# 1 Preambule

**Theory box**

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**Formula box**

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**Lab/examples box**

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# 2 Fluids as energy carriers

## 2.1 Fluid state variables and properties

**Formulas**

### 2.1.1 State variables

**Density**

$$\rho \triangleq \frac{m}{V} \left[\frac{kg}{m^3}\right] \tag{1}$$

**Specific volume**

$$v \triangleq \frac{V}{m} = \frac{1}{\rho} \left[\frac{m^3}{kg}\right] \tag{2}$$

### 2.1.2 Viscosity

**Kinematic viscosity**

$$\nu \triangleq \frac{\eta}{\rho} \left[\frac{m^2}{s}\right] \tag{3}$$

**Dynamic viscosity**

$$\eta \triangleq \nu \cdot \rho \left[Pa \cdot s = \frac{Ns}{m^2} = \frac{kg}{m \cdot s}\right] \tag{4}$$

### 2.1.3 Real and ideal fluid

| **Real fluid** | **Ideal fluid** |
|---|---|
| variable density ($\Delta\rho \neq 0$) | incompressible ($\Delta\rho = 0$) |
| friction ($\eta > 0, \nu > 0$) | frictionless ($\eta = 0, \nu = 0$) |

### 2.1.4 Compressibility

**Mach number**

$$M \triangleq \frac{u}{c} \tag{5}$$

where:
- $M$ is the Mach number [-]
  $M \lesssim 0.3$: incompressible flow
- $u$ is the flow velocity [m/s]
- $c$ is the speed of sound in the fluid [m/s]

and:
- $c_{\mathrm{w}}^{20^\circ} = 1484$ m/s
- $c_{\mathrm{a}}^{20^\circ} = 343$ m/s

## 2.2 Laminar and turbulent flow

**Reynolds number**

$$Re = \frac{v \cdot L}{\nu} = \frac{\rho \cdot v \cdot L}{\eta} \, [-] \tag{6}$$

where:
- $v$ is the mean flow velocity [m/s]
- $L$ is the characteristic length [m]

**Re values**
- $Re < 2000$: laminar flow
- $Re \simeq 2300$: critical point
- $2000 < Re < 4000$: transitional regime
- $Re \geq 4000$: turbulent flow

## 2.3 Pressure and velocity

**Pressure**

### 2.3.1 Total pressure

Added to the static pressure $p_{\mathrm{stat}}$, there is also the dynamic pressure $p_{\mathrm{dyn}}$ and the total pressure $p_{\mathrm{tot}}$:

$$p_{\mathrm{tot}} = p_{\mathrm{stat}} + p_{\mathrm{dyn}} = \rho \left(gh + \frac{v^2}{2}\right) \tag{7}$$

### 2.3.2 Absolute pressure

Absolute pressure $p_{\mathrm{abs}}$ refers to the pressure in a vacuum $p_{\mathrm{vaacum}} = 0$ Pa while relative pressure $p_{\mathrm{rel}}$ can refer to any chosen reference pressure $p_{\mathrm{ref}}$.

$$p_{\mathrm{abs}} = p_{\mathrm{rel}} - p_{\mathrm{ref}} \tag{8}$$

### 2.3.3 Velocity

Velocity is a vector quantity:

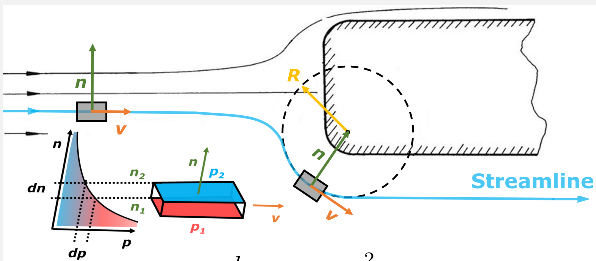
$$\vec{v} = (v_x v_y v_z) \tag{9}$$

The magnitude is given by:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{10}$$

## 2.4 Curvature pressure formula

**Deflection motion of a fluid element around a blunt body**

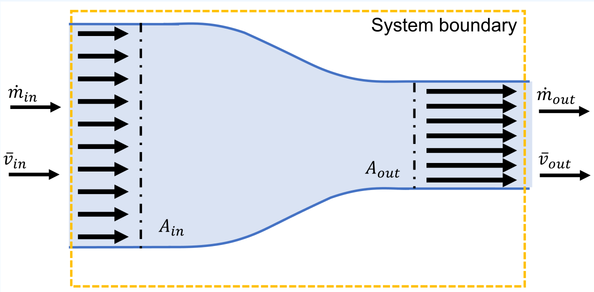



$$\frac{dp}{dn} = -\rho \cdot \frac{v^2}{R} \tag{11}$$

# 3 Mass conservation

## 3.1 Continuity equation / Mass conservation

**Continuity equation**




### 3.1.1 Steady mass-flow

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} \tag{12}$$

### 3.1.2 Incompressible fluid

$$\dot{m} = \rho \dot{V} \implies \dot{V}_{\text{in}} = \dot{V}_{\text{out}} \tag{13}$$

