

## 1 Preamble

### Theory box

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### Formula box

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### Lab/examples box

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## 2 Fluids as energy carriers

### 2.1 Fluid state variables and properties

#### Formulas

##### 2.1.1 State variables

###### Density

$$\rho \triangleq \frac{m}{V} \left[ \frac{\text{kg}}{\text{m}^3} \right] \quad (1)$$

###### Specific volume

$$v \triangleq \frac{V}{m} = \frac{1}{\rho} \left[ \frac{\text{m}^3}{\text{kg}} \right] \quad (2)$$

##### 2.1.2 Viscosity

###### Kinematic viscosity

$$\nu \triangleq \frac{\eta}{\rho} \left[ \frac{\text{m}^2}{\text{s}} \right] \quad (3)$$

###### Dynamic viscosity

$$\eta \triangleq \nu \cdot \rho \left[ \text{Pa} \cdot \text{s} = \frac{\text{Ns}}{\text{m}^2} = \frac{\text{kg}}{\text{m} \cdot \text{s}} \right] \quad (4)$$

##### 2.1.3 Real and ideal fluid

###### Real fluid

variable density ( $\Delta\rho \neq 0$ )  
friction ( $\eta > 0, \nu > 0$ )

###### Ideal fluid

incompressible ( $\Delta\rho = 0$ )  
frictionless ( $\eta = 0, \nu = 0$ )

##### 2.1.4 Compressibility

###### Mach number

$$M \triangleq \frac{u}{c} \quad (5)$$

where:

- $M$  is the Mach number [-]
- $M \lesssim 0.3$ : incompressible flow
- $u$  is the flow velocity [m/s]
- $c$  is the speed of sound in the fluid [m/s]

and:

- $c_w^{20^\circ} = 1484 \text{ m/s}$
- $c_a^{20^\circ} = 343 \text{ m/s}$

### 2.2 Laminar and turbulent flow

#### Reynolds number

$$Re = \frac{v \cdot L}{\nu} = \frac{\rho \cdot v \cdot L}{\eta} [-] \quad (6)$$

where:

- $v$  is the mean flow velocity [m/s]
- $L$  is the characteristic length [m]

#### Re values

- $Re < 2000$ : laminar flow
- $Re \approx 2300$ : critical point
- $2000 < Re < 4000$ : transitional regime
- $Re \geq 4000$ : turbulent flow

### 2.3 Pressure and velocity

#### Pressure

##### 2.3.1 Total pressure

Added to the static pressure  $p_{\text{stat}}$ , there is also the dynamic pressure  $p_{\text{dyn}}$  and the total pressure  $p_{\text{tot}}$ :

$$p_{\text{tot}} = p_{\text{stat}} + p_{\text{dyn}} = \rho \left( gh + \frac{v^2}{2} \right) \quad (7)$$

##### 2.3.2 Absolute pressure

Absolute pressure  $p_{\text{abs}}$  refers to the pressure in a vacuum  $p_{\text{vacuum}} = 0 \text{ Pa}$  while relative pressure  $p_{\text{rel}}$  can refer to any chosen reference pressure  $p_{\text{ref}}$ .

$$p_{\text{abs}} = p_{\text{rel}} - p_{\text{ref}} \quad (8)$$

##### 2.3.3 Velocity

Velocity is a vector quantity:

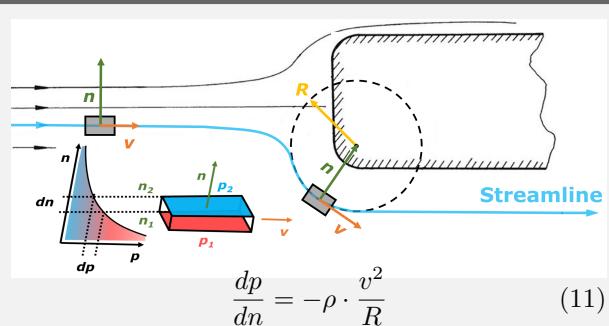
$$\vec{v} = (v_x v_y v_z) \quad (9)$$

The magnitude is given by:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad (10)$$

