

# Mathematics 1A

## HSLU, Semester 1

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## Part I

# Logic

## 1 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

### 1.1 Logical connectives

| A | B | $\neg B$ | $A \wedge B$ | $A \vee B$ | $A \implies B$ | $A \Leftrightarrow B$ |
|---|---|----------|--------------|------------|----------------|-----------------------|
| T | T | F        | T            | T          | T              | T                     |
| T | F | T        | F            | T          | F              | F                     |
| F | T | F        | F            | T          | T              | F                     |
| F | F | T        | F            | F          | T              | T                     |

#### 1.1.1 Logical conjunction $\wedge$

Given two statements  $P$  and  $Q$ ,  $P \wedge Q$  is true if both  $P$  and  $Q$  are true.

Let  $P = (x > 0)$  and  $Q = (y > 0)$ , then:

$$P \wedge Q = (x > 0 \wedge y > 0)$$

#### 1.1.2 Logical disjunction $\vee$

Given two statements  $P$  and  $Q$ ,  $P \vee Q$  is true if at least one of  $P$  or  $Q$  is true.

Let  $P = (x = 0)$  and  $Q = (y \neq 0)$ , then:

$$P \vee Q = (x = 0 \vee y \neq 0)$$

#### 1.1.3 Logical negation $\neg$

The negation of a statement  $P$ , denoted as  $\neg P$ , is true if  $P$  is false, and false if  $P$  is true.

Let  $P = (x \geq 5)$ , then:

$$\neg P = (x < 5)$$

#### 1.1.4 Implication $\implies$

The symbol  $\implies$  indicates that if statement  $P$  is true, then statement  $Q$  must also be true (i.e.,  $P$  implies  $Q$ ).

Warning: It does not require that  $Q$  implies  $P$ .

$$P = (x = 1) \implies Q = (x \in \mathbb{N})$$

#### 1.1.5 Inference $\Leftarrow$

The symbol  $\Leftarrow$  means that a conclusion or result implies the truth of an earlier statement.

If  $Q$  is true, then  $P$  must be true.

$$Q = (x > 0) \Leftarrow P = (x \in \mathbb{R}^+)$$

### 1.1.6 If and only if $\Leftrightarrow$

The symbol  $\Leftrightarrow$  indicates that two statements  $P$  and  $Q$  are logically equivalent, meaning  $P$  is true if and only if  $Q$  is true.

$$P = (x \in \mathbb{N}, x \neq 0) \Leftrightarrow Q = (x \in \mathbb{N}^*)$$

## Part II

# Set Theory

## 2 The set theory

### 2.1 Logical symbols

#### 2.1.1 Definition

Braces and the definition symbol “:=” are used to define a set giving all its elements:

$$A := \{a, b, c, d, e\}$$

#### 2.1.2 Equal

In this case, the equal symbol means that the set  $A$  is equal to the set  $B$ :

$$A = B$$

#### 2.1.3 Belongs to

The symbols  $\in$  and  $\ni$  describe an element which is part of the set:

$$a \in A \Leftrightarrow A \ni a$$

#### 2.1.4 Does not belong to

The symbols  $\notin$  mean that an element does not belong to the set:

$$f \notin A$$

#### 2.1.5 Inclusion and contains

The symbols  $\subset$  and  $\supset$  mean that a set has another set included in its set:

$$\mathbb{N} \subset \mathbb{Z} \Leftrightarrow \mathbb{Z} \supset \mathbb{N}$$

#### 2.1.6 For all/any

The symbol  $\forall$  means that we are considering any type of element:

$$\forall x \in \mathbb{R}, x > 0$$

In this case, we've defined a new set.

## 2.2 Numerical sets

- $\mathbb{N} :=$  Natural numbers (including 0);
- $\mathbb{Z} :=$  Integer numbers;
- $\mathbb{Q} :=$  Rational numbers;
- $\mathbb{R} :=$  Real numbers  $:= \mathbb{Q} \cup \{\text{irrational numbers}\}$ .

Notation: The “\*” symbol means that the set does not include 0.

### 2.2.1 Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$$B := \{\pi, 1, -1, 0\};$$

$$C := \{\pi, 1\};$$

$$D := \{\pi\}.$$

Then we write some examples:  $\pi \in B$ ,  $D \subset B$ ,  $C \subset B$ ,  $B \not\subset C$ ,  $0 \in B$ ,  $0 \notin C$ .

## 3 Union $\cup$ and Intersection $\cap$

### 3.1 Universe symbol

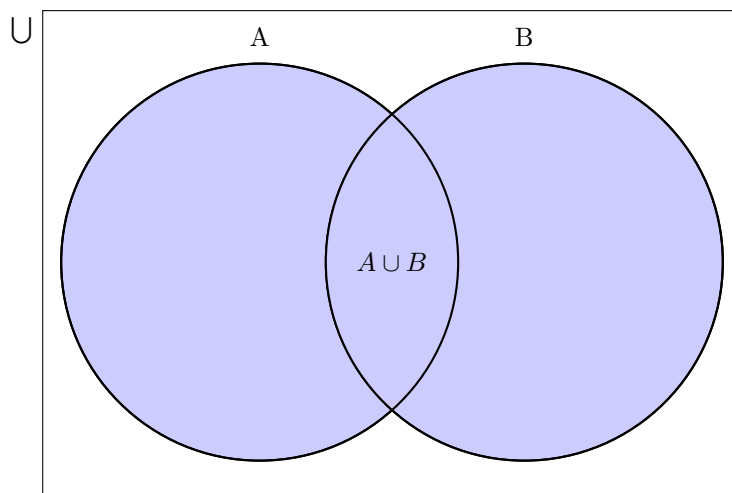
The symbol  $\bigcup :=$  Universe describes a big set which contains all sets involved in our discussions (not always).

### 3.2 Venn diagram

#### 3.2.1 Union $A \cup B$

If  $A$  and  $B$  are sets, then their union is:

$$\boxed{A \cup B = \{\forall x \in \bigcup \mid x \in A \vee x \in B\}}$$

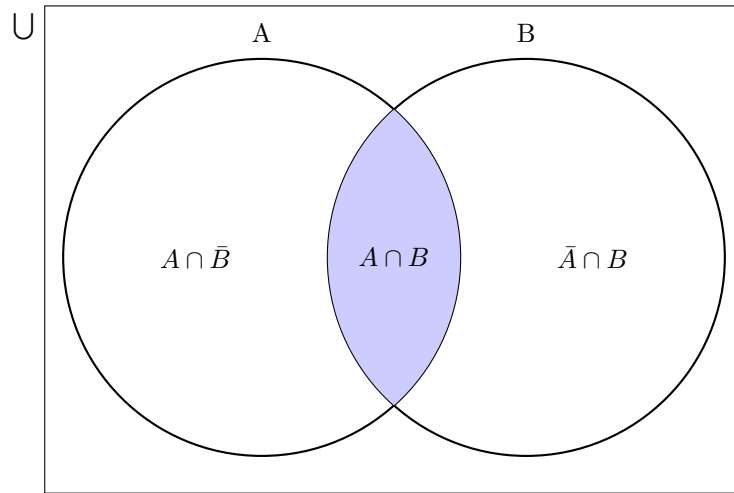




### 3.2.2 Intersection $A \cap B$

If  $A$  and  $B$  are sets, then their intersection is:

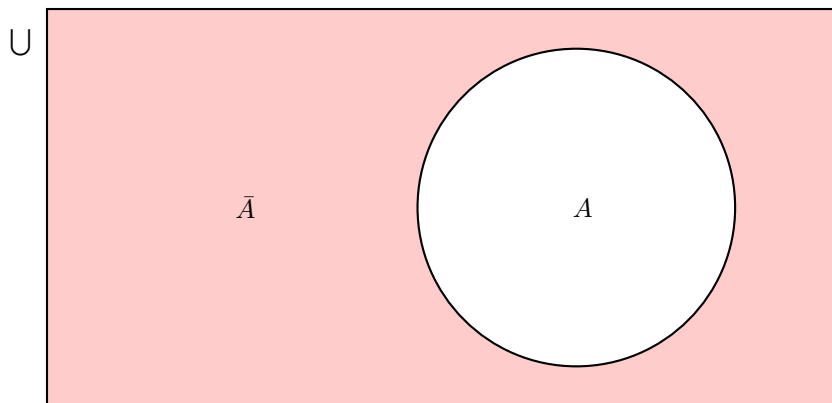
$$A \cap B = \{\forall x \in \mathcal{U} \mid x \in A \wedge x \in B\}$$



### 3.2.3 Complement $\bar{A}$

If  $A$  is a set, its complement is:

