Maths refreshing course
HSLU, Semester 1

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Contents

Part I

Lesson 1

1 Algebraic definitions

- $\mathbb{N}$:= Natural numbers (including 0)
- $\mathbb{Z}$:= Integer numbers
- $\mathbb{Q}$:= Rational numbers
- $\mathbb{R}$:= Real numbers

Notation: The "*" symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

2 Prime numbers

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself.

$$\boxed{n \in \mathbb{N},\ n \neq \{0, 1\}}$$

3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{R}^*$ and $a \subset \mathbb{R}$, then

$$\boxed{a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}}$$

3.1 Property 1

Let $a, b \in \mathbb{R},\ n, m \in \mathbb{N}$, then

$$\boxed{a^n \cdot a^m = a^{n+m}}$$

3.2 Property 2

Let $a, b \in \mathbb{R},\ n \in \mathbb{N}$, then

$$\boxed{(a \cdot b)^n = a^n \cdot b^n}$$

Notation: The power a^n, a is the base and n is the exponent.

3.3 Property 3

Let $a \in \mathbb{R},\ m, n \in \mathbb{N}^*$, then

$$\boxed{(a^n)^m = a^{n \cdot m}, \text{ which is} \neq a^{(n^m)}}$$

4 Fractions

Notation 1: $a \cdot b = a \times b = ab \quad | \quad \frac{a}{b} = a \div b = a : b$

Notation 2: "a" is called numerator, "b" is called denominator.

Notation 3: $\frac{a}{b},\ a, b \in \mathbb{R},\ b \neq 0$

4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

5 Negative powers

5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

5.2 Property 4

Let $\forall n \in \mathbb{N},\ \forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z},\ \forall a \in \mathbb{R},\ z \neq 0$
We can compute a^z

5.3 Property 5

Let $\forall a \in \mathbb{R},\ a \neq 0,\ \forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:
   - $\forall a \in \mathbb{R},\ \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
   - $\forall b \in \mathbb{R},\ (a \cdot b)^n = a^n \cdot b^n$
   - $(a^n)^m = a^{n \cdot m}$
2. $\forall a \in \mathbb{R}^*,\ a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

6 Fractions and percentages (and back)

$\alpha \in \mathbb{R},\ n\%$ of $\alpha \Longleftrightarrow \frac{n}{100} \cdot \alpha$

Part II

Lesson 2

7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \to$ equality;
- $a \neq b \to$ inequality (a is not equal to b);
- $a < b \to$ less than (a is strictly less than b);
- $a \leq b \to$ less than or equal to (a is less than or equal to b);
- $a > b \to$ greater than (a is strictly greater than b);
- $a \geq b \to$ greater than or equal to (a is greater than or equal to b).

Example: $x \in \mathbb{R},\ x \geq 2 \to 2 \leq x < \infty$

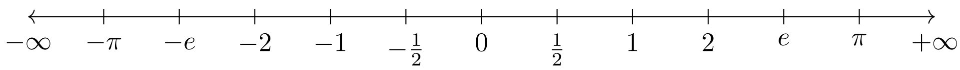
8 Brackets

( ) Parenthesis (round brackets)
[ ] Square brackets
{ } Braces

9 Latin notations

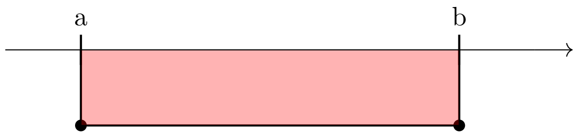
- e.g. = for example;
- i.e. = that is / that implies;
- q.e.d. ($\square$)= quod erat demonstrandum (we finally prove it).

10 The real line

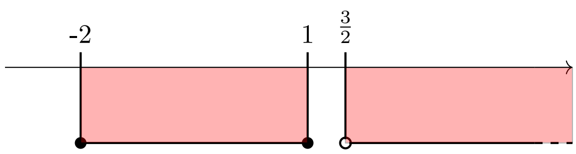


10.1 Exercises

1) $\forall a, b, x \in \mathbb{R},\ a \leq x \leq b$



2) $\forall x \in \mathbb{R},\ x \in\ ]-2, -1] \cup]\frac{3}{2}, +\infty]$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$ or $\vee$.

11 Properties of real numbers

11.1 Property 1 - Closure of "+" and "·"

$\forall x, y \in \mathbb{R}$
$x + y \in \mathbb{R}$
$x \cdot y \in \mathbb{R}$

Remark: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

11.2 Property 2 - Commutativity

$\forall x, y \in \mathbb{R}$
$x + y = y + x$
$x \cdot y = y \cdot x$

Remark: commutativity does not hold for divisions and subtractions.

11.3 Property 3 - Associative

$\forall x, y, z \in \mathbb{R}$
$x + (y + z) = (x + y) + z$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Remark: associativity does not hold for divisions and subtractions.

11.4 Property 4 - Distributive

$\forall x, y, z \in \mathbb{R}$
$x(y \pm z) = xy \pm xz$

11.5 Property 5 - Identity

$\forall x \in \mathbb{R}$

a) $0 + x = x$

b) $1 \cdot x = x$

Remark: $\forall x \in \mathbb{R},\ x \cdot 0 = 0$ is not an identity property.

11.6 Property 6 - Inverses and opposites

$\forall x \in \mathbb{R}$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0,\ x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

Remark 1: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

Remark 2: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
   ( ) inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
   x^a;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
   "·" and "÷";
4. Perform all additions and subtractions as you come to them, moving left-to-right:
   "+" and "−";
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

13 Signed numbers

A number is denoted as positive if it is directly preceded by a $+$ sign or no sign at all.
A number is denoted as negative if it is directly preceded by a $-$ sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

14 Absolute value

Let $x \in \mathbb{R}$, then

$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

14.1 Property

$\forall x \in \mathbb{R}$
$|x| > 0 \quad \text{if } y \neq 0$
$|x| = 0 \quad \text{if } x = 0$