Numerical Methods for Mathematical Finance

Lecture 2: Numerical Methods for SDE Simulation

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From Lesson 1...

A standard Brownian motion W_t is a stochastic process with:

1. $W_0 = 0$
2. Independent increments: $W_{t+s} - W_t$ is independent of the past values of W_t.
3. Gaussian increments: $W_{t+s} - W_t \sim \mathcal{N}(0, s)$.
4. Continuous paths, though they are almost surely non-differentiable.

For us: In SDEs, dW_t represents an "infinitesimal" increment in Brownian motion, satisfying:

$$dW_t \sim \mathcal{N}(0, dt)$$

GBM

The stock price $S(t)$ follows a Geometric Brownian Motion (GBM) process, satisfying the following SDE:

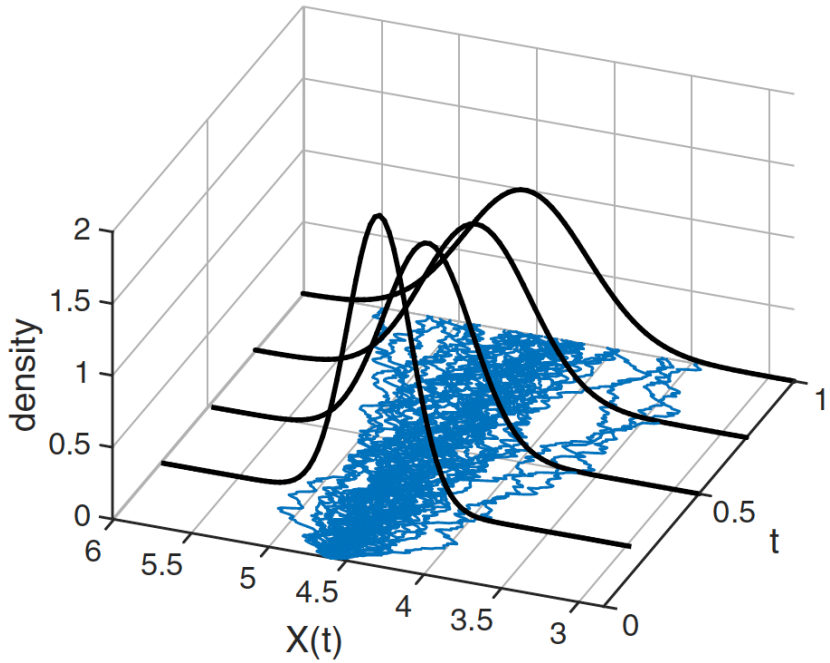
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(t_0) = S_0$$

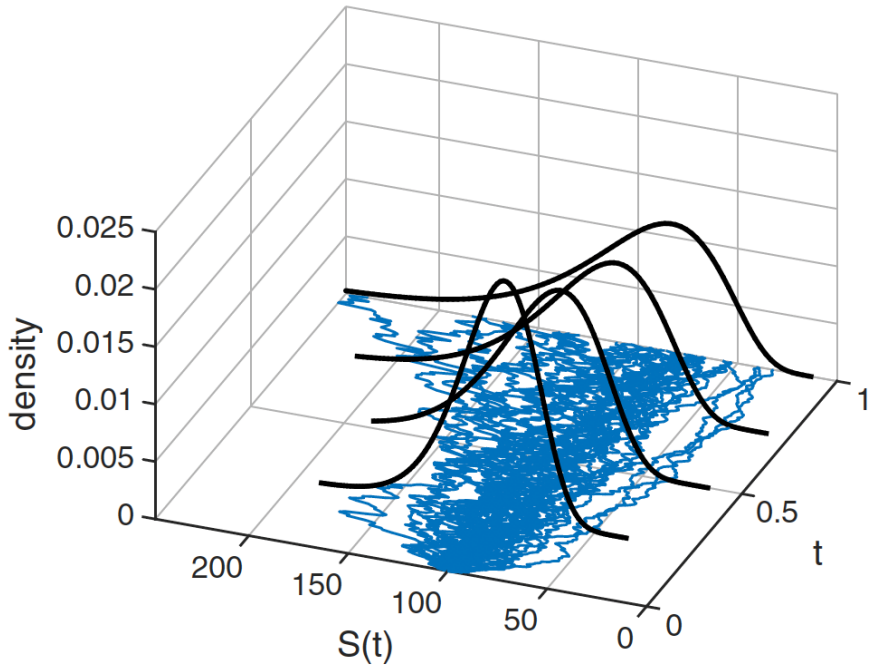
More generally, we consider the Itô process:

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

Paths and Corresponding Densities for GBM

$S_0 = 100,\ \mu = 0.05,\ \sigma = 0.4;\ T = 1$





$X(t) = \log S(t)$ follows a Gaussian distribution, ideal for modelling stock prices: log returns align with the normality assumption in financial models.

