Numerical Methods for Mathematical Finance

Lecture 3: Numerical Methods for Option Pricing

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What We Are Going to See Today

1. Introduction to Financial Derivatives
   - Overview of financial products based on underlying assets
2. $\mathbb{P}$ vs $\mathbb{Q}$
   - Real World Measure vs Risk Neutral Measure
3. The Black-Scholes Equation
   - Derivation of the fundamental PDE for European options
4. Solution Approach using the Feynman-Kac Theorem
   - Application of the Feynman-Kac theorem for solving the Black-Scholes equation
   - Calculation of the discounted expected payoff under $\mathbb{Q}$
5. MonteCarlo Simulation

Introduction

Assuming a GBM process for the asset price $S(t)$, the SDE is:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Black and Scholes derived a PDE for the price $V(t, S)$ of a European option:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

where r is the risk-free interest rate.

Derivative Pricing

Financial derivatives are products based on the performance of an underlying asset (e.g., stock, interest rate, or commodity price). A fundamental question in quantitative finance is determining the fair value of an option at the time of sale.

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Two macro categories:

- Future Contracts
- Options

Presentation Outline

1. Financial Derivatives
2. $\mathbb{P}$ vs $\mathbb{Q}$
3. The Black–Scholes PDE
4. Solving the Option Pricing: the Feynman-Kac Theorem
5. Monte Carlo Methods

Option Contract Definitions

An option contract is a financial contract that gives the holder the right to trade (buy or sell) an underlying asset in the future at a predetermined price.

- The holder has the *right*, but not the obligation, to execute the trade.
- The contract provides the *optionality* to buy or sell the asset.

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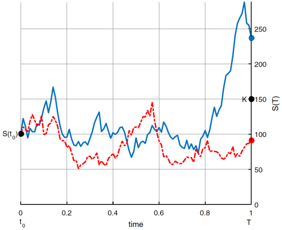
Two kind of options:

1. A **call** option gives an option holder the right to purchase an asset at some time in the future $t = T$, for a prescribed amount, the strike price denoted by K_C.
2. A **put** option gives a holder the right to sell an asset, at some time in the future $t = T$, for a prescribed amount, the strike price denoted by K_P.

Call Option Example



- $S(T) > K$: Holder exercises the option, pays K to acquire the asset. Profit $= S(T) - K = 100$.
- $S(T) < K$. Option is not exercised as the market price is lower than K.

  No need to exercise when $S(T) < K$ (why buy something for a price K when it is cheaper in the market?).

Decision to Exercise a European Call Option

1. Exercise if $S(T) > K$:

$$\text{Profit} = S(T) - K$$

   - Holder pays K to obtain asset worth $S(T)$.
   - Asset can be sold immediately for profit.

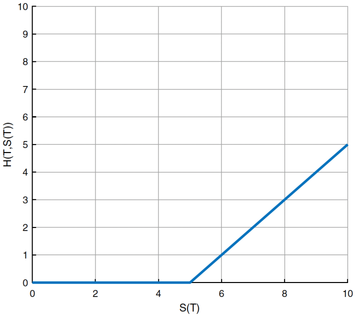
2. Do not exercise if $S(T) < K$:
   - Option is valueless.
   - Buying asset directly in the market is cheaper than exercising.
   - No advantage in using the option right.

Payoff Function for a Call Option

For a call option, the payoff function $H(T, S)$ is defined by

$$V_c(T, S) = H(T, S) := \max(S(T) - K, 0)$$

where the value of the call option at time t, for a certain asset value $S = S(t)$, is denoted by $V_c(t, S)$

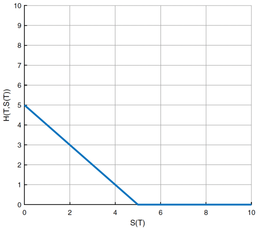


Payoff Function for a Put Option

At the maturity time $t = T$, a European put option has the value $K - S(T)$ if $S(T) < K$.
- The put option expires worthless if $S(T) > K$.
- The value of the put option at maturity can be defined as:

$$V_p(T, S(T)) = H(T, S(T)) := \max(K - S(T), 0)$$

- Here, $V_p(t, S(t))$ represents the value of the put option at time t, and $H(T, S(T))$ is the option's payoff function.



European vs. American Options

European Options:

1. Can only be exercised at the maturity date T.
2. Typically simpler to price and analyze due to fixed exercise point.
3. Commonly used for stock indices, bonds, and other standardized assets.
4. Rely on closed-form solutions or basic simulations

American Options:

1. Can be exercised at any time before or on the maturity date T.
2. More flexibility to the holder, potentially increasing the option's value.
3. Frequently used for individual stocks, commodities, and other assets where early exercise may be advantageous.
4. Require methods that can handle early exercise, like binomial trees.

