The Master Equation in a Bounded Domain with Absorption

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Abstract

We analyze the Master Equation and the convergence problem within Mean Field Games (MFG) theory considering a bounded domain with homogeneous Dirichlet conditions. Concerning the N-players differential game, the player's dynamic ends when touching the boundary. We analyze the well-posedness of the Master Equation and the regularity of its solutions for a suitable class of parabolic equations. Such results are then exploited to consider the convergence problem of the Nash equilibria towards the Master Equation solution, also proving the convergence of optimal trajectories. Eventually, we apply our findings to solve a toy model related to an optimal liquidation scenario.

Keywords: Master Equation, first order MFG, Dirichlet conditions, Absorption, Optimal Liquidations

1. Introduction

Mean Field Games (MFG) theory was introduced in 2006 by J.-M. Lasry and P.-L. Lions ([55, 56, 57, 58]) to describe the asymptotic behaviour of differential games with a large number of players, also called agents. A similar definition was given in the same years by Caines, Huang and Malhamé [47]. The term *mean field* refers to the highly symmetric form of interaction and the indistinguishability of the players whose number is usually denoted by *N*.

A formal description of this limit problem is obtained by analyzing the so-called *MFG system*, namely a forward-backward PDEs system. The latter has been widely studied, and most of the usual topics as the existence, uniqueness and stability of solutions have been proved in many different frameworks. Conversely, results about the convergence problem, namely when the *N*-players differential game approaches the MFG system, are considerably fewer. In particular, most of the literature mainly considers the case where the players' state space is the torus \mathbb{T}^d , hence a periodic framework, or the whole space \mathbb{R}^d .

Nevertheless when dealing with *real world problems* as those related to biological or financial models, see, e.g., [1], [15], [14], [13] [45], boundary conditions become unavoidable characteristics we have to deal with, being obliged to consider non periodic settings in bounded domains.

In this article we study the convergence problem in a bounded domain $\Omega \subset \mathbb{R}^d$, via *Master Equation*, whose precise definition will be given later, requiring an *absorbing* condition at the boundary $\partial\Omega$, namely we ask for each player to remain in the game as long as his dynamic lies in the interior $int(\Omega)$ of the domain Ω . Within the latter framework, the dynamic of the player i, $1 \le i \le N$, is described by the following Stochastic Differential Equation (SDE), with initial time $t_0 > 0$ and final time $T > t_0$:

$$\begin{cases} dX_t^{i,t_0} = b(t, X_t^{i,t_0}, \alpha_t^i)dt + \sqrt{2}\sigma(X_t^{i,t_0})dB_t^i \\ X_{t_0}^{i,t_0} = x_0^i. \end{cases}$$
(1)

 α^i_t being the control of the player i, i.e. a progressively measurable process taking values in a certain set A, while $b:[0,T]\times\Omega\times A\to\mathbb{R}^d$ is the *drift* function and $\sigma:\Omega\to\mathbb{R}^{d\times d}$ represents the *diffusion matrix*. Henceforth, for simplicity, we will shortly write X^i_t .

The process X^i is Markovian providing standard hypotheses on the drift and the coefficient terms. Therefore, we are allowed to define the hitting time τ at $\partial\Omega$ in [0, T] as

$$\tau(X_{\cdot}^{i}) = \inf \left\{ t \in [t_{0}, T] \mid X_{t}^{i} \in \partial \Omega \right\} \wedge T, \qquad (2)$$

where $a \wedge b := \min\{a, b\}$ and we abbreviate $\tau(X_t^i)$ by τ^i , whenever no confusion is possible.

From now on, we indicate a vector of \mathbb{R}^{Nd} with $v := (v_1, \dots, v_N)$, with $v_i \in \mathbb{R}^d$, $i \in \{1, \dots, N\}$. A *cost functional* is associated with each player i, which chooses his own strategy in order to minimize it. To obtain a convergence result, we have to require the cost functional to have a symmetric structure. Namely, we assume that the cost for the player i depends on both its position x^i and on the other players' empirical distribution, defined as follows:

$$m_{\mathbf{x}}^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j} \mathbb{1}_{\{x_j \in int(\Omega)\}}, \qquad (3)$$

where δ_x is the Dirac mass at x. The indicator function specifies that we are interested only in those players in the interior of Ω . Let us underline that the presence of such an indicator function slightly modifies the classic theoretical setting of the problem: for the rest of the paper, $m_x^{N,i}$ belongs to \mathcal{P}^{sub} , the space of subprobability measure, as in-depth analyzed in Section 2.2.

Consequently, the cost functional is defined in the following way

$$J_i^N(t_0, \mathbf{x}_0, \boldsymbol{\alpha}) = \mathbb{E}\left[\int_{t_0}^{\tau^i} \left(L\left(s, X_s^i, \alpha_s^i\right) + F\left(s, X_s^i, m_{\mathbf{X}_s}^{N,i}\right)\right) ds + G\left(X_{\tau^i}^i, m_{\mathbf{X}}^{N,i}\right)\right],\tag{4}$$

where L, F and G are, respectively, the Lagrangian term, the running cost and the final cost to pay at the hitting time, or at the final time when no boundary hitting happens.

Each player wishes to minimize its cost functional, which also depends on the strategies of the other players. The set of strategies α^* which allows the agents to play their optimal strategy, in relation to the other agents, is called *Nash equilibrium*. Namely,

$$J_i^N(t_0, x_0, \alpha^*) \leq J_i^N(t_0, x_0, \alpha_i, (\alpha_i^*)_{i \neq i}).$$

We define $v_i^N(t, \mathbf{x}) := J_i^N(t, \mathbf{x}, \alpha^*)$. Exploiting both the Itô-Doeblin formula and the dynamic programming principle, we have that α^* is a Nash equilibrium if v_i^N solves a coupled system of parabolic equations, traditionally called *Nash system*. Namely, for $(t, \mathbf{x}) \in (0, T) \times \Omega^N$ and for i = 1, ..., N, we have

$$\begin{cases}
-\partial_{t}v_{i}^{N} - \sum_{j=1}^{N} \operatorname{tr}(a(x_{j})D_{x_{j}x_{j}}^{2}v_{i}^{N}) + H(t, x_{i}, D_{x_{i}}v_{i}^{N}) \\
+ \sum_{j\neq i}^{N} H_{p}(t, x_{j}, D_{x_{j}}v_{j}^{N}) \cdot D_{x_{j}}v_{i}^{N} = F(t, x_{i}, m_{\mathbf{x}}^{N,i}), \\
v_{i}^{N}(T, \mathbf{x}) = G(x_{i}, m_{\mathbf{x}}^{N,i}), \\
v_{i}^{N}(t, \mathbf{x})|_{x_{i} \in \partial\Omega} = 0, \forall t \in [0, T]
\end{cases}$$
(5)

 $H(t, x, p) = \sup_{\alpha \in A} (-b(t, x, \alpha) \cdot p - L(t, x, \alpha))$ being the Hamiltonian of the system, and with H_p its derivative in p and $a = \sigma \sigma^*$.

Knowing the initial distribution, independently from the individual starting position on $\partial\Omega$, implies that these players suddenly stop their dynamic at the initial time, hence making no contribution to the evolution of other players. Therefore, for all i, j, the following equation holds:

$$v_i^N(t, x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_N) = v_i^N(t, x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_N),$$
(6)

for all $x, y \in \partial \Omega$. Hence, the value function v_i^N does not depend on x_j when $x_j \in \partial \Omega$. Roughly speaking, the degrees of freedom of the solution v_i^N decreases from N to $N-(number\ of\ players\ on\ \partial\Omega)$. This fact has a direct correspondence also in the dynamic of the game in terms of a leak of mass in the Fokker-Planck equation, as we will see in (10) and in the \mathcal{P}^{sub} formulation of the Master Equation w.r.t. to a *new* boundary condition as proved in Corollary 12.

It is possible to provide a suitable approximation for the asymptotic configuration when $N \to +\infty$ by introducing the MFG system. Roughly speaking, a generic player behaves according to the following dynamic:

$$\begin{cases} dX_t = b(X_t, \alpha_t) dt + \sqrt{2}\sigma(X_t) dB_t, \\ X_{t_0} = x_0, \end{cases}$$

and the cost functional is given by (here $\tau = \tau(X_{\cdot})$)

$$J(t_0,x_0,\alpha_{\cdot}) = \mathbb{E}\left[\int_{t_0}^{\tau} \left(L(s,X_s,\alpha_s) + F(X_s,m(s))\right)ds + G(X_\tau,m(\tau))\right].$$

The solution of a MFG represents the analogous to a Nash equilibrium for a non-cooperative game, see [16, 18, 23, 55, 56] for more details. Moreover, the optimal strategies in the MFG system provide approximated ε -Nash equilibria in the Nash system, as proved, e.g., in [22, 48, 50]. However, the lack of compactness properties of the problem creates many difficulties in the "very" convergence proof, i.e. in proving the convergence of the Nash equilibria in the N-players game towards the optimal strategies of the MFG system.

In order to handle this problem, Lasry and Lions in [57] proved that the solutions of the Mean Field Games are just the trajectories of a non-local transport equation, called the *Master Equation*, in the space of measures. In particular, if

 $(t, x, m) \in (0, T) \times \Omega \times \mathcal{P}^{sub}(\Omega)$, then the Master Equation is defined as follows:

$$\Omega \times \mathcal{P}^{sub}(\Omega), \text{ then the Master Equation is defined as follows:}
- \partial_t U(t, x, m) - \text{tr} \left(a(x) D_x^2 U(t, x, m) \right) + H(t, x, D_x U(t, x, m))
- \int_{\Omega} \text{tr} \left(a(y) D_y D_m U(t, x, m, y) \right) dm(y)
+ \int_{\Omega} D_m U(t, x, m, y) \cdot H_p(t, y, D_x U(t, y, m)) dm(y) = F(x, m)
& in $[0, T] \times \Omega \times \mathcal{P}^{sub}(\Omega),$

$$U(T, x, m) = G(x, m) \qquad \text{in } \Omega \times \mathcal{P}^{sub}(\Omega),$$

$$U(t, x, m) = 0 \qquad \text{for } (t, x, m) \in [0, T] \times \partial \Omega \times \mathcal{P}^{sub}(\Omega),$$

$$\frac{\delta U}{\delta m}(t, x, m, y) = 0 \qquad \text{for } (t, x, m, y) \in [0, T] \times \Omega \times \mathcal{P}^{sub}(\Omega) \times \partial \Omega,$$$$

where $D_m U$ and $\frac{\delta U}{\delta m}$ are two derivatives, considered with respect to the measure variable m, whose precise definition will be given later.

Furthermore, when we consider the limiting procedure by taking the number of agents N to $+\infty$, the solution of the Nash system will converge to the solution of the Master equation, as proved by Cardaliguet, Delarue, Lasry and Lions in [18] for the periodic setting $(\Omega = \mathbb{T}^d)$.

For the sake of completeness, let us underline that our formulation of (7), is typically referred to as *First order Master Equation*, or *Master Equation without common noise*, appearing when the dynamic of the generic player has the form (1). Besides, in the *Second order Master Equation*, or *Master Equation with common noise*, the dynamic of the generic player presents an additional Brownian term dW_t , common to all the players. In the latter case, the Master Equation presents other terms depending on $D_{mm}U$, a second order derivative of U with respect to the measure, and the overall system difficulty significantly grows.

After being introduced in [57], the Master Equation has been studied in many papers, almost always in the periodic case $\Omega = \mathbb{T}^d$, or in the whole space $\Omega = \mathbb{R}^d$. In [12], Buckdahn, Li, Peng proved the well-posedness of the first order Master equation without coupling terms, by means of probabilistic arguments, while Chassagneux, Crisan and Delarue, in [28], provided a first exhaustive existence and uniqueness result of solution, still without common noise. Moreover, Gangbo and Swiech, in [41], gave a short time existence for the Master Equation in the presence of common noise.

The most relevant result was certainly given in [18], where Cardaliaguet, Delarue, Lasry and Lions proved the well posedness of the Master Equation, with and without common noise, in a periodic setting. As regards other boundary conditions, in [66], Ricciardi proved existence and uniqueness results for the first order Master Equation with Neumann boundary conditions.

Moreover, Carmona and Delarue in [23] derived convergence results in the whole space, while in [29, 30], Delarue, Lacker and Ramanan used the Master Equation to analyse the large deviation problem as well as the central limit theorem. Concerning the major-minor problem, Cardaliaguet, Cirant and Porretta studied a convergence result in [17]. For finite state problems, we refer to the works of Bayraktar and Cohen in [5] and by Cecchin and Pelino in [26]. Finally, Bertucci, Lasry and Lions in [10] studied the Master Equation for the Finite State Space Planning Problem. More recently, in the setting of MFG with exhaustible resources, i.e. the continuum limit of a dynamic game of exhaustible resources modelling Cournot competition between producers, see e.g. [43, 42] for more details, Graber and Sircar study the Master Equation for a MFG of controls with absorption in [44]. Differently from the classical setting studied in this paper where the interaction is through the mean of the state variable, in [44] the authors studied a model of Cournot competition between producers whose states depend on the empirical measure over controls. Concerning a theoretical overview of MFG of control, we refer to [20].

Other important papers about the Master Equation and the convergence problem are given by [4, 24, 25, 27, 33, 34, 62, 52, 53, 39, 49, 40, 60].

