Derivatives (FIN-404)

Report: Assignment

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PART I

Futures are a type of financial derivative that consists of a contract to buy or sell an asset at a predetermined future date and price. If you will have to buy the asset you have a long position, while if you will have to sell it you have a short position. The future price by definition is such that the value of the future is zero. This implies that an investor can enter a long or a short position without any cost. The future price is computed as follows:

$$f_{it}(T) = E_t^Q[S_{it}]$$

Since futures are traded, an investor can close every day his position receiving (or paying) a settlement. For example, an investor that had a long position will receive:

$$f_{it_k}(T) - f_{it_{k-1}}(T)$$

The last important characteristic, which prevents arbitrage is that:

$$f_{iT}(T) = S_{iT}{}^{1}$$

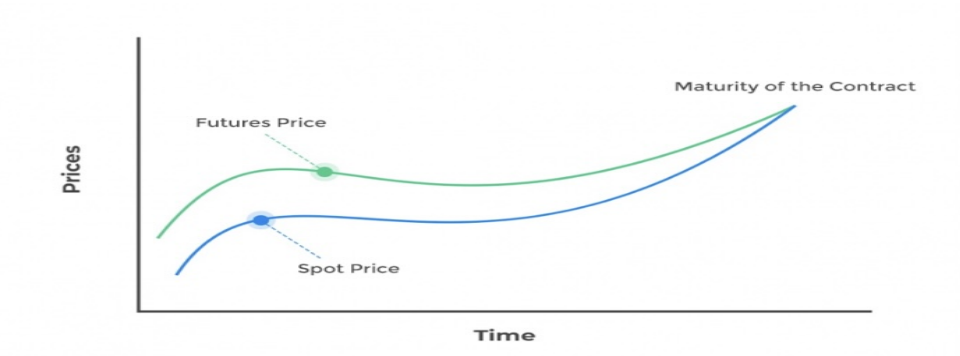

Figure 1: Convergence of Futures and Spot Prices

Futures contracts have both pros and cons. They can be used to speculate on price movements of the underlying or by companies to hedge products they sell or buy. These although can lead to missing out on potential gains, while investors can lose more due to the fact that futures use leverage. This means that the trader that enters a trade does not have to put up all the contract's value amount. Indeed, only an initial margin amount is required, which consists of a fraction of the total contract value. The fraction required by the broker depends on the broker's terms, the size of the contract, and the borrower's creditworthiness. [2]
The main difference between perpetual futures and futures is that the former has no expiration date. So there is no need to roll over to hold them permanently. To assure that the future price F_t does not diverge from the spot price S_t perpetual futures contracts are settled every 1, 4 or 8 hours through a funding mechanism. The mechanism used is based on periodic founding payments between long and short traders. If the contract is trading at a premium compared to

[1] the graph: https://analystprep.com/cfa-level-1-exam/derivatives/types-derivative-contracts/
[2] https://www.investopedia.com/terms/f/futures.asp

the spot price, long positions will pay funding to short positions, and vice versa if the contract is trading at a discount.

$$FundingPayment = FundingRate * TotalPositionValue$$

The funding rate is a function of the premium and interest rates. In particular, "the premium is the relative-spot price difference, and the interest-rate term captures the difference of risk-free rate between the quote and base assets". [3] If the perpetual is trading above the spot price the funding rate will be positive and long traders pay a fee to short traders. Vice versa, if the perpetual is trading below the spot price, short traders will pay a fee to long ones. There are

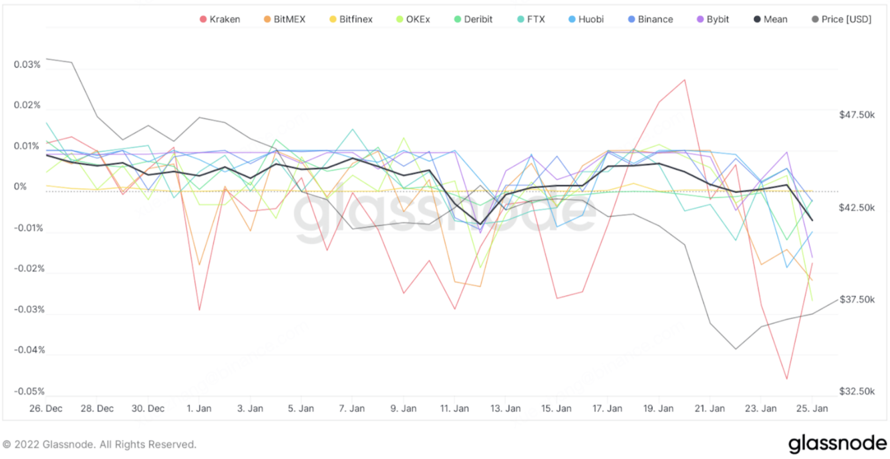

Figure 2: Funding rates on different platforms

three types of perpetual futures: linear, inverse, and quanto. Introducing the terminology:

$$future\ contract : BASE/QUOTE : Settlement - expiry$$

Perpetual linear futures maintain a constant linear relationship with the spot price of the underlying cryptocurrency. Each contract will represent a quantity of the COIN, and the margining and the settlement will be in USD. Perpetual linear futures have high liquidity and are interesting for investors that want to gain exposure to the asset's price movements. an example (COIN = BTC):

$$BTC/USD : USD - YYMMDD$$

Perpetual inverse futures try to track the inverse of the value of the underlying. In this case, the settlement and the margining are in the underlying, while each contract represents a quantity of the USD (or the fiat currency chosen). These contracts are of interest to traders that want to bet on a downward movement. An example (COIN = BTC):

$$BTC/USD : BTC - YYMMDD$$

Lastly, a perpetual Quanto future represents a quantity of COIN, while the margining and settlement are in BTC. In these contracts a fixed exchange rate for BTC/USD. Traders are so able to have long or short positions in the COIN/USD exchange rate without ever touching COIN or

[3] Crypto futures presentation by D.Ackerer, slide 22

USD. The margins are in Bitcoin, while traders gain or lose Bitcoin as the COIN/USD exchange rate changes. [4] An example (general COIN):

$$COIN/USD : BTC - YYMMDD$$

The first difference between perpetual futures and futures is the absence of maturity. Traders can maintain their positions for an undetermined period of time without worrying about continuously closing or rolling over their position. Furthermore, perpetual contracts have also higher leverage and lower trading costs. The absence of maturity makes it possible to enter or exit the position anywhere in time, so traders have to be worried only about the movements of the underlying. So perpetual futures enable taking exposure or hedge against the underlying without the need to roll over, with high leverage and without the delivery of it. Indeed, PERPs match spot index pricing more closely. Furthermore, dealing with digital assets makes some pros of normal futures vanish. Indeed, there is no need in the crypto market to hedge delivery costs or prices of products. [5]

We want to illustrate the cash flows associated to a long position in a perpetual linear future and a perpetual inverse future. We are then interested in understanding the pros and cons of linear vs inverse if we want to ultimately hold USD.

Let $(f_t)_{t=0}^{\infty}$ be a perpetual linear FERP [6]. So for any $T \in N$ the ex-dividend value of the security with a-dominated cash flow will be zero. The underlying we are going to consider is BTC/USD. The cash flows are:

$$\Delta f_t - \lambda(f_t - (BTC/USD)_t) - \xi(BTC/USD)_t \; ; \quad t = 1, 2, ..T$$

where the constants $\lambda > \xi$ are part of the contract specification. While Δf_t is common to normal futures, $\lambda(f_t - (BTC/USD)_t) + \xi(BTC/USD)_t$ is the funding payment. The founding rate is defined as follows:

$$\phi_t = \xi + \lambda(\frac{f_t}{(BTC/USD)_t} - 1)$$

In particular, ξ represents the interest rates, while $\lambda(\frac{f_t}{(BTC/USD)_t} - 1)$ the premium.

