

Maths refresher course

HSLU, Semester 1

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Contents

I	Lesson 1	4
1	Numerical sets	4
2	Prime numbers	4
3	Positive powers	4
3.1	Property 1	4
3.2	Property 2	4
3.3	Property 3	4
4	Fractions	5
4.1	Property 1	5
4.2	Property 2	5
4.3	Property 3	5
5	Negative powers	5
5.1	Definition	5
5.2	Property 4	5
5.3	Property 5	5
6	Fractions and percentages (and back)	6
II	Lesson 2	7
7	Symbols	7
8	Brackets	7
9	Latin notations	7
10	The real line	7
10.1	Exercises	7
11	Properties of real numbers	8
11.1	Property 1 - Closure under “+” and “.”	8
11.2	Property 2 - Commutativity	8
11.3	Property 3 - Associative	8
11.4	Property 4 - Distributive	8
11.5	Property 5 - Identity	8
11.6	Property 6 - Inverses and opposites	8
12	The order of operations	8
13	Signed numbers	9

14 Absolute value	9
14.1 Property	9
III Lesson 3	10
15 Polynomials	10
15.1 Terms and factors	10
15.1.1 Variables	10
15.1.2 Sets	10
15.2 Expressions, terms and factors	10
15.2.1 Expressions	10
15.2.2 Terms	10
15.2.3 Factors	10
16 Common factor	11
17 Notable products	11
18 Classification of polynomials	11
18.1 Definition	11
18.2 Degree	11
18.2.1 Monomials	11
18.2.2 Polynomials	11
IV Lesson 4	12
19 Operations between polynomials	12
19.1 Polynomials with one independent variable	12
19.1.1 Sum	12
19.1.2 Multiplications	12
19.2 Polynomials with two or more variables	12
19.2.1 Sum	12
20 Equations	12
20.1 Identities	13
20.2 Contradictions	13
20.3 Conditional equations	13
21 Fundamental theorem of algebra	13
22 Linear equations with one variable	13
22.1 Simple tools	13
22.1.1 Tool 1	13
22.1.2 Tool 2	13
23 Linear inequalities with one variable	14
23.1 Negative sign	14
24 Equations and inequalities with absolute values	14
V Lesson 5	15
25 Division of polynomials	15
25.1 Division algorithm for polynomials by monomials	15
26 Second degree polynomials	15
26.1 Quadratic formula	15
26.1.1 Discriminant of the polynomial	16
26.1.2 Evident solutions	16
26.2 Extraction of a root	16

VI	Lesson 6	17
27	Lines and parabolas	17
27.1	Cartesian diagram	17
27.2	Straight line	17
27.3	Slope-intercept equation	17
27.3.1	Slope	17
27.3.2	Drawing	18
27.4	Vertical lines	18
28	Equation of a line	18
28.1	General equation in a cartesian diagram	18
29	Vertical parabolas	19
29.1	Function of parabolas	19
29.2	Drawing example	19
29.3	Concavity of a parabola	19
29.4	Vertex of a parabola	19
30	Powers with \mathbb{Z} and \mathbb{R} exponents	20
VII	Lesson 7	21
31	Concept of functions	21
32	Trigonometry	21
32.1	Conversion table of degrees and radians	21
32.2	Trigonometric functions in the unit circle	22
32.2.1	Property 1	22
32.2.2	Property 2	22
32.2.3	Example with 45°	22
32.3	Tangent	22

Part I

Lesson 1

1 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0)
- $\mathbb{Z} :=$ Integer numbers
- $\mathbb{Q} :=$ Rational numbers
- $\mathbb{R} :=$ Real numbers

Notation: The “*” symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

2 Prime numbers

A prime number is a number $n \in \mathbb{N} \setminus \{0, 1\}$ such that, for every divisor $d \in \mathbb{N}$, if $d \mid n$, then $d = 1$ or $d = n$.

$$n \in \mathbb{N} \setminus \{0, 1\} \text{ is prime} \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)$$

3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{N}^*$ and $a \in \mathbb{R}$, then

$$a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

3.1 Property 1

Let $a, b \in \mathbb{R}, n, m \in \mathbb{N}$, then

$$a^n \cdot a^m = a^{n+m}$$

3.2 Property 2

Let $a, b \in \mathbb{R}, n \in \mathbb{N}$, then

$$(a \cdot b)^n = a^n \cdot b^n$$

Notation: The power a^n , a is the base and n is the exponent.