Put-Call Parity for European Options

1. Portfolio 1: Contains a European **put option** and a stock

$$\Pi_1(t, S) = V_p(t, S) + S(t)$$

2. Portfolio 2: Contains a European **call option** and a cash amount K

$$\Pi_2(t, S) = V_c(t, S) + Ke^{-r(T-t)}$$

The term $e^{-r(T-t)}$ is the discount factor, representing the value of €1 payable at time T from today, with a continuously compounding interest rate r.

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At maturity time T, we find:

$$\Pi_1(T, S) = \max(K - S(T), 0) + S(T)$$

This equals K if $S(T) < K$, and $S(T)$ if $S(T) > K$ and the same value holds for $\Pi_2(T, S)$.

$$\Pi_1(T, S) = \Pi_2(T, S) = \max(K, S(T))$$

.

Arbitrage-Free Condition

If $\Pi_1(T, S) = \Pi_2(T, S)$, then this equality must hold *at any time prior to the maturity time* T. Otherwise, arbitrage opportunities arise.

- Strategy: Buy the cheaper portfolio and sell the more expensive one, resulting in a positive cash flow at $t < T$.
- At $t = T$: Both portfolios have the same value.
- Thus, buying one portfolio and selling the other at $t = T$ yields no profit or loss.
- Profit achieved at $t < T$ remains, resulting in risk-free profit, which exceeds returns from simply holding cash.

Conclusion: For any $t < T$, the put-call parity relationship holds:

$$V_c(t, S) = V_p(t, S) + S(t) - Ke^{-r(T-t)}$$

Numerical Methods for European Options

1. Black-Scholes Model: Analytical solution for pricing European options in continuous time, assuming constant volatility and no dividends.
2. Monte Carlo Simulation: Generates asset price paths to estimate option price through averaging, particularly useful for high-dimensional problems or path-dependent options.
3. Finite Difference Methods: Solves the PDE associated with option pricing, ideal for modeling changes over time and space.
4. COS Method: Fourier-based method offering fast and accurate pricing by expanding the payoff function in a Fourier cosine series. Efficient for European options, especially with exotic payoffs.

Numerical Methods for American Options

1. Binomial Tree Model: Constructs a lattice to model the possible asset prices over time, allowing for early exercise at each node.
2. Finite Difference Methods: Extended to handle the free boundary problem introduced by early exercise (e.g., implicit, explicit, and Crank-Nicolson schemes).
3. Least-Squares Monte Carlo (LSM): Uses regression techniques to estimate the optimal exercise strategy, suitable for high-dimensional American options and complex payoffs.

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$\mathbb{P}$ vs $\mathbb{Q}$

Real-World Probability ($\mathbb{P}$):

- Actual probabilities of events based on historical data.
- Used to model the true dynamics of asset prices.
- Under $\mathbb{P}$, the asset price S_t evolves as:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}},$$

where μ is the real drift (expected return) and $W_t^{\mathbb{P}}$ a Bm under $\mathbb{P}$.

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Risk-Neutral Probability ($\mathbb{Q}$):

- A hypothetical measure where all assets earn the risk-free rate r.
- Used in pricing derivatives to ensure no-arbitrage conditions.
- Under $\mathbb{Q}$, the asset price S_t evolves as:

$$dS_t = rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}},$$

where r is the risk-free rate and $W_t^{\mathbb{Q}}$ is a Bm under $\mathbb{Q}$.

Pricing under $\mathbb{Q}$

- $\mathbb{P}$ incorporates risk premiums and higher volatility.
- $\mathbb{Q}$ is focused on pricing with risk-free growth.

This reflects the assumption in $\mathbb{Q}$ that investors are indifferent to risk, enabling derivative pricing using discounted expected values:

$$\text{Price of a derivative} = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\,\text{Payoff}\right].$$

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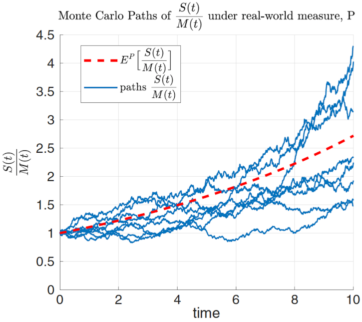
In the $\mathbb{Q}$-world:

- Investors are assumed to accept the same return as a risk-free investment (r);
- Investors behave as if they require no additional compensation (risk premium) for taking on risky investments;
- All assets are priced as though their expected return equals the risk-free rate, making pricing simpler and consistent with the no-arbitrage principle.