### 3.1.3 Streamline theory

$$\dot{V} = \bar{v}\, A \implies \bar{v}_{\text{in}}\, A_{\text{in}} = \bar{v}_{\text{out}}\, A_{\text{out}} \tag{14}$$

# 4 Energy conservation

## 4.1 Fluid mechanical energy conservation

**Derivation of the Bernoulli equation**

$$\dot{m}_1 \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 \right) = \dot{m}_2 \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2 \right) \tag{15}$$

This derivation is based on the assumption that the system has:

- steady flow
- ideal fluid
- adiabatic process
- no work in or out of the system
- 1D streamline flow

### 4.1.1 Energy flow

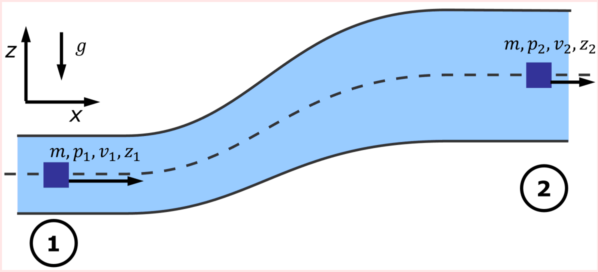
$$\frac{dE}{dt} = \underbrace{\sum P + \sum \dot{Q}}_{\substack{\text{Energy flow} \\ \text{across system boundary}}} + \underbrace{\sum_{in} \left[ \dot{m}^{\swarrow} \cdot \left( h^{\swarrow} + \frac{v^{2\swarrow}}{2} + g z^{\swarrow} \right) \right]}_{\substack{\text{Energy transfer} \\ \text{mass in}}} - \underbrace{\sum_{out} \left[ \dot{m}^{\nearrow} \cdot \left( h^{\nearrow} + \frac{v^{2\nearrow}}{2} + g z^{\nearrow} \right) \right]}_{\substack{\text{Energy transfer} \\ \text{mass out}}} \tag{16}$$

### 4.1.2 Outflow formula according to Torricelli

$$g z_1 = \frac{v_2^2}{2} \implies v_2 = \sqrt{2 g \Delta z} \tag{17}$$

## 4.2 Bernoulli equation

**Specific energy equation**




$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2 = \text{const.} \left[ \frac{J}{kg} \right] \tag{18}$$

### 4.2.1 Alternative forms

**Pressure equation**

$$p_1 + \frac{\rho v_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g z_2 = \text{const.}\, [Pa] \tag{19}$$

**Height equation**

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = \text{const.}\, [m] \tag{20}$$

**True energy equation**

The Bernoulli equation states that the sum of these energies is constant along a streamline.

### 4.2.2 Pressure energy

$$E_p = m \cdot \frac{p}{\rho}\, [J] \tag{21}$$

### 4.2.3 Kinetic energy

$$E_{\text{kin}} = m \cdot \frac{v^2}{2}\, [J] \tag{22}$$

### 4.2.4 Potential energy

$$E_{\text{pot}} = m \cdot g \cdot z\, [J] \tag{23}$$

### 4.2.5 Energy conservation

$$E_{p,1} + E_{\text{kin},1} + E_{\text{pot},1} = E_{p,2} + E_{\text{kin},2} + E_{\text{pot},2}$$

$$m \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + g z_1 \right) = m \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + g z_2 \right) \tag{24}$$

## 4.3 Hydrostatics

**Fundamental law of hydrostatics**

$$p = p_0 + \rho g h = \text{const.}\, [\text{Pa}] \tag{25}$$

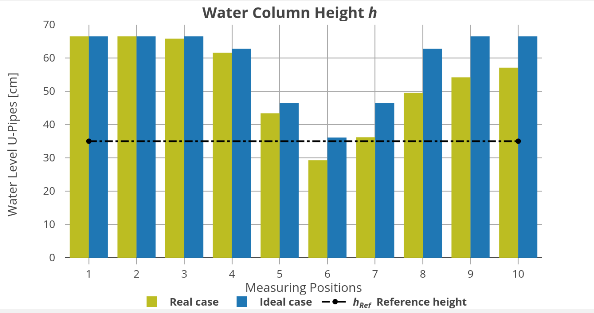
derived from:

$$p = p_0 + \frac{F_g}{A} = p_0 + \frac{mg}{A} = p_0 + \frac{\rho h A g}{A} \tag{26}$$