3.3 Property 3

Let $a \in \mathbb{R}, m, n \in \mathbb{N}^*$, then

$$(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}$$

4 Fractions

Notation 1: $a \cdot b = a \times b = ab$ | $\frac{a}{b} = a \div b = a : b$

Notation 2: “ a ” is called numerator, “ b ” is called denominator.

Notation 3: $\frac{a}{b}$, $a, b \in \mathbb{R}$, $b \neq 0$

4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

5 Negative powers

5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

5.2 Property 4

Let $\forall n \in \mathbb{N}$, $\forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z}$, $\forall a \in \mathbb{R}$, $z \neq 0$
We can compute a^z

5.3 Property 5

Let $\forall a \in \mathbb{R}$, $a \neq 0$, $\forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:

- $\forall a \in \mathbb{R}, \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
- $\forall b \in \mathbb{R}, (a \cdot b)^n = a^n \cdot b^n$
- $(a^n)^m = a^{n \cdot m}$

2. $\forall a \in \mathbb{R}^*, a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

6 Fractions and percentages (and back)

$$\alpha \in \mathbb{R}, n\% \text{ of } \alpha \iff \frac{n}{100} \cdot \alpha$$

Part II

Lesson 2

7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \rightarrow$ equality;
- $a \neq b \rightarrow$ inequality (a is not equal to b);
- $a < b \rightarrow$ less than (a is strictly less than b);
- $a \leq b \rightarrow$ less than or equal to (a is less than or equal to b);
- $a > b \rightarrow$ greater than (a is strictly greater than b);
- $a \geq b \rightarrow$ greater than or equal to (a is greater than or equal to b).

Example: $x \in \mathbb{R}$, $x \geq 2 \rightarrow 2 \leq x < \infty$

8 Brackets

() Parenthesis (round brackets)

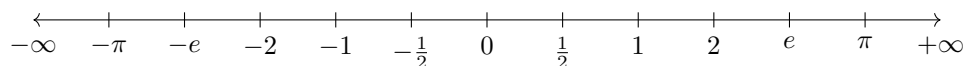
[] Square brackets

{ } Braces

9 Latin notations

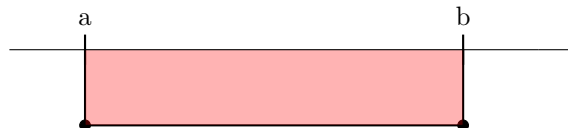
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. (\square)= quod erat demonstrandum (we finally prove it).

10 The real line

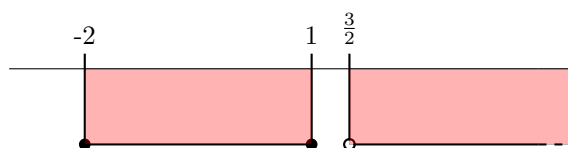


10.1 Exercises

1) $\forall a, b, x \in \mathbb{R}$, $a \leq x \leq b$



2) $\forall x \in \mathbb{R}$, $x \in]-2, -1] \cup]\frac{3}{2}, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol \cup .

11 Properties of real numbers

11.1 Property 1 - Closure under “+” and “.”

$$\forall x, y \in \mathbb{R}$$

$$x + y \in \mathbb{R}$$

$$x \cdot y \in \mathbb{R}$$

Remark: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

11.2 Property 2 - Commutativity

$$\forall x, y \in \mathbb{R}$$

$$x + y = y + x$$

$$x \cdot y = y \cdot x$$

Remark: commutativity does not hold for divisions and subtractions.