Today

1. Introduction to SDEs
   - Definitions
   - Example of SDEs
   - Numerical Solution of SDEs
2. Euler-Maruyama scheme
   - Pathwise Approximation
   - Weak Convergence
3. Milstein Scheme
4. SDE Simulations
   - Simulation of the OH Process
   - Cox–Ingersoll–Ross model
   - Heston model

Aims

1. Simulate a GBM with Euler–Maruyama method and study weak and strong convergence
2. Simulate an OH process with Euler and Milstein methods
3. Simulate CIR process with Taylor-based simulations
4. Simulate Heston models with Truncated Euler and Almost Exact Method

Presentation Outline

What are Stochastic Differential Equations?

- Stochastic Differential Equations (SDEs) describe systems that evolve over time with inherent randomness.
- SDEs combine deterministic and stochastic components, often using **Brownian motion** or other stochastic processes to model randomness.

Deterministic vs. Stochastic Differential Equations

Deterministic Differential Equation:

$$\frac{dx}{dt} = f(x, t)$$

- Solution is completely determined by the initial condition.
- Predictable evolution over time.

Stochastic Differential Equation:

$$dX_t = f(X_t, t)\,dt + g(X_t, t)\,dW_t$$

- Solution includes randomness, represented by dW_t, where W_t is Brownian motion.
- Evolution over time is unpredictable, with random fluctuations.

Form of an SDE

A typical SDE is written as:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

where:

- $\mu(X_t, t)$ is the **drift term**, representing the deterministic part that describes the average rate of change.
- $\sigma(X_t, t)$ is the **diffusion term**, representing the random fluctuation that adds stochastic variation, scaling with standard BM dW_t.

Existence Theorem

Drift $\mu(t,x)$ and volatility $\sigma(t,x)$ functions cannot be "just any" functions. They need to satisfy the following:

1. Lipschitz condition:

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K_1|x-y|,$$

2. Linear Growth

$$|\mu(t,x)| + |\sigma(t,x)| \leq K_2(1+|x|),$$

for some constants $K_1, K_2 \in \mathbb{R}^+$ and x and $y \in \mathbb{R}$.

When these conditions hold, then, with probability one, a continuous, adapted solution of SDE exists, and the solution satisfies

$$\sup_{0\leq t\leq T} \mathbb{E}[X^2(t)] < \infty.$$

Ito's Lemma

Ito's Lemma is the stochastic calculus equivalent of the chain rule for differentiating functions of stochastic processes.

Statement: Let X_t follow

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

and $f(X_t, t)$ be a twice differentiable function in X and once differentiable in t.

Then:

$$df(X_t, t) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma \frac{\partial f}{\partial X} dW_t$$

Geometric Brownian Motion (GBM)

GBM exhibits exponential growth with random fluctuations, characterized by drift and volatility.
The process X_t representing the asset price at time t follows

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t$$

where μ is the Drift rate, σ: the Volatility and W_t a Standard Brownian motion.

- **Expectation:** $\mathbb{E}[X_t] = X_0 e^{\mu t}$
- **Variance:** $\mathrm{Var}(X_t) = X_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$
- **Analytical Solution:** $X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$

Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process models mean-reverting behavior, commonly used in finance and physics.
It is a Gaussian process with a stationary distribution, reverting to a long-term mean over time.

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\,dW_t$$

where μ is the Long-term mean and θ is the Mean Reversion Speed.

- **Expectation:** $\mathbb{E}[X_t] = \mu + (X_0 - \mu)e^{-\theta t}$
- **Variance:** $\mathrm{Var}(X_t) = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$
- **Stationary Distribution:** $X_t \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2\theta}\right)$

Cox-Ingersoll-Ross (CIR) Model

The CIR model is widely used in finance for modeling interest rates and stochastic volatility.
It is a mean-reverting process that remains non-negative, making it suitable for quantities that cannot be negative.

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t$$

- **Mean Reversion:** $\mathbb{E}[X_t] \to \mu$ as $t \to \infty$
- **Variance Condition:** Feller condition $2\theta\mu \geq \sigma^2$ ensures non-negativity.
- **Stationary Distribution:** $X_t \sim \text{Gamma}\left(\frac{2\theta\mu}{\sigma^2}, \frac{\sigma^2}{2\theta}\right)$

Heston Model for Stochastic Volatility

The Heston model describes asset prices with stochastic volatility. It introduces a two-dimensional SDE system, modeling both price and volatility with correlated noise.

The asset price S_t at time t

$$dS_t = \mu S_t \, dt + \sqrt{V_t} S_t \, dW_{1,t}$$

whose stochastic volatility V_t

$$dV_t = \kappa(\mu - V_t) \, dt + \gamma \sqrt{V_t} \, dW_{2,t}$$

where ξ is the volatility of volatility, $W_{1,t}$, $W_{2,t}$ are two Correlated BMs.

- Mean Reversion:$\mathbb{E}[V_t] \to \mu$ as $t \to \infty$
- The correlation introduces dependencies between price and volatility.