### 2.4 Curvature pressure formula

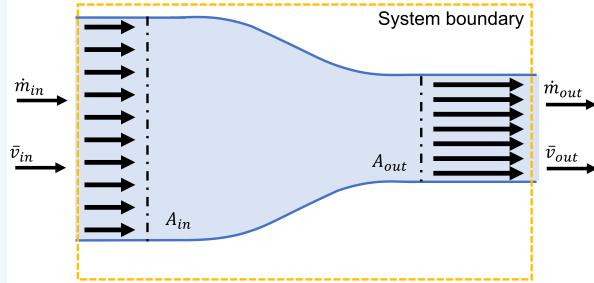
#### Deflection motion of a fluid element around a blunt body



### 3 Mass conservation

#### 3.1 Continuity equation / Mass conservation

##### Continuity equation



##### 3.1.1 Steady mass-flow

$$\dot{m}_{in} = \dot{m}_{out} \quad (12)$$

##### 3.1.2 Incompressible fluid

$$\dot{m} = \rho \dot{V} \implies \dot{V}_{in} = \dot{V}_{out} \quad (13)$$

##### 3.1.3 Streamline theory

$$\dot{V} = \bar{v} A \implies \bar{v}_{in} A_{in} = \bar{v}_{out} A_{out} \quad (14)$$

### 4 Energy conservation

#### 4.1 Fluid mechanical energy conservation

##### Derivation of the Bernoulli equation

$$\dot{m}_1 \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 \right) = \dot{m}_2 \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \right) \quad (15)$$

This derivation is based on the assumption that the system has:

- steady flow
- ideal fluid
- adiabatic process
- no work in or out of the system
- 1D streamline flow

##### 4.1.1 Energy flow

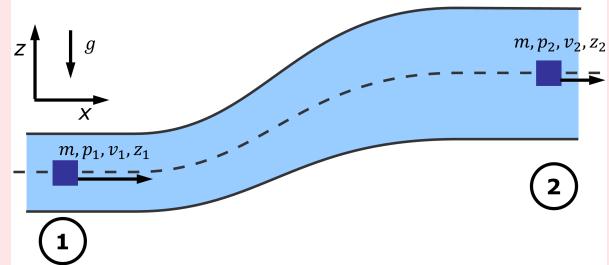
$$\begin{aligned} \frac{dE}{dt} = & \underbrace{\sum P + \sum \dot{Q}}_{\text{Energy flow across system boundary}} \\ & + \underbrace{\sum_{in} \left[ \dot{m}^\leftarrow \cdot \left( h^\leftarrow + \frac{v^{2\leftarrow}}{2} + gz^\leftarrow \right) \right] }_{\text{Energy transfer mass in}} \\ & - \underbrace{\sum_{out} \left[ \dot{m}^\rightarrow \cdot \left( h^\rightarrow + \frac{v^{2\rightarrow}}{2} + gz^\rightarrow \right) \right]}_{\text{Energy transfer mass out}} \end{aligned} \quad (16)$$

##### 4.1.2 Outflow formula according to Torricelli

$$gz_1 = \frac{v_2^2}{2} \implies v_2 = \sqrt{2g\Delta z} \quad (17)$$

#### 4.2 Bernoulli equation

##### Specific energy equation



$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = \text{const.} \left[ \frac{J}{kg} \right] \quad (18)$$

##### 4.2.1 Alternative forms

###### Pressure equation

$$p_1 + \frac{\rho v_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g z_2 = \text{const. [Pa]} \quad (19)$$

###### Height equation

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = \text{const. [m]} \quad (20)$$

##### True energy equation

The Bernoulli equation states that the sum of these energies is constant along a streamline.

##### 4.2.2 Pressure energy

$$E_p = m \cdot \frac{p}{\rho} [J] \quad (21)$$

##### 4.2.3 Kinetic energy

$$E_{kin} = m \cdot \frac{v^2}{2} [J] \quad (22)$$

##### 4.2.4 Potential energy

$$E_{pot} = m \cdot g \cdot z [J] \quad (23)$$

##### 4.2.5 Energy conservation

$$E_{p,1} + E_{kin,1} + E_{pot,1} = E_{p,2} + E_{kin,2} + E_{pot,2}$$

$$m \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 \right) = m \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \right) \quad (24)$$