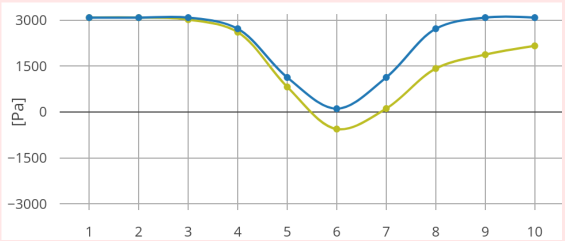
## 4.4 Venturi effect experiment

### Venturi effect

**Height – pressure difference at $\dot{V} = 6$ l/s**

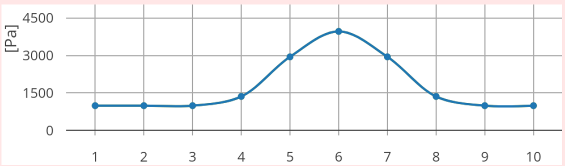



### Relative static pressure $p_{\text{rel}}$



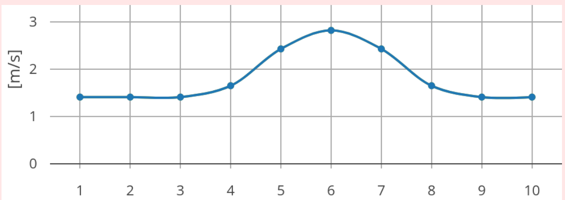

$$p_{\text{rel}} = p_{\text{hydro}} = \rho g \left(h - h_{\text{ref}}\right) \tag{27}$$

### Dynamic pressure $p_{\text{dyn}}$



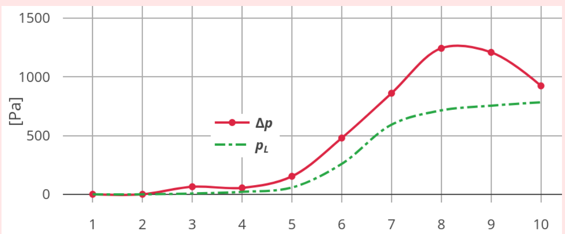

$$p_{\text{dyn}} = \rho \frac{v^2}{2} \tag{28}$$

### Dynamic pressure $v$




$$v = \frac{\dot{V}}{A} \tag{29}$$

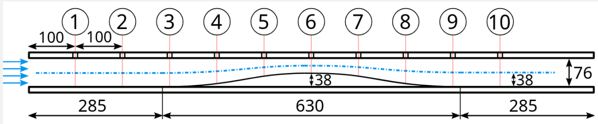
### Pressure difference $\Delta p$



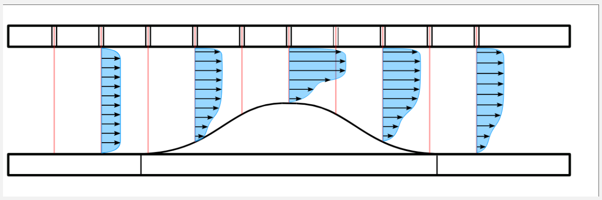

$$\Delta p = p_{\text{NoFric}} - p_{\text{real}} \implies p_V \sim v^2 \tag{30}$$

### Venturi effect

**Measurament points**

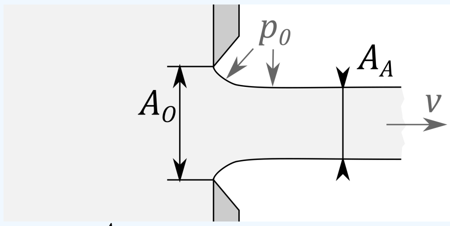



**Measurament shear flow**



## 4.5 Contraction coefficient

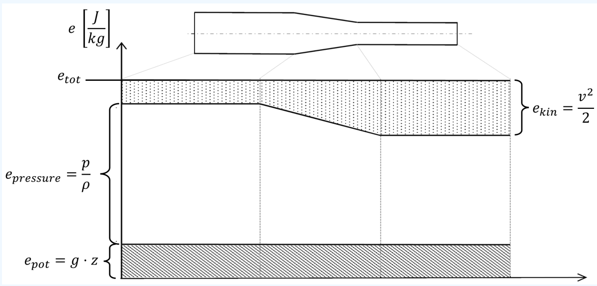
### Outflow contraction coefficient $\alpha$



