

Mathematics 1A

HSLU, Semester 1

Matteo Frongillo

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Part I

Week 1

1 The set theory

1.1 Definition of a set

A set is a collection of objects or elements.

Remark: The collection of all sets is not a set.

1.2 Logical symbols

1.2.1 Definition

Braces and the definition symbol “:=” are used to define a set giving all its elements:

$$A := \{a, b, c, d, e\}$$

1.2.2 Equal

In this case, the equal symbol means that the set A is equal to the set B :

$$A = B$$

1.2.3 Belongs to

The symbols \in and \ni describe an element which is part of the set:

$$a \in A \iff A \ni a$$

1.2.4 Does not belong to

The symbols \notin mean that an element does not belong to the set:

$$f \notin A$$

1.2.5 Inclusion and contains

The symbols \subset and \supset mean that a set has another set included in its set:

$$\mathbb{N} \subset \mathbb{Z} \iff \mathbb{Z} \supset \mathbb{N}$$

1.2.6 For all/any

The symbol \forall means that we are considering any type of element:

$$\forall x \in \mathbb{R}, x > 0$$

In this case, we've defined a new set.

1.3 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0);
- $\mathbb{Z} :=$ Integer numbers;
- $\mathbb{Q} :=$ Rational numbers;
- $\mathbb{R} :=$ Real numbers $:= \mathbb{Q} \cup \{\text{irrational numbers}\}$.

Notation: The “*” symbol means that the set does not include 0.

1.3.1 Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$$B := \{\pi, 1, -1, 0\};$$

$$C := \{\pi, 1\};$$

$$D := \{\pi\}.$$

Then we write some examples: $\pi \in B$, $D \subset B$, $C \subset B$, $B \not\subset C$, $0 \in B$, $0 \notin C$.

2 Intervals in the real line

Intervals describe what happens between two or more elements.

2.1 Examples

2.1.1 Interval sets

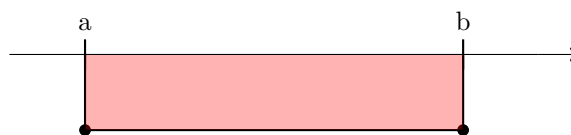
We have 4 cases:

- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\};$
- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\};$
- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\};$
- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}.$

Notation: a and b are often called the “end points” of the interval;

2.1.2 Graphical examples

$$\forall x \in \mathbb{R}, x \in [a, b]$$

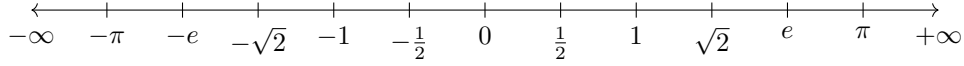


3 The extended line

In the real line \mathbb{R} we add $\pm\infty$.

Real line: $(-\infty, +\infty) = \mathbb{R}$

Extended real line: $[-\infty, +\infty] = \overline{\mathbb{R}}$



Remark: $\pm\infty \notin \mathbb{R}$

3.1 Properties

$$\boxed{\forall x \in \mathbb{R} \mid \infty > x \mid -\infty < 0}$$

3.2 Operation in the extended line

If $a, b \in \mathbb{R}$, then $a + b$, $a - b$, $a \cdot b$, $\frac{a}{b}$ (with $b \neq 0$) stay the same

3.2.1 Additions

Let $\forall a \in \mathbb{R}$:

- $a + \infty := \infty$;
- $a - \infty := -\infty$;
- $+\infty + \infty := +\infty$;
- $-\infty - \infty := -\infty$;
- $+\infty - \infty := \text{undefined}$.

3.2.2 Multiplications

Let $\forall a \in \mathbb{R}$:

- $+\infty \cdot +\infty := +\infty$;
- $-\infty \cdot +\infty := -\infty$;
- $-\infty \cdot (-\infty) := \infty$;
- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$
- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$
- $\frac{a}{+\infty} = \frac{a}{-\infty} := 0$;
- $\frac{+\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$
- $\frac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$
- $\frac{\infty}{\infty} := \text{undefined}$.

