

Maths refresher course

HSLU, Semester 1

Matteo Frongillo

September 5, 2024

Contents

I	Lesson 1	3
1	Numerical sets	3
2	Prime numbers	3
3	Positive powers	3
3.1	Property 1	3
3.2	Property 2	3
3.3	Property 3	3
4	Fractions	4
4.1	Property 1	4
4.2	Property 2	4
4.3	Property 3	4
5	Negative powers	4
5.1	Definition	4
5.2	Property 4	4
5.3	Property 5	4
6	Fractions and percentages (and back)	5
II	Lesson 2	6
7	Symbols	6
8	Brackets	6
9	Latin notations	6
10	The real line	6
10.1	Exercises	6
11	Properties of real numbers	7
11.1	Property 1 - Closure under “+” and “.”	7
11.2	Property 2 - Commutativity	7
11.3	Property 3 - Associative	7
11.4	Property 4 - Distributive	7
11.5	Property 5 - Identity	7
11.6	Property 6 - Inverses and opposites	7
12	The order of operations	7
13	Signed numbers	8

14 Absolute value	8
14.1 Property	8
III Lesson 3	9
15 Polynomials	9
15.1 Terms and factors	9
15.1.1 Variables	9
15.1.2 Sets	9
15.2 Expressions, terms and factors	9
15.2.1 Expressions	9
15.2.2 Terms	9
15.2.3 Factors	9
16 Common factor	10
17 Notable products	10
18 Classification of polynomials	10
18.1 Definition	10
18.2 Degree	10
18.2.1 Monomials	10
18.2.2 Polynomials	10
IV Lesson 4	11
19 Operations between polynomials	11
19.1 Polynomials with one independent variable	11
19.1.1 Sum	11
19.1.2 Multiplications	11
19.2 Polynomials with two or more variables	11
19.2.1 Sum	11
20 Equations	11
20.1 Identities	12
20.2 Contradictions	12
20.3 Conditional equations	12
21 Fundamental theorem of algebra	12
22 Linear equations with one variable	12
22.1 Simple tools	12
22.1.1 Tool 1	12
22.1.2 Tool 2	12
23 –	13

Part I

Lesson 1

1 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0)
- $\mathbb{Z} :=$ Integer numbers
- $\mathbb{Q} :=$ Rational numbers
- $\mathbb{R} :=$ Real numbers

Notation: The “*” symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

2 Prime numbers

A prime number is a number $n \in \mathbb{N} \setminus \{0, 1\}$ such that, for every divisor $d \in \mathbb{N}$, if $d \mid n$, then $d = 1$ or $d = n$.

$$n \in \mathbb{N} \setminus \{0, 1\} \text{ is prime} \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)$$

3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{N}^*$ and $a \in \mathbb{R}$, then

$$a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$$

3.1 Property 1

Let $a, b \in \mathbb{R}, n, m \in \mathbb{N}$, then

$$a^n \cdot a^m = a^{n+m}$$

3.2 Property 2

Let $a, b \in \mathbb{R}, n \in \mathbb{N}$, then

$$(a \cdot b)^n = a^n \cdot b^n$$

Notation: The power a^n , a is the base and n is the exponent.

3.3 Property 3

Let $a \in \mathbb{R}, m, n \in \mathbb{N}^*$, then

$$(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}$$

4 Fractions

Notation 1: $a \cdot b = a \times b = ab$ | $\frac{a}{b} = a \div b = a : b$

Notation 2: “ a ” is called numerator, “ b ” is called denominator.

Notation 3: $\frac{a}{b}$, $a, b \in \mathbb{R}$, $b \neq 0$

4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

5 Negative powers

5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

5.2 Property 4

Let $\forall n \in \mathbb{N}$, $\forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z}$, $\forall a \in \mathbb{R}$, $z \neq 0$
We can compute a^z

5.3 Property 5

Let $\forall a \in \mathbb{R}$, $a \neq 0$, $\forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:

- $\forall a \in \mathbb{R}, \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
- $\forall b \in \mathbb{R}, (a \cdot b)^n = a^n \cdot b^n$
- $(a^n)^m = a^{n \cdot m}$

2. $\forall a \in \mathbb{R}^*, a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

6 Fractions and percentages (and back)

$\alpha \in \mathbb{R}, n\% \text{ of } \alpha \iff \frac{n}{100} \cdot \alpha$

Part II

Lesson 2

7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \rightarrow$ equality;
- $a \neq b \rightarrow$ inequality (a is not equal to b);
- $a < b \rightarrow$ less than (a is strictly less than b);
- $a \leq b \rightarrow$ less than or equal to (a is less than or equal to b);
- $a > b \rightarrow$ greater than (a is strictly greater than b);
- $a \geq b \rightarrow$ greater than or equal to (a is greater than or equal to b).