For the perpetual inverse FERP, we have that for any $T \in N$ the ex-dividend value of the security with b-denominated cash flows is 0. We are still interested in BTC/USD as underlying. So, let $(i_t)_{t=0}^{\infty}$ be a perpetual inverse FERP, then cash flows are:

$$\Delta\left[\frac{1}{i_t}\right] - \lambda_i(\frac{1}{i_t} - \frac{1}{(BTC/USD)_t}) - \frac{\xi}{(BTC/USD)_t}$$

where the constants $\lambda_i > \xi > 0$ are part of the contract specification. In this case, $\Delta\left[\frac{1}{i_t}\right]$ is common to normal futures, $\lambda_i(\frac{1}{i_t} - \frac{1}{(BTC/USD)_t}) + \frac{\xi}{(BTC/USD)_t}$ is the funding payment. The founding rate is defined as follows:

$$\lambda_i(\frac{(BTC/USD)_t}{i_t} - 1) + \xi$$

In particular, ξ represents the interest rates, while $\lambda_i(\frac{(BTC/USD)_t}{i_t} - 1)$ the premium. Since our goal is to ultimately have USD, the linear PERP seems to be preferable. Indeed, its cash flows

[4] https://www.bitmex.com/app/quantoPerpetualsGuide

[5] https://insights.glassnode.com/the-week-onchain-week-17-2022/

[6] future exchanged rate process

are in USD, while the inverse PERP is in BTC. So, the inverse PERP exposes us to the exchange risk from BTC to USD. On the other hand, being exposed to the exchange risk can also improve our profits if BTCs increase their value with respect to USD dollars. The formulations of cash flows presented for both the linear and inverse FERP are a simplification relative to the real implementation used by trading platforms.

Perpetual futures have become the most traded derivative for crypto, reaching a daily volume of over 100 billion. [7] For BTC the trend had a stop. Over the last five years, BTC perpetual have been increasing, starting from a little volume spot traded and becoming a lot larger with the introduction of perpetual futures. Although, while during the first half the aggregate future trade volume was ranging daily between $70B and $80B, in 2022 reached around $31B experiencing a downside of 59%. [8] In the following graph [9], we can see the overtake of perpetual future over spot trading in the first years of their introduction.

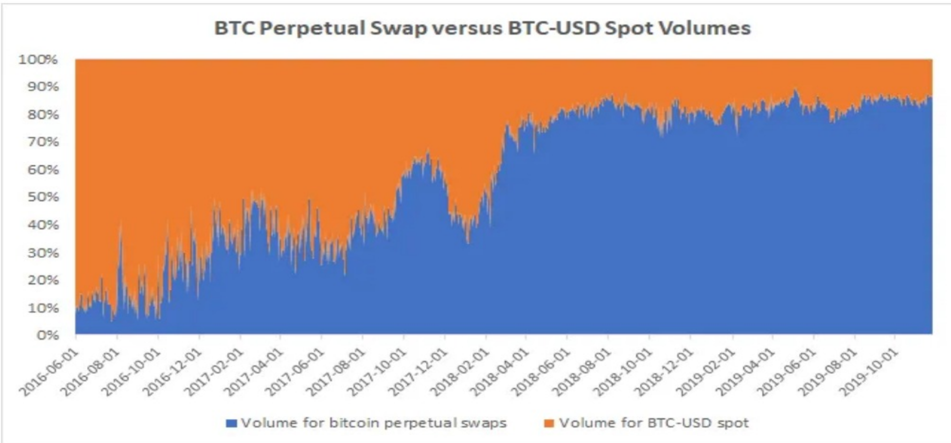

Figure 3: BTC: Perpetual Future vs Spot

For BTC we can see the dominance of PERPs also looking at their open interests evolution[10]. Open interest keeps record of all the open contracts and so differs from traded volume which may contain closed or netted positions.

[7] Fundamentals of Perpetual Futures by Songrun He, Asaf Manela, Omri Ross, Victor von Wachter
[8] https://insights.glassnode.com/the-week-onchain-week-17-2022/
[9] https://medium.com/interdax/an-overview-of-the-evolution-of-bitcoin-derivatives-c10080c00d29
[10] https://insights.glassnode.com/the-week-onchain-week-17-2022/

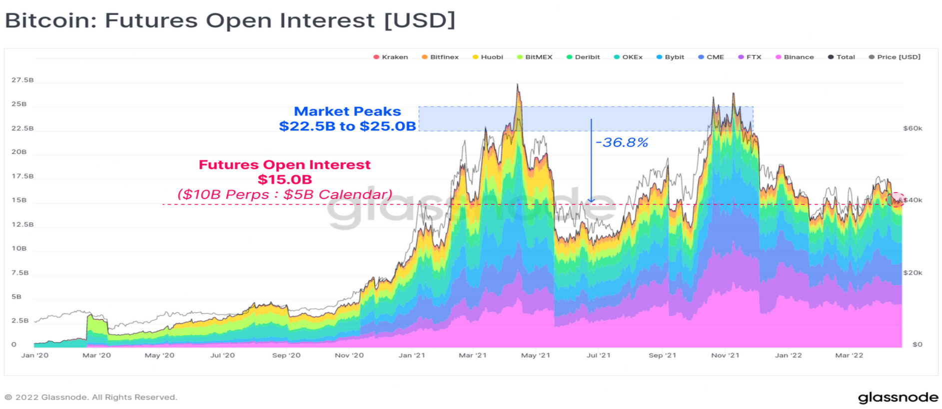

Figure 4: ETH: Perpetual Future vs Spot

Blue represents the trade volume of PERPs compared to total future, reaching 92.4%. Pink instead represents the ETH experienced the same overtake by perpetual against spot trades, and the PERPs are dominating the nowadays market as we can see from the following graph[11]:

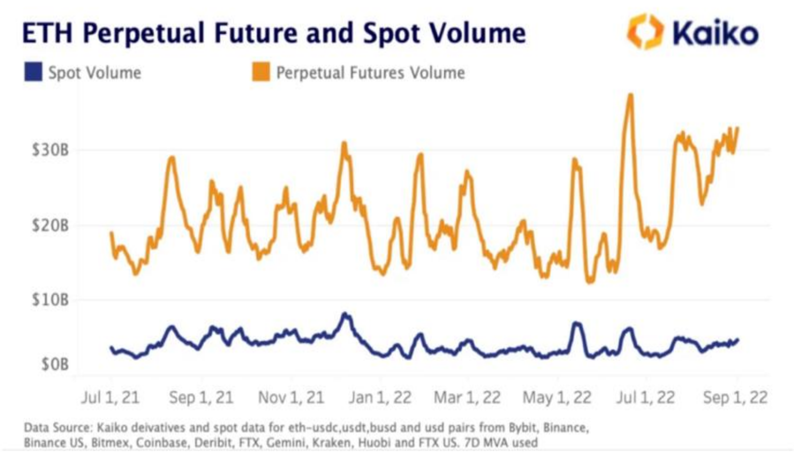

Figure 5: ETH: Perpetual Future vs Spot

Perpetual futures are without a doubt used mainly in the crypto market since their characteristics are well-suited to the needs of this market. Perpetual futures could be useful in markets with similar problems to the ones crypto markets had, like low liquidity and leverage.

