Maths refreshing course
HSLU, Semester 1

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September 4, 2024

Contents

Part I

Lesson 1

1 Numerical sets

- $\mathbb{N} :=$ Natural numbers (including 0)
- $\mathbb{Z} :=$ Integer numbers
- $\mathbb{Q} :=$ Rational numbers
- $\mathbb{R} :=$ Real numbers

Notation: The "$*$" symbol means that the set does not include 0.

We have that:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

2 Prime numbers

A prime number is a number $n \in \mathbb{N} \setminus \{0,1\}$ such that, for every divisor $d \in \mathbb{N}$, if $d \mid n$, then $d = 1$ or $d = n$.

$$\boxed{n \in \mathbb{N} \setminus \{0,1\} \text{ is prime } \iff \forall d \in \mathbb{N}, (d \mid n) \Rightarrow (d = 1 \text{ or } d = n)}$$

3 Positive powers

Let $a \in \mathbb{R}, n \in \mathbb{R}^*$ and $a \subset \mathbb{R}$, then

$$\boxed{a^1 := a \quad | \quad a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}}$$

3.1 Property 1

Let $a, b \in \mathbb{R},\ n, m \in \mathbb{N}$, then

$$\boxed{a^n \cdot a^m = a^{n+m}}$$

3.2 Property 2

Let $a, b \in \mathbb{R},\ n \in \mathbb{N}$, then

$$\boxed{(a \cdot b)^n = a^n \cdot b^n}$$

Notation: The power a^n, a is the base and n is the exponent.

3.3 Property 3

Let $a \in \mathbb{R},\ m, n \in \mathbb{N}^*$, then

$$\boxed{(a^n)^m = a^{n \cdot m}, \text{ which is } \neq a^{(n^m)}}$$

4 Fractions

Notation 1: $a \cdot b = a \times b = ab \quad | \quad \frac{a}{b} = a \div b = a : b$

Notation 2: "a" is called numerator, "b" is called denominator.

Notation 3: $\frac{a}{b},\ a, b \in \mathbb{R},\ b \neq 0$

4.1 Property 1

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

4.2 Property 2

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

4.3 Property 3

Let $a, b \in \mathbb{R}^*$ and $c, d \in \mathbb{R}$, then

$$\boxed{\frac{a}{b} \pm \frac{c}{d} = \frac{a \cdot d \pm c \cdot b}{b \cdot d}}$$

5 Negative powers

5.1 Definition

$$\boxed{\forall a \in \mathbb{R}^*; \quad a^{-1} := \frac{1}{a}}$$

5.2 Property 4

Let $\forall n \in \mathbb{N},\ \forall a \in \mathbb{R}$, then

$$\boxed{a^{-n} = \left(\frac{1}{a}\right)^n}$$

This property implies that $\forall z \in \mathbb{Z},\ \forall a \in \mathbb{R},\ z \neq 0$
We can compute a^z

5.3 Property 5

Let $\forall a \in \mathbb{R},\ a \neq 0,\ \forall n, m \in \mathbb{Z}$, then

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Consequences:

1. Properties 1, 2 and 3 also hold for integer exponents:
   - $\forall a \in \mathbb{R},\ \forall n, m \in \mathbb{Z} \Rightarrow a^n \cdot a^m = a^{n+m}$
   - $\forall b \in \mathbb{R},\ (a \cdot b)^n = a^n \cdot b^n$
   - $(a^n)^m = a^{n \cdot m}$
2. $\forall a \in \mathbb{R}^*,\ a^0 = a^{1-1} = \frac{a^1}{a^1} = 1 \Rightarrow a^0 = 1$

6 Fractions and percentages (and back)

$\alpha \in \mathbb{R},\ n\%$ of $\alpha \Longleftrightarrow \frac{n}{100} \cdot \alpha$

Part II

Lesson 2

7 Symbols

Let $a, b \in \mathbb{R}$, then

- $a = b \rightarrow$ equality;
- $a \neq b \rightarrow$ inequality (a is not equal to b);
- $a < b \rightarrow$ less than (a is strictly less than b);
- $a \leq b \rightarrow$ less than or equal to (a is less than or equal to b);
- $a > b \rightarrow$ greater than (a is strictly greater than b);
- $a \geq b \rightarrow$ greater than or equal to (a is greater than or equal to b).

Example: $x \in \mathbb{R},\ x \geq 2 \rightarrow 2 \leq x < \infty$

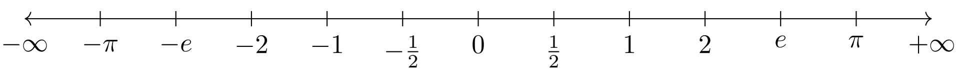
8 Brackets

( ) Parenthesis (round brackets)
[ ] Square brackets
{ } Braces

9 Latin notations

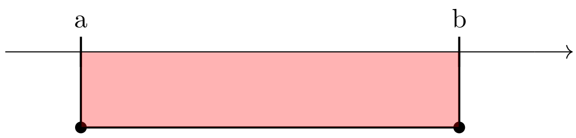
- e.g. = for example;
- i.e. = that is / that implies;
- Q.E.D. (□)= quod erat demonstrandum (we finally prove it).

