Mathematics 3A
HSLU, Semester 3

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Contents

Part I

Just stuff I have to explain, wait few days

Let π denote the plane:

$$s_x \in \pi, s_y \in \pi, s_z \in \pi$$

$$\pi : ax + by + cz + d = 0$$

For $S_x \in \pi \implies 1a + 0b + 0c + d = 0$, hence
$a + d = 0$

For $S_y \in \pi \implies 0a + 2b + 0c + d = 0$, hence
$2b + d = 0$

for $S_z \in \pi \implies 0a + 0b + 3c + d = 0$, hence
$3c + d = 0$

$$\begin{cases} a + d = 0 \\ 2b + d = 0 \\ 3c + d = 0 \end{cases} \implies \begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases}$$

Case 1:

$$d = 0 \implies a = 0, b = 0, c = 0 \implies \pi : 0 = 0 \implies \text{NOT a plane!}$$

Case 2:

$$d \neq 0 \implies \pi : \frac{ax + by + cz + d}{d} = 0 \implies \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

Hence:

$$\begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases} \implies \begin{cases} \frac{a}{d} = -1 \\ \frac{b}{d} = -\frac{1}{2} \\ \frac{c}{d} = -\frac{1}{3} \end{cases}$$

Which leads to:

$$\pi : -x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0$$

<u>Remark</u>: the equation of a plane is defined up to a multiplication by a real number different from 0

e.g.: the same planed is shared between those 3 equations
ex 1)

$$z = 0 \iff 5z = 0 \iff -10z = 0$$

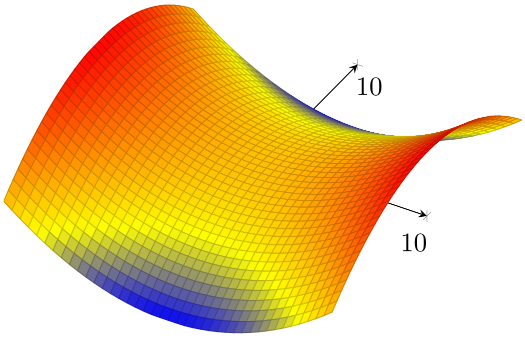
ex 2)

$$-x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0 \iff 6x + 3y + 2z + 6 = 0$$

1 Functions in two variables x and y

Let us take $\pi : x^2 - y^2 = 0$ as example.

The plot would look like this:



1.1 Spheres

2 Linear functions of two variables

We say that z is a *linear function* of x and y, if there are constant a, b and d such that:

$$\boxed{z = ax + by + d}$$

holds. Alternatively: if there are constant A, B, C, D, with $C \neq 0$, such that:

$$\boxed{Ax + By + Cz + D = 0}$$

holds. Since $C \neq 0$, we can rearrange this equation into:

$$\boxed{z = -\frac{Ax}{C} - \frac{By}{C} - \frac{D}{C}}$$

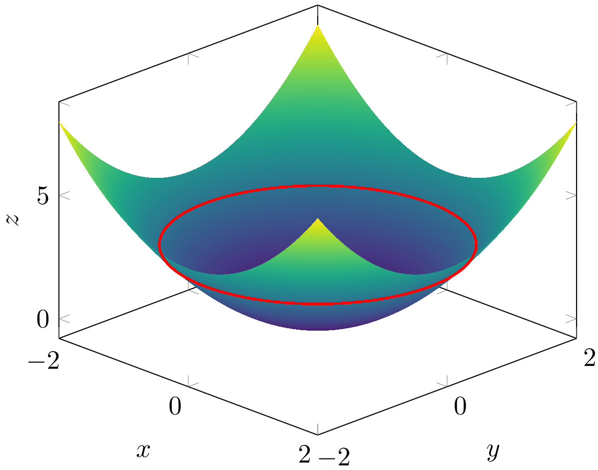
3 Contour lines

$$\begin{cases} z = f(x,y) \\ z = k \qquad k \in \mathbb{R} \end{cases}$$

$z = k$ represents all the possible horizontal planes

Ex:

$$\begin{cases} z = x^2 - y^2 \\ z = k \end{cases} \implies \begin{cases} k = x^2 - y^2 \\ z = k \end{cases}$$



All the planes with equation $z = k$ are parallel to the coordinate planes $z = 0$.

When $z = k = 0$, the circle is reduced to a point, the origin.

When $k < 0$, the equation $x^2 + y^2 = k$ has no solution in $\mathbb{R}$.

When $k > 0$, the equation $x^2 + y^2 = k$ represents a circle with radius $\sqrt{k}$ centered at the origin.