4 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including $\pm\infty$.

4.1 Examples

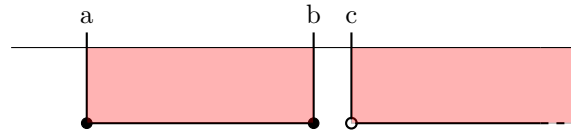
4.1.1 Interval sets

Let $a \in \mathbb{R}$, then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$;
- $[a, +\infty) = \{\forall x \in \mathbb{R} \mid x \geq a\}$;
- $(-\infty, +\infty) = \mathbb{R}$;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$.

4.1.2 Graphical examples

$\forall x \in \mathbb{R}, x \in [a, b] \cup]c, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol \cup .

5 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

5.1 Logical connectives

A	B	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

5.1.1 Logical conjunction \wedge

Given two statements P and Q , $P \wedge Q$ is true if both P and Q are true.

Let $P = (x > 0)$ and $Q = (y > 0)$, then:

$$P \wedge Q = (x > 0 \wedge y > 0)$$

5.1.2 Logical disjunction \vee

Given two statements P and Q , $P \vee Q$ is true if at least one of P or Q is true.

Let $P = (x = 0)$ and $Q = (y \neq 0)$, then:

$$P \vee Q = (x = 0 \vee y \neq 0)$$

5.1.3 Logical negation \neg

The negation of a statement P , denoted as $\neg P$, is true if P is false, and false if P is true.

Let $P = (x \geq 5)$, then:

$$\neg P = (x < 5)$$

5.1.4 Implication \Rightarrow

The symbol \Rightarrow indicates that if statement P is true, then statement Q must also be true (i.e., P implies Q).

Warning: It does not require that Q implies P .

$$P = (x = 1) \Rightarrow Q = (x \in \mathbb{N})$$

5.1.5 Inference \Leftarrow

The symbol \Leftarrow means that a conclusion or result implies the truth of an earlier statement.

If Q is true, then P must be true.

$$Q = (x > 0) \Leftarrow P = (x \in \mathbb{R}^+)$$

5.1.6 If and only if \Leftrightarrow

The symbol \Leftrightarrow indicates that two statements P and Q are logically equivalent, meaning P is true if and only if Q is true.

$$P = (x \in \mathbb{N}, x \neq 0) \Leftrightarrow Q = (x \in \mathbb{N}^*)$$

6 Union \cup and Intersection \cap

6.1 Universe symbol

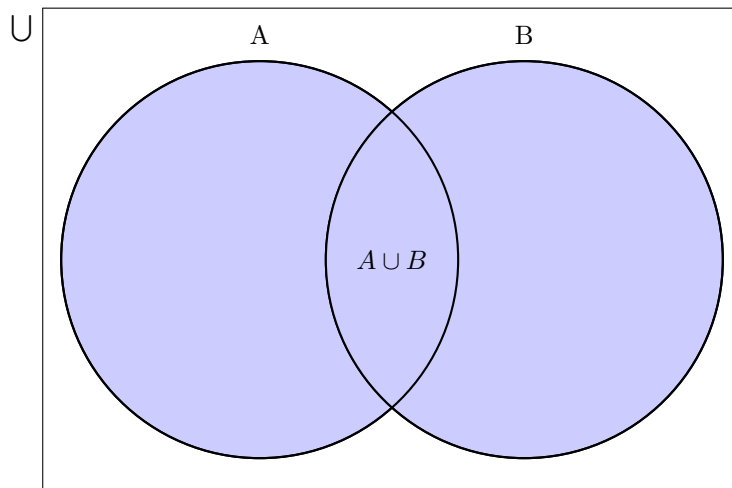
The symbol $\cup := \text{Universe}$ describes a big set which contains all sets involved in our discussions (not always).