Numerical Solution of SDEs

Solving SDEs analytically is challenging, so numerical methods are often used.

1. **Euler-Maruyama Method:**

$$X_{t+\Delta t} = X_t + \mu(X_t, t)\Delta t + \sigma(X_t, t)\Delta W_t$$

where $\Delta W_t \sim \mathcal{N}(0, \Delta t)$.

2. **Milstein Method (Higher Accuracy):**

$$X_{t+\Delta t} = X_t + \mu(X_t, t)\Delta t + \sigma(X_t, t)\Delta W_t + \frac{1}{2}\sigma(X_t, t)\frac{\partial \sigma}{\partial x}(X_t, t)((\Delta W_t)^2 - \Delta t)$$

Presentation Outline

Goal of Numerical Methods for SDEs

Generally, when both sides contain the term $X(t)$, the SDE needs to be solved numerically. An explicit solution is available only for a few standard processes.

Numerical methods are required when both sides of the SDE contain the term $X(t)$ and there is no explicit solution.

Our goal is to construct and analyze numerical methods for SDEs. We define a partition of the time interval:

$$0 = t_0 < t_1 < \cdots < t_m = T$$

where $\Delta t = \frac{T}{m}$, and use the Euler-Maruyama scheme for approximation.

Euler-Maruyama Scheme

The Euler-Maruyama method approximates the SDE by:

$$X(t_{i+1}) = X(t_i) + \mu(X(t_i))\Delta t + \sigma(X(t_i))\Delta W_i$$

where $\Delta W_i \sim \mathcal{N}(0, \Delta t)$. The recursion gives us the approximate values at discrete time steps, and intermediate values can be interpolated.

Euler Scheme for GBM

The Euler discretization for GBM is:

$$S(t_{i+1}) = S(t_i) + \mu S(t_i)\Delta t + \sigma S(t_i)(W(t_{i+1}) - W(t_i))$$

This method can be coded with $m = 2^4$ time steps. We observe that increasing the number of time steps improves the approximation.

Example: EM for GBM

The GBM process with dynamics, $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$, has an exact solution in the time interval $[t_i, t_{i+1}]$:

$$S(t_{i+1}) = S(t_i) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right)$$

The Euler discretization for the i-th time step is given by:

$$\begin{cases} S_{i+1} = S_i + \mu S_i \Delta t + \sigma S_i (W_{i+1} - W_i), \quad i = 0, \ldots, m-1 \\ S_0 = S(t_0) \end{cases}$$

Code: The exact solution should use $m = 2^9$ discretization points.

We consider the Euler scheme with $m = 2^2$ points; then $m = 2^4$ points.

Accuracy of the Euler Approximation

There are two main tasks in SDE simulations:

- Pathwise approximation: approximating the trajectory of a process

  Pathwise approximation methods aim to create discrete-time approximations of these continuous-time processes, producing paths that resemble the true stochastic trajectories as closely as possible.

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There are two main tasks in SDE simulations:

- Pathwise approximation: approximating the trajectory of a process

  Pathwise approximation methods aim to create discrete-time approximations of these continuous-time processes, producing paths that resemble the true stochastic trajectories as closely as possible.

- Approximating expectations: estimating functionals of the process

  Estimating expectations (or averages) of functionals of the process, especially when analytical solutions are unavailable. A functional could be something like the average or the maximum of the process over time, or any quantity derived from the stochastic process.

Defining a Criterion for Approximation Accuracy

We need to define a criterion to judge the quality or accuracy of the approximation, such a criterion must reflect the main goal of a practical simulation:

1. The first occurs in situations where a good pathwise approximation is required (e.g., direct simulations, filtering problems, testing statistical estimators, ...)
2. The second focuses on approximating expectations of functionals of the Itô process, e.g., its distribution or its moments. This is relevant when such functionals cannot be determined analytically.

Pathwise Approximation Error

When the exact solution is known, the pathwise error is computed using the Absolute Error Criterion:

$$\mathcal{E}^{strong} = \mathbb{E}\left[|X(T) - U(T)|\right]$$

where $X(T)$ is the exact solution and $U(T)$ is the Euler approximation.

Strong Convergence Error: The error is averaged over multiple simulations.

Pathwise Approximation

In cases where we know the explicit solution of an SDE, we can compute the approximation error using the Absolute Error Criterion:

$$\mathcal{E}^m = \mathbb{E}\left[|X(T) - x_m|\right]$$

This criterion provides a measure of the pathwise closeness at the end of the time interval $[0, T]$.

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For different Δt values, the empirical absolute error criterion over j paths:

$$\hat{\mathcal{E}}^m(\Delta t) = \frac{1}{N}\sum_{j=1}^{N} |S_j(T) - s_{m,j}| = \frac{1}{N}\sum_{j=1}^{N} \left| S(t_0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_{m,j}} - s_{m,j} \right|$$

where $s_{i,j}$ is the Euler discretization for the i-th timestep and j-th path, $i = 0, \ldots, m$ and $j = 1, \ldots, N$.