[11]Crypto futures presentation by D.Ackerer

PART II

1. What is the value in a of one share of the b-denominated asset?

$$S_t^a = X_t S_{0t}^b$$

2. Show that from the point of view (pov) of an investor who uses currency a as his unit of account the market consists in one riskless asset with rate r_a and one risky asset with price S_t^a to be determined.

$$S_{0t}^a$$

is the risk-free asset for an investor who uses currency a.

$$\frac{S_{t+1}^a}{S_t^a} = \frac{X_{t+1} S_{0t+1}^b}{X_t S_{0t}^b} = e^{r_b \Delta} U^{\theta_{t+1}} D^{1-\theta_{t+1}}$$

is the risky asset.

3. Provide a condition on the parameters (ra, rb, Δ, U, D) under which the market is arbitrage free and complete from the pov of an investor who uses currency a as his unit of account.

We impose the condition that discounted price is a martingale under an EMM.

$$\widehat{S_t^a} = E_t^{Q_a}[\widehat{S_{t+1}^a}]$$

$$1 = E_t^{Q_a}\left[\frac{\widehat{S_{t+1}^a}}{\widehat{S_t^a}}\right] = e^{(r_b - r_a)\Delta} E_t^{Q_a}\left[U^{\theta_{t+1}} D^{1-\theta_{t+1}}\right]$$

$$E_t^{Q_a}\left[U^{\theta_{t+1}} D^{1-\theta_{t+1}}\right] = q^a U + (1-q^a)\Delta = q^a(U-D) + D$$

$$q^a = \frac{e^{(r_a - r_b)\Delta} - D}{U - D}$$

If $U > e^{(r_b - r_a)\Delta > D}$, Q_a is unique and $0 < q^a < 1$ hence the model is arbitrage-free and complete since there exists a unique EMM. Hence, we have a binomial model with $U^1 = U e^{r_a \Delta}$ and $D^1 = D e^{r_b \Delta}$.

4. What is the value in b of one share of the a-denominated asset?

$$S_t^b = \frac{1}{X_t} S_{0t}^a$$

5. Show that from the pov of an investor who uses currency b as his unit of account the market consists one riskless asset with rate rb and one risky asset with price S_t^b to be determined.

$$S_{0t}^b$$

is the risk-free asset for an investor who uses currency b.

$$\frac{S_{t+1}^b}{S_t^b} = \frac{X_t S_{0t+1}^a}{X_{t+1} S_{0t}^a} = e^{r_a \Delta} \frac{1}{U^{\theta_{t+1}} D^{1-\theta_{t+1}}}$$

is the risky asset.

6. Show that the condition you derived in 3 above also guarantees that the market is arbitrage free and complete from the pov of an investor who uses currency b as his unit of account.
As in point 3 we impose the condition that discounted price is a martingale under an EMM.

$$\widehat{S_t^b} = E_t^{Q_b}[\widehat{S_{t+1}^b}]$$

$$1 = E_t^{Q_b}\left[\frac{\widehat{S_{t+1}^b}}{\widehat{S_t^b}}\right] = e^{\Delta(r_a - r_b)} E_t^{Q_b}\left[\frac{1}{U^{\theta_{t+1}} D^{1-\theta_{t+1}}}\right]$$

$$E_t^{Q_b}\left[\frac{1}{U^{\theta_{t+1}} D^{1-\theta_{t+1}}}\right] = q^b \frac{1}{U} + (1-q^b)\frac{1}{D} = q^b\left(\frac{1}{U} - \frac{1}{D}\right) + \frac{1}{D}$$

$$q^b = \frac{e^{(r_b - r_a)\Delta} - \frac{1}{D}}{\frac{1}{U} - \frac{1}{D}} = \frac{UDe^{(r_b - r_a)\Delta} - U}{D - U}$$

In order to have:

$$0 < q^b < 1$$

we need:

$$\frac{UDe^{(r_b - r_a)\Delta} - U}{D - U} > 0$$

$$De^{(r_b - r_a)\Delta} - 1 < 0 \;\; hence \;\; e^{(r_a - r_b)\Delta} > D$$

and

$$\frac{UDe^{(r_b - r_a)\Delta} - U}{D - U} < 1$$

$$UDe^{(r_b - r_a)}\Delta - U > D - U \;\; hence \;\; e^{(r_a - r_b)\Delta} < U$$

This is the same condition we derived in 3.
Hence we have a binomial model with $D^2 = \frac{1}{Ue^{-r_a\Delta}}$ and $U^2 = \frac{1}{De^{r_a\Delta}}$. But in this case q^b is the probability of a down move since it is the probability that the random variable θ assumes value 1.

7. Show that the futures exchange rate for date T is uniquely given by

$$f_t(T) = E^{Q_a}[X_T|F_t] = \Psi^{T-t} x_t$$

for some $\Psi > 0$ to be determined.

$$f_t(T) = E_t^{Q_a}[X_T] = E_t^{Q_a}\left[X_{T-1}\left(\frac{U}{D}\right)^{\theta_T}\right] D = E_t^{Q_a}\left[\left(\frac{U}{D}\right)^{\Sigma_{i=t+1}^T \theta_i}\right] x_t D^{T-1} =$$

$$(1 - q_a + q_a\frac{U}{D})^{T-t} D^{T-t} x_t = [D(1-q_a) + q_a U]^{T-t} x_t = \psi^{T-t} x_t$$

with

$$\psi = D(1-q_a) + q_a U = e^{(r_a - r_b)\Delta}$$

since

$$E[S^x] = (1-q+qS)^n \quad with \quad x \sim Bin(n,q)$$

8. Assume that $(f_t)_{t=0}^{\infty}$ is a perpetual FERP. Show that

$$M_t \equiv \sum_{\tau=1}^{t} e^{-r_a \Delta \tau}(\Delta f_\tau - \lambda(f_\tau - x_\tau) - \xi x_\tau)$$

is a martingale under Q_a.
We check that

$$M_t = E_t^{Q_a}[M_{t+1}]$$

Starting from the fact that for a perpetual FERP the value of a security with a-denominated cash flows

$$\Delta f_t - \lambda(f_t - x_t) - \xi x_t$$

at dates t = 1, 2, ..., T is zero at all times s = 0, 1, ..., T we have:

$$E_t^{Q_a}\left[\sum_{\tau=t+1}^{T} \frac{S_{0t}^a}{S_{0\tau}^a}(\Delta f_\tau - \lambda(f_\tau - x_\tau) - \xi x_\tau)\right] = 0 \quad \forall T, \forall t - 1 \tag{1}$$

$$\iff E_t^{Q_a}\left[\sum_{\tau=t+1}^{T} \frac{(\Delta f_\tau - \lambda(f_\tau - x_\tau) - \xi x_\tau)}{S_{0\tau}^a}\right] = 0$$

$$\iff E_t^{Q_a}\left[\sum_{\tau=1}^{T} \frac{(\Delta f_\tau - \lambda(f_\tau - x_\tau) - \xi x_\tau)}{S_{0\tau}^a}\right] = \sum_{\tau=1}^{t} \frac{(\Delta f_\tau - \lambda(f_\tau - x_\tau) - \xi x_\tau)}{S_{0\tau}^a}$$

$$\iff E_t^{Q_a}[M_T] = M_t$$

and M_t is a martingale under Q_a.