Concerning the MFG model with absorption, the first-order system has been widely studied, see for example [15], where the authors studied MFG with absorption through the introduction of a renormalized empirical measure, or [36, 35], where a monotone regularized version of the problem is considered. In [14] a model of bank run is considered within the framework of a MFG model with absorption and common noise. In [6], the authors consider a jump-diffusion dynamic with controlled jumps for the players, providing an example of an illiquid inter-bank market model, where the banks can change their reserves only at the jump times of some exogenous Poisson processes with a common constant intensity. Similar problems, such as minimal-time MFGs where agents want to leave a given bounded domain through a part of its boundary in minimal time with application to crowd motion, have been studied in [61]. Similarly, in [8, 9, 11], optimal stopping MFGs were studied for a model where a representative agent chooses both the optimal control and the optimal time to exit the game. A very general result about the Mean Field Games system, which includes the Dirichlet case as well as the periodic one and the Neumann case, was given by Porretta in [65].

Our aim is to both treat MFG with absorption and to study the related Master Equation (without common noise) for a bounded domain, assuming homogeneous Dirichlet boundary conditions. We refer to [66] and [67] for an equivalent result in the Neumann case.

The main contributions of this paper can be summarized as follows:

- Existence of the Master Equation solution for a bounded domain with Dirichlet setting: the solution corresponds to an equilibrium of a (random) flow of sub-probability measures. Here, the main novelty relies upon the study of the Master Equation (7) in $\mathcal{P}^{sub}(\Omega)$ instead of $\mathcal{P}(\Omega)$, the space of Borel probability measures in Ω . This particular measure space is essential to handle the dissipation of mass related to the framework of Dirichlet's boundary conditions. Furthermore, we use specific tools such as the generalized Wasserstein distance introduced in [63] to deal with subprobability measures.
- A new boundary condition in the Master Equation: we stress the fact that the last boundary condition $\frac{\delta U}{\delta m} = 0$ in (7) is completely new in the literature. It relies on the fact that the Dirichlet boundary condition in the Nash system (5) provides a smoothing effect near the boundary while approaching the asymptotic configuration.

The article is basically divided into three parts: first, we study the Master Equation (7), proving the existence, uniqueness and regularity of solutions; then we use previously obtained results on the Master Equation to prove the convergence problem, in a suitable sense; while in the third part, we provide an economical application, inspired by the models proposed by [20, 46, 54]. More precisely, in Section 2, we introduce the basic notations and hypotheses we will need throughout the article, and we state the two main results we want to prove. In Section 3 we give a formal proof for the well-posedness of the Master Equation, which becomes rigorous provided some strong regularity results of the function U. In Sections 4 and 5 we state the latter regularity results for U: in particular, Section 4 is devoted to the study of the Fokker-Planck equation and the Mean Field Games, whereas in Section 5 we analyze a MFG linearized system which provides a suitable regularity for the function U. In Section 6 we derive results about the convergence of the solution and we conclude with 7, presenting an optimal liquidation toy model.

The function U is defined as in (9), and various estimates, such as global bounds and global Lipschitz regularity, are established. Notably, one of the key challenges in demonstrating that U satisfies (7) is establishing its C^1 continuity with respect to m. This step necessitates a meticulous analysis of the linearized mean field game system (refer to [18, 66]) to establish strong regularity of U in both the spatial and measure variables.

It is worth highlighting that these regularity estimates demand strong regularity not only in the spatial domain but also in the measure variable.

In the spatial domain, the regularity is derived in [18] through differentiation of the equation with respect to x. However, in the case of Dirichlet boundary conditions, as well as in general for any boundary conditions, such methods are not directly applicable. Instead, we obtain these bounds by employing a distinct set of space-time estimates that require careful handling.

Moreover, it's essential to note that regularity estimates for the Dirichlet parabolic equation necessitate compatibility conditions between initial and boundary data. Unfortunately, these compatibility conditions cannot always be guaranteed within this context. Consequently, we extend the estimates obtained in [18] through an in-depth investigation of the regularity of solutions to the Fokker-Planck equation.

A noteworthy novelty in this work is the meticulous study of the Fokker-Planck equation and the linearized Mean Field Games system. This study is conducted within negative-order Hölder spaces, and the well-posedness is established via approximation using smooth functions. However, it's important to emphasize that smooth functions are not dense in the dual of Hölder spaces, as pointed out in [49]. As a result, we must confine our analysis to appropriate subsets of these dual spaces. Notably, these ideas can be applied also to adjust the Neumann case [66].

2. Problem Formulation and Main Results

In the first part of the article, we analyze the well-posedness of the Master Equation defined on $[0, T] \times \Omega \times \mathcal{P}^{sub}(\Omega)$ with a homogeneous Dirichlet condition on $\partial\Omega$.

Our aim is to find a solution $U: [0,T] \times \Omega \times \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$ which solves strongly the equation (7). Following the classical framework, described in [18], for each initial data $(t_0, m_0) \in [0, T] \times \mathcal{P}^{sub}(\Omega)$, we consider the MFG system in $[t_0, T] \times \Omega$ with a homogeneous Dirichlet conditions:

$$\begin{cases} -u_{t} - \operatorname{tr}(a(x)D^{2}u) + H(t, x, Du) = F(t, x, m(t)), \\ m_{t} - \sum_{i,j} \partial_{i,j}^{2}(a_{i,j}(x)m) - \operatorname{div}(mH_{p}(t, x, Du)) = 0, \\ m(t_{0}) = m_{0}, \qquad u(x, T) = G(x, m(T)), \\ u_{|\partial\Omega} = 0, \qquad m_{|\partial\Omega} = 0, \end{cases}$$

$$(8)$$

where, as always, a backward Hamilton-Jacobi-Bellman equation for the value function u of the generic player is coupled with a forward Fokker-Planck equation for the density m of the population. Then we define

$$U(t_0, x, m_0) = u(t_0, x), (9)$$

and we want to prove that, under certain assumptions, U is the solution of the Master Equation (7).

First and foremost, we point out that assuming a Dirichlet boundary condition implies the following preliminary estimate on m:

$$||m(t)||_{L^1} \le 1 \qquad \forall \, m_0 \in \mathcal{P}^{sub}(\Omega) \,. \tag{10}$$

This is a classical result, under certain assumptions on H (which will be included in our hypotheses), and it was proved, for example, in *Proposition 3.10* of [65] for $m_0 \in L^1(\Omega)$; by a density argument, it can be easily extended for $m_0 \in \mathcal{P}^{sub}(\Omega)$.

Hence, m can not be represented as a probability measure as it occurs in the periodic setting [18] or in the Neumann formulation [66]. This is why we have to work with $m \in \mathcal{P}^{sub}(\Omega)$.

Actually, this is not a strange result: following the stochastic interpretation and the N-players game, the agents exit the bounded boundary when they hit Ω and, thus, the corresponding distribution $m_x^{N,i}$ should decrease in terms of mass.

2.1. Hölder spaces

In what follows we briefly recall basic notions about the Banach functions spaces then used throughout the paper, see, e.g., [51, 66], for more details.

Let T > 0 and $\Omega \subset \mathbb{R}^d$ be the closure of an open, bounded and connected set, with the boundary of class $C^{2+\alpha}$, for some $\alpha > 0$; we denote with Q_T the set $Q_T := [0, T] \times \Omega$. We call $d(\cdot)$ the oriented distance function from the boundary of Ω , defined as

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial \Omega) & x \in \Omega, \\ -\operatorname{dist}(x, \partial \Omega) & x \notin \Omega. \end{cases}$$

Thanks to [31], we have $d(\cdot) \in C^{2+\alpha}$ in a neighbourhood of the boundary. Since we are only interested in the local character of d near $\partial\Omega$, when we write $d(\cdot)$ in this paper, we mean a $C^{2+\alpha}$ function coinciding with d in a neighbourhood of the boundary.

For $n \ge 0$ and $\alpha \in (0, 1)$, $C^{n+\alpha}(\Omega)$ is defined as the space of functions *n*-times differentiable, with derivatives α -Hölder continuous. Being $\phi \in C^{n+\alpha}(\Omega)$, its norm is defined in the following way:

$$\|\phi\|_{n+\alpha} := \sum_{|\ell| \le n} \|D^{\ell}\phi\|_{\infty} + \sum_{|\ell| = n} \sup_{x \ne y} \frac{|D^{\ell}\phi(x) - D^{\ell}\phi(y)|}{|x - y|^{\alpha}}.$$

Similarly, the parabolic space $C^{\frac{n+\alpha}{2},n+\alpha}(Q_T)$ consists of functions ϕ admitting derivatives $D_t^r D_x^s \phi$, with $2r + s \le n$, and with norm

$$\|\phi\|_{\frac{n+\alpha}{2},n+\alpha} := \sum_{2r+s \le n} \|D_t^r D_x^s \phi\|_{\infty} + \sum_{2r+s=n} \sup_{t} \|D_t^r D_x^s \phi(t,\cdot)\|_{\alpha} + \sum_{0 \le n+\alpha-2r-s \le 2} \sup_{x} \|D_t^r D_x^s \phi(\cdot,x)\|_{\frac{n+\alpha-2r-s}{2}}.$$

In order to work with Dirichlet boundary conditions, we define the spaces $C^{n+\alpha,D}(\Omega)$ and $C^{\frac{n+\alpha}{2},n+\alpha,D}(Q_T)$ as the subspaces of functions belonging to $C^{n+\alpha}(\Omega)$, resp. to $C^{\frac{n+\alpha}{2},n+\alpha}(Q_T)$, vanishing at $\partial\Omega$. When there is no risk of confusion, we will omit the set Ω or Q_T .

Analogously, we define the spaces $C^{0,\alpha}$, $C^{\alpha,0}$, $C^{1,2+\alpha}$. For the sake of completeness, also because of its relevance throughout the paper, let us specify the norm equipping the latter functional space:

$$\|\phi\|_{1,2+\alpha} := \|\phi\|_{\infty} + \|\phi_t\|_{0,\alpha} + \|D_x\phi\|_{\infty} + \|D_x^2\phi\|_{0,\alpha}$$

Eventually, we have to work with suitable subsets of the dual spaces of $C^{n+\alpha}$ and $C^{n+\alpha,D}$. In particular, we denote as $C^{-(n+\alpha)}$ the set

$$C^{-(n+\alpha)} := \left\{ \rho \in \left(C^{(n+\alpha)} \right)' \, \middle| \, \langle \rho, \phi \rangle = \sum_{|\gamma| \le n} \int_{\Omega} \partial^{\gamma} \phi(x) \, \rho_{\gamma}(dx) \quad \forall \phi \in C^{n+\alpha} \,, \quad (\rho_{\gamma})_{\gamma} \text{ measures on } \Omega \right\} \,.$$

In the same way we define the set $C^{-(n+\alpha),D}$ and $C^{-\frac{\alpha}{2},-\alpha}$. The norms on these spaces are inherited by the classical dual spaces norm:

$$\begin{split} ||\rho||_{-(n+\alpha)} &= \sup_{||\phi||_{n+\alpha} \le 1} \langle \rho, \phi \rangle \,, \qquad ||\rho||_{-(n+\alpha),D} = \sup_{||\phi||_{n+\alpha,D} \le 1} \langle \rho, \phi \rangle \,, \\ &||\rho||_{-\left(\frac{\alpha}{2},\alpha\right)} = \sup_{||\phi||_{\mathfrak{P},\alpha} \le 1} \langle \rho, \phi \rangle \,. \end{split}$$

The importance of these subspaces is given by the following result

Lemma 1. The space $C^{-(n+\alpha)}$ is a norm-closed subset of $(C^{n+\alpha})'$. Moreover, if $\rho \in C^{-n}$, then there exist a sequence $\{\rho_k\}_k \subset C^{2+\alpha}$ such that $\rho_k \to \rho$ in $C^{-(n+\alpha)}$ and $\rho_{k|\partial\Omega} = 0$.

Proof. Thanks to [49, Lemma 2.6], we know that $C^{-(n+\alpha)}$ is a norm-closed subset of $(C^{n+\alpha})'$. Let $\rho \in C^{-n}$. Then there exist measures $\{\rho_\gamma\}_{|\gamma| \le n}$ such that $\rho = \sum\limits_{|\gamma| \le n} \partial^\gamma \rho_\gamma(dx)$. Defining $\rho_\gamma(\mathbb{R}^d \setminus \Omega) = 0$, we can consider $\{\rho_\gamma\}_\gamma$ as measures on \mathbb{R}^d .

Let $\{n_{\varepsilon}\}_{\varepsilon}$ be smooth mollifiers. We define $\rho_{\gamma}^{k} := \rho_{\gamma} * \eta_{\frac{1}{k}}$ and $\tilde{\rho}_{k} := \sum_{|\gamma| \leq n} \partial^{\gamma} \rho_{\gamma}^{k}(dx)$. Thanks to [49, Lemma 2.3, Lemma 2.7], we have $\tilde{\rho}_{k} \in C^{\infty}$ and $\tilde{\rho}_{k} \to \rho$ in $C^{-(n+\beta)}$ for all $\beta > 0$.