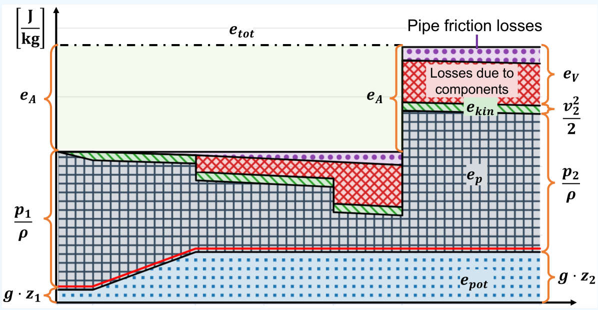

$$\alpha = \frac{A_{\text{actual}}}{A_{\text{opening}}} = \frac{\pi}{2+\pi} \approx 0.611[-] \tag{31}$$

## 4.6 Energy line diagram

### Ideal fluid energy line diagram




### Extended energy line diagram



## 4.7 Extended Bernoulli equation

**Extension of the Bernoulli equation**

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 + e_A = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 + e_V \left[\frac{J}{kg}\right]$$
$$E_{p,1} + K_1 + U_1 + E_A = E_{p,2} + K_2 + U_2 + E_V \, [J] \tag{32}$$

### 4.7.1 Additional terms

**Work term $e_A$**

$$e_A = \frac{p_A}{\rho} = gz_A = \frac{E_A}{m} = \frac{P_A}{\dot{m}} \left[\frac{J}{kg}\right] \tag{33}$$

where:

| | |
|---|---|
| $e_A$: work term [J/kg] | $E_A$: energy difference [J] |
| $p_A$: pressure diff [Pa] | $P_A$: power difference [W] |
| $z_A$: height difference [m] | |

If energy is added to the fluid along a streamline from point 1 to point 2 (eg. a pump), the total energy at point 2 becomes higher than at point 1.

**Sign convention**

$\mathbf{e_A > 0}$: work is done on the fluid
→ energy is added to the fluid (eg. pump);

$\mathbf{e_A < 0}$: work is done by the fluid
→ energy is extracted from the fluid (eg. turbine).

**Pump and turbine work $Y$**

In the pressure equation, the pressure $p_A$ increase (or decrease with a turbine) can be read directly at the working term, hence:

$$e_w = Y = \frac{W_A}{\dot{m}} = \frac{E_A}{m} = H \cdot g = \frac{p_A}{\rho} \left[\frac{J}{kg}\right] \tag{34}$$

The hydraulic power $P_{\text{hyd}}$ is then given by:

$$P_{\text{hyd}} = \dot{m} \cdot Y = \dot{V} \cdot \rho \cdot Y = \rho \cdot \dot{V} \cdot g \cdot H \, [W] \tag{35}$$

**Specific loss term $e_V$**

$$e_V = \frac{p_V}{\rho} = gz_V = \frac{E_V}{m} = \frac{P_V}{\dot{m}} \left[\frac{J}{kg}\right] \tag{36}$$

where:

| | |
|---|---|
| $e_V$: loss term [J/kg] | $E_V$: energy loss [J] |
| $p_V$: pressure diff [Pa] | $P_V$: power loss [W] |
| $z_V$: height loss [m] | |

The effects of a viscous fluid along a stramline from point 1 to point 2 are taken into account by $e_V$.

**Pressure loss $\Delta p_V$**

$$\Delta p_V = e_V \cdot \rho = \frac{E_V \cdot \rho}{m} = g \cdot z_V \cdot \rho = \zeta \cdot \rho \cdot \frac{v^2}{2} \, [Pa] \tag{37}$$

## 4.8 Loss behavior in turbolent flows

**Zeta value**

$$\zeta = \frac{2 \cdot \Delta p_V}{\rho \cdot v^2} \tag{38}$$

**Total pressure loss**

If multiple losses occur in a system due to sequentially connected hydraulic components, the ttal loss $\Delta p_{V,\text{tot}}$ is given by the sum of the individual losses:

$$\Delta p_{V,\text{tot}} = \sum_i \Delta p_{V,i} = \sum_i \zeta_i \cdot \rho \cdot \frac{v_i^2}{2} \, [Pa] \tag{39}$$