4 Cylinders

A cylinder is a surface generated by all the lines parallel to a given line d and passing through a given curve $\mathcal{C}$.

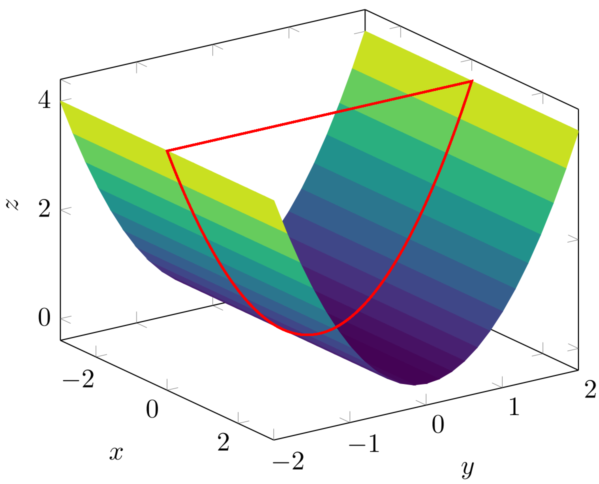
4.1 Property

Whenever you have a polynomial equation of degree at least 2 with a missing variable, then you have a cylinder (up to few exceptions).

Ex:

$$z = y^2 \implies y^2 - z = 0$$

This is a cylinder with generatrix parallel to the x axis and directrix the parabola $y^2 - z = 0$ in the yz plane.



Part II

Partial derivatives

For a multivariable function $f(x, y, ...)$, the partial derivative to one variable measures the instantaneous rate of change of f when that variable changes and the others are held constant:

$$\boxed{\frac{\partial z}{\partial x} = f_x(x, y)}$$

If z is a function of x and y, we define:

The rate of change of z with respect to x, with y fixed, at the point (x, y)=(a, b) as

$$\boxed{\frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{Z\big|_{(x,y)=(a+h,b)} - Z\big|_{(x,y)=(a,b)}}{h}}$$

The rate of change of z with respect to y, with x fixed, at the point $(x, y) = (a, b)$ as

$$\boxed{\frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{Z\big|_{(x,y)=(a,b+h)} - Z\big|_{(x,y)=(a,b)}}{h}}$$

For the lectures, we will be using the formula with 2-steps difference $(\Delta z_a = (a + h, b) - (a - h, b))$:

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} &= \frac{Z\big|_{(x,y)=(a+h,b)} - Z\big|_{(x,y)=(a-h,b)}}{2h} \\ \frac{\partial z}{\partial y}\Big|_{(x,y)=(a,b)} &= \frac{Z\big|_{(x,y)=(a,b+h)} - Z\big|_{(x,y)=(a,b-h)}}{2h} \end{aligned}}$$

5 Local linearization

5.1 Tangent plane of a function at point P

Let $f(x, y)$ be our function and $P(a, b)$ a point, $P \in f$:

$$\boxed{f(x, y) \approx f(a, b) + \frac{\partial}{\partial x} f(a, b)(x - a) + \frac{\partial}{\partial y} f(a, b)(y - b)}$$

6 Gradient

The gradient of a function $z = f(x, y)$ is defined by:

$$\boxed{\begin{aligned} &\text{grad } f = \nabla f = f_x \overrightarrow{e_x} + f_y \overrightarrow{e_y} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &\text{where } f_x = \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y} \end{aligned}}$$

6.1 Geometrical properties of the gradient vector ∇ in the plane

If f is differentiable at the point (a, b) and $\nabla f \neq \overrightarrow{0}$, then the following holds:

$\mathbf{\nabla f(a, b)}$:

- is perpendicular to the contour line of f through (a, b)
- points in the direction of the maximum rate of change f

The length $\|\nabla \mathbf{f(a, b)}\|$ of the gradient vector is:

- the maximum rate of change f at this point
- large when the contour lines are close together
- small when the contour lines are far apart

6.2 Gradient of a function of three variables

The gradient of a function $w = f(x, y, z)$ is defined by:

$$\boxed{\begin{array}{l} \text{grad } f = \nabla f = f_x \overrightarrow{e_x} + f_y \overrightarrow{e_y} + f_z \overrightarrow{e_z} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \\ \\ \text{where } f_x = \dfrac{\partial f}{\partial x},\ f_y = \dfrac{\partial f}{\partial y},\text{ and } f_z = \dfrac{\partial f}{\partial z} \end{array}}$$

6.3 Second-order partial derivatives of $z = f(x, y)$

A function $z = f(x, y)$ has two first-order partial derivatives, f_x and f_y, and four second-order partial derivatives:

1. $\dfrac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = (f_x)_x(x, y)$,
2. $\dfrac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = (f_y)_x(x, y)$,
3. $\dfrac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = (f_x)_y(x, y)$,
4. $\dfrac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = (f_y)_y(x, y)$

Usually, parenthesis are omitted, writing directly f_{xy} instead of $(f_x)_y$, and $\dfrac{\partial^2 z}{\partial y \partial x}$ instead of $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$.