Then, we consider a nonnegative smooth function $\xi_k(s) \in C^{\infty}(\mathbb{R})$ such that $\xi_k(s) = 1$ for $|s| \ge \frac{1}{k}$ and $\xi_k(0) = 0$. We define $\rho_k(x) = \tilde{\rho}_k(x) \, \xi_k(d(x))$. Then we have $\rho_k \in C^{2+\alpha}$ and the convergence to ρ in $C^{-(n+\alpha)}$ is preserved.

2.2. Subprobability Measures and Generalized Wasserstein Measure

Let us start the present subsection with a *proper* notion of distance for elements belonging to $\mathcal{P}^{sub}(\Omega)$. Since the usual Wasserstein distance $W_p(\mu, \nu)$ is defined only when the two measures μ, ν have the same mass, in what follows we rely on the so-called *generalized Wasserstein distance* which allows computing distance between measures with different masses, see e.g. [63, 64] for a complete description and [37, 68] for the corresponding probabilistic interpretation based on unbalanced optimal transport.

According to [64], we introduce also the following definition which is the one we will use throughout the article.

Definition 1. Let $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$ be two Borel sub-probability measures on Ω . We call the generalized Wasserstein distance between m_1 and m_2 , and we write $\mathbf{d}_1(m_1, m_2)$ the quantity

$$\mathbf{d}_{1}(m_{1}, m_{2}) := \sup_{Lip(\phi) \le 1, ||\phi||_{C^{0}} \le 1} \int_{\Omega} \phi(x) d(m_{1} - m_{2})(x), \tag{11}$$

We remark that $d_1(m_1, m_2)$ is equivalent to the well-known *flat metric* over the space of Radon measures with finite mass on Ω .

Furthermore, we define a suitable derivation of U with respect to the (sub)measure m as it appears in the Master Equation (7). Namely, we have:

Definition 2. Let $U: \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$. We say that U is of class C^1 if there exists a continuous map $K: \mathcal{P}^{sub}(\Omega) \times \Omega \to \mathbb{R}$ such that, for all $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$ we have

$$\lim_{s \to 0} \frac{U(m_1 + s(m_2 - m_1)) - U(m_1)}{s} = \int_{\Omega} K(m_1, x)(m_2(dx) - m_1(dx)). \tag{12}$$

We define $\frac{\delta U}{\delta m}(m, x) := K(m, x)$.

Remark 1. Condition (12) defines uniquely the derivative K. Conversely, in the classical setting, where $U: \mathcal{P}(\Omega) \to \mathbb{R}$, see e.g. [18, 66], the function K is defined up to an additive constant, since, for $c \in \mathbb{R}$ and $m_1, m_2 \in \mathcal{P}(\Omega)$, $\int_{\Omega} c \, d(m_2 - m_1) = 0$. Hence, in those cases a normalization condition is needed to uniquely identify K.

Let us note that, exploiting Eq. (12), we have the following equality for $m_1, m_2 \in \mathcal{P}^{sub}(\Omega)$,

$$U(m_2) - U(m_1) = \int_0^1 \int_{\Omega} \frac{\delta U}{\delta m} (m_1 + s(m_2 - m_1), x) (m_2(dx) - m_1(dx)) ds.$$

which turns out to be very useful in our computations. Moreover, we note that, if $\frac{\delta U}{\delta m}$ is C^1 in the space variable, we have

$$|U(m_2) - U(m_1)| \le \sup_{m} \left\| D_x \frac{\delta U}{\delta m}(m, \cdot) \right\|_{\infty} \mathbf{d}_1(m_1, m_2),$$

suggesting us define the *intrinsic derivative* of U with respect to m, which directly appears in the Master Equation (7).

Definition 3. Let $U: \mathcal{P}^{sub}(\Omega) \to \mathbb{R}$. If U is of class C^1 and $\frac{\delta U}{\delta m}$ is of class C^1 with respect to the last variable, we define the intrinsic derivative $D_m U: \mathcal{P}^{sub}(\Omega) \times \Omega \to \mathbb{R}^d$ as

$$D_m U(m, x) := D_x \frac{\delta U}{\delta m}(m, x).$$

2.3. Assumptions

We require the following hypotheses throughout the paper:

Hypotheses 1. *Let* $0 < \alpha < 1$. *Assume that*

(i) $||a(\cdot)||_{1+\alpha} < \infty$ and $\exists \mu > \lambda > 0$ s.t. $\forall \xi \in \mathbb{R}^d$

$$\lambda |\xi|^2 \le \langle a(x)\xi, \xi \rangle \le \mu |\xi|^2$$
;

- (ii) $H:[0,T]\times\Omega\times\mathbb{R}^d\to\mathbb{R}$, $G:\Omega\times\mathcal{P}^{sub}(\Omega)\to\mathbb{R}$ and $F:[0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega)\to\mathbb{R}$ are smooth functions with Hlocally Lipschitz with respect to the last variable;
- (iii) $\exists C > 0 \text{ s.t.}$

$$0 < H_{pp}(t, x, p) \le CI_{d \times d};$$

(iv) F is increasing in the last variable, i.e.

$$\int_{\Omega} \left(F(t,x,m) - F(t,x,m') \right) d(m-m')(x) \ge 0;$$

moreover

$$\sup_{m \in \mathcal{P}^{\text{Sub}}(\Omega)} \left(\|F(\cdot, \cdot, m)\|_{\frac{\alpha}{2}, \alpha} + \left\| \frac{\delta F}{\delta m}(\cdot, \cdot, m, \cdot) \right\|_{\frac{\alpha}{2}, \alpha, 2 + \alpha} \right) + \operatorname{Lip}\left(\frac{\delta F}{\delta m} \right) \leq C_F,$$

with

$$\operatorname{Lip}\left(\frac{\delta F}{\delta m}\right) := \sup_{m_1 \neq m_2} \left(\|m_1 - m_2\|_{-(2+\alpha), D}^{-1} \left\| \frac{\delta F}{\delta m}(\cdot, \cdot, m_1, \cdot) - \frac{\delta F}{\delta m}(\cdot, \cdot, m_2, \cdot) \right\|_{\frac{\alpha}{2}, \alpha, 2+\alpha} \right);$$

(v) G satisfies the same estimates as F with α and $1 + \alpha$ replaced by $2 + \alpha$, i.e.

$$\sup_{m \in \mathcal{P}^{\mathrm{Sub}}(\Omega)} \left(\|G(\cdot, m)\|_{2+\alpha} + \left\| \frac{\delta G}{\delta m}(\cdot, m, \cdot) \right\|_{2+\alpha, 2+\alpha} \right) + \operatorname{Lip}\left(\frac{\delta G}{\delta m} \right) \leq C_G ,$$

with

$$\begin{split} \operatorname{Lip}\left(\frac{\delta G}{\delta m}\right) &:= \\ &\sup_{m_1 \neq m_2} \left(\|m_1 - m_2\|_{-(2+\alpha), D}^{-1} \, \left\| \frac{\delta G}{\delta m}(\cdot, m_1, \cdot) - \frac{\delta G}{\delta m}(\cdot, m_2, \cdot) \right\|_{2+\alpha, 2+\alpha} \right) \, ; \end{split}$$

(vi) Moreover, we require the following Dirichlet boundary conditions:

$$\frac{\delta F}{\delta m}(t,x,m,y)_{|y\in\partial\Omega}=0\,,\qquad \frac{\delta G}{\delta m}(x,m,y)_{|y\in\partial\Omega}=0\,,\qquad G(x,m)_{|x\in\partial\Omega}=0\,,$$

for all $m \in \mathcal{P}^{sub}(\Omega)$.

We note that in hypotheses (vi) we have two standard compatibility assumptions for G and $\frac{\delta G}{\delta m}$, naturally linked to the boundary conditions framework. In particular, the condition on G is essential to have a classical solution for both the Master Equation (7) and the MFG system (8), whereas the condition on $\frac{\delta G}{\delta m}$ is a compatibility condition for the linearized MFG system, see e.g. Corollary 12.

We underline that condition on F is a novelty, and it will play a crucial role when considering the Dirichlet boundary condition of $\frac{\delta U}{\delta m}$

2.4. Convergence in the Nash system

The second part of the article is focused on the convergence problem. As already said, the value function v_i^N is defined as the cost function related to the optimal control, and functions $\{v_i^N\}_i$ solve the Nash system (5).

The idea is to define proper finite dimensional projections U along trajectories $m_x^{N,i}$. To this end, we define the following functions u_i^N :

$$u_i^N(t,\mathbf{x}) = U(t,x_i,m_{\mathbf{x}}^{N,i}),$$

then proving that they *almost* solve the Nash system (5), with an error of order $\frac{1}{N}$, and such that, within suitable spaces, $|u_i^N - v_i^N| \to 0$, for $N \to +\infty$. The main Theorem of this part is the following:

Theorem 2. Assume Hypotheses 1 hold true and let $(v^{N,i})_{i \in \{1,\dots,N\}}$ be the solution of (5) and U the solution of the master equation (7). Fix $N \ge 1$ and $(t_0, m_0) \in [0, T] \times \mathcal{P}^{sub}(\Omega)$.

(i) For any $\mathbf{x} \in (\Omega)^N$, let $m_{\mathbf{x}}^N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \mathbb{1}_{\{x_j \in int(\Omega)\}}$. Then,

$$\sup_{i=1,...,N} \left| v^{N,i}(t_0, \boldsymbol{x}) - U(t_0, x_i, m_{\boldsymbol{x}}^N) \right| \le \frac{C}{N}.$$

(ii) For any $i \in \{1, ..., N\}$ and $x_i \in \Omega$, we define

$$w^{N,i}(t_0,x_i,m_0) := \int_{\Omega^{N-1}} v^{N,i}(t_0,\boldsymbol{x}) \prod_{i \neq i} m_0(dx_i),$$

where $x = (x_1, ..., x_N)$. Then,

$$||w^{N,i}(t_0,\cdot,m_0) - U(t_0,\cdot,m_0)||_{L^1(m_0)} \le \begin{cases} CN^{-1/d} & \text{if } d \ge 3, \\ CN^{-1/2} \log(N) & \text{if } d = 2, \\ CN^{-1/2} & \text{if } d = 1. \end{cases}$$
(13)

3. Well-posedness of the Master Equation

A formal proof of existence and uniqueness of solutions for (7) can be easily given directly using the very definition of U. The difficulty here relies on the proof of the C^1 character of U with respect to m. First, we establish the required regularity for U and a related compatibility condition based on the boundary conditions. The existence and uniqueness theorem are rigorously proved, as we see in detail in the following theorem.

Theorem 3. Suppose hypotheses 1 are satisfied. Take U defined as in (9), suppose that $\frac{\delta U}{\delta m}$ exists, being also bounded in $C_x^{2+\alpha} \times C_y^{2+\alpha}$, uniformly in t and continuously in m, and that boundary conditions of (7) hold for U. Then U is the unique classical solution U of the Master Equation (7).

Proof. Existence. By a density argument, we only need to prove that U solves (7) at (t_0, x, m_0) , where m_0 is a smooth and positive function satisfying $m_{0|\partial\Omega} = 0$ and $\int_{\Omega} m_0(x) dx \le 1$. Taking (u, m) as the solution of the MFG system starting from m_0 at time t_0 , by the definition of U, we have that for all $s > t_0$

$$U(s, x, m(s)) = u(s, x),$$

therefore, we can write

$$\partial_t U(t_0, x, m_0) = \lim_{h \to 0} \frac{U(t_0 + h, x, m_0) - U(t_0 + h, x, m(t_0 + h))}{h} + u_t(t_0, x).$$

Again by the very definition of U:

$$u_t(t_0, x) = -\operatorname{tr}(a(x)D_x^2 U(t_0, x, m_0)) + H(t_0, x, D_x U(t_0, x, m_0)) - F(t_0, x, m_0).$$

As regards the limit part, we define $m_s := (1 - s)m(t_0) + sm(t_0 + h)$. Since U is C^1 with respect to m, then:

$$\begin{split} -\lim_{h\to 0} \int_0^1 \int_\Omega \frac{\delta U}{\delta m}(t_0+h,x,m_s,y) \frac{(m(t_0+h,y)-m(t_0,y))}{h} \, dy ds &= \\ -\int_\Omega \frac{\delta U}{\delta m}(t_0,x,m_0,y) \bigg(\sum_{ij} \partial_{ij}^2 (a_{ij}(x)m(t_0,y)) \\ &+ \operatorname{div} \big(m(t_0,y) H_p(t_0,y,Du(t_0,y)) \big) \, dy \, . \end{split}$$

Integrating by parts, and using the boundary conditions of $\frac{\delta U}{\delta m}$ and m, we can rewrite the right-hand side as

$$\begin{split} \int_{\Omega} \left[H_p(t_0, y, D_x U(t_0, y, m_0)) D_m U(t_0, x, m_0, y) \right. \\ \left. - \, \text{tr} \Big(a(y) D_y D_m U(t_0, x, m_0, y) \Big) \right] dm_0(y) \,, \end{split}$$

therefore

$$\begin{split} \partial_t U(t,x,m) &= -\mathrm{tr} \left(a(x) D_x^2 U(t,x,m) \right) + H(t,x,D_x U(t,x,m)) \\ &- \int_{\Omega} \mathrm{tr} \left(a(y) D_y D_m U(t,x,m,y) \right) dm(y) \\ &+ \int_{\Omega} D_m U(t,x,m,y) \cdot H_p(t,y,D_x U(t,y,m)) dm(y) - F(t,x,m) \,, \end{split}$$

hence concluding the existence part.

