# Maths refresher course
# HSLU, Semester 1

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## Contents

# Part I

# Lesson 1

## 1 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0)
- $\mathbb{Z} :=$ Integer numbers
- $\mathbb{Q} :=$ Rational numbers
- $\mathbb{R} :=$ Real numbers

Notation: The "$*$" symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

## 2 Prime numbers

A prime number is a number $n \in \mathbb{N} \setminus \{0, 1\}$ such that, for every divisor $d \in \mathbb{N}$, if $d \mid n$, then $d = 1$ or $d = n$.

$$\boxed{n \in \mathbb{N} \setminus \{0, 1\} \text{ is prime } \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)}$$

## 3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{R}^*$ and $a \subset \mathbb{R}$, then

$$\boxed{a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}}$$

### 3.1 Property 1

Let $a, b \in \mathbb{R},\ n, m \in \mathbb{N}$, then

$$\boxed{a^n \cdot a^m = a^{n+m}}$$

### 3.2 Property 2

Let $a, b \in \mathbb{R},\ n \in \mathbb{N}$, then

$$\boxed{(a \cdot b)^n = a^n \cdot b^n}$$

Notation: The power $a^n$, $a$ is the base and $n$ is the exponent.

### 3.3 Property 3

Let $a \in \mathbb{R},\ m, n \in \mathbb{N}^*$, then

$$\boxed{(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}}$$

# 4 Fractions

Notation 1: $a \cdot b = a \times b = ab \quad | \quad \frac{a}{b} = a \div b = a : b$

Notation 2: "$a$" is called numerator, "$b$" is called denominator.

Notation 3: $\frac{a}{b},\ a, b \in \mathbb{R},\ b \neq 0$

## 4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

## 4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

## 4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

# 5 Negative powers

## 5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

## 5.2 Property 4

Let $\forall n \in \mathbb{N},\ \forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z},\ \forall a \in \mathbb{R},\ z \neq 0$
We can compute $a^z$

## 5.3 Property 5

Let $\forall a \in \mathbb{R},\ a \neq 0,\ \forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:
   - $\forall a \in \mathbb{R},\ \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
   - $\forall b \in \mathbb{R},\ (a \cdot b)^n = a^n \cdot b^n$
   - $(a^n)^m = a^{n \cdot m}$
2. $\forall a \in \mathbb{R}^*,\ a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

# 6 Fractions and percentages (and back)

$\alpha \in \mathbb{R},\ n\%$ of $\alpha \Longleftrightarrow \frac{n}{100} \cdot \alpha$

# Part II

# Lesson 2

## 7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \rightarrow$ equality;
- $a \neq b \rightarrow$ inequality ($a$ is not equal to $b$);
- $a < b \rightarrow$ less than (a is strictly less than b);
- $a \leq b \rightarrow$ less than or equal to ($a$ is less than or equal to $b$);
- $a > b \rightarrow$ greater than ($a$ is strictly greater than $b$);
- $a \geq b \rightarrow$ greater than or equal to ($a$ is greater than or equal to $b$).

Example: $x \in \mathbb{R},\ x \geq 2 \rightarrow 2 \leq x < \infty$

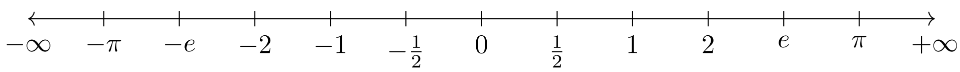
## 8 Brackets

( ) Parenthesis (round brackets)
[ ] Square brackets
{ } Braces

## 9 Latin notations

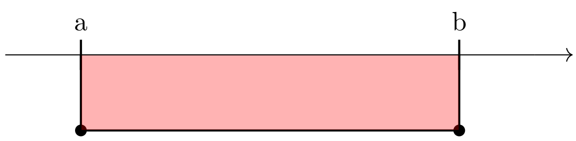
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. (□)= quod erat demonstrandum (we finally prove it).

## 10 The real line

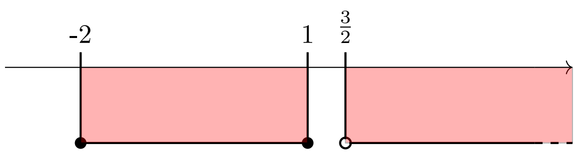



### 10.1 Exercises

1) $\forall a, b, x \in \mathbb{R},\ a \leq x \leq b$




2) $\forall x \in \mathbb{R},\ x \in\ ]-2, -1] \cup ]\frac{3}{2}, +\infty[$




Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

# 11 Properties of real numbers

## 11.1 Property 1 - Closure under "+" and "·"

$\forall x, y \in \mathbb{R}$
$x + y \in \mathbb{R}$
$x \cdot y \in \mathbb{R}$

<u>Remark</u>: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

## 11.2 Property 2 - Commutativity

$\forall x, y \in \mathbb{R}$
$x + y = y + x$
$x \cdot y = y \cdot x$

<u>Remark</u>: commutativity does not hold for divisions and subtractions.