9. Assume that $(f_t)_{t=0}^{\infty}$ is a perpetual FERP. Show that

$$f_t = E^{Q_a}[f_{t+1}|F_t] + \beta E^{Q_a}[x_{t+1}|F_t]$$

for all t = 0,1,... and some α, $\beta > 0$ to be determined.
Since M_t is a martingale

$$E_t^{Q_a}[M_{t+1}] = M_t^{Q_a} \Rightarrow E_t^{Q_a}[\Delta M_{t+1}] = 0$$

Therefore

$$\mathrm{E}_t^{Q_a}[e^{-r_a\Delta(t+1)}(\Delta f_{t+1} - \lambda(f_{t+1} - x_{t+1}) - \xi x_{t+1})] = 0 \Rightarrow f_t = E_t^{Q_a}[(f_{t+1}(1-\lambda) + x_{t+1}(\lambda - \xi)]$$

where $\alpha = 1 - \lambda$ and $\beta = \lambda - \xi$

10. Assume that $(f_t)_{t=0}^{\infty}$ is a perpetual FERP. Show that

$$f_t = \alpha^{T-t} E^{Q_a}[f_T|F_t] + \sum_{\tau=t+1}^{T} \alpha^{\tau-(t+1)} \beta E^{Q_a}[x_\tau|F_t]$$

for all $T \geq t = 0, 1, \ldots$
Starting from the result of previous point

$$f_t = \alpha E_t^{Q_a}[f_{t+1}] + \beta E_t^{Q_a}[x_{t+1}]$$

$$f_{t+1} = \alpha E_{t+1}^{Q_a}[f_{t+2}] + \beta E_{t+1}^{Q_a}[x_{t+2}]$$

$$f_t = \alpha E_t^{Q_a}[\alpha E_{t+1}^{Q_a}[f_{t+2}] + \beta E_{t+1}^{Q_a}[x_{t+2}]] + \beta E_t^{Q_a}[x_{t+1}] =$$

$$= \alpha^2 E_t^{Q_a}[f_{t+2}] + \alpha\beta E_t^{Q_a}[x_{t+2}] + \beta E_t^{Q_a}[x_{t+1}]$$

Iterating this procedure:

$$f_t = \alpha^{T-t} E_t^{Q_a}[f_T] + \sum_{\tau=t+1}^{T} \alpha^{\tau-(t+1)} \beta E^{Q_a}[x_\tau]$$

11. Show that if $(f_t)_{t=0}^{\infty}$ is a perpetual FERP and y is a constant then $(y\alpha^{-t} + f_t)_{t=0}^{\infty}$ is also a perpetual FERP. Explain intuitively where this non-uniqueness of the perpetual futures price come from?
To show that $(y\alpha^{-t} + f_t)_{t=0}^{\infty}$ is a FERP we check that 1 holds :

$$E_t^{Q_a}[\sum_{\tau=t+1}^{T} e^{-r_a\Delta(T-t)}(y\alpha^{-\tau} + f_\tau - y\alpha^{-(\tau-1)} - f_{\tau-1} - \lambda(y\alpha^{-\tau} + f_\tau - x_\tau) - \xi x_\tau] =$$

In this passage we use that 1 holds for a perpetual FERP

$$= E_t^{Q_a}[\sum_{\tau=t+1}^{T} e^{-r_a\Delta(T-t)}(y\alpha^{-\tau} - y\alpha^{-(\tau-1)} - \lambda(y\alpha^{-\tau})] =$$

$$= E_t^{Q_a}\left[\sum_{\tau=t+1}^{T} e^{-r_a\Delta(T-t)}\left(\frac{y}{(1-\lambda)^\tau} - \frac{y}{(1-\lambda)^{\tau-1}} - \lambda\frac{y}{(1-\lambda)^\tau}\right)\right] =$$

$$= E_t^{Q_a}\left[\sum_{\tau=t+1}^{T} e^{-r_a\Delta(T-t)}\left(\frac{y - y(1-\lambda) - \lambda y}{(1-\lambda)^\tau}\right)\right] = 0 \qquad \forall T, \forall t \leq T-1$$

$\Rightarrow (y\alpha^{-t} + f_t)_{t=0}^{\infty}$ is a FERP.
This non uniqueness can come from the fact that the perpetual FERP does not have a terminal date and therefore a terminal condition but the perpetual future price and the stock price are kept close to each other through the funding rate mechanism. In addition, at any point in time, there is no certainty about the fact that the two prices will converge.

12. Show that if

$$0 < (1-\lambda)e^{(r_a - r_b)\Delta} < 1$$

then

$$f_t^* \equiv \sum_{n=1}^{\infty} \alpha^{n-1} \beta E^{Q_a}[x_{t+n}|F_t] = \phi x_t$$

for some $\phi > 0$ to be determined and that in this case $(f_t^*)_{t=0}^{\infty}$ is the unique perpetual FERP that satisfies:

$$\lim_{N\to\infty} \alpha^N E^{Q_a}[f_{t+N}|F_t] = 0$$

for all t = 0, 1, ... In an infinite horizon problem this requirement plays the same role as a terminal boundary condition in a finite horizon problem. It is known as a transversality condition.

$$E_t^{Q_a}[X_{t+n}] = E_t^{Q_a}\left[x_t \left(\frac{U}{D}\right)^{\sum_{i=t+1}^{t+n}\theta_i} D^n\right] = x_t D^n E_t^{Q_a}\left[\left(\frac{U}{D}\right)^{\sum_{i=t+1}^{t+n}\theta_i}\right] =$$

$$since \quad \sum_{i=t+1}^{t+n} \theta_i \sim Bin(n, q_a)$$

$$= x_t D^n \left(1 - q_a + q_a \frac{U}{D}\right)^n = x_t \psi^n$$

Then

$$\sum_{n=1}^{\infty} \alpha^{n-1}\beta E^{Q_a}[x_{t+n}|F_t] = \sum_{n=1}^{\infty} \alpha^{n-1)}\beta x_t \psi^n = \beta x_t \sum_{n=1}^{\infty} \alpha^{n-1}\psi^n =$$

$$= \beta x_t \psi \sum_{n=1}^{\infty} (\alpha\psi)^{n-1}\psi^{n-1} = \beta x_t \psi \frac{1}{1-\alpha\psi} = \frac{\beta x_t \psi}{1-\alpha\psi}$$

Therefore

$$\phi = \frac{\beta\psi}{1-\alpha\psi}$$

To check that the series converges we need $|\alpha\psi| < 1$

$$e^{(r_a - r_b)\Delta} = \psi \quad \alpha = (1-\lambda)$$

$\Rightarrow$ if $0 < (1-\lambda)e^{(r_a-r_b)\Delta} < 1$ the series converges.
Finally we show that the limit condition holds for f_{t+n}^*

$$E_t^{Q_a}[f_{t+n}^*] = E_t^{Q_a}[\phi x_{t+n}] = x_t \phi D^n E_t^{Q_a}\left[\left(\frac{U}{D}\right)^{\sum_{i=t+1}^{t+n}\theta_i}\right] = x_t \phi \psi^n$$

And $\lim_{N\to\infty} \alpha^N \phi x_t \psi^n = 0$ for all t if $0 < (1-\lambda)e^{(r_a-r_b)\Delta} < 1$ holds.
If there exists another FERP $f_t$then we can write f_{t+N} as:

$$f_{t+N} = \alpha^{T-t-N}E^{Q_a}[f_T|F_{t+N}] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)}\beta E^{Q_a}[x_\tau|F_{t*N}]$$

Plugging this formula in the transversality condition we get:

$$\lim_{N\to\infty} \alpha^N E_t^{Q_a}[f_{t+N}|F_t] =$$

$$= lim_{N\to\infty}\alpha^N E_t^{Q_a}[\alpha^{T-t-N}E^{Q_a}[f_T|F_{t+N}] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)}\beta E^{Q_a}[x_\tau|F_{t+N}]] =$$

$$= \lim_{N\to\infty} \alpha^{T-t} E^{Q_a}[f_T] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)} \beta E^{Q_a}[x_\tau] =$$

$$= \lim_{N\to\infty} \alpha^{T-t} E^{Q_a}[f_T] = \alpha^{T-t} E^{Q_a}[f_T]$$

The second sum goes to zero since as N goes to infinity since τ becomes bigger than T. The first term does not depend on N and therefore the limit is not 0. Therefore, by contradiction we have proven that the only FERP that satisfies the transversality condition is the one with $T \to \infty$.

