Maths refresher course
HSLU, Semester 1

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Contents

Part I

Lesson 1

1 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0)
- $\mathbb{Z} :=$ Integer numbers
- $\mathbb{Q} :=$ Rational numbers
- $\mathbb{R} :=$ Real numbers

<u>Notation</u>: The "$*$" symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

2 Prime numbers

A prime number is a number $n \in \mathbb{N} \setminus \{0, 1\}$ such that, for every divisor $d \in \mathbb{N}$, if $d \mid n$, then $d = 1$ or $d = n$.

$$\boxed{n \in \mathbb{N} \setminus \{0, 1\} \text{ is prime } \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)}$$

3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{R}^*$ and $a \subset \mathbb{R}$, then

$$\boxed{a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}}$$

3.1 Property 1

Let $a, b \in \mathbb{R},\ n, m \in \mathbb{N}$, then

$$\boxed{a^n \cdot a^m = a^{n+m}}$$

3.2 Property 2

Let $a, b \in \mathbb{R},\ n \in \mathbb{N}$, then

$$\boxed{(a \cdot b)^n = a^n \cdot b^n}$$

<u>Notation</u>: The power a^n, a is the base and n is the exponent.

3.3 Property 3

Let $a \in \mathbb{R},\ m, n \in \mathbb{N}^*$, then

$$\boxed{(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}}$$

4 Fractions

<u>Notation 1</u>: $a \cdot b = a \times b = ab \quad | \quad \frac{a}{b} = a \div b = a : b$

<u>Notation 2</u>: "a" is called numerator, "b" is called denominator.

<u>Notation 3</u>: $\frac{a}{b},\ a, b \in \mathbb{R},\ b \neq 0$

4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

5 Negative powers

5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

5.2 Property 4

Let $\forall n \in \mathbb{N},\ \forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z},\ \forall a \in \mathbb{R},\ z \neq 0$
We can compute a^z

5.3 Property 5

Let $\forall a \in \mathbb{R},\ a \neq 0,\ \forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:
   - $\forall a \in \mathbb{R},\ \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
   - $\forall b \in \mathbb{R},\ (a \cdot b)^n = a^n \cdot b^n$
   - $(a^n)^m = a^{n \cdot m}$
2. $\forall a \in \mathbb{R}^*,\ a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

6 Fractions and percentages (and back)

$\alpha \in \mathbb{R},\ n\%$ of $\alpha \Longleftrightarrow \frac{n}{100} \cdot \alpha$

Part II

Lesson 2

7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \rightarrow$ equality;
- $a \neq b \rightarrow$ inequality (a is not equal to b);
- $a < b \rightarrow$ less than (a is strictly less than b);
- $a \leq b \rightarrow$ less than or equal to (a is less than or equal to b);
- $a > b \rightarrow$ greater than (a is strictly greater than b);
- $a \geq b \rightarrow$ greater than or equal to (a is greater than or equal to b).

Example: $x \in \mathbb{R},\ x \geq 2 \rightarrow 2 \leq x < \infty$

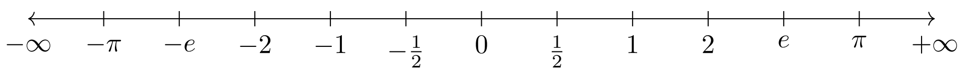
8 Brackets

( ) Parenthesis (round brackets)
[ ] Square brackets
{ } Braces

9 Latin notations

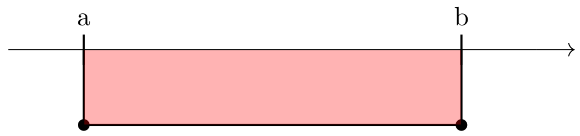
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. (□)= quod erat demonstrandum (we finally prove it).

10 The real line

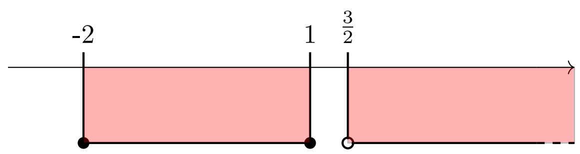


10.1 Exercises

1) $\forall a, b, x \in \mathbb{R},\ a \leq x \leq b$



2) $\forall x \in \mathbb{R},\ x \in\ ]-2, -1] \cup]\frac{3}{2}, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

11 Properties of real numbers

11.1 Property 1 - Closure under "+" and "·"

$\forall x, y \in \mathbb{R}$
$x + y \in \mathbb{R}$
$x \cdot y \in \mathbb{R}$

Remark: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

11.2 Property 2 - Commutativity

$\forall x, y \in \mathbb{R}$
$x + y = y + x$
$x \cdot y = y \cdot x$

Remark: commutativity does not hold for divisions and subtractions.