Absolute Error Criterion for Different Δt Values

We examine the absolute error criterion numerically. We repeat N different simulations denoted by $s_{m,j}$ for the Euler scheme and $S_j(T)$ for the exact one, with $j = 1, \ldots, N$.

Code: Simulate $N = 25$ trajectories for the Itô process and their corresponding Euler approximations with time steps: $\Delta t = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$. Repeat the experiment with several seeds.

Note: For this experiment, it is crucial to use the same Brownian motion for both the exact and approximated solution.

Δ	2^{-4}	2^{-5}	2^{-6}	2^{-7}
$\hat{\varepsilon}_1$	0.5093	0.4446	0.3265	0.2292
$\hat{\varepsilon}_2$	0.4692	0.3788	0.2234	0.1477

Table: Absolute errors $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ for different step lengths Δ.

Comments on the Absolute Error Estimate

Comments:

- The estimate of the absolute error decreases with decreasing time step Δt.
- However, these estimates are random variables.
- By the Central Limit Theorem, for large N, the error $\hat{\mathcal{E}}^m$ becomes asymptotically a Gaussian random variable and converges in distribution to the non-random value $\mathcal{E}^m$ as $N \to \infty$.

Weak Convergence

We define the weak convergence error for the expectation:

$$\mathcal{E}^{weak} = \left|\mathbb{E}[U(T)] - \mathbb{E}[X(T)]\right|$$

The goal is to ensure that the distributions of the Euler scheme and the exact solution are sufficiently close.

Weak Convergence

Let

$$\mu^m = \left|\mathbb{E}[S_m] - \mathbb{E}[S(T)]\right|$$

be the mean (weak) error.

Notice that to estimate μ, we do not need to use the same sample paths of the Wiener process when generating S and S_m. We require that the probability distributions of $S(T)$ and S_m are sufficiently close to each other.

Weak Convergence Error at Maturity for GBM

Then, for different Δt-values, the weak convergence error at the maturity time T is:

$$\hat{\mu}^m(\Delta t) = \left| \frac{1}{N} \sum_{j=1}^{N} S_j(T) - \frac{1}{N} \sum_{j=1}^{N} s_{m,j} \right|$$

$$= \left| S(t_0) \frac{1}{N} \sum_{j=1}^{N} e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_{m,j}} - \frac{1}{N} \sum_{j=1}^{N} s_{m,j} \right|,$$

where, as before, $\hat{\mu}^m$ is a random variable.

Order of Convergence

Denote by x_m the approximation for $X(T)$, where Δt is the time step size with m steps.

The approximation x_m converges in a **strong** sense to $X(T)$, with order $\alpha > 0$, if there exists $C > 0$, independent of Δt, such that

$$\varepsilon^{strong}(\Delta t) := \mathbb{E}\left[|x_m - X(T)|\right] \leq C\,\Delta t^{\alpha} \quad \text{for all sufficiently small } \Delta t.$$

For a sufficiently smooth function $g(\cdot)$, the approximation x_m converges in a **weak** sense to $X(T)$, with respect to $g(\cdot)$, with order $\beta > 0$, if there exists $C > 0$, independent of Δt, such that

$$\varepsilon^{weak}(\Delta t) := |\mathbb{E}[g(x_m)] - \mathbb{E}[g(X(T))]| \leq C\,\Delta t^{\beta} \quad \text{for all s.s. } \Delta t.$$

Order of Convergence for Euler scheme

For the Euler scheme, we have:

- Strong convergence of order $1/2$

$$\mathcal{E}^{strong}(\Delta t) \leq C \cdot (\Delta t)^{\frac{1}{2}} = \mathcal{O}\left((\Delta t)^{\frac{1}{2}}\right).$$

- Weak convergence of order 1

$$\mathcal{E}^{weak}(\Delta t) \leq C \cdot (\Delta t) = \mathcal{O}\left((\Delta t)\right).$$

These results hold under Lipschitz and linear growth conditions on the SDE coefficients.

Presentation Outline

Milstein Scheme

The discretization under the Milstein scheme is obtained by adding a third term to the Euler discretization.

The SDE is approximated by adding higher-order terms:

$$X(t_{i+1}) = X(t_i) + \mu(X(t_i))\Delta t + \sigma(X(t_i))\Delta W_i + \frac{1}{2}\sigma(X(t_i))\sigma'(X(t_i))(\Delta W_i^2 - \Delta t)$$

- This scheme achieves higher accuracy than Euler's method.
- Milstein scheme improves the convergence order.

