Variables and Constants

A *variable* can assume different values in different contexts, a *constant* only one.

Functions

If x, y are variables, y is called a *function of* x, if in every context in which x is constant also y is constant.

Direct proportionality

y is *directly proportional* to x, if $\frac{y}{x}$ is constant. $\frac{y}{x}$ is then called the *constant of proportionality of* y *with* x.

Linear Functions

y is a *linear function of* x, if $\frac{dy}{dx}$ is constant. $\frac{dy}{dx}$ is called the *slope* of y with x.

If m is the slope of y with x and y_0 the value of $y|_{x=x_0}$, then

$$y = y_0 + m \cdot (x - x_0) \qquad \text{(slope-point-formula)}$$

Exponential functions

y is an *exponential function of* x, if $\frac{dy}{dx}$ is directly proportional to y. The constant of proportionality is called the *continuous growth rate* of y with x. If k is the continuous growth rate of y with x and y_0 the value of $y|_{x=x_0}$, then

$$y = y_0 e^{k(x-x_0)}$$

Power functions

y is a *power function of* x, if $\frac{dy}{dx}$ is directly proportional to $\frac{y}{x}$. The constant of proportionality is called the *exponent* or the *elasticity* of y with x. If k is the exponent of y with x and y_0 the value of $y|_{x=x_0}$, then

$$y = y_0 \left(\frac{x}{x_0}\right)^k$$

Identities involving Powers

$$a^0 = 1 \qquad a^1 = a \qquad a^x a^t = a^{x+t}$$

$$a^{-x} = \frac{1}{a^x} \qquad \frac{a^x}{a^t} = a^{x-t} \qquad (a^x)^t = a^{xt}$$

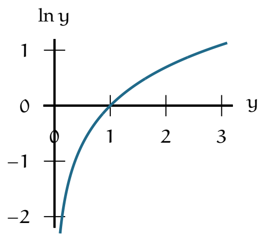
$$(ab)^x = a^x b^x$$

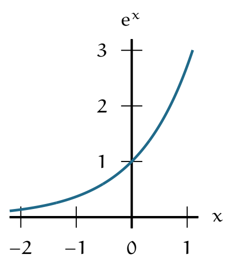
Natural Logarithm

$$y = \ln x \quad \text{means} \quad e^y = x$$

Example: $\ln 1 = 0$ since $e^0 = 1$

$$\ln e^x = x \qquad\qquad e^{\ln x} = x$$





Identities involving Logarithms

$$\ln(AB) = \ln A + \ln B \qquad \ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

$$\ln A^p = p \ln A$$

Distance and Midpoint Formulas

Distance between two points (x_1, y_1) and (x_2, y_2):

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint between (x_1, y_1) and (x_2, y_2):

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Solving quadratic equations

If $ax^2 + bx + c = 0$ then:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factoring Special Polynomials

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Binomial Formula

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \quad \text{for } n \in \mathbb{N}$$

where $\binom{n}{r}$ is the *binomial coefficient* "n chose r":

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)}{r \cdot (r-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}$$

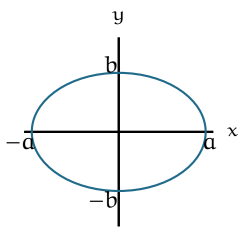
Equation of a circle in the xy-plane

Center $(x, y) = (x_0, y_0)$ and radius r:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

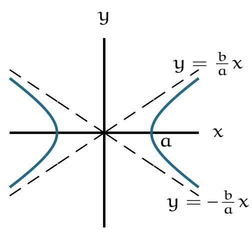
Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Geometric Formulas

Triangle

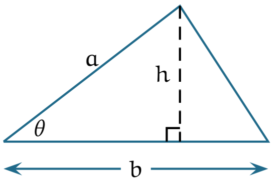
$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$

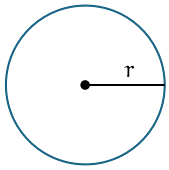
Circle

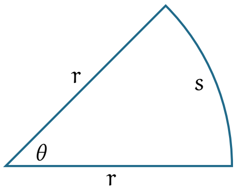
$$\text{Area} = \pi r^2$$
$$\text{Circumference} = 2\pi r$$

Sector of Circle

$$\text{Area} = \frac{1}{2}r^2\theta$$
$$s = r\theta$$







Sphere

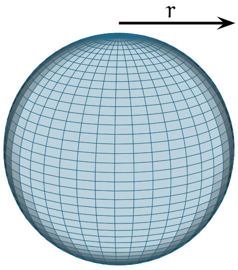
$$\text{Volume} = \frac{4}{3}\pi r^3$$
$$\text{Surface Area} = 4\pi r^2$$

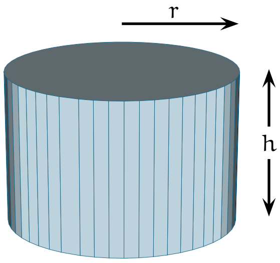
Cylinder

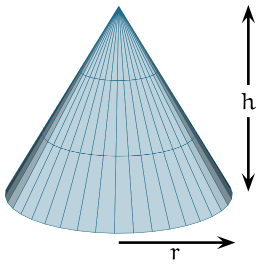
$$\text{Volume} = \pi r^2 h$$

Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$



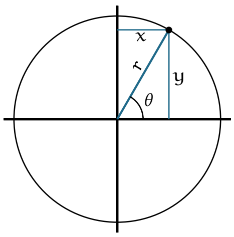




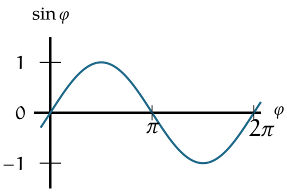
Trigonometric functions

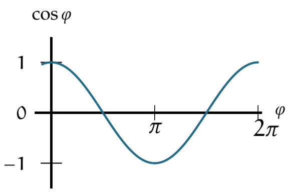
Relation Radians and Degrees: $\pi = 180^\circ$