11.3 Property 3 - Associative

$\forall x, y, z \in \mathbb{R}$
$x + (y + z) = (x + y) + z$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Remark: associativity does not hold for divisions and subtractions.

11.4 Property 4 - Distributive

$\forall x, y, z \in \mathbb{R}$
$x(y \pm z) = xy \pm xz$

11.5 Property 5 - Identity

$\forall x \in \mathbb{R}$

a) $0 + x = x$

b) $1 \cdot x = x$

Remark: $\forall x \in \mathbb{R},\ x \cdot 0 = 0$ is not an identity property.

11.6 Property 6 - Inverses and opposites

$\forall x \in \mathbb{R}$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0,\ x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

Remark 1: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

Remark 2: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
   ( ) inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
   x^a;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
   "·" and "÷";
4. Perform all additions and subtractions as you come to them, moving left-to-right:
   "+" and "−";
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

13 Signed numbers

A number is denoted as positive if it is directly preceded by a $+$ sign or no sign at all.
A number is denoted as negative if it is directly preceded by a $-$ sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

14 Absolute value

Let $x \in \mathbb{R}$, then

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

14.1 Property

$\forall x \in \mathbb{R}$
$|x| > 0 \quad \text{if } y \neq 0$
$|x| = 0 \quad \text{if } x = 0$

Part III

Lesson 3

15 Polynomials

15.1 Terms and factors

15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are a, b, x, y.

When we have an equality $y = x + a$, $\forall x \in \mathbb{R}$, x can assume any value in the set of real numbers (x is an independent variable), while y strictly depends on the value that we decide to give to x.

Notice: we can write $y = x + a$ as $y - a = x$, changing which variable is independent and which is dependent.

15.1.2 Sets

Consider the set $A = [a, b]$, where $a \leq b$. Then:

$$\boxed{\forall x \in A,\ a \leq x \leq b}$$

15.2 Expressions, terms and factors

15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

15.2.2 Terms

A term is any part of the expression separated by "+" or "−".

$$\boxed{y = \underbrace{ax^2}_{term} + \underbrace{bx \cdot c}_{term}}$$

15.2.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called "factorization".
The goal of a factorization is to factorize an expression as much as possible.

16 Common factor

Any expression made of terms is composed of several factors.

$$\boxed{x^2 + x^3 + x = x(x + x^2 + 1),\ \forall x \in \mathbb{R}}$$

17 Notable products

- $(a+b)^2 = a^2 + 2ab + b^2$ (difference of two squares);
- $(a-b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a-b)(a+b) = a^2 - b^2$ (square of a binomial);
- $(a-b)(a^2 + b^2 + ab) = a^3 - b^3$ (difference of two cubes);
- $(a+b)(a^2 + b^2 - ab) = a^3 + a^3$ (sum of two cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

18 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

Number of Terms	Name	Example	Comment
One	Monomial	ax^2	Mono means “one” in Greek
Two	Binomial	$ax^2 - bx$	Bi means “two” in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means “three” in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means “many” in Greek

(1)

18.1 Definition

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

18.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

18.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$p(x) = x^2 + 1 \rightarrow$ the degree is 2.
$\forall x \in \mathbb{R},\ p(0) = 0^2 + 1 = 1 \rightarrow 1$ is a polynomial with degree 0.

18.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$p(x) = x^3 + 1 + x^5 + x^2 1 \rightarrow \deg(p(x)) = 21$
$q(x) = 12\underbrace{abcd}_{\text{deg}=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$

Notation: Let $f(x) = ax^2 + bx + c$, a and b are called coefficient.
The coefficient of the monomial with highest coefficient is called **leading coefficient**.

Part IV

Lesson 4

19 Operations between polynomials

19.1 Polynomials with one independent variable

The order of the monomials is not important, but it is preferable to write the highest degree monomials in decreasing order.

$$\boxed{p(x) = ax^2 - bx + c}$$

19.1.1 Sum

We have to sum all the monomials of the same degree.

$p(x) = x^2 + x - 1$
$q(x) = 5 - x + x^5 - x^2$

$p(x) + q(x) = x^2 + x - 1 + 5 - x + x^5 - x^2 = x^5 + 4$

Definition: in a polynomial with one variable, monomials of same degree are called **similar terms**.

Remark: when there is a difference between polynomials, the minus MUST be distributed throughout the next monomial.

19.1.2 Multiplications

We have to multiply the factors with each other using the distributive property.