Uniqueness. Suppose that there exists a second solution V of (7). We fix (t_0, m_0) , with m_0 smooth s.t. $m_{0|\partial\Omega} = 0$, and we take \tilde{m} as the solution of

$$\begin{cases} \tilde{m}_t - \sum_{ij} \partial_{ij}^2(a_{ij}(x)\tilde{m}) - \operatorname{div}(\tilde{m}H_p(t, x, D_xV(t, x, \tilde{m}))) = 0, \\ \tilde{m}(t_0) = m_0, \\ \tilde{m}_{l\partial\Omega} = 0 \end{cases}$$

which is well-posed since $D_x V$ is Lipschitz continuous with respect to the measure. Then we define $\tilde{u}(t, x) = V(t, x, \tilde{m}(t))$ and we want to prove that \tilde{u} solves a Hamilton-Jacobi equation. Using the equations of V and \tilde{m} , we get

$$\begin{split} \tilde{u}_t(t,x) &= V_t(t,x,\tilde{m}(t)) + \int_{\Omega} \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y) \, \tilde{m}_t(t,y) \, dy \\ &= V_t(t,x,\tilde{m}(t)) + \int_{\Omega} \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y) \, \sum_{ij} \partial_{ij}^2(a_{ij}(x)\tilde{m}(t,y)) \, dy \\ &+ \int_{\Omega} \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y) \, \mathrm{div}(\tilde{m}H_p(t,x,D_xV(t,x,\tilde{m}))) \, dy \, . \end{split}$$

Previous integrals can be easily estimated with an integration by parts, and taking into account the boundary condition for V. As regards the first, we have

$$\int_{\Omega} \frac{\delta V}{\delta m}(t, x, \tilde{m}(t), y) \sum_{ij} \partial_{ij}^{2}(a_{ij}(x)\tilde{m}(t, y)) dy =$$

$$\int_{\Omega} \operatorname{tr}(a(y)D_{y}D_{m}V(t, x, \tilde{m}(t), y)) \tilde{m}(t, y) dy,$$

while for the second

$$\begin{split} &\int_{\Omega} \frac{\delta V}{\delta m}(t,x,\tilde{m}(t),y) \operatorname{div}(\tilde{m}H_{p}(t,x,D_{x}V(t,x,\tilde{m}))) \, dy \\ &= -\int_{\Omega} H_{p}(t,x,D_{x}V(t,x,\tilde{m})) D_{m}V(t,x,\tilde{m},y) \tilde{m}(t,y) dy \, . \end{split}$$

Previous estimates allow us to write

$$\begin{split} \tilde{u}_t(t,x) &= V_t(t,x,\tilde{m}(t)) + \int_{\Omega} \operatorname{tr}(a(y)D_yD_mV(t,x,\tilde{m},y)) \, d\tilde{m}(y) \\ &- \int_{\Omega} H_p(t,y,D_xV(t,y,\tilde{m}))D_mV(t,x,\tilde{m},y) \, d\tilde{m}(y) \\ &= -\operatorname{tr}(a(x)D^2\tilde{u}(t,x)) + H(t,x,D\tilde{u}(t,x)) - F(t,x,\tilde{m}(t)) \,, \end{split}$$

hence (\tilde{u}, \tilde{m}) is a solution of the MFG system (8). Then, by uniqueness of solutions for the MFG system, we get $(\tilde{u}, \tilde{m}) = (u, m)$, implying that $V(t_0, x, m_0) = U(t_0, x, m_0)$, whenever m_0 is smooth. Therefore, by density, the uniqueness is proved.

In the next two sections, we will prove the C^1 character of U with respect to m, the C^2 regularity in x and y of $\frac{\delta U}{\delta m}$ and the boundary conditions for U, hence justify the hypotheses previously given, also allowing us to then apply Theorem 3 and finally prove the well-posedness of (7).

4. The Fokker-Planck equation and the MFG system

Let us first give the following technical lemma, providing a regularity result for a linear PDE with non-homogeneous drift.

Lemma 4. Suppose a satisfies hypothesis (i) of 1, let $q = \frac{d+2}{1-\alpha}$, $b, f \in L^q(Q_T)$, $\psi \in C^{1+\alpha,D}(\Omega)$, and z be the solution of

$$\begin{cases} -z_t - \operatorname{tr}(a(x)D^2z) + b(t,x) \cdot Dz = f(t,x), \\ z(T) = \psi, \\ z_{|\partial\Omega} = 0. \end{cases}$$

Then z satisfies

$$||z||_{\frac{1+\alpha}{2},1+\alpha} \le C(||f||_{L^q} + ||\psi||_{1+\alpha}). \tag{14}$$

Moreover, if $b, f \in C^{0,\alpha}(Q_T)$ and $\psi \in C^{2+\alpha,D}(\Omega)$, it holds

$$||z||_{1,2+\alpha} \le C\left(||f||_{0,\alpha} + ||\psi||_{2+\alpha}\right). \tag{15}$$

Proof. We exploit ideas stated in [66]. In particular, if $f \in C(Q_T)$, $b \in C(\Omega)$, then we conclude with *Theorem 5.1.1* of [59]. While, in the general case, we write $z = z_1 + z_2$, where z_1 and z_2 respectively satisfy:

$$\begin{cases} -(z_1)_t - \operatorname{tr}(a(x)D^2z_1) = 0, \\ z_1(T) = \psi, \\ z_{1|\partial\Omega} = 0, \end{cases}$$

and

$$\begin{cases} -(z_2)_t - \text{tr}(a(x)D^2z_2) + b \cdot Dz_2 = f - b \cdot Dz_1, \\ z_2(T) = 0, \\ z_{2|\partial\Omega} = 0. \end{cases}$$

For z_1 , we apply *Theorem 5.1.11* of [59] to obtain $||z_1||_{\frac{1+\alpha}{2},1+\alpha} \le C ||\psi||_{1+\alpha}$. For z_2 , by Corollary of *Theorem IV.9.1* in [51], we have

$$||z_2||_{\frac{1+\alpha}{2},1+\alpha} \le C ||f-bDz_1||_{L^q} \le C (||f||_{L^q} + ||\psi||_{1+\alpha}),$$

concluding the first part. Then, if $b, f \in C^{0,\alpha}(Q_T)$ and $\psi \in C^{2+\alpha,D}$, by (14), we have $f - b \cdot Dz \in C^{0,\alpha}$, allowing to apply *Theorem 5.1.13* in [59] to conclude.

4.1. The Fokker-Planck equation

From now on, q will denote the quantity $q:=\frac{d+2}{1-\alpha}$, and p will be the conjugate exponent of q, i.e. $p:=\frac{d+2}{d+1+\alpha}$. The present section is focused on the study of the following Fokker-Planck equation

$$\begin{cases} \mu_t - \operatorname{div}(a(x)D\mu) - \operatorname{div}(\mu b) = \operatorname{div}(c), \\ \mu(t_0) = \mu_0, \\ \mu_{|\partial\Omega} = 0, \end{cases}$$
(16)

where $c \in L^1(Q_T)$, $\mu_0 \in C^{-(1+\alpha)}$, and $b \in L^q(Q_T)$. Let us underline that the latter equation plays a fundamental role in studying both the Mean Field Games system and the linearized one.

The main difficulty here relies on low regularity of both data and coefficients. Hence, the main idea is to start with the regular case, where we have the existence and uniqueness of solutions, to then obtain estimates that we will then exploit to pass to the limit in the general case.

Let us start by giving a suitable definition of a solution for the system (16):

Definition 4. Let $c \in L^1$, $\mu_0 \in C^{-1}$, $b \in L^q$. We say that a function $\mu \in C([0,T];C^{-(1+\alpha),D}) \cap L^1(Q_T)$ is a weak solution of (16) if, for all $t \in (t_0,T]$ and ϕ satisfying in $[t_0,t] \times \Omega$ the following linear equation

$$\begin{cases} -\phi_t - \operatorname{div}(aD\phi) + bD\phi = \psi, \\ \phi(t) = \xi, \\ \phi_{|\partial\Omega} = 0, \end{cases}$$
(17)

with $\psi \in L^{\infty}(\Omega)$ and $\xi \in C^{1+\alpha,D}$, it holds true:

$$\langle \mu(t), \xi \rangle + \int_{t_0}^t \int_{\Omega} \mu(s, x) \psi(s, x) \, dx ds = \langle \mu_0, \phi(t_0, \cdot) \rangle - \int_{t_0}^t \int_{\Omega} c(s, x) \cdot D\phi(s, x) \, dx ds \,, \tag{18}$$

where $\langle \cdot, \cdot \rangle$ denotes, respectively, the duality between $C^{-(1+\alpha),D}$ and $C^{1+\alpha,D}$, $C^{-(1+\alpha)}$ and $C^{1+\alpha}$.

Note that Lemma 4 implies $\phi \in C^{\frac{1+\alpha}{2},1+\alpha}$, therefore the terms on the right-hand side are well-defined and the definition is well-posed.

Proposition 5. Let $c \in L^1$, $\mu_0 \in C^{-1}$, $b \in L^q$. Then there exists a unique solution for (16), which satisfies, for a certain C depending on a and b,

$$\sup_{t \in [t_0, T]} ||\mu(t)||_{-(1+\alpha), D} + ||\mu||_{L^p} \le C \left(||\mu_0||_{-(1+\alpha)} + ||c||_{L^1} \right), \tag{19}$$

Moreover, if $\mu_0^n \to \mu_0$ in $C^{-(1+\alpha)}$, $b^n \to b$ in L^q , $c^n \to c$ in L^1 , we have $\mu^n \to \mu$ in $C([0,T];C^{-(1+\alpha),D}) \cap L^p(Q_T)$, where μ^n and μ are the solutions related to (μ_0^n, b^n, f^n) and (μ_0, b, f) .

Proof. Without loss of generality, we can consider $t_0 = 0$.

Existence: smooth case. Assume that f, b and μ_0 are smooth functions, with μ_0 satisfying $\mu_0(x)_{|x\in\partial\Omega}=0$. Then, splitting the divergence terms in (16), we obtain a linear equation and the existence of solutions is guaranteed by [51, 59]. Let ϕ be a solution of (17), with $\psi=0$ and $\xi\in C^{1+\alpha,D}$. Using ϕ as a test function for μ , we get by Lemma 4

$$\langle \mu(t), \xi \rangle = \langle \mu_0, \phi(0, \cdot) \rangle - \int_0^t \int_{\Omega} c(s, x) \cdot D\phi(s, x) \, dx ds$$

$$\leq C \, ||\xi||_{1+\alpha} \left(||\mu_0||_{-(1+\alpha)} + ||c||_{L^1} \right).$$

Passing to the *sup* for $\|\xi\|_{1+\alpha} \le 1$, we obtain

$$\sup_{t \in [0,T]} \|\mu(t)\|_{-(1+\alpha),D} \le C \left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1} \right), \tag{20}$$

and we are left to prove the L^p bound for μ . To this end, we take ϕ as the solution of (17), with t = T, $\xi = 0$ and $\psi \in L^q$. Then, again by Lemma 4, we have

$$\int_{0}^{T} \int_{\Omega} \mu \psi \, dx ds = \langle \mu_{0}, \phi(0, \cdot) \rangle - \int_{0}^{T} \int_{\Omega} c(s, x) \cdot D\phi(s, x) \, dx ds$$

$$\leq C \|\psi\|_{L^{q}} \left(\|\mu_{0}\|_{-(1+\alpha)} + \|c\|_{L^{1}} \right).$$

Passing to the sup for $\|\mu\|_{L^q} \le 1$, we get

$$\|\mu\|_{L^p} \le C \left(\|\mu_0\|_{-(1+\alpha)} + \|c\|_{L^1}\right)$$

which concludes the proof in the regular case.

Existence: general case. We proceed adapting and adjusting [66, Proposition 5.3]. By Proposition 1, we take smooth data μ_0^k , c_k , b_k , with $\mu_0^k(x) = 0$ for $x \in \partial \Omega$, and converging to μ_0 , c, b, respectively in $C^{-(1+\alpha),D}$, L^1 and L^q .

