

# Mathematics 3A

## HSLU, Semester 3

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## Part I

# Planes and surfaces in space

## 1 Plane $\pi$ in space

Let  $\pi$  denote the plane:

$$s_x \in \pi, s_y \in \pi, s_z \in \pi$$

$$\pi : ax + by + cz + d = 0$$

For  $S_x \in \pi \implies 1a + 0b + 0c + d = 0$ , hence  
 $a + d = 0$

For  $S_y \in \pi \implies 0a + 2b + 0c + d = 0$ , hence  
 $2b + d = 0$

for  $S_z \in \pi \implies 0a + 0b + 3c + d = 0$ , hence  
 $3c + d = 0$

$$\begin{cases} a + d = 0 \\ 2b + d = 0 \\ 3c + d = 0 \end{cases} \implies \begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases}$$

Case 1:

$$d = 0 \implies a = 0, b = 0, c = 0 \implies \pi : 0 = 0 \implies \text{NOT a plane!}$$

Case 2:

$$d \neq 0 \implies \pi : \frac{ax + by + cz + d}{d} = 0 \implies \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

Hence:

$$\begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases} \implies \begin{cases} \frac{a}{d} = -1 \\ \frac{b}{d} = -\frac{1}{2} \\ \frac{c}{d} = -\frac{1}{3} \end{cases}$$

Which leads to:

$$\pi : -x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0$$

Remark: the equation of a plane is defined up to a multiplication by a real number different from 0

e.g.: the same planed is shared between those 3 equations  
ex 1)

$$z = 0 \iff 5z = 0 \iff -10z = 0$$

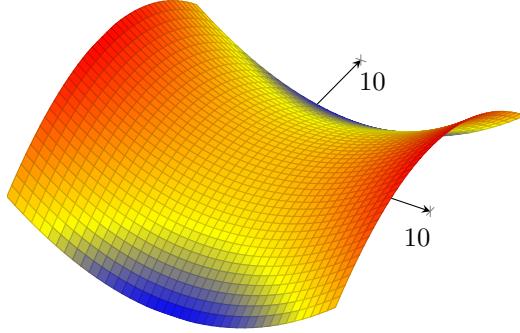
ex 2)

$$-x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0 \iff 6x + 3y + 2z + 6 = 0$$

## 2 Functions in two variables $x$ and $y$

Let us take  $\pi : x^2 - y^2 = 0$  as example.

The plot would look like this:



### 2.1 Spheres

## 3 Linear functions of two variables

We say that  $z$  is a *linear function* of  $x$  and  $y$ , if there are constant  $a, b$  and  $d$  such that:

$$z = ax + by + d$$

holds. Alternatively: if there are constant  $A, B, C, D$ , with  $C \neq 0$ , such that:

$$Ax + By + Cz + D = 0$$

holds. Since  $C \neq 0$ , we can rearrange this equation into:

$$z = -\frac{Ax}{C} - \frac{By}{C} - \frac{D}{C}$$

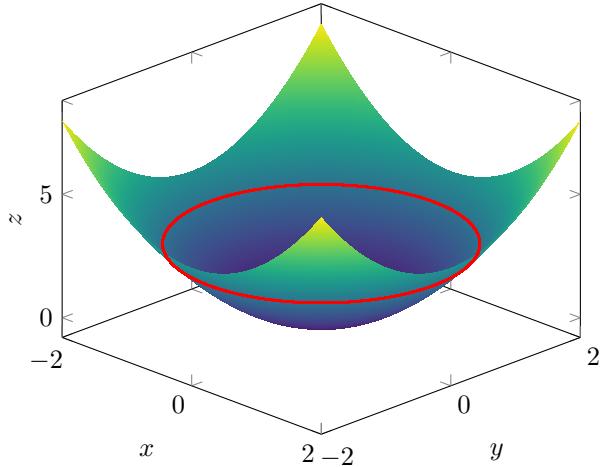
## 4 Contour lines

$$\begin{cases} z = f(x, y) \\ z = k \quad k \in \mathbb{R} \end{cases}$$

$z = k$  represents all the possible horizontal planes

Ex:

$$\begin{cases} z = x^2 - y^2 \\ z = k \end{cases} \implies \begin{cases} k = x^2 - y^2 \\ z = k \end{cases}$$



All the planes with equation  $z = k$  are parallel to the coordinate planes  $z = 0$ .