13. Assume that $\lambda > \xi$ satisfies (1). Show that over one period the perpetual futures contract with parameters (λ, ξ) and perpetual FERP $(f_t^*)_{t=0}^{\infty}$ can be replicated by borrowing π units of currency a at rate r_a over one period and investing this amount transformed in units of currency b at rate r_b over one period.

Over 1 period we have:

	t	t+1
Future	0	$f_{t+1}^* - f_t^* - \lambda(f_{t+1}^* - x_{t+1}) - \xi x_{t+1} =$ $\phi x_{t+1} - \phi x_t - \lambda(\phi x_{t+1} - x_{t+1}) - \xi x_{t+1} =$ $x_{t+1}(\phi - \lambda\phi + \lambda - \xi) - \phi x_t$
Portfolio	0	$-\pi e^{r_a\Delta} + \pi \frac{1}{x_t} e^{r_b\Delta} x_{t+1}$

The cashflows associated with the future are the ones used in previous points where we substitute the value of f_{t+1}^* found in previous points. The cashflows of the porfolio come from the fact that at time t we borrow π at rate r_a, we convert this in currency b dividing by x_t, we invest it at rate r_b and in t+1 we convert it back to currency a.

Matching coefficients:

$$\pi e^{r_a\Delta} = \phi x_t \Rightarrow \pi = e^{-r_a\Delta} \phi x_t$$

Substituting in:

$$\pi \frac{1}{x_t} e^{r_b\Delta} x_{t+1} = e^{(r_a - r_b)\Delta} \phi = \frac{\phi}{\psi}$$

Finally we see that the following equation always holds:

$$\frac{\phi}{\psi} = (\phi - \lambda\phi + \lambda - \xi)$$

since

$$\phi\left(\frac{1}{\psi} + \lambda - 1\right) = \lambda - \xi \quad with \quad \alpha = 1 - \lambda \quad and \quad \beta = \lambda - \xi$$

$$\frac{\psi\beta}{1 - \alpha\psi}\left(\frac{1}{\psi} - \alpha\right) = \beta$$

$$\frac{\psi\beta}{1 - \alpha\psi}\left(\frac{1 - \alpha\psi}{\psi}\right) = \beta$$

$$\beta = \beta$$

14. Show that the inverse futures exchange rate for date T is uniquely given by:

$$i_t(T) = E^{Q_b}\left[\frac{1}{x_T}\middle|F_t\right]^{-1} = \psi^{T-t}x_t$$

with the same $\Psi > 0$ as in statement 1 above.

$$i_t(T) = E^{Q_b}\left[\frac{1}{x_T}\middle|F_t\right]^{-1} = \left(E^{Q_b}\left[\frac{1}{x_{T-1}}\left(\frac{D}{U}\right)^{\theta_T}\right]\frac{1}{D}\right)^{-1} =$$

iterating

$$= \left(E_t^{Q_b}\left[\left(\frac{D}{U}\right)^{\Sigma_{i=t+1}^{T}\theta_i}\right]\frac{1}{x_t}\frac{1}{D^{T-t}}\right)^{-1} = \left[\left(1-q_b+q_b\frac{D}{U}\right)^{T-t}\frac{1}{x_t}\frac{1}{D^{T-t}}\right]^{-1} =$$

$$= \left[\left[(1-q_b)\frac{1}{D}+q_b\frac{1}{U}\right]^{T-t}\right]^{-1}x_t = \left[\left[(1-q_b)\frac{1}{D}+q_b\frac{1}{U}\right]^{-1}\right]^{T-t}x_t$$

Therefore

$$\varphi = \left[(1-q_b)\frac{1}{D}+q_b\frac{1}{U}\right]^{-1} = \left[\frac{-Ue^{(r_b-r_a)\Delta}}{D-U}+\frac{De^{(r_b-r_a)\Delta}-1}{D-U}\right]^{-1} =$$

$$= \left[e^{(r_b-r_a)\Delta}\right]^{-1} = e^{(r_a-r_b)\Delta}$$

That is the same as in point 7.

15. Use the same steps as in the linear case to show that if $(i_t)_{t=0}^{\infty}$ is a perpetual inverse FERP then $g_t \equiv \frac{1}{i_t}$

$$g_t = \alpha_i^{T-t}E^{Q_b}[g_T|F_t] + \sum_{\tau=t+1}^{T}\alpha^{\tau-(t+1)}\beta_i E^{Q_b}\left[\frac{1}{x_\tau}\middle|F_t\right]$$

$\forall T \geq t = 0,1,\ldots$ **and some $\alpha_i, \beta_i > 0$ to be determined.**

Firstly we show that:

$$\sum_{\tau=1}^{T}e^{-r_b\Delta\tau}\left(\Delta\left[\frac{1}{i_\tau}\right]-\lambda_i\left(\frac{1}{i_t}-\frac{1}{x_t}\right)-\frac{\xi_i}{x_t}\right)$$

is a martingale under Q_b.

Starting from the fact that for perpetual inverse FERP the value of a security with b-denominated cash flows

$$\Delta\left(\frac{1}{i_t(T)}\right) = \frac{1}{i_t(T)} - \frac{1}{i_{t-1}(T)}$$

at dates $t = 1, 2, \ldots, T$ is equal to zero at all times we have:

$$E_t^{Q_b}\left[\sum_{\tau=t+1}^{T}\frac{S_{0t}^b}{S_{0\tau}^b}\left(\Delta\left[\frac{1}{i_\tau}\right]-\lambda_i\left(\frac{1}{i_\tau}-\frac{1}{x_\tau}\right)-\frac{\xi_i}{x_\tau}\right)\right]=0 \quad \forall T, \forall t \leq T-1$$

$$\iff E_t^{Q_b}\left[\sum_{\tau=t+1}^{T}\frac{1}{S_{0\tau}^b}\left(\Delta\left[\frac{1}{i_\tau}\right]-\lambda_i\left(\frac{1}{i_\tau}-\frac{1}{x_\tau}\right)-\frac{\xi_i}{x_\tau}\right)\right]=0$$

$$\iff E_t^{Q_b}\left[\sum_{\tau=1}^{T}\frac{1}{S_{0\tau}^b}\left(\Delta\left[\frac{1}{i_\tau}\right]-\lambda_i\left(\frac{1}{i_\tau}-\frac{1}{x_\tau}\right)-\frac{\xi_i}{x_\tau}\right)\right]=\sum_{\tau=1}^{T}\frac{1}{S_{0\tau}^b}\left(\Delta\left[\frac{1}{i_\tau}\right]-\lambda_i\left(\frac{1}{i_\tau}-\frac{1}{x_\tau}\right)-\frac{\xi_i}{x_\tau}\right)$$

$$\iff E_t^{Q_b}[M_T]=M_t$$

Therefore, M_t is a martingale under Q_b

$$E_t^{Q_b}[M_{t+1}]=M_t \Rightarrow E_t^{Q_b}[\Delta M_{t+1}]=0$$

$$E_t^{Q_b}\left[e^{-r_b\Delta(t+1)}\left(\Delta\left[\frac{1}{i_{t+1}}\right]-\lambda_i\left(\frac{1}{i_{t+1}}-\frac{1}{x_{t+1}}\right)-\frac{\xi_i}{x_{t+1}}\right)\right]=0$$

$$\Rightarrow \frac{1}{i_t}=E_t^{Q_b}\left[\frac{1}{i_{t+1}}(1-\lambda_i)+\frac{1}{x_{t+1}}(\lambda_i-\xi_i)\right]$$

$$\Rightarrow \frac{1}{i_t}=E_t^{Q_b}\left[\frac{1}{i_{t+1}}\right]+\beta_i E_t^{Q_b}\left[\frac{1}{x_{t+1}}\right] \quad with \quad \alpha_i=1-\lambda_i \quad and \quad \beta_i=\lambda_i-\xi_i$$

