

# Mathematics 1A

## HSLU, Semester 1

Matteo Frongillo

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## Part I

# Week 1

## 1 The set theory

### 1.1 Definition of a set

A set is a collection of objects or elements.

Remark: The collection of all sets is not a set.

### 1.2 Logical symbols

#### 1.2.1 Definition

Braces and the definition symbol “:=” are used to define a set giving all its elements:

$$A := \{a, b, c, d, e\}$$

#### 1.2.2 Equal

In this case, the equal symbol means that the set  $A$  is equal to the set  $B$ :

$$A = B$$

#### 1.2.3 Belongs to

The symbols  $\in$  and  $\ni$  describe an element which is part of the set:

$$a \in A \iff A \ni a$$

#### 1.2.4 Does not belong to

The symbols  $\notin$  mean that an element does not belong to the set:

$$f \notin A$$

#### 1.2.5 Inclusion and contains

The symbols  $\subset$  and  $\supset$  mean that a set has another set included in its set:

$$\mathbb{N} \subset \mathbb{Z} \iff \mathbb{Z} \supset \mathbb{N}$$

#### 1.2.6 For all/any

The symbol  $\forall$  means that we are considering any type of element:

$$\forall x \in \mathbb{R}, x > 0$$

In this case, we've defined a new set.

### 1.3 Numerical sets

- $\mathbb{N} :=$  Natural numbers (including 0);
- $\mathbb{Z} :=$  Integer numbers;
- $\mathbb{Q} :=$  Rational numbers;
- $\mathbb{R} :=$  Real numbers  $:= \mathbb{Q} \cup \{\text{irrational numbers}\}$ .

Notation: The “\*” symbol means that the set does not include 0.

#### 1.3.1 Inclusion of sets

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

$$B := \{\pi, 1, -1, 0\};$$

$$C := \{\pi, 1\};$$

$$D := \{\pi\}.$$

Then we write some examples:  $\pi \in B$ ,  $D \subset B$ ,  $C \subset B$ ,  $B \not\subset C$ ,  $0 \in B$ ,  $0 \notin C$ .

## 2 Intervals in the real line

Intervals describe what happens between two or more elements.

### 2.1 Examples

#### 2.1.1 Interval sets

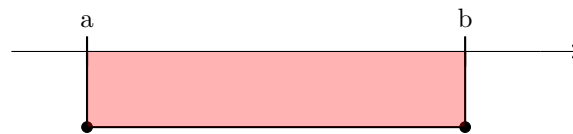
We have 4 cases:

- $(a, b) = \{\forall x \in \mathbb{R} \mid a < x < b\};$
- $[a, b) = \{\forall x \in \mathbb{R} \mid a \leq x < b\};$
- $(a, b] = \{\forall x \in \mathbb{R} \mid a < x \leq b\};$
- $[a, b] = \{\forall x \in \mathbb{R} \mid a \leq x \leq b\}.$

Notation:  $a$  and  $b$  are often called the “end points” of the interval;

#### 2.1.2 Graphical examples

$$\forall x \in \mathbb{R}, x \in [a, b]$$

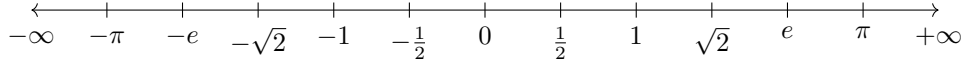


### 3 The extended line

In the real line  $\mathbb{R}$  we add  $\pm\infty$ .

**Real line:**  $(-\infty, +\infty) = \mathbb{R}$

**Extended real line:**  $[-\infty, +\infty] = \overline{\mathbb{R}}$



Remark:  $\pm\infty \notin \mathbb{R}$

#### 3.1 Properties

$$\boxed{\forall x \in \mathbb{R} \mid \infty > x \mid -\infty < 0}$$

#### 3.2 Operation in the extended line

If  $a, b \in \mathbb{R}$ , then  $a + b$ ,  $a - b$ ,  $a \cdot b$ ,  $\frac{a}{b}$  (with  $b \neq 0$ ) stay the same

##### 3.2.1 Additions

Let  $\forall a \in \mathbb{R}$ :

- $a + \infty := \infty$ ;
- $a - \infty := -\infty$ ;
- $+\infty + \infty := +\infty$ ;
- $-\infty - \infty := -\infty$ ;
- $+\infty - \infty := \text{undefined}$ .

