Mathematics 1A
HSLU, Semester 1

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Contents

Part I

Logic

1 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

1.1 Logical connectives

A	B	$\neg B$	$A \wedge B$	$A \vee B$	$A \implies B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

1.1.1 Logical conjunction $\wedge$

Given two statements P and Q, $P \wedge Q$ is true if both P and Q are true.

Let $P = (x > 0)$ and $Q = (y > 0)$, then:

$$\boxed{P \wedge Q = (x > 0 \wedge y > 0)}$$

1.1.2 Logical disjunction $\vee$

Given two statements P and Q, $P \vee Q$ is true if at least one of P or Q is true.

Let $P = (x = 0)$ and $Q = (y \neq 0)$, then:

$$\boxed{P \vee Q = (x = 0 \vee y \neq 0)}$$

1.1.3 Logical negation $\neg$

The negation of a statement P, denoted as $\neg P$, is true if P is false, and false if P is true.

Let $P = (x \geq 5)$, then:

$$\boxed{\neg P = (x < 5)}$$

1.1.4 Implication $\implies$

The symbol $\implies$ indicates that if statement P is true, then statement Q must also be true (i.e., P implies Q).
Warning: It does not require that Q implies P.

$$\boxed{P = (x = 1) \implies Q = (x \in \mathbb{N})}$$

1.1.5 Inference $\impliedby$

The symbol $\Leftarrow$ means that a conclusion or result implies the truth of an earlier statement.
If Q is true, then P must be true.

$$\boxed{Q = (x > 0) \impliedby P = (x \in \mathbb{R}^+)}$$

1.1.6 If and only if $\Leftrightarrow$

The symbol $\Leftrightarrow$ indicates that two statements P and Q are logically equivalent, meaning P is true if and only if Q is true.

$$\boxed{P = (x \in \mathbb{N},\ x \neq 0) \iff Q = (x \in \mathbb{N}^*)}$$

Part II

Set Theory

2 The set theory

2.1 Logical symbols

2.1.1 Definition

Braces and the definition symbol ":=" are used to define a set giving all its elements:

$$\boxed{A := \{a, b, c, d, e\}}$$

2.1.2 Equal

In this case, the equal symbol means that the set A is equal to the set B:

$$\boxed{A = B}$$

2.1.3 Belongs to

The symbols $\in$ and $\ni$ describe an element which is part of the set:

$$\boxed{a \in A \iff A \ni a}$$

2.1.4 Does not belong to

The symbols $\notin$ mean that an element does not belong to the set:

$$\boxed{f \notin A}$$

2.1.5 Inclusion and contains

The symbols $\subset$ and $\supset$ mean that a set has another set included in its set:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \iff \mathbb{Z} \supset \mathbb{N}}$$

2.1.6 For all/any

The symbol $\forall$ means that we are considering any type of element:

$$\boxed{\forall x \in \mathbb{R},\ x > 0}$$

In this case, we've defined a new set.

2.2 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0);
- $\mathbb{Z} :=$ Integer numbers;
- $\mathbb{Q} :=$ Rational numbers;
- $\mathbb{R} :=$ Real numbers $:= \mathbb{Q} \cup \{\text{irrational numbers}\}$.

Notation: The "*" symbol means that the set does not include 0.

2.2.1 Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$B := \{\pi, 1, -1, 0\}$;
$C := \{\pi, 1\}$;
$D := \{\pi\}$.

Then we write some examples: $\pi \in B,\ D \subset B,\ C \subset B,\ B \not\subset C,\ 0 \in B,\ 0 \notin C$.

3 Union $\cup$ and Intersection $\cap$

3.1 Universe symbol

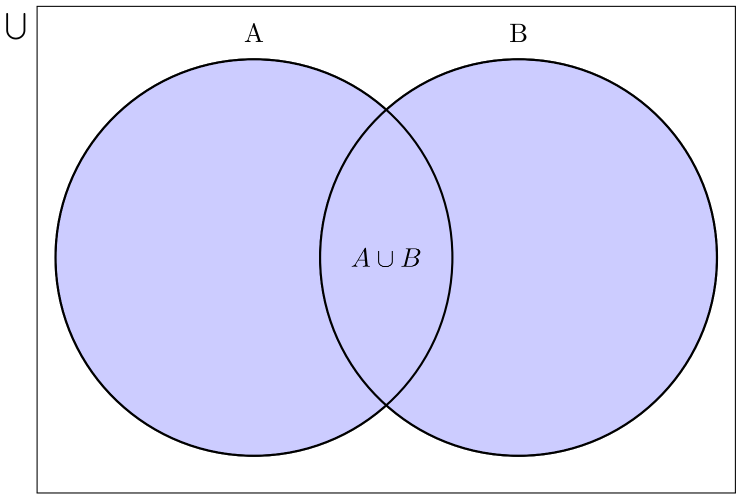
The symbol $\bigcup :=$ Universe describes a big set which contains all sets involved in our discussions (not always).

3.2 Venn diagram

3.2.1 Union $A \cup B$

If A and B are sets, then their union is:

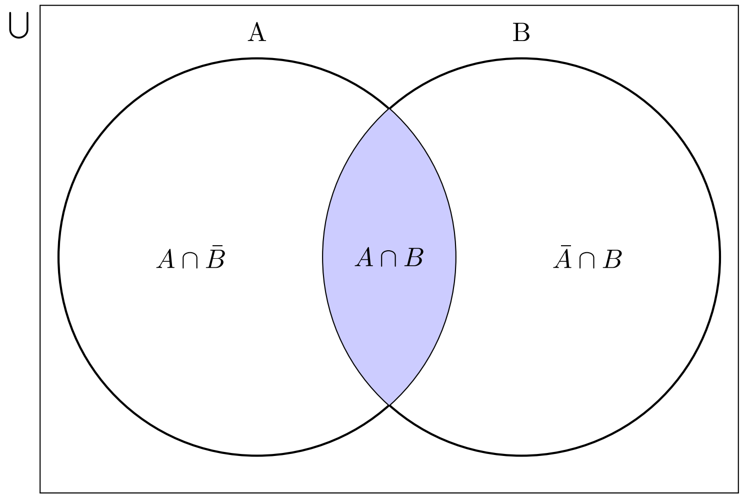
$$\boxed{A \cup U = \{\forall x \in \bigcup \mid x \in A \vee x \in B\}}$$



3.2.2 Intersection $A \cap B$

If A and B are sets, then their intersection is:

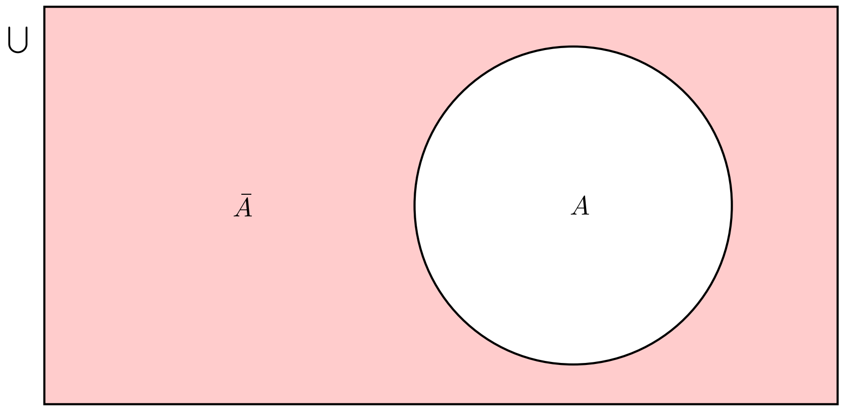
$$\boxed{A \cap B = \{\forall x \in \bigcup \mid x \in A \wedge x \in B\}}$$



3.2.3 Complement $\bar{A}$

If A is a set, its complement is:

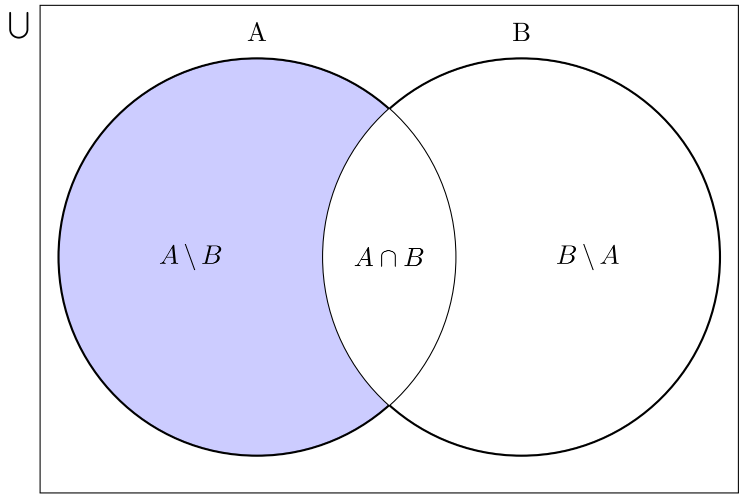
$$\boxed{\bar{A} = \{\forall x \in \bigcup \mid x \notin A\}}$$



3.2.4 Difference between sets $\setminus$

If A and B are sets, then their difference is:

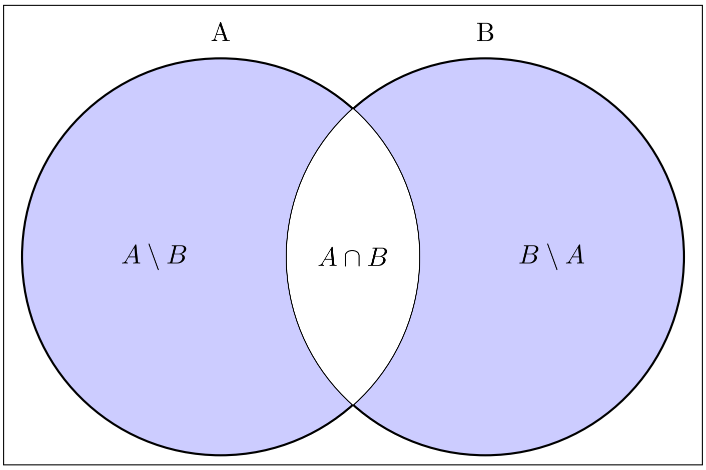
$$\boxed{A \setminus B = \{\forall x \in \bigcup \mid x \in A,\ x \notin B\}}$$



3.2.5 Symmetrical difference $\triangle$

If A and B are sets, then their symmetrical difference is:

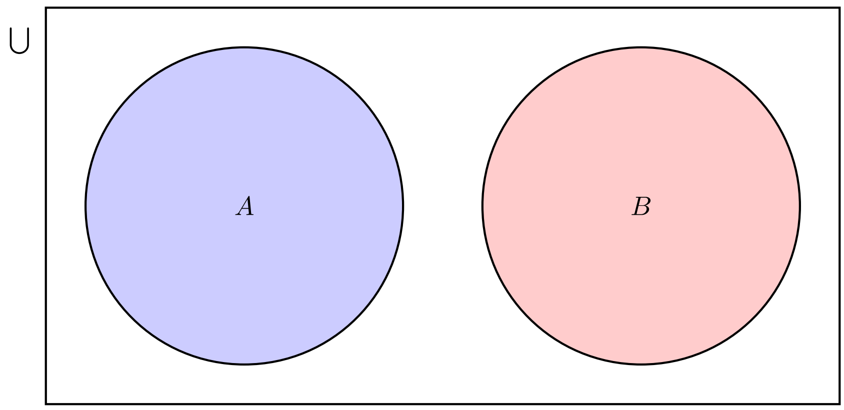
$$\boxed{A \triangle B = (A \setminus B) \cup (B \setminus A)}$$



3.2.6 Disjoined sets (Empty sets) $\emptyset$

$\emptyset :=$ the set containing zero elements:

$$\boxed{A \cap B = \emptyset}$$



Part III

Algebra

4 Intervals in the real line

Intervals describe what happens between two or more elements.

4.1 Examples

4.1.1 Interval sets

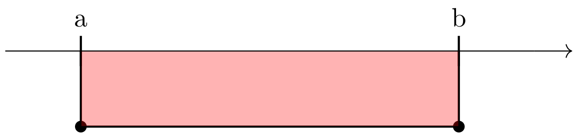
We have 4 cases:

- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\}$;
- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\}$;
- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\}$;
- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}$.

Notation: a and b are often called the "end points" of the interval;

4.1.2 Graphical examples

$\forall x \in \mathbb{R},\ x \in [a, b]$

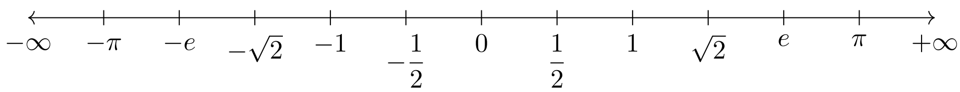


5 The extended line

In the real line $\mathbb{R}$ we add $\pm\infty$.

Real line: $(-\infty, +\infty) = \mathbb{R}$

Extended real line: $[-\infty, +\infty] = \overline{\mathbb{R}}$



Remark: $\pm\infty \notin \mathbb{R}$

5.1 Properties

$$\boxed{\forall x \in \mathbb{R} \mid \infty > x \mid -\infty < 0}$$

5.2 Operation in the extended line

If $a, b \in \mathbb{R}$, then $a + b,\ a - b,\ a \cdot b,\ \frac{a}{b}$ (with $b \neq 0$) stay the same

5.2.1 Additions

Let $\forall a \in \mathbb{R}$:

- $a + \infty := \infty$;
- $a - \infty := -\infty$;
- $+\infty + \infty := +\infty$;
- $-\infty - \infty := -\infty$;
- $+\infty - \infty :=$ undefined.

5.2.2 Moltiplications

Let $\forall a \in \mathbb{R}$:

- $+\infty \cdot +\infty := +\infty$;
- $-\infty \cdot +\infty := -\infty$;
- $-\infty \cdot (-\infty) := \infty$;
- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$
- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$
- $\dfrac{a}{+\infty} = \dfrac{a}{-\infty} := 0$;
- $\dfrac{+\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$
- $\dfrac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$
- $\dfrac{\infty}{\infty} :=$ undefined.

6 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including $\pm\infty$.

6.1 Examples

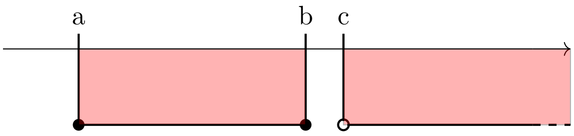
6.1.1 Interval sets

Let $a \in \mathbb{R}$, then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$;
- $[a, +\infty] = \{\forall x \in \mathbb{R} \mid x \geq a\}$;
- $(-\infty, +\infty) = \mathbb{R}$;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$.

6.1.2 Graphical examples

$\forall x \in \mathbb{R},\ x \in\ [a, b]\ \cup\ ]c, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

7 The absolute value function

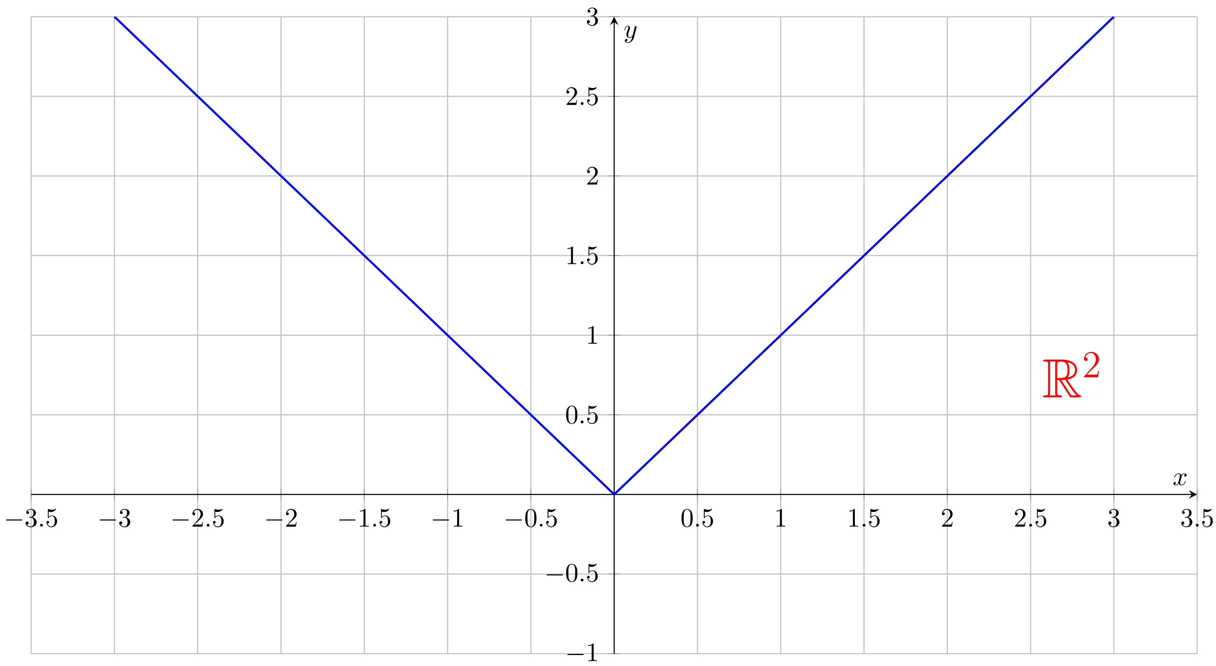
The absolute value is an operator that returns the positive value of a number, regardless of its original sign.