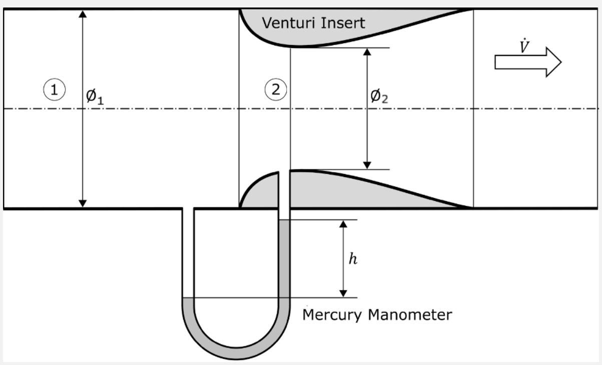
$$\Delta p_{V,\text{tot}} = \rho \cdot \frac{v^2}{2} \cdot \sum_i \zeta_i = \rho \cdot \frac{v^2}{2} \cdot \zeta_{\text{tot}} \, [Pa] \tag{40}$$

**Pressure head (prevalenza)**

The pressure head $H$ is the (energy) height corresponding to its specific potential energy $e_A$:

$$H = \frac{e_A}{g} = \frac{\Delta p_A}{\rho \cdot g} \, [m] \tag{41}$$

**U-Tube manometer**




$$h = \frac{\rho \left(v_2^2 - v_1^2\right)}{2g \left(\rho_{\text{Hg}} - \rho_w\right)} \tag{42}$$

## 4.9 Efficiency

**Efficiency factor $\eta$**

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\text{Benefit}}{\text{Effort}} \tag{43}$$

$$\eta_{\text{hyd}} = \frac{P_{\text{real}}}{P_{\text{ideal}}} = \frac{\dot{m} \cdot e_{\text{real}}}{\dot{m} \cdot e_{\text{ideal}}} = \frac{e_A - e_V}{e_A}$$
$$\eta_{\text{hyd}} = \left( = \frac{\Delta e_k + \Delta e_{\text{pot}} + \Delta e_p}{e_A} \right) \tag{44}$$

### 4.9.1 Volumetric efficiency $\eta_{\text{vol}}$

$$\eta_{\text{vol}} = \frac{\dot{m}_{\text{real}}}{\dot{m}_{\text{ideal}}} = \frac{\dot{V}_{\text{real}}}{\dot{V}_{\text{ideal}}} \tag{45}$$

**Efficiency factor $\eta$**

### 4.9.2 Efficiency of a pump-driven system

$$\eta_{\text{pump}} = \frac{P_{\text{hyd}}}{P_{\text{mech}}} = \frac{\dot{m}\cdot Y}{M\cdot\omega} \tag{46}$$

$$\eta_{\text{tot}} = \underbrace{\eta_{\text{el}}\cdot\eta_{\text{mech}}\cdot\eta_{\text{vol}}}_{\text{Pump}}\cdot\eta_{\text{hyd}}^{\text{system}} \tag{47}$$

In the case of an eletrically driven pump, the effective power transferred to the fluid is thus:

$$P_{\text{eff}} = P_{\text{el}}\cdot\eta_{\text{tot}} \tag{48}$$

### 4.9.3 Efficiency of a turbine-driven system

$$\eta_{\text{turbine}} = \frac{P_{\text{mech}}}{P_{\text{hyd}}} = \eta_{\text{mech}}\cdot\eta_{\text{hyd}} \tag{49}$$

$$\eta_{\text{tot}} = \eta_{\text{turbine}}\cdot\eta_{\text{el}} = \eta_{\text{mech}}\cdot\eta_{\text{hyd}}\cdot\eta_{\text{el}} \tag{50}$$

# 5 Pipe flows

## 5.1 Flow characteristics

**Reynolds number in pipes**

$$Re = \frac{v_m\cdot d}{\nu} \tag{51}$$

**Pipe flows**

### 5.1.1 Laminar pipe flow

The pressure loss of a laminar pipe flow is described by the Hagen-Poiseuille:

$$v(r) = \frac{p_1-p_2}{4\eta\cdot l}\left(R^2-r^2\right) \tag{52}$$

$$v_m = \frac{v_{\max}}{2} = \frac{p_1-p_2}{8\eta\cdot l}\cdot R^2$$

$$v_m = \frac{p_1-p_2}{32\eta\cdot l}\cdot d^2$$

$$\Delta p = 32\eta\cdot v_m\cdot\frac{l}{d^2} \tag{53}$$

### 5.1.2 Turbolent flow / Pressure lost in pipelines

Flow losses in pipeline systems consist of pressure losses in straight or curved pipes as well as in fittings.