Example: $x \in \mathbb{R}$, $x \geq 2 \rightarrow 2 \leq x < \infty$

8 Brackets

() Parenthesis (round brackets)

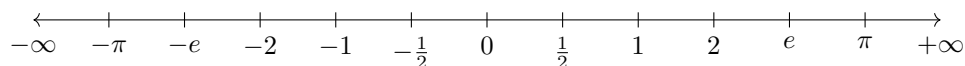
[] Square brackets

{ } Braces

9 Latin notations

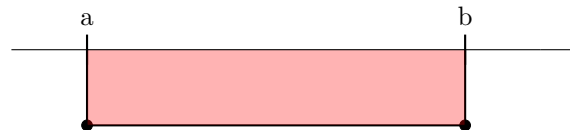
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. (\square)= quod erat demonstrandum (we finally prove it).

10 The real line

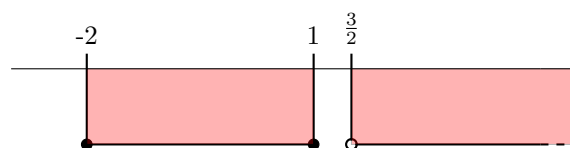


10.1 Exercises

1) $\forall a, b, x \in \mathbb{R}$, $a \leq x \leq b$



2) $\forall x \in \mathbb{R}$, $x \in]-2, -1] \cup]\frac{3}{2}, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol \cup .

11 Properties of real numbers

11.1 Property 1 - Closure under “+” and “.”

$$\forall x, y \in \mathbb{R}$$

$$x + y \in \mathbb{R}$$

$$x \cdot y \in \mathbb{R}$$

Remark: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

11.2 Property 2 - Commutativity

$$\forall x, y \in \mathbb{R}$$

$$x + y = y + x$$

$$x \cdot y = y \cdot x$$

Remark: commutativity does not hold for divisions and subtractions.

11.3 Property 3 - Associative

$$\forall x, y, z \in \mathbb{R}$$

$$x + (y + z) = (x + y) + z$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Remark: associativity does not hold for divisions and subtractions.

11.4 Property 4 - Distributive

$$\forall x, y, z \in \mathbb{R}$$

$$x(y \pm z) = xy \pm xz$$

11.5 Property 5 - Identity

$$\forall x \in \mathbb{R}$$

a) $0 + x = x$

b) $1 \cdot x = x$

Remark: $\forall x \in \mathbb{R}$, $x \cdot 0 = 0$ is not an identity property.

11.6 Property 6 - Inverses and opposites

$$\forall x \in \mathbb{R}$$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0$, $x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

Remark 1: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

Remark 2: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
() inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
 x^a ;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
“.” and “÷”;
4. Perform all additions and subtractions as you come to them, moving left-to-right:
“+” and “-”;
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

13 Signed numbers

A number is denoted as positive if it is directly preceded by a $+$ sign or no sign at all.

A number is denoted as negative if it is directly preceded by a $-$ sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

14 Absolute value

Let $x \in \mathbb{R}$, then

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

14.1 Property

$\forall x \in \mathbb{R}$

$$|x| > 0 \quad \text{if } x \neq 0$$

$$|x| = 0 \quad \text{if } x = 0$$

Part III

Lesson 3

15 Polynomials

15.1 Terms and factors

15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are a , b , x , y .

When we have an equality $y = x + a$, $\forall x \in \mathbb{R}$, x can assume any value in the set of real numbers (x is an independent variable), while y strictly depends on the value that we decide to give to x .

Notice: we can write $y = x + a$ as $y - a = x$, changing which variable is independent and which is dependent.

15.1.2 Sets

Consider the set $A = [a, b]$, where $a \leq b$. Then:

$$\boxed{\forall x \in A, a \leq x \leq b}$$

15.2 Expressions, terms and factors

15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

15.2.2 Terms

A term is any part of the expression separated by “+” or “−”.

$$\boxed{y = \underbrace{ax^2}_{\text{term}} + \underbrace{bx \cdot c}_{\text{term}}}$$

15.2.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called “factorization”.

The goal of a factorization is to factorize an expression as much as possible.