Example: Milstein Scheme for GBM

In the case of the GBM process, the discretization reads:

$$s_{i+1} \approx s_i + \mu s_i \Delta t + \sigma s_i \left(W(t_{i+1}) - W(t_i)\right) + \frac{1}{2}\sigma^2 s_i \left(\left(W(t_{i+1}) - W(t_i)\right)^2 - \Delta t\right),$$

or equivalently,

$$s_{i+1} = s_i + \mu s_i \Delta t + \sigma s_i \sqrt{\Delta t} Z + \frac{1}{2}\sigma^2 s_i \left(\Delta t Z^2 - \Delta t\right),$$

where Z is a standard normal random variable.

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where Z is a standard normal random variable.

The Milstein scheme for GBM improves convergence speed:

- Strong convergence of order 1
- Weak convergence of order 1

Presentation Outline

Ornstein-Uhlenbeck Process

Ornstein-Uhlenbeck Process (OU) that solves

$$dX_t = \theta(\mu - X_t)\,dt + \sigma\,dW_t$$

has an analytical (closed) formula

$$X_t = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma \int_0^t e^{-\theta(t-s)}\,dW_s$$

Applications:

- Finance (e.g., interest rate model).
- Physics (e.g., modeling Brownian motion with drift).

Discretization Schemes for the OH Process

1. Euler Discretization Scheme:

$$X_{t+\Delta t} = X_t + \theta(\mu - X_t)\Delta t + \sigma\sqrt{\Delta t}\, Z_t$$

where $Z_t \sim \mathcal{N}(0,1)$.

2. Milstein Scheme:

$$X_{t+\Delta t} = X_t + \theta(\mu - X_t)\Delta t + \sigma\sqrt{\Delta t}\, Z_t + \frac{1}{2}\sigma^2\Delta t(Z_t^2 - 1)$$

3. Numerical scheme for the Exact Solution:

$$X_{t+\Delta t} = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma\sqrt{\frac{1 - e^{-2\theta t}}{2\theta}} \cdot Z$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable.

Code

Simulation of the CIR Process

The CIR (Cox-Ingersoll-Ross) process is used to model variance:

$$dV(t) = \kappa(\bar{v} - V(t))dt + \gamma\sqrt{V(t)}dW(t)$$

This process is always non-negative under the Feller condition $2\kappa\bar{v} \geq \gamma^2$.

Principal use cases:

1. Pricing Interest Rate and Bonds;
2. Stochastic Volatility (for the Heston model);
3. Population growth, biological processes.

Feller Condition

The Feller condition is given by: $2\kappa\bar{v} \geq \gamma^2$

- If this condition is satisfied, then the process $V(t)$ cannot reach zero.
- If it is not satisfied, then the origin is accessible but reflecting.

Hence, in both cases, the process $V(t)$ cannot become negative.

The non–negativity problem arises when a standard discretization is employed.

Nonnegativity Problem in Euler Discretization

Apply the Euler discretization to the process:

$$v_{i+1} = v_i + \kappa(\bar{v} - v_i)\Delta t + \gamma\sqrt{v_i \Delta t}Z,$$

Assume $v_i > 0$ and calculate the $\mathbb{P}$ that a next realization, v_{i+1}, becomes negative, i.e., $\mathbb{P}[v_{i+1} < 0]$:

$$\mathbb{P}[v_{i+1} < 0 \mid v_i > 0] = \mathbb{P}\left[v_i + \kappa(\bar{v} - v_i)\Delta t + \gamma\sqrt{v_i \Delta t}Z < 0 \mid v_i > 0\right]$$

$$= \mathbb{P}\left[\gamma\sqrt{v_i \Delta t}Z < -v_i - \kappa(\bar{v} - v_i)\Delta t \mid v_i > 0\right],$$