We consider μ^k as the solution of (16). The previous convergences imply, for a certain C > 0,

$$\begin{aligned} ||\mu_0^k||_{-(1+\alpha)} &\leq C \, ||\mu_0||_{-(1+\alpha)} , \\ ||b_k||_{L^q} &\leq C \, ||b||_{L^q} , \\ ||c_k||_{L^1} &\leq C \, ||c||_{L^1} . \end{aligned}$$

Considering the function $\mu^{k,h} := \mu^k - \mu^h$, which satisfies (16) with data $b = b_k$, $c = c_k - c_h + \mu_h(b_k - b_h)$, $\mu^0 = \mu_0^k - \mu_0^h$, then, by estimates (19), we have

$$\begin{split} \sup_t ||\mu^{k,h}(t)||_{-(1+\alpha),D} \; + \; ||\mu^{k,h}||_{L^p} \\ \leq C \left(||\mu_0^k - \mu_0^h||_{-(1+\alpha)} + ||c_k - c_h||_{L^1} + ||\mu_h(b_k - b_h)||_{L^1} \right) \,, \end{split}$$

By the uniform bound of $\{\mu_k\}_k$ in L^p , the last term can be estimated as follows

$$\|\mu^h(b_k - b_h)\|_{L^1} \le C\|b_k - b_h\|_{L^q}, \tag{21}$$

and since $\{\mu_0^k\}_k$, $\{c_k\}_k$ and $\{b_k\}_k$ are Cauchy sequences, the right-hand side goes to 0 and $\{\mu^k\}_k$ is a Cauchy sequence, too. Then $\mu^k \to \mu$ in $C([0,T];C^{-(1+\alpha),N}) \cap L^p(Q_T)$, for a certain μ also satisfying (19). We are then left to prove that μ satisfies (18), for ϕ satisfying (17). To this end, we consider the test function ϕ_k related to μ^k , so that

$$\langle \mu^k(t), \xi \rangle + \int_0^t \int_{\Omega} \mu^k(s, x) \psi(s, x) \, dx ds = \langle \mu_0^k, \phi_k(0, \cdot) \rangle - \int_0^t \int_{\Omega} c_k(s, x) \cdot D\phi_k(s, x) \, dx ds \,, \tag{22}$$

Concerning convergence of ϕ_k towards ϕ , since $\phi_k - \phi$ satisfies

$$\begin{cases} -(\phi_k - \phi)_t - \operatorname{div}(aD(\phi_k - \phi)) + b_k D(\phi_k - \phi) = (b_k - b)D\phi, \\ (\phi_k - \phi)(t) = 0, \\ (\phi_k - \phi)_{|\partial\Omega} = 0, \end{cases}$$

by Lemma 4, we have

$$\|\phi_k - \phi\|_{\frac{1+\alpha}{2}, 1+\alpha} \le C\|(b_k - b)D\phi\|_{L^q} \to 0$$
,

allowing us to pass to the limit in (22), hence ending the existence part.

Uniqueness and stability. Let μ_1 and μ_2 be two solutions. Then $\mu := \mu_1 - \mu_2$ solves the same problem with $c = \mu_0 = 0$, and by (19) we have $\|\mu\|_{L^1} = \sup_{t \in [0,T]} \|\mu(t)\|_{-(1+\alpha),D} = 0$, concluding the uniqueness part.

For the stability part, we consider $c_n \to c$, $\mu_0^n \to \mu_0$ and $b_n \to b$. Then the function $\mu^n - \mu$ satisfies (16), with b, μ_0 and f respectively replaced by b^n , $\mu_0^n - \mu_0$, and $c^n - c + \mu(b^n - b)$. Hence, (19) implies

$$\begin{split} \sup_{t \in [0,T]} ||\mu^n - \mu||_{-(1+\alpha),D} \ + ||\mu^n - \mu||_{L^p} \\ \leq C \left(||\mu^n_0 - \mu_0||_{-(1+\alpha)} + ||c^n - c||_{L^1} + ||\mu(b^n - b)||_{L^1} \right) \,. \end{split}$$

Proceeding as done for the existence in the general case, we derive the right-hand side convergence to 0, which implies $\mu^n \to \mu$ in $C([0,T];C^{-(1+\alpha),D}) \cap L^p(Q_T)$, hence concluding the proof.

As a direct consequence of Prop. (5), we can derive a further estimate in the space $C^{-(2+\alpha),D}$.

Corollary 6. Suppose hypotheses of Proposition 5 are satisfied together with $b \in C^{\frac{\alpha}{2},\alpha}$. Then the function μ satisfies

$$\sup_{t \in [0,T]} \|\mu(t)\|_{-(2+\alpha),D} + \|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)} \le C\left(\|\mu_0\|_{-(2+\alpha)} + \|c\|_{L^1}\right),\tag{23}$$

(24)

Proof. Since $b \in C^{\frac{\alpha}{2},\alpha}$, we can take ϕ as the solution of (17), with $\xi \in C^{2+\alpha,D}(\Omega)$, $\psi = 0$, so that, applying the results of [51], [59], we have $\|\phi\|_{1+\frac{\alpha}{2},2+\alpha} \le C \|\xi\|_{2+\alpha}$, and (18) implies

$$\langle \mu(t), \xi \rangle = \langle \mu_0, \phi(0, \cdot) \rangle - \int_0^T \int_{\Omega} c(s, x) \cdot D\phi(s, x) \, dx ds$$

$$\leq C \left(||\mu_0||_{-(2+\alpha)} + ||c||_{L^1} \right) ||\xi||_{2+\alpha} .$$

Now, as in Proposition 5, choose ϕ as the solution of (17), with t = T, $\xi = 0$ and $\psi \in C^{\frac{\alpha}{2},\alpha}$. Then, again by Lemma 4, we have

$$\begin{split} \int_0^T \int_{\Omega} \mu \psi \, dx ds &= \langle \mu_0, \phi(0, \cdot) \rangle - \int_0^T \int_{\Omega} c(s, x) \cdot D\phi(s, x) \, dx ds \\ &\leq C \, \|\psi\|_{\frac{\alpha}{3}, \alpha} \left(\|\mu_0\|_{-(2+\alpha)} + \|c\|_{L^1} \right) \, . \end{split}$$

Passing to the sup in the two inequalities for $\|\xi\|_{1+\frac{\alpha}{2},2+\alpha} \le 1$ and $\|\mu\|_{L^q} \le 1$, we get (23).

4.2. Mean Field Games system and Lipschitz regularity of U

Now, we turn out to the study of the Mean Field Games system and its properties to derive essential results we will then exploit to obtain some first regularity estimates of the solution U of the Master Equation.

In particular, the MFG system has the following form:

$$\begin{cases}
-\partial_{t}u - \operatorname{tr}(a(x)D^{2}u) + H(t, x, Du) = F(t, x, m(t)), \\
\partial_{t}m - \sum_{i,j} \partial_{ij}^{2}(a_{ij}(x)m) - \operatorname{div}(mH_{p}(t, x, Du)) = 0, \\
u(T) = G(x, m(T)), \qquad m(0) = m_{0}, \\
u_{|\partial\Omega=0}, \qquad m_{|\partial\Omega=0}.
\end{cases}$$
(25)

The first result is obtained by studying some regularity properties of (25) uniformly in m_0 .

Proposition 7. The system (25) has a unique classical solution (u, m), with $u \in C^{1+\frac{\alpha}{2},2+\alpha}$ and $m \in C([t_0,T];\mathcal{P}^{sub}(\Omega))$. Moreover, m(t) has a positive density for $t > t_0$, and the following estimates hold:

$$||m||_{\frac{\alpha}{3},-(1+\alpha),D} + ||m||_{L^p} + ||u||_{1+\frac{\alpha}{3},2+\alpha} \le C.$$
(26)

Furthermore, if (u_1, m_1) and (u_2, m_2) are two solutions of (25), with $m_1(t_0) = m_{01}$, $m_2(t_0) = m_{02}$, then it holds

$$||u_{1} - u_{2}||_{1,2+\alpha} + ||m_{1} - m_{2}||_{L^{p}(Q_{T})} \le C \mathbf{d}_{1}(m_{01}, m_{02}),$$

$$\sup_{t \in [0,T]} ||m_{1}(t) - m_{2}(t)||_{-(1+\alpha),D} \le C \mathbf{d}_{1}(m_{01}, m_{02}).$$
(27)

Proof. Step 1: Existence and uniqueness of solutions. We want to apply Schauder's fixed point Theorem. Let *X* be the convex set, closed for the uniform distance, defined as follows:

$$X := \left\{ \gamma \in C([t_0, T]; \mathcal{P}^{sub}(\Omega)) \text{ s.t. } \|\gamma\|_{\frac{\alpha}{2}, -(1+\alpha), D} \le M \ \forall s, t \in [t_0, T] \right\},$$

where M will be specified later. We define a map $\Phi: X \to X$ as follows.

For $\gamma \in X$, let $u = u_{\gamma}$ be the solution of the HJB equation:

$$\begin{cases}
-u_t - \operatorname{tr}(a(x)D^2u) + H(t, x, Du) = F(t, x, \gamma(t)), \\
u(T) = G(x, \gamma(T)), \\
u_{|\partial\Omega} = 0.
\end{cases}$$
(28)

Using hypotheses on F and G and $Theorem\ V.6.1$ of [51], we know that there exists a unique classical solution $u \in C^{1+\frac{\alpha}{2},\alpha}$. Moreover, by Taylor formula, we can write $H(t,x,Du) = H(t,x,0) + V(t,x) \cdot Du$ for a certain $V \in L^{\infty}$, thanks to the global boundedness of Du, obtaining a linear equation satisfied by u. Then, exploiting both the Corollary of *Theorem IV.9.1* and *Theorem IV.5.2* of [51] we get

$$||u||_{1+\frac{\alpha}{2},2+\alpha} \le C\left(||F(\cdot,\cdot,\gamma(\cdot))||_{\frac{\alpha}{2},\alpha} + ||G(\cdot,\gamma(T))||_{2+\alpha}\right),$$

Define $\Phi(\beta) = m$, where $m \in C([t_0, T]; \mathcal{P}^{sub}(\Omega))$ is the solution of the FP equation

$$\begin{cases} m_{t} - \sum_{i,j} \partial_{ij}^{2}(a_{ij}(x)m) - \operatorname{div}(mH_{p}(t, x, Du)) = 0, \\ m(t_{0}) = m_{0}, \\ m_{|\partial\Omega} = 0. \end{cases}$$
(29)

The existence of a unique solution for (29) is guaranteed by Proposition 5. Moreover, we know from (19) that

$$\sup_{t \in [t_0, T]} ||m(t)||_{-(1+\alpha), D} + ||m||_{L^p} \le C ||m_0||_{-(1+\alpha)},$$

for a certain p > 1. Therefore, to check that $m \in X$, we are left with proving that, for some C > 0, it holds

$$||m(t) - m(s)||_{-(1+\alpha)D} \le C|t - s|^{\frac{\alpha}{2}}, \quad \text{for all } t \ne s.$$

Subtracting the distributional formulation (18) in t and s, we have

$$\int_{\Omega} \xi(x)m(t,dx) - \int_{\Omega} \phi(s,x)m(s,dx) + \int_{s}^{t} \int_{\Omega} \psi(r,x)m(r,dx)dr = 0,$$
(30)

for each $\xi \in C^{1+\alpha,D}$, $\psi \in L^{\infty}$ and ϕ satisfying

$$\begin{cases} -\phi_t - \operatorname{tr}(a(x)D^2\phi) + H_p(t, x, Du) \cdot D\phi = \psi, \\ \phi(t) = \xi, \\ \phi_{|\partial\Omega} = 0. \end{cases}$$
(31)

We choose $\psi = 0$. Thanks to Lemma 4 we know that $\phi \in C^{1+\frac{\alpha}{2},1+\alpha}$ and its norm is bounded according to (14). Coming back to (30), we obtain

$$\begin{split} \int_{\Omega} \xi(x) (m(t,dx) - m(s,dx)) &= \int_{\Omega} (\phi(t,x) - \phi(s,x)) m(s,dx) \leq C \, \|\phi(t) - \phi(s)\|_{1+\alpha} \\ & \|m(s)\|_{-(1+\alpha),D} \leq C |t-s|^{\frac{\alpha}{2}} \, \|m_0\|_{-(1+\alpha)} \, \|\xi\|_{1+\alpha} \ , \end{split}$$

and taking the sup over the $\xi \in C^{1+\alpha,D}$ with $\|\xi\|_{1+\alpha} \le 1$,

$$||m(t) - m(s)||_{-(1+\alpha),D} \le C|t - s|^{\frac{\alpha}{2}} ||m_0||_{-(1+\alpha)}$$
.

Choosing $M = C ||m_0||_{-(1+\alpha)}$, we have proved that $m \in X$.

Since *X* is convex and closed and $\Phi(X) \subseteq X$, to apply Schauder's theorem we need to show that:

- $\Phi(X)$ is relatively compact;
- Φ is continuous.

In this way, the closure of the convex envelope of $\Phi(X)$, say $\hat{X} := \overline{\operatorname{inv}(\Phi(X))}$, is compact and convex, and for the closure and the convexity of X we have $\Phi(X) \subseteq \hat{X} \subseteq X$. So we can consider the restriction $\Phi_{|}: \hat{X} \to \hat{X}$, which satisfies the classical hypotheses of the Schauder's Theorem. Hence, the existence of a fixed point remains guaranteed.

We start proving the relatively compactness of $\Phi(X)$. Let $\{\gamma_n\}_n \subset X$ and let u_n and m_n be the related solutions. Applying Ascoli-Arzelà's Theorem we have $u_{n_k} \to u$ in $C^{1,2}$, for a certain subsequence $\{u_{n_k}\}_k$ and $u \in X$.

To prove the convergence of $\{m_{n_k}\}_n$, we take ϕ_{n_k} as the solution of (31) with Du replaced by Du_{n_k} and $\psi = 0$. The difference $\phi_{k,h} := \phi_{n_k} - \phi_{n_h}$ satisfies

$$\begin{cases} -(\phi_{k,h})_t - \operatorname{tr}(a(x)D^2\phi_{k,h}) + H_p(t, x, Du_{n_k}) \cdot D\phi_{k,h} \\ &= (H_p(t, x, Du_{n_h}) - H_p(t, x, Du_{n_k})) \cdot D\phi_{n_h}, \\ \phi_{k,h}(t) = 0, \\ \phi_{k,h|\partial\Omega} = 0, \end{cases}$$

then Lemma 4 implies

$$\|\phi_{k,h}\|_{\frac{1+\alpha}{2},1+\alpha} \le C \|(H_p(t,x,Du_{n_h}) - H_p(t,x,Du_{n_k})) \cdot D\phi_{n_h}\|_{\infty}$$

$$\le C \|Du_{n_h} - Du_{n_k}\|_{\infty} \le \omega(k,h),$$

where $\omega(k, h) \to 0$ when $k, h \to \infty$.

