Mathematics 3A
HSLU, Semester 3

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Last update: November 29, 2025

Contents

Part I

Planes and surfaces in space

1   Plane π in space

Let π denote the plane:

$$s_x \in \pi, s_y \in \pi, s_z \in \pi$$

$$\pi : ax + by + cz + d = 0$$

For $S_x \in \pi \Longrightarrow 1a + 0b + 0c + d = 0$, hence
$a + d = 0$

For $S_y \in \pi \Longrightarrow 0a + 2b + 0c + d = 0$, hence
$2b + d = 0$

for $S_z \in \pi \Longrightarrow 0a + 0b + 3c + d = 0$, hence
$3c + d = 0$

$$\begin{cases} a + d = 0 \\ 2b + d = 0 \\ 3c + d = 0 \end{cases} \implies \begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases}$$

Case 1:

$$d = 0 \Longrightarrow a = 0, b = 0, c = 0 \Longrightarrow \pi : 0 = 0 \Longrightarrow \text{NOT a plane!}$$

Case 2:

$$d \neq 0 \Longrightarrow \pi : \frac{ax + by + cz + d}{d} = 0 \Longrightarrow \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

Hence:

$$\begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases} \implies \begin{cases} \frac{a}{d} = -1 \\ \frac{b}{d} = -\frac{1}{2} \\ \frac{c}{d} = -\frac{1}{3} \end{cases}$$

Which leads to:

$$\pi : -x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0$$

<u>Remark</u>: the equation of a plane is defined up to a multiplication by a real number different from 0

e.g.: the same planed is shared between those 3 equations
ex 1)

$$z = 0 \Longleftrightarrow 5z = 0 \Longleftrightarrow -10z = 0$$

ex 2)

$$-x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0 \Longleftrightarrow 6x + 3y + 2z + 6 = 0$$

2   Functions in two variables x and y

Let us take $\pi : x^2 - y^2 = 0$ as example.

The plot would look like this:
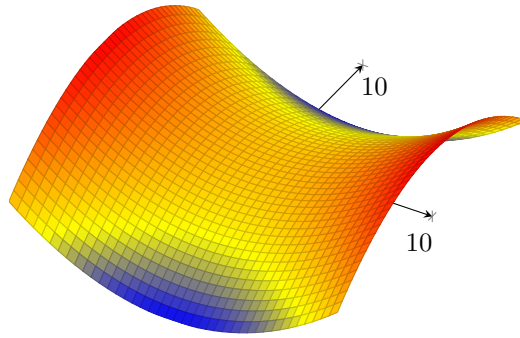
2.1   Spheres

3   Linear functions of two variables

We say that z is a *linear function* of x and y, if there are constant a, b and d such that:

$$\boxed{z = ax + by + d}$$

holds. Alternatively: if there are constant A, B, C, D, with $C \neq 0$, such that:

$$\boxed{Ax + By + Cz + D = 0}$$

holds. Since $C \neq 0$, we can rearrange this equation into:

$$\boxed{z = -\frac{Ax}{C} - \frac{By}{C} - \frac{D}{C}}$$

4 Contour lines

$$\begin{cases} z = f(x,y) \\ z = k \qquad k \in \mathbb{R} \end{cases}$$

$z = k$ represents all the possible horizontal planes

Ex:

$$\begin{cases} z = x^2 - y^2 \\ z = k \end{cases} \implies \begin{cases} k = x^2 - y^2 \\ z = k \end{cases}$$
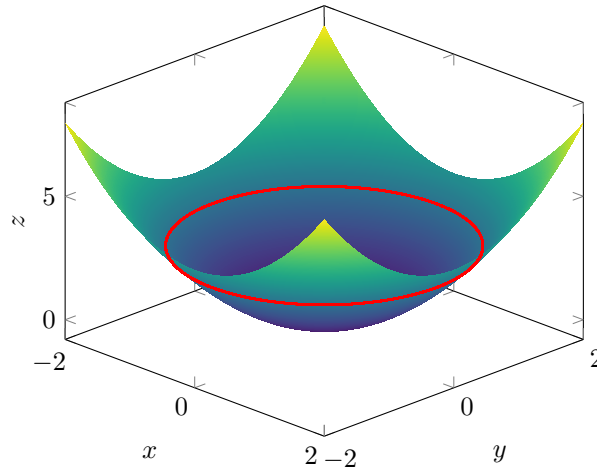
All the planes with equation $z = k$ are parallel to the coordinate planes $z = 0$.