$$\Delta p = \lambda\cdot\frac{l}{d}\cdot\rho\cdot\frac{v_m^2}{2} \tag{54}$$

where:
$\lambda$: resistance coeff. [-] $d$: pipe diameter [m]
$l$: pipe length [m] $v_m$: mean flow velocity [m/s]

**Resistance coefficient $\lambda$**

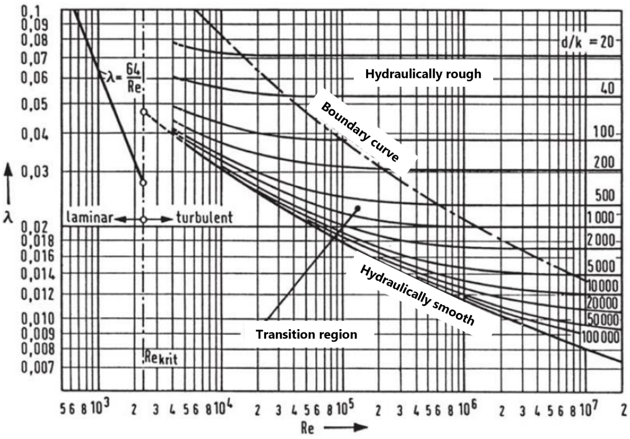
$$\lambda\cdot\frac{l}{d}\cdot\rho\cdot\frac{v_m^2}{2} = 32\eta\cdot v_m\cdot\frac{l}{d^2}$$

$$\lambda = \frac{64\eta}{v_m\cdot d\cdot\rho} = \frac{64}{Re}$$

## 5.2 Straight pipes

### 5.2.1 Moody diagram

The resistance coefficient $\lambda$ depends on the flow characteristics (quantified by the Reynolds number $Re$) and the relative wall roughness.

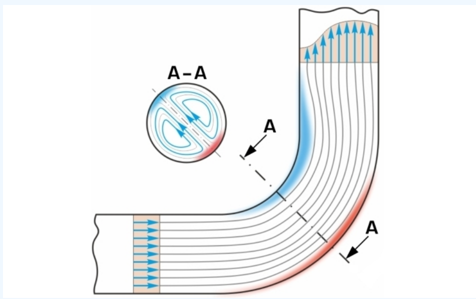



**Pipe fittings**

In pipeline systems, a portion of the pressure losses is caused by fittings:

$$\Delta p = \zeta\cdot\rho\cdot\frac{v_m^2}{2} \tag{55}$$

### 5.2.2 Elbows




$$\Delta p = \zeta\cdot\rho\cdot\frac{v^2}{2} \tag{56}$$

$$\zeta = f_{Re}\cdot\zeta_u \tag{57}$$

where (given from individual diagrams):

- $\zeta_u$ is the geometric resistance coefficient;
- $f_{Re}$ is the Reynolds correction factor.

### 5.2.3 Diffuser

A diffuser is a section in a pipeline with a continuous increase in cross-sectional area.

The frictional losses $\Delta p_v$ in a diffuser are given by:

$$\Delta p_v = \frac{\zeta\rho v_1^2}{2} \tag{58}$$

$$p_2 - p_1 = \Delta p_B - \Delta p_v \tag{59}$$

where $\Delta p_B$ is the Bernoulli pressure (frictionless).

**Pipe fittings**

The diffuser efficiency $\eta_D$ according to Bernoulli:

$$\eta_D = \frac{p_2 - p_1}{\Delta p_B} = 1 - \zeta \frac{1}{1 - \left(\frac{A_1}{A_2}\right)^2} \tag{60}$$

The various coefficients are stated as:

$$c_p = \frac{2\,(p_2 - p_1)}{\rho v_1^2} = \eta_D \cdot c_{p,id} \tag{61}$$

$$c_{p,id} = 1 - \left(\frac{A_1}{A_2}\right)^2 \tag{62}$$

$$\zeta_1 = c_{p,id} - c_p \tag{63}$$

The opening angle of the diffuser can be calculated as:

$$\tan(\theta) = \frac{d_2 - d_1}{2L} \tag{64}$$

$$\varphi = 2\theta \tag{65}$$

The optimal angle $\varphi_{\text{opt}}$ is between 6-20 degrees.

#### 5.2.4 Inlets and outlets