##### 3.2.2 Multiplications

Let  $\forall a \in \mathbb{R}$ :

- $+\infty \cdot +\infty := +\infty$ ;
- $-\infty \cdot +\infty := -\infty$ ;
- $-\infty \cdot (-\infty) := \infty$ ;
- $a \cdot \infty := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & \text{undefined} \end{cases}$
- $a \cdot (-\infty) := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & \text{undefined} \end{cases}$
- $\frac{a}{+\infty} = \frac{a}{-\infty} := 0$ ;
- $\frac{\pm\infty}{a} := \begin{cases} a > 0 & +\infty \\ a < 0 & -\infty \\ a = 0 & +\infty \end{cases}$
- $\frac{-\infty}{a} := \begin{cases} a > 0 & -\infty \\ a < 0 & +\infty \\ a = 0 & -\infty \end{cases}$
- $\frac{\infty}{\infty} := \text{undefined}$ .

## 4 Intervals including $\pm\infty$

Intervals describe what happens between two or more elements, including  $\pm\infty$ .

### 4.1 Examples

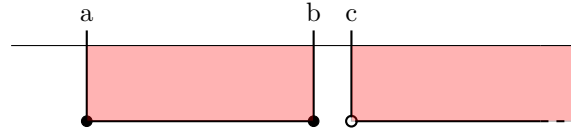
#### 4.1.1 Interval sets

Let  $a \in \mathbb{R}$ , then:

- $(-\infty, a) = \{\forall x \in \mathbb{R} \mid x < a\}$ ;
- $(a, +\infty) = \{\forall x \in \mathbb{R} \mid x > a\}$ ;
- $(-\infty, a] = \{\forall x \in \mathbb{R} \mid x \leq a\}$ ;
- $[a, +\infty) = \{\forall x \in \mathbb{R} \mid x \geq a\}$ ;
- $(-\infty, +\infty) = \mathbb{R}$ ;
- $[-\infty, +\infty] = \overline{\mathbb{R}}$ .

#### 4.1.2 Graphical examples

$\forall x \in \mathbb{R}, x \in [a, b] \cup ]c, +\infty[$



Notation: The union of two or more intervals where  $x \in \mathbb{R}$  is denoted by the symbol  $\cup$ .

## 5 Propositional logic

Propositional logic is a branch of mathematics that deals with propositions and logical operations.

### 5.1 Logical connectives

A	B	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

#### 5.1.1 Logical conjunction $\wedge$

Given two statements  $P$  and  $Q$ ,  $P \wedge Q$  is true if both  $P$  and  $Q$  are true.

Let  $P = (x > 0)$  and  $Q = (y > 0)$ , then:

$$P \wedge Q = (x > 0 \wedge y > 0)$$

#### 5.1.2 Logical disjunction $\vee$

Given two statements  $P$  and  $Q$ ,  $P \vee Q$  is true if at least one of  $P$  or  $Q$  is true.

Let  $P = (x = 0)$  and  $Q = (y \neq 0)$ , then:

$$P \vee Q = (x = 0 \vee y \neq 0)$$

### 5.1.3 Logical negation $\neg$

The negation of a statement  $P$ , denoted as  $\neg P$ , is true if  $P$  is false, and false if  $P$  is true.

Let  $P = (x \geq 5)$ , then:

$$\neg P = (x < 5)$$

### 5.1.4 Implication $\Rightarrow$

The symbol  $\Rightarrow$  indicates that if statement  $P$  is true, then statement  $Q$  must also be true (i.e.,  $P$  implies  $Q$ ).

Warning: It does not require that  $Q$  implies  $P$ .

$$P = (x = 1) \Rightarrow Q = (x \in \mathbb{N})$$

### 5.1.5 Inference $\Leftarrow$

The symbol  $\Leftarrow$  means that a conclusion or result implies the truth of an earlier statement.

If  $Q$  is true, then  $P$  must be true.

$$Q = (x > 0) \Leftarrow P = (x \in \mathbb{R}^+)$$

### 5.1.6 If and only if $\Leftrightarrow$

The symbol  $\Leftrightarrow$  indicates that two statements  $P$  and  $Q$  are logically equivalent, meaning  $P$  is true if and only if  $Q$  is true.