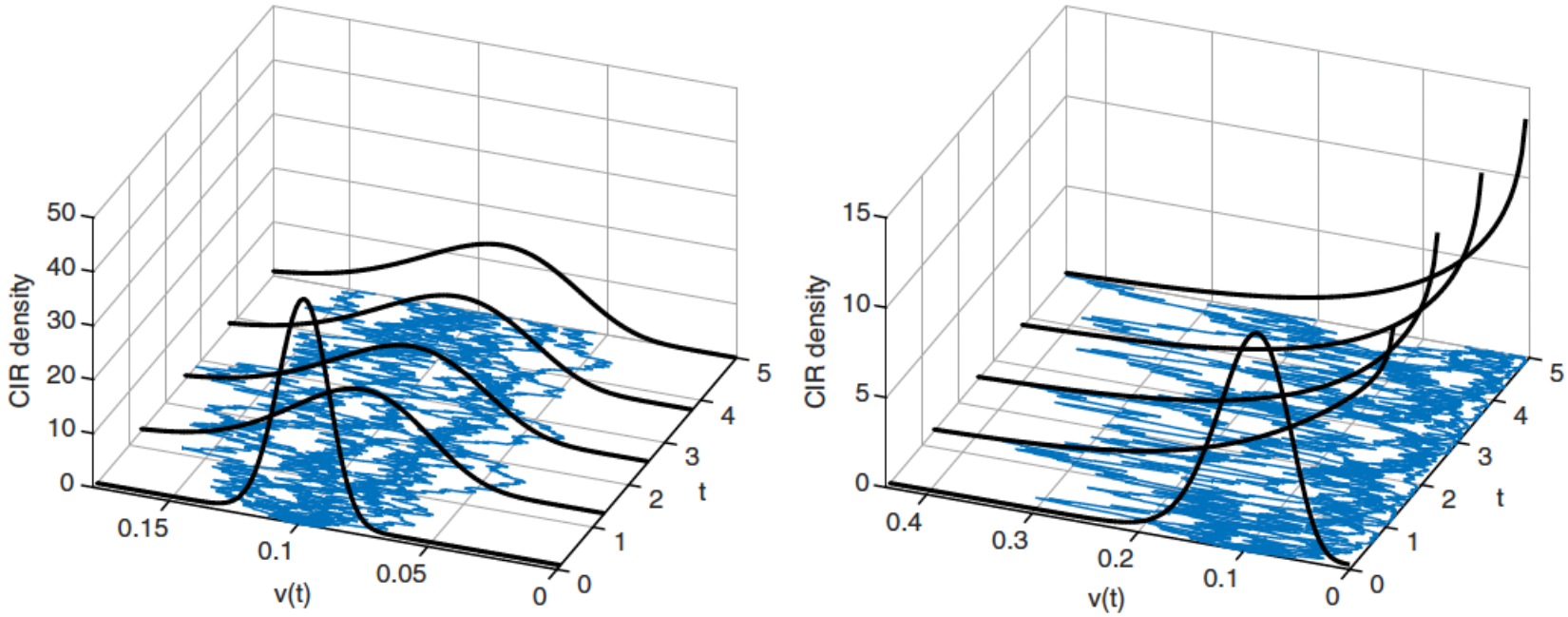
which equals:

$$\mathbb{P}[v_{i+1} < 0 \mid v_i > 0] = \mathbb{P}\left[Z < \frac{-v_i - \kappa(\bar{v} - v_i)\Delta t}{\gamma\sqrt{v_i \Delta t}} \middle| v_i > 0\right] > 0.$$

But Z is a normally distributed random variable (so, it is unbounded). Hence,

$$\mathbb{P}[v_{i+1} < 0 \mid v_i > 0] > 0$$

Density of the CIR process



Paths and the corresponding PDF for the CIR process in the cases where the Feller condition is satisfied and is not satisfied

Parameters:

- $\kappa = 0.5,\ V(0) = 0.1,\ \bar{v} = 0.1;$
- Left: $\gamma = 0.1$; Right: $\gamma = 0.35$.

Continuity of the CIR Process and Issues with Euler Discretization

Why?

- The CIR process is continuous, hence it can only become negative after first reaching zero.
- For $V(t) = 0$, the CIR Eq. becomes deterministic. As soon as the variance reaches zero, it will immediately be positive afterwards.
- In contrast, the Euler discretization is not continuous.

Remark: Application of the Euler scheme to the CIR-type process may lead to undesired and unrealistic path realizations. Whereas the mean-reverting CIR process is guaranteed to be nonnegative, the Euler discretization is not.

Handling Non-Negativity in the CIR Process

Three possible methods:

1. Truncated Euler scheme: Project negative values to zero
2. Reflecting Euler scheme: Force paths to move upwards when they hit zero
3. Exact simulation: Simulate from the noncentral chi-squared distribution

1. Truncated Euler Scheme

It can be summarized as follows:

$$\begin{cases} \tilde{v}_{i+1} = v_i + \kappa(\bar{v} - v_i)\Delta t + \gamma\sqrt{v_i \Delta t} Z, \\ v_{i+1} = \max(\tilde{v}_{i+1}, 0). \end{cases}$$

In words: the paths that attain negative values are projected to the origin.

Notice that, by this truncation, a different process than the original CIR is represented numerically.

The accuracy of this scheme is parameter-dependent: if the Feller condition is not satisfied, then the density accumulates around zero, and the adjusted paths may be highly biased.

2. Reflecting Euler Scheme

The reflecting scheme is given by the following adjustment of the Euler scheme:

$$\begin{cases} \tilde{v}_{i+1} = v_i + \kappa(\bar{v} - v_i)\Delta t + \gamma\sqrt{v_i \Delta t} Z, \\ v_{i+1} = |\tilde{v}_{i+1}|. \end{cases} \tag{9.29}$$

In this case, the paths are forced to move upward. This is particularly useful when the Feller condition is not satisfied.

Code

Non-Central Chi-Squared Distribution: Definition

The non-central arises when dealing with sums of squared normal random variables with non-zero means.

