

# Maths refresher course

## HSLU, Semester 1

Matteo Frongillo

September 6, 2024

### Contents

<b>I</b>	<b>Lesson 1</b>	<b>3</b>
1	Numerical sets	3
2	Prime numbers	3
3	Positive powers	3
3.1	Property 1 . . . . .	3
3.2	Property 2 . . . . .	3
3.3	Property 3 . . . . .	3
4	Fractions	4
4.1	Property 1 . . . . .	4
4.2	Property 2 . . . . .	4
4.3	Property 3 . . . . .	4
5	Negative powers	4
5.1	Definition . . . . .	4
5.2	Property 4 . . . . .	4
5.3	Property 5 . . . . .	4
6	Fractions and percentages (and back)	5
<b>II</b>	<b>Lesson 2</b>	<b>6</b>
7	Symbols	6
8	Brackets	6
9	Latin notations	6
10	The real line	6
10.1	Exercises . . . . .	6
11	Properties of real numbers	7
11.1	Property 1 - Closure under “+” and “.” . . . .	7
11.2	Property 2 - Commutativity . . . . .	7
11.3	Property 3 - Associative . . . . .	7
11.4	Property 4 - Distributive . . . . .	7
11.5	Property 5 - Identity . . . . .	7
11.6	Property 6 - Inverses and opposites . . . . .	7
12	The order of operations	7
13	Signed numbers	8

<b>14 Absolute value</b>	<b>8</b>
14.1 Property . . . . .	8
<b>III Lesson 3</b>	<b>9</b>
<b>15 Polynomials</b>	<b>9</b>
15.1 Terms and factors . . . . .	9
15.1.1 Variables . . . . .	9
15.1.2 Sets . . . . .	9
15.2 Expressions, terms and factors . . . . .	9
15.2.1 Expressions . . . . .	9
15.2.2 Terms . . . . .	9
15.2.3 Factors . . . . .	9
<b>16 Common factor</b>	<b>10</b>
<b>17 Notable products</b>	<b>10</b>
<b>18 Classification of polynomials</b>	<b>10</b>
18.1 Definition . . . . .	10
18.2 Degree . . . . .	10
18.2.1 Monomials . . . . .	10
18.2.2 Polynomials . . . . .	10
<b>IV Lesson 4</b>	<b>11</b>
<b>19 Operations between polynomials</b>	<b>11</b>
19.1 Polynomials with one independent variable . . . . .	11
19.1.1 Sum . . . . .	11
19.1.2 Multiplications . . . . .	11
19.2 Polynomials with two or more variables . . . . .	11
19.2.1 Sum . . . . .	11
<b>20 Equations</b>	<b>11</b>
20.1 Identities . . . . .	12
20.2 Contradictions . . . . .	12
20.3 Conditional equations . . . . .	12
<b>21 Fundamental theorem of algebra</b>	<b>12</b>
<b>22 Linear equations with one variable</b>	<b>12</b>
22.1 Simple tools . . . . .	12
22.1.1 Tool 1 . . . . .	12
22.1.2 Tool 2 . . . . .	12
<b>23 Linear inequalities with one variable</b>	<b>13</b>
23.1 Negative sign . . . . .	13
<b>24 TO CHECK Equations and inequalities with absolute values</b>	<b>13</b>
<b>V Lesson 5</b>	<b>14</b>
<b>25 Division of polynomials</b>	<b>14</b>
25.1 Division algorithm for polynomials . . . . .	14
<b>26 Polynomials with <math>\deg = 2</math></b>	<b>14</b>
26.1 Quadratic formula . . . . .	14
26.1.1 Evident solutions . . . . .	14
26.2 Extraction of a root . . . . .	14

## Part I

# Lesson 1

## 1 Numerical sets

- $\mathbb{N} :=$  Natural numbers (including 0)
- $\mathbb{Z} :=$  Integer numbers
- $\mathbb{Q} :=$  Rational numbers
- $\mathbb{R} :=$  Real numbers