$$P = (x \in \mathbb{N}, x \neq 0) \Leftrightarrow Q = (x \in \mathbb{N}^*)$$

## 6 Union $\cup$ and Intersection $\cap$

### 6.1 Universe symbol

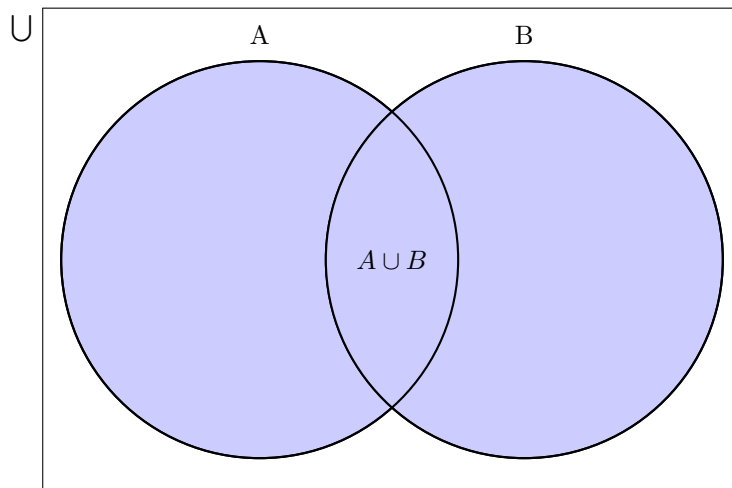
The symbol  $\cup :=$  Universe describes a big set which contains all sets involved in our discussions (not always).

### 6.2 Venn diagram

#### 6.2.1 Union $A \cup B$

If  $A$  and  $B$  are sets, then their union is:

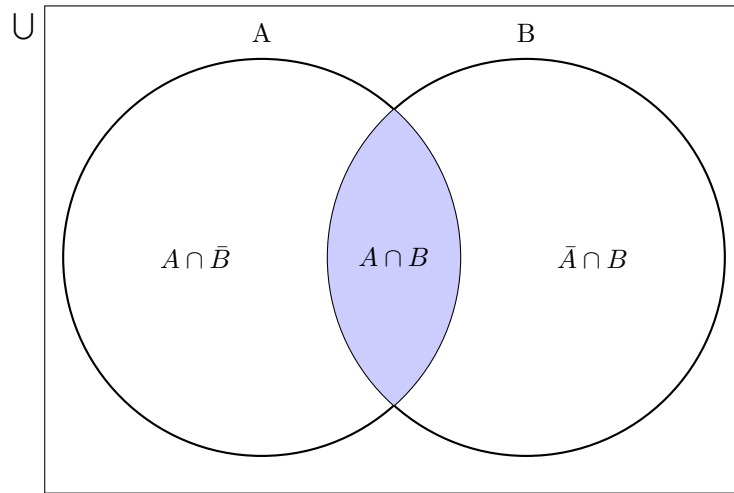
$$A \cup B = \{\forall x \in \cup \mid x \in A \vee x \in B\}$$



### 6.2.2 Intersection $A \cap B$

If  $A$  and  $B$  are sets, then their intersection is:

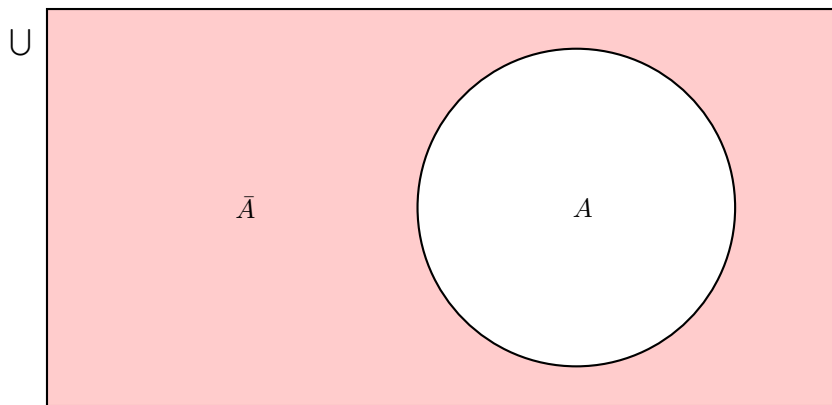
$$A \cap B = \{\forall x \in \mathcal{U} \mid x \in A \wedge x \in B\}$$



### 6.2.3 Complement $\bar{A}$

If  $A$  is a set, its complement is:

$$\bar{A} = \{\forall x \in \mathcal{U} \mid x \notin A\}$$

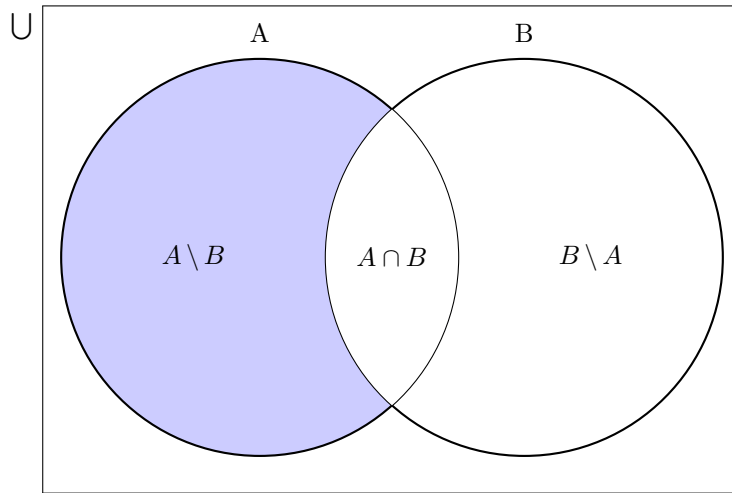




#### 6.2.4 Difference between sets $\setminus$

If  $A$  and  $B$  are sets, then their difference is:

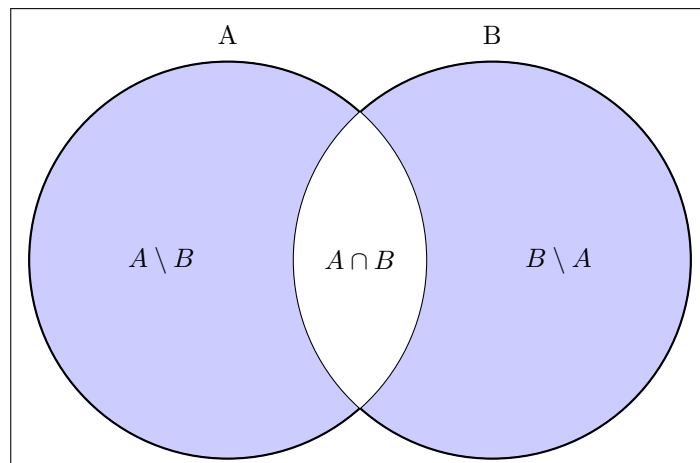
$$A \setminus B = \{\forall x \in \cup \mid x \in A, x \notin B\}$$



#### 6.2.5 Symmetrical difference $\triangle$

If  $A$  and  $B$  are sets, then their symmetrical difference is:

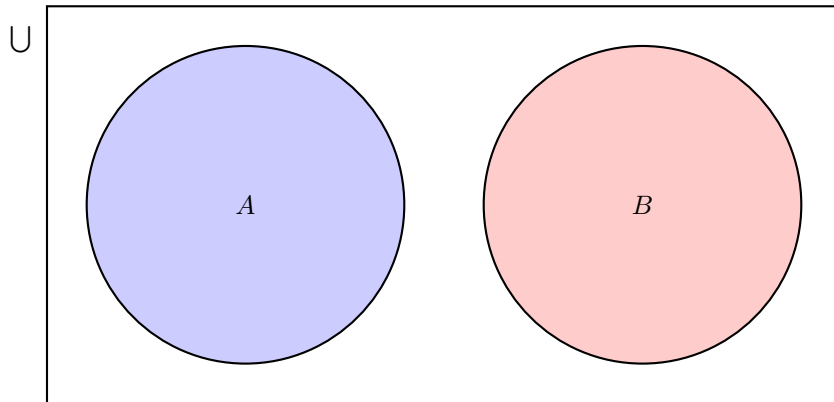
$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$



### 6.2.6 Disjoined sets (Empty sets) $\emptyset$

$\emptyset$  := the set containing zero elements:

$$A \cap B = \emptyset$$



## 7 The absolute value function

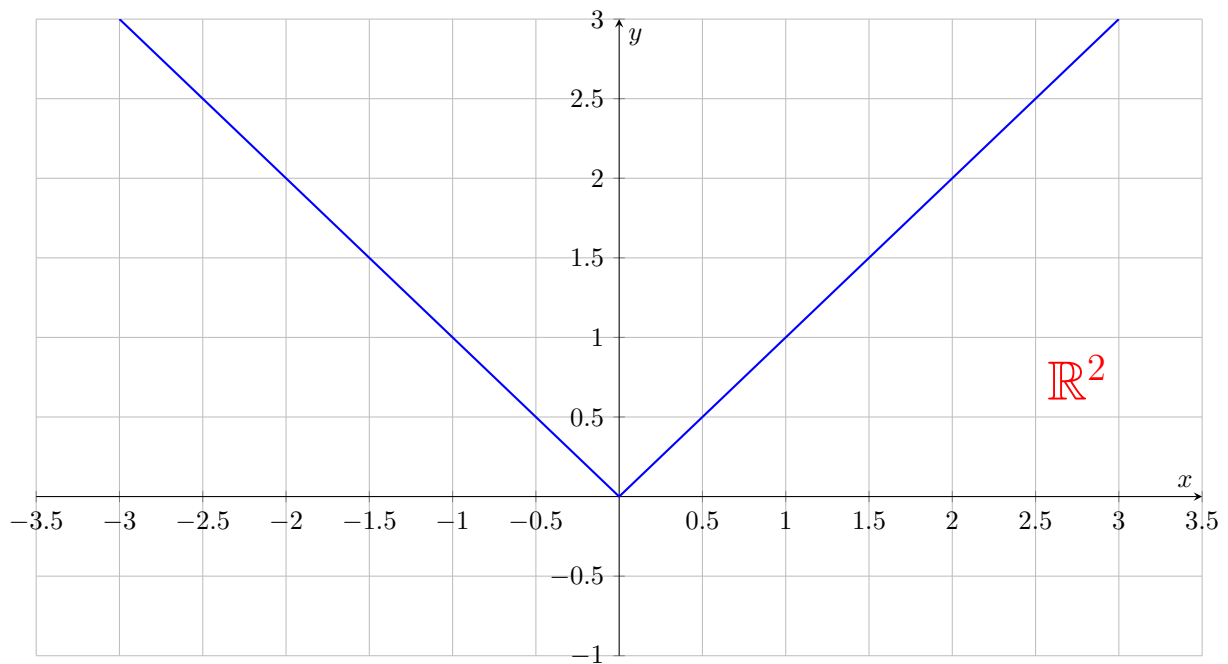
The absolute value is an operator that returns the positive value of a number, regardless of its original sign.