Real-World Measure $\mathbb{P}$ vs Risk-Neutral Measure $\mathbb{Q}$

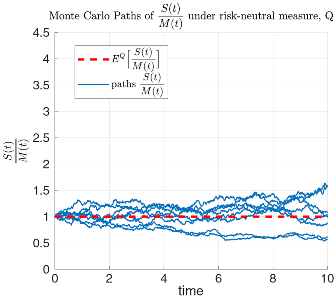
Real-World Measure $\mathbb{P}$	**Risk-Neutral Measure $\mathbb{Q}$**
Reflects actual probabilities of real-world events	Assumes investors are indifferent to risk
Used for modeling and forecasting asset price movements based on historical data	Primarily used for pricing derivatives (e.g., options)
Calibration is based on historical stock prices and observed data	Calibration is based on market prices of options or other derivatives
Focused on long-term trends and risk premiums	Asset drift equals the risk-free rate (no risk premium)
Example: Estimating stock price volatility	Example: Pricing European options using the Black-Scholes model

Real-World Measure P



- Simulated paths of $\frac{S(t)}{M(t)}$ show upward stock growth due to the risk premium.
- The red dashed line $E^P\left[\frac{S(t)}{M(t)}\right]$ reflects the expected growth over time.
- Higher volatility in the paths is evident, indicating the real-world risks and uncertainty.

Risk-Neutral Measure Q



- Simulated paths of $\frac{S(t)}{M(t)}$ are more stable and flatter.
- The red dashed line $E^Q\left[\frac{S(t)}{M(t)}\right]$ remains flat, showing no excess returns above the risk-free rate.
- Paths are used in derivative pricing, assuming risk neutrality.

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The Black-Scholes Option Pricing Equation

Assumptions:

- The stock price process is assumed to follow a Geometric Brownian Motion (GBM).
- We consider the dynamics under the real-world measure $\mathbb{P}$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t)$$

The financial derivative $V(t, S)$, represents the value of a European option

The option price $V(t, S)$ depends on t and the stochastic process $S(t)$, we apply Itô's lemma to derive its dynamics:

$$dV(t, S) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$

Constructing a Portfolio

Substituting dS and simplifying, we get:

$$dV(t,S) = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW^{\mathbb{P}}(t)$$

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We construct a portfolio $\Pi(t,S)$, consisting of one option with value $V(t,S)$ and an amount, $-\Delta$, of stocks with value $S(t)$,

$$\Pi(t,S) = V(t,S) - \Delta S(t)$$

This portfolio thus consists of one long position in the option $V(t,S)$, and a short position of size Δ in the underlying $S(t)$.

A *deterministic* portfolio

By Itô's lemma, the infinitesimal change in portfolio value $\Pi(t, S)$

$$
\begin{aligned}
d\Pi &= dV - \Delta dS \\
&= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW^{\mathbb{P}} - \Delta \left(\mu S dt + \sigma S dW^{\mathbb{P}}\right) \\
&= \left[\frac{\partial V}{\partial t} + \mu S \left(\frac{\partial V}{\partial S} - \Delta\right) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right) dW^{\mathbb{P}}
\end{aligned}
$$

To eliminate the randomness, we set:

$$\Delta = \frac{\partial V}{\partial S}$$

The infinitesimal change in the portfolio $\Pi(t, S)$ is then:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{1}$$

- The portfolio becomes deterministic as the $dW^{\mathbb{P}}$-terms cancel out.
- With Δ chosen as above, the portfolio dynamics are independent of the drift parameter μ under the real-world measure $\mathbb{P}$.

Portfolio Growth at the Risk-Free Rate

The portfolio's value should grow, on average, at the same *speed* (i.e., generate the same return) as money placed in a risk-free savings account.

$$d\Pi = r\Pi dt$$

where r represents the constant interest rate for a risk-free savings account.

Bank Account Growth Model:

The bank account $M(t) = M(t_0)e^{r(t-t_0)}$ is modeled by:

$$dM = rMdt$$

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The change in portfolio value as:

$$d\Pi = r\left(V - S\frac{\partial V}{\partial S}\right)dt \tag{2}$$

The Black-Scholes Partial Differential Equation

By comparing Eq. (1) and (2), the Black-Scholes Equation for the option value $V(t, S)$ is the following parabolic PDE:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

- The **diffusion term**, $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ is positive, ensuring that the problem is well-posed when accompanied by a final condition.
- The natural condition for the Black-Scholes PDE is given by:

  $$V(T, S) = H(T, S),$$

  where $H(T, S)$ represents the payoff function.

