Mathematics 1A
HSLU, Semester 1

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Contents

Part I

Week 1

1 The set theory

1.1 Definition of a set

A set is a collection of objects or elements.

Remark: The collection of all sets is not a set.

1.2 Logical symbols

1.2.1 Definition

Braces and the definition symbol ":=" are used to define a set giving all its elements:

$$\boxed{A := \{a, b, c, d, e\}}$$

1.2.2 Equal

In this case, the equal symbol means that the set A is equal to the set B:

$$\boxed{A = B}$$

1.2.3 Belongs to

The symbols $\in$ and $\ni$ describe an element which is part of the set:

$$\boxed{a \in A \iff A \ni a}$$

1.2.4 Does not belong to

The symbols $\notin$ mean that an element does not belong to the set:

$$\boxed{f \notin A}$$

1.2.5 Inclusion and contains

The symbols $\subset$ and $\supset$ mean that a set has another set included in its set:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \iff \mathbb{Z} \supset \mathbb{N}}$$

1.2.6 For all/any

The symbol $\forall$ means that we are considering any type of element:

$$\boxed{\forall x \in \mathbb{R},\ x > 0}$$

In this case, we've defined a new set.

1.3 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0);
- $\mathbb{Z} :=$ Integer numbers;
- $\mathbb{Q} :=$ Rational numbers;
- $\mathbb{R} :=$ Real numbers $:= \mathbb{Q} \cup \{\text{irrational numbers}\}$.

Notation: The "$*$" symbol means that the set does not include 0.

1.3.1 Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$B := \{\pi, 1, -1, 0\}$;
$C := \{\pi, 1\}$;
$D := \{\pi\}$.

Then we write some examples: $\pi \in B,\ D \subset B,\ C \subset B,\ B \not\subset C,\ 0 \in B,\ 0 \notin C$.

2 Intervals in the real line

Intervals describe what happens between two or more elements.

2.1 Examples

2.1.1 Interval sets

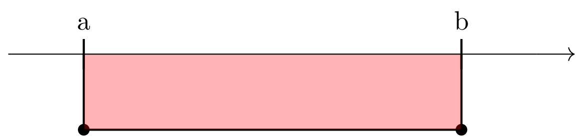
We have 4 cases:

- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\}$;
- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\}$;
- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\}$;
- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}$.

Notation: a and b are often called the "end points" of the interval;

2.1.2 Graphical examples

$\forall x \in \mathbb{R},\ x \in [a, b]$

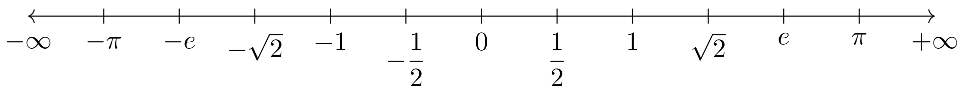


3 The extended line

In the real line $\mathbb{R}$ we add $\pm\infty$.

Real line: $(-\infty, +\infty) = \mathbb{R}$

Extended real line: $[-\infty, +\infty] = \overline{\mathbb{R}}$



Remark: $\pm\infty \notin \mathbb{R}$

3.1 Properties

$$\boxed{\forall x \in \mathbb{R} \;\mid\; \infty > x \;\mid\; -\infty < 0}$$

3.2 Operation in the extended line

If $a, b \in \mathbb{R}$, then $a + b,\ a - b,\ a \cdot b,\ \frac{a}{b}$ (with $b \neq 0$) stay the same

3.2.1 Additions

Let $\forall a \in \mathbb{R}$:

- $a + \infty := \infty;$
- $a - \infty := -\infty;$
- $+\infty + \infty := +\infty;$
- $-\infty - \infty := -\infty;$
- $+\infty - \infty :=$ undefined.

3.2.2 Moltiplications

Let $\forall a \in \mathbb{R}$:

- $+\infty \cdot +\infty := +\infty;$
- $-\infty \cdot +\infty := -\infty;$
- $-\infty \cdot (-\infty) := \infty;$
- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$
- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$
- $\frac{a}{+\infty} = \frac{a}{-\infty} := 0;$
- $\frac{+\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$
- $\frac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$

- $\frac{\infty}{\infty}$:= undefined.

4 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including $\pm\infty$.

4.1 Examples

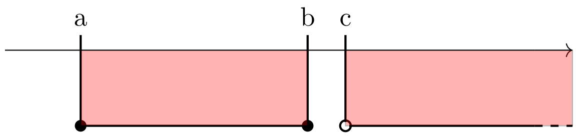
4.1.1 Interval sets

Let $a \in \mathbb{R}$, then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$;
- $[a, +\infty] = \{\forall x \in \mathbb{R} \mid x \geq a\}$;
- $(-\infty, +\infty) = \mathbb{R}$;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$.