Let  $x \in \mathbb{R}$ , then:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } -x < 0 \end{cases}$$

### 7.1 Graph of absolute value functions

Let's plot the function  $y = |x|$ :



## 7.2 Properties

Let  $a, b \in \mathbb{R}$ , then:

- $|a \cdot b| = |a| \cdot |b|$ ;
- $|\frac{a}{b}| = \frac{|a|}{|b|}$  for  $b \neq 0$ ;
- $|a \pm b| \neq |a| \pm |b|$ .

## 7.3 Triangular inequalities

Let  $a, b \in \mathbb{R}$ , then:

$$\begin{array}{l} |a| + |b| \geq |a + b| \\ |a| - |b| \leq |a - b| \end{array}$$

## Part II

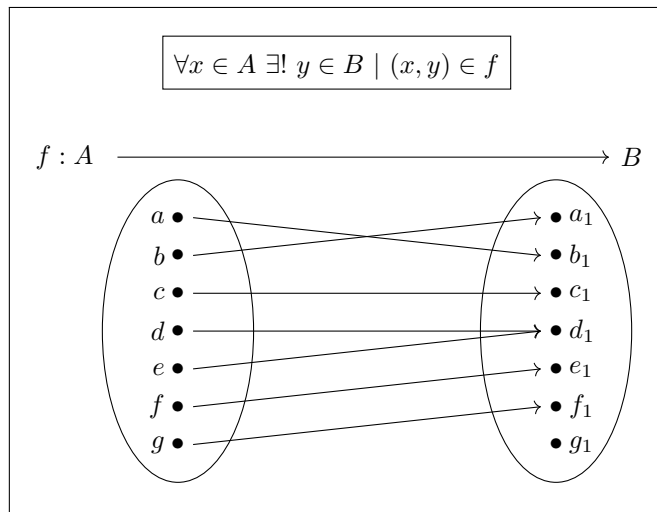
## Week 2

### 8 Concept of functions

Let's take any two sets  $A = \{a, b, c, d, e, f, g\}$  and  $B = \{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$ .

$$\begin{array}{l} f : A \rightarrow B \\ a \mapsto f(a) \end{array}$$

A function is a relation between the sets  $A$  and  $B$ , according to which we associate to each element of  $A$  one and only one element of  $B$ :



Notation:  $f(a) = b_1, f(b) = a_1, f(c) = c_1, f(d) = d_1, \dots$

Each point in set  $A$  is associated with one element of  $B$ . However, it is possible for more than two elements of  $A$  to point to the same element of  $B$ .

The set  $A$  is called *domain* of  $f$ . The set  $B$  is called the *codomain* of  $f$ .