6.4 Equality of mixed partial derivatives (Schwarz's Theorem)

If f_{xy} and f_{yx} are continuous at a point (a, b) inside the domain, then:

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

7 Directional derivatives in the plane

7.1 Directional derivative of f at $P(a,b)$ in the direction of $\overrightarrow{u}$

If $\overrightarrow{e_u} = \overrightarrow{u} = u_1\overrightarrow{e_x} + u_2\overrightarrow{e_y}$ is a unit vector $\|u\| = 1$, we define the directional derivative $\dfrac{\partial f}{\partial \overrightarrow{u}} = f_{\overrightarrow{u}}$ by

$$\boxed{\frac{\partial f}{\partial \overrightarrow{u}}(a,b) = f_{\overrightarrow{u}}(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

7.2 Gradient and directional derivative

If f is differentiable and $\overrightarrow{e_u} = u_1\overrightarrow{e_x} + u_2\overrightarrow{e_y}$ is the unit vector in the direction of $\overrightarrow{u}$, then:

$$\boxed{\frac{\partial f}{\partial \overrightarrow{u}}(a,b) = f_{\overrightarrow{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \nabla f(a,b) \cdot \overrightarrow{e_u}}$$

8 Critical points

8.1 Discriminant

Let (x_0, y_0) be a critical point. Furthermore, let

$$\boxed{D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \big(f_{xy}(x_0,y_0)\big)^2}$$

Then the following holds:

- If $D > 0$ and $f_{xx} > 0$, then f has a local minimum at (x_0, y_0)
- If $D > 0$ and $f_{xx} < 0$, then f has a local maximum at (x_0, y_0)
- If $D < 0$, then f has a saddle point at (x_0, y_0)
- If $D = 0$, no conclusion can be made

9 Constraints and Lagrange Multipliers

9.1 Graphical representation

9.2 Lagrange multiplier λ

9.3 Lagrange function $\mathcal{L}$

When optimizing $f(x,y)$ under the constraint $g(x,y) = c$, the Lagrange function is used:

$$\boxed{\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)}$$

The partial derivatives must be calculated:

$$\boxed{\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -\big(g(x,y) - c\big) \end{aligned}}$$

Part III

Integration of functions with multiple variables

10 Domain of integration Ω

Let $f : \Omega \subset \mathbb{R}^2 \to R$. The set Ω is a region in the xy-plane over which the double integral

$$\boxed{\iint_\Omega f(x,y)\ dx\ dy}$$

is taken.

11 Double integrals as iterated integrals

If the region R is a rectangle with $a \le x \le b$ and $c \le y \le d$ and if f is continuous in the region R, then the integral of f over R is equal to the iterated integral

$$\boxed{\int\limits_R f\,dA = \int\limits_{y=c}^{d} \int\limits_{x=a}^{b} f(x,y)\,dx\,dy}$$

The iterated integrals can also be written as

$$\boxed{\int\limits_c^d \int\limits_a^b f(x,y)\,dx\,dy}$$

11.1 Double integral over rectangles

$$\boxed{\int\limits_R f(x,y)\,dA = \int\limits_c^d \int\limits_a^b f(x,y)\,dx\,dy = \int\limits_a^b \int\limits_c^d f(x,y)\,dy\,dx}$$

11.2 Triangular regions

For the triangle with vertices $(0,0)$, $(1,0)$, $(0,1) \implies 0 \le y \le 1,\ 0 \le x \le 1-y$:

$$\boxed{\int\limits_{y=0}^{1} \int\limits_{x=0}^{1-y} f(x,y)\,dx\,dy}$$

Equivalently $0 \le x \le 1,\ 0 \le y \le 1-x$:

$$\boxed{\int\limits_{x=0}^{1} \int\limits_{y=0}^{1-x} f(x,y)\,dy\,dx}$$

11.3 Double integral over general regions

If the region Ω is not a rectangle, one must describe it using variable limits that follow the boundary of Ω

11.3.1 x-simple region

If the region $\Omega = \{(x, y) \mid a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$, then

$$\boxed{\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy\, dx}$$

11.3.2 y-simple region

If the region $\Omega = \{(x, y) \mid c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$, then

$$\boxed{\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy}$$