6.2 Venn diagram

6.2.1 Union $A \cup B$

If A and B are sets, then their union is:

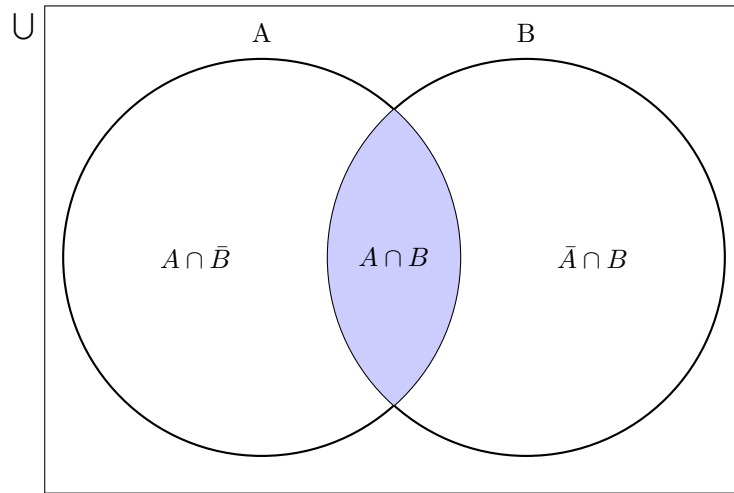
$$A \cup B = \{\forall x \in \cup \mid x \in A \vee x \in B\}$$



6.2.2 Intersection $A \cap B$

If A and B are sets, then their intersection is:

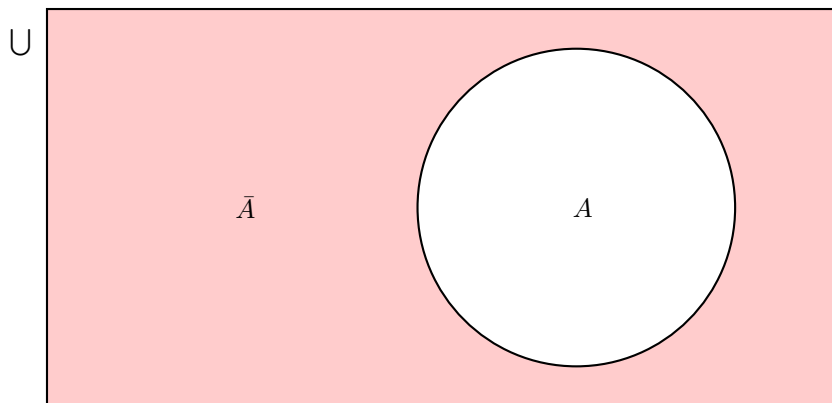
$$A \cap B = \{\forall x \in \mathcal{U} \mid x \in A \wedge x \in B\}$$



6.2.3 Complement \bar{A}

If A is a set, its complement is:

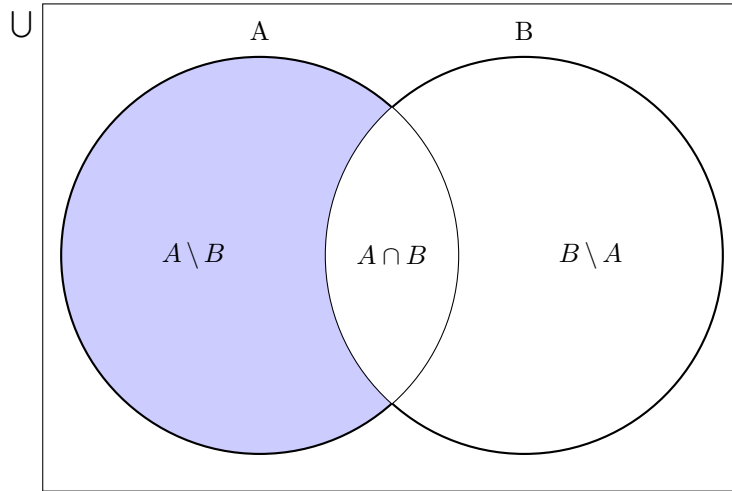
$$\bar{A} = \{\forall x \in \mathcal{U} \mid x \notin A\}$$