#### 8.1 Image (Range)

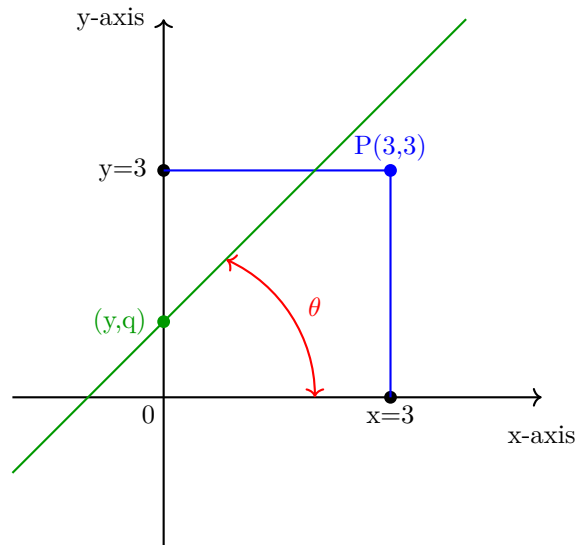
Let  $f : X \rightarrow Y$  be a function. The image of  $f$  is defined as:

$$\text{Im}(f) = \{y \in Y \mid y = f(x), x \in X\}$$

Easily, the image is the set containing all the elements of the set  $B$  associated with the elements of the set  $A$ .

## 9 Linear function

### 9.1 Cartesian diagram



### 9.2 Straight line

Let A and B be any two distinct points, then there is one and only one line passing through A and B.

### 9.3 Slope-intercept equation

Let  $m, q \in \mathbb{R}$ , then

$$y = mx + q$$

- $m$ : slope;
- $q$ : vertical intercept.

#### 9.3.1 Slope

The slope of a line can be calculated with the equation

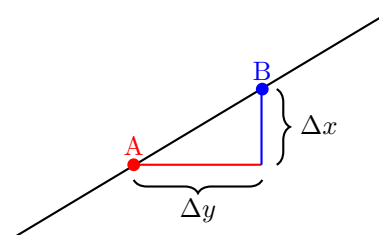
$$m = \frac{y_B - y_A}{x_B - x_A} = \frac{\Delta y}{\Delta x} = \tan(\theta)$$

We have three different slope outcomes:

- $m > 0$ , the line is increasing;
- $m = 0$ , the line is stable;
- $m < 0$ , the line is decreasing.

Warning: This works only if  $x_B \neq x_A$ .

#### 9.3.2 Drawing



## 9.4 Vertical lines

The more the value of  $m$  increases, the closer the line will get to the vertical, without ever reaching it.

Let  $c \in \mathbb{R}$ , then  $x = c$ .

Vertical lines cannot be written as a function.

## 10 Equation of a line

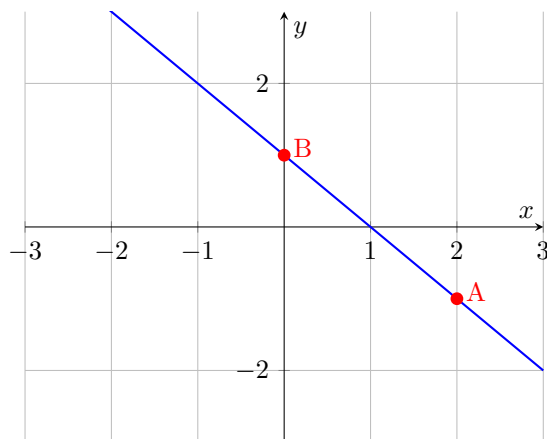
Let  $m, x_A, y_A \in \mathbb{R}$  and  $A(x_A, y_A)$ , then

$$y - y_A = m(x - x_A)$$

e.g.: Find the line with  $m = -1$  and  $A(2, -1)$ .

$$y - 1 = -1(x + 2) \Rightarrow y = -x + 1$$

Points:  $A(2, -1)$ ;  $B(0, 1)$



### 10.1 General equation in a cartesian diagram

$$ax + by + c = 0$$

Remark:

- All the lines can be described with this kind of equation;
- When  $b = 0$ ,  $a \neq 0$ , then  $ax = -c \Rightarrow x = \frac{-c}{a} \in \mathbb{R}$ ;
- When  $b \neq 0$ , then  $y = -\frac{a}{b}x - \frac{c}{b}$ , where  $m = -\frac{a}{b}$  and  $q = -\frac{c}{b}$ .

## 11 Increasing and decreasing functions

Let  $f : [a, b] \rightarrow \mathbb{R}$

Notation: if you replace  $[a, b]$  with  $\mathbb{R}$ , you obtain the definition in the whole  $\mathbb{R}$ .

### 11.1 Increasing functions

- $f$  is increasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) \geq f(x_1)$ ;
- $f$  is strictly increasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) > f(x_1)$ .

### 11.2 Decreasing functions

- $f$  is decreasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) \leq f(x_1)$ ;
- $f$  is strictly decreasing if  $\forall x_1, x_2 \in [a, b] \mid x_2 > x_1$ , then  $f(x_2) < f(x_1)$ .

## 12 Inverse function

Let's take any two sets  $A$  and  $B$ .