Let $x \in \mathbb{R}$, then:

$$\boxed{|x| = \begin{cases} x \text{ if } & x \geq 0 \\ x \text{ if } & -x < 0 \end{cases}}$$

7.1 Graph of absolute value functions

Let's plot the function $y = |x|$:



7.2 Properties

Let $a, b \in \mathbb{R}$, then:

- $|a \cdot b| = |a| \cdot |b|$;
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$;
- $|a \pm b| \neq |a| \pm |b|$.

7.3 Triangular inequalities

Let $a, b \in \mathbb{R}$, then:

$$\boxed{\begin{array}{l} |a|+|b| \geq |a+b| \\ |a|-|b| \leq |a-b| \end{array}}$$

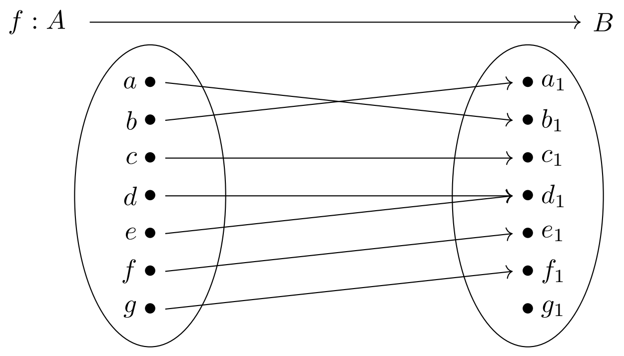
8 Concept of functions

Let's take any two sets $A\{a, b, c, d, e, f, g\}$ and $B\{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$.

$$\boxed{\begin{aligned} f : A &\implies B \\ a &\longmapsto f(a) \end{aligned}}$$

A function is a relation between the sets A and B, according to which we associate to each element of A one and only one element of B:

$$\boxed{\forall x \in A \; \exists!\, y \in B \mid (x, y) \in f}$$



<u>Notation</u>: $f(a) = b_1,\ f(b) = a_1,\ f(c) = c_1,\ f(d) = d_1,$ ...

Each point in set A is associated with one element of B. However, it is possible for more than two elements of A to point to the same element of B.

$$\boxed{\text{The set } A \text{ is called } \textit{domain} \text{ of } f. \text{ The set } B \text{ is called the } \textit{codomain} \text{ of } f.}$$

8.1 Image (Range)

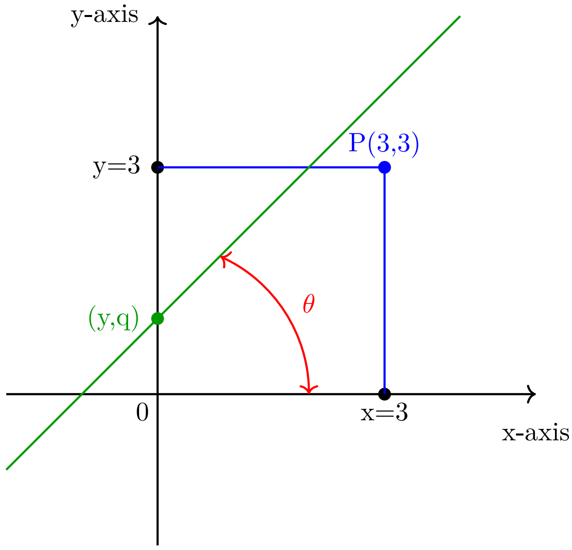
Let $f : X \implies Y$ be a function. The image of f is defined as:

$$\boxed{\mathrm{Im}(f) = \{y \in Y \mid y = f(x),\ x \in X\}}$$

Easily, the image is the set containing all the elements of the set B associated with the elements of the set A.

9 Linear function

9.1 Cartesian diagram



9.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

9.3 Slope-intercept equation

Let $m, q \in \mathbb{R}$, then

$$\boxed{y = mx + q}$$

- m: slope;
- q: vertical intercept.

9.3.1 Slope

The slope of a line can be calculated with the equation

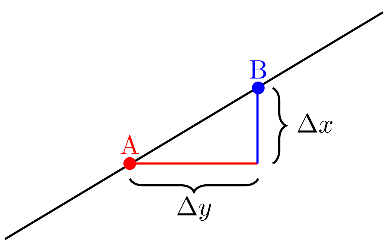
$$\boxed{m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)}$$

We have three different slope outcomes:

- $m > 0$, the line is increasing;
- $m = 0$, the line is stable;
- $m < 0$, the line is decreasing.

Warning: This works only if $x_B \neq x_A$.

9.3.2 Drawing



9.4 Vertical lines

The more the value of m increases, the closer the line will get to the vertical, without ever reaching it.

Let $c \in \mathbb{R}$, then $x = c$.

Vertical lines cannot be written as a function.

10 Equation of a line

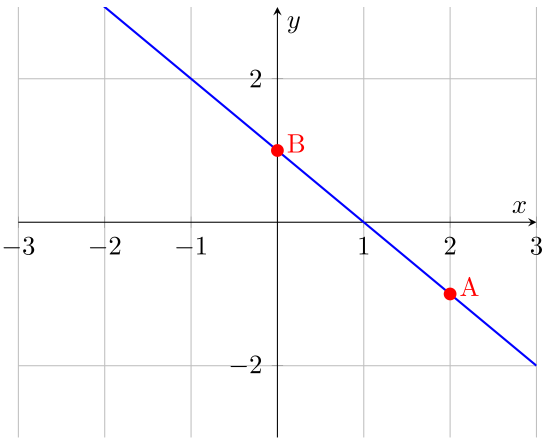
Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$, then

$$\boxed{y - y_A = m(x - x_A)}$$

e.g.: Find the line with $m = -1$ and $A(2, -1)$.

$$y - 1 = -1(x + 2) \implies y = -x + 1$$

Points: $A(2, -1)$; $B(0, 1)$



10.1 General equation in a cartesian diagram

$$\boxed{ax + by + c = 0}$$

<u>Remark</u>:

- All the lines can be described with this kind of equation;
- When $b = 0$, $a \neq 0$, then $ax = -c \implies x = \dfrac{-c}{a} \in \mathbb{R}$;
- When $b \neq 0$, then $y = -\dfrac{a}{b}x - \dfrac{c}{b}$, where $m = -\dfrac{a}{b}$ and $q = -\dfrac{c}{b}$.

11 Increasing and decreasing functions

Let $f : [a, b] \longrightarrow \mathbb{R}$

Notation: if your replace $[a, b]$ with $\mathbb{R}$, you obtain the definition in the whote $\mathbb{R}$.

11.1 Increasing functions

- f is increasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) \geq f(x_1)$;
- f is strictly increasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) > f(x_1)$.

11.2 Decreasing functions

- f is decreasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) \leq f(x_1)$;
- f is strictly decreasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) < f(x_1)$.

12 Inverse function

Let's take any two sets A and B.

A function $f : A \implies B$ is invertible if there exists another function $f^{-1} : B \implies A$, called the inverse function, such that:

$$\boxed{\begin{array}{l} \forall x \in A,\ f^{-1}(f(x)) = x \\ \forall y \in B,\ f(f^{-1}(y)) = y \end{array}}$$

Warning: A function is invertible if and only if it is bijective.

12.1 Facts about inverse functions

1)
Let $f : D \implies \mathbb{R}$

f is invertible in D when:

- f is strictly increasing;
- f is strictly decreasing.