11.3 Property 3 - Associative

$$\forall x, y, z \in \mathbb{R}$$

$$x + (y + z) = (x + y) + z$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Remark: associativity does not hold for divisions and subtractions.

11.4 Property 4 - Distributive

$$\forall x, y, z \in \mathbb{R}$$

$$x(y \pm z) = xy \pm xz$$

11.5 Property 5 - Identity

$$\forall x \in \mathbb{R}$$

a) $0 + x = x$

b) $1 \cdot x = x$

Remark: $\forall x \in \mathbb{R}$, $x \cdot 0 = 0$ is not an identity property.

11.6 Property 6 - Inverses and opposites

$$\forall x \in \mathbb{R}$$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0$, $x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

Remark 1: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

Remark 2: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
() inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
 x^a ;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
“.” and “÷”;
4. Perform all additions and subtractions as you come to them, moving left-to-right:
“+” and “-”;
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

13 Signed numbers

A number is denoted as positive if it is directly preceded by a $+$ sign or no sign at all.

A number is denoted as negative if it is directly preceded by a $-$ sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

14 Absolute value

Let $x \in \mathbb{R}$, then

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

14.1 Property

$\forall x \in \mathbb{R}$

$$|x| > 0 \quad \text{if } x \neq 0$$

$$|x| = 0 \quad \text{if } x = 0$$

Part III

Lesson 3

15 Polynomials

15.1 Terms and factors

15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are a , b , x , y .

When we have an equality $y = x + a$, $\forall x \in \mathbb{R}$, x can assume any value in the set of real numbers (x is an independent variable), while y strictly depends on the value that we decide to give to x .

Notice: we can write $y = x + a$ as $y - a = x$, changing which variable is independent and which is dependent.

15.1.2 Sets

Consider the set $A = [a, b]$, where $a \leq b$. Then:

$$\boxed{\forall x \in A, a \leq x \leq b}$$

15.2 Expressions, terms and factors

15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

15.2.2 Terms

A term is any part of the expression separated by “+” or “−”.

$$\boxed{y = \underbrace{ax^2}_{\text{term}} + \underbrace{bx \cdot c}_{\text{term}}}$$

15.2.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called “factorization”.

The goal of a factorization is to factorize an expression as much as possible.

16 Common factor

Any expression made of terms is composed of several factors.

$$x^2 + x^3 + x = x(x + x^2 + 1), \forall x \in \mathbb{R}$$

17 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$ (difference of two squares);
- $(a - b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$ (square of a binomial);
- $(a - b)(a^2 + b^2 + ab) = a^3 - b^3$ (difference of two cubes);
- $(a + b)(a^2 + b^2 - ab) = a^3 + b^3$ (sum of two cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

18 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

Number of Terms	Name	Example	Comment
One	Monomial	ax^2	Mono means "one" in Greek
Two	Binomial	$ax^2 - bx$	Bi means "two" in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means "three" in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means "many" in Greek

(1)

18.1 Definition

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

18.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

18.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$$p(x) = x^2 + 1 \rightarrow \text{the degree is } 2.$$

$$\forall x \in \mathbb{R}, p(0) = 0^2 + 1 = 1 \rightarrow 1 \text{ is a polynomial with degree } 0.$$

18.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$$p(x) = x^3 + 1 + x^5 + x^2 \rightarrow \deg(p(x)) = 5$$

$$q(x) = 12 \underbrace{abcd}_{\deg=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$$

Notation: Let $f(x) = ax^2 + bx + c$, a and b are called coefficient.

The coefficient of the monomial with highest coefficient is called **leading coefficient**.