$p(x) = (x - 1)$
$q(x) = (x^2 + 2x)$

$p(x) \cdot q(x) = (x - 1)(x^2 + 2x) = x^3 + 2x^2 - x^2 - 2x = x^3 + x^2 - 2x = x(x^2 + x - 2)$

19.2 Polynomials with two or more variables

19.2.1 Sum

$p(x) = ab + a^2b$
$q(x) = 4ab - 3ab^2$

$p(x) + q(x) = ab + a^2b + 4ab - 3ab^2 = a^2b - 3ab^2 + 5ab = ab(a - b + 5)$

Remark: $5a^3b^4 + 7a^3b^4 = 12a^3b^4$, but with $5a^3b^4 + 7a^4b^3$ we can't go further with the sum.

20 Equations

An equation is a formula given by the equality of expressions.

Symbol notations:

- $\exists$ = there exist(s);
- $\nexists$ = there does not exists;
- $\exists!$ = it exists and it is unique;
- : or | = such that.

Equations are the main topic, then we have

- Identities;
- Contradictions;
- Conditional equations.

20.1 Identities

An identity is an equality that holds true regardless of the values chosen for its variables:

$$\boxed{\forall x \in \mathbb{R},\ \exists y \in \mathbb{R} \mid f(x,y) = 0}$$

e.g.

- $1 = 1$;
- $x - 1 = -1 + x$;
- $\sin^2(x) + \cos^2(x) = 1$.

20.2 Contradictions

A contradiction occurs when we get a statement p, such that p is true and its negation $\sim p$ is also true:

$$\boxed{\forall x \in \mathbb{R},\ \neg(\exists y \in \mathbb{R} \mid f(x,y) = 0)}$$

e.g.

- $0 = 1$, false;
- $x^2 = -1$ it is always positive or zero;
- $|a| = -3$ it is always positive or zero;
- $\sqrt{-(x^2+1)} = 1$ it is never defined ($\nexists$).

20.3 Conditional equations

In general, we want to find a solution for each equation, i.e. all the real numbers that, when they replace a variable inside the equation, give an identity:

$$\boxed{\forall x \in \mathbb{R},\ (x > 0 \Rightarrow \exists y \in \mathbb{R} \mid f(x,y) = 0)}$$

e.g.

- $x = 1$;
- $x + y = 3$;
- $\sin(\alpha) = 0.5$.

21 Fundamental theorem of algebra

Let $p(x)$ be a polynomial with one variable and real coefficients.
Assume that $\deg(p(x)) = n \in \mathbb{N}$, then:

$$\boxed{p(x) = 0 \text{ has at most } n \text{ solutions}}$$

22 Linear equations with one variable

$p(x) = q(x)$ where $\deg(0,(x)) = 1$

22.1 Simple tools

22.1.1 Tool 1

$a, b \in \mathbb{R},\ x + a = b$, let's isolate the variable x: $x - a - a = b - a \Rightarrow x = b - a$

22.1.2 Tool 2

$a, b \in \mathbb{R},\ ax = b$, let's isolate the variable x: $\frac{ax}{a} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

23 Linear inequalities with one variable

The inequality is a relation between two or more sets.
Let $a, b, x \in \mathbb{R},\ a < x,\ b > x$, then:

$$\boxed{a < x < b}$$

23.1 Negative sign

In solving the inequality we have to move a negative factor from one side to the other, so we need to reverse the sign of the inequality:

$$\boxed{-ax < b \Rightarrow x > -\frac{b}{a}}$$

24 Equations and inequalities with absolute values

To solve absolute values we need to consider two cases.
Let's take this equation: $|x+2| = -x+4$, then

$$\boxed{\begin{cases} \text{case 1: } x+2 = -x+4 \Rightarrow 2x = 2 \Rightarrow x_1 = 1 \\ \text{case 2: } -x-2 = -x+4 \Rightarrow -2 = 4 \text{ (contradiction)} \end{cases} \Longrightarrow \text{Sol: } x = \begin{cases} 1 & \text{if } x+2 \geq 0 \\ \text{no solution} & \text{if } x+2 < 0 \end{cases}}$$

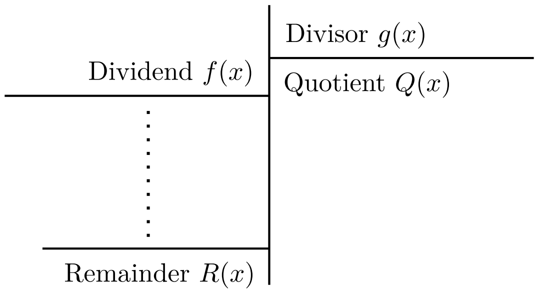
Part V

Lesson 5

25 Division of polynomials

25.1 Division algorithm for polynomials by monomials

Let $f(x)$ be a polynomial and $g(x)$ a monomial such that $g(x) \neq 0$. Consider the rational expression $\frac{f(x)}{g(x)}$, then:



- Divide the highest degree term in $f(x)$ (the dividend) by the highest degree term in $g(x)$ (the divisor). This gives the first partial quotient $q_1(x)$.
- Multiply the partial quotient $R_1(x)$ by the entire divisor $g(x)$. This product represents the part of the dividend that can be "cancelled" in this step.
- Subtract the product obtained in step 2 from the original dividend $f(x)$. This subtraction gives a new polynomial, often called the remainder $R_1(x)$, which is of a lower degree than the original dividend.
- Now divide the leading term of the new remainder $R_1(x)$ by the leading term of $g(x)$. This gives the next partial quotient $Q_2(x)$.
- Multiply $Q_2(x)$ by $g(x)$ and subtract it from the current remainder. This process generates a new remainder $R_2(x)$.
- Keep repeating the division, multiplication, and subtraction steps until the degree of the remainder is less than the degree of the divisor $g(x)$. At this point, you cannot continue dividing.
- The final quotient $Q(x)$ is the sum of all the partial quotients: $Q(x) = Q_1(x) + Q_2(x) + \cdots + Q_n(x)$.
- The remainder $R_n(x)$ is the result after all subtractions are completed. If the remainder is zero, the division is exact. If not, the remainder is the leftover part of the division.

Tip: When the sum of the coefficients is equal to 0, then the polynomial is always divisible by $x - 1$.

26 Second degree polynomials

Let $a, b, c \in \mathbb{R}$, then

$$\boxed{ax^2 + bx + c = 0}$$

The three possible outcomes we can have when solving this 2nd-degree polynomial are:

- 2 solutions;
- 1 solution;
- 0 solutions.

26.1 Quadratic formula

$$\boxed{x_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}}$$

26.1.1 Discriminant of the polynomial

$$\boxed{\Delta = b^2 - 4ac}$$

From the discriminant we can determine how many solutions the equation will have:

- $\Delta > 0 \Rightarrow 2$ real solutions;
- $\Delta = 0 \Rightarrow 1$ real solution;
- $\Delta < 0 \Rightarrow 0$ real solutions (2 complex solutions).

26.1.2 Evident solutions

When we have a 2nd-degree equation $(x - a)(x - b) = 0$, we have two obvious solutions in $\mathbb{R}$. In this case, $x_1 = a,\ x_2 = b$

This factorization can be obtained using notable products.

e.g. Let $x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0$, then $x = -2$.

26.2 Extraction of a root

Let $a \in \mathbb{R},\ a \geq 0$, then:

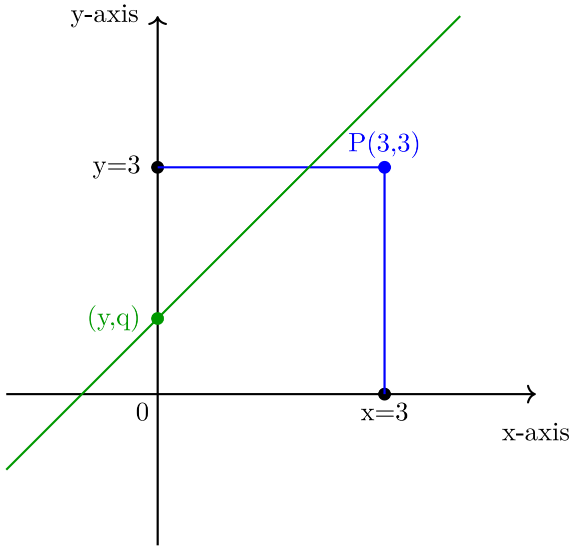
$$\boxed{x^2 - a = 0 \Rightarrow x = \pm\sqrt{a}}$$

Part VI

Lesson 6

27 Lines and parabolas

27.1 Cartesian diagram



27.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

27.3 Slope-intercept equation

Let $m, q \in \mathbb{R}$, then

$$\boxed{y = mx + q}$$

- m: slope ($\tan(\alpha)$);
- q: vertical intercept.