Using (30) with (m_{n_k}, ϕ_{n_k}) and (m_{n_h}, ϕ_{n_h}) , for $k, h \in \mathbb{N}$, $s = t_0$, subtracting the two equalities, we get

$$\sup_{t\in[t_0,T]} \|m_{n_k}(t) - m_{n_h}(t)\|_{-(1+\alpha),D} \le \omega(k,h) \implies \exists m \in X \text{ s.t. } m_{n_k} \to m \text{ in } X,$$

hence concluding the compactness part. The continuity is an easy consequence of the previous arguments.

In particular, we apply Schauder's theorem and obtain a classical solution of the problem (25). The estimate (26) follows from the above estimates for (28) and (29).

We skip the uniqueness part, which is a standard argument, see, e.g., Proposition 3.3 of [66].

Step 2. Let (u_1, m_1) and (u_2, m_2) be two classical solutions of (25), with $m_1(t_0) = m_{01}$, $m_2(t_0) = m_{02}$. We take ϕ as the solution of (31) related to u_1 , with $\psi = 0$, and we note that ϕ is also a good test function for the equation of m_2 , since it satisfies, for $\psi = (H_p(t, x, Du_2) - H_p(t, x, Du_1)) \cdot D\phi \in L^{\infty}$,

$$\begin{cases} -\phi_t - \operatorname{tr}(a(x)D^2\phi) + H_p(t, x, Du_2) \cdot D\phi = \psi \,, \\ \phi(t) = \xi \,, \\ \phi_{|\partial\Omega} = 0 \,, \end{cases}$$

Then, subtracting the weak formulations of m_1 and m_2 related to ψ , we find

$$\int_{\Omega} \xi(x) (m_1(t) - m_2(t)) \, dx = \tag{32}$$

$$\int_{t_0}^{t} \int_{\Omega} (H_p(t, x, Du_1) - H_p(t, x, Du_2)) D\phi \, m_2(s, x) dx \, ds + \langle \phi(0, \cdot), m_{01} - m_{02} \rangle \,. \tag{33}$$

By Lipschitz continuity of both ϕ and H_p , with respect to p, we get

$$\int_{\Omega} \xi(x) (m_1(t) - m_2(t)) dx \le C \int_{t_0}^t \int_{\Omega} |Du_1 - Du_2| m_2(s, x) dx ds + C \mathbf{d}_1(m_{01}, m_{02}),$$

We want to estimate the first term in the right-hand side with Young's inequality. To this end, we consider the quantities

$$\tilde{c} = \int_{t_0}^t \int_{\Omega} m_2(s, x) \, dx ds, \qquad \tilde{m}_2(s, x) = \frac{m_2(s, x)}{\tilde{c}}.$$

Then \tilde{m}_2 is a probability measure in $[0, t] \times \Omega$, and Young's inequality implies that

$$\int_{t_0}^{t} \int_{\Omega} |Du_1 - Du_2| m_2(s, x) dx ds = \tilde{c} \int_{t_0}^{t} \int_{\Omega} |Du_1 - Du_2| \tilde{m}_2(s, x) dx ds$$

$$\leq \tilde{c} \left(\int_{t_0}^{t} \int_{\Omega} |Du_1 - Du_2|^2 \tilde{m}_2(s, x) dx \right)^{\frac{1}{2}} \leq C \left(\int_{t_0}^{t} \int_{\Omega} |Du_1 - Du_2|^2 m_2(s, x) dx \right)^{\frac{1}{2}},$$

since \tilde{c} is bounded thanks to the L^p bound of m_2 .

Using the Lasry-Lions monotonicity argument (see Lemma 3.1.2 of [18]), we have

$$\int_{t_0}^{T} \int_{\Omega} |Du_1 - Du_2|^2 (m_1(t, dx) + m_2(t, dx)) dt \le$$

$$\le C \int_{\Omega} (u_1(0, x) - u_2(0, x)) (m_{01}(dx) - m_{02}(dx))$$

$$\le C \|u_1 - u_2\|_{\frac{1+\alpha}{2}, 1+\alpha} \, \mathbf{d}_1(m_{01}, m_{02}) .$$

Hence, we obtain

$$\int_{\Omega} \xi(x)(m_1(t) - m_2(t)) dx$$

$$\leq C \left(||u_1 - u_2||_{\frac{1+\alpha}{2}, 1+\alpha}^{\frac{1}{2}} d_1(m_{01}, m_{02})^{\frac{1}{2}} + d_1(m_{01}, m_{02}) \right).$$

and finally, taking the sup over the ξ with $\|\xi\|_{-(1+\alpha),D} \le 1$ and over $t \in [0,T]$,

$$\sup_{t \in [0,T]} ||m_1(t) - m_2(t)||_{-(1+\alpha),D} \tag{34}$$

$$\leq C\left(\|u_1 - u_2\|_{\frac{1+\alpha}{2}, 1+\alpha}^{\frac{1}{2}} \mathbf{d}_1(m_{01}, m_{02})^{\frac{1}{2}} + \mathbf{d}_1(m_{01}, m_{02})\right). \tag{35}$$

Let us now call $u := u_1 - u_2$, then u solves:

$$\begin{cases} -u_t - \operatorname{tr}(a(x)D^2u) + V(t, x)Du = f(t, x), \\ u(T) = g(x), \qquad u_{|\partial\Omega} = 0, \end{cases}$$

where

$$V(t,x) = \int_0^1 H_p(t,x,\lambda Du_1(t,x) + (1-\lambda)Du_2(t,x)d\lambda;$$

$$f(t,x) = \int_0^1 \int_{\Omega} \frac{\delta F}{\delta m}(t,x,m_{\lambda}(t),y)(m_1(t,dy) - m_2(t,dy))d\lambda;$$

$$g(x) = \int_0^1 \int_{\Omega} \frac{\delta G}{\delta m}(x,m_{\lambda}(T),y)(m_1(T,dy) - m_2(T,dy))d\lambda,$$

where m_{λ} is defined as follows

$$m_{\lambda}(\cdot) := \lambda m_1(\cdot) + (1 - \lambda) m_2(\cdot)$$
.

As to apply Lemma 4, we estimate the regularity of the data:

$$\begin{split} \sup_{t \in [0,T]} & \| f(t,\cdot) \|_{\alpha} \\ & \leq \sup_{t \in [0,T]} \int_{0}^{1} \left\| \frac{\delta F}{\delta m}(t,\cdot,m_{\lambda}(t),\cdot) \right\|_{\alpha,(1+\alpha,D)} d\lambda \ \| m_{1}(t) - m_{2}(t) \|_{-(1+\alpha),D} \\ & \leq C \sup_{t \in [0,T]} \| m_{1}(t) - m_{2}(t) \|_{-(1+\alpha),D} \ , \end{split}$$

analogously

$$||g(\cdot)||_{2+\alpha} \le C \sup_{t \in [0,T]} ||m_1(t) - m_2(t)||_{-(1+\alpha),D}$$
 (36)

So, Eq. (15) implies

$$||u_1 - u_2||_{1,2+\alpha} \le C \sup_{t \in [0,T]} ||m_1(t) - m_2(t)||_{-(1+\alpha),D} .$$
(37)

Coming back to (34), we have

$$\sup_{t \in [0,T]} ||m_1(t) - m_2(t)||_{-(1+\alpha),D}
\leq C \left(\sup_{t \in [0,T]} ||m_1(t) - m_2(t)||_{-(1+\alpha),D} \right)^{\frac{1}{2}} \mathbf{d}_1(m_{01}, m_{02})^{\frac{1}{2}} + \mathbf{d}_1(m_{01}, m_{02}) \right),$$

hence, by a generalized Young's inequality:

$$\sup_{t \in [0,T]} \|m_1(t) - m_2(t)\|_{-(1+\alpha),D} \le C \mathbf{d}_1(m_{01}, m_{02}). \tag{38}$$

Plugging this estimate in (37), we finally obtain

$$||u_1 - u_2||_{1,2+\alpha} \le C\mathbf{d}_1(m_{01}, m_{02}).$$

For the L^p inequality, we consider $m := m_1 - m_2$. Then m solves the equation

$$\begin{cases} m_t - \sum_{i,j} \partial_{ij}^2(a_{ij}(x)m) - \operatorname{div}(m(H_p(t, x, Du_1))) = \\ \operatorname{div}(m_2(H_p(t, x, Du_2) - H_p(t, x, Du_1))), \\ m(t_0) = m_{01} - m_{02}, \\ [m_1 - m_2]_{|\partial\Omega} = 0, \end{cases}$$

i.e. m is a solution of (16) with $f = \text{div}(m_2(H_p(t, x, Du_2) - H_p(t, x, Du_1))), \mu_0 = m_{01} - m_{02}, b = H_p(t, x, Du_1)$. Then estimates (19) imply

$$||m_1 - m_2||_{L^p(Q_T)} \le C(||\mu_0||_{-(1+\alpha)} + ||f||_{L^1}).$$

We estimate the right-hand side term. As regards μ_0 we have

$$||\mu_0||_{-(1+\alpha)} = \sup_{\|\phi\|_{1+\alpha} \le 1} \int_{\Omega} \phi(x) (m_{01} - m_{02}) (dx) \le C \mathbf{d}_1(m_{01}, m_{02}).$$

For the f term we argue in the following way:

$$\begin{split} \|f\|_{L^{1}} &= \int_{0}^{T} \sup_{\|\phi\|_{W^{1,\infty}} \le 1} \left(\int_{\Omega} H_{p}(x, Du_{2}) - H_{p}(x, Du_{1}) D\phi \, m_{2}(t, dx) \right) dt \\ &\leq C \, \|u_{1} - u_{2}\|_{\frac{1+\alpha}{2}, 1+\alpha} \le C \mathbf{d}_{1}(m_{01}, m_{02}) \,, \end{split}$$

which allows us to conclude.

Previous proposition (7) allows to state that

$$\sup_{t \in [0,T]} \sup_{m \in \mathcal{P}^{sub}(\Omega)} ||U(t,\cdot,m)||_{2+\alpha} \le C,$$

$$\sup_{t \in [0,T]} \sup_{m_1 \ne m_2} \left[(\mathbf{d}_1(m_1,m_2))^{-1} ||U(t,\cdot,m_1) - U(t,\cdot,m_2)||_{2+\alpha} \right] \le C,$$
(39)

which are two initial regularity results for the function U.

5. Linearized system and differentiability of U with respect to the measure

This section is devoted to the study of the following *linearized MFG system*:

$$\begin{cases}
-z_{t} - \operatorname{tr}(a(x)D^{2}z) + H_{p}(t, x, Du)Dz = \frac{\delta F}{\delta m}(t, x, m(t))(\rho(t)) + h(t, x), \\
\rho_{t} - \sum_{ij} \partial_{ij}^{2}(a_{ij}(x)\rho) - \operatorname{div}(\rho(H_{p}(t, x, Du))) + \\
- \operatorname{div}(mH_{pp}(t, x, Du)Dz + c) = 0, \\
z(T, x) = \frac{\delta G}{\delta m}(x, m(T))(\rho(T)) + z_{T}(x), \qquad \rho(t_{0}) = \rho_{0}, \\
z_{|\partial\Omega} = 0, \qquad \rho_{|\partial\Omega} = 0,
\end{cases}$$
(40)

where $z_T \in C^{2+\alpha,D}$, $\rho_0 \in C^{-(1+\alpha)}$, $h \in C^{0,\alpha}([t_0,T] \times \Omega)$, $c \in L^1([t_0,T] \times \Omega)$, and where we define for F (and for G)

$$\frac{\delta F}{\delta m}(t,x,m(t))(\rho(t)) := \left\langle \frac{\delta F}{\delta m}(t,x,m(t),\cdot),\rho(t) \right\rangle,$$

where the duality is between $C^{-(1+\alpha),D}$ and $C^{(1+\alpha),D}$.

The study of this system plays a crucial role in proving the C^1 character of U in terms of m. In particular, if we define the couple (v, μ) as the solution of (40) with $h = c = z_T = 0$ and $\mu(t_0) = \mu_0$, we obtain

$$v(t_0, x) = \left\langle \frac{\delta U}{\delta m}(t_0, x, m_0, \cdot), \mu_0 \right\rangle. \tag{41}$$

Let us start giving a suitable definition of solution for the previous system.

Definition 5. A couple (z, ρ) is a solution of (40) if $z \in C^{1,2+\alpha}$, is a classical solution of the linear HJB equation and $\rho \in C([0,T];C^{-(1+\alpha),D}(\Omega)) \cap L^1(Q_T)$ solves the Fokker-Planck equation, accordingly with Definition 4.

Let us underline that $c \in L^1([t_0, T] \times \Omega) \implies \operatorname{div}(c) \in L^1([t_0, T]; W^{-1,\infty}(\Omega))$. Therefore the well-posedness of the Fokker-Planck equation is included in Proposition 5.