4.1.2 Graphical examples

$\forall x \in \mathbb{R},\ x \in\ [a, b]\ \cup\ ]c, +\infty[$



<u>Notation</u>: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

5 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

5.1 Logical connectives

A	B	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

5.1.1 Logical conjunction $\wedge$

Given two statements P and Q, $P \wedge Q$ is true if both P and Q are true.

Let $P = (x > 0)$ and $Q = (y > 0)$, then:

$$\boxed{P \wedge Q = (x > 0 \wedge y > 0)}$$

5.1.2 Logical disjunction $\vee$

Given two statements P and Q, $P \vee Q$ is true if at least one of P or Q is true.

Let $P = (x = 0)$ and $Q = (y \neq 0)$, then:

$$\boxed{P \vee Q = (x = 0 \vee y \neq 0)}$$

5.1.3 Logical negation $\neg$

The negation of a statement P, denoted as $\neg P$, is true if P is false, and false if P is true.

Let $P = (x \geq 5)$, then:

$$\boxed{\neg P = (x < 5)}$$

5.1.4 Implication $\Rightarrow$

The symbol $\Rightarrow$ indicates that if statement P is true, then statement Q must also be true (i.e., P implies Q). Warning: It does not require that Q implies P.

$$\boxed{P = (x = 1) \Rightarrow Q = (x \in \mathbb{N})}$$

5.1.5 Inference $\Leftarrow$

The symbol $\Leftarrow$ means that a conclusion or result implies the truth of an earlier statement.
If Q is true, then P must be true.

$$\boxed{Q = (x > 0) \Leftarrow P = (x \in \mathbb{R}^{+})}$$

5.1.6 If and only if $\Leftrightarrow$

The symbol $\Leftrightarrow$ indicates that two statements P and Q are logically equivalent, meaning P is true if and only if Q is true.

$$\boxed{P = (x \in \mathbb{N},\ x \neq 0) \Longleftrightarrow Q = (x \in \mathbb{N}^{*})}$$

6 Union $\cup$ and Intersection $\cap$

6.1 Universe symbol

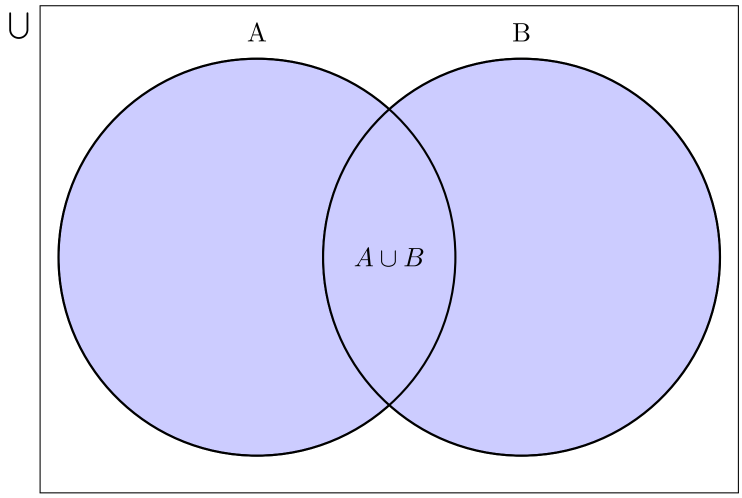
The symbol $\bigcup :=$ Universe describes a big set which contains all sets involved in our discussions (not always).

6.2 Venn diagram

6.2.1 Union $A \cup B$

If A and B are sets, then their union is:

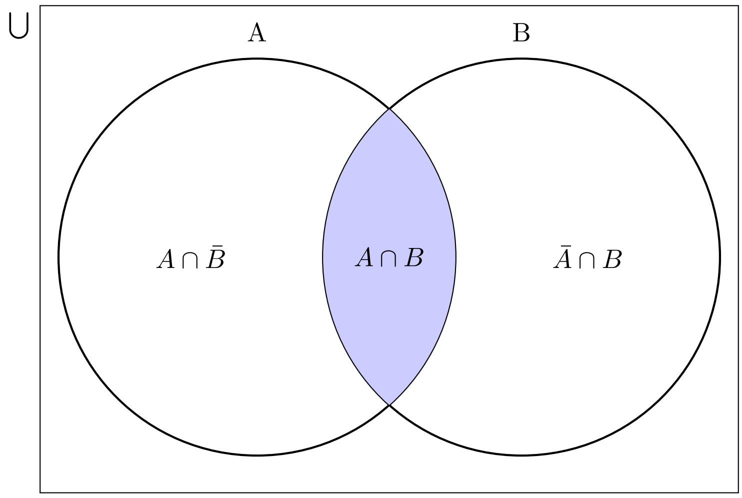
$$\boxed{A \cup U = \{\forall x \in \bigcup \mid x \in A \vee x \in B\}}$$