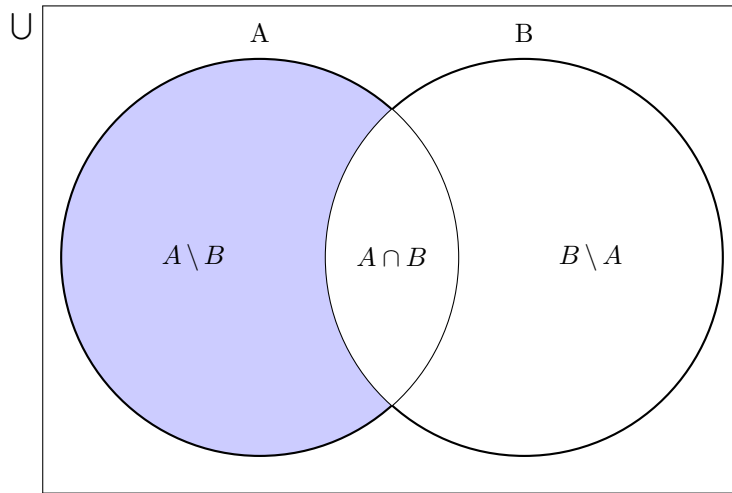
$$\bar{A} = \{\forall x \in \mathcal{U} \mid x \notin A\}$$



### 3.2.4 Difference between sets $\setminus$

If  $A$  and  $B$  are sets, then their difference is:

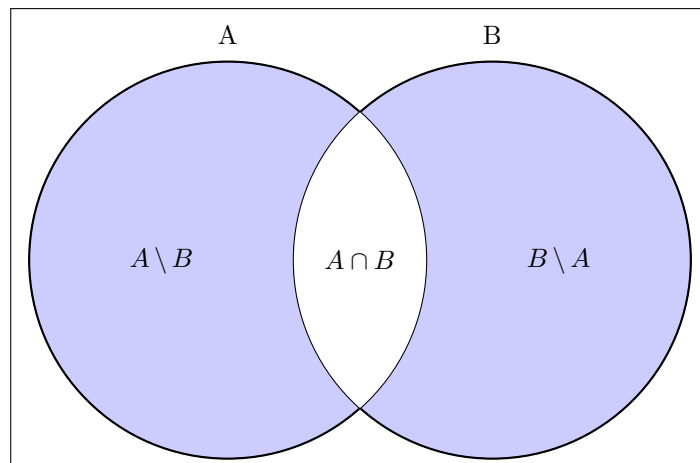
$$A \setminus B = \{\forall x \in \bigcup \mid x \in A, x \notin B\}$$



### 3.2.5 Symmetrical difference $\triangle$

If  $A$  and  $B$  are sets, then their symmetrical difference is:

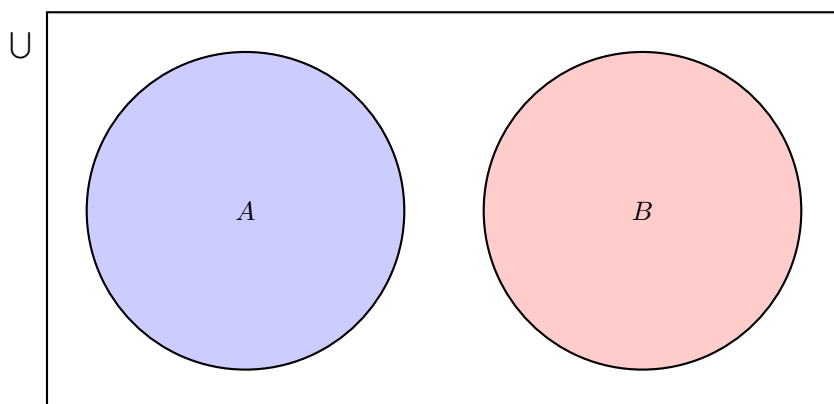
$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



### 3.2.6 Disjoined sets (Empty sets) $\emptyset$

$\emptyset$  := the set containing zero elements:

$$A \cap B = \emptyset$$



## Part III

# Algebra

## 4 Intervals in the real line

Intervals describe what happens between two or more elements.

### 4.1 Examples

#### 4.1.1 Interval sets

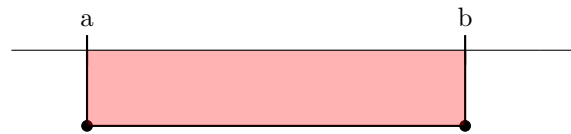
We have 4 cases:

- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\};$
- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\};$
- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\};$
- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}.$

Notation:  $a$  and  $b$  are often called the “end points” of the interval;

#### 4.1.2 Graphical examples

$\forall x \in \mathbb{R}, x \in [a, b]$



## 5 The extended line

In the real line  $\mathbb{R}$  we add  $\pm\infty$ .

**Real line:**  $(-\infty, +\infty) = \mathbb{R}$

**Extended real line:**  $[-\infty, +\infty] = \overline{\mathbb{R}}$



Remark:  $\pm\infty \notin \mathbb{R}$

### 5.1 Properties

$$\boxed{\forall x \in \mathbb{R} \mid \infty > x \mid -\infty < 0}$$

### 5.2 Operation in the extended line

If  $a, b \in \mathbb{R}$ , then  $a + b$ ,  $a - b$ ,  $a \cdot b$ ,  $\frac{a}{b}$  (with  $b \neq 0$ ) stay the same

### 5.2.1 Additions

Let  $\forall a \in \mathbb{R}$ :

- $a + \infty := \infty$ ;
- $a - \infty := -\infty$ ;
- $+\infty + \infty := +\infty$ ;
- $-\infty - \infty := -\infty$ ;
- $+\infty - \infty := \text{undefined}$ .

### 5.2.2 Multiplications

Let  $\forall a \in \mathbb{R}$ :

- $+\infty \cdot +\infty := +\infty$ ;
- $-\infty \cdot +\infty := -\infty$ ;
- $-\infty \cdot (-\infty) := \infty$ ;
- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$
- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$
- $\frac{a}{+\infty} = \frac{a}{-\infty} := 0$ ;
- $\frac{+\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$
- $\frac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$
- $\frac{\infty}{\infty} := \text{undefined}$ .

## 6 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including  $\pm\infty$ .

### 6.1 Examples

#### 6.1.1 Interval sets

Let  $a \in \mathbb{R}$ , then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$ ;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$ ;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$ ;
- $[a, +\infty] = \{\forall x \in \mathbb{R} \mid x \geq a\}$ ;
- $(-\infty, +\infty) = \mathbb{R}$ ;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$ .

### 6.1.2 Graphical examples

$\forall x \in \mathbb{R}, x \in [a, b] \cup ]c, +\infty[$



Notation: The union of two or more intervals where  $x \in \mathbb{R}$  is denoted by the symbol  $\cup$ .

## 7 The absolute value function

The absolute value is an operator that returns the positive value of a number, regardless of its original sign.

Let  $x \in \mathbb{R}$ , then:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } -x < 0 \end{cases}$$

### 7.1 Graph of absolute value functions

Let's plot the function  $y = |x|$ :



### 7.2 Properties

Let  $a, b \in \mathbb{R}$ , then:

- $|a \cdot b| = |a| \cdot |b|$ ;
- $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$  for  $b \neq 0$ ;
- $|a \pm b| \neq |a| \pm |b|$ .

### 7.3 Triangular inequalities

Let  $a, b \in \mathbb{R}$ , then:

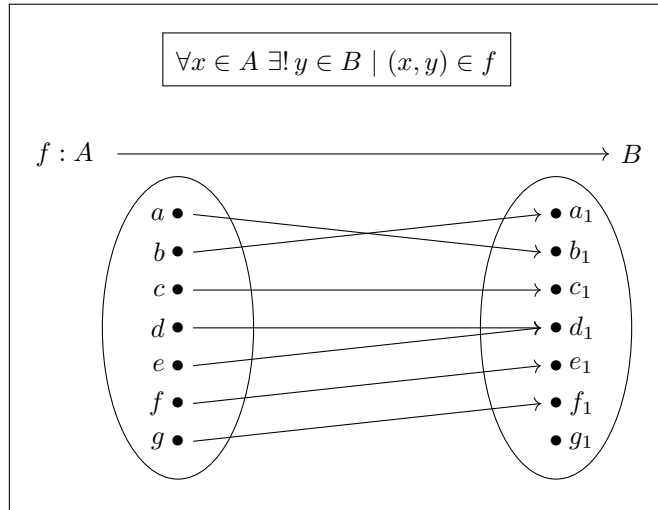
$$\begin{array}{l} |a| + |b| \geq |a+b| \\ |a| - |b| \leq |a-b| \end{array}$$

## 8 Concept of functions

Let's take any two sets  $A \{a, b, c, d, e, f, g\}$  and  $B \{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$ .

$$\begin{array}{l} f : A \Rightarrow B \\ a \mapsto f(a) \end{array}$$

A function is a relation between the sets  $A$  and  $B$ , according to which we associate to each element of  $A$  one and only one element of  $B$ :



Notation:  $f(a) = b_1, f(b) = a_1, f(c) = c_1, f(d) = d_1, \dots$

Each point in set  $A$  is associated with one element of  $B$ . However, it is possible for more than two elements of  $A$  to point to the same element of  $B$ .