6.2.4 Difference between sets \setminus

If A and B are sets, then their difference is:

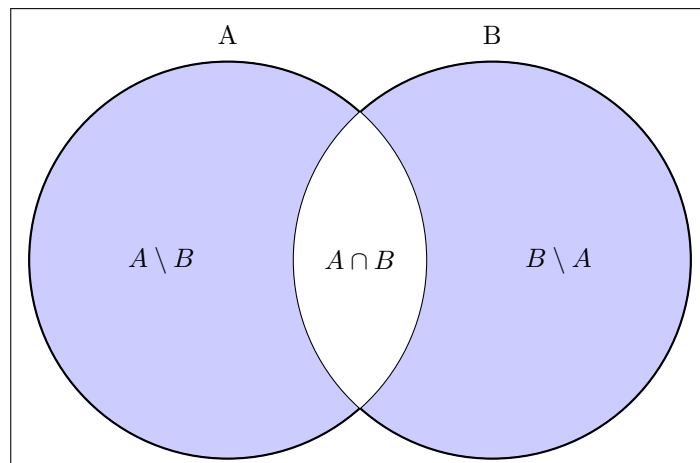
$$A \setminus B = \{\forall x \in \bigcup \mid x \in A, x \notin B\}$$



6.2.5 Symmetrical difference \triangle

If A and B are sets, then their symmetrical difference is:

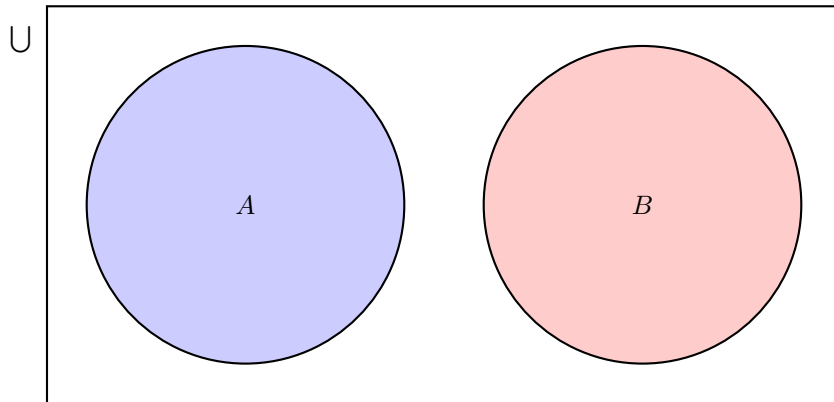
$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



6.2.6 Disjoined sets (Empty sets) \emptyset

\emptyset := the set containing zero elements:

$$A \cap B = \emptyset$$



7 The absolute value function

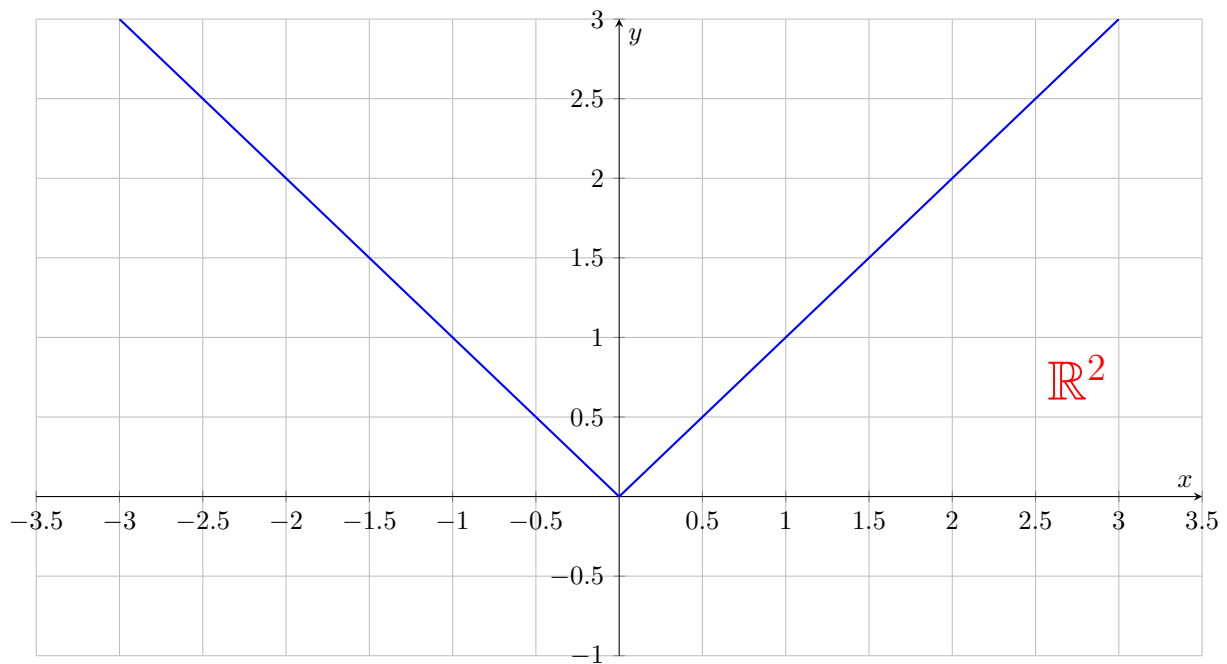
The absolute value is an operator that returns the positive value of a number, regardless of its original sign.

Let $x \in \mathbb{R}$, then:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } -x < 0 \end{cases}$$

7.1 Graph of absolute value functions

Let's plot the function $y = |x|$:



7.2 Properties

Let $a, b \in \mathbb{R}$, then:

- $|a \cdot b| = |a| \cdot |b|$;
- $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$;
- $|a \pm b| \neq |a| \pm |b|$.

7.3 Triangular inequalities

Let $a, b \in \mathbb{R}$, then:

$$\begin{array}{l} |a| + |b| \geq |a + b| \\ |a| - |b| \leq |a - b| \end{array}$$

Part II

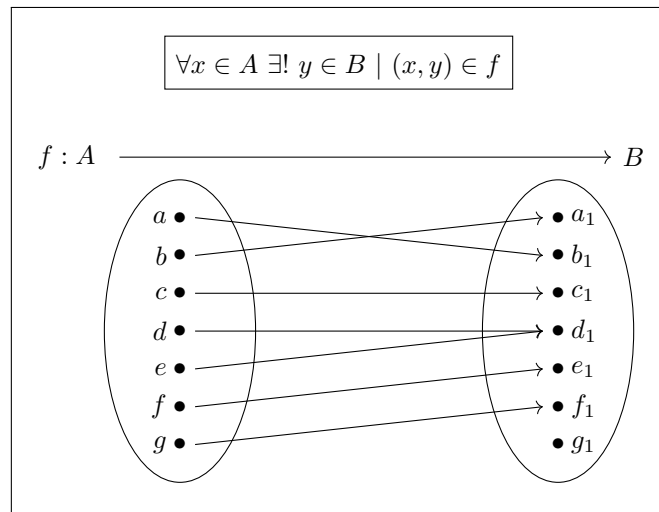
Week 2

8 Concept of functions

Let's take any two sets $A = \{a, b, c, d, e, f, g\}$ and $B = \{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$.

$$\begin{array}{l} f : A \rightarrow B \\ a \mapsto f(a) \end{array}$$

A function is a relation between the sets A and B , according to which we associate to each element of A one and only one element of B :



Notation: $f(a) = b_1, f(b) = a_1, f(c) = c_1, f(d) = d_1, \dots$

Each point in set A is associated with one element of B . However, it is possible for more than two elements of A to point to the same element of B .