The following Proposition states existence, uniqueness and regularity for the problem defined by eq. (40). The proof relies on results previously obtained for the Fokker-Planck equation, and it proceeds as the Neumann case, see Propositions 5.8 and 5.11 of [66]), therefore we omit it.

Proposition 8. Let hypotheses 1 hold for a certain $0 < \alpha < 1$, and let $\rho_0 \in C^{-1}$. Then there exists a unique solution $(z,\rho) \in C^{1,2+\alpha} \times \left(C([0,T];C^{-(1+\alpha),D}(\Omega)) \cap L^1(Q_T)\right)$ for the system (40), satisfying

$$||z||_{1,2+\alpha} + \sup_{t} ||\rho(t)||_{-(1+\alpha),D} + ||\rho||_{L^p} \le CM,$$
(42)

where C depends on H, p is defined as in section 4.1, and M is given by

$$M := \|z_T\|_{2+\alpha} + \|\rho_0\|_{-(1+\alpha)} + \|h\|_{0,\alpha} + \|c\|_{L^1} . \tag{43}$$

Moreover, the solution (v, μ) related to $h = c = z_T = 0$ satisfies

$$\|v\|_{1,2+\alpha} + \sup_{t \in [0,T]} \|\mu(t)\|_{-(2+\alpha),D} + \|\mu\|_{-\left(\frac{\alpha}{2},\alpha\right)} \le C \|\mu_0\|_{-(2+\alpha)} . \tag{44}$$

Throughout the rest of the paper, we will denote with (v, μ) a solution to the system (40), with $h = c = z_T = 0$ and $\mu_0 := \rho_0$. We will refer to this system as the *pure linearized system*. Instead, the general system (40), with solution (z, ρ) , will be called the *general linearized system*.

To prove $U \in C^1$ as well as the related representation formula (41), we have to prove that the pure linearized system has a fundamental solution, which is the content of the next proposition.

Proposition 9. Let hypotheses 1 hold. Then there exists a function $K:[0,T]\times\Omega\times\mathcal{P}^{sub}(\Omega)\times\Omega\to\mathbb{R}$ such that, for any solution (v, μ) , with initial data (t_0, m_0, μ_0) , we have

$$v(t_0, x) =_{-(1+\alpha)} \langle \mu_0, K(t_0, x, m_0, \cdot) \rangle_{1+\alpha}$$
(45)

Moreover, K is twice differentiable with respect to both x and y, and it holds

$$\sup_{(t,m)\in[0,T]\times\mathcal{P}^{sub}(\Omega)} ||K(t,\cdot,m,\cdot)||_{2+\alpha,2+\alpha} \le C,$$
(46)

Proof. Let $\mu_0 = \delta_y$ be the Dirac function at $y \in \Omega$. We define $K(t_0, x, m_0, y) = v(t_0, x; \delta_y)$, where $v(\cdot, \cdot; \mu_0)$ indicates the function v related to the pure linearized system with initial data μ_0 . Exploiting (42), we have that K is twice differentiable w.r.t. x, and it holds

$$||K(t_0, \cdot, m_0, y)||_{2+\alpha} \le C ||\delta_y||_{-(1+\alpha)} = C$$

Furthermore, the linearity character of (40) implies

$$\frac{K(t_0, x, m_0, y + he_i) - K(t_0, x, m_0, y)}{h} = v(t_0, x; \Delta_h^i \delta_y),$$

where we use the notation $\Delta_h^i \delta_y = \frac{1}{h} (\delta_{y+he_i} - \delta_y)$. The linear character of the pure linearized system directly implies that the solution (v, μ) is stable with respect to the initial condition μ_0 , allowing to pass to the limit and obtain

$$\frac{\partial K}{\partial v_i}(t_0, x, m_0, y) = v(t_0, x; -\partial_{y_i} \delta_y),$$

with the derivative that has to be intended in a distributional sense. As to prove the existence and second derivatives' bounds, we consider the incremental ratio

$$R_{i,j}^{h}(x,y) := \frac{\partial_{y_i} K(t_0, x, m_0, y + he_j) - \partial_{y_i} K(t_0, x, m_0, y)}{h}.$$
 (47)

Hence, estimate (44) together with Lagrange Theorem imply that, for $|l| \le 2$, we have

$$\begin{split} \left| D_x^l R_{i,j}^h(x,y) - D_x^l R_{i,j}^k(x,y) \right| &= \left| D_x^l v \left(t_0, x; \Delta_h^j(-\partial_{y_i} \delta_y) - \Delta_k^j(-\partial_{y_i} \delta_y) \right) \right| \\ &\leq C \left\| \Delta_h^j(-\partial_{y_i} \delta_y) - \Delta_k^j(-\partial_{y_i} \delta_y) \right\|_{-(2+\alpha)} \\ &= \sup_{\|\phi\|_{2+\alpha} \leq 1} \left(\frac{\partial_{y_i} \phi(y + he_j) - \partial_{y_i} \phi(y)}{h} - \frac{\partial_{y_i} \phi(y + ke_j) - \partial_{y_i} \phi(y)}{k} \right) \\ &= \sup_{\|\phi\|_{2+\alpha} \leq 1} \left(\partial_{y_i y_j}^2 \phi(y_{\phi,h}) - \partial_{y_i y_j}^2 \phi(y_{\phi,k}) \right) \leq \sup_{\|\phi\|_{2+\alpha} \leq 1} \left| y_{\phi,h} - y_{\phi,k} \right|^{\alpha} \leq |h|^{\alpha} + |k|^{\alpha}, \end{split}$$

for a certain $y_{\phi,h}$, resp. $y_{\phi,k}$ in the line segment between y and $y + he_j$ (resp. $y + ke_j$), and where we have used the same notation as seen above for $\Delta_h^j(-\partial_{y_i}\delta_y)$.

The latter proves that (47), together with its first and second derivative w.r.t x, are Cauchy sequences in h, implying that $D_x^l \frac{\delta U}{\delta m}$ is twice differentiable w.r.t. y and for all $|l| \le 2$.

As to conclude estimate (46), we prove the estimate for the second order derivatives w.r.t. y, the first derivative being simpler. By estimate (44) on v, if $y, y' \in \Omega$, then

$$\begin{split} \left\| R_{i,j}^h(\cdot,y) - R_{i,j}^h(\cdot,y') \right\|_{2+\alpha} &= \left\| \nu(t_0,\cdot;\Delta_h^j(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'})) \right\|_{2+\alpha} \\ &\leq C \left\| \Delta_h^j(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'}) \right\|_{-(2+\alpha)}. \end{split}$$

Passing to the limit for $h \to 0$, we have

$$\Delta_h^j(-\partial_{y_i}\delta_y + \partial_{y_i}\delta_{y'}) \longrightarrow \partial_{y_j}\partial_{y_i}\delta_y - \partial_{y_j}\partial_{y_i}\delta_{y'}, \text{ in } C^{-(2+\alpha)}.$$

Then, by Ascoli-Arzelà and previously obtained convergence result for R_{i}^{h} , we have

$$\begin{split} \left\| \partial_{y_i y_j}^2 \left(K(t_0, \cdot, m_0, y) - K(t, \cdot, m_0, y') \right) \right\|_{2+\alpha} \\ & \leq C \left\| \partial_{y_i y_j}^2 (\delta_y - \delta_{y'}) \right\|_{-(2+\alpha)} \leq C |y - y'|^{\alpha} \,, \end{split}$$

which proves (46). The representation formula (45) is an immediate consequence of the linear character of the equation and of the density of the Dirac functions generated set, hence concluding the proof. \Box

We are now ready to consider the main topic of this section: we want to prove that the function K is actually the derivative of U with respect to the measure. Let us underline that the differentiability with respect to the measure m will be the key for proving U is indeed a classical solution of the Master Equation (7).

Theorem 10. Suppose hypotheses 1 hold. Then the function U defined by (9) is differentiable with respect to m, with the derivative given by

$$\frac{\delta U}{\delta m}(t, x, m, y) = K(t, x, m, y).$$

Proof. Within the present proof, given two functions a_1, a_2 , we define $a_{1+\tau} := \tau a_2 + (1-\tau)a_1$, for $\tau \in [0,1]$.

We will prove a more general fact, the representation formula for U being then a direct consequence of it.

In particular, if (u_1, m_1) and (u_2, m_2) are two solutions of (25), with initial conditions (t_0, m_{01}) and (t_0, m_{02}) , and (v, μ) is the solution of the pure linearized system related to (u_2, m_2) , with initial condition $(t_0, m_{01} - m_{02})$, then a sort of first-order Taylor expansion of U with respect to m holds, namely:

$$||u_1 - u_2 - v||_{1,2+\alpha} + \sup_{t \in [0,T]} ||m_1(t) - m_2(t) - \mu(t)||_{-(1+\alpha),D} \le C\mathbf{d}_1(m_{01}, m_{02})^2.$$
(48)

As to prove above *expansion*, let us start defining $(z, \rho) = (u_1 - u_2 - v, m_1 - m_2 - \mu)$. Then (z, ρ) satisfies the general linearized system (40) related to (u_2, m_2) , with data $h = h_1 + h_2$, $c = c_1 + c_2$ and z_T given by:

$$\begin{split} h_1 &= -\int_0^1 (H_p(t,x,Du_{1+s}) - H_p(t,x,Du_2)) \cdot D(u_1 - u_2) \, ds \,, \\ h_2 &= \int_0^1 \int_\Omega \left(\frac{\delta F}{\delta m}(t,x,m_{1+s}(t),y) - \frac{\delta F}{\delta m}(t,x,m_2(t),y) \right) (m_1(t) - m_2(t)) (dy) ds \,, \\ c_1(t) &= (m_1(t) - m_2(t)) H_{pp}(t,x,Du_2) (Du_1 - Du_2) \,, \\ c_2(t) &= m_1 \int_0^1 \left(H_{pp}(t,x,Du_{1+s}) - H_{pp}(t,x,Du_2) \right) (Du_1 - Du_2) \, ds \,, \\ z_T &= \int_0^1 \int_\Omega \left(\frac{\delta G}{\delta m}(x,m_{1+s}(T),y) - \frac{\delta G}{\delta m}(x,m_2(T),y) \right) (m_1(T) - m_2(T)) (dy) ds \,. \end{split}$$

Applying (42) one has

$$||z||_{1,2+\alpha} + \sup_{t \in [0,T]} ||\rho(t)||_{-(1+\alpha),D} \le C \left(||h||_{0,\alpha} + ||c||_{L^1} + ||z_T||_{2+\alpha} \right). \tag{49}$$

We want to estimate the right-hand side term to obtain the desired Taylor expansion. As regards h, exploiting eq. (27) and Hölder norm properties, we have:

$$||h_1||_{0,\alpha} = \left\| \int_0^1 \int_0^1 s \langle H_{pp}(t, x, Du_{1+rs}) (Du_1 - Du_2), (Du_1 - Du_2) \rangle dr ds \right\|_{0,\alpha}$$

$$\leq C \mathbf{d}_1(m_{01}, m_{02})^2,$$

Analogously, again by eq. (27) and exploiting regularity of both F and G, the same estimate also holds for h_2 and z_T . For the function c, we can write

$$||c_1||_{L^1} = \int_0^T \int_{\Omega} H_{pp}(t, x, Du_2)(Du_1 - Du_2)(m_1(t, dx) - m_2(t, dx)) dt$$

$$\leq C ||u_1 - u_2||_{1, 2+\alpha} ||m_1(t) - m_2(t)|| \leq C \mathbf{d}_1(m_{01}, m_{02})^2,$$

then, proceeding as above, we have

$$\begin{aligned} &\|c_2\|_{L^1} = \\ &\int_0^1 \int_0^T \int_{\Omega} \left(H_{pp}(t, x, Du_{1+s}) - H_{pp}(t, x, Du_2) \right) (Du_1 - Du_2) m_1(t, dx) \, dt ds \\ &\leq C \|Du_1 - Du_2\|_{\infty}^2 \leq C \mathbf{d}_1 (m_0^1, m_0^2)^2 \, . \end{aligned}$$

Substituting these estimates in (49), we obtain (48). Using the representation (45) for v, we get

$$\left\| U(t_0, \cdot, m_{01}) - U(t_0, \cdot, m_{02}) - \int_{\Omega} K(t_0, \cdot, m_{02}, y)(m_{01} - m_{02})(dy) \right\|_{\infty} \\ \leq C \mathbf{d}_1(m_{01}, m_{02})^2.$$

As a straightforward consequence, we have that U is differentiable with respect to m and

$$\frac{\delta U}{\delta m}(t, x, m, y) = K(t, x, m, y),$$

hence concluding the proof.

Using (46) we also obtain the following strong regularity estimate for $\frac{\delta U}{\delta m}$:

$$\sup_{t} \left\| \frac{\delta U}{\delta m}(t, \cdot, m, \cdot) \right\|_{2+\alpha, 2+\alpha} \le C. \tag{50}$$

This gives sense to the Master Equation (7), since the quantity $D_y D_m U$ is now well-defined. However, to apply Theorem 3 we still need to prove the continuity of $D_v D_m U$ in the measure variable.

In the next result we prove a Lipschitz bound for $\frac{\delta U}{\delta m}$, with respect to the measure m. This bound plays a key role, since not only it implies the continuity of $D_y D_m U$ with respect to m, but it is also used in the next section to apply Lemma 13.