The set  $A$  is called *domain* of  $f$ . The set  $B$  is called the *codomain* of  $f$ .

### 8.1 Image (Range)

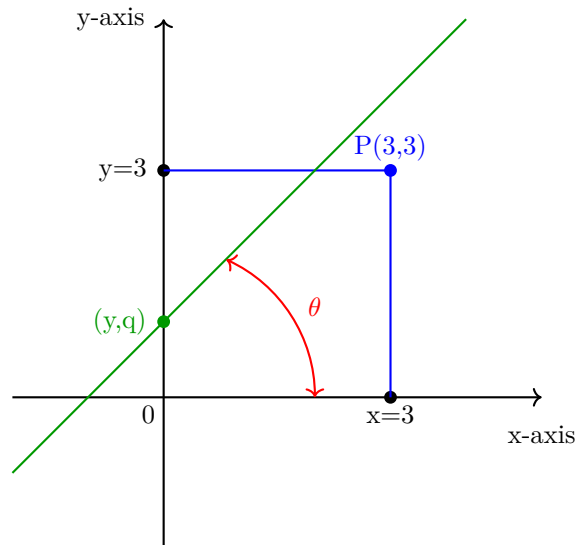
Let  $f : X \Rightarrow Y$  be a function. The image of  $f$  is defined as:

$$\text{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$$

Easily, the image is the set containing all the elements of the set  $B$  associated with the elements of the set  $A$ .

## 9 Linear function

### 9.1 Cartesian diagram



### 9.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

### 9.3 Slope-intercept equation

Let  $m, q \in \mathbb{R}$ , then

$$y = mx + q$$

- $m$ : slope;
- $q$ : vertical intercept.

#### 9.3.1 Slope

The slope of a line can be calculated with the equation

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

We have three different slope outcomes:

- $m > 0$ , the line is increasing;
- $m = 0$ , the line is stable;
- $m < 0$ , the line is decreasing.

Warning: This works only if  $x_B \neq x_A$ .

#### 9.3.2 Drawing





## 9.4 Vertical lines

The more the value of  $m$  increases, the closer the line will get to the vertical, without ever reaching it.

Let  $c \in \mathbb{R}$ , then  $x = c$ .

Vertical lines cannot be written as a function.

## 10 Equation of a line

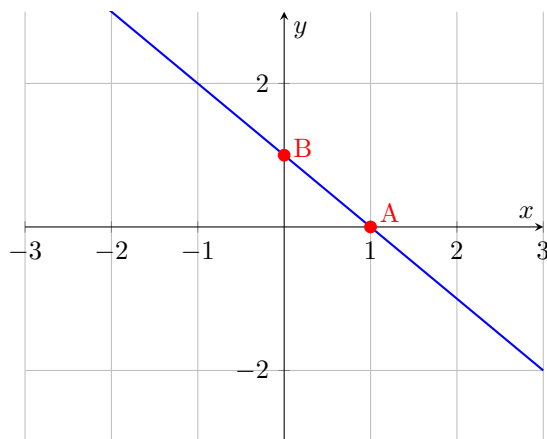
Let  $m, x_A, y_A \in \mathbb{R}$  and  $A(x_A, y_A)$ , then

$$y - y_A = m(x - x_A)$$

e.g.: Find the line with  $m = -1$  and  $A(1, 0)$ .

$$y - 0 = -1(x - 1) \implies y = -x + 1$$

Points:  $A(1, 0)$ ;  $B(0, 1)$



### 10.1 General equation in a cartesian diagram

$$ax + by + c = 0$$

Remark:

- All the lines can be described with this kind of equation;
- When  $b = 0$ ,  $a \neq 0$ , then  $ax = -c \implies x = \frac{-c}{a} \in \mathbb{R}$ ;
- When  $b \neq 0$ , then  $y = -\frac{a}{b}x - \frac{c}{b}$ , where  $m = -\frac{a}{b}$  and  $q = -\frac{c}{b}$ .

## 11 Increasing and decreasing functions

Let  $f : [a, b] \rightarrow \mathbb{R}$

Notation: if you replace  $[a, b]$  with  $\mathbb{R}$ , you obtain the definition in the whole  $\mathbb{R}$ .

### 11.1 Increasing functions

- $f$  is increasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) \geq f(x_1)$ ;
- $f$  is strictly increasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) > f(x_1)$ .

### 11.2 Decreasing functions

- $f$  is decreasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) \leq f(x_1)$ ;
- $f$  is strictly decreasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) < f(x_1)$ .

## 12 Inverse function

Let's take any two sets  $A$  and  $B$ .

A function  $f : A \Rightarrow B$  is invertible if there exists another function  $f^{-1} : B \Rightarrow A$ , called the inverse function, such that:

$$\boxed{\begin{array}{l} \forall x \in A, f^{-1}(f(x)) = x \\ \forall y \in B, f(f^{-1}(y)) = y \end{array}}$$

Warning: A function is invertible if and only if it is bijective.

### 12.1 Facts about inverse functions

1)

Let  $f : D \Rightarrow \mathbb{R}$

$f$  is invertible in  $D$  when:

- $f$  is strictly increasing;
- $f$  is strictly decreasing.

2)

Let  $f : D \Rightarrow \mathbb{R}$

$f$  is invertible when  $f^{-1} : \text{Im}(f) \Rightarrow D$ .

## 13 Expressions and factorization

### 13.1 Expressions, terms and factors

#### 13.1.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$y = ax^2 + bx \cdot c$$

#### 13.2 Terms

A term is any part of the expression separated by “+” or “-”.

$$y = \underbrace{ax^2}_{\text{term}} + \underbrace{bx \cdot c}_{\text{term}}$$

##### 13.2.1 Factors

Each term can be split into a product of factors.

$$x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

Notice: the process of splitting a term into several factors is called “factorization”.

The goal of a factorization is to factorize an expression as much as possible.

##### 13.2.2 Common factor

Any expression made of terms is composed of several factors.

$$x^2 + x^3 + x = x(x + x^2 + 1), \forall x \in \mathbb{R}$$

### 13.3 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$  (square of a binomial);
- $(a - b)^2 = a^2 - 2ab + b^2$  (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$  (difference of squares);
- $(a + b)(a^2 - ab + b^2) = a^3 + b^3$  (sum of cubes);
- $(a - b)(a^2 + ab + b^2) = a^3 - b^3$  (difference of cubes).

Remark: notable products are useful to factorize expressions when we don’t know a common factor.

## 14 Polynomial function

Let  $n \in \mathbb{N}^*$ , then a polynomial is the sum or difference of  $n$ -monomials.

## 15 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of **terms**;
2. the **degree** of the polynomial.

| Number of Terms | Name       | Example                                      | Degree   |
|-----------------|------------|--|----------|
| One             | Monomial   | $ax^2$                                       | 1        |
| Two             | Binomial   | $ax^2 - bx$                                  | 2        |
| Three           | Trinomial  | $ax^2 - bx + c$                              | 3        |
| Four or more    | Polynomial | $a_nx^n - a_1x^{n-1} + a_2x^{n-2} \dots a_0$ | n-degree |

Remark: The degree of a polynomial is the largest exponent of its monomials.

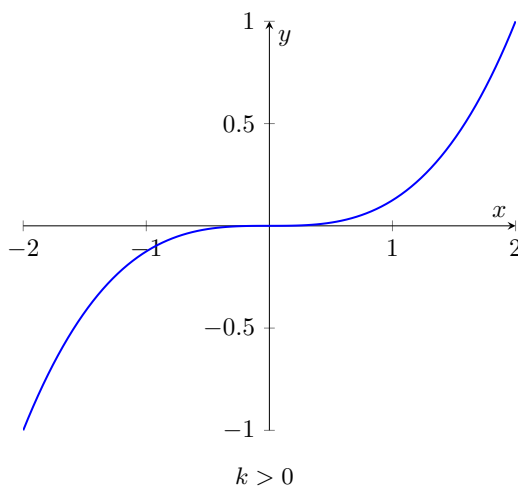
## 16 Symmetrical functions

Let  $y = kx^n$ , then we plot:

### 16.1 $n$ odd

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}$$

#### 16.1.1 Graph examples



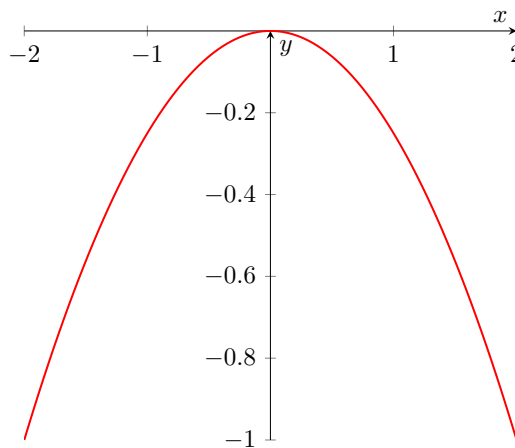
## 16.2 $n$ even

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}$$

### 16.2.1 Graph examples



$k > 0$   
Concave up



$k < 0$   
Concave down

Definition:

- a function  $y = f(x)$  is called **odd** if it is symmetric with respect to the origin;
- a function  $y = f(x)$  is called **even** if it is symmetric with respect to the y-axis.