When $z = k = 0$, the circle is reduced to a point, the origin.

When $k < 0$, the equation $x^2 + y^2 = k$ has no solution in $\mathbb{R}$.

When $k > 0$, the equation $x^2 + y^2 = k$ represents a circle with radius $\sqrt{k}$ centered at the origin.

5 Cylinders

A cylinder is a surface generated by all the lines parallel to a given line d and passing through a given curve $\mathcal{C}$.

5.1 Property

Whenever you have a polynomial equation of degree at least 2 with a missing variable, then you have a cylinder (up to few exceptions).

Ex:

$$z = y^2 \implies y^2 - z = 0$$

This is a cylinder with generatrix parallel to the x axis and directrix the parabola $y^2 - z = 0$ in the yz plane.

Part II

Partial derivatives

For a multivariable function $f(x, y, ...)$, the partial derivative to one variable measures the instantaneous rate of change of f when that variable changes and the others are held constant:
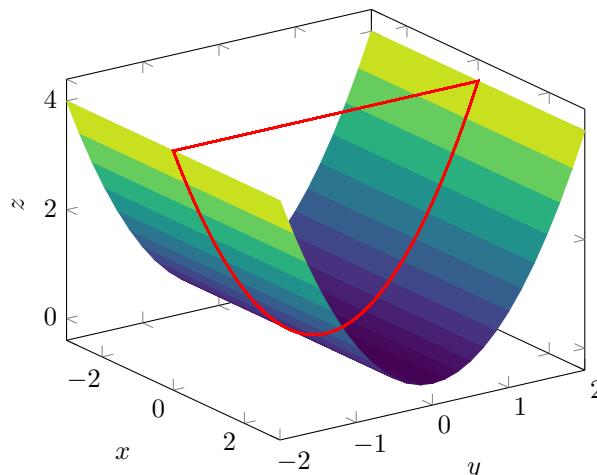
$$\boxed{\frac{\partial z}{\partial x} = f_x(x, y)}$$

If z is a function of x and y, we define:

The rate of change of z with respect to x, with y fixed, at the point $(x, y)=(a, b)$ as

$$\boxed{\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(a,b)} = \lim_{h \to 0} \frac{\left.Z\right|_{(x,y)=(a+h,b)} - \left.Z\right|_{(x,y)=(a,b)}}{h}}$$

The rate of change of z with respect to y, with x fixed, at the point $(x, y) = (a, b)$ as

$$\boxed{\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(a,b)} = \lim_{h \to 0} \frac{\left.Z\right|_{(x,y)=(a,b+h)} - \left.Z\right|_{(x,y)=(a,b)}}{h}}$$

For the lectures, we will be using the formula with 2-steps difference ($\Delta z_a = (a + h, b) - (a - h, b)$):

$$\boxed{\begin{aligned} \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(a,b)} &= \frac{\left.Z\right|_{(x,y)=(a+h,b)} - \left.Z\right|_{(x,y)=(a-h,b)}}{2h} \\ \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(a,b)} &= \frac{\left.Z\right|_{(x,y)=(a,b+h)} - \left.Z\right|_{(x,y)=(a,b-h)}}{2h} \end{aligned}}$$

6 Local linearization

6.1 Tangent plane of a function at point P

Let $f(x, y)$ be our function and $P(a, b)$ a point, $P \in f$:

$$\boxed{f(x, y) \approx f(a, b) + \frac{\partial}{\partial x} f(a, b)(x - a) + \frac{\partial}{\partial y} f(a, b)(y - b)}$$