Part IV

Lesson 4

19 Operations between polynomials

19.1 Polynomials with one independent variable

The order of the monomials is not important, but it is preferable to write the highest degree monomials in decreasing order.

$$p(x) = ax^2 - bx + c$$

19.1.1 Sum

We have to sum all the monomials of the same degree.

$$\begin{aligned} p(x) &= x^2 + x - 1 \\ q(x) &= 5 - x + x^5 - x^2 \end{aligned}$$

$$p(x) + q(x) = x^2 + x - 1 + 5 - x + x^5 - x^2 = x^5 + 4$$

Definition: in a polynomial with one variable, monomials of same degree are called **similar terms**.

Remark: when there is a difference between polynomials, the minus MUST be distributed throughout the next monomial.

19.1.2 Multiplications

We have to multiply the factors with each other using the distributive property.

$$\begin{aligned} p(x) &= (x - 1) \\ q(x) &= (x^2 + 2x) \end{aligned}$$

$$p(x) \cdot q(x) = (x - 1)(x^2 + 2x) = x^3 + 2x^2 - x^2 - 2x = x^3 + x^2 - 2x = x(x^2 + x - 2)$$

19.2 Polynomials with two or more variables

19.2.1 Sum

$$\begin{aligned} p(x) &= ab + a^2b \\ q(x) &= 4ab - 3ab^2 \end{aligned}$$

$$p(x) + q(x) = ab + a^2b + 4ab - 3ab^2 = a^2b - 3ab^2 + 5ab = ab(a - b + 5)$$

Remark: $5a^3b^4 + 7a^3b^4 = 12a^3b^4$, but with $5a^3b^4 + 7a^4b^3$ we can't go further with the sum.

20 Equations

An equation is a formula given by the equality of expressions.

Symbol notations:

- \exists = there exist(s);
- \nexists = there does not exist(s);
- $\exists!$ = it exists and it is unique;
- $:$ or $|$ = such that.

Equations are the main topic, then we have

- Identities;
- Contradictions;
- Conditional equations.

20.1 Identities

An identity is an equality that holds true regardless of the values chosen for its variables:

$$\boxed{\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \mid f(x, y) = 0}$$

e.g.

- $1 = 1$;
- $x - 1 = -1 + x$;
- $\sin^2(x) + \cos^2(x) = 1$.

20.2 Contradictions

A contradiction occurs when we get a statement p , such that p is true and its negation $\sim p$ is also true:

$$\boxed{\forall x \in \mathbb{R}, \neg(\exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $0 = 1$, false;
- $x^2 = -1$ it is always positive or zero;
- $|a| = -3$ it is always positive or zero;
- $\sqrt{-(x^2 + 1)} = 1$ it is never defined (\nexists).

20.3 Conditional equations

In general, we want to find a solution for each equation, i.e. all the real numbers that, when they replace a variable inside the equation, give an identity:

$$\boxed{\forall x \in \mathbb{R}, (x > 0 \Rightarrow \exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $x = 1$;
- $x + y = 3$;
- $\sin(\alpha) = 0.5$.

21 Fundamental theorem of algebra

Let $p(x)$ be a polynomial with one variable and real coefficients.

Assume that $\deg(p(x)) = n \in \mathbb{N}$, then:

$$\boxed{p(x) = 0 \text{ has at most } n \text{ solutions}}$$

22 Linear equations with one variable

$p(x) = q(x)$ where $\deg(0, (x)) = 1$

22.1 Simple tools

22.1.1 Tool 1

$a, b \in \mathbb{R}$, $x + a = b$, let's isolate the variable x : $x + a - a = b - a \Rightarrow x = b - a$

22.1.2 Tool 2

$a, b \in \mathbb{R}$, $ax = b$, let's isolate the variable x : $\frac{ax}{a} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

23 Linear inequalities with one variable

The inequality is a relation between two or more sets.

Let $a, b, x \in \mathbb{R}$, $a < x$, $b > x$, then:

$$a < x < b$$

23.1 Negative sign

In solving the inequality we have to move a negative factor from one side to the other, so we need to reverse the sign of the inequality:

$$-ax < b \Rightarrow x > -\frac{b}{a}$$

24 Equations and inequalities with absolute values

To solve absolute values we need to consider two cases.