## 16.3 General case

Let  $y = p(x)$ , where  $p(x)$  is any polynomial with real coefficients:

$$p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \dots + a_2 \cdot x^2 + a_1 \cdot x^1 + a_0$$

where:

- $n \in \mathbb{N}$ ;
- $n = \deg(p(x))$ ;
- $a_n = \text{leading coefficient}$ .

$$p(x) = \sum_{i=0}^n a_i \cdot x^i$$

## 16.4 Symmetry of a polynomial

Let  $y = p(x)$  be a polynomial function, then:

1)

$y = p(x)$  is odd iff all the degrees of all the terms of  $p(x)$  are odd;

2)

$y = p(x)$  is even iff all the degrees of all the terms of  $p(x)$  are even;

3)

$y = p(x)$  has mixed degrees,  $p(x)$  is neither odd nor even.

## 17 Intersection with axis

### 17.1 Vertical intersection

Let  $y = f(x)$  be any function, then we solve for  $y$ :

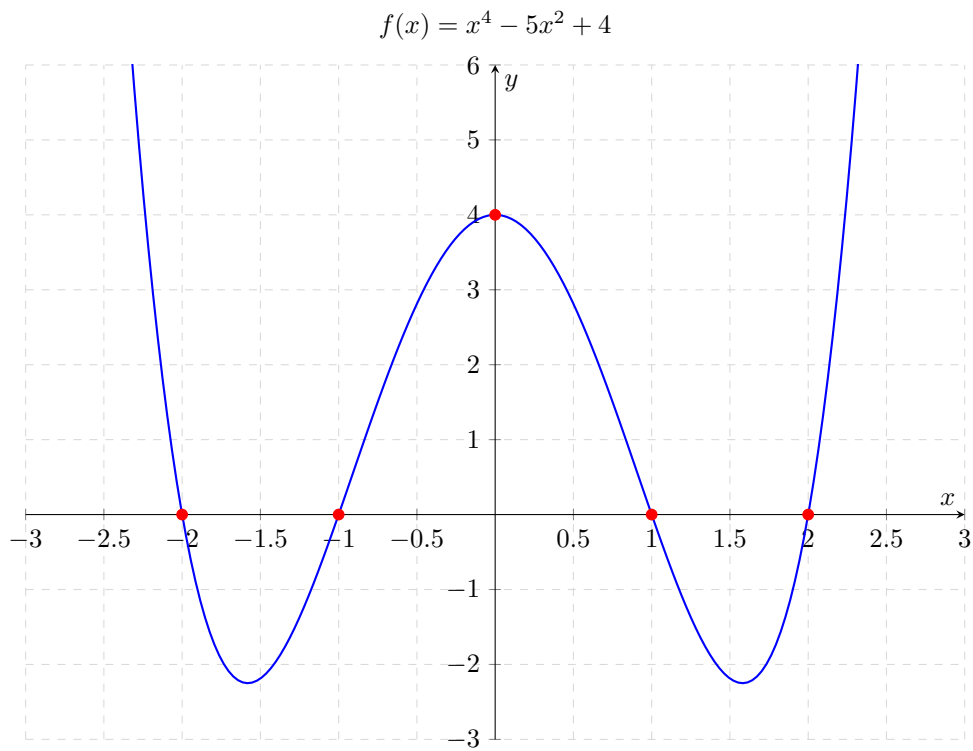
$$\begin{cases} x = 0 \\ y = f(0) \end{cases}$$

### 17.2 Zeros of a function

Let  $y = f(x)$  be any function, then we solve for  $x$ :

$$\begin{cases} y = 0 \\ 0 = f(x) \end{cases}$$

### 17.3 Graph example



## 18 Dominant elements in a function approaching $\pm\infty$

As  $x$  approaches  $\pm\infty$ , the term with the highest degree in a polynomial function dominates the behavior of the function.

$$p(x) \text{ has, as a dominant, the element } a_n \text{ with the highest degree } x^n$$

### 18.1 Order of dominance

#### 18.1.1 Approaching to $+\infty$

Let  $n \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $2 < n < m$ , then:

$$\ln(x) < x < x^n < x^m < n^x < m^x < x^x$$

In these cases, we always have  $x \Rightarrow +\infty \Rightarrow p(x) \Rightarrow +\infty$

#### 18.1.2 Approaching to $-\infty$

Let  $\lambda > 2$  and odd,  $k > 2$  and even.

$$\begin{array}{l} x^\lambda < -x^2 < x^1 < 0 \\ -x^k < -x^2 < x^1 < 0 \end{array}$$

Functions like  $x^\lambda$  (with  $\lambda$  odd) and  $-x^k$  (with  $k$  even) both approach  $-\infty$ , but at different rates.

#### 18.1.3 Dominance in rational functions

When the dominant element is at the numerator:

$$\lim_{x \rightarrow \infty} \frac{x^n}{x^{n-1}} = \infty$$

When the dominant element is at the denominator:

$$\lim_{x \rightarrow \infty} \frac{x^{n-1}}{x^n} = 0$$

When we have the same degree either in the numerator and in the denominator:

$$\lim_{x \rightarrow \infty} \frac{ax^n}{bx^n} = \frac{a}{b}$$

Definition: **horizontal asymptote** appears when  $x$  approaches to  $\infty$ , which implies that  $y$  approaches to a number  $A$  different from  $\pm\infty$

## 19 Exponential and logarithm functions

The relationship between exponentials and logarithms is based on the following formula:

$$a^{\log_a(x)} = x \iff \log_a(a^x) = x$$

### 19.1 Exponentials

#### 19.1.1 General equation

Let  $\alpha \in \mathbb{R}_+^*$ ,  $x \in \mathbb{R}$ , and  $a > 1$ , then:

$$y = \alpha \cdot a^x$$

#### 19.1.2 Euler's number

Euler's number is defined by the limit:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718 \dots$$

Alternatively, it can be expressed as:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

### 19.2 Logarithms

#### 19.2.1 Natural logarithm

The inverse function of the Euler's exponential function:

$$f(x) = e^x \iff h(x) = \ln(x)$$

Remark: the domain of  $\ln(x)$  is  $D_n : \forall x \in \mathbb{R}_+^*$

#### 19.2.2 Logarithms with arbitrary bases

The inverse function of any arbitrary exponential function:

$$f(x) = n^x \iff h(x) = \log_n(x)$$

Alternatively, it can be expressed as:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

#### 19.2.3 Common logarithm

The common logarithm uses base 10:

$$\log_{10}(x) = \frac{\ln(x)}{\ln(10)}$$



### 19.3 Exponential growth

$$N(t) = N_0 \cdot e^{kt}$$

## 20 Composite functions

Let  $y = f(x)$  and  $z = g(y)$  be two functions, then:

$$z = g(f(x))$$

### 20.1 Examples

1)

Let  $f(x) = x^2 + 4x$  and  $g(y) = y^2 + \cos(y)$  be two functions, then:

$$g(f(x)) = (x^2 + 4x)^2 + \cos(x^2 + 4x)$$

2)

Let  $f(x) = x^3$ ,  $h(x) = \arctan(x)$  and  $g(x) = \ln(x)$  be functions, then:

$$g(h(f(x))) = \ln(\arctan(x^3))$$

## Part IV

# Trigonometry

## 21 Trigonometry

### 21.1 Conversion table of degrees and radians

| Angles (in Degrees) | 0°             | 30°          | 45°          | 60°          | 90°       | 180°    | 270°       | 360°     |
|---------------------|----------------|--------------|--------------|--------------|-----------|---------|------------|----------|
| Angles (in Radians) | 0 <sup>c</sup> | $\pi/6^c$    | $\pi/4^c$    | $\pi/3^c$    | $\pi/2^c$ | $\pi^c$ | $3\pi/2^c$ | $2\pi^c$ |
| $\sin(\theta)$      | 0              | 1/2          | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1         | 0       | -1         | 0        |
| $\cos(\theta)$      | 1              | $\sqrt{3}/2$ | $\sqrt{2}/2$ | 1/2          | 0         | -1      | 0          | 1        |
| $\tan(\theta)$      | 0              | $\sqrt{3}/3$ | 1            | $\sqrt{3}$   | $\infty$  | 0       | $\infty$   | 0        |

