Numerical Methods for Mathematical Finance

Lecture 1: Brownian Motion and Geometric BM

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Presentation Outline

About This Mini-Course

- Timetable: **November's Tuesdays**

  8 hours in room L
  - Tuesday 5/11: 16:30-18:30
  - Tuesday 12/11: 16:30-18:30
  - Tuesday 19/11: 16:30-18:30
  - Tuesday 26/11: 16:30-18:30

- Prerequisites: Stochastic Calculus and Mathematical Finance

- We will use Python for simulations, specifically Jupyter Notebook

- Exam: Individual project (Report + Code + Presentation in a seminar)

Syllabus and Material

- Main Text: *Mathematical Modeling and Computation in Finance* by C. W. Oosterlee and L. A. Grzelak

  First 150 pages:

  `https://www.researchgate.net/publication/334748386_Mathematical_Modeling_and_Computation_in_Finance_With_Exercises_and_Python_and_MATLAB_Computer_Codes`

  Extra:

  `https://github.com/LechGrzelak/QuantFinanceBook`

- Slide + notebooks available on personal GitHub page for the course `https://github.com/matteogarbelli/NM4MF24`

Plan for the Course

1. Simulations of:
   - Brownian motion (BM),
   - Geometric BM,
   - Correlated (G)BMs
2. Simulations for SDEs using:
   - Euler-Maruyama scheme
   - Milstein scheme
3. Pricing of European options via:
   - Monte Carlo approach
4. Introduction to:
   - Energy markets (day-ahead electricity market)
   - Application of a Reinforcement Learning Algorithm to the electricity market

Presentation Outline

Stochastic Processes in Finance

- Stochastic processes model uncertain or random events that fluctuate through time
- We aim to create processes resembling financial and economic dynamics
- These processes help answer questions related to the behaviour of financial phenomena

Definition of a Stochastic Process

A **stochastic process** $\{X(t), t \in \mathcal{I}\}$ is a collection of random variables indexed by t taking values in some index set $\mathcal{I}$:

- If the stochastic process is **discrete in time**, then $\mathcal{I} = \{t_0, t_1, \dots, t_N\}$ (or $\mathcal{I} = \{t_0, t_1, \dots\}$). For discrete processes, we often write $\{X_n = X(t_n), t_n \in \mathcal{I}\}$.
- If the stochastic process is **continuous in time**, then $\mathcal{I} = [a, b]$ or $\mathcal{I} = \mathbb{R}_+$.

We usually think of the index t as time, hence we assume $t_0 < t_1 < t_2 < \dots$. Then $X(t)$ is the value of the process at time t.

Definition of a Probability Space

We call the tuple $(\Omega, \mathscr{F}, \mathbb{P})$ a **probability space** where:

- *Sample Space* Ω is the set of all possible outcomes;
- *Filtration* $(\mathcal{F}_t)$: a sequence of increasing sets of information, represented by collections of subsets of the sample space (events);
- *Probability measure* $\mathbb{P}$: a function that assigns a probability to each event within the filtration, with values between 0 and 1.

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Example. (A fair six-sided dice)

- Sample Space (Ω): $\{1, 2, 3, 4, 5, 6\}$
- Filtration: $A = \{2, 4, 6\}$, $B = \{5, 6\}$
- Probability Function P:
  - $P(A) = 0.5$
  - $P(B) = \frac{1}{3}$

Natural Filtration and Adaptation

- We call $\mathscr{F} = \{\mathscr{F}_t, t \in \mathcal{I}\}$ the **natural filtration** generated by the process X.
- In words: $\{\mathscr{F}_t, t \in \mathcal{I}\}$ is the **filtration** associated to a stochastic process $\{X(t), t \in \mathcal{I}\}$, if for all $t_i \in \mathcal{I}$, $\mathscr{F}_{t_i}$ contains all the information about the history of the stochastic process X up to time t_i.
- A stochastic process X is **adapted to the filtration** $\mathscr{F}$ if $\sigma(X_{t_j}, 1 \leq j \leq i) \subseteq \mathscr{F}_{t_i}$.
- In words: the process X cannot "look into the future".

Filtration in a Stochastic Process

- Suppose we have a set of calendar dates/days, $t_1, t_2, \ldots, t_m$;
- Up to today, we have observed certain state values of the stochastic process $X(t)$, hence the past is known;
- For the future, we do not know the precise path, but we may simulate the future according to increasing information.