### 4.3 Hydrostatics

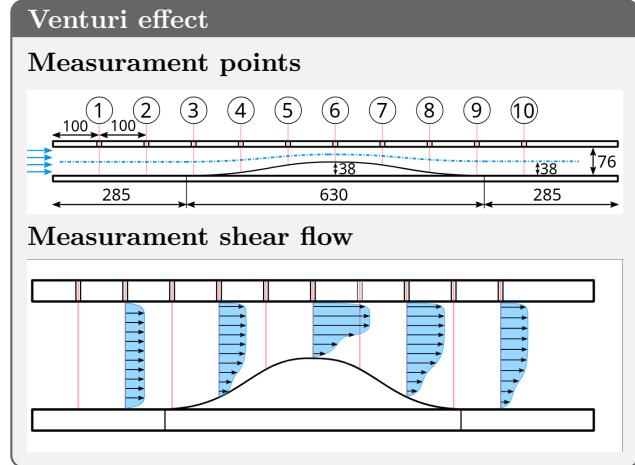
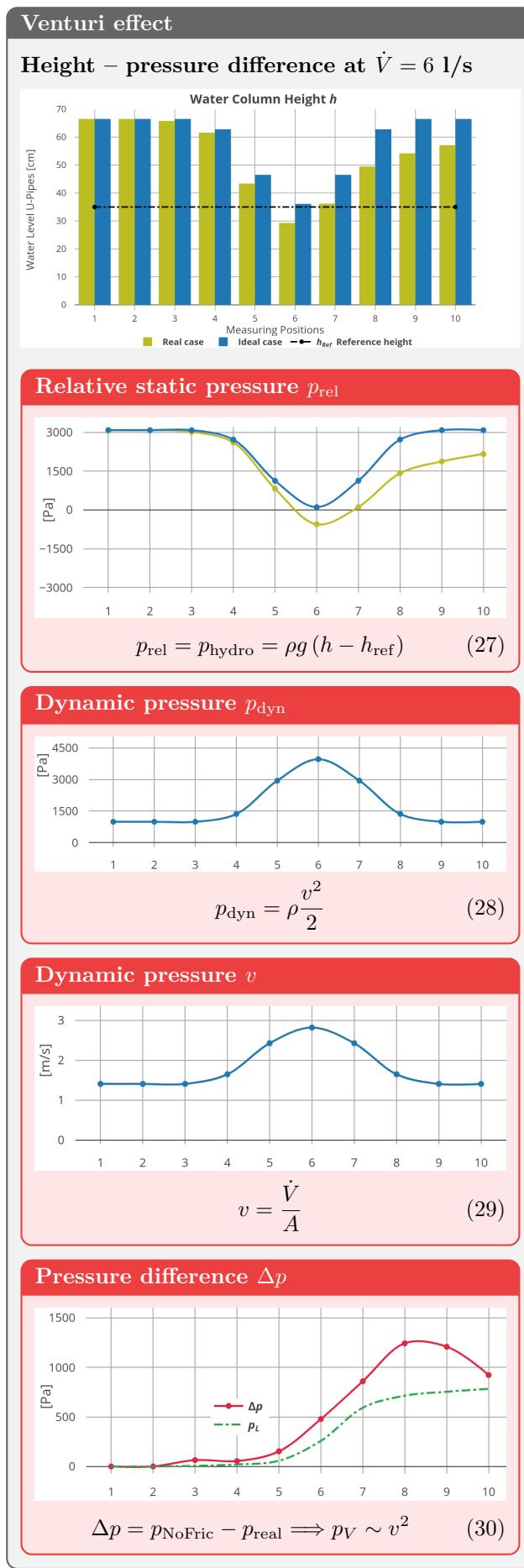
##### Fundamental law of hydrostatics

$$p = p_0 + \rho gh = \text{const. [Pa]} \quad (25)$$

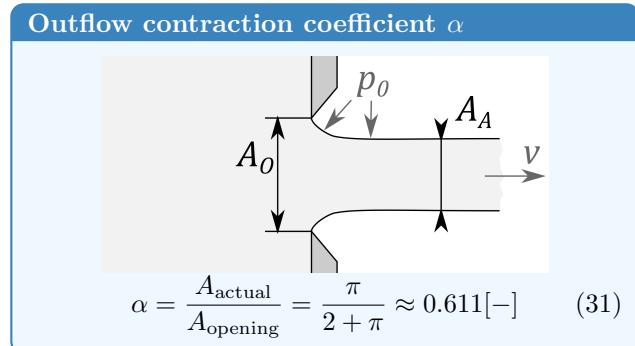
derived from:

$$p = p_0 + \frac{F_g}{A} = p_0 + \frac{mg}{A} = p_0 + \frac{\rho h Ag}{A} \quad (26)$$