16 Common factor

Any expression made of terms is composed of several factors.

$$x^2 + x^3 + x = x(x + x^2 + 1), \forall x \in \mathbb{R}$$

17 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$ (difference of two squares);
- $(a - b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$ (square of a binomial);
- $(a - b)(a^2 + b^2 + ab) = a^3 - b^3$ (difference of two cubes);
- $(a + b)(a^2 + b^2 - ab) = a^3 + b^3$ (sum of two cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

18 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

Number of Terms	Name	Example	Comment
One	Monomial	ax^2	Mono means "one" in Greek
Two	Binomial	$ax^2 - bx$	Bi means "two" in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means "three" in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means "many" in Greek

(1)

18.1 Definition

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

18.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

18.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$$p(x) = x^2 + 1 \rightarrow \text{the degree is } 2.$$

$$\forall x \in \mathbb{R}, p(0) = 0^2 + 1 = 1 \rightarrow 1 \text{ is a polynomial with degree } 0.$$

18.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$$p(x) = x^3 + 1 + x^5 + x^2 \rightarrow \deg(p(x)) = 5$$

$$q(x) = 12 \underbrace{abcd}_{\deg=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$$

Notation: Let $f(x) = ax^2 + bx + c$, a and b are called coefficient.

The coefficient of the monomial with highest coefficient is called **leading coefficient**.

Part IV

Lesson 4

19 Operations between polynomials

19.1 Polynomials with one independent variable

The order of the monomials is not important, but it is preferable to write the highest degree monomial in decreasing order.

$$p(x) = ax^2 - bx + c$$

19.1.1 Sum

We have to sum all the same degree monomials.

$$\begin{aligned} p(x) &= x^2 + x - 1 \\ q(x) &= 5 - x + x^5 - x^2 \end{aligned}$$

$$p(x) + q(x) = x^2 + x - 1 + 5 - x + x^5 - x^2 = x^5 + 4$$

Definition: in a polynomial with one variable, monomials of same degree are called **similar terms**.

Remark: when there is a difference between polynomials, the minus MUST be distributed throughout the next monomial.

19.1.2 Multiplications

We have to multiply factors each other with the distributive property.

$$\begin{aligned} p(x) &= (x - 1) \\ q(x) &= (x^2 + 2x) \end{aligned}$$

$$p(x) \cdot q(x) = (x - 1)(x^2 + 2x) = x^3 + 2x^2 - x^2 - 2x = x^3 + x^2 - 2x = x(x^2 + x - 2)$$

19.2 Polynomials with two or more variables

19.2.1 Sum

$$\begin{aligned} p(x) &= ab + a^2b \\ q(x) &= 4ab - 3ab^2 \end{aligned}$$

$$p(x) + q(x) = ab + a^2b + 4ab - 3ab^2 = a^2b - 3ab^2 + 5ab = ab(a - b + 5)$$

Remark: $5a^3b^4 + 7a^3b^4 = 12a^3b^4$, but with $5a^3b^4 + 7a^4b^3$ we can't go further with the sum.

20 Equations

An equation is a formula given by the equality of the expressions.

Symbol notations:

- \exists = there exist(s);
- \nexists = not exists;
- $\exists!$ = it exists and it is unique;
- $:$ or $|$ = such that.

Equations are the main topic, then we have

- Identities;
- Contradictions;
- Conditional equations.

20.1 Identities

An identity is an equality that holds true regardless of the values chosen for its variables

$$\boxed{\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \mid f(x, y) = 0}$$

e.g.

- $1 = 1$;
- $x - 1 = -1 + x$;
- $\sin^2(x) + \cos^2(x) = 1$.

20.2 Contradictions

A contradiction occurs when we get a statement p , such that p is true and its negation $\sim p$ is also true.

$$\boxed{\forall x \in \mathbb{R}, \neg(\exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $0 = 1$, false;
- $x^2 = -1$ it is always positive or zero;
- $|a| = -3$ it is always positive or zero;
- $\sqrt{-(x^2 + 1)} = 1$ it is never defined (\nexists).

20.3 Conditional equations

In general, we want to find a solution for each equation, i.e. all the real number that, when they replace a variable inside the equation are giving an identity.

$$\boxed{\forall x \in \mathbb{R}, (x > 0 \Rightarrow \exists y \in \mathbb{R} \mid f(x, y) = 0)}$$

e.g.

- $x = 1$;
- $x + y = 3$;
- $\sin(\alpha) = 0.5$.

21 Fundamental theorem of algebra

Let $p(x)$ be a polynomial with one variable and real coefficients.

Assume that $\deg(p(x)) = n \in \mathbb{N}$, then:

$$\boxed{p(x) = 0 \text{ has at most } n \text{ solutions}}$$

22 Linear equations with one variable

$p(x) = q(x)$ where $\deg(0, (x)) = 1$

22.1 Simple tools

22.1.1 Tool 1

$a, b \in \mathbb{R}$, $x + a = b$, let's isolate the variable $x - a - a = b - a \Rightarrow x = b - a$

22.1.2 Tool 2

$a, b \in \mathbb{R}$, $ax = b$, let's isolate the variable $\frac{ax}{a} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