2)
Let $f : D \implies \mathbb{R}$

f is invertible when $f^{-1} : \mathrm{Im}(f) \implies D$.

13 Expressions and factorization

13.1 Expressions, terms and factors

13.1.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

13.2 Terms

A term is any part of the expression separated by "+" or "−".

$$\boxed{y = \underbrace{ax^2}_{term} + \underbrace{bx \cdot c}_{term}}$$

13.2.1 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called "factorization".
The goal of a factorization is to factorize an expression as much as possible.

13.2.2 Common factor

Any expression made of terms is composed of several factors.

$$\boxed{x^2 + x^3 + x = x(x + x^2 + 1),\ \forall x \in \mathbb{R}}$$

13.3 Notable producs

- $(a + b)^2 = a^2 + 2ab + b^2$ (square of a binomial);
- $(a - b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$ (difference of squares);
- $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ (sum of cubes);
- $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ (difference of cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

14 Polynomial function

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

15 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of **terms**;
2. the **degree** of the polynomial.

Number of Terms	Name	Example	Degree
One	Monomial	ax^2	1
Two	Binomial	$ax^2 - bx$	2
Three	Trinomial	$ax^2 - bx + c$	3
Four or more	Polynomial	$a_n x^n - a_1 x^{n-1} + a_2 x^{n-2} \cdots a_0$	n-degree

Remark: The degree of a polynomial is the largest exponent of its monomials.

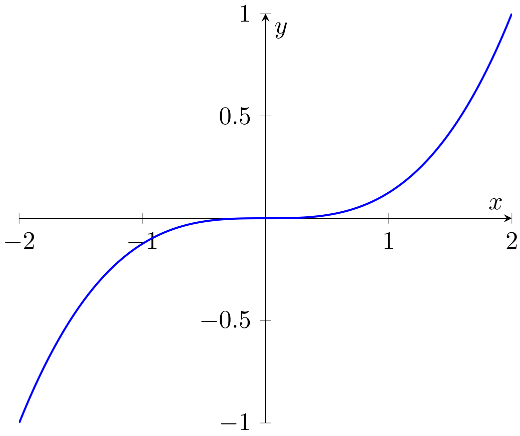
16 Symmetrical functions

Let $y = kx^n$, then we plot:

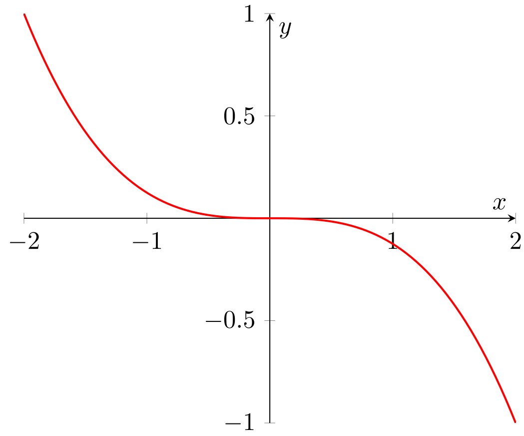
16.1 n odd

$$\boxed{f(-x) = -f(x), \quad \forall x \in \mathbb{R}}$$

16.1.1 Graph examples



$k > 0$

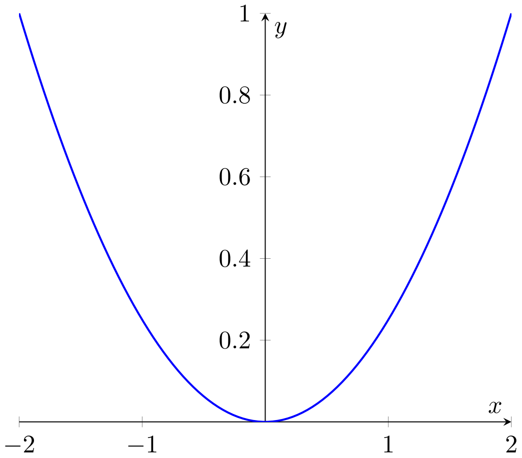


$k < 0$

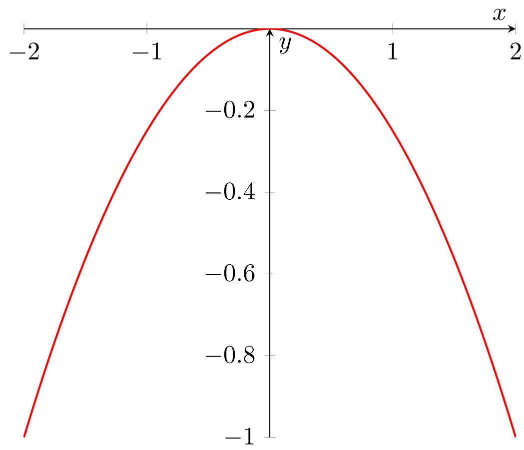
16.2 n even

$$\boxed{f(-x) = f(x), \quad \forall x \in \mathbb{R}}$$

16.2.1 Graph examples



$k > 0$
Concave up



$k < 0$
Concave down

Definition:

- a function $y = f(x)$ is called **odd** if it is symmetric with respect to the origin;
- a function $y = f(x)$ is called **even** if it is symmetric with respect to the y-axis.

16.3 General case

Let $y = p(x)$, where $p(x)$ is any polynomial with real coefficients:

$$\boxed{p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + ... + a_2 \cdot x^2 + a_1 \cdot x^1 + a_0}$$

where:

- $n \in \mathbb{N}$;
- $n = \deg(p(x))$;
- a_n = leading coefficient.

$$\boxed{p(x) = \sum_{i=0}^{n} a_i \cdot x^i}$$

16.4 Symmetry of a polynomial

Let $y = p(x)$ be a polynomial function, then:

1)
$y = p(x)$ is odd iff all the degrees of all the terms of $p(x)$ are odd;

2)
$y = p(x)$ is even iff all the degrees of all the terms of $p(x)$ are even;

3)
$y = p(x)$ has mixed degrees, $p(x)$ is neither odd nor even.

17 Intersection with axis

17.1 Vertical intersection

Let $y = f(x)$ be any function, then we solve for y:

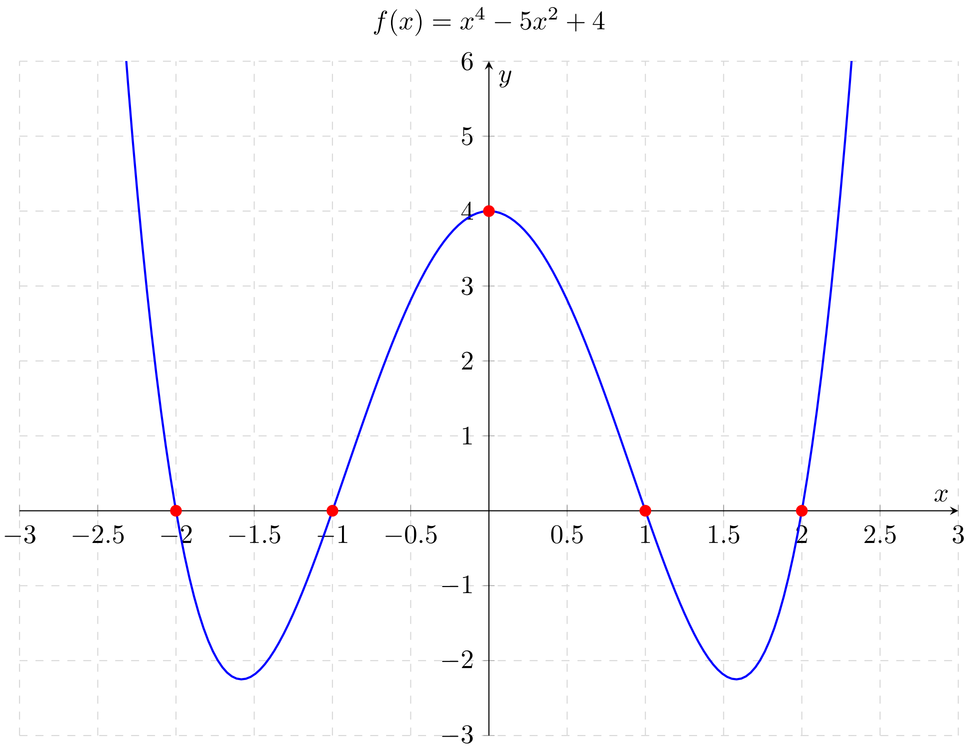
$$\boxed{\begin{cases} x = 0 \\ y = f(0) \end{cases}}$$