When  $z = k = 0$ , the circle is reduced to a point, the origin.

When  $k < 0$ , the equation  $x^2 + y^2 = k$  has no solution in  $\mathbb{R}$ .

When  $k > 0$ , the equation  $x^2 + y^2 = k$  represents a circle with radius  $\sqrt{k}$  centered at the origin.

## 5 Cylinders

A cylinder is a surface generated by all the lines parallel to a given line  $d$  and passing through a given curve  $\mathcal{C}$ .

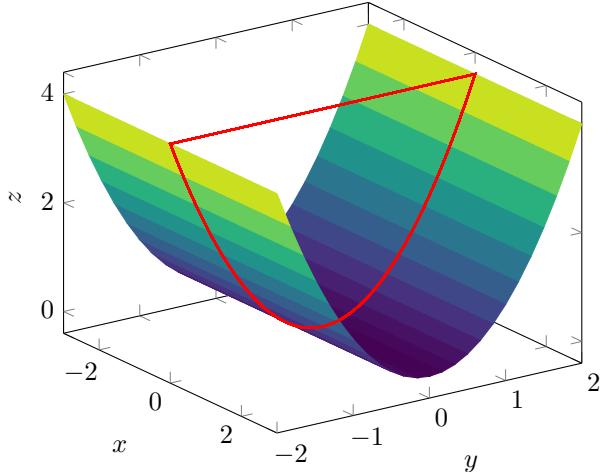
### 5.1 Property

Whenever you have a polynomial equation of degree at least 2 with a missing variable, then you have a cylinder (up to few exceptions).

Ex:

$$z = y^2 \implies y^2 - z = 0$$

This is a cylinder with generatrix parallel to the  $x$  axis and directrix the parabola  $y^2 - z = 0$  in the  $yz$  plane.



## Part II

# Partial derivatives

For a multivariable function  $f(x, y, \dots)$ , the partial derivative to one variable measures the instantaneous rate of change of  $f$  when that variable changes and the others are held constant:

$$\boxed{\frac{\partial z}{\partial x} = f_x(x, y)}$$

If  $z$  is a function of  $x$  and  $y$ , we define:

The rate of change of  $z$  with respect to  $x$ , with  $y$  fixed, at the point  $(x, y) = (a, b)$  as

$$\boxed{\frac{\partial z}{\partial x}|_{(x,y)=(a,b)} = \lim_{h \rightarrow 0} \frac{Z|_{(x,y)=(a+h,b)} - Z|_{(x,y)=(a,b)}}{h}}$$

The rate of change of  $z$  with respect to  $y$ , with  $x$  fixed, at the point  $(x, y) = (a, b)$  as

$$\boxed{\frac{\partial z}{\partial y}|_{(x,y)=(a,b)} = \lim_{h \rightarrow 0} \frac{Z|_{(x,y)=(a,b+h)} - Z|_{(x,y)=(a,b)}}{h}}$$

For the lectures, we will be using the formula with 2-steps difference ( $\Delta z_a = (a + h, b) - (a - h, b)$ ):

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x}|_{(x,y)=(a,b)} &= \frac{Z|_{(x,y)=(a+h,b)} - Z|_{(x,y)=(a-h,b)}}{2h} \\ \frac{\partial z}{\partial y}|_{(x,y)=(a,b)} &= \frac{Z|_{(x,y)=(a,b+h)} - Z|_{(x,y)=(a,b-h)}}{2h}\end{aligned}}$$

## 6 Local linearization

### 6.1 Tangent plane of a function at point P

Let  $f(x, y)$  be our function and  $P(a, b)$  a point,  $P \in f$ :

$$\boxed{f(x, y) \approx f(a, b) + \frac{\partial}{\partial x} f(a, b)(x - a) + \frac{\partial}{\partial y} f(a, b)(y - b)}$$

## 7 Gradient

The gradient of a function  $z = f(x, y)$  is defined by:

$$\boxed{\begin{aligned}\text{grad } f = \nabla f &= f_x \vec{e}_x + f_y \vec{e}_y = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ \text{where } f_x &= \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y}\end{aligned}}$$