A function  $f : A \rightarrow B$  is invertible if there exists another function  $f^{-1} : B \rightarrow A$ , called the inverse function, such that:

$$\begin{array}{l} \forall x \in A, f^{-1}(f(x)) = x \\ \forall y \in B, f(f^{-1}(y)) = y \end{array}$$

Warning: A function is invertible if and only if it is bijective.

### 12.1 Facts about inverse functions

1)

Let  $f : D \rightarrow \mathbb{R}$

$f$  is invertible in  $D$  when:

- $f$  is strictly increasing;
- $f$  is strictly decreasing.

2)

Let  $f : D \rightarrow \mathbb{R}$

$f$  is invertible when  $f^{-1} : \text{Im}(f) \rightarrow D$ .

## Part III

# Week 3

## 13 Polynomial function

### 13.1 Expressions, terms and factors

#### 13.1.1 Expressions

An expression is any formula containing numbers, variables, operations, and brackets.

$$y = ax^2 + bx \cdot c$$

#### 13.1.2 Terms

A term is any part of the expression separated by “+” or “−”.

$$y = \underbrace{ax^2}_{\text{term}} + \underbrace{bx \cdot c}_{\text{term}}$$

#### 13.1.3 Factors

Each term can be split into a product of factors.

$$x \cdot y \cdot (a - b) \cdot 24 = x \cdot y \cdot (a - b) \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

Notice: the process of splitting a term into several factors is called “factorization”.  
The goal of a factorization is to factorize an expression as much as possible.

## 14 Common factor

Any expression made of terms is composed of several factors.

$$x^2 + x^3 + x = x(x + x^2 + 1), \forall x \in \mathbb{R}$$

## 15 Notable products

- $(a + b)^2 = a^2 + 2ab + b^2$  (square of a binomial);
- $(a - b)^2 = a^2 - 2ab + b^2$  (square of a binomial);
- $(a - b)(a + b) = a^2 - b^2$  (difference of squares);
- $(a + b)(a^2 - ab + b^2) = a^3 + b^3$  (sum of cubes);
- $(a - b)(a^2 + ab + b^2) = a^3 - b^3$  (difference of cubes).

Remark: notable products are useful to factorize expressions when we don’t know a common factor.

## 16 Classification of polynomials

Polynomials can be classified using two criteria:

1. the number of terms;
2. the degree of the polynomial.



Number of Terms	Name	Example	Comment
One	Monomial	$ax^2$	Mono means “one” in Greek
Two	Binomial	$ax^2 - bx$	Bi means “two” in Latin
Three	Trinomial	$ax^2 - bx + c$	Tri means “three” in Greek
Four or more	Polynomial	$ax^3 - bx^2 + cx - d$	Poly means “many” in Greek

## 16.1 Definition

Let  $n \in \mathbb{N}^*$ , then a polynomial is the sum or difference of n-monomials.

## 16.2 Degree

The degree of a polynomial is the largest exponent of its monomials.

### 16.2.1 Monomials

The degree of a monomial is the sum of all the exponents of all the variables.

$$p(x) = x^2 + 1 \rightarrow \text{the degree is 2.}$$

$$\forall x \in \mathbb{R}, p(0) = 0^2 + 1 = 1 \rightarrow 1 \text{ is a polynomial with degree 0.}$$

### 16.2.2 Polynomials

The degree of a polynomial is the highest of all the degrees of all the monomials which compose the polynomial.

$$p(x) = x^3 + 1 + x^5 + x^2 \rightarrow \deg(p(x)) = 5$$

$$q(x) = 12 \underbrace{abcd}_{\deg=4} - 31x^3 + 2xy \rightarrow \deg(q(x)) = 4$$

Notation: Let  $f(x) = ax^2 + bx + c$ ,  $a$  and  $b$  are called coefficient.

The coefficient of the monomial with highest coefficient is called **leading coefficient**.

