

# Mathematics 3A

## HSLU, Semester 3

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## Part I

# Just stuff I have to explain, wait few days

Let  $\pi$  denote the plane:

$$s_x \in \pi, s_y \in \pi, s_z \in \pi$$

$$\pi : ax + by + cz + d = 0$$

For  $S_x \in \pi \implies 1a + 0b + 0c + d = 0$ , hence  
 $a + d = 0$

For  $S_y \in \pi \implies 0a + 2b + 0c + d = 0$ , hence  
 $2b + d = 0$

for  $S_z \in \pi \implies 0a + 0b + 3c + d = 0$ , hence  
 $3c + d = 0$

$$\begin{cases} a + d = 0 \\ 2b + d = 0 \\ 3c + d = 0 \end{cases} \implies \begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases}$$

Case 1:

$$d = 0 \implies a = 0, b = 0, c = 0 \implies \pi : 0 = 0 \implies \text{NOT a plane!}$$

Case 2:

$$d \neq 0 \implies \pi : \frac{ax + by + cz + d}{d} = 0 \implies \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

Hence:

$$\begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases} \implies \begin{cases} \frac{a}{d} = -1 \\ \frac{b}{d} = -\frac{1}{2} \\ \frac{c}{d} = -\frac{1}{3} \end{cases}$$

Which leads to:

$$\pi : -x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0$$

Remark: the equation of a plane is defined up to a multiplication by a real number different from 0

e.g.: the same plane is shared between those 3 equations  
ex 1)

$$z = 0 \iff 5z = 0 \iff -10z = 0$$

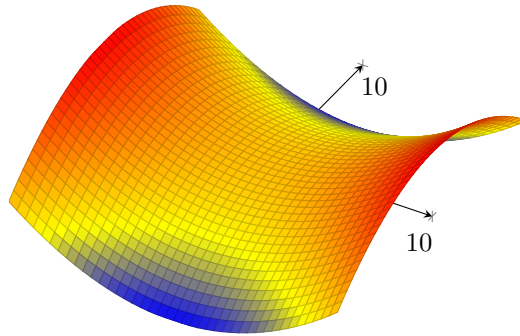
ex 2)

$$-x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0 \iff 6x + 3y + 2z + 6 = 0$$

# 1 Functions in two variables $x$ and $y$

Let us take  $\pi : x^2 - y^2 = 0$  as example.

The plot would look like this:



## 1.1 Spheres

## 2 Linear functions of two variables

We say that  $z$  is a *linear function* of  $x$  and  $y$ , if there are constant  $a, b$  and  $d$  such that:

$$z = ax + by + d$$

holds. Alternatively: if there are constant  $A, B, C, D$ , with  $C \neq 0$ , such that:

$$Ax + By + Cz + D = 0$$

holds. Since  $C \neq 0$ , we can rearrange this equation into:

$$z = -\frac{Ax}{C} - \frac{By}{C} - \frac{D}{C}$$

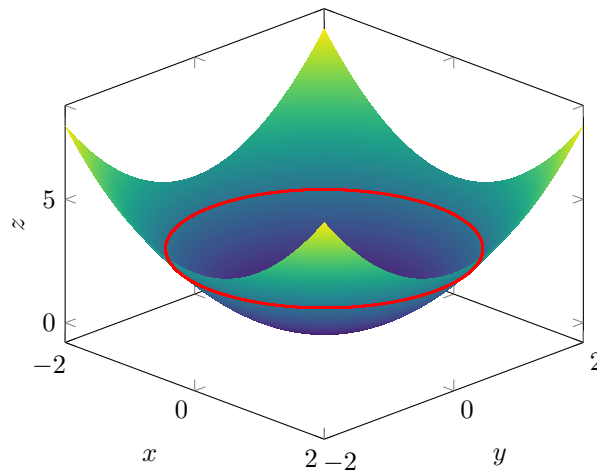
### 3 Contour lines

$$\begin{cases} z = f(x, y) \\ z = k \quad k \in \mathbb{R} \end{cases}$$

$z = k$  represents all the possible horizontal planes

Ex:

$$\begin{cases} z = x^2 - y^2 \\ z = k \end{cases} \implies \begin{cases} k = x^2 - y^2 \\ z = k \end{cases}$$



All the planes with equation  $z = k$  are parallel to the coordinate planes  $z = 0$ .