## 7.1 Geometrical properties of the gradient vector $\nabla$ in the plane

If  $f$  is differentiable at the point  $(a, b)$  and  $\nabla f \neq \vec{0}$ , then the following holds:

$\nabla f(\mathbf{a}, \mathbf{b})$ :

- is perpendicular to the contour line of  $f$  through  $(a, b)$
- points in the direction of the maximum rate of change  $f$

The length  $\|\nabla f(\mathbf{a}, \mathbf{b})\|$  of the gradient vector is:

- the maximum rate of change  $f$  at this point
- large when the contour lines are close together
- small when the contour lines are far apart

## 7.2 Gradient of a function of three variables

The gradient of a function  $w = f(x, y, z)$  is defined by:

$$\text{grad } f = \nabla f = f_x \vec{e}_x + f_y \vec{e}_y + f_z \vec{e}_z = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

where  $f_x = \frac{\partial f}{\partial x}$ ,  $f_y = \frac{\partial f}{\partial y}$ , and  $f_z = \frac{\partial f}{\partial z}$

## 7.3 Second-order partial derivatives of $z = f(x, y)$

A function  $z = f(x, y)$  has two first-order partial derivatives,  $f_x$  and  $f_y$ , and four second-order partial derivatives:

1.  $\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = (f_x)_x(x, y),$
2.  $\frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = (f_y)_x(x, y),$
3.  $\frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = (f_x)_y(x, y),$
4.  $\frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = (f_y)_y(x, y)$

Usually, parenthesis are omitted, writing directly  $f_{xy}$  instead of  $(f_x)_y$ , and  $\frac{\partial^2 z}{\partial y \partial x}$  instead of  $\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right)$ .

## 7.4 Equality of mixed partial derivatives (Schwarz's Theorem)

If  $f_{xy}$  and  $f_{yx}$  are continuous at a point  $(a, b)$  inside the domain, then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

## 8 Directional derivatives in the plane

### 8.1 Directional derivative of $f$ at $P(a, b)$ in the direction of $\vec{u}$

If  $\vec{e}_u = \vec{u} = u_1 \vec{e}_x + u_2 \vec{e}_y$  is a unit vector  $\|u\| = 1$ , we define the directional derivative  $\frac{\partial f}{\partial \vec{u}} = f_{\vec{u}}$  by

$$\boxed{\frac{\partial f}{\partial \vec{u}}(a, b) = f_{\vec{u}}(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}}$$

### 8.2 Gradient and directional derivative

If  $f$  is differentiable and  $\vec{e}_u = u_1 \vec{e}_x + u_2 \vec{e}_y$  is the unit vector in the direction of  $\vec{u}$ , then:

$$\boxed{\frac{\partial f}{\partial \vec{u}}(a, b) = f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \nabla f(a, b) \cdot \vec{e}_u}$$

## 9 Critical points

### 9.1 Discriminant

Let  $(x_0, y_0)$  be a critical point. Furthermore, let

$$\boxed{D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2}$$

Then the following holds:

- If  $D > 0$  and  $f_{xx} > 0$ , then  $f$  has a local minimum at  $(x_0, y_0)$
- If  $D > 0$  and  $f_{xx} < 0$ , then  $f$  has a local maximum at  $(x_0, y_0)$
- If  $D < 0$ , then  $f$  has a saddle point at  $(x_0, y_0)$
- If  $D = 0$ , no conclusion can be made

## 10 Constraints and Lagrange Multipliers

### 10.1 Lagrange multiplier $\lambda$

The scalar  $\lambda$  measures how sensitive the optimal value of  $f$  is with respect to small changes in the constraint level  $c$ . Formally,

$$\boxed{\lambda = \frac{\partial f^*}{\partial c}}$$

where  $f^*$  denotes the optimal value of  $f$ . A positive  $\lambda$  indicates that relaxing the constraint ( $c$  larger) increases the optimal value of  $f$ .