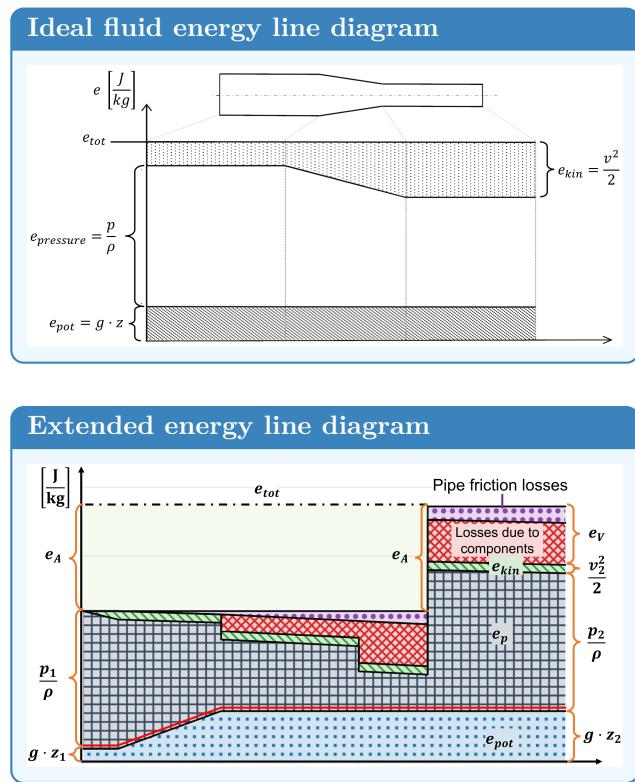
#### 4.4 Venturi effect experiment



#### 4.5 Contraction coefficient



#### 4.6 Energy line diagram



## 4.7 Extended Bernoulli equation

### Extension of the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 + e_A = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 + e_V \left[ \frac{J}{kg} \right]$$

$$E_{p,1} + K_1 + U_1 + E_A = E_{p,2} + K_2 + U_2 + E_V [J] \quad (32)$$

### 4.7.1 Additional terms

#### Work term $e_A$

$$e_A = \frac{p_A}{\rho} = gz_A = \frac{E_A}{m} = \frac{P_A}{\dot{m}} \left[ \frac{J}{kg} \right] \quad (33)$$

where:

$e_A$ : work term [J/kg]       $E_A$ : energy difference [J]  
 $p_A$ : pressure diff [Pa]       $P_A$ : power difference [W]  
 $z_A$ : height difference [m]

If energy is added to the fluid along a streamline from point 1 to point 2 (eg. a pump), the total energy at point 2 becomes higher than at point 1.

#### Sign convention

$e_A > 0$ : work is done on the fluid  
 → energy is added to the fluid (eg. pump);

$e_A < 0$ : work is done by the fluid  
 → energy is extracted from the fluid (eg. turbine).

### Pump and turbine work $Y$

In the pressure equation, the pressure  $p_A$  increase (or decrease with a turbine) can be read directly at the working term, hence:

$$e_w = Y = \frac{W_A}{\dot{m}} = \frac{E_A}{m} = H \cdot g = \frac{p_A}{\rho} \left[ \frac{J}{kg} \right] \quad (34)$$

The hydraulic power  $P_{hyd}$  is then given by:

$$P_{hyd} = \dot{m} \cdot Y = \dot{V} \cdot \rho \cdot Y = \rho \cdot \dot{V} \cdot g \cdot H [W] \quad (35)$$

### Specific loss term $e_V$

$$e_V = \frac{p_V}{\rho} = gz_V = \frac{E_V}{m} = \frac{P_V}{\dot{m}} \left[ \frac{J}{kg} \right] \quad (36)$$

where:

$e_V$ : loss term [J/kg]       $E_V$ : energy loss [J]  
 $p_V$ : pressure diff [Pa]       $P_V$ : power loss [W]  
 $z_V$ : height loss [m]

The effects of a viscous fluid along a streamline from point 1 to point 2 are taken into account by  $e_V$ .