17.2 Zeros of a function

Let $y = f(x)$ be any function, then we solve for x:

$$\boxed{\begin{cases} y = 0 \\ 0 = f(x) \end{cases}}$$

17.3 Graph example



18 Dominant elements in a function approaching $\pm\infty$

As x approaches $\pm\infty$, the term with the highest degree in a polynomial function dominates the behavior of the function.

$$\boxed{p(x) \text{ has, as a dominant, the element } a_n \text{ with the highest degree } x^n}$$

18.1 Order of dominance

18.1.1 Approaching to $+\infty$

Let $n \in \mathbb{N},\ m \in \mathbb{N},\ 2 < n < m$, then:

$$\boxed{\ln(x) < x < x^n < x^m < n^x < m^x < x^x}$$

In these cases, we always have $x \implies +\infty \implies p(x) \implies +\infty$

18.1.2 Approaching to -∞

Let $\lambda > 2$ and odd, $k > 2$ and even.

$$\boxed{\begin{array}{l} x^\lambda < -x^2 < x^1 < 0 \\ -x^k < -x^2 < x^1 < 0 \end{array}}$$

Functions like x^λ (with λ odd) and $-x^k$ (with k even) both approach $-\infty$, but at different rates.

18.1.3 Dominance in rational functions

When the dominant element is at the numerator:

$$\boxed{\lim_{x\to\infty} \frac{x^n}{x^{n-1}} = \infty}$$

When the dominant element is at the denominator:

$$\boxed{\lim_{x\to\infty} \frac{x^{n-1}}{x^n} = 0}$$

When we have the same degree either in the numerator and in the denominator:

$$\boxed{\lim_{x\to\infty} \frac{ax^n}{bx^n} = \frac{a}{b}}$$

Definition: **horizontal asymptote** appears when x approaches to ∞, which implies that y approaches to a number A different from $\pm\infty$

19 Exponential and logarithm functions

The relationship between exponentials and logarithms is based on the following formula:

$$\boxed{a^{\log_a(x)} = x \Longleftrightarrow \log_a(a^x) = x}$$

19.1 Exponentials

19.1.1 General equation

Let $\alpha \in \mathbb{R}_+^*$, $x \in \mathbb{R}$, and $a > 1$, then:

$$\boxed{y = \alpha \cdot a^x}$$

19.1.2 Euler's number

Euler's number is defined by the limit:

$$\boxed{e = \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718 \cdots}$$

Alternatively, it can be expressed as:

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!}}$$

19.2 Logarithms

19.2.1 Natural logarithm

The inverse function of the Euler's exponential function:

$$\boxed{f(x) = e^x \Longleftrightarrow h(x) = \ln(x)}$$

<u>Remark</u>: the domain of $\ln(x)$ is $\mathrm{D}_n : \ \forall x \in \mathbb{R}_+^*$

19.2.2 Logarithms with arbitrary bases

The inverse function of any arbitrary exponential function:

$$\boxed{f(x) = n^x \Longleftrightarrow h(x) = \log_n(x)}$$

Alternatively, it can be expressed as:

$$\boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

19.2.3 Common logarithm

The common logarithm uses base 10:

$$\boxed{\log_{10}(x) = \frac{\ln(x)}{\ln(10)}}$$

19.3 Exponential growth

$$\boxed{N(t) = N_0 \cdot e^{kt}}$$

20 Composite functions

Let $y = f(x)$ and $z = g(y)$ be two functions, then:

$$\boxed{z = g(f(x))}$$

20.1 Examples

1)
Let $f(x) = x^2 + 4x$ and $g(y) = y^2 + \cos(y)$ be two functions, then:

$$g(f(x)) = \left(x^2 + 4x\right)^2 + \cos(x^2 + 4x)$$

2)
Let $f(x) = x^3,\ h(x) = \arctan(x)$ and $g(x) = \ln(x)$ be functions, then:

$$g(h(f(x))) = \ln(\arctan(x^3))$$

Part IV
Trigonometry

21 Trigonometry

21.1 Conversion table of degrees and radians

Angles (in Degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Angles (in Radians)	0^c	$\pi/6^c$	$\pi/4^c$	$\pi/3^c$	$\pi/2^c$	π^c	$3\pi/2^c$	$2\pi^c$
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0	1
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0	∞	0

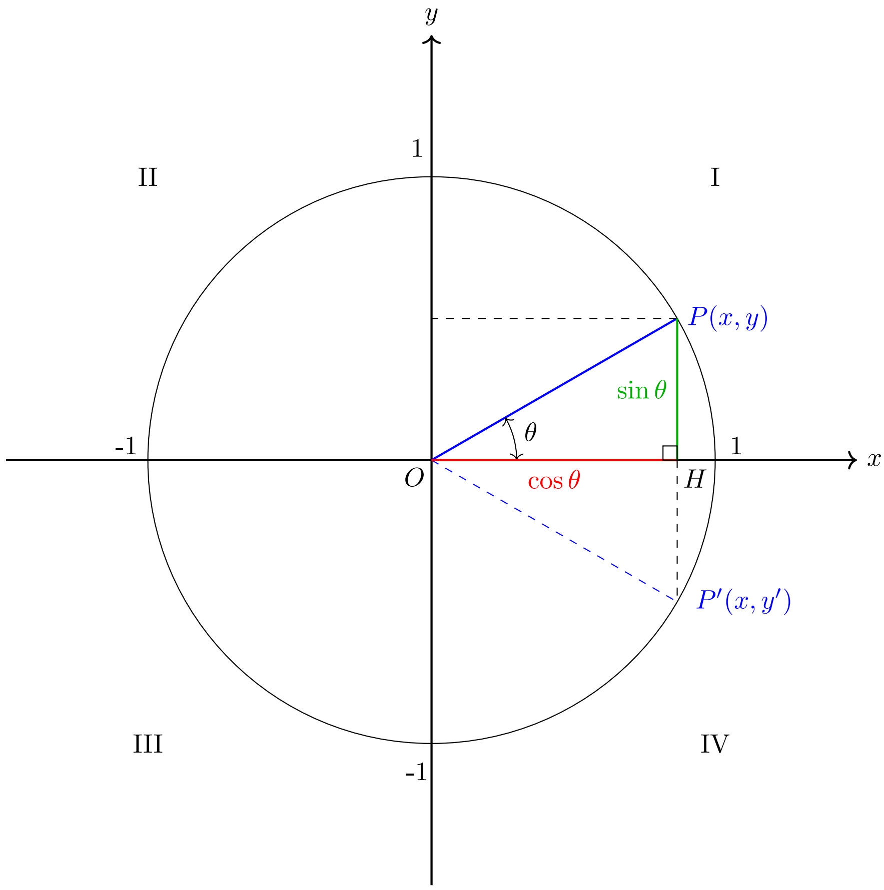
Remark:

$$\cos(2\pi + \theta) = \cos(\theta) \qquad | \qquad \sin(2\pi + \theta) = \sin(\theta)$$

Remark: Let $\forall k \in \mathbb{Z}$, $\forall \theta \in \mathbb{R}$, then:

$$\boxed{\cos(\theta + 2\pi k) = \cos(\theta)}$$

21.2 Trigonometric functions in the unit circle



Remark: the circle has center in the origin O, radius $= 1$ and function $x^2 + y^2 = 1$

Trigonometric functions can be extended to angles beyond 0 and 90° using the unit circle. For an angle θ in the unit circle:

$$\boxed{\sin\theta := y \ \mid\ \cos\theta := x \ \mid\ \tan\theta := \frac{y}{x}}$$

21.2.1 Property 1 – Domain and range

Because we are inside a circle of radius 1:

- $-1 \leq \cos(\theta) \leq 1$;
- $-1 \leq \sin(\theta) \leq 1$.

21.2.2 Property 2 – Trigonometric identity

Because we have a 90° angle, we can use Pythagoras:

$$\boxed{\overrightarrow{OH}^2 + \overrightarrow{PH}^2 = \overrightarrow{OP}^2}$$

Let $\forall\theta \in \mathbb{R}$, then we can compute the following trigonometric identity:

$$\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$$

21.3 Tangent

A tangent of an angle is exactly the slope of a line:

$$\boxed{m = \frac{\Delta y}{\Delta x} = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}}$$

<u>Remark</u>: the tangent is not defined when the angle is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, that is when we have a vertical line.