Using the recursion relation found we finally get:

$$\frac{1}{i_t}=\alpha_i^{T-t}E_t^{Q_b}\left[\frac{1}{i_T}\right]+\sum_{\tau=t+1}^{T}\alpha^{\tau-(t+1)}\beta_i E_t^{Q_b}\left[\frac{1}{x_\tau}\right]$$

$$\frac{1}{i_t}=g_t \quad \frac{1}{i_T}=g_T$$

and the relation is proved

16. **Show that if:**

$$0<(1-\lambda_i)e^{(r_b-r_a)\Delta}<1$$

then:

$$g_t^* \equiv \sum_{n=1}^{\infty}\alpha_i^{n-1}\beta_i E^{Q_b}\left[\frac{1}{x_{t+n}}\middle| F_t\right]=\frac{1}{\Phi_i x_t}$$

for some $\phi > 0$ to be determined and that $i_t^* = 1/g_t^* = \phi_i x_t$ is the unique perpetual inverse FERP that satisfies the transversality condition.

$$\lim_{N\to\infty}\alpha^N E^{Q_b}\left[\frac{1}{i_{t+N}^*}\middle| F_t\right]=0$$

for all t = 0,1,...

$$E_t^{Q_b}\left[\frac{1}{x_{t+n}}\right]=E_t^{Q_b}\left[\frac{1}{x_t}\left(\frac{D}{U}\right)^{\sum_{i=t+1}^{t+n}\theta_i}\frac{1}{D^n}\right]=\frac{1}{x_t}D^n E^{Q_b}\left[\left(\frac{D}{U}\right)^{\sum_{i=t+1}^{t+n}\theta_i}\right]=$$

$$with \quad \sum_{i=t+1}^{t+n} \theta_i \sim Bin(n, q_b)$$

$$\frac{1}{x_t}\frac{1}{D^n}\left(1-q_b+q_b\frac{D}{U}\right)^n = \frac{1}{x_t}\left((1-q_b)\frac{1}{D}+q_b\frac{1}{U}\right)^n = \frac{1}{x_t}\frac{1}{\psi^n}$$

Then

$$\sum_{n=1}^{\infty} \alpha_i^{n-1}\beta_i E^{Q_b}\left[\frac{1}{x_{t+n}}\right] = \sum_{n=1}^{\infty} \alpha_i^{n-1}\beta_i \frac{1}{x_t}\frac{1}{\psi^n} = \beta_i \sum_{n=1}^{\infty} \alpha_i^{n-1}\frac{1}{\psi^n} =$$

$$\beta_i \frac{1}{x_t}\frac{1}{\psi}\sum_{n=1}^{\infty}\left(\frac{\alpha_i}{\psi}\right)^{n-1} = \beta_i \frac{1}{x_t}\frac{1}{\psi}\frac{1}{1-\frac{\alpha_i}{\psi}} = \frac{\beta_i}{x_t}\frac{1}{\psi-\alpha_i}$$

$$\Rightarrow \phi_i = \frac{\psi-\alpha_i}{\beta_i}$$

To check that the series converges we need $|\frac{\alpha_i}{\psi}| < 1 \quad |\frac{1-\lambda_i}{\psi}| < 1 \quad |\frac{1-\lambda_i}{e^{(r_a-r_b)\Delta}}| < 1$
If $0 < (1-\lambda_i)e^{(r_b-r_a)\Delta} < 1$ The series converges.
Finally we show that the limit condition holds for $\frac{1}{i^*_{t+n}}$

$$E^{Q_b}\left[\frac{1}{i^*_{t+n}}\right] = E^{Q_b}\left[\frac{1}{\phi_i}\frac{1}{x_{t+n}}\right] = \frac{1}{\phi_i x_t}\frac{1}{D^n}E^{Q_b}\left[\left(\frac{D}{U}\right)^{\Sigma_{i=t+1}^{t+n}\theta_i}\right] = \frac{1}{\phi_i \psi^n x_t}$$

$$\lim_{N\to\infty} \alpha_i^n \frac{1}{\phi_i \psi^n x_t} = 0 \quad if\ 0 < (1-\lambda_i)e^{(r_b-r_a)\Delta} < 1$$

If there exists another inverse FERP i_t then we can write it as:

$$\frac{1}{i_{t+N}} = \alpha^{T-t-N}E^{Q_a}[\frac{1}{i_T}|F_{t+N}] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)}\beta E^{Q_a}[\frac{1}{x_\tau}|F_{t*N}]$$

Plugging this formula in the transversality condition we get:

$$\lim_{N\to\infty} \alpha^N E^{Q_a}[\frac{1}{i_{t+N}}|F_t] =$$

$$= lim_{N\to\infty}\alpha^N E^{Q_a}[\alpha^{T-t-N}E^{Q_a}[\frac{1}{i_T}|F_{t+N}] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)}\beta E^{Q_a}[\frac{1}{x_\tau}|F_{t+N}]] =$$

$$= \lim_{N\to\infty} \alpha^{T-t}E^{Q_a}[\frac{1}{i_T}] + \sum_{\tau=t+N+1}^{T} \alpha^{\tau-(t+N+1)}\beta E^{Q_a}[\frac{1}{x_\tau}] =$$

$$= \lim_{N\to\infty} \alpha^{T-t}E^{Q_a}[\frac{1}{i_T}] = \alpha^{T-t}E^{Q_a}[\frac{1}{i_T}]$$

The second sum goes to zero since as N goes to infinity since τ becomes bigger than T. The first term does not depend on N and therefore the limit is not 0. Therefore, by contradiction we have proven that the only inverse FERP that satisfies the transversality condition is the one with $T \to \infty$.

17. **Show that there are unique values of ξ and ξ_i for which $\phi = \phi_i = 1$ irrespective of the choice of λ and λ_i satisfying**

$$0 < (1-\lambda)e^{(r_a - r_b)\Delta} < 1$$

and

$$0 < (1-\lambda_i)e^{(r_b - r_a)\Delta} < 1$$

.