### Pressure loss $\Delta p_V$

$$\Delta p_V = e_V \cdot \rho = \frac{E_V \cdot \rho}{m} = g \cdot z_V \cdot \rho = \zeta \cdot \rho \cdot \frac{v^2}{2} [Pa] \quad (37)$$

## 4.8 Loss behavior in turbulent flows

### Zeta value

$$\zeta = \frac{2 \cdot \Delta p_V}{\rho \cdot v^2} \quad (38)$$

### Total pressure loss

If multiple losses occur in a system due to sequentially connected hydraulic components, the total loss  $\Delta p_{V,tot}$  is given by the sum of the individual losses:

$$\Delta p_{V,tot} = \sum_i \Delta p_{V,i} = \sum_i \zeta_i \cdot \rho \cdot \frac{v_i^2}{2} [Pa] \quad (39)$$

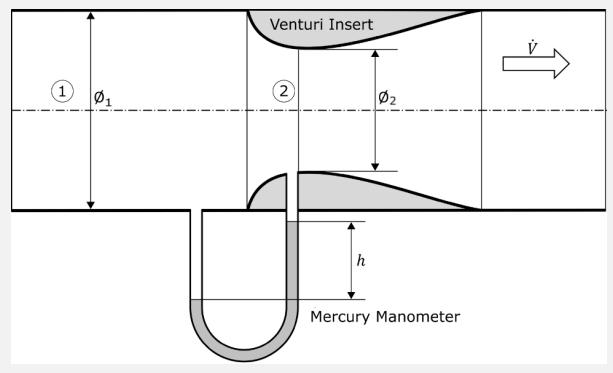
$$\Delta p_{V,tot} = \rho \cdot \frac{v^2}{2} \cdot \sum_i \zeta_i = \rho \cdot \frac{v^2}{2} \cdot \zeta_{tot} [Pa] \quad (40)$$

### Pressure head (prevalenza)

The pressure head  $H$  is the (energy) height corresponding to its specific potential energy  $e_A$ :

$$H = \frac{e_A}{g} = \frac{\Delta p_A}{\rho \cdot g} [m] \quad (41)$$

### U-Tube manometer



$$h = \frac{\rho (v_2^2 - v_1^2)}{2g (\rho_{Hg} - \rho_w)} \quad (42)$$

## 4.9 Efficiency

### Efficiency factor $\eta$

$$\eta = \frac{P_{out}}{P_{in}} = \frac{\text{Benefit}}{\text{Effort}} \quad (43)$$

$$\eta_{hyd} = \frac{P_{real}}{P_{ideal}} = \frac{\dot{m} \cdot e_{real}}{\dot{m} \cdot e_{ideal}} = \frac{e_A - e_V}{e_A}$$

$$\eta_{hyd} = \left( = \frac{\Delta e_k + \Delta e_{pot} + \Delta e_p}{e_A} \right) \quad (44)$$

### 4.9.1 Volumetric efficiency $\eta_{vol}$

$$\eta_{vol} = \frac{\dot{m}_{real}}{\dot{m}_{ideal}} = \frac{\dot{V}_{real}}{\dot{V}_{ideal}} \quad (45)$$

## Efficiency factor $\eta$

### 4.9.2 Efficiency of a pump-driven system

$$\eta_{\text{pump}} = \frac{P_{\text{hyd}}}{P_{\text{mech}}} = \frac{\dot{m} \cdot Y}{M \cdot \omega} \quad (46)$$

$$\eta_{\text{tot}} = \underbrace{\eta_{\text{el}} \cdot \eta_{\text{mech}} \cdot \eta_{\text{vol}}}_{\text{Pump}} \cdot \eta_{\text{hyd}}^{\text{system}} \quad (47)$$

In the case of an electrically driven pump, the effective power transferred to the fluid is thus:

$$P_{\text{eff}} = P_{\text{el}} \cdot \eta_{\text{tot}} \quad (48)$$

### 4.9.3 Efficiency of a turbine-driven system

$$\eta_{\text{turbine}} = \frac{P_{\text{mech}}}{P_{\text{hyd}}} = \eta_{\text{mech}} \cdot \eta_{\text{hyd}} \quad (49)$$

$$\eta_{\text{tot}} = \eta_{\text{turbine}} \cdot \eta_{\text{el}} = \eta_{\text{mech}} \cdot \eta_{\text{hyd}} \cdot \eta_{\text{el}} \quad (50)$$