6.2.2 Intersection $A \cap B$

If A and B are sets, then their intersection is:

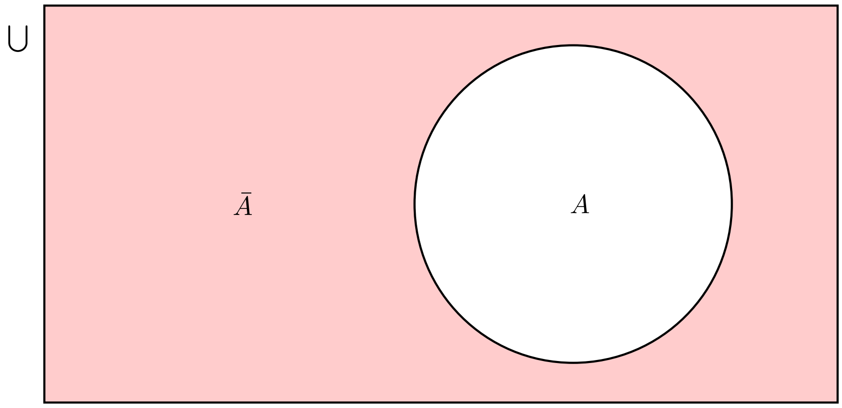
$$\boxed{A \cap B = \{\forall x \in \bigcup \mid x \in A \wedge x \in B\}}$$



6.2.3 Complement $\bar{A}$

If A is a set, its complement is:

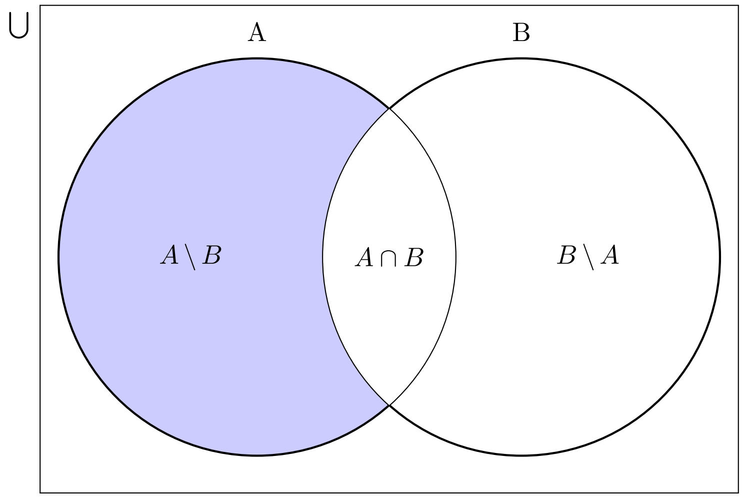
$$\boxed{\bar{A} = \{\forall x \in \bigcup \mid x \notin A\}}$$



6.2.4 Difference between sets $\setminus$

If A and B are sets, then their difference is:

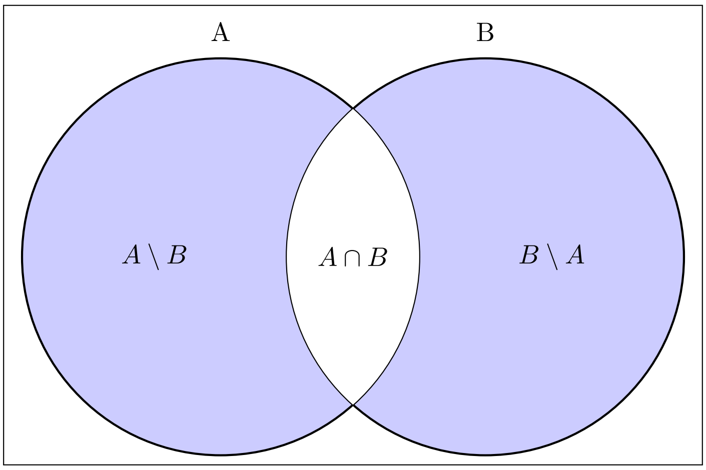
$$\boxed{A \setminus B = \{\forall x \in \bigcup \mid x \in A,\ x \notin B\}}$$