Notation: The “\*” symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

## 2 Prime numbers

A prime number is a number  $n \in \mathbb{N} \setminus \{0, 1\}$  such that, for every divisor  $d \in \mathbb{N}$ , if  $d \mid n$ , then  $d = 1$  or  $d = n$ .

$$n \in \mathbb{N} \setminus \{0, 1\} \text{ is prime} \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)$$

## 3 Positive powers

Let  $a \in \mathbb{R}, n \in \mathbb{N}^*$  and  $a \in \mathbb{R}$ , then

$$a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

### 3.1 Property 1

Let  $a, b \in \mathbb{R}, n, m \in \mathbb{N}$ , then

$$a^n \cdot a^m = a^{n+m}$$

### 3.2 Property 2

Let  $a, b \in \mathbb{R}, n \in \mathbb{N}$ , then

$$(a \cdot b)^n = a^n \cdot b^n$$

Notation: The power  $a^n$ ,  $a$  is the base and  $n$  is the exponent.

### 3.3 Property 3

Let  $a \in \mathbb{R}, m, n \in \mathbb{N}^*$ , then

$$(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}$$

## 4 Fractions

Notation 1:  $a \cdot b = a \times b = ab$     |     $\frac{a}{b} = a \div b = a : b$

Notation 2: “ $a$ ” is called numerator, “ $b$ ” is called denominator.

Notation 3:  $\frac{a}{b}$ ,  $a, b \in \mathbb{R}$ ,  $b \neq 0$

### 4.1 Property 1

Let  $a, b \in \mathbb{R}^*$  and  $c, d \in \mathbb{R}$ , then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

### 4.2 Property 2

Let  $a, b \in \mathbb{R}^*$  and  $c, d \in \mathbb{R}$ , then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

### 4.3 Property 3

Let  $a, b \in \mathbb{R}^*$  and  $c, d \in \mathbb{R}$ , then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

## 5 Negative powers

### 5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

### 5.2 Property 4

Let  $\forall n \in \mathbb{N}$ ,  $\forall a \in \mathbb{R}$ , then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that  $\forall z \in \mathbb{Z}$ ,  $\forall a \in \mathbb{R}$ ,  $z \neq 0$   
We can compute  $a^z$

### 5.3 Property 5

Let  $\forall a \in \mathbb{R}$ ,  $a \neq 0$ ,  $\forall n, m \in \mathbb{Z}$ , then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:

- $\forall a \in \mathbb{R}, \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
- $\forall b \in \mathbb{R}, (a \cdot b)^n = a^n \cdot b^n$
- $(a^n)^m = a^{n \cdot m}$

2.  $\forall a \in \mathbb{R}^*, a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

## 6 Fractions and percentages (and back)

$\alpha \in \mathbb{R}, n\% \text{ of } \alpha \iff \frac{n}{100} \cdot \alpha$

## Part II

# Lesson 2

## 7 Symbols

Let  $a, b \in \mathbb{R}$ , then

- $a = b \rightarrow$  equality;
- $a \neq b \rightarrow$  inequality ( $a$  is not equal to  $b$ );
- $a < b \rightarrow$  less than ( $a$  is strictly less than  $b$ );
- $a \leq b \rightarrow$  less than or equal to ( $a$  is less than or equal to  $b$ );
- $a > b \rightarrow$  greater than ( $a$  is strictly greater than  $b$ );
- $a \geq b \rightarrow$  greater than or equal to ( $a$  is greater than or equal to  $b$ ).

Example:  $x \in \mathbb{R}$ ,  $x \geq 2 \rightarrow 2 \leq x < \infty$

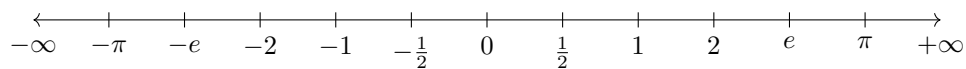
## 8 Brackets

- ( ) Parenthesis (round brackets)
- [ ] Square brackets
- { } Braces

## 9 Latin notations

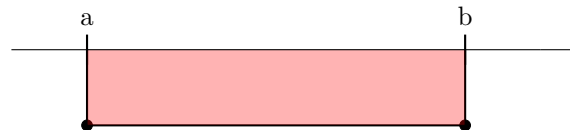
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. ( $\square$ )= quod erat demonstrandum (we finally prove it).