Let Z_i be independent standard normal random variables (with mean 0 and variance 1), and let μ_i be constants. Then a random variable

$$X \sim \chi^2(k, \lambda),$$

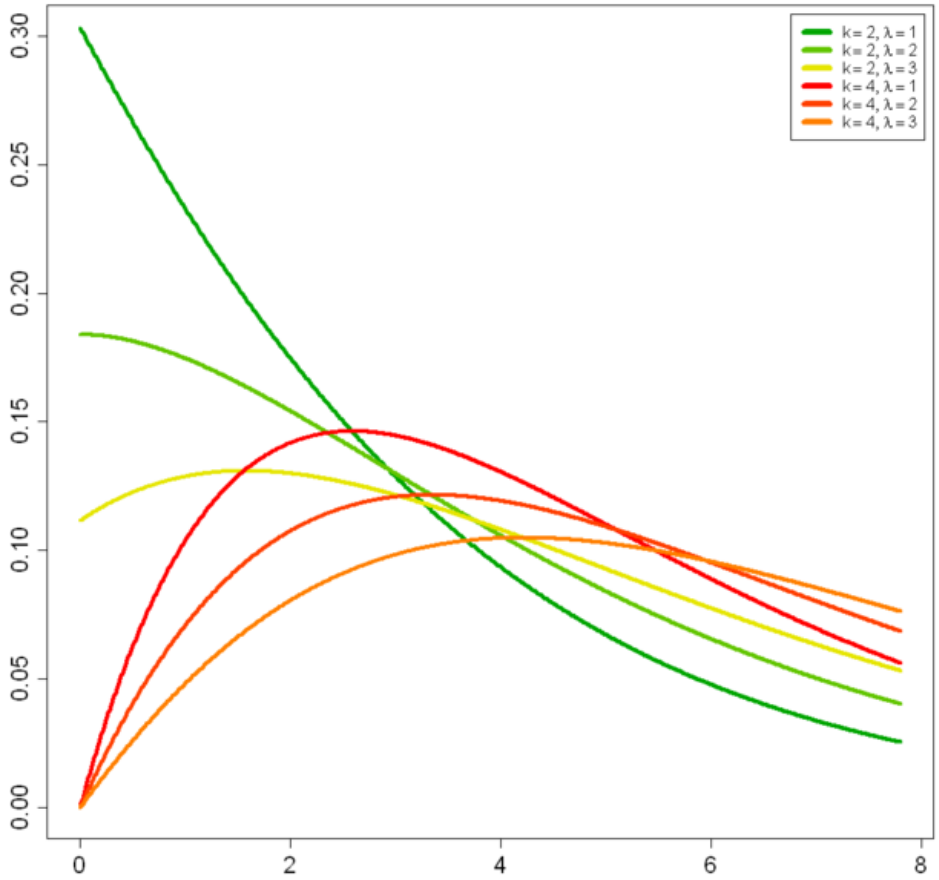
with k degrees of freedom and non-centrality parameter λ if:

$$X = \sum_{i=1}^{k} (Z_i + \mu_i)^2,$$

where the non-centrality parameter λ is given by:

$$\lambda = \sum_{i=1}^{k} \mu_i^2.$$

PDF for Non-Central Chi-Squared Distribution



For applications where sum of squared non-zero mean normal variables appear...

3.Exact Simulation of the CIR Model

We can use the fact that the variance process $C(t)$ follows a non–central chi-squared distribution:

Process $V(t) \mid V(s)$, $t > s > 0$, under the CIR dynamics, is known to be distributed as $\bar{c}(t,s)$ times a non–central chi-squared random variable, $\chi^2(\delta, \bar{\kappa}(t,s))$, where δ is the "degrees of freedom" parameter and the noncentrality parameter is $\bar{\kappa}(t,s)$, i.e.,

$$V(t) \mid V(s) \sim \bar{c}(t,s)\, \chi^2(\delta, \bar{\kappa}(t,s)), \quad t > s > 0, \tag{8.4}$$

with

$$\bar{c}(t,s) = \frac{1}{4\kappa}\gamma^2 \left(1 - e^{-\kappa(t-s)}\right), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t,s) = \frac{4\kappa v(s) e^{-\kappa(t-s)}}{\gamma^2 \left(1 - e^{-\kappa(t-s)}\right)}. \tag{8.5}$$

3. Exact Simulation Scheme for the CIR Process

Exact simulation scheme for the path realizations of the CIR process, as, for $i = 0, \ldots, m-1$,

$$\bar{c}(t_{i+1}, t_i) = \frac{\gamma^2}{4\kappa}\left(1 - e^{-\kappa(t_{i+1}-t_i)}\right),$$

$$\bar{\kappa}(t_{i+1}, t_i) = \frac{4\kappa e^{-\kappa(t_{i+1}-t_i)}}{\gamma^2\left(1 - e^{-\kappa(t_{i+1}-t_i)}\right)} v_i,$$

$$v_{i+1} = \bar{c}(t_{i+1}, t_i)\, \chi^2\left(\delta, \bar{\kappa}(t_{i+1}, t_i)\right),$$

with a constant parameter $\delta = \frac{4\kappa\bar{v}}{\gamma^2}$, and some initial value $v(t_0) = v_0$.