6.2.5 Symmetrical difference $\triangle$

If A and B are sets, then their symmetrical difference is:

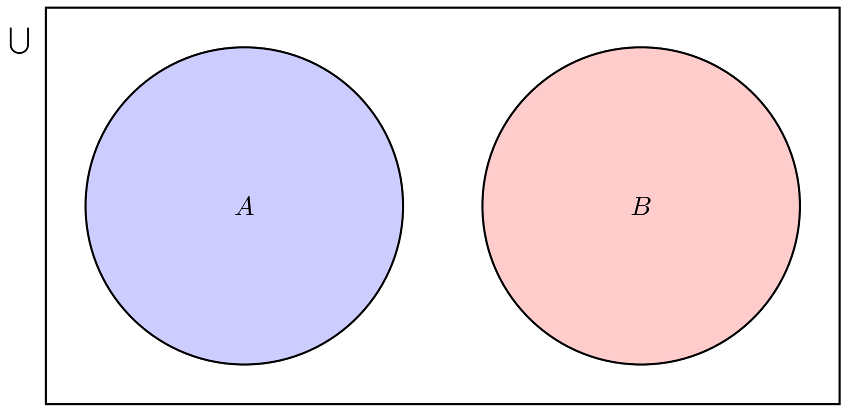
$$\boxed{A \triangle B = (A \setminus B) \cup (B \setminus A)}$$



6.2.6 Disjoined sets (Empty sets) $\emptyset$

$\emptyset :=$ the set containing zero elements:

$$\boxed{A \cap B = \emptyset}$$



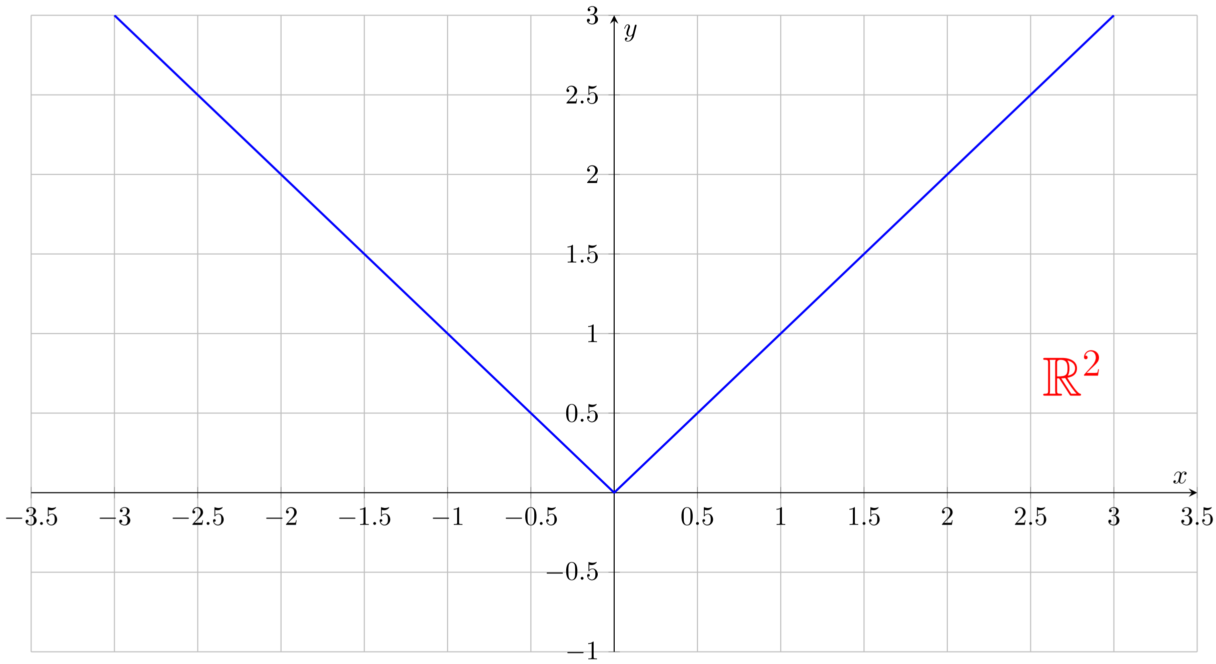
7 The absolute value function

The absolute value is an operator that returns the positive value of a number, regardless of its original sign. Let $x \in \mathbb{R}$, then:

$$\boxed{|x| = \begin{cases} x \text{ if } & x \geq 0 \\ x \text{ if } & -x < 0 \end{cases}}$$

7.1 Graph of absolute value functions

Let's plot the function $y = |x|$:



7.2 Properties

Let $a, b \in \mathbb{R}$, then:

- $|a \cdot b| = |a| \cdot |b|$;
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$;
- $|a \pm b| \neq |a| \pm |b|$.