Filtration in a Stochastic Process

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The information available at time t_i is described by a filtration:

We say that $\{\mathscr{F}_t, t \in \mathcal{I}\}$ is the **filtration** associated to a stochastic process $\{X(t), t \in \mathcal{I}\}$, if for all $t_i \in \mathcal{I}$, $\mathscr{F}_{t_i}$ is the sigma algebra $\mathscr{F}_{t_i} = \sigma(X_{t_j}, 1 \leq j \leq i)$ generated by the sequence X_{t_j} for $1 \leq j \leq i$.

Filtration in a Stochastic Process

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A filtration is a family of σ-algebras that are ordered non-decreasingly.
Note that for $s < t$, we have that $\mathscr{F}_s \subset \mathscr{F}_t$.

Martingale

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we say that a (right-continuous with left limit) stochastic process $\{M(t), t \in \mathcal{I}\}$ is a **martingale** associated to the filtration $\mathscr{F} = \{\mathscr{F}_t, t \in \mathcal{I}\}$ if the following properties hold:

1. M is adapted to the filtration $\mathscr{F}$;
2. $\mathbb{E}[|M(t)|] < \infty$, for all $t \in \mathcal{I}$;
3. For all $s < t \in \mathcal{I}$, we have

$$\mathbb{E}[M(t)|\mathscr{F}_s] = M(s).$$

The best prediction of the expectation of a martingale's future value is its present value.

Presentation Outline

Aims

1. Simulate a BM in Python
2. Simulate a GBM in Python
3. Simulate correlated BMs in Python
4. Simulate correlated GBMs in Python

Brownian Motions

1. Standard BM
2. Arithmetic (Generalized) BM
3. Geometric BM

Brownian Motion

A Brownian motion is a continuous-time stochastic process that starts at zero and has independent and normally distributed increments, and satisfies the stationarity of increments.

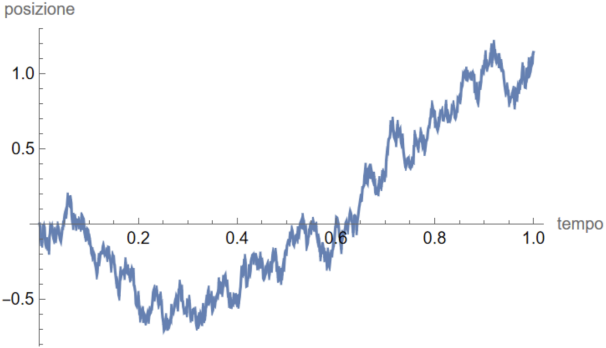
A stochastic process $\{W(t), t \in [0, \infty)\}$ is a **(standard) Brownian motion** if:

- $W(0) = 0$;
- $W(t) - W(s)$ is independent from $W(t') - W(s')$ for $[s, t] \cap [s', t'] = \emptyset$
- $W(t) - W(s) \sim N(0, t - s)$
- $W(t)$ is almost surely continuous

Standard Brownian Motion $W(t)$

Let us consider the process $\{W(t), 0 \leq t \leq T\}$, a standard Brownian motion. Since $W(t) = W(t) - W(0)$, it follows that:

$$W(t) \sim \mathcal{N}(0, t).$$



Arithmetic Brownian Motion $B(t)$

We say that a stochastic process $\{B(t), 0 \leq t \leq T\}$ is a **Arithmetic (or Generalized) Brownian motion**, with drift $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma \in \mathbb{R}^+$, if the process $\frac{B(t)-\mu t}{\sigma}$ is a standard Brownian motion.

Consider $B(t)$, a Arithmetic Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma \in \mathbb{R}^+$. From the definition, it is known that:

$$W(t) := \frac{B(t) - \mu t}{\sigma}$$

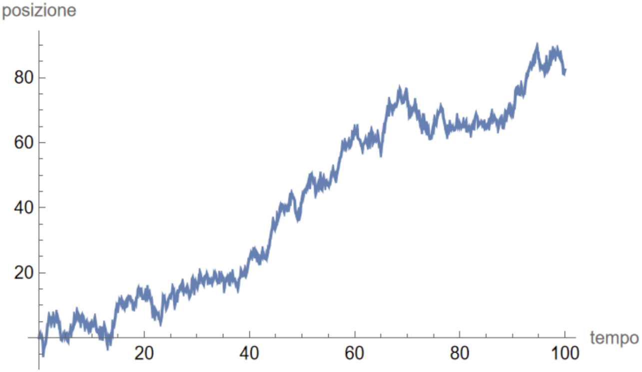
is a standard Brownian motion, and for each $t \in [0, T]$, we have:

$$B(t) = \sigma W(t) + \mu t.$$

Arithmetic Brownian Motion $B(t)$

The distribution of $B(t)$ with $t \in [0, T]$ is given by:

$$B(t) \sim N(\mu t, \sigma^2 t).$$



Drift $\mu = 1$, diffusion $\sigma = 10$

Applications of Arithmetic BM

- **Finance:**
  - Modeling short-term asset prices and returns with linear trends.
  - Interest rate modeling, especially for mean-reverting rates.
  - Simplified option pricing (e.g., Bachelier model for non-negative prices).
  - Simulating P&L for hedged portfolios with minimal compounding.
- **Economics:**
  - Forecasting economic indicators like unemployment or exchange rates.
  - Public debt and budget deficit forecasting.
- **Energy Market:**
  - Modeling electricity and gas prices with mean-reverting tendencies.
  - Short-term commodity price forecasting (e.g., oil, coal).

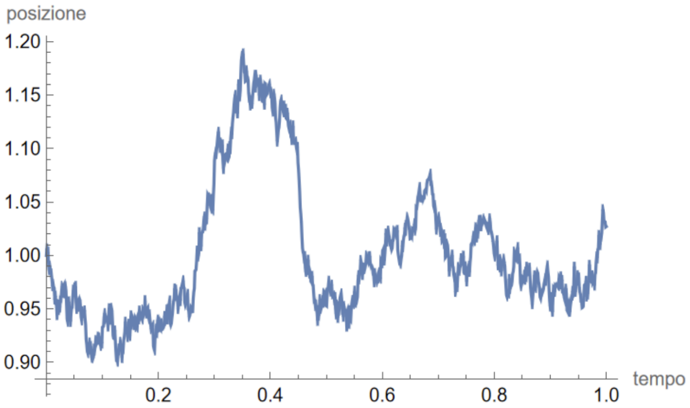
Geometric Brownian Motion

- The most common asset price process in finance is the Geometric Brownian Motion (GBM):

$$S(t) = S(0)e^{\sigma W(t)+\mu t}, \quad t \geq 0$$

- GBM models stock prices assuming percentage changes are independent and identically distributed

Geometric Brownian Motion $S(t)$



Geometric Brownian Motion

A stochastic process s_t is said to follow a GBM if it satisfies the following SDE:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \tag{1}$$

where:

- W_t is a Brownian motion;
- **Drift** μ: Represents the average rate of return of the process.;
- **Volatility** σ: Measures the variability or dispersion of the process.

Drift and Volatility are assumed to be constant!

Applying Ito's Lemma to $Y_t = \ln(X_t)$

Ito's Formula

$$df(X_t) = \left(\frac{\partial f}{\partial X}\mu X_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2 X_t^2\right) dt + \frac{\partial f}{\partial X}\sigma X_t \, dW_t.$$

Applying Ito's Lemma to $Y_t = \ln(X_t)$

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We define $Y_t = \ln(X_t)$ and seek the differential dY_t. Since $f(X_t) = \ln(X_t)$, then:

$$dY_t = \left(\frac{1}{X_t} \cdot \mu X_t + \frac{1}{2} \cdot \left(-\frac{1}{X_t^2} \right) \sigma^2 X_t^2 \right) dt + \frac{1}{X_t} \sigma X_t \, dW_t.$$

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Thus, the SDE for $Y_t = \ln(X_t)$ is:

$$dY_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma \, dW_t.$$

Integrating + Exponential

Now we integrate both sides over the interval $[0, t]$:

$$Y_t = Y_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma \, dW_s.$$

Since $Y_0 = \ln(X_0)$, we have:

$$Y_t = \ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

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$$Y_t = \ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Since $X_t = e^{Y_t}$, we get:

$$X_t = \exp\left(\ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Analytic Solution of GBM

For an arbitrary initial value X_0, the solution to the GBM stochastic differential equation (SDE) is:

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{2}$$

where μ is the drift, σ is the volatility, and W_t is a Wiener process (Brownian motion).