Let us consider

$$\phi = 1 \Rightarrow \frac{\beta\psi}{1-\alpha\psi} = 1 \Rightarrow \frac{(\lambda-\xi)\psi}{1-(1-\lambda)\psi} = 1 \Rightarrow$$

$$\Rightarrow (\lambda-\xi)\psi = 1-\psi+\lambda\psi \Rightarrow -\xi\psi = 1-\psi \Rightarrow \xi = \frac{\psi-1}{\psi} \quad independent\ on\ \lambda$$

$$\phi_i = 1 \Rightarrow \frac{\psi-\alpha_i}{\beta_i} = 1 \Rightarrow \frac{\psi-(1-\lambda_i)}{\lambda_i-\xi_i} = 1 \Rightarrow$$

$$\psi - 1 = -\xi_i \Rightarrow \xi_i = 1-\psi \quad independent\ on\ \lambda_i$$

18. Assume from now on that

$$(\lambda, \xi; \lambda_i, \xi_i) = (\Lambda+\Gamma, \Gamma; \Lambda_i+\Gamma_i, \Gamma_i)\Delta$$

for some Λ,$\Lambda_i > 0$ such that

$$0 < (1-\lambda)e^{(r_a - r_b)\Delta} < 1$$

and

$$0 < (1-\lambda_i)e^{(r_b - r_a)\Delta} < 1$$

hold for Δ small enough. Show that the associated perpetual futures exchange rates satisfy:

$$\lim_{\Delta\to 0}\left(\frac{f_t^*}{x_t}\right) = \Theta \equiv \frac{\Lambda}{\Lambda+\Gamma-\delta}$$

and

$$\lim_{\Delta\to 0}\left(\frac{i_t^*}{x_t}\right) = \Theta_i \equiv \frac{\Lambda_i+\Gamma_i+\delta}{\Lambda_i}$$

for some constant $\delta \in R$ to be determined. Show that there are unique values of Γ and Γ_i such that $\Theta = \Theta_i = 1$ for any Λ and Λ_i . How does the specification of the implied funding rates compare to those actually used by exchanges such as Binance or BitMEX?

$$\lim_{\Delta\to 0}\left(\frac{f_t^*}{x_t}\right) = \lim_{\Delta\to 0}\phi = \lim_{\Delta\to 0}\frac{(\lambda-\xi)e^{(r_a-r_b)\Delta}}{1-(1-\lambda)e^{(r_a-r_b)\Delta}} = \frac{\lambda-\xi}{\lambda} = \frac{\Lambda+\Gamma-\Gamma}{\Lambda+\Gamma} = \frac{\Lambda}{\Lambda+\Gamma} = \Theta \Rightarrow \delta = 0$$

$$\lim_{\Delta\to 0}\left(\frac{i_t^*}{x_t}\right) = \lim_{\Delta\to 0}\frac{\phi_i x_t}{x_t} == \lim_{\Delta\to 0}\phi_i = \lim_{\Delta\to 0}\frac{e^{(r_a-r_b)\Delta}-(1-\lambda_i)}{\lambda_i-\xi_i} = \frac{\lambda_i}{\xi_i-\lambda_i} = \frac{\Lambda_i+\Gamma_i}{\Lambda_i} = \Theta_i \Rightarrow \delta = 0$$

$\Theta = 1$ if $\Gamma = 0$ independent on Λ
$\Theta_i = 1$ if $\Gamma_i = 0$ independent on Λ_i
This implies $\xi = 0$ and $\xi_i = 0$, and therefore, the interest rate component of the funding rate is set to 0. This differs from the specification of the implied funding rates actually used by exhanges such as Binance or BitMEX (e.g. Binance uses an interest rate of 0.01%). In addition,

our model does not present restrictions on λ and λ_i, only assuming $0 < \lambda < 1$ and $0 < \lambda_i < 1$. Also this is different from what exchanges do. In fact, the funding rate is defined by:
Funding Rate (F) = Premium Index (P) + clamp (Interest Rate (I) – Premium Index (P), 0.05%, -0.05%) [12]
where:
Interest Rate (I) = (Interest Quote Index – Interest Base Index)/Funding Interval
Interest Base Index = The Interest Rate for borrowing the Base currency
Interest Quote Index = The Interest Rate for borrowing the Quote currency
Funding Interval = 3 (Since funding occurs every 8 hours)
Premium Index (P) = [Max (0, Impact Bid Price – Price Index) – Max (0, Price Index – Impact Ask Price)] / Price Index

19. The above suggests that to prevent arbitrages the perpetual and perpetual inverse exchange rates on a currency pair should both be equal to the spot exchange rate, at least if the exchange uses an appropriate specification of the interest rate parts (ξ, ξ_i) of the funding rate. Use the data in DataSample.csv and a statistical method of your choice to confirm or infirm these relations for the currency pairs BTC/\$ and ETH/\$. In case of failure propose and implement a trading strategy designed to benefit from the discrepancy.
Firstly, we check whether the discrepancy between perpetual exchange rates on a currency pair and the spot exchange rates is significantly different from 0. In order to do that, we implemented two different methods:

- Run a regression of the spot exchange rate on the perpetual exchange rate and check if the two are equal through a F-test with the null hypothesis: $\alpha = 0$ $\beta = 1$
- Perform a paired t-test between the spot exchange rate and the perpetual exchange rate

For all the pair spot exchange rates and perpetual exchange rates (for both BTC and ETH) the p-values of both tests are largely below the 5% threshold and close to 0. Therefore, we can reject the null that spot exchange rates and perpetual exchange rates are equal.
Then, the main idea of our strategy for linear FERP, is to go long in the future contract and sell the crypto-currency in the spot market if the perpetual exchange rate is lower than the spot exchange rate and, vice-versa, if the perpetual exchange rate is higher than the spot exchange rate, we short the future contract and buy the crypto-currency in the spot market.
The main idea of our strategy for inverse FERP, is to go long in the future contract and sell the currency (in this case USD) in exchange for the crypto if the perpetual exchange rate is higher than the spot exchange rate and, vice-versa, if the perpetual exchange rate is lower than the spot exchange rate, we short the future contract and buy the currency (USD) in exchange for the crypto.
This is because the previous point suggests that the spot exchange rate and the perpetual linear exchange rate should be equal and, therefore, if the second one is higher than the first one we expect the first one to decrease or the second to increase until a day when they will be the same (or better when there will be switch of sign in the difference between spot exchange rate and future exchange rate).
On the other hand the same reasoning is done for the inverse FERP, but due to the different cash flows of this contract we should invest in the opposite way.

[12] https://phemex.com/user-guides/how-are-funding-rates-calculated

To model the cash flows of the contract we use the simple definition given in text of the project. We set $\xi = 0$ and $\xi_i = 0$ as found in the previous point and since $0 < \lambda < 1$ and $0 < \lambda_i < 1$ we try both $\lambda = 0.01$ and $\lambda = 0.99$, and, $\lambda_i = 0.01$ and $\lambda_i = 0.99$. We use these extremes values since the cash flows are linear in λ and λ_i and therefore, in this way we could model the best and worst scenarios.
Since the data set provides daily values we implement strategies with a daily frequency. To evaluate the performance of the strategies we sum the cash flows that we obtain each day. We decide not to capitalize all of them to the last, nor to discount them to the initial date, since we have a short period of time.
To implement these strategies we need the possibility to use leverage in the future exchange and also the possibility to invest a given amount of money in the spot market to buy the cryptocurrency. This amount of money could be obtained, for example, borrowing at the risk free rate. The results of the strategies are shown in the table 1

Table 1: Results of the strategies

	$\lambda = 0.01$	$\lambda = 0.99$
BTC linear	281.9891 USD	3642.3209 USD
BTC inverse	9.2029e-07 BTC	1.2268e-05 BTC
ETH linear	16.9775 USD	262.7125 USD
ETH inverse	1.3811e-05 ETH	0.00019456 ETH

The plots 6,7,8,9 show the cash flows of the strategies in each day.