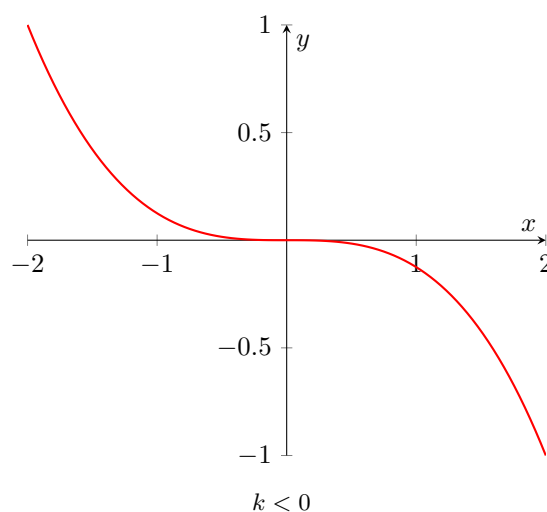
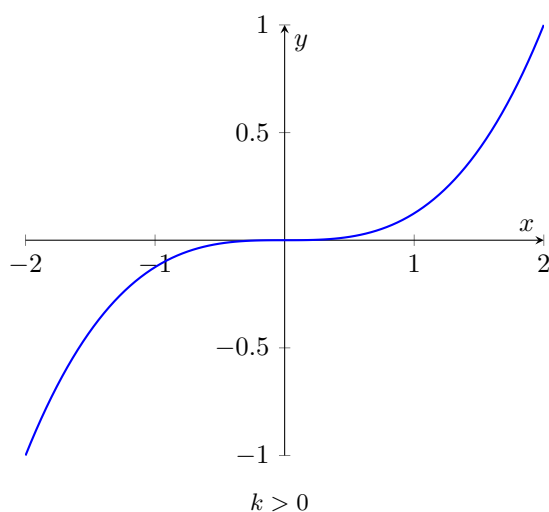
## 17 Symmetrical functions

Let  $y = kx^n$ , then we plot:

### 17.1 $n$ odd

$$f(-x) = -f(x), \quad \forall x \in \mathbb{R}$$

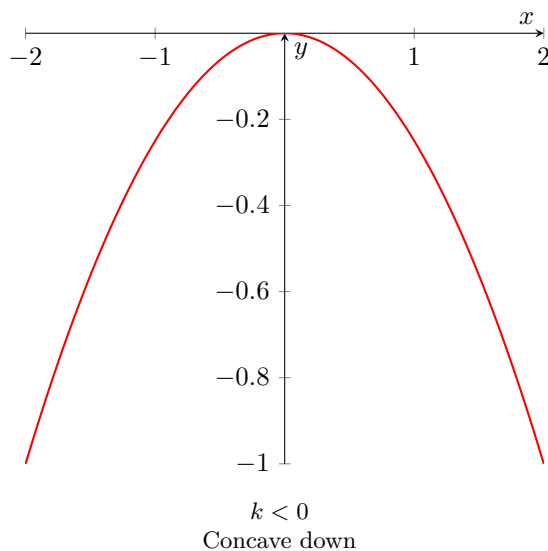
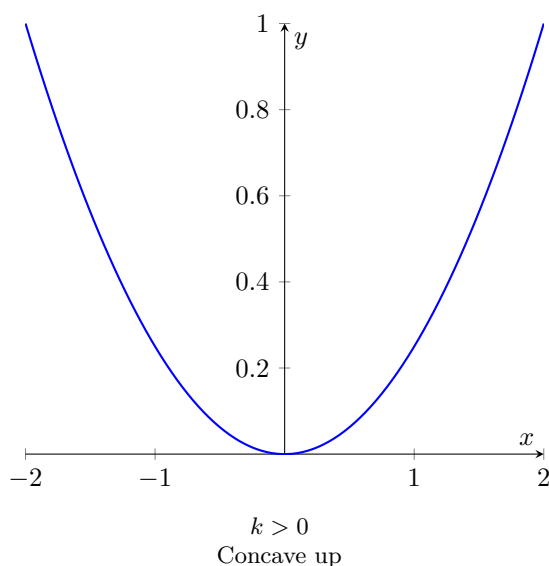
#### 17.1.1 Graph examples



## 17.2 $n$ even

$$f(-x) = f(x), \quad \forall x \in \mathbb{R}$$

### 17.2.1 Graph examples



Definition:

- a function  $y = f(x)$  is called **odd** if it is symmetric with respect to the origin;
- a function  $y = f(x)$  is called **even** if it is symmetric with respect to the y-axis.

## 17.3 General case

Let  $y = p(x)$ , where  $p(x)$  is any polynomial with real coefficients:

$$p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-2} + a_{n-2} \cdot x^{n-1} + \cdots + a_2 \cdot x^2 + a_1 \cdot x^1 + a_0 \cdot x$$

where:

- $n \in \mathbb{N}$ ;
- $n = \deg(p(x))$ ;
- $a_n =$  leading coefficient.

$$p(x) = \sum_{i=0}^n a_i \cdot x^i$$

## 17.4 Symmetry of a polynomial

Let  $y = p(x)$  be a polynomial function, then:

1)

$y = p(x)$  is odd iff all the degrees of all the terms of  $p(x)$  are odd;

2)

$y = p(x)$  is even iff all the degrees of all the terms of  $p(x)$  are even;

3)

$y = p(x)$  has mixed degrees,  $p(x)$  is neither odd nor even.