Remark: This final condition specifies the option type, but the Black-Scholes equation itself is derived independently of option type and is valid for calls, puts, and even some other option forms.

Closed-Form Solution to the Black-Scholes PDE

The closed-form solution for a European call option is:

$$C(S_0, K, T, r, \sigma) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

and for a European put option:

$$P(S_0, K, T, r, \sigma) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1),$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

$\Phi(x)$ is the cumulative distribution function of the standard normal distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy.$$

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Different Approaches to European Option Pricing

We can use different methods to solve the pricing problem:

- 1) PDE methods
- Using the Feynman-Kac formula:
  - 2) Numerical integration via Fourier methods
  - 3) Monte Carlo methods

The Feynman-Kac formula translates the PDE problem into an expectation of a discounted payoff under the risk-neutral measure:

$$V(t, S) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H(S_T) \mid S_t = S\right]$$

where $H(S_T)$ is the payoff function.

Feynman-Kac Theorem

Given the money-savings account $dM(t) = rM(t)\,dt$, with constant interest rate r, let $V(t,S)$ be a sufficiently differentiable function of time t and stock price $S(t)$.
Suppose $V(t,S)$ satisfies a PDE with general $\bar{\mu}(t,S)$ and $\bar{\sigma}(t,S)$:

$$\frac{\partial V}{\partial t} + \bar{\mu}(t,S)\frac{\partial V}{\partial S} + \frac{1}{2}\bar{\sigma}^2(t,S)\frac{\partial^2 V}{\partial S^2} - rV = 0,$$

with a final condition given by $V(T,S) = H(T,S)$.

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The solution $V(t, S)$ at any time $t < T$ is then given by:

$$V(t, S) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[H(T, S)|\mathcal{F}(t)\right] = M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{H(T, S)}{M(T)}\middle|\mathcal{F}(t)\right]$$

where $\mathbb{E}$ is taken under the measure $\mathbb{Q}$, w.r.t. a process S

$$dS(t) = \bar{\mu}(t, S)\,dt + \bar{\sigma}(t, S)\,dW^{\mathbb{Q}}(t), \quad t > t_0.$$

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Monte Carlo Methods for Pricing

Monte Carlo methods rely on probability theory:

- Simulate many random asset price paths
- For each path, compute the payoff
- Approximate the option price by the discounted average of payoffs:

$$V(t,S) \approx e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} H(S_T^{(i)})$$

Monte Carlo methods work due to the Central Limit Theorem and the Law of Large Numbers, and we can compute the standard error for the estimated price:

$$\mathrm{SE} = \frac{\sigma_{\mathrm{MC}}}{\sqrt{N}}$$

where σ_{MC} is the sample standard deviation.

Monte Carlo Algorithm for Option Valuation 1

A Monte Carlo algorithm to approximate an option value can be summarized as follows:

1. Partition the time interval $[0, T]$, so that the following time points, $0 = t_0 < t_1 < \cdots < t_m = T$, are obtained. The time points are defined by $t_i = \frac{i \cdot T}{m}$, $i = 0, \ldots, m$, where $m + 1$ represents the number of time steps. The time step is denoted by $\Delta t = t_{i+1} - t_i$, where this time partitioning does not need to be uniform.

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2. Generate asset values, $s_{i,j}$, taking the risk-neutral dynamics of the underlying model into account. Note that asset path value $s_{i,j}$ has two indices, $i = 1, \ldots, m$ (the time steps) and $j = 0, \ldots, N$, with N being the number of generated Monte Carlo asset paths.

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3. Compute the N payoff values, H_j, and store these results. In the case of European options, we have $H_j = H(T, s_{m,j})$.

Monte Carlo Algorithm for Option Valuation 2

4 Compute the average:

$$\mathbb{E}^{\mathbb{Q}}\left[H(T,S)|\mathcal{F}(t_0)\right] \approx \frac{1}{N}\sum_{j=1}^{N} H_j.$$

The right-hand side is known as the *Monte Carlo estimate*.