## 10 The real line

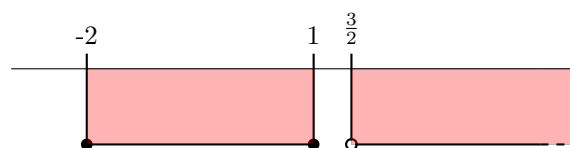


### 10.1 Exercises

1)  $\forall a, b, x \in \mathbb{R}$ ,  $a \leq x \leq b$



2)  $\forall x \in \mathbb{R}$ ,  $x \in ]-2, -1] \cup ]\frac{3}{2}, +\infty[$



Notation: The union of two or more intervals where  $x \in \mathbb{R}$  is denoted by the symbol  $\cup$ .

## 11 Properties of real numbers

### 11.1 Property 1 - Closure under “+” and “.”

$$\forall x, y \in \mathbb{R}$$

$$x + y \in \mathbb{R}$$

$$x \cdot y \in \mathbb{R}$$

Remark: for  $\forall x \in \mathbb{Z}$ , closure does not hold for division.

### 11.2 Property 2 - Commutativity

$$\forall x, y \in \mathbb{R}$$

$$x + y = y + x$$

$$x \cdot y = y \cdot x$$

Remark: commutativity does not hold for divisions and subtractions.

### 11.3 Property 3 - Associative

$$\forall x, y, z \in \mathbb{R}$$

$$x + (y + z) = (x + y) + z$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Remark: associativity does not hold for divisions and subtractions.

### 11.4 Property 4 - Distributive

$$\forall x, y, z \in \mathbb{R}$$

$$x(y \pm z) = xy \pm xz$$

### 11.5 Property 5 - Identity

$$\forall x \in \mathbb{R}$$

a)  $0 + x = x$

b)  $1 \cdot x = x$

Remark:  $\forall x \in \mathbb{R}$ ,  $x \cdot 0 = 0$  is not an identity property.

### 11.6 Property 6 - Inverses and opposites

$$\forall x \in \mathbb{R}$$

a)  $x + (-x) = 0$  (additive inverse)

b) when  $x \neq 0$ ,  $x \cdot \frac{1}{x} = 1$  (multiplicative inverse or opposite)

Remark 1:  $\forall x \in \mathbb{N}$  does not exist either inverse nor opposite.

Remark 2:  $\forall x \in \mathbb{Z}$  has inverses, but not opposites.

## 12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:  
( ) inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:  
 $x^a$ ;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:  
“.” and “÷”;
4. Perform all additions and subtractions as you come to them, moving left-to-right:  
“+” and “-”;
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

## 13 Signed numbers

A number is denoted as positive if it is directly preceded by a + sign or no sign at all.

A number is denoted as negative if it is directly preceded by a - sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

## 14 Absolute value

Let  $x \in \mathbb{R}$ , then

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

### 14.1 Property

$\forall x \in \mathbb{R}$

$$|x| > 0 \quad \text{if } x \neq 0$$

$$|x| = 0 \quad \text{if } x = 0$$



## Part III

# Lesson 3

## 15 Polynomials

### 15.1 Terms and factors

#### 15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are  $a$ ,  $b$ ,  $x$ ,  $y$ .

When we have an equality  $y = x + a$ ,  $\forall x \in \mathbb{R}$ ,  $x$  can assume any value in the set of real numbers ( $x$  is an independent variable), while  $y$  strictly depends on the value that we decide to give to  $x$ .

Notice: we can write  $y = x + a$  as  $y - a = x$ , changing which variable is independent and which is dependent.