7.3 Triangular inequalities

Let $a, b \in \mathbb{R}$, then:

$$\boxed{\begin{array}{l} |a|+|b| \geq |a+b| \\ |a|-|b| \leq |a-b| \end{array}}$$

Part II

Week 2

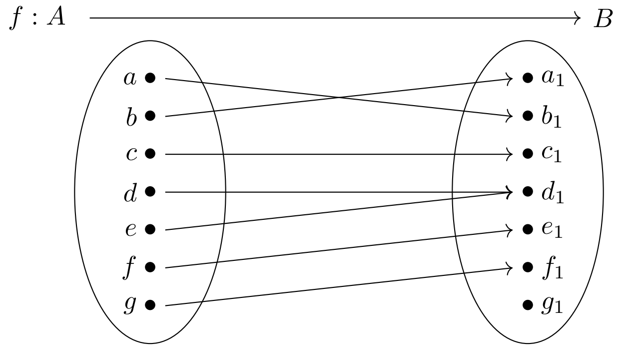
8 Concept of functions

Let's take any two sets $A\{a,b,c,d,e,f,g\}$ and $B\{a_1,b_1,c_1,d_1,e_1,f_1,g_1\}$.

$$\boxed{\begin{aligned} f : A &\to B \\ a &\longmapsto f(a) \end{aligned}}$$

A function is a relation between the sets A and B, according to which we associate to each element of A one and only one element of B:

$$\boxed{\forall x \in A\ \exists!\, y \in B \mid (x,y) \in f}$$



Notation: $f(a)=b_1,\ f(b)=a_1,\ f(c)=c_1,\ f(d)=d_1$, ...

Each point in set A is associated with one element of B. However, it is possible for more than two elements of A to point to the same element of B.

The set A is called *domain* of f. The set B is called the *codomain* of f.

8.1 Image (Range)

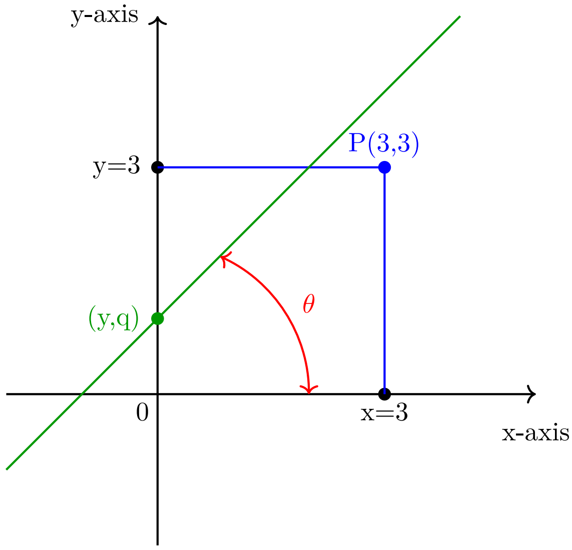
Let $f : X \to Y$ be a function. The image of f is defined as:

$$\boxed{\mathrm{Im}(f) = \{y \in Y \mid y = f(x),\ x \in X\}}$$

Easily, the image is the set containing all the elements of the set B associated with the elements of the set A.

9 Linear function

9.1 Cartesian diagram



9.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

9.3 Slope-intercept equation

Let $m, q \in \mathbb{R}$, then

$$\boxed{y = mx + q}$$

- m: slope;
- q: vertical intercept.

9.3.1 Slope

The slope of a line can be calculated with the equation

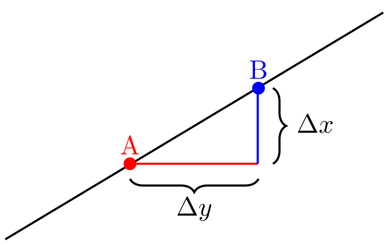
$$\boxed{m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)}$$

We have three different slope outcomes:

- $m > 0$, the line is increasing;
- $m = 0$, the line is stable;
- $m < 0$, the line is decreasing.

Warning: This works only if $x_B \neq x_A$.

9.3.2 Drawing



9.4 Vertical lines

The more the value of m increases, the closer the line will get to the vertical, without ever reaching it.

Let $c \in \mathbb{R}$, then $x = c$.

Vertical lines cannot be written as a function.