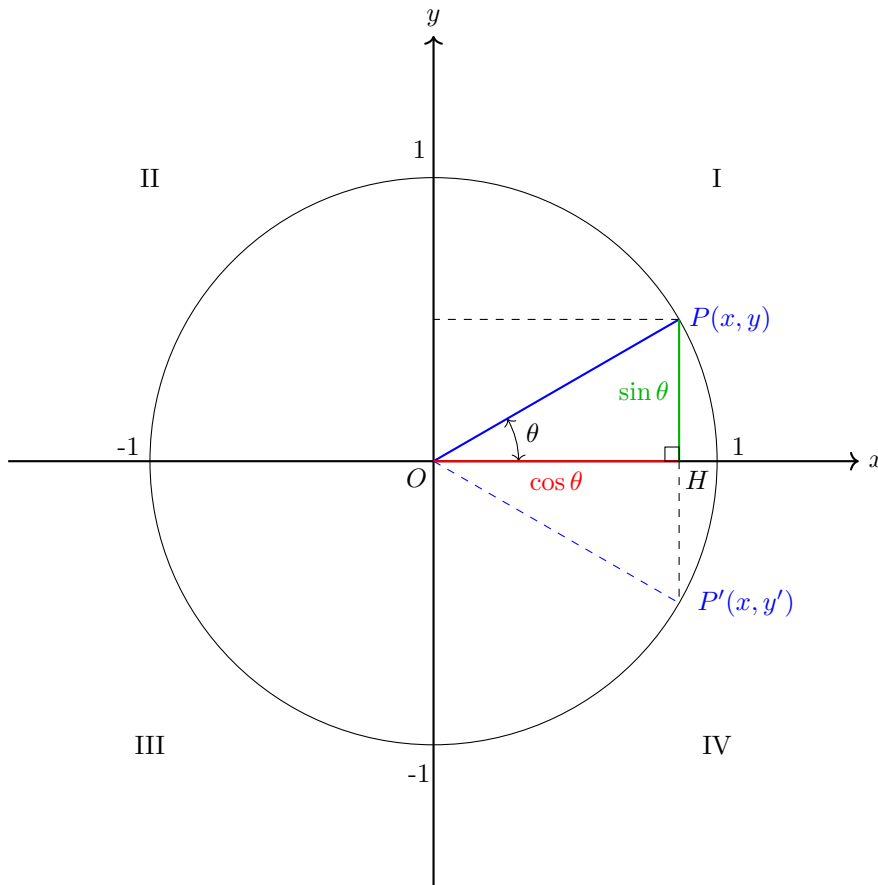
Remark:

$$\cos(2\pi + \theta) = \cos(\theta) \quad | \quad \sin(2\pi + \theta) = \sin(\theta)$$

Remark: Let  $\forall k \in \mathbb{Z}$ ,  $\forall \theta \in \mathbb{R}$ , then:

$$\cos(\theta + 2\pi k) = \cos(\theta)$$

### 21.2 Trigonometric functions in the unit circle



Remark: the circle has center in the origin  $O$ , radius = 1 and function  $x^2 + y^2 = 1$

Trigonometric functions can be extended to angles beyond  $0$  and  $90^\circ$  using the unit circle. For an angle  $\theta$  in the unit circle:

$$\sin \theta := y \quad | \quad \cos \theta := x \quad | \quad \tan \theta := \frac{y}{x}$$

### 21.2.1 Property 1 – Domain and range

Because we are inside a circle of radius 1:

- $-1 \leq \cos(\theta) \leq 1$ ;
- $-1 \leq \sin(\theta) \leq 1$ .

### 21.2.2 Property 2 – Trigonometric identity

Because we have a  $90^\circ$  angle, we can use Pythagoras:

$$\overrightarrow{OH}^2 + \overrightarrow{PH}^2 = \overrightarrow{OP}^2$$

Let  $\forall \theta \in \mathbb{R}$ , then we can compute the following trigonometric identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

## 21.3 Tangent

A tangent of an angle is exactly the slope of a line:

$$m = \frac{\Delta y}{\Delta x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Remark: the tangent is not defined when the angle is  $\frac{\pi}{2}$  or  $\frac{3\pi}{2}$ , that is when we have a vertical line.

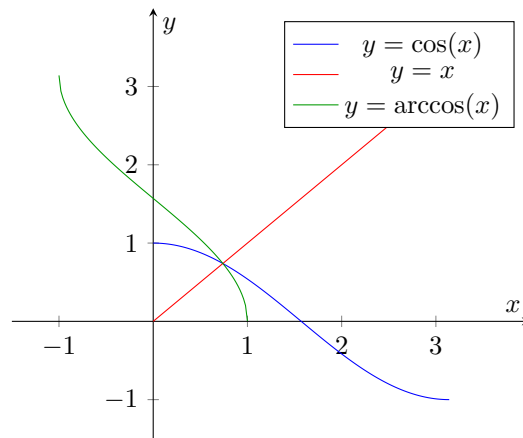
## 21.4 Domain of trigonometric functions

$$\begin{aligned} y &= \cos(x), & x &\in \mathbb{R} \\ y &= \sin(x), & x &\in \mathbb{R} \\ y &= \tan(x), & x &\in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\} \end{aligned}$$

## 21.5 Inverse trigonometric functions

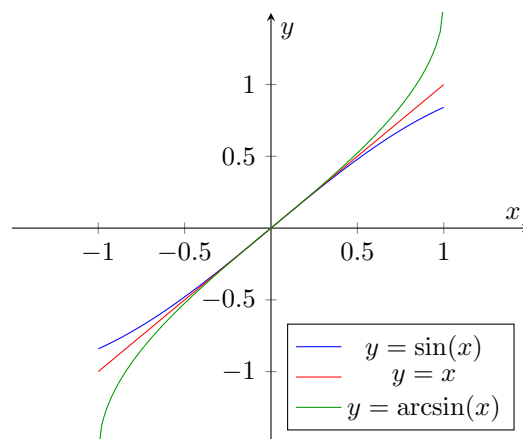
Warning: in order to be invertible, a function should be either always strictly increasing or always strictly decreasing.

### 21.5.1 Arccosine



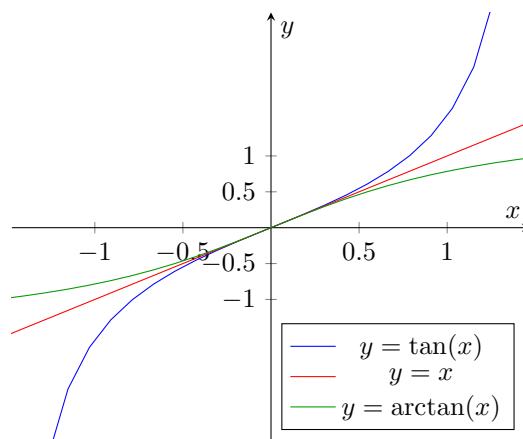
### 21.5.2 Arcsine

The domain of the arcsine is  $\forall x \in [-1, 1]$  and the range is  $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



### 21.5.3 Arctan

The domain is  $\forall x \in \mathbb{R}$  and the range is  $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



## 21.6 Harmonic oscillation

Let  $A, B > 0$ , then the function is oscillating harmonically with  $t$  around  $D$ :

$$y = D + A \cdot \sin(Bt + \varphi)$$



## Part V

# Limits

## 22 Concept of limit of a real function

### 22.1 Definition

Let  $f : \mathcal{D} \rightarrow \mathbb{R}$  be a function and  $c$  a point, the limit  $L = \lim_{x \rightarrow c} f(x)$  with  $x$  tending to  $c$  exists only if in a given  $\epsilon > 0$  arbitrarily small, exists another  $\delta > 0$  such that:

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

### 22.2 Graphic interpretation



## 23 Limit value at a finite point

The notion of the “limit of  $f(x)$  as  $x$  approaches a (finite) point  $a \in \mathbb{R}$ ” is only meaningful if the point  $a$  can be approximated by points from the domain of definition of  $f$ . We can precisely formulate this concept with the notion of an “accumulation point”.

### Definition

Given a set  $A \subset \mathbb{R}$  and a real number  $a \in \mathbb{R}$ , the real number  $a$  is called an *accumulation point* of the set  $A$  if every open interval of the form  $(a - \delta, a + \delta)$  with  $\delta > 0$  contains infinitely many points of  $A$ .

In the above definition, it is not required that  $a$  lies in  $A$ . Often, we will consider functions whose domains are unions of intervals of the form:

$$(b, a) \cup (a, c)$$

For example, consider the function defined by  $f(x) = \frac{1}{x}$ , defined on  $(-\infty, 0) \cup (0, \infty)$ . The point 0 is an accumulation point of the domain of definition of  $\frac{1}{x}$ .

### Definition

Given a real function  $f$ , an accumulation point  $x_0$  of  $\mathcal{D}_f$ , and  $L \in \mathbb{R} = \mathbb{R} \cup \{\pm\infty\}$ , we say that the function  $f$  has the limit  $L$  as  $x \rightarrow x_0$  if  $f(x)$  gets arbitrarily close to  $L$ , provided  $x$  is sufficiently close to (but never equal to)  $x_0$ .