## 5 Pipe flows

### 5.1 Flow characteristics

#### Reynolds number in pipes

$$Re = \frac{v_m \cdot d}{\nu} \quad (51)$$

## Pipe flows

### 5.1.1 Laminar pipe flow

The pressure loss of a laminar pipe flow is described by the Hagen-Poiseuille:

$$v(r) = \frac{p_1 - p_2}{4\eta \cdot l} (R^2 - r^2) \quad (52)$$

$$v_m = \frac{v_{\max}}{2} = \frac{p_1 - p_2}{8\eta \cdot l} \cdot R^2$$

$$v_m = \frac{p_1 - p_2}{32\eta \cdot l} \cdot d^2$$

$$\Delta p = 32\eta \cdot v_m \cdot \frac{l}{d^2} \quad (53)$$

### 5.1.2 Turbulent flow / Pressure lost in pipelines

Flow losses in pipeline systems consist of pressure losses in straight or curved pipes as well as in fittings.

$$\Delta p = \lambda \cdot \frac{l}{d} \cdot \rho \cdot \frac{v_m^2}{2} \quad (54)$$

### 5.1.3 Resistance coefficient $\lambda$

$$\lambda \cdot \frac{l}{d} \cdot \rho \cdot \frac{v_m^2}{2} = 32\eta \cdot v_m \cdot \frac{l}{d^2}$$

$$\lambda = \frac{64\eta}{v_m \cdot d \cdot \rho} = \frac{64}{Re} \quad (55)$$

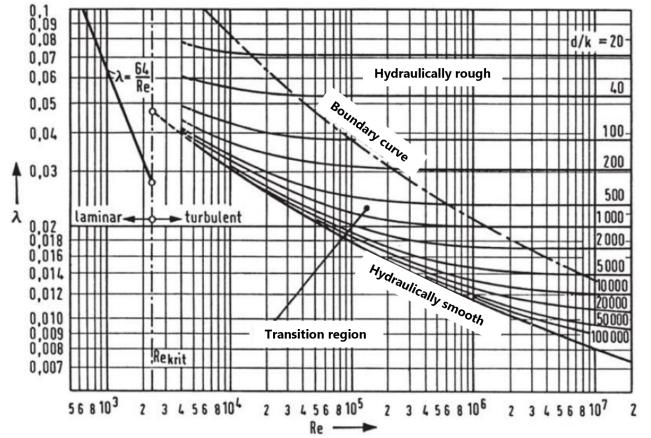
### 5.1.4 Loss coefficient $\zeta$ of a pipe

$$\zeta = \frac{l}{d} \cdot \lambda \quad (56)$$

## 5.2 Straight pipes

### 5.2.1 Moody diagram

The resistance coefficient  $\lambda$  depends on the flow characteristics (quantified by the Reynolds number  $Re$ ) and the relative wall roughness.

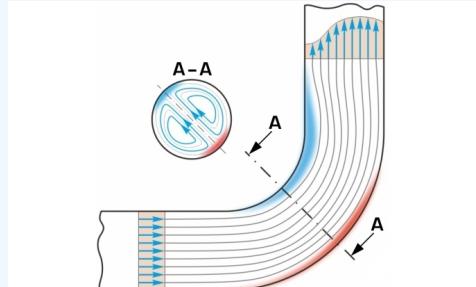


## Pipe fittings

In pipeline systems, a portion of the pressure losses is caused by fittings:

$$\Delta p = \zeta \cdot \rho \cdot \frac{v_m^2}{2} \quad (57)$$

### 5.2.2 Elbows



$$\Delta p = \zeta \cdot \rho \cdot \frac{v^2}{2} \quad (58)$$

$$\zeta = f_{Re} \cdot \zeta_u \quad (59)$$

where (given from individual diagrams):

- $\zeta_u$  is the geometric resistance coefficient;
- $f_{Re}$  is the Reynolds correction factor.

### 5.2.3 Diffuser

A diffuser is a section in a pipeline with a continuous increase in cross-sectional area.

The frictional losses  $\Delta p_v$  in a diffuser are given by:

$$\Delta p_v = \frac{\zeta \rho v_1^2}{2} \quad (60)$$

$$\Delta p_{v,\text{ideal}} = \rho \frac{v_2^2 - v_1^2}{2} \quad (61)$$

## Pipe fittings

The diffuser efficiency  $\eta_D$  according to Bernoulli:

$$\eta_D = \frac{p_2 - p_1}{\Delta p_B} = 1 - \zeta \frac{1}{1 - \left(\frac{A_1}{A_2}\right)^2} = \frac{c_p}{c_{p,\text{ideal}}} \quad (62)$$

The various coefficients are stated as:

$$c_p = \frac{2(p_2 - p_1)}{\rho v_1^2} = \eta_D \cdot c_{p,\text{ideal}} \quad (63)$$

$$c_{p,\text{ideal}} = 1 - \left(\frac{A_1}{A_2}\right)^2 \quad (64)$$

$$\zeta_1 = c_{p,\text{ideal}} - c_p \quad (65)$$

The opening angle of a diffuser can be calculated as:

$$\tan(\theta) = \frac{d_2 - d_1}{2L} \quad (66)$$

$$\varphi = 2\theta \quad (67)$$

The optimal angle  $\varphi_{\text{opt}}$  is between 6-20 degrees.