Let's take of this equation: $|x + 2| = -x + 4$, then

$$\begin{cases} \text{case 1: } x + 2 = -x + 4 \Rightarrow 2x = 2 \Rightarrow x_1 = 1 \\ \text{case 2: } -x - 2 = -x + 4 \Rightarrow -2 = 4 \text{ (contradiction)} \end{cases} \Rightarrow \text{Sol: } x = \begin{cases} 1 & \text{if } x + 2 \geq 0 \\ \text{no solution} & \text{if } x + 2 < 0 \end{cases}$$

Part V

Lesson 5

25 Division of polynomials

25.1 Division algorithm for polynomials by monomials

Let $f(x)$ be a polynomial and $g(x)$ a monomial such that $g(x) \neq 0$. Consider the rational expression $\frac{f(x)}{g(x)}$, then:

	Divisor $g(x)$
Dividend $f(x)$	Quotient $Q(x)$
⋮	
⋮	
⋮	
⋮	
⋮	
Remainder $R(x)$	

- Divide the highest degree term in $f(x)$ (the dividend) by the highest degree term in $g(x)$ (the divisor). This gives the first partial quotient $q_1(x)$.
- Multiply the partial quotient $q_1(x)$ by the entire divisor $g(x)$. This product represents the part of the dividend that can be "cancelled" in this step.
- Subtract the product obtained in step 2 from the original dividend $f(x)$. This subtraction gives a new polynomial, often called the remainder $R_1(x)$, which is of a lower degree than the original dividend.
- Now divide the leading term of the new remainder $R_1(x)$ by the leading term of $g(x)$. This gives the next partial quotient $Q_2(x)$.
- Multiply $Q_2(x)$ by $g(x)$ and subtract it from the current remainder. This process generates a new remainder $R_2(x)$.
- Keep repeating the division, multiplication, and subtraction steps until the degree of the remainder is less than the degree of the divisor $g(x)$. At this point, you cannot continue dividing.
- The final quotient $Q(x)$ is the sum of all the partial quotients: $Q(x) = Q_1(x) + Q_2(x) + \cdots + Q_n(x)$.
- The remainder $R_n(x)$ is the result after all subtractions are completed. If the remainder is zero, the division is exact. If not, the remainder is the leftover part of the division.

Tip: When the sum of the coefficients is equal to 0, then the polynomial is always divisible by $x - 1$.

26 Second degree polynomials

Let $a, b, c \in \mathbb{R}$, then

$$ax^2 + bx + c = 0$$

The three possible outcomes we can have when solving this 2nd-degree polynomial are:

- 2 solutions;
- 1 solution;
- 0 solutions.

26.1 Quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

26.1.1 Discriminant of the polynomial

$$\Delta = b^2 - 4ac$$

From the discriminant we can determine how many solutions the equation will have:

- $\Delta > 0 \Rightarrow 2$ real solutions;
- $\Delta = 0 \Rightarrow 1$ real solution;
- $\Delta < 0 \Rightarrow 0$ real solutions (2 complex solutions).

26.1.2 Evident solutions

When we have a 2nd-degree equation $(x - a)(x - b) = 0$, we have two obvious solutions in \mathbb{R} . In this case, $x_1 = a$, $x_2 = b$

This factorization can be obtained using notable products.

e.g. Let $x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0$, then $x = -2$.