## 23.1 One-sided limits

Often, one considers limits where  $x$  approaches  $x_0$  from only one direction, either from the right or from the left. In these cases, we refer to a right-sided or left-sided limit and use the following notations:

$$\lim_{x \rightarrow x_0^+} f(x) \quad \text{or} \quad \lim_{x \rightarrow x_0, x > x_0} f(x) \quad \text{or} \quad \lim_{x \rightarrow x_0} f(x)$$

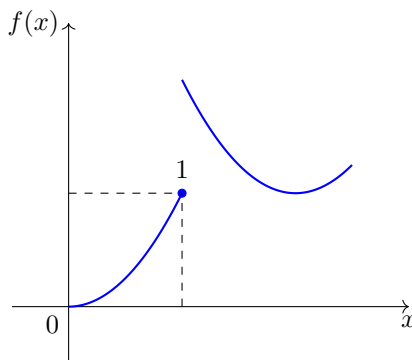
for a right-sided limit, and:

$$\lim_{x \rightarrow x_0^-} f(x) \quad \text{or} \quad \lim_{x \rightarrow x_0, x < x_0} f(x) \quad \text{or} \quad \lim_{x \rightarrow x_0} f(x)$$

for a left-sided limit.

$\lim_{x \rightarrow a}$  can indicate a limit as  $x$  approaches an arbitrary point (e.g.,  $a = x_0$  for  $x_0 \in \mathbb{R}$ ), as well as a one-sided limit ( $a = x_0^+$  or  $a = x_0^-$  for  $x_0 \in \mathbb{R}$ ), or a limit at infinity ( $a = \pm\infty$ ).

### 23.1.1 Graph example



## 24 Continuity of a function

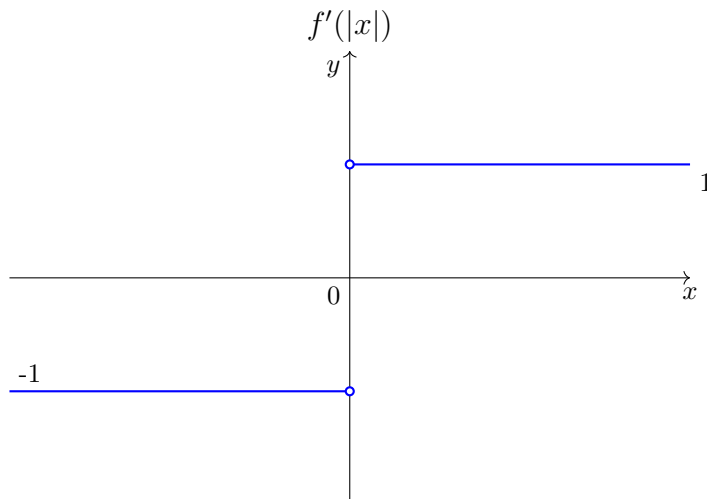
### **Definition** Continuity of a real function

Given a real function  $f : D \rightarrow C$ , the function is continuous at the point  $x = c$  where  $c \in D$  if:

$$\lim_{x \rightarrow c} f(x) = f(c)$$

and therefore, if the limit exists and is equal to the value of the function at that point.

In other words, the function is continuous at the point if the limit exists (from both the left and right, coinciding) and the value of the function at that point is equivalent to the value of the limit.



## 24.1 Continuity in short

- A function is said to be continuous at a point if the limit at that point exists and is equal to the value of the function at that point;
- A function is said to be continuous on a subinterval of the domain if it is continuous at all points in that subinterval;
- A function is said to be continuous if it is continuous at all points in its interval.



## Part VI

# Derivatives

Assume that  $y = f(x)$  is a differentiable function in some interval  $(a, b)$ , then we have defined the derivative function.

## 25 Derivative notations

| Type of derivative                   | First derivative                | Second derivative                   | n-th derivative                     |
|--------------------------------------|---------------------------------|-------------------------------------|-------------------------------------|
| Lagrange's notation                  | $f'(x)$                         | $f''(x)$                            | $f^{(n)}(x)$                        |
| Leibniz's notation                   | $\frac{d}{dx}f(x)$              | $\frac{d^2}{dx^2}f(x)$              | $\frac{d^n}{dx^n}f(x)$              |
| Leibniz's notation<br>in a point $a$ | $\frac{d}{dx}[f(x)] \mid_{x=a}$ | $\frac{d^2}{dx^2}[f(x)] \mid_{x=a}$ | $\frac{d^n}{dx^n}[f(x)] \mid_{x=a}$ |
| Newton's notation                    | $\dot{f}$                       | $\ddot{f}$                          | $\overset{(n)}{f}$                  |

## 26 Definition of derivative

The derivative of a real function  $f(x)$  is defined as:

$$f'(x) = \lim_{\Delta x \rightarrow 0^\pm} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

if the limit exists.

Definition:  $f'(a)$ , if it exists, is called derivative of  $f(x)$  at  $x = a$ .

It corresponds to the slope of the tangent line at  $x = a$  to the function  $y = f(x)$

### 26.1 Simplified definition (Exponentiation rule)

Let  $\forall \alpha \in \mathbb{R}$ , then:

$$f(x) = x^\alpha \Rightarrow f'(x) = \alpha \cdot x^{\alpha-1}$$

### 26.2 Existence of the derivative

The derivative exists if and only if:

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Remark: If a function is differentiable, then it is continuous:

$$\text{Differentiable} \Rightarrow \text{Continuous}$$

## 27 Geometric meaning of the derivative



The secant of a function  $f(x)$  between a point  $A$  and  $B$  is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}$$

The closer we bring  $A$  and  $B$ , the smaller  $\Delta x$  becomes. As  $\Delta x$  decreases, the slope of the secant becomes more representative of the rate of change of  $f$  in the interval  $[A; B]$ .

When the  $\Delta x$  of the slope becomes infinitesimally small, we obtain the exact slope at a point (instantaneous). This slope is represented by the tangent line:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$



The derivative of a function  $f(x)$  is therefore another function,  $f'(x)$ , which represents the rate of change of  $f(x)$  at every point. In other words,  $f'(x)$  represents the slope of the tangent at each  $x$  of  $f(x)$ . This is precisely represented by the definition of the derivative, which is the slope  $\frac{\Delta y}{\Delta x}$  calculated with the limit of  $\Delta x \rightarrow 0$ .

### 27.1 Equation of the tangent line

When  $f'(x)$  is defined,  $p = (x, f(x))$ , and  $p \in$  tangent line, then:

$$y - f(a) = f'(a) \cdot (x - a)$$

## 28 Bernoulli – de l'Hôpital Theorem

Bernoulli – de l'Hôpital theorem is applicable **only if** the function results in an indeterminate form.

### 28.1 The 7 indeterminate forms

The seven indeterminate forms are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

## 28.2 Statement of the theorem

Let us consider two real functions  $f(x)$  and  $g(x)$  that are differentiable in a neighborhood of  $x_0 \in \mathbb{R}$  (not necessarily at  $x_0$ ).

If  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$  results in an indeterminate form, then:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

if the limit exists.

## 29 Derivation rules

### 29.1 Normal cases

Let  $y = f(x)$  and  $y = g(x)$  be two derivable functions, then:

#### 29.1.1 Linearity

Let  $c \in \mathbb{R}$ , then:

$$\frac{d}{dx} [c \cdot f(x)] = c \cdot \frac{d}{dx} [f(x)]$$

#### 29.1.2 Sum and subtraction

$$f'(x) \pm g'(x) = \frac{d}{dx} [f(x) \pm g(x)]$$

#### 29.1.3 Multiplication

$$\frac{d}{dx} [f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

#### 29.1.4 Difference

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

#### 29.1.5 Exponential

Let  $a > 0$ , then:

$$\frac{d}{dx} [a^x] = a^x \cdot \ln(a)$$

#### 29.1.6 Composite function (Chain rule)

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

### 29.1.7 Inverse function

$$\frac{d}{dx} [f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

### 29.1.8 Trigonometric functions



|   |   |
|---|---|
| $(\sec x)' = \sec x \tan x$             | $(\csc x)' = -\csc x \cot x$                    |
| $(\tan x)' = \frac{1}{\cos^2(x)}$       | $(\cot x)' = \frac{-1}{\sin^2(x)}$              |
| $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ | $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$        |
| $(\arctan x)' = \frac{1}{1+x^2}$        | $(\operatorname{arccot} x)' = \frac{-1}{1+x^2}$ |

## 29.2 Particular cases

1.

$$f(x) = g(x)^\alpha \implies f'(x) = \alpha \cdot g'(x) \cdot g(x)^{\alpha-1} \quad \text{i.g.: } (x^2 + 1)^4 \implies 8x \cdot (x^2 + 1)^3$$

2.

$$f(x) = e^{g(x)} \implies f'(x) = g'(x) \cdot e^{g(x)}$$

3.

$$f(x) = \frac{1}{g(x)} \implies f'(x) = \frac{-g'(x)}{(g(x))^2}$$

## 29.3 Physical application

### 29.3.1 Average acceleration $a_{av}$

$$a_{av} := \frac{v(t_f) - v(t_i)}{t_f - t_i}$$

### 29.3.2 Instant acceleration $a(t)$

$$a(t) := v'(t) = \lim_{t \rightarrow 0} \frac{v(t+h) - v(t)}{\Delta t}$$

## 30 Linearization

### 30.1 The linearization principle

In a very small neighborhood around a point  $a$ , we can assume that the function is linear at that point.