- We can simulate CIR paths without paying special attention to the Feller condition.
- The scheme relies on an efficient sampling from the χ^2 distribution (we need a package with a good implementation)

The Heston Model

What if we need to simulate a system of SDEs?
In Stochastic Volatility Models (SVMs), where the volatility of the stock process is itself modelled by a diffusion process.

In the Heston SVM, we deal with two SDEs: one for the underlying asset price $S(t)$ and one for the volatilty process $v(t)$:

$$\begin{cases} dS(t) = \mu S(t)\,dt + \sqrt{v(t)}S(t)\,dW_x(t), & S(t_0) = S_0 > 0, \\ dv(t) = \kappa(\bar{v} - v(t))\,dt + \gamma\sqrt{v(t)}\,dW_v(t), & v(t_0) = v_0 > 0, \end{cases}$$

where $W_x(t)$ and $W_v(t)$ are Wiener processes that may be correlated, allowing the modeling of stochastic volatility as a separate process alongside the asset price.

The Cholesky Decomposition in the Heston Model

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With the help of the Cholesky decomposition, the model can be presented in terms of independent BMs as:

$$\begin{cases} dS(t) = \mu S(t)\, dt + \sqrt{v(t)} S(t) \left(\rho\, d\tilde{W}_v(t) + \sqrt{1-\rho^2}\, d\tilde{W}_x(t) \right), \\ dv(t) = \kappa(\bar{v} - v(t))\, dt + \gamma \sqrt{v(t)}\, d\tilde{W}_v(t), \end{cases}$$

where $\tilde{W}_x(t)$ and $\tilde{W}_v(t)$ are independent BMs.

Euler Simulation of the Heston Model

The Heston SVM under the log transformation reads:

$$\begin{cases} dX(t) = \left(\mu - \frac{1}{2}v(t)\right)dt + \sqrt{v(t)}\left[\rho\, d\tilde{W}_v(t) + \sqrt{1-\rho^2}\, d\tilde{W}_x(t)\right], \\ dv(t) = \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}\, d\tilde{W}_v(t), \end{cases}$$

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After integration over $[t_i, t_{i+1}]$, the following discretization scheme

$$x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} \left(\mu - \frac{1}{2}v(t)\right) dt + \rho \int_{t_i}^{t_{i+1}} \sqrt{v(t)} \, d\tilde{W}_v(t) + \sqrt{1-\rho^2} \int_{t_i}^{t_{i+1}} \sqrt{v(t)} \, d\tilde{W}_x(t),$$

$$v_{i+1} = v_i + \kappa \int_{t_i}^{t_{i+1}} (\bar{v} - v(t)) \, dt + \gamma \int_{t_i}^{t_{i+1}} \sqrt{v(t)} \, d\tilde{W}_v(t).$$

holds.

Discretization for x_{i+1} in the Heston Model

Notice that the two integrals with $\tilde{W}_v(t)$ in the SDEs above are the same.

The discretization for x_{i+1} is

$$x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} \left(\mu - \frac{1}{2}v(t)\right) dt + \frac{\rho}{\gamma}\left(v_{i+1} - v_i - \kappa \int_{t_i}^{t_{i+1}} (\bar{v} - v(t))\, dt\right) + \sqrt{1-\rho^2} \int_{t_i}^{t_{i+1}} \sqrt{v(t)} d\tilde{W}_x(t).$$

We discretize all the integrals

$$x_{i+1} \approx x_i + \left(\mu - \frac{1}{2}v_i\right)\Delta t + \frac{\rho}{\gamma}\left(v_{i+1} - v_i - \kappa\left(\bar{v} - v_i\right)\Delta t\right) + \sqrt{1-\rho^2}\sqrt{v_i}\left(\tilde{W}_x(t_{i+1}) - \tilde{W}_x(t_i)\right).$$

Heston Model Solutions

After collecting all terms and using that $\tilde{W}_x(t_{i+1}) - \tilde{W}_x(t_i) \overset{d}{=} \sqrt{\Delta t}\, Z_x$ with $Z_x \sim N(0,1)$, we find:

$$x_{i+1} \approx x_i + \left(\mu - \frac{\rho\kappa\bar{v}}{\gamma}\right)\Delta t + \left(\frac{\rho\kappa}{\gamma} - \frac{1}{2}\right)\Delta t - \frac{\rho}{\gamma}v_i + \frac{\rho}{\gamma}v_{i+1} + \sqrt{1-\rho^2}\sqrt{\Delta t}\, v_i Z_x.$$

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Alternatively...

The Almost Exact Simulation Scheme

The variance v_{i+1} can be simulated, for a given value of v_i, employing the Euler scheme or by the noncentral chi-squared distribution. In this latter case, the algorithm is called the *Almost Exact Simulation Scheme*.