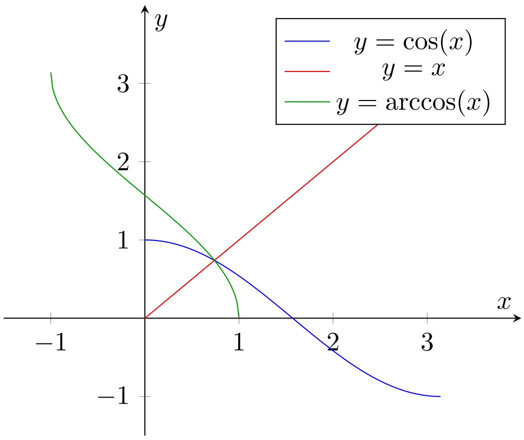
21.4 Domain of trigonometric functions

$$\boxed{\begin{array}{rcl} y &=& \cos(x), \quad x^c \in \mathbb{R} \\ y &=& \sin(x), \quad x^c \in \mathbb{R} \\ y &=& \tan(x), \quad x^c \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\right\} \end{array}}$$

21.5 Inverse trigonometric functions

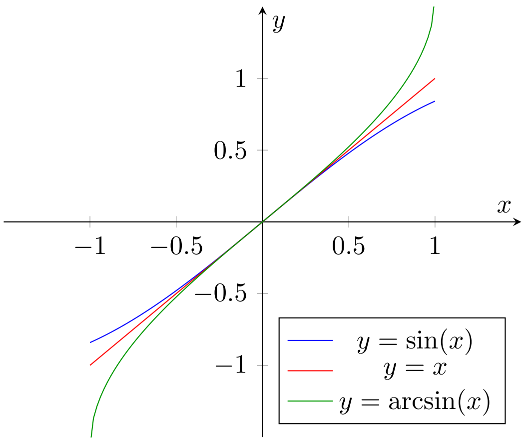
Warning: in order to be invertible, a function should be either always strictly increasing or always strictly decreasing.

21.5.1 Arccosine



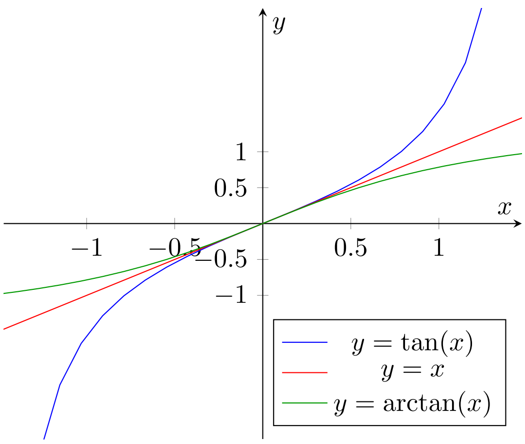
21.5.2 Arcsine

The domain of the arcsine is $\forall x \in [-1, 1]$ and the range is $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



21.5.3 Arctan

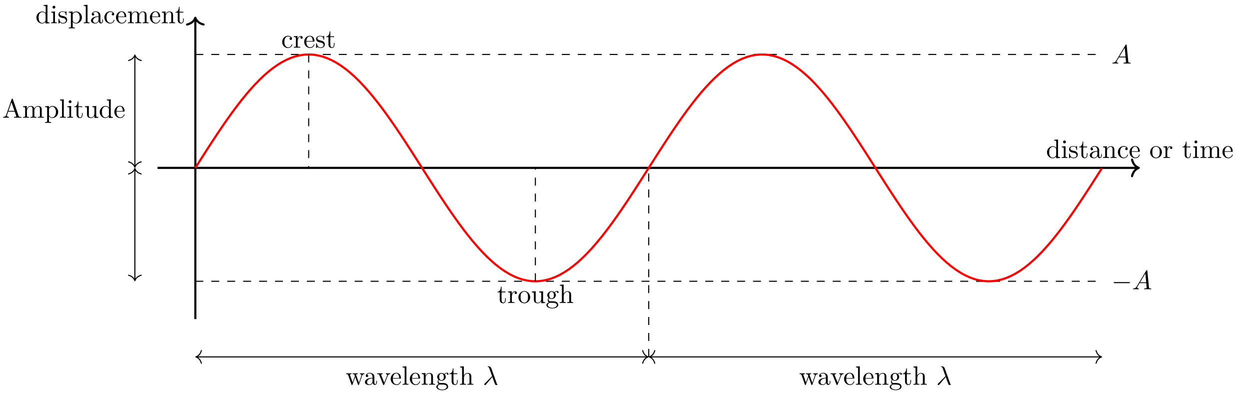
The domain is $\forall x \in \mathbb{R}$ and the range is $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



21.6 Harmonic oscillation

Let $A, B > 0$, then the function is oscillating harmonically with t around D:

$$\boxed{y = D + A \cdot \sin(Bt + \varphi)}$$



Part V

Calculus I

22 Limits

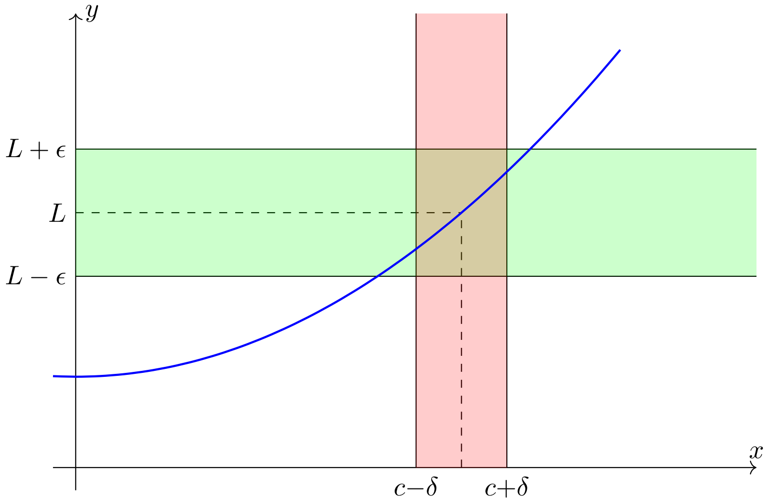
22.1 Concept of limit of a real function

22.1.1 Definition

Let $f : \mathcal{D} \to \mathbb{R}$ be a function and c a point, the limit $L = \lim_{x \to c} f(x)$ with x tending to c exists only if in a given $\epsilon > 0$ arbitrarily small, exists another $\delta > 0$ such that:

$$\boxed{0 < |x - c| < \delta \implies |f(x) - L| < \epsilon}$$

22.1.2 Graphic interpretation



22.2 Limit Value at a finite point

The notion of the "limit of $f(x)$ as x approaches a (finite) point $a \in \mathbb{R}$" is only meaningful if the point a can be approximated by points from the domain of definition of f. We can precisely formulate this concept with the notion of an "accumulation point".

> **Definition**
> Given a set $A \subset \mathbb{R}$ and a real number $a \in \mathbb{R}$, the real number a is called an *accumulation point* of the set A if every open interval of the form $(a - \delta, a + \delta)$ with $\delta > 0$ contains infinitely many points of A.

In the above definition, it is not required that a lies in A. Often, we will consider functions whose domains are unions of intervals of the form:

$$\boxed{(b, a) \cup (a, c)}$$

For example, consider the function defined by $f(x) = \frac{1}{x}$, defined on $(-\infty, 0) \cup (0, \infty)$. The point 0 is an accumulation point of the domain of definition of $\frac{1}{x}$.

> **Definition**
> Given a real function f, an accumulation point x_0 of $\mathcal{D}_f$, and $L \in \mathbb{R} = \mathbb{R} \cup \{\pm\infty\}$, we say that the function f has the limit L as $x \to x_0$ if $f(x)$ gets arbitrarily close to L, provided x is sufficiently close to (but never equal to) x_0.