10 Equation of a line

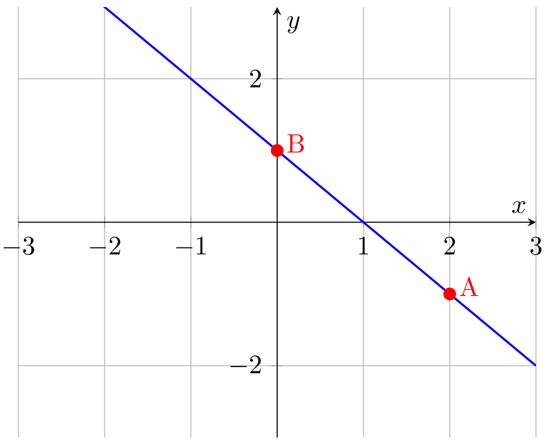
Let $m, x_A, y_A \in \mathbb{R}$ and $A(x_A, y_A)$, then

$$\boxed{y - y_A = m(x - x_A)}$$

e.g.: Find the line with $m = -1$ and $A(2, -1)$.

$y - 1 = -1(x + 2) \Rightarrow y = -x + 1$

Points: $A(2, -1)$; $B(0, 1)$



10.1 General equation in a cartesian diagram

$$\boxed{ax + by + c = 0}$$

Remark:

- All the lines can be described with this kind of equation;
- When $b = 0$, $a \neq 0$, then $ax = -c \Rightarrow x = \dfrac{-c}{a} \in \mathbb{R}$;
- When $b \neq 0$, then $y = -\dfrac{a}{b}x - \dfrac{c}{b}$, where $m = -\dfrac{a}{b}$ and $q = -\dfrac{c}{b}$.

11 Increasing and decreasing functions

Let $f : [a, b] \longrightarrow \mathbb{R}$

Notation: if your replace $[a, b]$ with $\mathbb{R}$, you obtain the definition in the whote $\mathbb{R}$.

11.1 Increasing functions

- f is increasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) \geq f(x_1)$;
- f is strictly increasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) > f(x_1)$.

11.2 Decreasing functions

- f is decreasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) \leq f(x_1)$;
- f is strictly decreasing if $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$, then $f(x_2) < f(x_1)$.

12 Inverse function

Let's take any two sets A and B.

A function $f : A \to B$ is invertible if there exists another function $f^{-1} : B \to A$, called the inverse function, such that:

$$\boxed{\begin{array}{l} \forall x \in A,\ f^{-1}(f(x)) = x \\ \forall y \in B,\ f(f^{-1}(y)) = y \end{array}}$$

Warning: A function is invertible if and only if it is bijective.

12.1 Facts about inverse functions

1)
Let $f : D \to \mathbb{R}$

f is invertible in D when:

- f is strictly increasing;
- f is strictly decreasing.

2)
Let $f : D \to \mathbb{R}$

f is invertible when $f^{-1} : \mathrm{Im}(f) \to D$.

Part III

Week 3

13 Polynomial function

13.1 Expressions, terms and factors

13.1.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

13.1.2 Terms

A term is any part of the expression separated by "+" or "−".

$$\boxed{y = \underbrace{ax^2}_{term} + \underbrace{bx \cdot c}_{term}}$$

13.1.3 Factors

Each term can be split into a product of factors.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called "factorization".
The goal of a factorization is to factorize an expression as much as possible.

14 Common factor

Any expression made of terms is composed of several factors.

$$\boxed{x^2 + x^3 + x = x(x + x^2 + 1),\ \forall x \in \mathbb{R}}$$

15 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$ (square of a binomial);
- $(a - b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$ (difference of squares);
- $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ (sum of cubes);
- $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ (difference of cubes).

Remark: notable products are useful to factorize expressions when we don't know a common factor.

16 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.

Number of Terms	Name	Example	Comment
One	Monomial	ax^2	Mono means “one” in Greek
Two	Binomial	$ax^2 - bx$	Bi means “two” in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means “three” in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means “many” in Greek

16.1 Definition

Let $n \in \mathbb{N}^*$, then a polynomial is the sum or difference of n-monomials.

16.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

16.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$p(x) = x^2 + 1 \rightarrow$ the degree is 2.

$\forall x \in \mathbb{R},\ p(0) = 0^2 + 1 = 1 \rightarrow$ A constant term, like 1, is a polynomial with degree 0.

16.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$p(x) = x^3 + 1 + x^5 + x^21 \rightarrow \deg(p(x)) = 21$

Let a, b, c, d, x, y be variables, then:

$q(a, b, c, d, x, y) = 12\underbrace{abcd}_{\text{deg}=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$

Notation: Let $f(x) = ax^2 + bx + c$, a, b and c are called coefficients.
The coefficient of the monomial with highest degree is called **leading coefficient**.