10 The real line

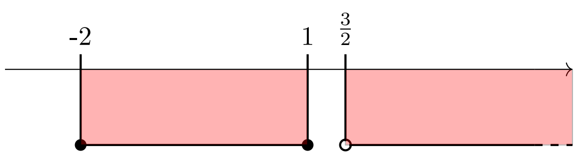


10.1 Exercises

1) $\forall a, b, x \in \mathbb{R},\ a \leq x \leq b$



2) $\forall x \in \mathbb{R},\ x \in\ ]-2, -1] \cup\ ]\frac{3}{2}, +\infty[$



Notation: The union of two or more intervals where $x \in \mathbb{R}$ is denoted by the symbol $\cup$.

11 Properties of real numbers

11.1 Property 1 - Closure under "+" and "·"

$\forall x, y \in \mathbb{R}$
$x + y \in \mathbb{R}$
$x \cdot y \in \mathbb{R}$

Remark: for $\forall x \in \mathbb{Z}$, closure does not hold for division.

11.2 Property 2 - Commutativity

$\forall x, y \in \mathbb{R}$
$x + y = y + x$
$x \cdot y = y \cdot x$

Remark: commutativity does not hold for divisions and subtractions.

11.3 Property 3 - Associative

$\forall x, y, z \in \mathbb{R}$
$x + (y + z) = (x + y) + z$
$x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Remark: associativity does not hold for divisions and subtractions.

11.4 Property 4 - Distributive

$\forall x, y, z \in \mathbb{R}$
$x(y \pm z) = xy \pm xz$

11.5 Property 5 - Identity

$\forall x \in \mathbb{R}$

a) $0 + x = x$

b) $1 \cdot x = x$

Remark: $\forall x \in \mathbb{R},\ x \cdot 0 = 0$ is not an identity property.

11.6 Property 6 - Inverses and opposites

$\forall x \in \mathbb{R}$

a) $x + (-x) = 0$ (additive inverse)

b) when $x \neq 0,\ x \cdot \frac{1}{x} = 1$ (multiplicative inverse or opposite)

Remark 1: $\forall x \in \mathbb{N}$ does not exist either inverse nor opposite.

Remark 2: $\forall x \in \mathbb{Z}$ has inverses, but not opposites.

12 The order of operations

1. Perform all operations inside grouping symbols beginning with the innermost set:
   ( ) inside brackets operations;
2. Perform all exponential operations as you come to them, moving left-to-right:
   x^a;
3. Perform all multiplications and divisions as you come to them, moving left-to-right:
   "·" and "÷";
4. Perform all additions and subtractions as you come to them, moving left-to-right:
   "+" and "−";
5. When the level of priority is the same (e.g. multiplications and divisions) solve them as you come to them.

13 Signed numbers

A number is denoted as positive if it is directly preceded by a + sign or no sign at all.
A number is denoted as negative if it is directly preceded by a − sign.

$\forall x \in \mathbb{R}$

$$-(-x) = x \qquad +(-x) = -x \qquad +(+x) = x \qquad -(+x) = -x$$

14 Absolute value

Let $x \in \mathbb{R}$, then

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

14.1 Property

$\forall x \in \mathbb{R}$
$|x| > 0 \quad \text{if } y \neq 0$
$|x| = 0 \quad \text{if } x = 0$

Part III

Lesson 3

15 Polynomials

15.1 Terms and factors

15.1.1 Variables

A variable is a letter or a symbol that can assume any value.

$$\boxed{\forall x \in \mathbb{R}}$$

The most common variables are a, b, x, y.

When we have an equality $y = x + a$, $\forall x \in \mathbb{R}$, x can assume any value in the set of real numbers (x is an independent variable), while y strictly depends on the value that we decide to give to x.

Notice: we can write $y = x + a$ as $y - a = x$, changing which variable is independent and which is dependent.

15.1.2 Sets

... ...

$$\boxed{\text{In the set } \forall x \in [a, b],\ x \text{ can assume any number between } a \leq x \leq b}$$

15.2 Expressions, terms and factors

15.2.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$\boxed{y = ax^2 + bx \cdot c}$$

15.2.2 Terms

A term is any part of the expression separated by "+" or "−".

$$\boxed{y = \underbrace{ax^2}_{term} + \underbrace{bx \cdot c}_{term}}$$

15.2.3 Factors

Each term can be split into several factors separated by a multiplication sign.

$$\boxed{x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Notice: the process of splitting a term into several factors is called "factorization".
The goal of a factorization is to factorize an expression as much as possible.

16 Common factor

Any expression made up terms is composed of several factors.

$$\boxed{\forall x \in \mathbb{R} \mapsto x^2 + x^3 + x = x(x + x^2 + 1)}$$

17 Notable products

- $(a+b)^2 = a^2 + 2ab + b^2$ (difference of two squares);
- $(a-b)^2 = a^2 - 2ab + b^2$ (square of a binomial);
- $(a-b)(a+b) = a^2 - b^2$ (square of a binomial);
- $(a-b)(a^2 + b^2 + ab) = a^3 - b^3$ (difference of two cubes);
- $(a+b)(a^2 + b^2 - ab) = a^3 + a^3$ (sum of two cubes).

<u>Remark</u>: notable products are useful to factorize expressions when we don't know a common factor.

18 Polynomials