### 5.2.4 Inlets

When a stationary fluid is introduced from a large container into a pipe, losses occur due to acceleration and the formation of separation bubbles.

- sharp edge:  $0.45 < \zeta < 0.50$
- broken edge:  $\zeta = 0.20$

### 5.2.5 Outlets

For an outlet into a large basin, the entire kinetic energy is converted into static pressure loss, meaning  $\zeta = 1$  can be assumed.

### 5.2.6 Valves and fittings

$\zeta$ -values are specified by different manufacturer.

## 6 Linear momentum theorem

### Linear momentum

$$\vec{I} = m \cdot \vec{v} [\text{Ns}] \quad (68)$$

### 6.1 Linear momentum balance

#### Momentum flux

The change in motion is a change in linear momentum over time:

$$\vec{F}_{\text{res}} = \frac{d\vec{I}}{dt} = \dot{\vec{I}} = \frac{d(m \cdot \vec{v})}{dt} \quad (69)$$

at constant mass:

$$\dot{\vec{I}} = m \cdot \vec{a} = \dot{m} \cdot \vec{v} \quad (70)$$

## 6.2 System of forces

### Force balance

The sum of all external forces on a control volume is the difference of momentum flow:

$$\vec{F}_{\text{res}} = \sum_i \vec{F}_i = \dot{\vec{I}}_{\text{out}} - \dot{\vec{I}}_{\text{in}} \quad (71)$$

expanded to:

$$\dot{\vec{I}}_{\text{out}} - \dot{\vec{I}}_{\text{in}} = \dot{m} (\vec{v}_2 - \vec{v}_1) = \rho \cdot \dot{V} \cdot (\vec{v}_2 - \vec{v}_1) \quad (72)$$

where:

$$m_1 = \rho_1 \cdot \dot{V}_1 \cdot \Delta t = \rho_1 \cdot v_1 \cdot A_1 \cdot \Delta t$$

$$m_2 = \rho_2 \cdot \dot{V}_2 \cdot \Delta t = \rho_2 \cdot v_2 \cdot A_2 \cdot \Delta t \quad (73)$$

$$\vec{I}_{\text{in}} = \vec{I}_1 = m_1 \cdot \vec{v}_1 = \rho_1 \cdot v_1 \cdot A_1 \cdot \Delta t \cdot \vec{v}_1$$

$$\vec{I}_{\text{out}} = \vec{I}_2 = m_2 \cdot \vec{v}_2 = \rho_2 \cdot v_2 \cdot A_2 \cdot \Delta t \cdot \vec{v}_2 \quad (74)$$

### 6.2.1 Momentum as vector

$$\begin{aligned} \dot{\vec{I}}_{xyz} &= (\dot{I}_x \dot{I}_y \dot{I}_z) \\ \dot{I} &= \sqrt{\dot{I}_x^2 + \dot{I}_y^2 + \dot{I}_z^2} \end{aligned} \quad (75)$$

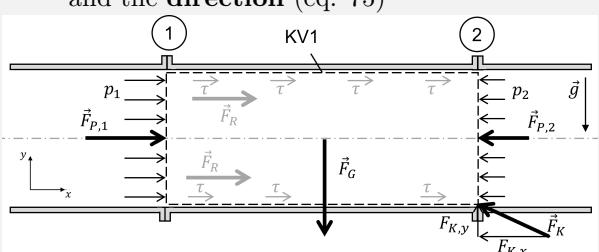
The external forces can consist of pressure forces  $F_P$ , body forces (support forces)  $F_B$ , gravitational forces  $F_G$ , and frictional forces  $F_F$ :

$$\vec{F}_{\text{res}} = \sum_i \vec{F}_i = \vec{F}_P + \vec{F}_B + \vec{F}_G + \vec{F}_F \quad (76)$$