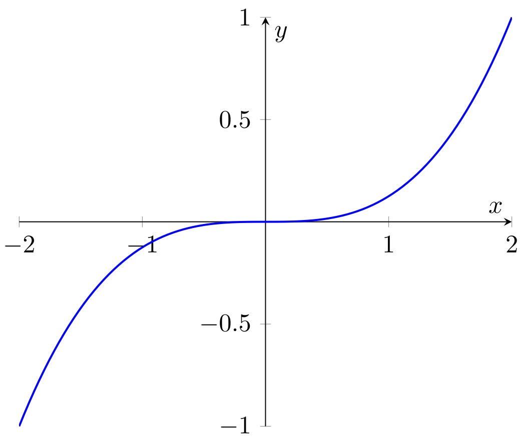
17 Symmetrical functions

Let $y = kx^n$, then we plot:

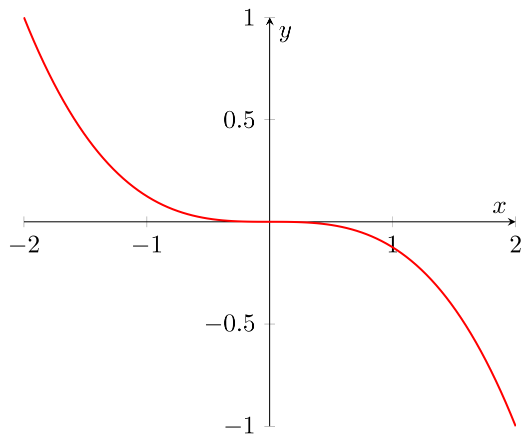
17.1 n odd

$$\boxed{f(-x) = -f(x), \quad \forall x \in \mathbb{R}}$$

17.1.1 Graph examples



$k > 0$

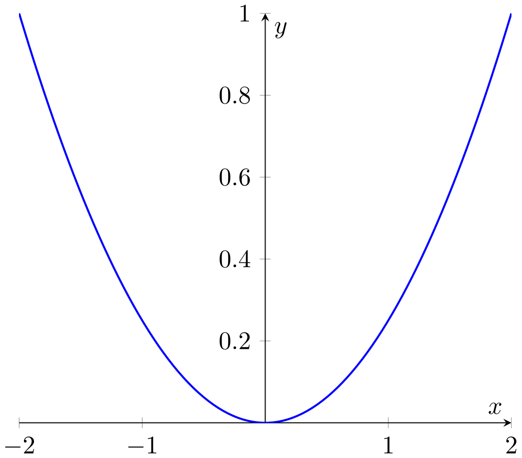


$k < 0$

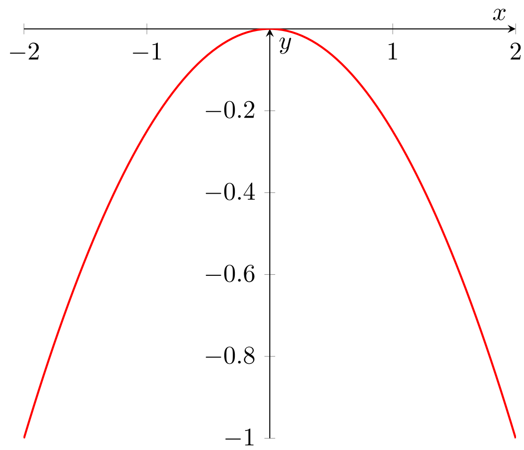
17.2 n even

$$\boxed{f(-x) = f(x), \quad \forall x \in \mathbb{R}}$$

17.2.1 Graph examples



$k > 0$
Concave up



$k < 0$
Concave down

Definition:

- a function $y = f(x)$ is called **odd** if it is symmetric with respect to the origin;
- a function $y = f(x)$ is called **even** if it is symmetric with respect to the y-axis.

17.3 General case

Let $y = p(x)$, where $p(x)$ is any polynomial with real coefficients:

$$\boxed{p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + ... + a_2 \cdot x^2 + a_1 \cdot x^1 + a_0}$$

where:

- $n \in \mathbb{N}$;
- $n = \deg(p(x))$;
- $a_n =$ leading coefficient.

$$\boxed{p(x) = \sum_{i=0}^{n} a_i \cdot x^i}$$

17.4 Symmetry of a polynomial

Let $y = p(x)$ be a polynomial function, then:

1)
$y = p(x)$ is odd iff all the degrees of all the terms of $p(x)$ are odd;

2)
$y = p(x)$ is even iff all the degrees of all the terms of $p(x)$ are even;

3)
$y = p(x)$ has mixed degrees, $p(x)$ is neither odd nor even.