#### 15.1.2 Sets

Consider the set  $A = [a, b]$ , where  $a \leq b$ . Then:

$$\boxed{\forall x \in A, a \leq x \leq b}$$

### 15.2 Expressions, terms and factors

#### 15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

#### 15.2.2 Terms

A term is any part of the expression separated by “+” or “−”.

$$\boxed{y = \underbrace{ax^2}_{\text{term}} + \underbrace{bx \cdot c}_{\text{term}}}$$

#### 15.2.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called “factorization”.

The goal of a factorization is to factorize an expression as much as possible.

## 16 Common factor

Any expression made of terms is composed of several factors.

$$x^2 + x^3 + x = x(x + x^2 + 1), \forall x \in \mathbb{R}$$

## 17 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$  (difference of two squares);
- $(a - b)^2 = a^2 - 2ab + b^2$  (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$  (square of a binomial);
- $(a - b)(a^2 + b^2 + ab) = a^3 - b^3$  (difference of two cubes);
- $(a + b)(a^2 + b^2 - ab) = a^3 + b^3$  (sum of two cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

## 18 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

Number of Terms	Name	Example	Comment
One	Monomial	$ax^2$	Mono means "one" in Greek
Two	Binomial	$ax^2 - bx$	Bi means "two" in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means "three" in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means "many" in Greek

(1)

### 18.1 Definition

Let  $n \in \mathbb{N}^*$ , then a polynomial is the sum or difference of n-monomials.

### 18.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

#### 18.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$p(x) = x^2 + 1 \rightarrow$  the degree is 2.

$\forall x \in \mathbb{R}, p(0) = 0^2 + 1 = 1 \rightarrow 1$  is a polynomial with degree 0.

#### 18.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$p(x) = x^3 + 1 + x^5 + x^2 \rightarrow \deg(p(x)) = 5$

$q(x) = 12 \underbrace{abcd}_{\deg=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$

Notation: Let  $f(x) = ax^2 + bx + c$ ,  $a$  and  $b$  are called coefficient.

The coefficient of the monomial with highest coefficient is called **leading coefficient**.

## Part IV

# Lesson 4

## 19 Operations between polynomials

### 19.1 Polynomials with one independent variable

The order of the monomials is not important, but it is preferable to write the highest degree monomials in decreasing order.

$$p(x) = ax^2 - bx + c$$

#### 19.1.1 Sum

We have to sum all the monomials of the same degree.

$$\begin{aligned} p(x) &= x^2 + x - 1 \\ q(x) &= 5 - x + x^5 - x^2 \end{aligned}$$

$$p(x) + q(x) = x^2 + x - 1 + 5 - x + x^5 - x^2 = x^5 + 4$$

Definition: in a polynomial with one variable, monomials of same degree are called **similar terms**.

Remark: when there is a difference between polynomials, the minus MUST be distributed throughout the next monomial.

#### 19.1.2 Multiplications

We have to multiply the factors with each other using the distributive property.

$$\begin{aligned} p(x) &= (x - 1) \\ q(x) &= (x^2 + 2x) \end{aligned}$$

$$p(x) \cdot q(x) = (x - 1)(x^2 + 2x) = x^3 + 2x^2 - x^2 - 2x = x^3 + x^2 - 2x = x(x^2 + x - 2)$$

## 19.2 Polynomials with two or more variables

### 19.2.1 Sum

$$\begin{aligned} p(x) &= ab + a^2b \\ q(x) &= 4ab - 3ab^2 \end{aligned}$$

$$p(x) + q(x) = ab + a^2b + 4ab - 3ab^2 = a^2b - 3ab^2 + 5ab = ab(a - b + 5)$$

Remark:  $5a^3b^4 + 7a^3b^4 = 12a^3b^4$ , but with  $5a^3b^4 + 7a^4b^3$  we can't go further with the sum.

## 20 Equations

An equation is a formula given by the equality of expressions.

Symbol notations:

- $\exists$  = there exist(s);
- $\nexists$  = there does not exist(s);
- $\exists!$  = it exists and it is unique;
- $:$  or  $|$  = such that.

Equations are the main topic, then we have

- Identities;
- Contradictions;
- Conditional equations.