The set A is called *domain* of f . The set B is called the *codomain* of f .

8.1 Image (Range)

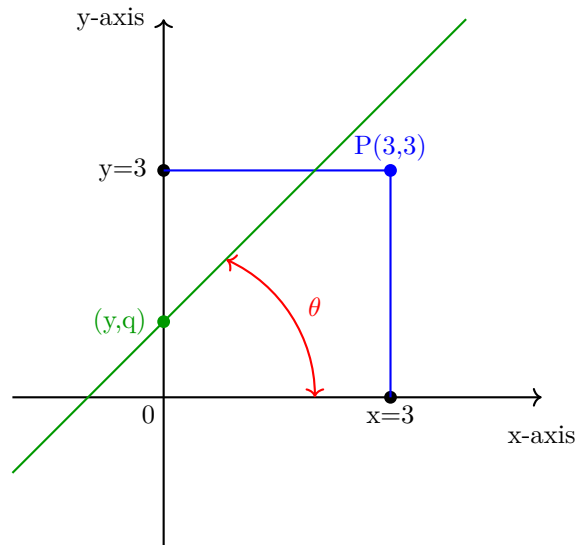
Let $f : X \rightarrow Y$ be a function. The image of f is defined as:

$$\text{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$$

Easily, the image is the set containing all the elements of the set B associated with the elements of the set A .

9 Linear function

9.1 Cartesian diagram



9.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

9.3 Slope-intercept equation

Let $m, q \in \mathbb{R}$, then

$$y = mx + q$$

- m : slope;
- q : vertical intercept.

9.3.1 Slope

The slope of a line can be calculated with the equation

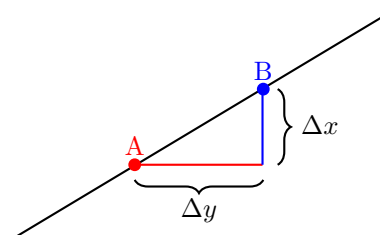
$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

We have three different slope outcomes:

- $m > 0$, the line is increasing;
- $m = 0$, the line is stable;
- $m < 0$, the line is decreasing.

Warning: This works only if $x_B \neq x_A$.

9.3.2 Drawing



9.4 Vertical lines

The more the value of m increases, the closer the line will get to the vertical, without ever reaching it.

Let $c \in \mathbb{R}$, then $x = c$.

Vertical lines cannot be written as a function.

10 Equation of a line

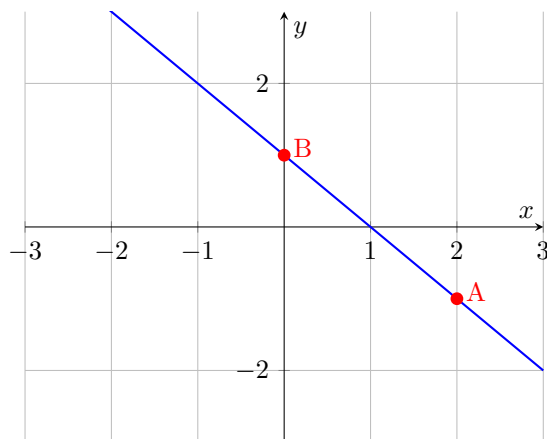
Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$, then

$$y - y_A = m(x - x_A)$$

e.g.: Find the line with $m = -1$ and $A(2, -1)$.

$$y - 1 = -1(x + 2) \Rightarrow y = -x + 1$$

Points: $A(2, -1)$; $B(0, 1)$



10.1 General equation in a cartesian diagram

$$ax + by + c = 0$$

Remark:

- All the lines can be described with this kind of equation;
- When $b = 0$, $a \neq 0$, then $ax = -c \Rightarrow x = \frac{-c}{a} \in \mathbb{R}$;
- When $b \neq 0$, then $y = -\frac{a}{b}x - \frac{c}{b}$, where $m = -\frac{a}{b}$ and $q = -\frac{c}{b}$.