Mathematics 1A
HSLU, Semester 1

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Contents

Part I

Week 1

1 The set theory

1.1 Definition of a set

A set is a collection of objects or elements.

<u>Remark</u>: The collection of all sets is not a set.

1.2 Logical symbols

1.2.1 Definition

Braces and the definition symbol ":=" are used to define a set giving all its elements:

$$\boxed{A := \{a, b, c, d, e\}}$$

1.2.2 Equal

In this case, the equal symbol means that the set A is equal to the set B:

$$\boxed{A = B}$$

1.2.3 Belongs to

The symbols $\in$ and $\ni$ describe an element which is part of the set:

$$\boxed{a \in A \Longleftrightarrow A \ni a}$$

1.2.4 Does not belong to

The symbols $\notin$ mean that an element does not belong to the set:

$$\boxed{f \notin A}$$

1.2.5 Inclusion and contains

The symbols $\subset$ and $\supset$ mean that a set has another set included in its set:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \Longleftrightarrow \mathbb{Z} \supset \mathbb{N}}$$

1.2.6 For all/any

The symbol $\forall$ means that we are considering any type of element:

$$\boxed{\forall x \in \mathbb{R}, \ x > 0}$$

In this case, we've defined a new set.

1.3   Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0);

- $\mathbb{Z} :=$ Integer numbers;

- $\mathbb{Q} :=$ Rational numbers;

- $\mathbb{R} :=$ Real numbers $:= \mathbb{Q} \cup \{\text{irrational numbers}\}.$

<u>Notation</u>: The "*" symbol means that the set does not include 0.

1.3.1   Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$B := \{\pi, 1, -1, 0\};$
$C := \{\pi, 1\};$
$D := \{\pi\}.$

Then we write some examples: $\pi \in B, \ D \subset B, \ C \subset B, \ B \not\subset C, \ 0 \in B, \ 0 \notin C.$

2   Intervals in the real line

Intervals describe what happens between two or more elements.

2.1   Examples

2.1.1   Interval sets

We have 4 cases:

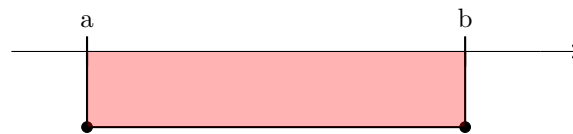- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\};$

- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\};$

- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\};$

- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}.$

<u>Notation</u>: a and b are often called the "end points" of the interval;

2.1.2   Graphical examples

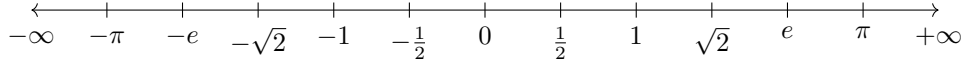$\forall x \in \mathbb{R}, \ x \in [a, b]$

3   The extended line

In the real line $\mathbb{R}$ we add $\pm\infty$.

Real line: $(-\infty, +\infty) = \mathbb{R}$

Extended real line: $[-\infty, +\infty] = \overline{\mathbb{R}}$

$$\xleftarrow{\quad}\ \underset{-\infty}{|}\ \underset{-\pi}{|}\ \underset{-e}{|}\ \underset{-\sqrt{2}}{|}\ \underset{-1}{|}\ \underset{-\frac{1}{2}}{|}\ \underset{0}{|}\ \underset{\frac{1}{2}}{|}\ \underset{1}{|}\ \underset{\sqrt{2}}{|}\ \underset{e}{|}\ \underset{\pi}{|}\ \xrightarrow{\quad}{+\infty}$$

<u>Remark</u>: $\pm\infty \notin \mathbb{R}$

3.1   Properties

$$\boxed{\forall x \in \mathbb{R}\ \mid\ \infty > x\ \mid\ -\infty < 0}$$

3.2   Operation in the extended line

If $a, b \in \mathbb{R}$, then $a + b$, $a - b$, $a \cdot b$, $\frac{a}{b}$ (with $b \neq 0$) stay the same

3.2.1   Additions

Let $\forall a \in \mathbb{R}$:

- $a + \infty := \infty$;

- $a - \infty := -\infty$;

- $+\infty + \infty := +\infty$;

- $-\infty - \infty := -\infty$;

- $+\infty - \infty := $ undefined.

3.2.2   Moltiplications

Let $\forall a \in \mathbb{R}$:

- $+\infty \cdot +\infty := +\infty$;

- $-\infty \cdot +\infty := -\infty$;

- $-\infty \cdot (-\infty) := \infty$;

- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$

- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$

- $\frac{a}{+\infty} = \frac{a}{-\infty} := 0$;

- $\frac{\pm\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$

- $\frac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$

- $\frac{\infty}{\infty} := $ undefined.

4 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including $\pm\infty$.