18 Intersection with axis

18.1 Vertical intersection

Let $y = f(x)$ be any function, then we solve for y:

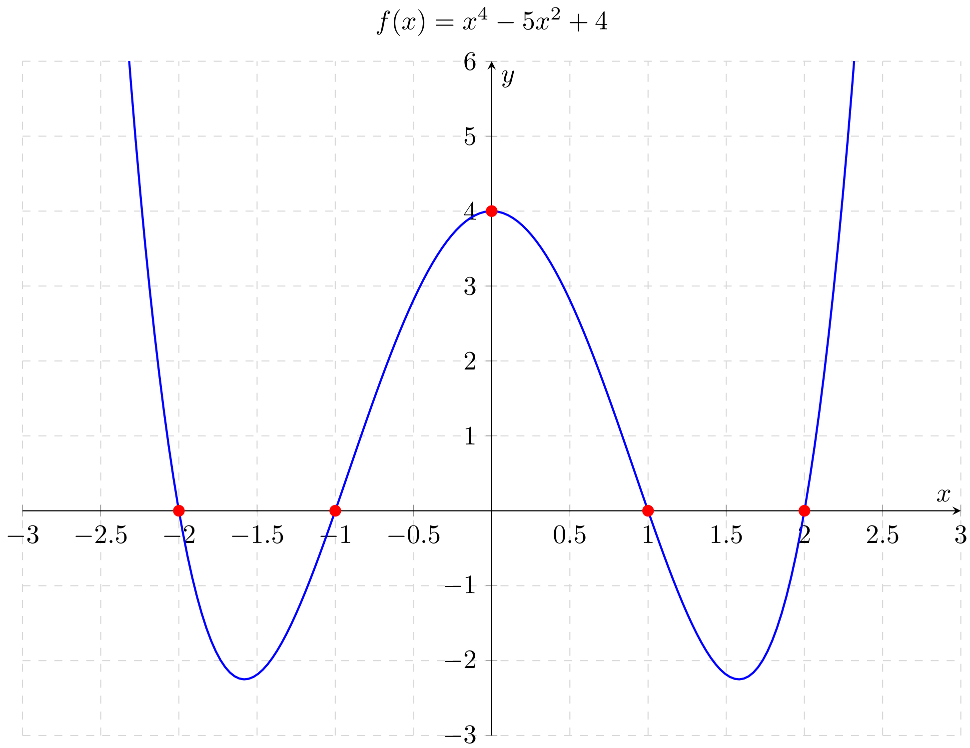
$$\boxed{\begin{cases} x = 0 \\ y = f(0) \end{cases}}$$

18.2 Zeros of a function

Let $y = f(x)$ be any function, then we solve for x:

$$\boxed{\begin{cases} y = 0 \\ 0 = f(x) \end{cases}}$$

18.3 Graph example



19 Dominant elements in a function approaching $\pm\infty$

As x approaches $\pm\infty$, the term with the highest degree in a polynomial function dominates the behavior of the function.

$$\boxed{p(x) \text{ has, as a dominant, the element } a_n \text{ with the highest degree } x^n}$$

19.1 Order of dominance

19.1.1 Approaching to $+\infty$

Let $n \in \mathbb{N},\ m \in \mathbb{N},\ 2 < n < m$, then:

$$\boxed{\ln(x) < x < x^n < x^m < n^x < m^x < x^x}$$

In these cases, we always have $x \to +\infty \Rightarrow p(x) \to +\infty$

19.1.2 Approaching to -∞

Let $\lambda > 2$ and odd, $k > 2$ and even.

$$\boxed{\begin{array}{l} x^\lambda < -x^2 < x^1 < 0 \\ -x^k < -x^2 < x^1 < 0 \end{array}}$$

Functions like x^λ (with λ odd) and $-x^k$ (with k even) both approach $-\infty$, but at different rates.

19.1.3 Dominance in rational functions

When the dominant element is at the numerator:

$$\boxed{\lim_{x\to\infty} \frac{x^n}{x^{n-1}} = \infty}$$

When the dominant element is at the denominator:

$$\boxed{\lim_{x\to\infty} \frac{x^{n-1}}{x^n} = 0}$$

When we have the same degree either in the numerator and in the denominator:

$$\boxed{\lim_{x\to\infty} \frac{ax^n}{bx^n} = \frac{a}{b}}$$

Definition: **horizontal asymptote** appears when x approaches to ∞, which implies that y approaches to a number A different from $\pm\infty$