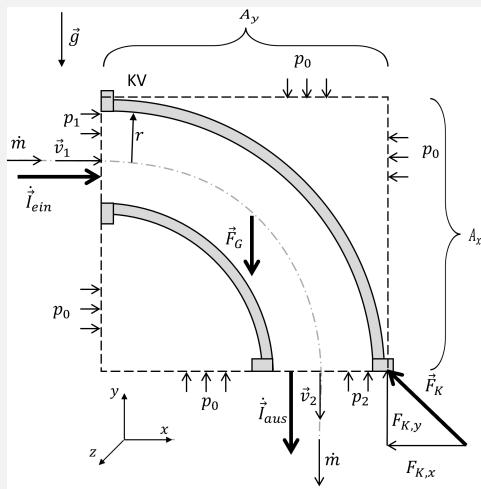
## 6.3 Control volume

### Linear momentum calculation steps

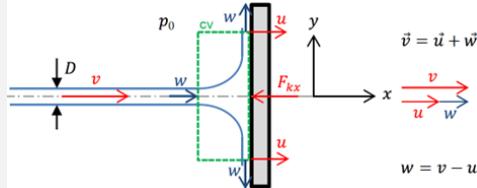
- I: Step: Select a suitable **coordinate system** and draw it in a sketch of the flow problem;
- II: Step: Select **control volume** sensibly and draw it in the sketch. The balance boundary should be set so that the external forces on its surface are known;
- III: Step: Draw in the **forces**  $\vec{F}_P, \vec{F}_B, \vec{F}_G, \vec{F}_F$  acting on the CV from the outside and calculate them from the known quantities.
- IV: Step: Calculate the **linear momentum fluxes** at the outlet and inlet and insert them as the resulting force (eq. 72)
- V: Step: Dissolve the momentum balance equations according to the **sought quantity** or its components and calculate them.
- VI: Step: If necessary, calculate the **magnitude** and the **direction** (eq. 75)



### Linear momentum calculation steps



#### 6.3.1 Plates



### 6.4 Momentum on x-direction

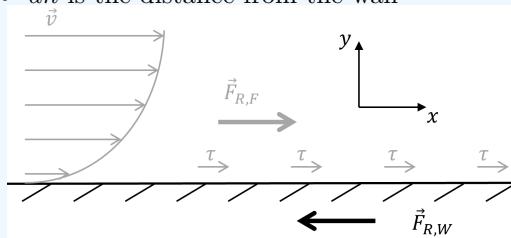
#### Wall shear stress

The shear stress  $\tau_w$  is the force per unit area acting on the pipe's walls:

$$\tau_w = \frac{dv}{dn} \cdot \eta \quad (77)$$

where:

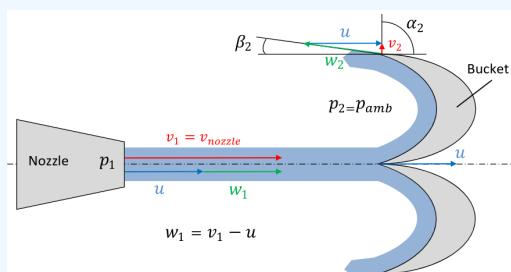
- $dv$  is the velocity difference;
- $dn$  is the distance from the wall



$$|F_{K,x}| = |A(p_{in} - p_{out})| = |\tau_w \cdot A \cdot l| \quad (78)$$

### 6.5 Pelton turbine

#### Pelton bucket



### Pelton bucket

#### 6.5.1 Velocities

- $v_1$ : absolute velocity [m/s]
- $u$ : radial velocity [m/s]
- $w_1$ : relative velocity (turbine POV) [m/s]

$$v_{nozzle} = \sqrt{2g\Delta h} = \sqrt{\frac{2\Delta p}{\rho}}$$

$$w_1 = v_{nozzle} - u = \sqrt{\frac{2\Delta p}{\rho}} - D_{wheel} \cdot \pi \cdot n_{wheel} \quad (79)$$

$$u = \frac{v_{nozzle}}{2} \quad (80)$$

#### 6.5.2 Rotational speed

$$k_u = \frac{u}{v_{nozzle}} \quad (81)$$

#### 6.5.3 Hydraulic power

$$P_{hyd} \approx \rho \cdot g \cdot H \cdot \dot{V} \approx \Delta p \cdot \dot{V} [W] \quad (82)$$

### 6.6 Borda-Carnot diffuser