### 10.2 Graphical representation

The optimization of  $f(x, y)$  under the constraint  $g(x, y) = c$  can be visualized as searching for points where a level curve of  $f$  is tangent to the constraint curve. At an optimum, the gradients are parallel:

$$\boxed{\nabla f(x, y) = \lambda \nabla g(x, y)}$$

### 10.3 Lagrange function $\mathcal{L}$

When optimizing  $f(x, y)$  under the constraint  $g(x, y) = c$ , the Lagrange function is used:

$$\boxed{\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)}$$

The partial derivatives must be calculated:

$$\boxed{\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(g(x, y) - c)\end{aligned}}$$

The stationary points of  $\mathcal{L}$  satisfy:

$$\boxed{\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 0 \\ g(x, y) &= c\end{aligned}}$$

Solutions  $(x, y, \lambda)$  of this system give the candidate extrema of  $f$  under the constraint  $g(x, y) = c$ .

## Part III

# Integration of functions with multiple variables

## 11 Domain of integration $\Omega$

Let  $f : \Omega \subset \mathbb{R}^2 \rightarrow R$ . The set  $\Omega$  is a region in the  $xy$ -plane over which the double integral

$$\iint_{\Omega} f(x, y) dx dy$$

is taken.

## 12 Double integrals as iterated integrals

If the region  $R$  is a rectangle with  $a \leq x \leq b$  and  $c \leq y \leq d$  and if  $f$  is continuous in the region  $R$ , then the integral of  $f$  over  $R$  is equal to the iterated integral

$$\int_R f dA = \int_{y=c}^d \int_{x=a}^b f(x, y) dx dy$$

The iterated integrals can also be written as

$$\int_c^d \int_a^b f(x, y) dx dy$$

### 12.1 Double integral over rectangles

$$\int_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

### 12.2 Triangular regions

For the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$   $\Rightarrow 0 \leq y \leq 1$ ,  $0 \leq x \leq 1 - y$ :

$$\int_{y=0}^1 \int_{x=0}^{1-y} f(x, y) dx dy$$

Equivalently  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1 - x$ :

$$\int_{x=0}^1 \int_{y=0}^{1-x} f(x, y) dy dx$$

### 12.3 Double integral over general regions

If the region  $\Omega$  is not a rectangle, one must describe it using variable limits that follow the boundary of  $\Omega$

#### 12.3.1 $x$ -simple region

If the region  $\Omega = \{(x, y) \mid a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ , then

$$\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy dx$$

#### 12.3.2 $y$ -simple region

If the region  $\Omega = \{(x, y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}$ , then

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy$$

### 12.4 Double integrals in Polar coordinates

#### 12.4.1 Polar coordinates

Polar coordinates are defined as the coordinate change

$$f : (0, +\infty) \times (-\pi, \pi] \rightarrow \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 \mid x \leq 0\}$$

given by

$$f(r, \varphi) = (r \cos \varphi, r \sin \varphi)$$

#### 12.4.2 Integration formula

To compute an integral in polar coordinates:

$$\begin{aligned} x &= r \cos \varphi, \\ y &= r \sin \varphi, \\ x^2 + y^2 &= r^2 \end{aligned}$$

and

$$dA = r \, d\varphi \, dr \quad \text{and} \quad dA = r \, dr \, d\varphi$$

### 13 Triple integrals as iterated integrals

If the region  $V$  is a box with  $a \leq x \leq b, c \leq y \leq d$ , and  $p \leq z \leq q$  and if  $f$  is continuous in the region  $V$ , then the integral of  $f$  over  $V$  is equal to the iterated integral

$$\int_W f \, dV = \int_p^q \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz$$

## 13.1 Triple integrals in Cylindrical coordinates

### 13.1.1 Cylindrical coordinates

Cylindrical coordinates are defined as the coordinate change

$$f : (0, +\infty) \times (-\pi, \pi] \times \mathbb{R} \rightarrow \mathbb{R}^3 \setminus \{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\}$$

given by

$$f(r, \varphi, z) = (r \cos \varphi, r \sin \varphi, z)$$

### 13.1.2 Integration formula

Each point  $(x, y, z)$  in a 3D space is represented by  $0 \leq r < \infty$ ,  $-\pi < \varphi \leq \pi$ , and  $-\infty < z < \infty$ . The following relations hold:

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \\ z &= z \\ x^2 + y^2 &= r^2 \end{aligned}$$

and

$$dV = r \ dr \ d\varphi \ dz$$