7 Gradient

The gradient of a function $z = f(x, y)$ is defined by:

$$\boxed{\begin{aligned} \text{grad } f = \nabla f &= f_x \overrightarrow{e_x} + f_y \overrightarrow{e_y} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ \text{where } f_x &= \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y} \end{aligned}}$$

7.1 Geometrical properties of the gradient vector ∇ in the plane

If f is differentiable at the point (a, b) and $\nabla f \neq \vec{0}$, then the following holds:

$\nabla \mathbf{f(a, b)}$:

- is perpendicular to the contour line of f through (a, b)
- points in the direction of the maximum rate of change f

The length $\|\nabla \mathbf{f(a, b)}\|$ of the gradient vector is:

- the maximum rate of change f at this point
- large when the contour lines are close together
- small when the contour lines are far apart

7.2 Gradient of a function of three variables

The gradient of a function $w = f(x, y, z)$ is defined by:

$$\text{grad } f = \nabla f = f_x \vec{e_x} + f_y \vec{e_y} + f_z \vec{e_z} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$\text{where } f_x = \frac{\partial f}{\partial x}, \ f_y = \frac{\partial f}{\partial y}, \text{ and } f_z = \frac{\partial f}{\partial z}$$

7.3 Second-order partial derivatives of $z = f(x, y)$

A function $z = f(x, y)$ has two first-order partial derivatives, f_x and f_y, and four second-order partial derivatives:

$$1. \quad \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = (f_x)_x(x, y),$$

$$2. \quad \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = (f_y)_x(x, y),$$

$$3. \quad \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = (f_x)_y(x, y),$$

$$4. \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = (f_y)_y(x, y)$$

Usually, parenthesis are omitted, writing directly f_{xy} instead of $(f_x)_y$, and $\frac{\partial^2 z}{\partial y \partial x}$ instead of $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$.

7.4 Equality of mixed partial derivatives (Schwarz's Theorem)

If f_{xy} and f_{yx} are continuous at a point (a, b) inside the domain, then:

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

8 Directional derivatives in the plane

8.1 Directional derivative of f at $P(a,b)$ in the direction of $\vec{u}$

If $\vec{e_u} = \vec{u} = u_1\vec{e_x} + u_2\vec{e_y}$ is a unit vector $\|u\| = 1$, we define the directional derivative $\dfrac{\partial f}{\partial \vec{u}} = f_{\vec{u}}$ by

$$\frac{\partial f}{\partial \vec{u}}(a,b) = f_{\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

8.2 Gradient and directional derivative

If f is differentiable and $\vec{e_u} = u_1\vec{e_x} + u_2\vec{e_y}$ is the unit vector in the direction of $\vec{u}$, then:

$$\frac{\partial f}{\partial \vec{u}}(a,b) = f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \nabla f(a,b) \cdot \vec{e_u}$$

9 Critical points

9.1 Discriminant

Let (x_0, y_0) be a critical point. Furthermore, let

$$D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$$

Then the following holds:

- If $D > 0$ and $f_{xx} > 0$, then f has a local minimum at (x_0, y_0)
- If $D > 0$ and $f_{xx} < 0$, then f has a local maximum at (x_0, y_0)
- If $D < 0$, then f has a saddle point at (x_0, y_0)
- If $D = 0$, no conclusion can be made

10 Constraints and Lagrange Multipliers

10.1 Lagrange multiplier λ

The scalar λ measures how sensitive the optimal value of f is with respect to small changes in the constraint level c. Formally,

$$\lambda = \frac{\partial f^*}{\partial c}$$

where f^* denotes the optimal value of f. A positive λ indicates that relaxing the constraint (c larger) increases the optimal value of f.