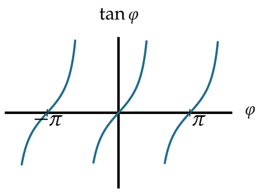
$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$



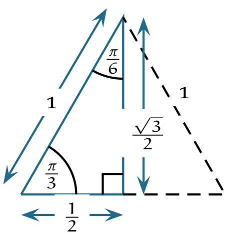
$$(\sin A)^2 + (\cos A)^2 = 1$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\sin(2A) = 2\sin A\cos A$$
$$\cos(2A) = 2(\cos A)^2 - 1 = 1 - 2(\sin A)^2$$

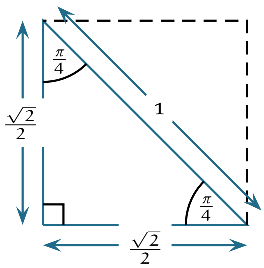






Values of $\sin\varphi$, $\cos\varphi$ and $\tan\varphi$ for special values of φ:





Rules of Differentiation

In the following y, u and v are functions of x.

1. $\frac{dk}{dx} = 0 \quad (k \text{ constant})$
2. $\frac{d(k\cdot y)}{dx} = k \cdot \frac{dy}{dx} \quad (k \text{ constant})$
3. $\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
4. $\frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$
5. $\frac{d(u/v)}{dx} = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$
6. $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ (If z is a function of y and y a function of x)
7. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ (If y is an invertible function of x)
8. $\frac{d}{dx}(x^k) = k \cdot x^{k-1} \quad (k \text{ constant})$
9. $\frac{d(e^{kx})}{dx} = ke^x \quad (k \text{ constant})$
10. $\frac{d(\ln x)}{dx} = \frac{1}{x}$
11. $\frac{d(\sin x)}{dx}(\sin x) = \cos x$
12. $\frac{d(\cos x)}{dx} = -\sin x$
13. $\frac{d(\tan x)}{dx} = \frac{1}{\cos^2 x}$
14. $\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$
16. $\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$

Fundamental Theorems of Differential and Integral Calculus

I. $$\int_{x=a}^{b} \frac{dy}{dx}\, dx = y|_{x=b} - y|_{x=a}$$

II. $$\frac{d}{db} \int_{x=a}^{b} y\, dx = y|_{x=b} \quad (a \text{ constant})$$

Rules of Integration

1. $$\int_{x=a}^{b} y\, dx = -\int_{x=b}^{a} y\, dx$$
2. $$\int_{x=a}^{c} y\, dx = \int_{x=a}^{b} y\, dx + \int_{x=b}^{c} y\, dx$$
3. $$\int_{x=a}^{b} (u \pm v)\, dx = \int_{x=a}^{b} u\, dx \pm \int_{x=a}^{b} v\, dx$$
4. $$\int_{x=a}^{b} k \cdot y\, dx = k \int_{x=a}^{b} y\, dx \quad (k \text{ constant})$$
5. For y a function of w and w a function of x:
$$\int_{x=a}^{b} y \frac{dw}{dx}\, dx = \int_{w=w|_{x=a}}^{w|_{x=b}} y\, dw$$
6. $$\int_{x=a}^{b} u \frac{dv}{dx}\, dx = (uv)\Big|_{x=a}^{b} - \int_{x=a}^{b} \frac{du}{dx} v\, dx$$

Table of indefinite integrals

I. Basic Functions

1. $\int x^k\, dx = \frac{1}{k+1} x^{k+1} + C, \ (k \neq -1 \text{ constant})$
2. $\int \frac{1}{x}\, dx = \ln|x| + C$
3. $\int e^{kx}\, dx = \frac{1}{k} e^{kx} + C, \quad (k \text{ constant})$
4. $\int (\ln x)\, dx = x(\ln x) - x + C$
5. $\int (\sin x)\, dx = -\cos x + C$
6. $\int (\cos x)\, dx = \sin x + C$
7. $\int (\tan x)\, dx = -\ln|\cos x| + C$

II. Products of e^x, $\cos x$ and $\sin x$

8. $\int e^{ax} \sin(bx)\, dx = \frac{1}{a^2+b^2} e^{ax} (a \sin(bx) - b\cos(bx)) + C$
9. $\int e^{ax} \cos(bx)\, dx = \frac{1}{a^2+b^2} e^{ax} (a \cos(bx) + b\sin(bx)) + C$

10. $\displaystyle\int \sin(ax)\sin(bx)\ dx = \frac{1}{b^2-a^2}\left(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)\right) + C,\ a \neq b$

11. $\displaystyle\int \cos(ax)\cos(bx)\ dx = \frac{1}{b^2-a^2}\left(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)\right) + C,\ a \neq b$

12. $\displaystyle\int \sin(ax)\cos(bx)\ dx = \frac{1}{b^2-a^2}\left(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)\right) + C,\ a \neq b$

III. Product of Polynomial $p(x)$ with $\ln x$, e^x, $\cos x$, $\sin x$

13. $\displaystyle\int x^n \ln x\ dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C,\ n \neq -1$

14. $\displaystyle\int p(x)e^{ax}\ dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a}\int p'(x)e^{ax}\ dx = \frac{1}{a}p(x)e^{ax} - \frac{1}{a^2}p'(x)e^{ax} + \frac{1}{a^3}p''(x)e^{ax} - \ldots$

    (*Signs alternate* $+-+-+-\ldots$)

15. $\displaystyle\int p(x)\sin(ax)\ dx = -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a}\int p'(x)\cos(ax)\ dx$

    $\displaystyle = -\tfrac{1}{a}p(x)\cos(ax) + \tfrac{1}{a^2}p'(x)\sin(ax) + \tfrac{1}{a^3}p''(x)\cos(ax) - \ldots$

    (*Signs alternate* $-++--++--\ldots$)

16. $\displaystyle\int p(x)\cos(ax)\ dx = \frac{1}{a}p(x)\sin(ax) - \frac{1}{a}\int p'(x)\sin(ax)\ dx$