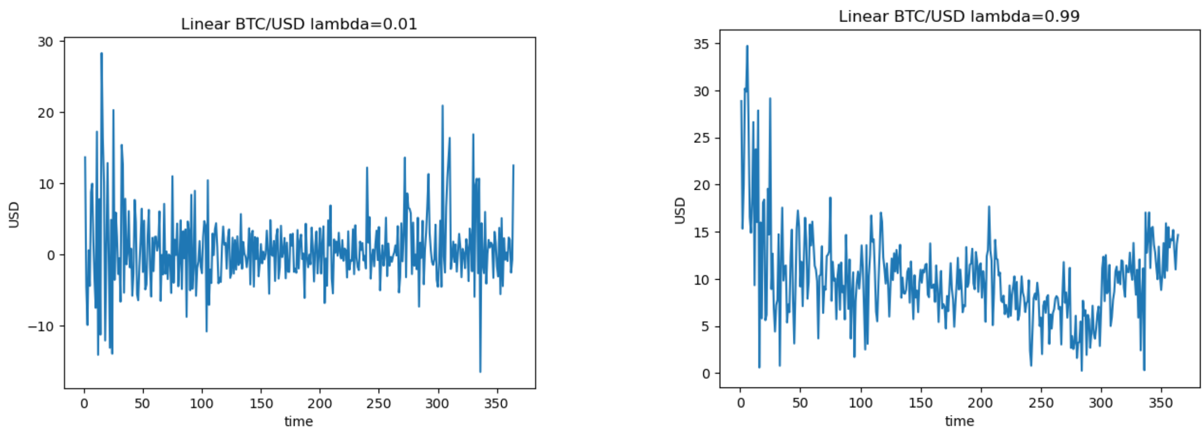

Figure 6: Daily cash flows for strategies on linear FERP on BTC/USD

It is possible to see that all the strategies make profit and that the performance of the strategies increase when λ increases as we expected since perpetual rate is more often undervalued. In conclusion, it seems to be possible to use the previous strategies to make an arbitrage that exploits the fact that the spot exchange rate and the future exchange rate will converge. It is important to underline that there could be still some risks related to these strategies[13] such as:

- If the prices take too much time to converge the position in the future market can be closed due to an amount of leverage higher than the maximum allowed.

[13]https://blog.hummingbot.org/2021-03-spot-perpetual-protocol-guide/

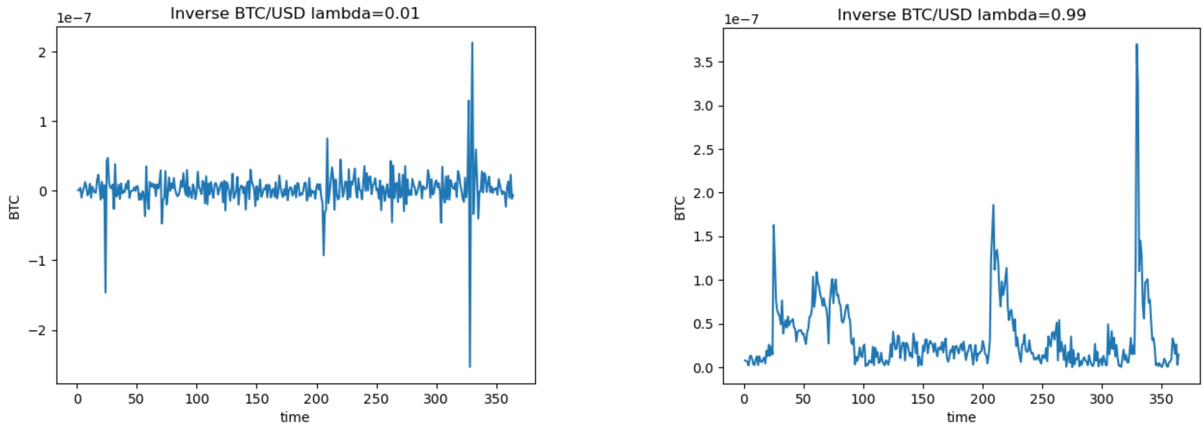

Figure 7: Daily cash flows for strategies on inverse FERP on BTC/USD

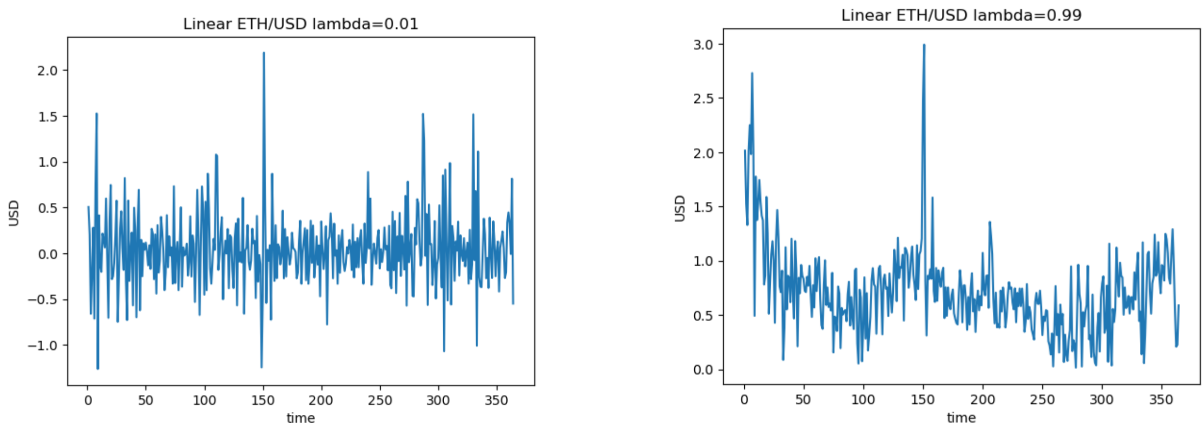

Figure 8: Daily cash flows for strategies on linear FERP on ETH/USD

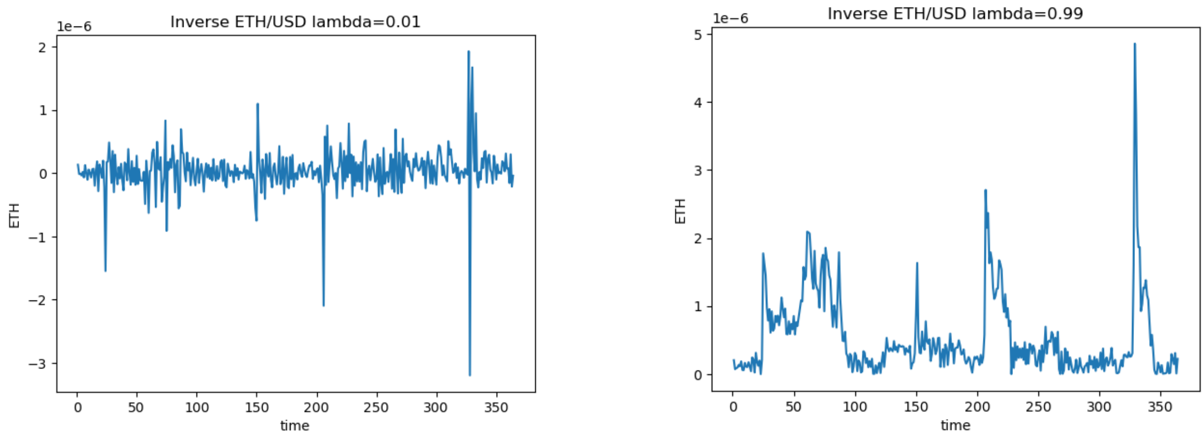

Figure 9: Daily cashflows for strategies on inverse FERP on ETH/USD

- Trades could take some time to be executed and therefore the arbitrage opportunity could no more be present