26.2 Extraction of a root

Let $a \in \mathbb{R}$, $a \geq 0$, then:

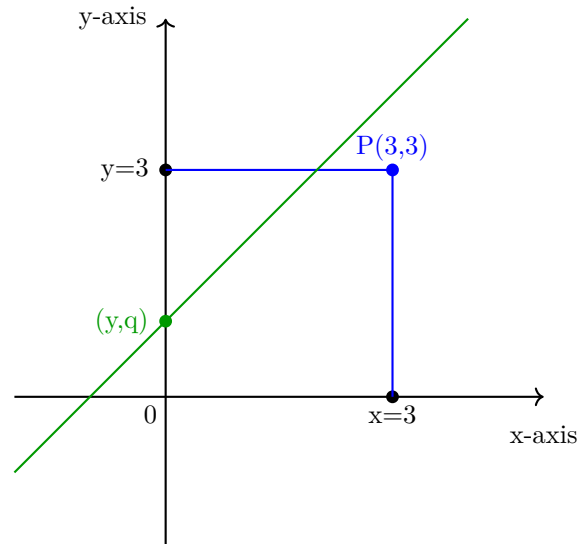
$$x^2 - a = 0 \Rightarrow x = \pm\sqrt{a}$$

Part VI

Lesson 6

27 Lines and parabolas

27.1 Cartesian diagram



27.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

27.3 Slope-intercept equation

Let $m, q \in \mathbb{R}$, then

$$y = mx + q$$

- m : slope ($\tan(\alpha)$);
- q : vertical intercept.

27.3.1 Slope

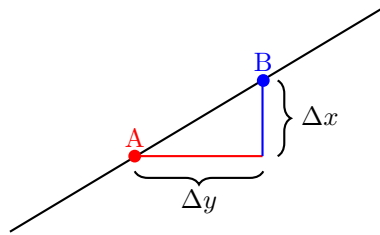
The slope of a line can be calculated with the equation

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x}$$

We have three different slope outcomes:

- $m > 0$, the line is increasing;
- $m = 0$, the line is stable;
- $m < 0$, the line is decreasing.

27.3.2 Drawing



27.4 Vertical lines

The more the value of m increases, the closer the line will get to the vertical, without ever reaching it.

Let $c \in \mathbb{R}$, then $x = c$.

Vertical lines cannot be written as a function.

28 Equation of a line

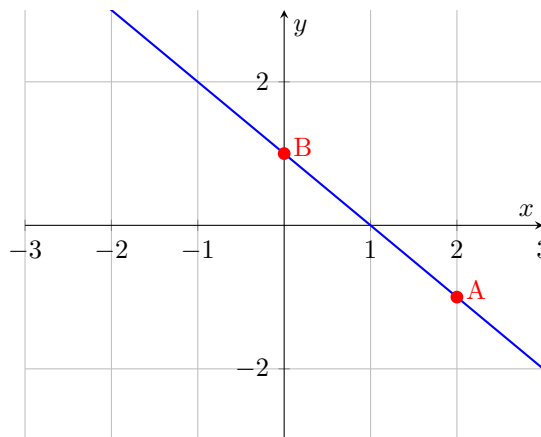
Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$, then

$$y - y_A = m(x - x_A)$$

e.g.: Find the line with $m = -1$ and $A(2, -1)$.

$$y - 1 = -1(x + 2) \Rightarrow y = -x + 1$$

Points: $A(2, -1)$; $B(0, 1)$



28.1 General equation in a cartesian diagram

$$ax + by + c = 0$$

Remarks:

- All the lines can be described with this kind of equation;
- When $b = 0$, $a \neq 0$, then $ax = -c \Rightarrow x = -\frac{c}{a} \in \mathbb{R}$;
- When $b \neq 0$, then $y = -\frac{a}{b}x - \frac{c}{b}$, where $m = -\frac{a}{b}$ and $q = -\frac{c}{b}$.

29 Vertical parabolas

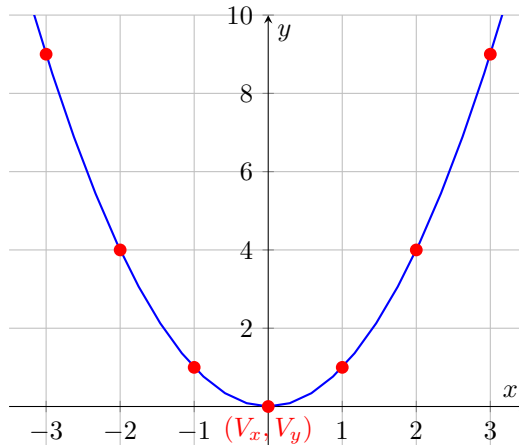
29.1 Function of parabolas

Let $a, b, c \in \mathbb{R}$, then

$$y = ax^2 + bx + c$$

29.2 Drawing example

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



29.3 Concavity of a parabola

We have three cases:

- $a > 0$, concave up;
- $a = 0$, not a parabola;
- $a < 0$, concave down.