## 11.3 Property 3 - Associative

$\forall x, y, z \in \mathbb{R}$
$x + (y + z) = (x + y) + z$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

<u>Remark</u>: associativity does not hold for divisions and subtractions.

## 11.4 Property 4 - Distributive

$\forall x, y, z \in \mathbb{R}$
$x(y \pm z) = xy \pm xz$

## 11.5 Property 5 - Identity

$\forall x \in \mathbb{R}$

a) $0 + x = x$

b) $1 \cdot x = x$

<u>Remark</u>: $\forall x \in \mathbb{R},\ x \cdot 0 = 0$ is <u>not</u> an identity property.

## 11.6 Property 6 - Inverses and opposites

$\forall x \in \mathbb{R}$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0,\ x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

<u>Remark 1</u>: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

<u>Remark 2</u>: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

# 12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
   ( ) inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
   $x^a$;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
   "·" and "÷";
4. Perform all additions and subtractions as you come to them, moving left-to-right:
   "+" and "−";
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

# 13 Signed numbers

A number is denoted as positive if it is directly preceded by a $+$ sign or no sign at all.
A number is denoted as negative if it is directly preceded by a $-$ sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

# 14 Absolute value

Let $x \in \mathbb{R}$, then

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

## 14.1 Property

$\forall x \in \mathbb{R}$
$|x| > 0 \quad \text{if } y \neq 0$
$|x| = 0 \quad \text{if } x = 0$

# Part III

# Lesson 3

## 15 Polynomials

### 15.1 Terms and factors

#### 15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are $a$, $b$, $x$, $y$.

When we have an equality $y = x + a$, $\forall x \in \mathbb{R}$, $x$ can assume any value in the set of real numbers ($x$ is an independent variable), while $y$ strictly depends on the value that we decide to give to x.

Notice: we can write $y = x + a$ as $y - a = x$, changing which variable is independent and which is dependent.

#### 15.1.2 Sets

Consider the set $A = [a, b]$, where $a \leq b$. Then:

$$\boxed{\forall x \in A,\ a \leq x \leq b}$$

### 15.2 Expressions, terms and factors

#### 15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

#### 15.2.2 Terms

A term is any part of the expression separated by "+" or "−".

$$\boxed{y = \underbrace{ax^2}_{term} + \underbrace{bx \cdot c}_{term}}$$

#### 15.2.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called "factorization".
The goal of a factorization is to factorize an expression as much as possible.

# 16 Common factor

Any expression made of terms is composed of several factors.

$$\boxed{x^2 + x^3 + x = x(x + x^2 + 1),\ \forall x \in \mathbb{R}}$$

# 17 Notable products

- $(a+b)^2 = a^2 + 2ab + b^2$ (difference of two squares);
- $(a-b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a-b)(a+b) = a^2 - b^2$ (square of a binomial);
- $(a-b)(a^2 + b^2 + ab) = a^3 - b^3$ (difference of two cubes);
- $(a+b)(a^2 + b^2 - ab) = a^3 + a^3$ (sum of two cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

# 18 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

| Number of Terms | Name | Example | Comment |
|---|---|---|---|
| One | Monomial | $ax^2$ | Mono means "one" in Greek |
| Two | Binomial | $ax^2 - bx$ | Bi means "two" in Latin |
| Three | Trinomial | $ax^2 - bx + c$ | Tri means "three" in Greek |
| Four or more | Polynomial | $ax^3 - bx^2 + cx - d$ | Poly means "many" in Greek |

(1)

## 18.1 Definition

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

## 18.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

### 18.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$p(x) = x^2 + 1 \rightarrow$ the degree is 2.
$\forall x \in \mathbb{R},\ p(0) = 0^2 + 1 = 1 \rightarrow 1$ is a polynomial with degree 0.

### 18.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$p(x) = x^3 + 1 + x^5 + x^2 1 \rightarrow \deg(p(x)) = 21$
$q(x) = 12\underbrace{abcd}_{\text{deg}=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$

Notation: Let $f(x) = ax^2 + bx + c$, $a$ and $b$ are called coefficient.
The coefficient of the monomial with highest coefficient is called **leading coefficient**.

# Part IV

# Lesson 4

## 19 Operations between polynomials

### 19.1 Polynomials with one independent variable

The order of the monomials is not important, but it is preferable to write the highest degree monomials in decreasing order.

$$\boxed{p(x) = ax^2 - bx + c}$$

#### 19.1.1 Sum

We have to sum all the monomials of the same degree.