### 30.2 Tangent line approximation

In this case, assuming that the function is linear, we can use the tangent line equation:

$$f(x) = f(a) + f'(a) \cdot (x - a)$$

### 30.3 Error function

The error of the approximation is given by the difference between the exact function and the linearization:

$$E(x) = d(f(x) | f_{\text{lin}}(x)) = f(x) - f(a) - f'(a) \cdot (x - a)$$

## 31 Monotonicity

### 31.1 Definition of monotonicity

A real function  $f$  defined on an interval  $I \subset \mathcal{D}_f$  is denoted as:

- *strictly monotonically increasing* on  $I$ , if  $f(x_2) > f(x_1)$  applies for all  $x_1, x_2 \in I$  with  $x_2 > x_1$ ;
- *monotonically increasing* on  $I$ , if  $f(x_2) \geq f(x_1)$  applies for all  $x_1, x_2 \in I$  with  $x_2 > x_1$ ;
- *strictly monotonically decreasing* on  $I$ , if  $f(x_2) < f(x_1)$  applies for all  $x_1, x_2 \in I$  with  $x_2 > x_1$ ;
- *monotonically decreasing* on  $I$ , if  $f(x_2) \leq f(x_1)$  applies for all  $x_1, x_2 \in I$  with  $x_2 > x_1$ .

#### 31.1.1 Monotonicity criterion

Let the function  $f$  be differentiable on the interval  $I$ :

- If  $f'(x) > 0$  (resp.  $\geq 0$ ) for all  $x \in I$ , then  $f$  is strictly monotonically increasing (resp. monotonically increasing) on  $I$ .
- If  $f'(x) < 0$  (resp.  $\leq 0$ ) for all  $x \in I$ , then  $f$  is strictly monotonically decreasing (resp. monotonically decreasing) on  $I$ .
- If  $f'(x) = 0$  for all  $x \in I$ , then  $f$  is constant on  $I$ .

#### 31.1.2 Monotonicity table

Let  $f(x)$  be differentiable,  $f'(x) < 0$  if  $a < x < b$ ,  $a, b, c \in \mathcal{D}_f$ , and  $a, b, c$  are critical points, then:

|         |            |            |            |            |
|---------|------------|------------|------------|------------|
| $f'(x)$ | +          | -          | +          | -          |
| $f(x)$  | $\nearrow$ | $\searrow$ | $\nearrow$ | $\searrow$ |
|         | a          | b          | c          |            |

### 31.2 Critical point

Let  $y_f(x)$  be a function, then we say that  $x \in \mathcal{D}_f$  is a critical point if  $f'(x) = 0$  or  $f'(x) \uparrow$

Warning: many critical points are local extrema, some aren't.

### 31.3 Darboux theorem

Let  $f$  be differentiable on an interval  $I$ :

1. find the critical points  $f'(a, b) = 0$ ,  $a < b$ ;
2. take a random point **between the critical points** in  $c \mid a < c < b$ ;
3. compute  $f'(c)$ .

|  |
|--|
| If $f'(c) > 0$ , then $f'(x) > 0$ , $\forall x \in (a, b)$<br>If $f'(c) < 0$ , then $f'(x) < 0$ , $\forall x \in (a, b)$ |
|--|

## 32 Minimum and maximum

Let a real function  $f$  and a point  $x_0 \in \mathcal{D}_f$  be given.



Local and global extrema of a function in the interval  $[-a, a]$

### 32.1 Local extrema

#### 32.1.1 Local maximum

The function  $f$  has a **local maximum point** at point  $x_0$  if there is an open neighborhood  $U(x_0)$  such that:

|   |
|---|
| $f(x) \leq f(x_0), \forall x \in U(x_0) \cap \mathcal{D}_f$ |
|---|

#### 32.1.2 Local minimum

The function  $f$  has a **local minimum point** at point  $x_0$  if there is an open neighborhood  $U(x_0)$  such that:

|   |
|---|
| $f(x) \geq f(x_0), \forall x \in U(x_0) \cap \mathcal{D}_f$ |
|---|

### 32.2 Global extrema

#### 32.2.1 Global maximum

The function  $f$  has a **global maximum** at point  $x_0$  if:

|   |
|---|
| $f(x) \leq f(x_0), \forall x \in \mathcal{D}_f$ |
|---|

#### 32.2.2 Global minimum

The function  $f$  has a **global minimum** at point  $x_0$  if:

|   |
|---|
| $f(x) \geq f(x_0), \forall x \in \mathcal{D}_f$ |
|---|

### 32.3 Extrema tricks

- If  $f'(x_0) = 0$  and  $f''(x_0) < 0$  are valid,  $f$  has a local maximum in  $x_0$ ;
- If  $f'(x_0) = 0$  and  $f''(x_0) > 0$  are valid,  $f$  has a local minimum in  $x_0$ .

Warning: This method does not work because if  $f''(x_0) = 0$ , then we may have either a local maximum, minimum or a stationary point.

## 33 Higher derivatives

We define  $y = f^{(n)}(x)$  as the derivative of  $y = f^{(n-1)}(x)$ .

Remark: Derivatives will be written with the Lagrange's notation and roman numbers, i.g.:  $f'''(x) \rightarrow f^{IV}(x)$

### 33.1 Concavity

#### 33.1.1 Definition of Concavity

The concavity of a function  $f(x)$  describes the direction of its curvature, which can be upward when  $f''(x) > 0$  or downward when  $f''(x) < 0$ .

Additionally, the concavity can be increasing when  $f'(x) > 0$  and decreasing when  $f'(x) < 0$ .

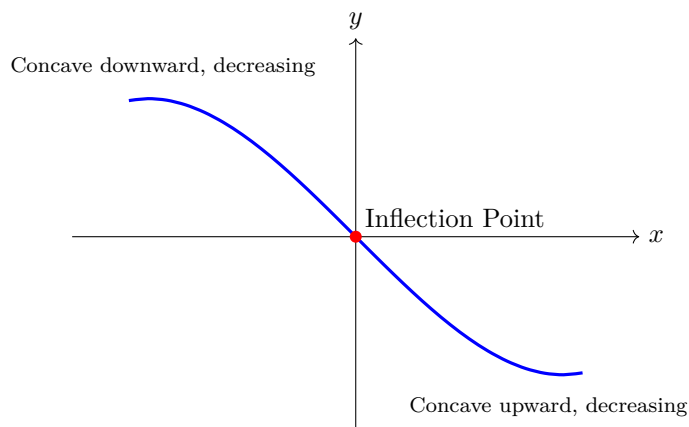


- $f(x)$  is **concave upward** in an interval  $I$  if all the tangents in  $I$  are below the graph.
- $f(x)$  is **concave downward** in an interval  $I$  if all the tangents in  $I$  are above the graph.
- If  $f''(x) > 0$  for all  $x$  in an interval  $I$ , then  $f(x)$  is concave upward in  $I$ .
- If  $f''(x) < 0$  for all  $x$  in an interval  $I$ , then  $f(x)$  is concave downward in  $I$ .

### 33.2 Inflection point

An inflection point for  $y = f(x)$  is a point  $x_0 \in \mathcal{D}_f$  where the function is continuous and its concavity changes.

For any inflection point,  $f''(x) = 0$ .



Warning:  $f''(x_0) = 0 \not\Rightarrow x_0$  is an inflection point.

### 33.3 Curvature of a function

Let  $y = f(x)$  be a derivable function in a point  $x_0$ , then:

$$\kappa = \frac{f''(x_0)}{(1 + (f'(x_0))^2)^{3/2}}$$

#### 33.3.1 Radius of curvature

The radius of curvature is given by the following formula:

$$r_p = \frac{1}{|\kappa|}$$

#### 33.3.2 Graphical example



- $C$  = Center of the circle
- $r_p$  = Radius of curvature
- $x_0$  = Specific point



## Part VII

# Integrals

### 34 Definite integral

Let  $f(x)$  be a continuous function defined on an interval  $[a, b]$ . The definite integral from  $a$  to  $b$  represents the net area under the curve of  $f(x)$  between  $x = a$  and  $x = b$ , considering areas above the x-axis as positive and those below as negative.