The Two Equations

Let's consider two equations describing stochastic processes:

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

vs

$$S_t = S_0 \exp\left(\mu t + \sigma W_t\right).$$

- The term $\mu - \frac{\sigma^2}{2}$ represents the **adjusted drift**, accounting for the variability introduced by the stochastic process.
- The adjustment $-\frac{\sigma^2}{2}$ arises from Ito's Lemma, reflecting the impact of volatility on the expected rate of growth.
- The naive drift of μ does not consider the mean-reducing effect of volatility.

Expected Value and Variance of X_t

The solution X_t for any value of t is a log-normally distributed random variable with expected value and variance given by:

$$\mathbb{E}(X_t) = X_0 e^{\mu t} \tag{3}$$

$$\mathrm{Var}(X_t) = X_0^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1 \right) \tag{4}$$

These properties are helpful in financial modeling, particularly for calculating the mean and risk (volatility) of assets over time.

Properties of GBM

GBM is widely used in financial modeling, especially for stock prices, as it assumes that returns are log-normally distributed.

- X_t is always positive, making GBM a popular choice for modeling asset prices.
- The logarithm of X_t follows an Arithmetic Brownian Motion, meaning $\ln(X_t)$ has normally distributed increments.
- At any time t, S_t is log-normally distributed.

Geometric Brownian Motion is a fundamental model for:

- Stock price modeling
- Option pricing (Black-Scholes model)
- Other financial assets that cannot take negative values

Why GBM to Model Stock Prices?

Some of the reasons for using GBM in stock price modeling include:

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we expect in reality.
- A GBM process only assumes positive values, similar to real stock prices.
- A GBM process shows the same type of "roughness" in its paths as seen in real stock prices.
- Calculations with GBM processes are relatively easy.

Limitations of GBM in Modeling Stock Prices

Despite its usefulness, GBM is not a completely realistic model and has the following limitations:

- In reality, stock price volatility changes over time (possibly stochastically), but in GBM, volatility is assumed constant.
- Real stock prices often experience jumps due to unpredictable events or news, but in GBM, the path is continuous (no discontinuities).

Presentation Outline

Aims

1. Simulate a BM in Python
2. Simulate a GBM in Python
3. Simulate correlated BMs in Python
4. Simulate correlated GBMs in Python

1) Simulating a Brownian Motion (BM)

For a Brownian motion $W(t)$, we know:

- $W(0) = 0$
- W has independent increments
- $W(t) - W(s) \sim N(0, t - s)$

Given a partition $0 = t_0 < t_1 < \cdots < t_m = T$, we have:

$$W(t_{i+1}) - W(t_i) \sim N(0, t_{i+1} - t_i)$$

1. Generate m i.i.d. unit normal variables $z_1, z_2, \ldots, z_m$
2. Compute the increments: $W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i} \cdot z_i$
3. Recursively compute $W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} \cdot z_i$

2) Simulating a Geometric Brownian Motion (GBM)

We want to simulate:

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

Let $X(t) = (\mu - \frac{\sigma^2}{2})t + \sigma W(t)$, then:

$$S(t) = S(0)e^{X(t)}$$

We simulate $X(t)$ by the recursion:

$$X(t_{i+1}) = X(t_i) + (\mu - \frac{\sigma^2}{2})(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i))$$

Finally, we compute:

$$S(t_{i+1}) = \exp(X(t_{i+1}))$$

3) Simulating Correlated Brownian Motions

Let

$$\boldsymbol{W}(t) := \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}.$$

To simulate $\boldsymbol{W}(t)$, we can first simulate two independent Brownian motions $\widetilde{\boldsymbol{W}}_1$ and $\widetilde{\boldsymbol{W}}_2$. For

$$\widetilde{\boldsymbol{W}}(t) := \begin{pmatrix} \widetilde{W}_1(t) \\ \widetilde{W}_2(t) \end{pmatrix},$$

we obtain $\boldsymbol{W}(t)$ by

$$\boldsymbol{W}(t) = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \widetilde{\boldsymbol{W}}(t) = \begin{pmatrix} \widetilde{W}_1(t) \\ \rho \widetilde{W}_1(t) + \sqrt{1-\rho^2}\, \widetilde{W}_2(t) \end{pmatrix} = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}.$$

Simulating Correlated BMs

- We generate m i.i.d. pairs of unit normal $\boldsymbol{z}_i \in \mathbb{R}^2$, for $i = 1, \ldots, m$.
- We obtain:

$$\begin{cases} \boldsymbol{W}(0) = 0 \in \mathbb{R}^2 \\ \boldsymbol{W}(t_{i+1}) = \boldsymbol{W}(t_i) + \sqrt{t_{i+1} - t_i} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \boldsymbol{z}_{i+1}, \end{cases}$$

for $i = 0, \ldots, m-1$.