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5 Calculate the option value as

$$V(t_0,S) \approx e^{-r(T-t_0)}\frac{1}{N}\sum_{j=1}^{N} H_j.$$

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5 Calculate the option value as

$$V(t_0,S) \approx e^{-r(T-t_0)}\frac{1}{N}\sum_{j=1}^{N} H_j.$$

6 Determine the standard error (standard deviation) related to the obtained values in Step 5.

Error Analysis

Monte Carlo methods have two sources of error:

- Error from approximating the expectation with a finite number of samples
- Error from approximating the asset price path numerically

The total error is measured by the Mean Squared Error (MSE):

$$\mathrm{MSE}(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] = \mathrm{Var}(\hat{\theta}) + \left(\mathbb{E}[\hat{\theta}] - \theta\right)^2$$

where $\hat{\theta}$ is the estimated price and θ is the true price.

MC approximation

With a Monte Carlo method, we approximate a solution by a sequence of random realizations. With N MC paths, we obtain the following approximation:

$$\mathbb{E}^{\mathbb{Q}}[H(T,S)] \approx \hat{H}_N(T,S) := \frac{1}{N}\sum_{j=1}^{N} H_j.$$

By the strong law of large numbers, we know that, for $N \to \infty$,

$$\lim_{N\to\infty} \hat{H}_N(T,S) = \mathbb{E}^{\mathbb{Q}}[H(T,S)],$$

with probability 1.

Actually, we have two sources of error:

- Estimating the expectation from finitely many samples.
- Approximating $S(T)$ by a numerical method.

Both errors are measured by the mean-squared error.

Notebook [3.1]: Analytic vs MC for B&S Model

Numerical methods for SDE to simulate the underlying trajectories.

We compare the Euler scheme and the Milstein scheme for European call options and digital call options.

1. We can price a European call option with payoff:

$$H(T, S_T) = \max(S_T - K, 0)$$

2. The payoff function for a cash-or-nothing option is:

$$H(T, S) = Q\mathbb{1}_{\{S(T)>K\}}$$

Code: Both methods require a similar number of paths to achieve satisfactory pricing results. This is because they have the same order of weak convergence.

Notebook [3.2]: Heston Model

We compare the Euler discretization scheme and the *Almost exact simulation scheme with truncated variance.*

$$\begin{aligned} x_{i+1} &\approx x_i + k_0 + k_1 v_i + k_2 v_i^2 + \sqrt{k_3 v_i} Z_x, \\ v_{i+1} &= \bar{c}(t_{i+1}, t_i)\chi^2\big(\delta, \bar{\kappa}(t_{i+1}, t_i)\big), \end{aligned} \qquad Z_x \sim \mathcal{N}(0,1)$$

with

$$\bar{c}(t_{i+1}, t_i) = \frac{\gamma^2}{4\kappa}\left(1 - e^{-\kappa(t_{i+1}-t_i)}\right), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2},$$

$$\bar{\kappa}(t_{i+1}, t_i) = \frac{4\kappa e^{-\kappa\Delta t}}{\gamma^2\left(1 - e^{-\kappa\Delta t}\right)} v_i,$$

$$k_0 = \left(r - \frac{\rho_{x,v}\kappa\bar{v}}{\gamma}\right)\Delta t, \quad k_1 = \left(\frac{\rho_{x,v}\kappa}{\gamma} - \frac{1}{2}\right)\Delta t - \frac{\rho_{x,v}}{\gamma},$$

$$k_2 = \frac{\rho_{x,v}}{\gamma}, \quad k_3 = (1 - \rho_{x,v}^2)\Delta t.$$

where $\chi^2(\delta, \cdot)$ the noncentral chi-squared distribution with δ degrees of freedom and noncentrality parameter $\bar{\kappa}$.

The COS Method for Option Pricing

The COS method is an efficient option pricing method based on the Fourier cosine series expansion of the probability density function. The payoff function is expressed as:

$$V(t, S) = e^{-r(T-t)} \int_a^b H(y) f_{X_T}(y)\, dy$$

where $f_{X_T}(y)$ is the probability density function of $X_T = \log S_T$. This density function is expanded in terms of a Fourier cosine series.

Steps in the COS Method

For pricing European options using the COS method:

1. Truncate the integration range $[a, b]$
2. Approximate the probability density function by a Fourier cosine expansion:

$$f_{X_T}(y) \approx \sum_{k=0}^{N-1} A_k \cos\left(\frac{k\pi(y-a)}{b-a}\right)$$

3. Compute the option price by summing a few terms of the series:

$$V(t, S) \approx e^{-r(T-t)} \sum_{k=0}^{N-1} A_k H_k$$

Error Analysis for the COS Method

There are three sources of error in the COS method:

- Truncation of the integration range
- Approximation of the density by a truncated cosine series
- Approximation of the series coefficients by the characteristic function

For smooth density functions, the error converges exponentially.