## 20.1 Identities

An identity is an equality that holds true regardless of the values chosen for its variables

$$\boxed{\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \mid f(x, y) = 0}$$

e.g.

- $1 = 1$ ;
- $x - 1 = -1 + x$ ;
- $\sin^2(x) + \cos^2(x) = 1$ .

## 20.2 Contradictions

A contradiction occurs when we get a statement  $p$ , such that  $p$  is true and its negation  $\sim p$  is also true.

$$\boxed{\forall x \in \mathbb{R}, \neg(\exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $0 = 1$ , false;
- $x^2 = -1$  it is always positive or zero;
- $|a| = -3$  it is always positive or zero;
- $\sqrt{-(x^2 + 1)} = 1$  it is never defined ( $\nexists$ ).

## 20.3 Conditional equations

In general, we want to find a solution for each equation, i.e. all the real number that, when they replace a variable inside the equation, give an identity.

$$\boxed{\forall x \in \mathbb{R}, (x > 0 \Rightarrow \exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $x = 1$ ;
- $x + y = 3$ ;
- $\sin(\alpha) = 0.5$ .

## 21 Fundamental theorem of algebra

Let  $p(x)$  be a polynomial with one variable and real coefficients.

Assume that  $\deg(p(x)) = n \in \mathbb{N}$ , then:

$$\boxed{p(x) = 0 \text{ has at most } n \text{ solutions}}$$

## 22 Linear equations with one variable

$p(x) = q(x)$  where  $\deg(0, (x)) = 1$

### 22.1 Simple tools

#### 22.1.1 Tool 1

$a, b \in \mathbb{R}$ ,  $x + a = b$ , let's isolate the variable  $x$ :  $x + a - a = b - a \Rightarrow x = b - a$

#### 22.1.2 Tool 2

$a, b \in \mathbb{R}$ ,  $ax = b$ , let's isolate the variable  $x$ :  $\frac{ax}{a} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

## 23 Linear inequalities with one variable

The inequality is a relation between two or more sets.

Let  $a, b, x \in \mathbb{R}$ ,  $a < x$ ,  $b > x$ , then:

$$a < x < b$$

### 23.1 Negative sign

In solving the inequality we have to move a negative factor from one side to the other, so we need to reverse the sign of the inequality:

$$-ax < b \Rightarrow x > -\frac{b}{a}$$

## 24 TO CHECK Equations and inequalities with absolute values

To solve absolute values we need to consider two cases.

Let's take this equation:  $|x + a| = -x + b$ , then

$$\begin{cases} \text{case 1: } x + a = -x + b \Rightarrow 2x = b - a \Rightarrow x_1 = \frac{b-a}{2} \\ \text{case 2: } -x - a = -x + b \Rightarrow x_2 = \frac{b-a}{2} \end{cases} \Rightarrow \text{Sol: } x \in \frac{b-a}{2}$$

## Part V

# Lesson 5

## 25 Division of polynomials

### 25.1 Division algorithm for polynomials

Write all steps

## 26 Polynomials with $\deg = 2$

Let  $a, b, c \in \mathbb{R}$ , then

$$ax^2 + bx + c = 0$$

The three possible outputs we can have solving this 2nd grade polynomial are:

- 2 solutions;
- 1 solution;
- 0 solutions.

### 26.1 Quadratic formula

Delta:  $\Delta = \sqrt{b^2 - 4ac}$

$$x_{1,2} = \frac{-b \pm \Delta}{2a}$$

#### 26.1.1 Evident solutions

When we have a 2nd grade equation  $(x - a)(x - b) = 0$ , we have two evident solutions in  $\mathbb{R}$ . In this case,  $x_1 = a$ ,  $x_2 = b$

This factorization can be reached with notable products.

e.g. Let  $x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0$ , then  $x = -2$ .

### 26.2 Extraction of a root

Let  $a \in \mathbb{R}$ ,  $a \geq 0$ , then:

$$x^2 - a = 0 \Rightarrow x = \pm\sqrt{a}$$