When  $z = k = 0$ , the circle is reduced to a point, the origin.

When  $k < 0$ , the equation  $x^2 + y^2 = k$  has no solution in  $\mathbb{R}$ .

When  $k > 0$ , the equation  $x^2 + y^2 = k$  represents a circle with radius  $\sqrt{k}$  centered at the origin.

### 4 Cylinders

A cylinder is a surface generated by all the lines parallel to a given line  $d$  and passing through a given curve  $\mathcal{C}$ .

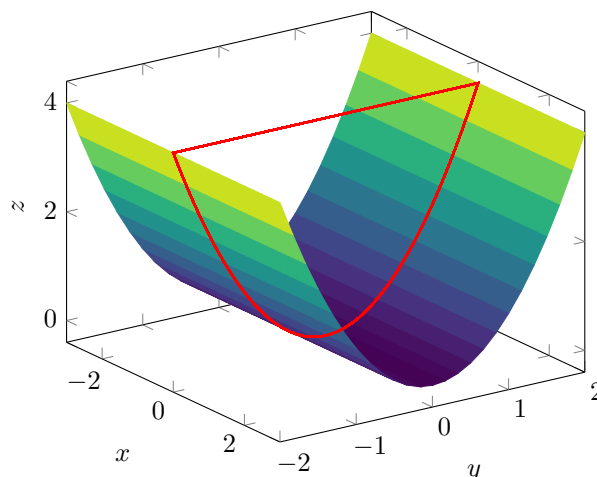
#### 4.1 Property

Whenever you have a polynomial equation of degree at least 2 with a missing variable, then you have a cylinder (up to few exceptions).

Ex:

$$z = y^2 \implies y^2 - z = 0$$

This is a cylinder with generatrix parallel to the  $x$  axis and directrix the parabola  $y^2 - z = 0$  in the  $yz$  plane.



## Part II

# Partial derivatives

For a multivariable function  $f(x, y, \dots)$ , the partial derivative to one variable measures the instantaneous rate of change of  $f$  when that variable changes and the others are held constant:

$$\frac{\partial z}{\partial x} = f_x(x, y)$$

If  $z$  is a function of  $x$  and  $y$ , we define:

The rate of change of  $z$  with respect to  $x$ , with  $y$  fixed, at the point  $(x, y) = (a, b)$  as

$$\left. \frac{\partial z}{\partial x} \right|_{(x,y)=(a,b)} = \lim_{h \rightarrow 0} \frac{Z|_{(x,y)=(a+h,b)} - Z|_{(x,y)=(a,b)}}{h}$$

The rate of change of  $z$  with respect to  $y$ , with  $x$  fixed, at the point  $(x, y) = (a, b)$  as

$$\left. \frac{\partial z}{\partial y} \right|_{(x,y)=(a,b)} = \lim_{h \rightarrow 0} \frac{Z|_{(x,y)=(a,b+h)} - Z|_{(x,y)=(a,b)}}{h}$$

For the lectures, we will be using the formula with 2-steps difference ( $\Delta z_a = (a + h, b) - (a - h, b)$ ):

$$\begin{aligned} \left. \frac{\partial z}{\partial x} \right|_{(x,y)=(a,b)} &= \frac{Z|_{(x,y)=(a+h,b)} - Z|_{(x,y)=(a-h,b)}}{2h} \\ \left. \frac{\partial z}{\partial y} \right|_{(x,y)=(a,b)} &= \frac{Z|_{(x,y)=(a,b+h)} - Z|_{(x,y)=(a,b-h)}}{2h} \end{aligned}$$

## 5 Local linearization

### 5.1 Tangent plane of a function at point $P$

Let  $f(x, y)$  be our function and  $P(a, b)$  a point,  $P \in f$ :

$$f(x, y) \approx f(a, b) + \frac{\partial}{\partial x} f(a, b)(x - a) + \frac{\partial}{\partial y} f(a, b)(y - b)$$