Covariance Calculation

To check the covariance between $W_1(t)$ and $W_2(t)$:

$$\begin{aligned}
\mathrm{Cov}(W_1(t), W_2(t)) &= \mathbb{E}[W_1(t)W_2(t)] - \mathbb{E}[W_1(t)]\mathbb{E}[W_2(t)] \\
&= \mathbb{E}\left[\widetilde{W}_1(t)\left(\rho\widetilde{W}_1(t) + \sqrt{1-\rho^2}\,\widetilde{W}_2(t)\right)\right] - 0 \\
&= \rho\mathbb{E}[\widetilde{W}_1(t)^2] + \sqrt{1-\rho^2}\,\mathbb{E}[\widetilde{W}_1(t)]\mathbb{E}[\widetilde{W}_2(t)] \\
&= \rho\mathbb{E}[\widetilde{W}_1(t)^2] = \rho\,\mathrm{Var}[\widetilde{W}_1(t)] = \rho t,
\end{aligned} \tag{5}$$

using $\mathbb{E}[W_i(t)] = \mathbb{E}[\widetilde{W}_i(t)] = 0$.

Thus, the correlation between $W_1(t)$ and $W_2(t)$ equals ρ, as desired.

4) Simulating Correlated GBMs in Python

In many applications, we may want to simulate several correlated assets, such as in a portfolio or for basket options. Assume two correlated GBMs:

$$S_1(t) = S_1(0)e^{X_1(t)}, \quad X_1(t) = \mu_1 t + \sigma_1 W_1(t)$$

$$S_2(t) = S_2(0)e^{X_2(t)}, \quad X_2(t) = \mu_2 t + \sigma_2 W_2(t)$$

where $W_1(t)$ and $W_2(t)$ are correlated Brownian motions, with:

$$\mathrm{Cov}(W_1(t), W_2(t)) = \rho$$

for every $t \geq 0$, with $-1 \leq \rho \leq 1$.

Simulating Correlated GBMs I

- We simulate a Brownian motion with drift $\boldsymbol{X} \in \mathbb{R}^2$:

$$\begin{cases} X(0) = \log(S(0)) \\ X(t_{i+1}) = X(t_i) + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} (t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} \boldsymbol{z}_{i+1}, \end{cases}$$

- We get:

$$S(t_{i+1}) = \exp(X(t_{i+1})), \quad i = 0, \ldots, m-1,$$

where the exponential function acts element-wise.

Simulating Correlated GBMs II

- The equations for simulating S_1 and S_2 are:

$$S_1(t_{i+1}) = S_1(t_i)\exp\left\{\mu_1(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\cdot\sigma_1\cdot z_{i+1,1}\right\}$$

$$S_2(t_{i+1}) = S_2(t_i)\exp\left\{\mu_2(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\cdot\sigma_2\cdot\left(\rho z_{i+1,1} + \sqrt{1-\rho^2}z_{i+1,2}\right)\right\}$$

- What is the matrix $L := \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$?
- The matrix L is the Cholesky decomposition of the correlation matrix C.

Cholesky Decomposition

Each symmetric positive definite matrix C has a unique factorization, the so-called Cholesky Decomposition , of the form

$$C = LL^\top,$$

where L is a lower triangular matrix with positive diagonal entries.

For two processes X_1 and X_2, the correlation matrix is:

$$\begin{pmatrix} \mathrm{corr}(X_1, X_1) & \mathrm{corr}(X_2, X_1) \\ \mathrm{corr}(X_1, X_2) & \mathrm{corr}(X_2, X_2) \end{pmatrix}.$$

For a given (2×2)-C, we find L as:

$$C \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \rho_{X_1 X_2} \\ \rho_{X_2 X_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho_{X_2 X_1} & \sqrt{1 - \rho^2_{X_2 X_1}} \end{pmatrix} \begin{pmatrix} 1 & \rho_{X_1 X_2} \\ 0 & \sqrt{1 - \rho^2_{X_1 X_2}} \end{pmatrix}.$$