22.3 One-sided limits

Often, one considers limits where x approaches x_0 from only one direction, either from the right or from the left. In these cases, we refer to a right-sided or left-sided limit and use the following notations:

$$\boxed{\lim_{x \to x_0^+} f(x) \quad \text{or} \quad \lim_{x \to x_0, x > x_0} f(x) \quad \text{or} \quad \lim_{x \to x_0} f(x)}$$

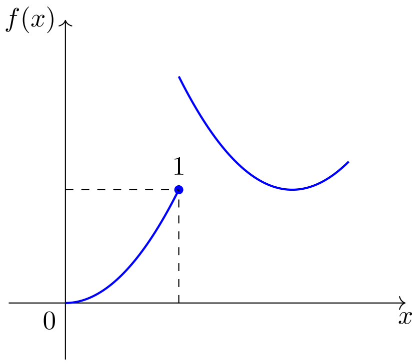
for a right-sided limit, and:

$$\boxed{\lim_{x \to x_0^-} f(x) \quad \text{or} \quad \lim_{x \to x_0, x < x_0} f(x) \quad \text{or} \quad \lim_{x \to x_0} f(x)}$$

for a left-sided limit.

$\lim\limits_{x \to a}$ can indicate a limit as x approaches an arbitrary point (e.g., $a = x_0$ for $x_0 \in \mathbb{R}$), as well as a one-sided limit ($a = x_0^+$ or $a = x_0^-$ for $x_0 \in \mathbb{R}$), or a limit at infinity ($a = \pm\infty$).

22.3.1 Graph example



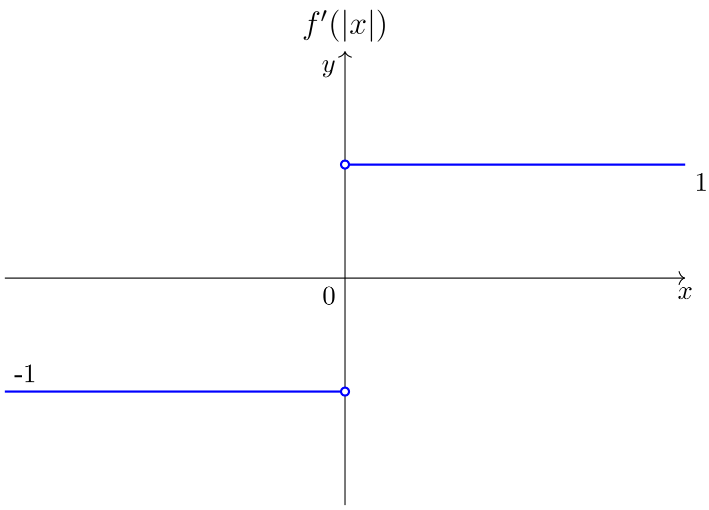
22.4 Continuity of a function

Definition *Continuity of a real function*
Given a real function $f : D \to C$, the function is continuous at the point $x = c$ where $c \in D$ if:

$$\lim_{x \to c} f(x) = f(c)$$

and therefore, if the limit exists and is equal to the value of the function at that point.

In other words, the function is continuous at the point if the limit exists (from both the left and right, coinciding) and the value of the function at that point is equivalent to the value of the limit.



22.4.1 Continuity in short

- A function is said to be continuous at a point if the limit at that point exists and is equal to the value of the function at that point;
- A function is said to be continuous on a subinterval of the domain if it is continuous at all points in that subinterval;
- A function is said to be continuous if it is continuous at all points in its interval.

23 Derivatives

23.1 Derivative notations

Type of derivate	First derivate	Second derivate	n-th derivate
Lagrange's notation	$f'(x)$	$f''(x)$	$f^{(n)}(x)$
Leibniz's notation	$\frac{d}{dx}f(x)$	$\frac{d^2}{dx^2}f(x)$	$\frac{d^n}{dx^n}f(x)$
Leibniz's notation in a point a	$\frac{d}{dx}[f(x)]\mid_{x=a}$	$\frac{d^2}{dx^2}[f(x)]\mid_{x=a}$	$\frac{d^n}{dx^n}[f(x)]\mid_{x=a}$
Newton's notation	$\dot{f}$	$\ddot{f}$	$\overset{(n)}{f}$

23.2 Definition of derivate

The derivate of a real function $f(x)$ is defined as:

$$\boxed{f'(x) = \lim_{\Delta x \to 0^{\pm}} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

if the limit exists.

Definition: $f'(a)$, if it exists, is calld derivative of $f(x)$ at $x = a$.
It corresponds to the slope of the tangent line at $x = a$ to the function $y = f(x)$

23.2.1 Simplified definition (Exponentiation rule)

Let $\forall \alpha \in \mathbb{R}$, then:

$$\boxed{f(x) = x^{\alpha} \Rightarrow f'(x) = \alpha \cdot x^{\alpha - 1}}$$

23.2.2 Existence of the derivative

The derivative exists if and only if:

$$\boxed{\lim_{\Delta x \to 0^{+}} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0^{-}} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Remark: If a function is differentiable, then it is continuous:

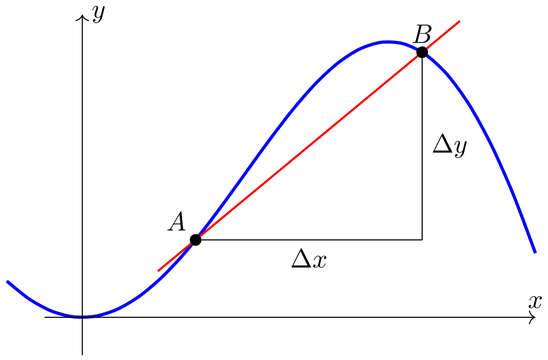
$$\boxed{\text{Differentiable} \Longrightarrow \text{Continuos}}$$

23.3 Equation of the tangent line

When $f'(x)$ is defined, $p = (x, f(x))$, and $p \in$ tangent line, then:

$$\boxed{y - f(x) = f'(x) \cdot (x - a)}$$

23.4 Geometric meaning of the derivative



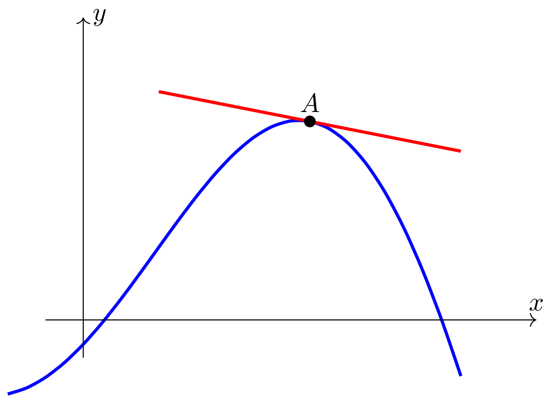
The secant of a function $f(x)$ between a point A and B is given by:

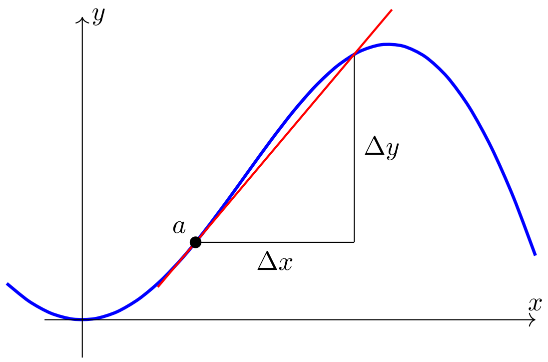
$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(B) - f(A)}{B - A}}$$

The closer we bring A and B, the smaller Δx becomes. As Δx decreases, the slope of the secant becomes more representative of the rate of change of f in the interval $[A; B]$.

When the Δx of the slope becomes infinitesimally small, we obtain the exact slope at a point (instantaneous). This slope is represented by the tangent line:

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$





The derivative of a function $f(x)$ is therefore another function, $f'(x)$, which represents the rate of change of $f(x)$ at every point. In other words, $f'(x)$ represents the slope of the tangent at each x of $f(x)$. This is precisely represented by the definition of the derivative, which is the slope $\frac{\Delta y}{\Delta x}$ calculated with the limit of $\Delta x \to 0$.

23.5 Bernoulli – de l'Hôpital Theorem

Bernulli – de l'Hôpital theorem is applicable only if the function results in an indeterminate form.

23.5.1 The 7 indeterminate forms

The seven indeterminate forms are

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

23.5.2 Statement of the theorem

Let us consider two real functions $f(x)$ and $g(x)$ that are differentiable in a neighborhood of $x_0 \in \mathbb{R}$ (not necessarily at x_0).