29.4 Vertex of a parabola

The vertex of a parabola $y = ax^2 + bx + c$ is the point given by the coordinates:

$$V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a} \right)$$

Remarks: we have two different cases:

- When $a > 0$, the vertex is the lower point of the parabola;
- When $a < 0$, the vertex is the highest point of the parabola.

e.g.: given $y = x^2$, find the vertex: $V = \left(-\frac{0}{2}, -\frac{0}{4} \right) \rightarrow V(0, 0)$

Alternative: solving the x coordinate V_x , we can substitute the x inside the given function $f(x)$.

30 Powers with \mathbb{Z} and \mathbb{R} exponents

Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, then:

$$\alpha^{\frac{1}{n}} = \sqrt[n]{\alpha}$$

Let $m, n \in \mathbb{Z}$, then

$$\alpha^{\frac{m}{n}} = \left(\alpha^{\frac{1}{n}} \right)^m$$

Let $a, c \in \mathbb{Z}$; $b, d \in \mathbb{Z}^*$ and $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. Then, we can approximate λ by a fraction:

$$\frac{a}{b} < \lambda < \frac{c}{d}$$

Part VII

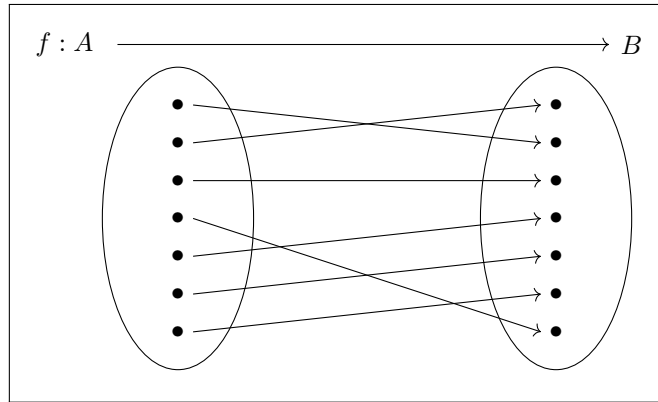
Lesson 7

31 Concept of functions

Let's take any two sets $A \{a, b, c, d, e, f, g\}$ and $B \{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$.

$$\begin{array}{l} f : \mathbb{R} \mapsto \mathbb{R} \\ x \mapsto mx + q \end{array}$$

A function is a relation between the sets A and B , according to which we associate to each element of A one and only one element of B :



Each point in set B is reached by at least one arrow. However, it is possible for more than two elements of A to point to the same element of B .

32 Trigonometry

Trigonometric functions can be extended to angles beyond 0 and 90° using the unit circle. For an angle θ in the unit circle:

$$\sin \theta = y \quad | \quad \cos \theta = x \quad | \quad \tan \theta = \frac{y}{x}$$

32.1 Conversion table of degrees and radians

Angles (in Degrees)	0°	30°	45°	60°	90°	180°	270°	360°
Angles (in Radians)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0	1
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0	∞	0