#### 34.1 Definite integral cases

Let  $f(x)$  be a continuous function in  $\mathbb{R} \rightarrow \mathbb{R}$ , then we have three possible cases:

##### 34.1.1 First case

When  $a < b$ :

$$\int_a^b f(x) dx = F(x) \Big|_a^b + C$$

##### 34.1.2 Second case

When  $a > b$ :

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

##### 34.1.3 Third case

When  $a = b$ ,  $\forall a \in \mathbb{R}$ :

$$\int_a^a f(x) dx = 0$$

## 34.2 Riemann sum

Let  $f : [a, b] \rightarrow \mathbb{R}$ , :

$$R_n := \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$$

with:

- $\Delta x = \frac{b-a}{n}$ ,  $n \in \mathbb{N}$
- $x_i = a + i \cdot \Delta x$

Warning: there are functions that are not integrable in the Riemann sense, such as  $\text{sgn}(x)$

### 34.2.1 Statement of the theorem

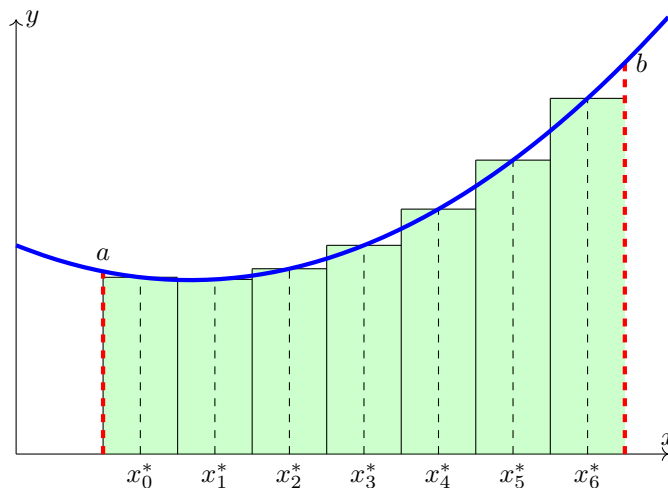
$$\int_a^b f(x) dx = \lim_{n \rightarrow +\infty} R_n$$

### 34.2.2 Sigma notation

$$R_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(1 + \frac{i}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} \left(1 + \frac{i}{n}\right)^2$$

### 34.2.3 Graphic interpretation

The definite integral can be considered as the Riemann integral, which can be defined with the infinitesimal sum of rectangles with a base tending to 0 and a height equal to a point within the base:



### 34.3 Integration rules

#### 34.3.1 Linearity

Let  $\lambda \in \mathbb{R}$ , then:

$$\int \lambda f(x) dx = \lambda \int f(x) dx$$

#### 34.3.2 Sum and subtraction

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

### 34.4 Integral with infinite bounds

#### 34.4.1 When upper bound is $+\infty$

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

#### 34.4.2 When lower bound is $-\infty$

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

#### 34.4.3 When bounds are $\infty$

$$\int_{-\infty}^\infty f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx + \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

## 35 Indefinite integral

We denote the **set** of all antiderivatives of a function  $f$  as an *indefinite integral* of  $f$ . This is written as:

$$\int f(x) dx$$

with no integration limits specified.

### 35.1 Fundamental theorem of calculus

Let  $F : [a, b] \rightarrow \mathbb{R}$  a continuous function, differentiable in  $(a, b)$ .

Let  $f(x) = F'(x)$  and  $\forall C \in \mathbb{R}$ , then:

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Remark: Since  $F'(x) = f(x)$ , we have infinite possible primitives, which are distinguished by the constant  $C$ .

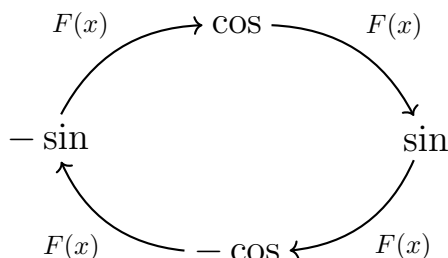
### 35.2 Second fundamental theorem of differential and integral calculus

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $x_0 \in [a, b]$ , then  $\forall x \in [a, b]$ :

$$F_0(x) = \int_x^{x_0} f(t) dt$$

### 35.3 Special cases

#### 35.3.1 Trigonometric functions



#### 35.3.2 Other rules

$$\begin{array}{ll} \int x^\alpha dx = \frac{1}{\alpha+1} \cdot x^{\alpha+1} + C & \int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C \\ \int e^x dx = e^x + C & \int \frac{1}{1+x^2} dx = \arctan(x) + C \end{array}$$

## 36 Integration with substitution rule

### 36.1 Indefinite integrals

According to the chain rule, we have:

$$\frac{d}{dx} [F(g(x))] = F'(g(x)) \cdot g'(x) = f(g(x))g'(x)$$

If  $g'(x)$  is continuous:

$$\int f(g(x))g'(x) dx = F(g(x)) + C$$

Now let:

$$u = g(x) \implies du = g'(x) dx$$

With this we obtain:

$$\int f(g(x))g'(x) dx = \int f(u) du$$

### 36.2 Definite integrals

The corresponding formula for definite integrals is:

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

## 37 Integration by parts

### 37.1 Indefinite integrals

Let  $f(x)$  and  $g(x)$  be two functions. Integration by parts descends from the product rule of differential calculus:

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

By integrating both sides:

$$f(x)g(x) = \int (f(x)g(x))' dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx$$

Rearranging this equation leads to the formula of integration by parts:

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

## 37.2 Definite integrals

The corresponding formula for definite integrals is:

$$\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) dx$$

## 37.3 Particular cases

1.

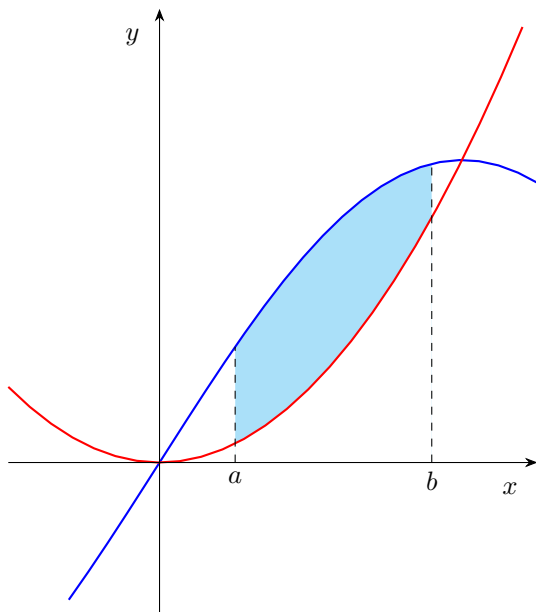
$$\frac{d}{dx} (e^{h(x)}) = h'(x) \cdot e^{h(x)} \implies \int h'(x) \cdot e^{h(x)} dx = e^{h(x)} + C$$

2.

$$\frac{d}{dx} (\ln(\pm f(x))) = \frac{f'(x)}{f(x)} \implies \int \frac{f'(x)}{f(x)} dx = \ln |f(x)|$$

## 38 Areas with integrals

### 38.1 Area between two curves



#### 38.1.1 Case 1

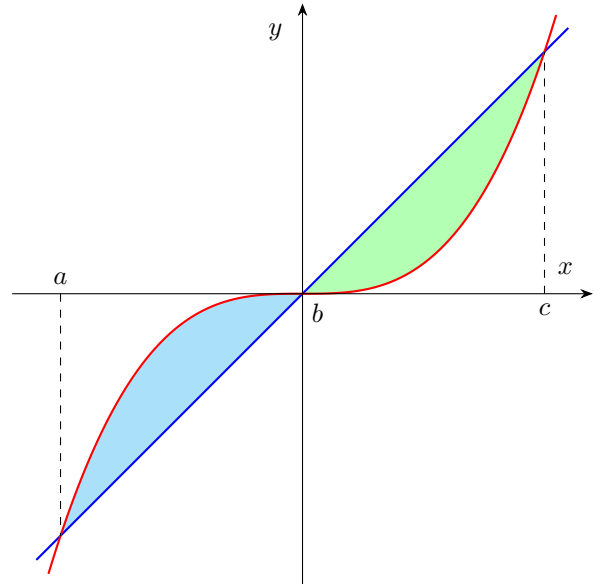
Given a function  $y = f(x)$  and  $y = g(x)$ , the area enclosed by the two functions in the interval  $I = [a; b]$  is given by:

$$A = \int_a^b |f(x) - g(x)| dx$$

### 38.1.2 Case 2

Given a function  $y = f(x)$  and  $y = g(x)$ , the area enclosed by the two functions in the interval  $I = [a; c]$  is given by:

$$A = \int_a^b f(x) - g(x) dx + \int_b^c g(x) - f(x) dx$$



## 38.2 Arc length

Let a plane curve be defined by a differentiable function  $f(x)$  whose derivative  $f'(x)$  is continuous over the interval  $a \leq x \leq b$ . Then, the arc length between the points  $(a, f(a))$  and  $(b, f(b))$  is:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$