_2-1} h_2 \end{pmatrix} g = A(x) \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix} = 0$$

$\downarrow$ non singular, therefore $\alpha_1 = \alpha_2 = 0$

and it is a contradiction.

$\rightarrow \dim\left(G^{\perp} + \operatorname{span}\{dH_1, dH_2\}\right) = n$

$\rightarrow \exists\ \phi_3$ in $\{\lambda_1, \dots, \lambda_{n-m}\} \ni L_{g_i}\phi_3 \equiv 0 \quad i = 1, \dots, m$

Prop: the normal form exists if the input vector fields generate an involutive distribution.

Looking again at the FL problem from the Normal Form: under feedback and coordinate transformation one has the structure shown previously.
If in addition there exists the NF ($G = \bar{G}$) and a coordinate transformation such that $L_g \phi_3 = 0$ is chosen then the unobservable dynamics will not depend directly

then the unobservable dynamics will not depend directly from the control.

$$\dot{z}^1 = \begin{pmatrix} z_2^1 \\ \cdots \\ z_{r_1}^1 \\ b_1(z,\eta) \end{pmatrix} + \begin{pmatrix} 0 \\ | \\ 0 \\ a_{11}(z,\eta) \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ | \\ 0 \\ a_{12}(z,\eta) \end{pmatrix} u_2$$

$$\dot{z}^2 = \begin{pmatrix} z_2^2 \\ \cdots \\ z_{r_2}^2 \\ b_2(z,\eta) \end{pmatrix} + \begin{pmatrix} 0 \\ | \\ 0 \\ a_{21}(z,\eta) \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ | \\ 0 \\ a_{22}(z,\eta) \end{pmatrix} u_2$$

$$\dot{\eta} = q(z,\eta) + p_1(z,\eta)u_1 + p_2(z,\eta)u_2$$

$$A(z,\eta) = \begin{pmatrix} a_{11}(z,\eta) & a_{12}(z,\eta) \\ a_{21}(z,\eta) & a_{22}(z,\eta) \end{pmatrix} \qquad b(z,\eta) = \begin{pmatrix} b_1(z,\eta) \\ b_2(z,\eta) \end{pmatrix} = \begin{pmatrix} L_f^{r_1} h \\ L_f^{r_2} h \end{pmatrix}$$

$$u = A^{-1}(z,\eta) \cdot \left(-b(z,\eta) + v\right)$$