$p(x) = x^2 + x - 1$
$q(x) = 5 - x + x^5 - x^2$

$p(x) + q(x) = x^2 + x - 1 + 5 - x + x^5 - x^2 = x^5 + 4$

Definition: in a polynomial with one variable, monomials of same degree are called **similar terms**.

Remark: when these is a difference between polynomials, the minus MUST be distributed throughout the next monomial.

#### 19.1.2 Multiplications

We have to multiply the factors with each other using the distributive property.

$p(x) = (x - 1)$
$q(x) = (x^2 + 2x)$

$p(x) \cdot q(x) = (x - 1)(x^2 + 2x) = x^3 + 2x^2 - x^2 - 2x = x^3 + x^2 - 2x = x(x^2 + x - 2)$

### 19.2 Polynomials with two or more variables

#### 19.2.1 Sum

$p(x) = ab + a^2b$
$q(x) = 4ab - 3ab^2$

$p(x) + q(x) = ab + a^2b + 4ab - 3ab^2 = a^2b - 3ab^2 + 5ab = ab(a - b + 5)$

Remark: $5a^3b^4 + 7a^3b^4 = 12a^3b^4$, but with $5a^3b^4 + 7a^4b^3$ we can't go further with the sum.

## 20 Equations

An equation is a formula given by the equality of expressions.

Symbol notations:

- $\exists$ = there exist(s);
- $\nexists$ = there does not exists;
- $\exists!$ = it exists and it is unique;
- : or $|$ = such that.

Equations are the main topic, then we have

- Identities;
- Contradictions;
- Conditional equations.

## 20.1 Identities

An identity is an equality that holds true regardless of the values chosen for its variables

$$\boxed{\forall x \in \mathbb{R},\ \exists y \in \mathbb{R} \mid f(x,y) = 0}$$

e.g.

- $1 = 1$;
- $x - 1 = -1 + x$;
- $sin^2(x) + cos^2(x) = 1$.

## 20.2 Contradictions

A contradiction occurs when we get a statement $p$, such that $p$ is true and its negation $\sim p$ is also true.

$$\boxed{\forall x \in \mathbb{R},\ \neg(\exists y \in \mathbb{R} \mid f(x,y) = 0)}$$

e.g.

- $0 = 1$, false;
- $x^2 = -1$ it is always positive or zero;
- $|a| = -3$ it is always positive or zero;
- $\sqrt{-(x^2+1)} = 1$ it is never defined ($\nexists$).

## 20.3 Conditional equations

In general, we want to find a solution for each equation, i.e. all the real number that, when they replace a variable inside the equation, give an identity.

$$\boxed{\forall x \in \mathbb{R},\ (x > 0 \Rightarrow \exists y \in \mathbb{R} \mid f(x,y) = 0)}$$

e.g.

- $x = 1$;
- $x + y = 3$;
- $sin(\alpha) = 0.5$.

# 21 Fundamental theorem of algebra

Let $p(x)$ be a polynomial with one variable and real coefficients.
Assume that $\deg(p(x)) = n \in \mathbb{N}$, then:

$$\boxed{p(x) = 0 \text{ has at most } n \text{ solutions}}$$

# 22 Linear equations with one variable

$p(x) = q(x)$ where $\deg(0,(x)) = 1$

## 22.1 Simple tools

### 22.1.1 Tool 1

$a, b \in \mathbb{R},\ x + a = b$, let's isolate the variable $x$: $x - a - a = b - a \Rightarrow x = b - a$

### 22.1.2 Tool 2

$a, b \in \mathbb{R},\ ax = b$, let's isolate the variable $x$: $\frac{ax}{a} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

# 23 Linear inequalities with one variable

The inequality is a relation between two or more sets.
Let $a, b, x \in \mathbb{R},\ a < x,\ b > x$, then:

$$\boxed{a < x < b}$$

## 23.1 Negative sign

In solving the inequality we have to move a negative factor from one side to the other, so we need to reverse the sign of the inequality:

$$\boxed{-ax < b \Rightarrow x > -\frac{b}{a}}$$

# 24 Equations and inequalities with absolute values

To solve absolute values we need to consider two cases.
Let's take this equation: $|x + a| = -x + b$, then

$$\boxed{\begin{cases} \text{case 1: } x + a = -x + b \Rightarrow 2x = b - a \Rightarrow x_1 = \frac{b-a}{2} \\ \text{case 2: } -x - a = x - b \Rightarrow -2x = a - b \Rightarrow x_2 = \frac{b-a}{2} \end{cases} \quad \Longrightarrow \text{Sol: } x \in \frac{b-a}{2}}$$