## 6 Gradient

The gradient of a function  $z = f(x, y)$  is defined by:

$$\begin{aligned} \text{grad } f &= \nabla f = f_x \vec{e}_x + f_y \vec{e}_y = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ \text{where } f_x &= \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y} \end{aligned}$$

## 6.1 Geometrical properties of the gradient vector $\nabla$ in the plane

If  $f$  is differentiable at the point  $(a, b)$  and  $\nabla f \neq \vec{0}$ , then the following holds:

$\nabla f(\mathbf{a}, \mathbf{b})$ :

- is perpendicular to the contour line of  $f$  through  $(a, b)$
- points in the direction of the maximum rate of change  $f$

The length  $\|\nabla f(\mathbf{a}, \mathbf{b})\|$  of the gradient vector is:

- the maximum rate of change  $f$  at this point
- large when the contour lines are close together
- small when the contour lines are far apart

## 6.2 Gradient of a function of three variables

The gradient of a function  $w = f(x, y, z)$  is defined by:

$$\text{grad } f = \nabla f = f_x \vec{e}_x + f_y \vec{e}_y + f_z \vec{e}_z = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

where  $f_x = \frac{\partial f}{\partial x}$ ,  $f_y = \frac{\partial f}{\partial y}$ , and  $f_z = \frac{\partial f}{\partial z}$

## 6.3 Second-order partial derivatives of $z = f(x, y)$

A function  $z = f(x, y)$  has two first-order partial derivatives,  $f_x$  and  $f_y$ , and four second-order partial derivatives:

$$\begin{aligned} 1. \quad & \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = (f_x)_x(x, y), \\ 2. \quad & \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = (f_y)_x(x, y), \\ 3. \quad & \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = (f_x)_y(x, y), \\ 4. \quad & \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = (f_y)_y(x, y) \end{aligned}$$

Usually, parenthesis are omitted, writing directly  $f_{xy}$  instead of  $(f_x)_y$ , and  $\frac{\partial^2 z}{\partial y \partial x}$  instead of  $\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)$ .

## 6.4 Equality of mixed partial derivatives (Schwarz's Theorem)

If  $f_{xy}$  and  $f_{yx}$  are continuous at a point  $(a, b)$  inside the domain, then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

## 7 Directional derivatives in the plane

### 7.1 Directional derivative of $f$ at $P(a, b)$ in the direction of $\vec{u}$

If  $\vec{e}_u = \vec{u} = u_1\vec{e}_x + u_2\vec{e}_y$  is a unit vector  $\|\vec{u}\| = 1$ , we define the directional derivative  $\frac{\partial f}{\partial \vec{u}} = f_{\vec{u}}$  by

$$\frac{\partial f}{\partial \vec{u}}(a, b) = f_{\vec{u}}(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

### 7.2 Gradient and directional derivative

If  $f$  is differentiable and  $\vec{e}_u = u_1\vec{e}_x + u_2\vec{e}_y$  is the unit vector in the direction of  $\vec{u}$ , then:

$$\frac{\partial f}{\partial \vec{u}}(a, b) = f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \nabla f(a, b) \cdot \vec{e}_u$$

## 8 Critical points

### 8.1 Discriminant

Let  $(x_0, y_0)$  be a critical point. Furthermore, let

$$D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Then the following holds:

- If  $D > 0$  and  $f_{xx} > 0$ , then  $f$  has a local minimum at  $(x_0, y_0)$
- If  $D > 0$  and  $f_{xx} < 0$ , then  $f$  has a local maximum at  $(x_0, y_0)$
- If  $D < 0$ , then  $f$  has a saddle point at  $(x_0, y_0)$
- If  $D = 0$ , no conclusion can be made

## 9 Constraints and Lagrange Multipliers

### 9.1 Graphical representation

### 9.2 Lagrange multiplier $\lambda$

### 9.3 Lagrange function $\mathcal{L}$

When optimizing  $f(x, y)$  under the constraint  $g(x, y) = c$ , the Lagrange function is used:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

The partial derivatives must be calculated:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(g(x, y) - c)\end{aligned}$$