10.2 Graphical representation

The optimization of $f(x,y)$ under the constraint $g(x,y) = c$ can be visualized as searching for points where a level curve of f is tangent to the constraint curve. At an optimum, the gradients are parallel:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

10.3 Lagrange function $\mathcal{L}$

When optimizing $f(x, y)$ under the constraint $g(x, y) = c$, the Lagrange function is used:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

The partial derivatives must be calculated:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\big(g(x, y) - c\big)$$

The stationary points of $\mathcal{L}$ satisfy:

$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0$$

$$g(x, y) = c$$

Solutions (x, y, λ) of this system give the candidate extrema of f under the constraint $g(x, y) = c$.

Part III

Integration of functions with multiple variables

11   Domain of integration Ω

Let $f : \Omega \subset \mathbb{R}^2 \to R$. The set Ω is a region in the xy-plane over which the double integral

$$\iint_\Omega f(x,y) \; dx \; dy$$

is taken.

12   Double integrals as iterated integrals

If the region R is a rectangle with $a \leq x \leq b$ and $c \leq y \leq d$ and if f is continuous in the region R, then the integral of f over R is equal to the iterated integral

$$\int_R f \, dA = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y) \, dx \, dy$$

The iterated integrals can also be written as

$$\int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy$$

12.1   Double integral over rectangles

$$\int_R f(x,y) \, dA = \int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy = \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx$$

12.2   Triangular regions

For the triangle with vertices $(0,0)$, $(1,0)$, $(0,1) \implies 0 \leq y \leq 1, \; 0 \leq x \leq 1 - y$:

$$\int_{y=0}^{1} \int_{x=0}^{1-y} f(x,y) \, dx \, dy$$

Equivalently $0 \leq x \leq 1, \; 0 \leq y \leq 1 - x$:

$$\int_{x=0}^{1} \int_{y=0}^{1-x} f(x,y) \, dy \, dx$$

12.3 Double integral over general regions

If the region Ω is not a rectangle, one must describe it using variable limits that follow the boundary of Ω

12.3.1 x-simple region

If the region $\Omega = \{(x,y) \mid a \leq x \leq b, \ \varphi_1(x) \leq y \leq \varphi_2(x)\}$, then

$$\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\, dx$$

12.3.2 y-simple region

If the region $\Omega = \{(x,y) \mid c \leq y \leq d, \ \psi_1(y) \leq x \leq \psi_2(y)\}$, then

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

12.4 Double integrals in Polar coordiates

12.4.1 Polar coordinates

Polar coordinates are defined as the coordinate change

$$f : (0, +\infty) \times (-\pi, \pi] \rightarrow \mathbb{R}^2 \setminus \left\{ (x,0) \in \mathbb{R}^2 \mid x \leq 0 \right\}$$

given by

$$f(r, \varphi) = (r\cos\varphi, r\sin\varphi)$$

12.4.2 Integration formula

To compute an integral in polar coordinates:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ x^2 + y^2 &= r^2 \end{aligned}$$

and

$$dA = r\, \mathrm{d}\varphi\, \mathrm{d}r \quad \text{and} \quad dA = r\, \mathrm{d}r\, \mathrm{d}\varphi$$

13 Triple integrals as iterated integrals

If the region V is a box with $a \leq x \leq b$, $c \leq y \leq d$, and $p \leq z \leq q$ and if f is continuous in the region V, then the integral of f over V is equal to the iterated integral

$$\int_W f\, dV = \int_p^q \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

13.1 Triple integrals in Cylindrical coordinates

13.1.1 Cylindrical coordinates

Cylindrical coordinates are defined as the coordinate change

$$f : (0, +\infty) \times (-\pi, \pi] \times \mathbb{R} \to \mathbb{R}^3 \setminus \left\{ (x, 0, z) \in \mathbb{R}^3 \mid x \leq 0 \right\}$$

given by

$$f(r, \varphi, z) = (r \cos \varphi, r \sin \varphi, z)$$

13.1.2 Integration formula

Each point (x, y, z) in a 3D space is represented by $0 \leq r < \infty$, $-\pi < \varphi \leq \pi$, and $-\infty < z < \infty$. The following relations hold:

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \\ z &= z \\ x^2 + y^2 &= r^2 \end{aligned}$$

and

$$\mathrm{d}V = r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z$$