    $\displaystyle = \tfrac{1}{a}p(x)\sin(ax) + \tfrac{1}{a^2}p'(x)\cos(ax) - \tfrac{1}{a^3}p''(x)\sin(ax) - \ldots$

    (*Signs alternate* $++--++--\ldots$)

IV. Integer powers of $\sin x$ and $\cos x$

17. $\displaystyle\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\ dx,$ n positive

18. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\ dx,$ n positive

19. $\displaystyle\int \frac{1}{\sin^m x}\ dx = \frac{-1}{m-1}\frac{\cos x}{\sin^{m-1}x} + \frac{m-2}{m-1}\int \frac{1}{\sin^{m-2}x}\ dx,\ m \neq 1,$ m positive

20. $\displaystyle\int \frac{1}{\sin x}\ dx = \frac{1}{2}\ln\left|\frac{(\cos x)-1}{(\cos x)+1}\right| + C$

21. $\displaystyle\int \frac{1}{\cos^m x}\ dx = \frac{1}{m-1}\frac{\sin x}{\cos^{m-1}x} + \frac{m-2}{m-1}\int \frac{1}{\cos^{m-2}x}\ dx,\ m \neq 1,$ m positive

22. $\displaystyle\int \frac{1}{\cos x}\ dx = \frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C$

23. $\displaystyle\int \sin^m x\cos^n x\ dx$:

    - If m is odd, let $w = \cos x$
    - If n is odd, let $w = \sin x$.
    - If both m and n are even and positive, convert all to $\sin x$ or all to $\cos x$ (using $\sin^2 x + \cos^2 x = 1$), and use IV-17 or IV-18.
    - If m and n are even and one of them is negative, convert to whichever function is in the denominator and use IV-19 or IV-21.
    - If both m and n are even and negative, substitute $w = \tan x$, which converts the integrand to a rational function that can be integrated by the method of partial fractions.

V. Rational Functions with Squares in the Denominator

24. $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C,\ a \neq 0$

25. $\displaystyle\int \frac{bx+c}{x^2+a^2}\,dx = \frac{b}{2}\ln\left|x^2+a^2\right| + \frac{c}{a}\arctan\frac{x}{a} + C,\ a \neq 0$

26. $\displaystyle\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\left(\ln|x-a| - \ln|x-b|\right) + C,\ a \neq b$

27. $\displaystyle\int \frac{cx+d}{(x-a)(x-b)}\,dx = \frac{1}{a-b}\left((ac+d)\ln|x-a| - (bc+d)\ln|x-b|\right) + C,\ a \neq b$

VI. Integrals containing $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $a>0$

28. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$

29. $\displaystyle\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C$

30. $\displaystyle\int \sqrt{a^2\pm x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2\pm x^2} + a^2\int\frac{1}{\sqrt{a^2\pm x^2}}\,dx\right) + C$

31. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx\right) + C$

Some functions of x, that have no elementary antiderivative

- e^{-x^2}
- $\dfrac{1}{\ln x}$
- $\ln(\ln x)$
- $\sin(x^2)$
- $\frac{\sin x}{x}$
- $\sqrt{1-x^4}$
- $\frac{e^x}{x}$
- $e^{(e^x)}$
- $\cos(x^2)$
- $\frac{\sin x}{\sqrt{x}}$

Taylor-Polynomials/Series

Expanding a quantity y in terms of powers of x, assuming it is a function of x that is infinitely often differentiable at $x=0$. (Taylor-Expansion)

$$y \approx \sum_{k=0}^{n} \frac{1}{k!}\frac{dk}{dy}x\bigg|_{x=0} \cdot x^k \quad \text{when } x \approx 0$$

where $k!$ is the *factorial* of $k \in \mathbb{N}$ defined by

$$k! = \begin{cases} 1 & \text{when } k=0 \\ k\cdot(k-1)! & \text{when } k>0 \end{cases}$$

Important expansions in terms of powers of x

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \qquad \text{(Geometric Series)}$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots \qquad \text{(Binomial Series)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

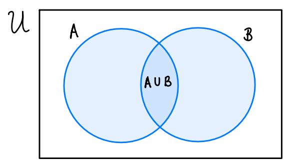
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The last four series only converge for $|x| < 1$.

Logic

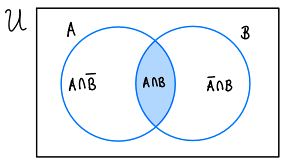
Union [∪]

$A \cup B = \{\forall x \in \mathcal{U} \mid x \in A \vee x \in B\}$



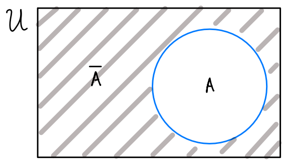
Intersection [∩]

$A \cap B = \{\forall x \in \mathcal{U} \mid x \in A \wedge x \in B\}$



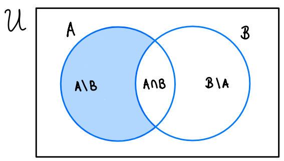
Complement [$\bar{A}$]

$\bar{A} = \{\forall x \in \mathcal{U} \mid x \notin A\}$



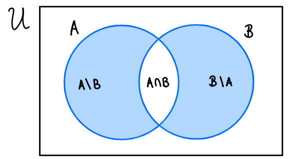
Difference between sets [\]

$A \setminus B = \{\forall x \in \mathcal{U} \mid x \in A, x \notin B\}$



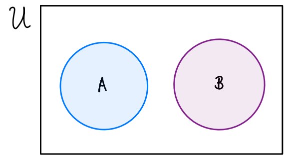
Symmetrical difference [Δ]

$A \Delta B = (A \setminus B) \cup (B \setminus A)$



Disjoined sets [$\emptyset$]

$A \cap B = \emptyset$



Algebra

Absolute values properties

Let $a, b \in \mathbb{R}$:

- $|ab| = |a| \cdot |b|$
- $|a/b| = |a| / |b|$ for $b \neq 0$
- ! $|a \pm b| \neq |a| \pm |b|$

Triangual inequalities

- $|a| + |b| \geq |a + b|$
- $|a| - |b| \leq |a - b|$

Concept of functions

Let's take any two sets A and B

$f: A \Rightarrow B$
$a \mapsto f(a)$

$$\boxed{\forall x \in A \ \exists! \, y \in B \mid (x, y) \in f}$$

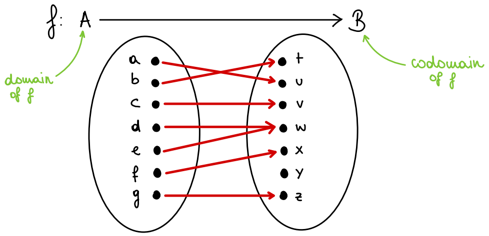


Image (Range)

Let $f: X \Rightarrow Y$ be a function:

$Im(f) = \{y \in Y \mid y = f(x), x \in X\}$

Linear functions

Slope-intercept equation: $y = mx + q$

m = slope ; q = vertical intercept

Slope

$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

$m > 0 \rightarrow$ line is increasing
$m = 0 \rightarrow$ line is stable
$m < 0 \rightarrow$ line is decreasing

Equation of a line

Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$:

$$\boxed{y - y_A = m(x - x_A)}$$

General equation of deg = 1

$$\boxed{ax + bx + c = 0}$$

Inverse function

$f: A \Rightarrow B$; $f^{-1}: B \Rightarrow A$

$$\boxed{\begin{array}{l} \forall x \in A, f^{-1}(f(x)) = x \\ \forall y \in A, f(f^{-1}(y)) = y \end{array}}$$

Good to know:

1) Let $f: D_f \Rightarrow \mathbb{R}$; f is invertible in D when:
   - f is strictly increasing
   - f is strictly decreasing
2) Let $f: D_f \Rightarrow \mathbb{R}$; f is invertible in D when:
   - $Im(f) \Rightarrow D_f$

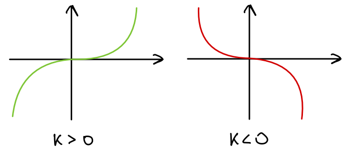
Symmetrical functions

Let $y = kx^n$

when n is odd

$f(-x) = -f(x), \ \forall x \in \mathbb{R}$

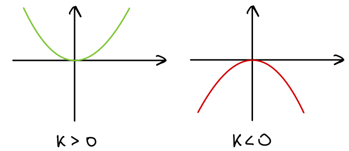
Plot:



when n is even

$f(-x) = f(x), \ \forall x \in \mathbb{R}$

Plot:



general case

Let $y = p(x)$

$$p(x) = \sum_{i=0}^{n} a_n \cdot x^n$$

where:
- $n = deg(p(x))$, $n \in \mathbb{N}$
- a_n = leading coefficient, $a_n \neq 0$

In a nutshell

- If $\forall n$ are odd, then $y = p(x)$ is odd
- If $\forall n$ are even, then $y = p(x)$ is even
- If $\forall n$ are both odd and even, then $y = p(x)$ is neither odd nor even

Dominant elements in function $\rightarrow \infty$

$\frac{p(x)}{q(x)} \rightarrow$
- $deg(p(x)) > deg(q(x))$, $p(x)$ is dominant
- $deg(p(x)) < deg(q(x))$, $q(x)$ is dominant
- $deg(p(x)) = deg(q(x))$, both have the same dominance

Dominance in rational functions

- $\lim_{x\to\infty} \frac{x^n}{x^{n-1}} = \infty$
- $\lim_{x\to\infty} \frac{x^{n-1}}{x^n} = 0$
- $\lim_{x\to\infty} \frac{ax^n}{bx^n} = \frac{a}{b}$

Exponential & logarithm functions

$$\boxed{a^{\log_a(x)} = x \iff \log_a(a^x) = x}$$

Exponentials

Let $\alpha \in \mathbb{R}^*_+$, $x \in \mathbb{R}$ and $a>1$:

$$y = \alpha \cdot a^x$$

Euler number

$$e = \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2{,}718\ldots$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Logarithms

with base e:

$$f(x) = e^x \iff f^{-1}(y) = \ln(x)$$

↳Tip: $Df^{-1}: \forall x \in \mathbb{R}^*$

with arbitrary bases:

$$f(x) = n^x \iff f^{-1}(y) = \log_n(x)$$

↳Tip: $\log_n(x) = \frac{\ln(x)}{\ln(n)}$

Exponential growth

$$\boxed{N(t) = N_0 e^{kt}}$$

Trigonometry

Degree – Radians conversion table

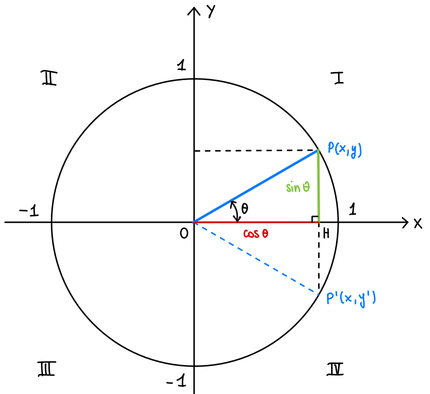
Angle (°)	0	30	45	60	90	180	270	360
Angle (c)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0	1
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0	∞	0

Remark:

Let $\forall k \in \mathbb{Z}, \forall \theta \in \mathbb{R}$:

$$\cos(\theta + 2\pi k) = \cos(\theta)$$
$$\sin(\theta + 2\pi k) = \sin(\theta)$$

Trigonometric functions in the unit circle



for the unit circle:

$\sin\theta := y$; $\cos\theta := x$; $\tan\theta := \frac{y}{x}$

Property 1

- $-1 \le \cos\theta \le 1$
- $-1 \le \sin\theta \le 1$

Property 2

- $\vec{OH}^2 + \vec{PH}^2 = \vec{OP}^2$
- $\sin^2\theta + \cos^2\theta = 1$

Tangent

$$m = \frac{\Delta y}{\Delta x} = \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \theta \neq \{90°, 270°\}$$

Domain of trigonometric functions

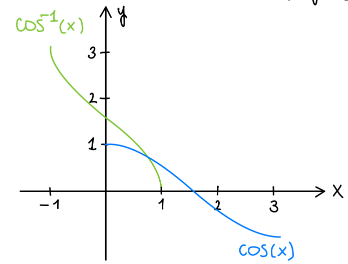
$y = \cos(x)$, $x^c \in \mathbb{R}$

$y = \sin(x)$, $x^c \in \mathbb{R}$

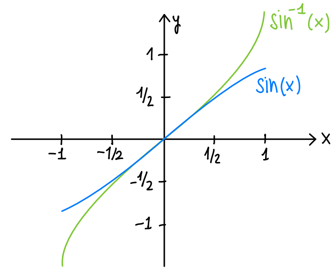
$y = \tan(x)$, $x^c \in \mathbb{R}\setminus\{\pi/2 + k\pi \mid k \in \mathbb{Z}\}$

Inverse trigonometric functions

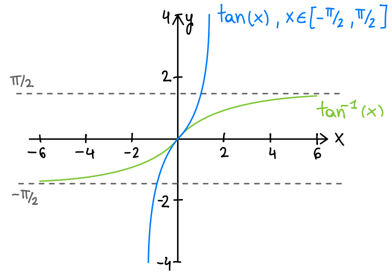
Arccosine: $Df: x \in [-1,1]$; $Imf: y \in [\pi, 0]$



Arcsine: $Df: x \in [-1,1]$; $Imf: y \in [-\pi/2, \pi/2]$



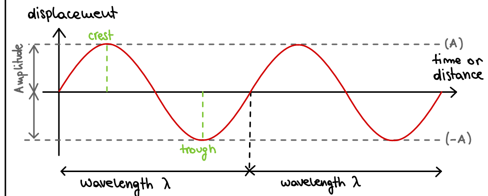
Arctan: $Df: x \in \mathbb{R}$; $Imf: x \in [-\pi/2, \pi/2]$



Harmonic oscillation

Let $A, B > 0$, then the function is oscillating harmonically with t around D

$$\boxed{y = D + A\sin(Bt + \varphi)}$$

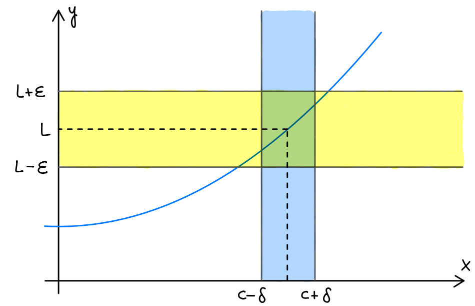


Limits

Definition

Let $f: D \to \mathbb{R}$ be a function and "c" a point, the limit $L = \lim_{x\to c} f(x)$:

$$\boxed{0 < |x-c| < \delta \implies |f(x) - L| < \varepsilon}$$



Accumulation point

Let $A \subset \mathbb{R}$, a is an accumulation point of A if $\forall \delta > 0$, $(a-\delta, a+\delta) \cap A$ contains infinitely many points

Limit value at a finite point

Let $f: Df \to \mathbb{R}$, x_0 an accumulation point of Df, and $L \in \mathbb{R} \cup \{\pm\infty\}$, then:

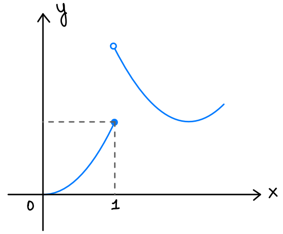
$$\boxed{\lim_{x\to x_0} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0; 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon}$$

One-sided limits

Right-sided: $\lim_{x\to x_0^+} f(x)$

Left-sided: $\lim_{x\to x_0^-} f(x)$

Graph example:



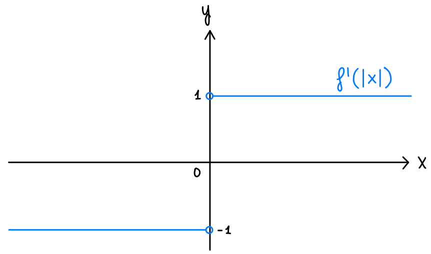
At point $x=1$, the limit does not exists because $\lim_{x\to 1^-} \neq \lim_{x\to 1^+}$

Continuity of a function

Given a real function $f: D\to C$, the function is continuous at the point $x=c$ where $c\in D$ if:

$$\lim_{x\to c} f(x) = f(c)$$

and therefore, if the limit exists and is equal to the value of the function at that point



In a nutshell

- A function $f(x)$ is said to be continuous at point $x=c$ if $\lim_{x\to c} f(x)=f(c)$
- A function $f(x)$ is said to be continuous on a subinterval $[a,b]$ of its domain if it is continuous at all points $x\in[a,b]$
- A function $f(x)$ is said to be continuous on an interval I if it is continuous at all the points $x\in I$

Derivatives

Derivative definition

If the limit exists:

$$\boxed{f'(x)=\lim_{\Delta x\to 0^\pm}\frac{f(x+\Delta x)-f(x)}{\Delta x}}$$

Rules:

Exponential rules

Let $\forall\alpha\in\mathbb{R}$:

$$f(x)=x^\alpha \Rightarrow f'(x)=\alpha x^{\alpha-1}$$

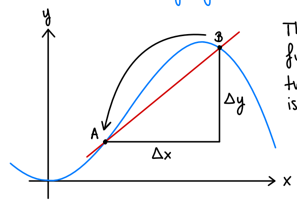
Existance of the derivative

$$\lim_{\Delta x\to 0^+}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Remark:

If a function is differentiable, then it's continuous

Geometric meaning of the derivative



The secant of a function $f(x)$ between two points A and B is given by:

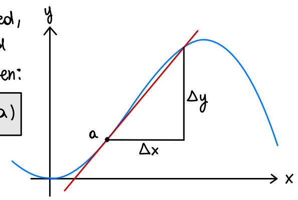
$$\boxed{\frac{\Delta y}{\Delta x}=\frac{f(B)-f(A)}{B-A}}$$

When the Δx of the slope becomes infinitely small:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

When $f'(x)$ is defined, $p=(x,f(x))$, and $p\in$ tangent line, then:

$$\boxed{y-f(a)=f'(a)(x-a)}$$



Bernoulli-de L'Hôpital theorem

If $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ results in an indeterminate form, then:

$$\boxed{\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}}$$

Derivation rules

Linearity Let $c\in\mathbb{R}$, then:

$$\frac{d}{dx}(cf(x))=cf'(x)$$

Sum and subtraction

$$\frac{d}{dx}(f(x)\pm g(x))=f'(x)\pm g'(x)$$

Multiplication

$$\frac{d}{dx}(f(x)\cdot g(x))=f'(x)g(x)+f(x)g'(x)$$

Quotient

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Exponential Let $a>0$, then:

$$\frac{d}{dx}(a^x)=a^x\ln(a)$$

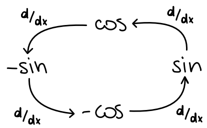
Composite function (chain rule)

$$\frac{d}{dx}(f(g(x)))=f'(g(x))\cdot g'(x)$$

Inverse function

$$\frac{d}{dx}(f^{-1}(x))=\frac{1}{f'(f^{-1}(x))}$$

Trigonometric functions



• $\sec'x=\sec x\tan x$	• $\csc'x=-\csc x\cot x$
• $\tan'x=\frac{1}{\csc^2 x}$	• $\cot'x=\frac{-1}{\sin^2 x}$
• $\arcsin'x=\frac{1}{\sqrt{1-x^2}}$	• $\arccos'x=\frac{-1}{\sqrt{1-x^2}}$
• $\arctan'x=\frac{1}{1+x^2}$	• $\text{arccot}'x=\frac{-1}{1+x^2}$

Particular cases

- $f(x)=g(x)^\alpha \Rightarrow f'(x)=\alpha\cdot g'(x)\cdot g(x)^{\alpha-1}$
- $f(x)=e^{g(x)} \Rightarrow f'(x)=g'(x)\cdot e^{g(x)}$
- $f(x)=\frac{1}{g(x)} \Rightarrow f'(x)=\frac{-g'(x)}{(g(x))^2}$
- $\frac{d}{dx}(\ln(\pm f(x)))=\frac{f'(x)}{f(x)}$

Average acceleration a_{av}

$$a_{av}:=\frac{v(t_f)-v(t_i)}{t_f-t_i}$$

Instant acceleration $a(t)$

$$a(t):=v'(t)=\lim_{t\to 0}\frac{v(t+h)-v(t)}{\Delta t}$$

Linearization

Tangent line approximation

$$f(x)=f(a)+f'(a)(x-a)$$

Error function

$$E(x)=d(f(x)|f_{lin}(x))$$

$$E(x)=f(x)-f(a)+f'(a)(x-a)$$

Monotonicity

Definitions:

- strictly increasing: $f(x_2)>f(x_1)$ for $x_2>x_1$
- increasing: $f(x_2)\geq f(x_1)$ for $x_2>x_1$
- strictly decreasing: $f(x_2)<f(x_1)$ for $x_2>x_1$
- decreasing: $f(x_2)\leq f(x_1)$ for $x_2>x_1$

Criterion:

Let f be differentiable on I:

- $f'>0 \Rightarrow$ strictly increasing
- $f'\geq 0 \Rightarrow$ increasing
- $f'<0 \Rightarrow$ strictly decreasing
- $f'\leq 0 \Rightarrow$ decreasing
- $f'=0 \Rightarrow$ constant

Critical point

Let $y_f(x)$ be a function, $x\in D_f$, then:

x is a critical point if $f'(x)=0$ or $\underbrace{f'(x)\uparrow}_{\uparrow\text{ means undefined}}$

Monotonicity table

Let $f(x)$ be differentiable, $f'(x)<0$ if $a<x<b$, $a,b,c \in D_f$ and a,b,c are critical points, then:

$f'(x)$	+	−	+	−
$f(x)$	↗	↘	↗	↘
	a	b	c	

Darboux theorem

Let f be differentiable on I, then follow this steps:

1) find the critical points $f'(a,b)=0$, $a<b$;
2) take an arbitrary point between the critical points in $c \mid a<c<b$;

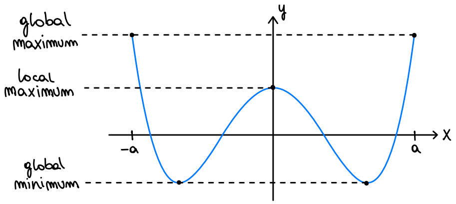
(!) 3) compute $f'(c)$.

NOT $f(c)$

- If $f'(c)>0$, then $f'(x)>0$, $\forall x \in (a,b)$
- If $f'(c)<0$, then $f'(x)<0$, $\forall x \in (a,b)$

Minimum and maximum

Let a real function f and a point $x_0 \in D_f$ be given:



Local maximum

The function f has a local maximum at point x_0 if there is an open neighborhood $U(x_0)$ s.t.:

$$f(x) \le f(x_0), \forall x \in U(x_0) \cap D_f$$

Local minimum

$$f(x) \ge f(x_0), \forall x \in U(x_0) \cap D_f$$

Global maximum

The function f has a global maximum at point x_0 if:

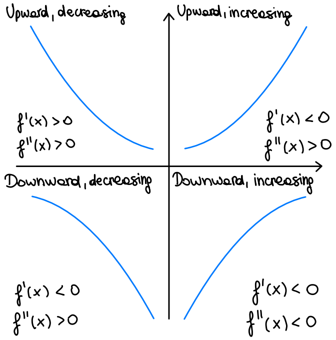
$$f(x) \le f(x_0), \forall x \in D_f$$

Global minimum

$$f(x) \ge f(x_0), \forall x \in D_f$$

Concavity

- $f(x)$ is concave up on I if:
  - Tangents lie below the graph;
  - $f''(x)>0$, $\forall x \in I$
- $f(x)$ is concave down on I if:
  - Tangents lie above the graph;
  - $f''(x)<0$, $\forall x \in I$

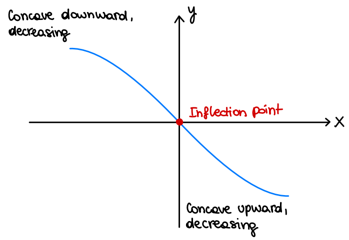


Inflection point

An inflection point for $y=f(x)$ is a point $x_0 \in D_f$ where the function is continuous and its concavity changes.

For any inflection point, $f''(x)=0$

↳ $f''(x_0)=0 \not\Rightarrow x_0$ is not an inflection point



Curvature of a function

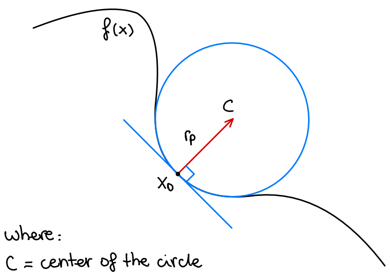
Let $y=f(x)$ be a derivable function in a point x_0, then:

$$\boxed{k = \frac{f''(x_0)}{(1+(f'(x_0))^2)^{3/2}}}$$

Radius of curvature

$$\boxed{r_p = \frac{1}{|k|}}$$

Graphical example

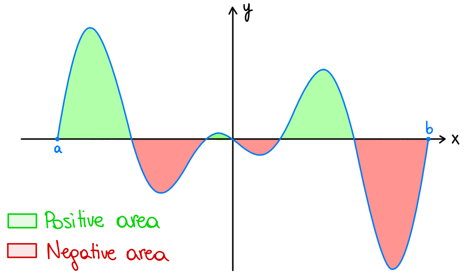


where:

- C = center of the circle
- r_p = radius of curvature
- X_0 = specific point

Integrals

Definite integral cases



Let $f(x)$ be a continuous function in $\mathbb{R} \to \mathbb{R}^+$, then we have 3 possible cases:

First case: when $a<b$: $\int_a^b f(x)\,dx = F(x)\Big|_a^b + C$

Second case: when $a>b$: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Third case: when $a=b$, $\forall a \in \mathbb{R}$: $\int_a^b f(x)\,dx = 0$

Riemann Sum

Let $f:[a,b] \to \mathbb{R}$:

$$\boxed{R_n := \sum_{i=0}^{n-1} f(x_i)\cdot \Delta x}$$

with:

- $\Delta x = \frac{b-a}{n}$, $n \in \mathbb{N}$
- $x_i = a + i \cdot \Delta x$

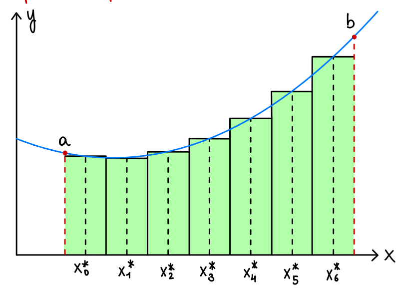
Statement of the theorem

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} R_n$$

Sigma notation

$$\boxed{R_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(1+\frac{i}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n}\left(1+\frac{i}{n}\right)^2}$$

Graphic interpretation



Integration rules

Linearity: Let $\lambda \in \mathbb{R}$, then:

$$\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$$

Sum and Subtraction:

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

With infinite bounds

When upper bound is $+\infty$

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

When lower bound is $-\infty$

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

When bounds are ∞

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx + \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

Indefinite integral

We denote the set of all antiderivatives of a function f as an indefinitive integral of f:

$$\boxed{\int f(x)\,dx}$$

Fundamental theorem of calculus

Let $F:[a,b]\to\mathbb{R}$ a continuous function, differentiable in (a,b).

Let $f(x)=F'(x)$ and $\forall C\in\mathbb{R}$, then:

$$\boxed{\int_a^b f'(x)\,dx = f(b)-f(a)}$$

Remark:

Since $f(x)=F'(x)$, we have infinite possibile primitives, which are distinguished by "$+C$"

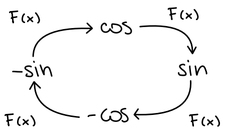
Second fundamental theorem of differential and integer calculus

If $f:[a,b]\to\mathbb{R}$ is continuous, and $x_0\in[a,b]$, then $\forall x\in[a,b]$:

$$\boxed{F_0(x)=\int_x^{x_0} f(t)\,dt}$$

Special cases

Trigonometric functions



$\int \sec x\,dx = \ln\lvert\sec x+\tan x\rvert + C$	$\int \csc x\,dx = -\ln\lvert\csc x+\cot x\rvert + C$
$\int \tan x\,dx = -\ln\lvert\cos x\rvert + C$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2}+C$	
$\int \arccos x\,dx = x\arccos x + \sqrt{1-x^2}+C$	
$\int \arctan x\,dx = x\arctan x + \frac{1}{2}\ln(1+x^2)+C$	
$\int \operatorname{arccot} x\,dx = x\operatorname{arccot} x + \frac{1}{2}\ln(1+x^2)+C$	

Other rules

- $\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}+C,\quad \alpha\neq 1$
- $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Integration with substitution rule

According to the chain rule:

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\cdot g'(x) = f(g(x))\,g'(x)$$

If $g'(x)$ is continuous:

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Now let:

$$u=g(x) \Rightarrow du = g'(x)\,dx$$

With this we obtain:

$$\boxed{\int f(g(x))\,g'(x)\,dx = \int f(u)\,du}$$

For definite integrals

$$\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

Integration by parts

According to the product rule:

$$(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)$$

By integrating both sides:

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Rearranging the formula:

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

For definite integrals:

$$\boxed{\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx}$$

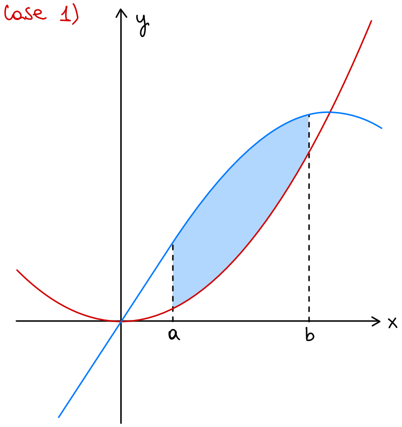
Particular cases

- $\int p'(x)\,e^{p(x)}\,dx = e^{p(x)}+C$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$

Area calculation

Between two functions

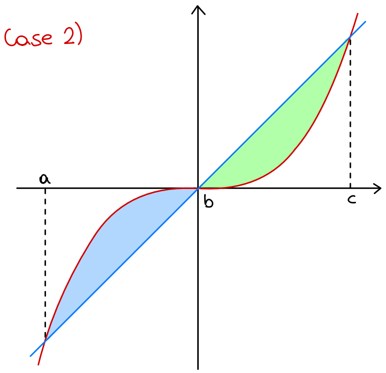
Case 1)



Given a function $y=f(x)$ and $y=g(x)$, the area inclosed by the two functions on $I=[a;b]$ is:

$$\boxed{A=\int_a^b f(x)-g(x)\,dx}$$

Case 2)



$$\boxed{A=\int_a^b f(x)-g(x)\,dx + \int_b^c g(x)-f(x)\,dx}$$

Arclength

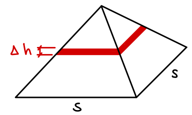
Let a plane curve be defined by a differentiable function $f(x)$ whose derivative $f'(x)$ is continuous on $I:[a,b]\to\mathbb{R}$.

Then, the arc length between the points $(a,f(a))$ and $(b,f(b))$ is:

$$\boxed{S=\int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

$![f'(x)]^2!$

Volume calculation on pyramids



Computing the pyramid for n-slices:

$$\boxed{V=\sum_{i=0}^{n-1} s_i^2\,\Delta h = \int_a^b s^2\,dh}$$

Since we don't know how to solve an integral in two variables yet:

$$\boxed{h:s = \text{proportion of }(h:s) \Rightarrow s = h\cdot(h:s)}$$

With this step:

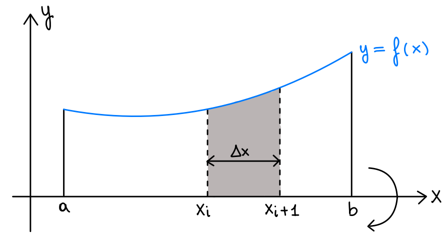
$$\boxed{V=\int_{h(a)}^{h(b)} (h\cdot(h:s))^2\cdot dh}$$

It is always better to keep dh as unique variable

Revolution around the x-axis

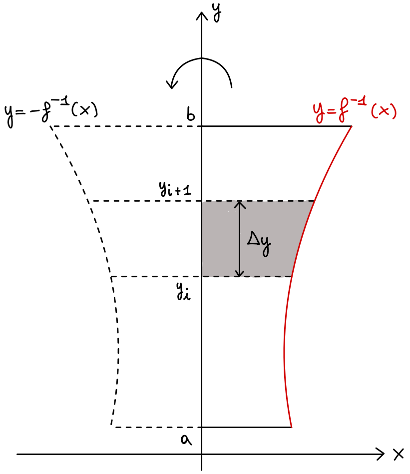
Let a plane curve be given by $y=f(x)$ with f continuous in the interval $[a,b]$. If the curve rotates around the x-axis, his volume is:

$$\boxed{V=\pi\int_a^b [f(x)]^2\,dx}$$



Revolution around the y-axis

$$\boxed{V = \pi \int_{f(a)}^{f(b)} [f^{-1}(x)]^2 \, dx}$$



Center of gravity

Surface of a homogeneous plane

Consider a surface in the xy-plane, bounded above by a continuous curve $y=f(x)$ and below by another continuous curve $y=g(x)$, on $I:[a,b]$.

First, we compute the area between the two functions:

$$\boxed{A = \int_a^b f(x) - g(x) \, dx}$$

The coordinates of its center of gravity $S=(S_x, S_y)$ are then defined by:

$$x \quad S_x = \frac{1}{A} \int_a^b x(f(x) - g(x)) \, dx$$

$$y \quad S_y = \frac{1}{A} \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$$

Mass point

For one surface

The total center of gravity of two homogeneous surfaces $S_1(x_1, y_1)$ and $S_2(x_2, y_2)$ and areas A_1, A_2 is $S=(x,y)$, where x and y are:

$$x = \frac{x_1 A_1 + x_2 A_2}{A_1 + A_2}$$

$$y = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2}$$

For n-surfaces

$$x = \frac{x_1 A_1 + \ldots + x_n A_n}{A_1 + \ldots + A_n} = \frac{\sum_{i=0}^{n} x_i A_i}{\sum_{i=0}^{n} A_i}$$

$$y = \frac{y_1 A_1 + \ldots + y_n A_n}{A_1 + \ldots + A_n} = \frac{\sum_{i=0}^{n} y_i A_i}{\sum_{i=0}^{n} A_i}$$

Leibniz integral rule

Let $f(t)$ be a continuous function, and let $a(z), b(z)$ be differentiable functions of z. To compute its derivative:

$$\boxed{\frac{d}{dz} \int_{a(z)}^{b(z)} f(t) \, dt = f(b(z)) \cdot b'(z) - f(a(z)) \cdot a'(z)}$$