If $\lim_{x \to x_0} \frac{f(x_0)}{g(x_0)}$ results in an indeterminate form, then:

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

if the limit exists.

23.6 Derivation rules

Let $y = f(x)$ and $y = g(x)$ be two derivable functions, then:

23.6.1 Linearity

Let $c \in \mathbb{R}$, then:

$$\boxed{\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]}$$

23.6.2 Sum and subtraction

$$\boxed{f'(x) \pm g'(x) = \frac{d}{dx}\left[f(x) \pm g(x)\right]}$$

23.6.3 Multiplication

$$\boxed{\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)}$$

23.6.4 Difference

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g(x)}{\left(g(x)\right)^2}}$$

23.6.5 Exponential

Let $a > 0$, then:

$$\boxed{\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln(a)}$$

23.6.6 Composite function (Chain rule)

$$\boxed{\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)}$$

23.6.7 Inverse function

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}}$$

23.6.8 Trigonometric functions

$$\begin{tikzcd}
& \cos \arrow[dl, bend right, "\frac{d}{dx}"'] & \\
-\sin \arrow[dr, bend right, "\frac{d}{dx}"'] & & \sin \arrow[ul, bend right, "\frac{d}{dx}"'] \\
& -\cos \arrow[ur, bend right, "\frac{d}{dx}"'] &
\end{tikzcd}$$

$$
\boxed{\begin{array}{ll}
(\sec x)' = \sec x \tan x & (\csc x)' = -\csc x \cot x \\[2ex]
(\tan x)' = \dfrac{1}{\cos^2(x)} & (\cot x)' = \dfrac{-1}{\sin^2(x)} \\[2ex]
(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}} & (\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}} \\[2ex]
(\arctan x)' = \dfrac{1}{1+x^2} & (\text{arccot } x)' = \dfrac{-1}{1+x^2}
\end{array}}
$$

23.7 Particular cases

1.

$$f(x) = g(x)^\alpha \implies f'(x) = \alpha \cdot g'(x) \cdot g(x)^{\alpha-1} \qquad \text{i.g.: } \left(x^2+1\right)^4 \implies 8x \cdot \left(x^2+1\right)^3$$

2.

$$f(x) = e^{g(x)} \implies f'(x) = g'(x) \cdot e^{g(x)}$$

3.

$$f(x) = \frac{1}{g(x)} \implies f'(x) = \frac{-g'(x)}{\left(g(x)\right)^2}$$

23.8 Physical application

Linear uniform motion

$$\boxed{v(t) = c \in \mathbb{R}}$$

Uniformly accelerated motion

$$\boxed{a(t) = c \in \mathbb{R}}$$

Average acceleration a_{av}

$$\boxed{a_{av} := \frac{v(t_f) - v(t_i)}{t_f - t_i}}$$

Instant acceleration $a(t)$

$$\boxed{a(t) := v'(t) = \lim_{t \to 0} \frac{v(t+h) - v(t)}{\Delta t}}$$

23.9 Linearization

23.9.1 The linearization principle

In a very small neighborhood around a point a, we can assume that the function is linear at that point.

23.9.2 Tangent line approximation

In this case, assuming that the function is linear, we can use the tangent line equation:

$$\boxed{f(x) = f(a) + f'(a) \cdot (x - a)}$$

23.9.3 Error function

The error of the approximation is given by the difference between the exact function and the linearization:

$$\boxed{E(x) = d(f(x) \mid f_{\text{lin}}(x)) = f(x) - f(a) - f'(a) \cdot (x - a)}$$

23.10 Monotonicity

23.10.1 Definition of monotonicity

A real function f defined on an interval $I \subset \mathcal{D}_f$ is denoted as:

- *strictly monotonically increasing* on I, if $f(x_2) > f(x_1)$ applies for all $x_1, x_2 \in I$ with $x_2 > x_1$;
- *monotonically increasing* on I, if $f(x_2) \geq f(x_1)$ applies for all $x_1, x_2 \in I$ with $x_2 > x_1$;
- *strictly monotonically decreasing* on I, if $f(x_2) < f(x_1)$ applies for all $x_1, x_2 \in I$ with $x_2 > x_1$;
- *monotonically decreasing* on I, if $f(x_2) \leq f(x_1)$ applies for all $x_1, x_2 \in I$ with $x_2 > x_1$.

23.10.2 Monotonicity criterion

Let the function f be differentiable on the interval I:

- If $f'(x) > 0$ (resp. ≥ 0) for all $x \in I$, then f is strictly monotonically increasing (resp. monotonically increasing) on I.
- If $f'(x) < 0$ (resp. ≤ 0) for all $x \in I$, then f is strictly monotonically decreasing (resp. monotonically decreasing) on I.
- If $f'(x) = 0$ for all $x \in I$, then f is constant on I.

23.10.3 Monotonicity table

Let $f(x)$ be differentiable, $f'(x) < 0$ if $a < x < b$, $a, b, c \in \mathcal{D}_f$, and a, b, c are critical points, then:

$f'(x)$	$+$	$-$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

a b c

23.10.4 Critical point

Let $y_f(x)$ be a function, then we say that $x \in \mathcal{D}_f$ is a critical point in $f'(x) = 0$

Warning: many critical points are local extrema, some aren't.

23.10.5 Darboux theorem

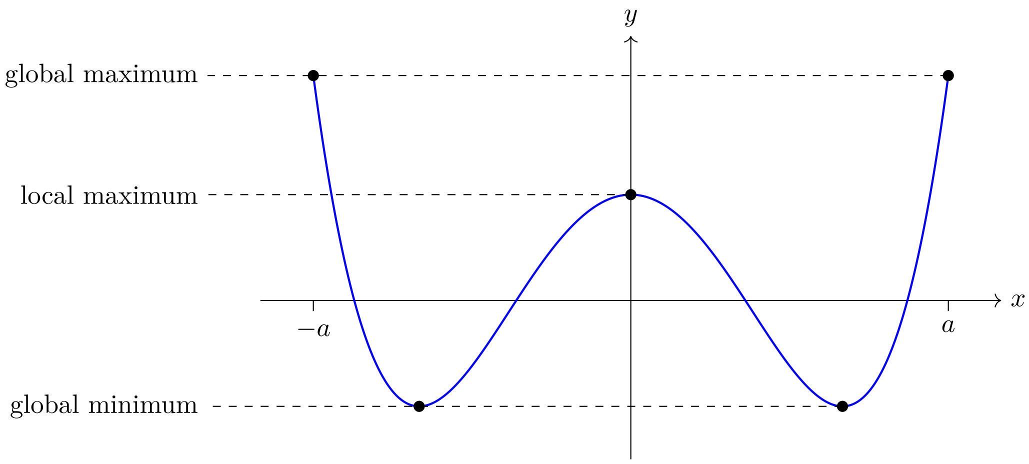
Let f be differentiable on an interval I:

1. find the critical points $f'(a,b) = 0,\ a < b$;
2. take a random point **between the critical points** in $c \mid a < c < b$;
3. compute $f'(c)$.

$$\boxed{\begin{array}{l} \text{If } f'(c) > 0, \text{ then } f'(x) > 0,\ \forall x \in (a,b) \\ \text{If } f'(c) < 0, \text{ then } f'(x) < 0,\ \forall x \in (a,b) \end{array}}$$

23.11 Minimum and maximum

Let a real function f and a point $x_0 \in \mathcal{D}_f$ be given.



Local and global extrema of a function in the interval [-a,a]

23.11.1 Local extrema

Local maximum
The function f has a **local maximum point** at point x_0 if there is an open neighborhood $U(x_o)$ such that:

$$\boxed{f(x) \leq f(x_0),\ \forall x \in U(x_0) \cap \mathcal{D}_f}$$

Local minimum
The function f has a **local minimum point** at point x_0 if there is an open neighborhood $U(x_o)$ such that:

$$\boxed{f(x) \geq f(x_0),\ \forall x \in U(x_0) \cap \mathcal{D}_f}$$

23.11.2 Global extrema

Global maximum
The function f has a **global maximum** at point x_0 if:

$$\boxed{f(x) \leq f(x_0),\ \forall x \in \mathcal{D}_f}$$

Global minimum
The function f has a **global minimum** at point x_0 if:

$$\boxed{f(x) \geq f(x_0),\ \forall x \in \mathcal{D}_f}$$