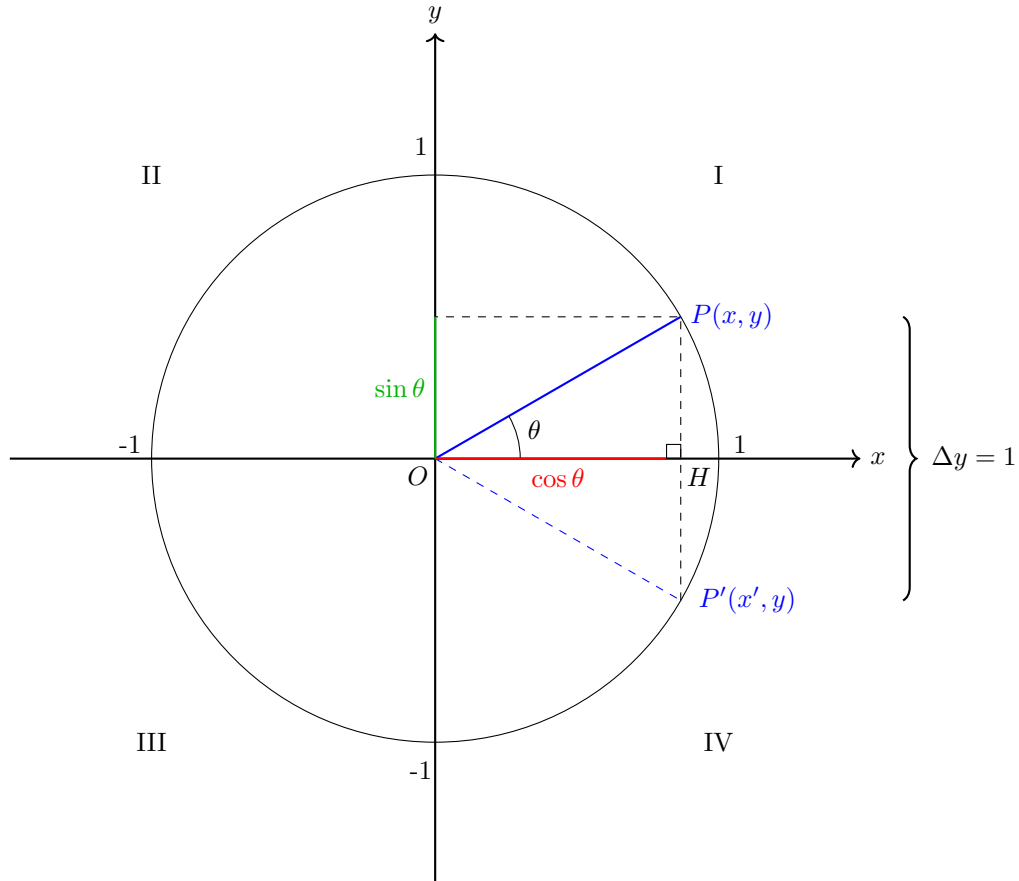
Remark:

$$\cos(360^\circ + \theta) = \cos(\theta) \quad | \quad \sin(360^\circ + \theta) = \sin(\theta)$$

Remark: Let $\forall k \in \mathbb{Z}, \forall \theta \in \mathbb{R}$, then:

$$\cos(\theta + k \cdot 360^\circ) = \cos(\theta)$$

32.2 Trigonometric functions in the unit circle



32.2.1 Property 1

Because we are inside a circle of radius 1:

- $-1 \leq \cos(\theta) \leq 1$;
- $-1 \leq \sin(\theta) \leq 1$.

32.2.2 Property 2

Because we have a 90° angle, we can use Pythagoras:

$$\overline{OH}^2 + \overline{PH}^2 = \overline{OP}^2$$

Then, we can compute that:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \forall \theta \in \mathbb{R}$$

32.2.3 Example with 45°

When $\theta = 45^\circ$, then $\sin(\theta) = \cos(\theta) \Rightarrow 2 \cos^2(\theta) = 1 \Rightarrow \cos(\theta) = \sqrt{\frac{1}{2}} \Rightarrow \sin(\theta) = \cos(\theta) = \frac{\sqrt{2}}{2}$

32.3 Tangent

A tangent of an angle is exactly the slope of a line:

$$m = \frac{\Delta y}{\Delta x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Remark: the tangent is not defined when the angle is 90° or 270° , that is when we have a vertical line.