4.1 Examples

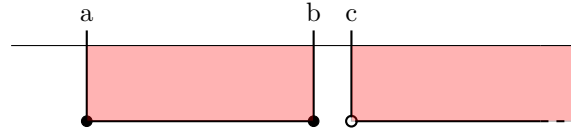### 4.1.1 Interval sets

Let $a \in \mathbb{R}$, then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$;
- $[a, +\infty] = \{\forall x \in \mathbb{R} \mid x \geq a\}$;
- $(-\infty, +\infty) = \mathbb{R}$;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$.

4.1.2 Graphical examples

$\forall x \in \mathbb{R}, \ x \in [a, b] \ \cup \ ]c, +\infty[$

Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

5 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

5.1 Logical connectives

A	B	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

5.1.1 Logical conjunction $\wedge$

Given two statements P and Q, $P \wedge Q$ is true if both P and Q are true.

Let $P = (x > 0)$ and $Q = (y > 0)$, then:

$$\boxed{P \wedge Q = (x > 0 \wedge y > 0)}$$

5.1.2 Logical disjunction $\vee$

Given two statements P and Q, $P \vee Q$ is true if at least one of P or Q is true.

Let $P = (x = 0)$ and $Q = (y \neq 0)$, then:

$$\boxed{P \vee Q = (x = 0 \vee y \neq 0)}$$

5.1.3 Logical negation ¬

The negation of a statement P, denoted as $\neg P$, is true if P is false, and false if P is true.

Let $P = (x \geq 5)$, then:

$$\boxed{\neg P = (x < 5)}$$

5.1.4 Implication ⇒

The symbol $\Rightarrow$ indicates that if statement P is true, then statement Q must also be true (i.e., P implies Q).
Warning: It does not require that Q implies P.

$$\boxed{P = (x = 1) \Rightarrow Q = (x \in \mathbb{N})}$$

5.1.5 Inference ⇐

The symbol $\Leftarrow$ means that a conclusion or result implies the truth of an earlier statement.
If Q is true, then P must be true.

$$\boxed{Q = (x > 0) \Leftarrow P = (x \in \mathbb{R}^+)}$$

5.1.6 If and only if ⇔

The symbol $\Leftrightarrow$ indicates that two statements P and Q are logically equivalent, meaning P is true if and only if Q is true.

$$\boxed{P = (x \in \mathbb{N},\ x \neq 0) \Longleftrightarrow Q = (x \in \mathbb{N}^*)}$$

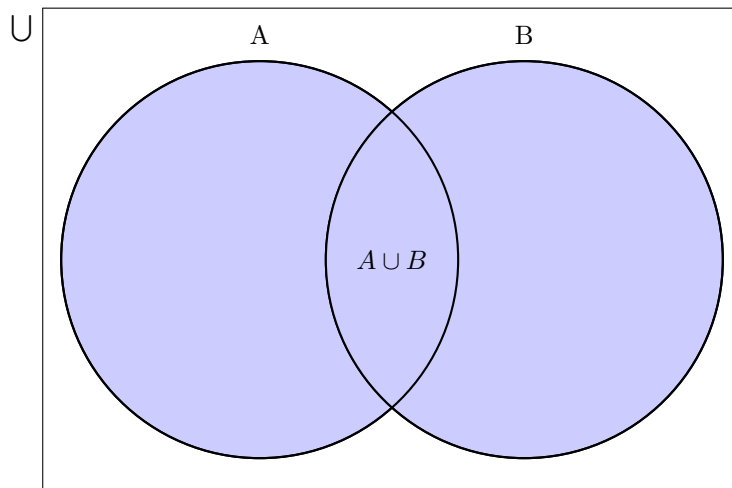# 6 Union ∪ and Intersection ∩

6.1 Universe symbol

The symbol $\bigcup :=$ Universe describes a big set which contains all sets involved in our discussions (not always).

6.2 Venn diagram

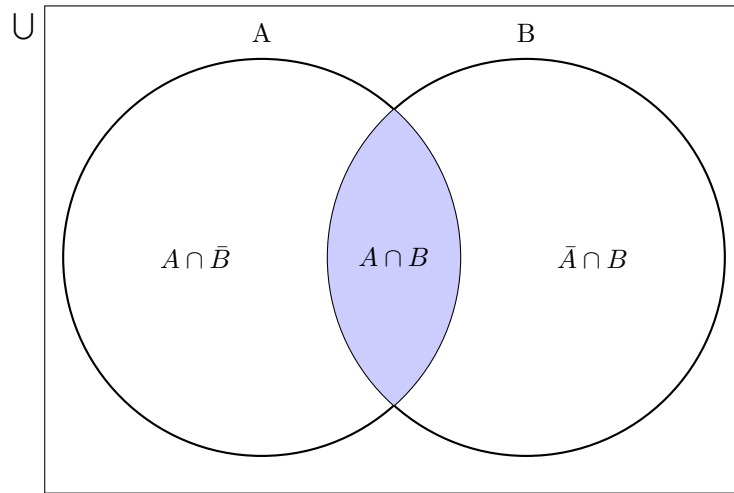### 6.2.1 Union $A \cup B$

If A and B are sets, then their union is:

$$\boxed{A \cup U = \{\forall x \in \bigcup \mid x \in A \lor x \in B\}}$$

6.2.2 Intersection $A \cap B$

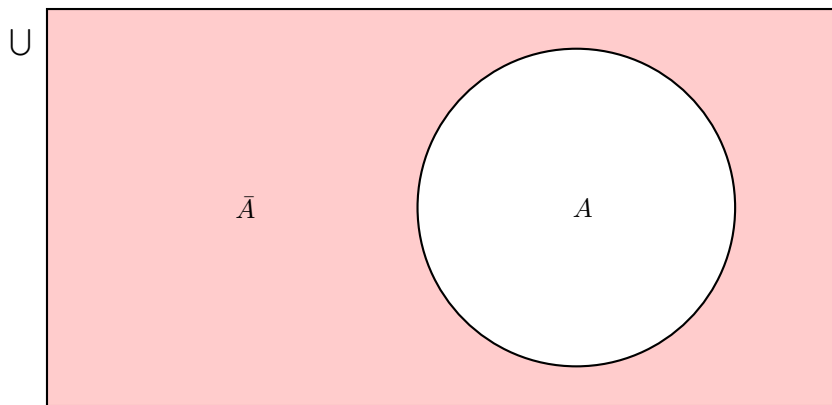If A and B are sets, then their intersection is:

$$A \cap B = \{\forall x \in \bigcup \mid x \in A \wedge x \in B\}$$

6.2.3 Complement $\bar{A}$
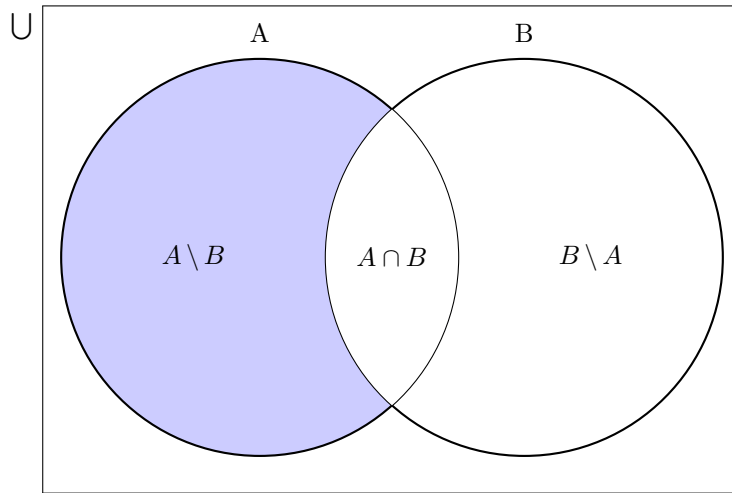
If A is a set, its complement is:

$$\bar{A} = \{\forall x \in \bigcup \mid x \notin A\}$$

6.2.4 Difference between sets \
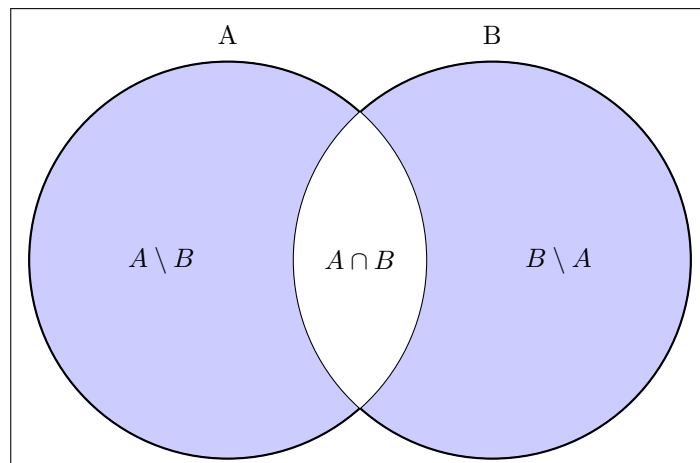
If A and B are sets, then their difference is:

$$\boxed{A \setminus B = \{\forall x \in \bigcup \mid x \in A, \ x \notin B\}}$$

6.2.5 Symmetrical difference $\triangle$

If A and B are sets, then their symmetrical difference is:
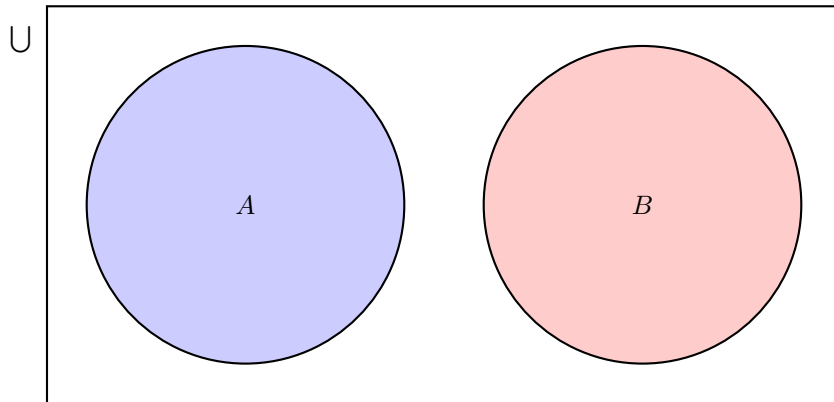
$$\boxed{A \triangle B = (A \setminus B) \cup (B \setminus A)}$$

6.2.6 Disjoined sets (Empty sets) $\emptyset$

$\emptyset :=$ the set containing zero elements:

$$\boxed{A \cap B = \emptyset}$$

7 The absolute value function

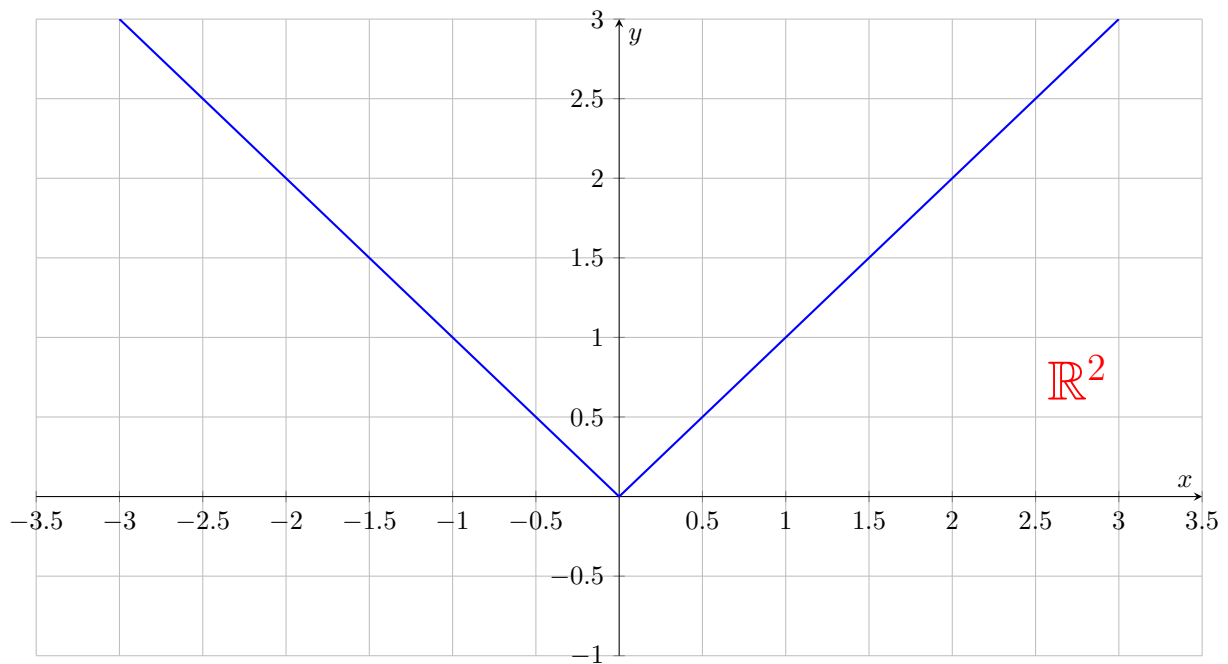The absolute value is an operator that returns the positive value of a number, regardless of its original sign.

Let $x \in \mathbb{R}$, then:

$$\boxed{|x| = \begin{cases} x \text{ if } & x \geq 0 \\ x \text{ if } & -x < 0 \end{cases}}$$

7.1 Graph of absolute value functions

Let's plot the function $y = |x|$:

7.2   Properties

Let $a, b \in \mathbb{R}$, then:

- $|a \cdot b| = |a| \cdot |b|$;

- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$;

- $|a \pm b| \neq |a| \pm |b|$.

7.3   Triangular inequalities

Let $a, b \in \mathbb{R}$, then:

$$
\begin{aligned}
|a| + |b| &\geq |a+b| \\
|a| - |b| &\leq |a-b|
\end{aligned}
$$

Part II

Week 2