27.3.1 Slope

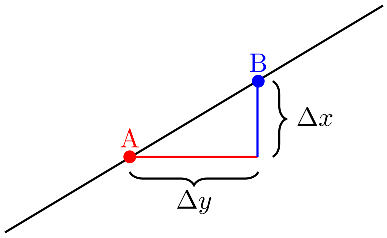
The slope of a line can be calculated with the equation

$$\boxed{m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x}}$$

We have three different slope outcomes:

- $m > 0$, the line is increasing;
- $m = 0$, the line is stable;
- $m < 0$, the line is decreasing.

27.3.2 Drawing



27.4 Vertical lines

The more the value of m increases, the closer the line will get to the vertical, without ever reaching it.

Let $c \in \mathbb{R}$, then $x = c$.

Vertical lines cannot be written as a function.

28 Equation of a line

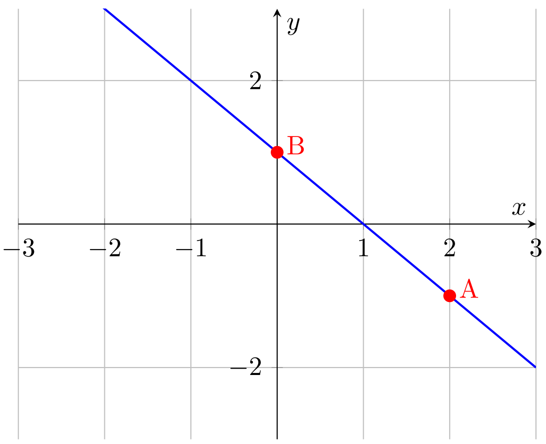
Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$, then

$$\boxed{y - y_A = m(x - x_A)}$$

e.g.: Find the line with $m = -1$ and $A(2, -1)$.

$y - 1 = -1(x + 2) \Rightarrow y = -x + 1$

Points: $A(2, -1)$; $B(0, 1)$



28.1 General equation in a cartesian diagram

$$\boxed{ax + by + c = 0}$$

Remarks:

- All the lines can be described with this kind of equation;
- When $b = 0$, $a \neq 0$, then $ax = -c \Rightarrow x = \frac{-c}{a} \in \mathbb{R}$;
- When $b \neq 0$, then $y = -\frac{a}{b}x - \frac{c}{b}$, where $m = -\frac{a}{b}$ and $q = -\frac{c}{b}$.

29 Vertical parabolas

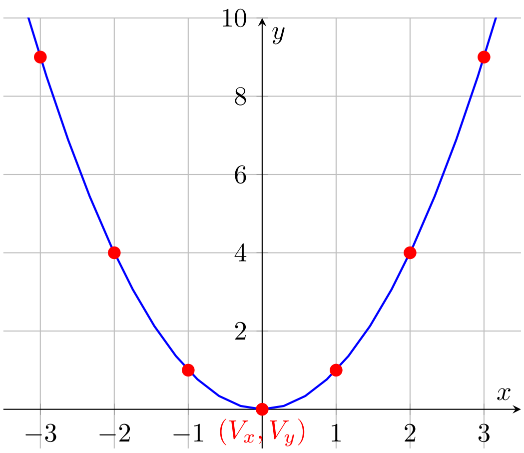
29.1 Function of parabolas

Let $a, b, c \in \mathbb{R}$, then

$$\boxed{y = a^2 + bx + c}$$

29.2 Drawing example

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



29.3 Concavity of a parabola

We have three cases:

- $a > 0$, concave up;
- $a = 0$, not a parabola;
- $a < 0$, concave down.

29.4 Vertex of a parabola

The vertex of a parabola $y = ax^2 + bx + c$ is the point given by the coordinates:

$$\boxed{V = \left(-\frac{b}{2a},\ -\frac{\Delta}{4a}\right)}$$

Remarks: we have two different cases:

- When $a > 0$, the vertex is the lower point of the parabola;
- When $a < 0$, the vertex is the highest point of the parabola.

e.g.: given $y = x^2$, find the vertex: $V = \left(-\frac{0}{2},\ -\frac{0}{4}\right) \to V(0, 0)$

Alternative: solving the x coordinate V_x, we can sostitute the x inside the given funcion $f(x)$.

30 Powers with $\mathbb{Z}$ and $\mathbb{R}$ exponents

Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, then:

$$\boxed{\alpha^{\frac{1}{n}} = \sqrt[n]{\alpha}}$$

Let $m, n \in \mathbb{Z}$, then

$$\boxed{\alpha^{\frac{m}{n}} = \left(\alpha^{\frac{1}{n}}\right)^{m}}$$

Let $a, c \in \mathbb{Z};\ b, d \in \mathbb{Z}^*$ and $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. Then, we can approximate λ by a fraction:

$$\boxed{\frac{a}{b} < \lambda < \frac{c}{d}}$$