Theorem 11. Let Assumptions 1 hold. Then the derivative of the solution of the Master Equation $\frac{\delta U}{\delta m}$ is Lipschitz continuous with respect to the measure m:

$$\sup_{t \in [0,T]} \sup_{m_1 \neq m_2} (\mathbf{d}_1(m_1, m_2))^{-1} \left\| \frac{\delta U}{\delta m}(t, \cdot, m_1, \cdot) - \frac{\delta U}{\delta m}(t, \cdot, m_2, \cdot) \right\|_{2+\alpha, 2+\alpha} \le C$$
(51)

where C depends on n, F, G, H and T.

Proof. We consider, for i = 1, 2, the solution (v_i, μ_i) of the linearized system (40) related to (u_i, m_i) . To avoid too heavy notations, we take $t_0 = 0$ and we define

$$\begin{split} H_i'(t,x) &:= H_p(t,x,Du_i(t,x))\,, \qquad \qquad H_i''(t,x) = H_{pp}(t,x,Du_i(t,x))\,, \\ F'(t,x,m,\mu) &= \int_{\Omega} \frac{\delta F}{\delta m}(t,x,m,y)\,\mu(dy)\,, \qquad G'(x,m,\mu) = \int_{\Omega} \frac{\delta G}{\delta m}(x,m,y)\,\mu(dy)\,. \end{split}$$

Then the couple $(z, \rho) := (v_1 - v_2, \mu_1 - \mu_2)$ satisfies the following linear system:

$$\begin{cases} -z_t - \text{tr}(a(x)D^2z) + H_1' \cdot Dz = F'(t, x, m_1(t), \rho(t)) + h, \\ \rho_t - \sum_{i,j} \partial_{ij}^2(a_{ij}(x)\rho) - \text{div}(\rho H_1') - \text{div}(m_1 H_1''Dz + c) = 0, \\ z(T, x) = G'(x, m_1(T), \rho(T)) + z_T, & \rho(t_0) = 0, \\ z_{|\partial\Omega} = 0, & \rho_{|\partial\Omega} = 0, \end{cases}$$

where

$$\begin{split} h(t,x) &= h_1(t,x) + h_2(t,x) \,, \\ h_1(t,x) &= F'(t,x,m_1(t),\mu_2(t)) - F'(t,x,m_2(t),\mu_2(t)) \,, \\ h_2(t,x) &= (H'_1(t,x) - H'_2(t,x)) \cdot Dv_2(t,x) \,, \\ c(t,x) &= \mu_2(t)(H'_1 - H'_2)(t,x) + \left[(m_1H''_1 - m_2H''_2) \right](t,x) \,, \\ z_T(x) &= G'(x,m_1(T),\mu_2(T)) - G'(x,m_2(T),\mu_2(T)) \,. \end{split}$$

Applying (42) we obtain this estimate on z:

$$||z||_{1,2+\alpha} \le C \left(||z_T||_{2+\alpha} + ||h||_{0,\alpha} + ||c||_{L^1} \right).$$

Now we estimate the terms in the right-hand side.

The term with z_T , thanks to (44) and the hypothesis (ν) of 1, is immediately estimated:

$$\|z_T\|_{2+\alpha} \leq \left\| \frac{\delta G}{\delta m}(\cdot, m_1(T), \cdot) - \frac{\delta G}{\delta m}(\cdot, m_2(T), \cdot) \right\|_{2+\alpha, 2+\alpha} \|\mu_2(T)\|_{-(2+\alpha), D} \leq C \mathbf{d}_1(m_{01}, m_{02}) \|\mu_0\|_{-(2+\alpha)} \ .$$

As regards the space estimate for h, we have

$$||h(t,\cdot)||_{\alpha} \leq ||F'(\cdot,m_1(t),\mu_2(t)) - F'(\cdot,m_2(t),\mu_2(t))||_{\alpha} + ||(H'_1 - H'_2)(t,\cdot)Dv_2(t,\cdot)||_{\alpha}$$

The first term is bounded as z_T :

$$||F'(\cdot, m_1(t), \mu_2(t)) - F'(\cdot, m_2(t), \mu_2(t))||_{\alpha} \le C\mathbf{d}_1(m_{01}, m_{02}) ||\mu_0||_{-(2+\alpha)}$$

The second term, using (27) and (44), can be estimated in this way:

$$\left\| (H_1' - H_2')(t, \cdot) D v_2(t, \cdot) \right\|_{\alpha} \leq C \left\| (u_1 - u_2)(t) \right\|_{1 + \alpha} \left\| v_2(t) \right\|_{1 + \alpha} \leq C \mathbf{d}_1(m_{01}, m_{02}) \left\| \mu_0 \right\|_{-(2 + \alpha)}.$$

In summary,

$$||h||_{0,\alpha} = \sup_{t \in [0,T]} ||h(t,\cdot)||_{\alpha} \le C \mathbf{d}_1(m_{01}, m_{02}) ||\mu_0||_{-(2+\alpha)}.$$

Finally, we estimate $||c||_{L^1}$. We have

$$\begin{split} \|c\|_{L^{1}} &= \int_{0}^{T} \int_{\Omega} (H'_{1} - H'_{2})(t, x) \, \mu_{2}(t, dx) \, dt + \int_{0}^{T} \int_{\Omega} H''_{1}(t, x) D v_{2}(t, x) (m_{1}(t) - m_{2}(t))(dx) \, dt \\ &+ \int_{0}^{T} \int_{\Omega} (H'_{1} - H'_{2})(t, x) \, D v_{2}(t, x) \, m_{2}(t, dx) \, dt \leq C \, \|u_{1} - u_{2}\|_{\frac{1+\alpha}{2}, 1+\alpha} \, \|\mu_{2}\|_{-\left(\frac{\alpha}{2}, \alpha\right)} \\ &+ C \, \|u_{1}\|_{\frac{1+\alpha}{2}, 1+\alpha} \, \|v_{2}\|_{\frac{1+\alpha}{2}, 1+\alpha} \, \|m_{1} - m_{2}\|_{L^{1}} + C \, \|u_{1} - u_{2}\|_{\frac{1+\alpha}{2}, 1+\alpha} \, \|v_{2}\|_{\frac{1+\alpha}{2}, 1+\alpha} \; . \end{split}$$

The first term in the right-hand side, thanks to (27) and (44), is bounded by

$$C \|u_1 - u_2\|_{\frac{1+\alpha}{2}, 1+\alpha} \|\mu_2\|_{-(\frac{\alpha}{2}, \alpha)} \le C \mathbf{d}_1(m_{01}, m_{02}) \|\mu_0\|_{-(2+\alpha)}$$
.

The second and the third term are estimated in the same way, using (26), (27) and (44). Then

$$||c||_{L^1} \leq C\mathbf{d}_1(m_{01}, m_{02}) ||\mu_0||_{-(2+\alpha)}$$
.

Putting together all these estimates, we finally obtain:

$$||z||_{1,2+\alpha} \le C\mathbf{d}_1(m_{01},m_{02}) ||\mu_0||_{-(2+\alpha)}$$

Since

$$z(t_0,x) = \int_{\Omega} \left(\frac{\delta U}{\delta m}(t_0,x,m_1,y) - \frac{\delta U}{\delta m}(t_0,x,m_2,y) \right) \mu_0(dy) \,,$$

we have proved (51).

We stress the fact that in the bound (51) a $2 + \alpha$ norm in the last variable is strongly required, to make this estimate valid also for the quantity $D_v D_m U$.

To conclude, we are still left proving the boundary condition for $\frac{\delta U}{\delta m}$ in (7) to make true all the hypotheses needed to apply Theorem 3. The latter will be the last result of this section.

Corollary 12. If hypotheses 1 hold true, then we have the following boundary conditions for U:

$$\begin{split} \frac{\delta U}{\delta m}(t,x,m,y) &= 0\,, \qquad \forall x \in \partial \Omega, y \in \Omega, t \in [0,T], m \in \mathcal{P}^{sub}(\Omega)\,, \\ \frac{\delta U}{\delta m}(t,x,m,y) &= 0\,, \qquad \forall x \in \Omega, y \in \partial \Omega, t \in [0,T], m \in \mathcal{P}^{sub}(\Omega)\,. \end{split}$$

Proof. The proof of the first boundary condition is trivial, since $\frac{\delta U}{\delta m}(t_0, x, m_0, y) = v(t_0, x; \delta_y)$ and v satisfies a Dirichlet boundary condition.

For the second condition, let us consider $y \in \partial \Omega$. To prove that $v(t_0, x; \delta_v) = 0$, it is enough to check that v = 0 solves

$$\begin{cases} -v_t - \operatorname{tr}(a(x)D^2v) + H_p(t, x, Du) \cdot Dv = \frac{\delta F}{\delta m}(t, x, m(t))(\mu(t)), \\ v(T, x) = \frac{\delta G}{\delta m}(x, m(T))(\mu(T)), \\ v_{|\partial\Omega} = 0, \end{cases}$$
(52)

where μ is the unique solution, in the sense of Definition 4, of

$$\begin{cases} \mu_t - \sum_{ij} \partial_{ij}^2 (a_{ij}(x)\mu) - \operatorname{div}(\mu H_p(t, x, Du)) = 0, \\ \mu(t_0) = \mu_0, \\ \mu_{l\partial\Omega} = 0. \end{cases}$$

In this way we have $mH_{pp}(t, x, Du)Dv = 0$, and so the couple (v, μ) turns out to be a solution of

$$\begin{cases} -v_t - \operatorname{tr}(a(x)D^2v) + H_p(t, x, Du)Dv = \frac{\delta F}{\delta m}(t, x, m(t))(\rho(t)), \\ \mu_t - \sum_{ij} \partial_{ij}^2(a_{ij}(x)\mu) - \operatorname{div}(\mu H_p(t, x, Du) + mH_{pp}(t, x, Du)Dv) = 0, \\ v(T, x) = \frac{\delta G}{\delta m}(x, m(T))(\mu(T)), \qquad \mu(t_0) = \mu_0, \\ v_{|\partial\Omega} = 0, \qquad \mu_{|\partial\Omega} = 0, \end{cases}$$

which is exactly the pure linearized system.

Due to the linearity character of (52), we only need to prove that

$$\frac{\delta F}{\delta m}(t, x, m(t))(\mu(t)) = \frac{\delta G}{\delta m}(x, m(T))(\mu(T)) = 0.$$

Thanks to the hypotheses 1, both $\frac{\delta F}{\delta m}(x,m(t),\cdot)$ and $\frac{\delta G}{\delta m}(x,m(T),\cdot)$ satisfy a Dirichlet boundary condition, being also elements of $C^{2+\alpha}$. Then, choosing $\phi(t,y)$ satisfying (17) with $\psi(t,y)=0$ and $\xi(y)=\frac{\delta F}{\delta m}(t,x,m(t),y)$, we have

$$\begin{split} \frac{\delta F}{\delta m}(t,x,m(t))(\mu(t)) &= \left\langle \mu(t), \frac{\delta F}{\delta m}(t,x,m(t),\cdot) \right\rangle \\ &= \left\langle \mu_0, \phi(0,\cdot) \right\rangle = \left\langle \delta_y, \phi(0,\cdot) \right\rangle = \phi(0,y) = 0 \,, \end{split}$$

since ψ satisfies a Dirichlet boundary condition. The same computations hold for $\frac{\delta G}{\delta m}$, therefore we have:

$$\frac{\delta U}{\delta m}(t_0, x, m_0, y) = \left\langle \frac{\delta U}{\delta m}(t_0, x, m_0, \cdot), \delta_y \right\rangle = v(t_0, x) = 0.$$

6. Convergence of the Nash System

The aim of this section is twofold: proving that, for an integer $N \ge 2$, a classical solution $(v^{N,i})_{i \in \{1,...,N\}}$ of the Nash system (5) converges in a suitable sense to the solution of the Master Equation (7), and to prove that also optimal trajectories converge.

6.1. Finite Dimensional Projections of U

The symmetrical structure of the problem suggests considering *suitable* finite dimensional projections of U, along the empirical distributions $m_{\mathbf{x}}^{N,i}$. Therefore, for $i \in \{1, ..., N\}$, we define:

$$u^{N,i}(t,\mathbf{x}) := U(t,x_i,m_{\mathbf{x}}^{N,i}),$$
 (53)

where $\mathbf{x} = (x_1, ..., x_N) \in \Omega^N$ and $m_x^{N,i}$ is defined as in (3). Exploiting the regularity of U, we already know that

$$u_i^N \in C^{1,2+\alpha}$$

with respect to the couple (t, x_i) .

In order to prove a regularity result for $(x_j)_{j\neq i}$, we need two technical results about the derivative of the solution of (7) with respect to m.

Theorem (11) implies the following lemma, which turns out to be crucial to prove the representation formula for the derivatives of u_i^N . Since the related proof is analogous to *Proposition 7.1, 7.4* of [18], we omit it.

Lemma 13. Suppose hypotheses 1 are satisfied, then, if $m \in \mathcal{P}^{sub}(\Omega)$ and $\phi \in L^2(m, \mathbb{R}^d)$ is a bounded vector field, we have

$$\left\| U(t,\cdot,(id+\phi)\sharp m) - U(t,\cdot,m) - \int_{\Omega} D_m U(t,\cdot,m,y) \cdot \phi(y) dm(y) \right\|_{1+\alpha} \le C \|\phi\|_{L^2(m)}^2$$

$$\tag{54}$$

Now we can prove the following representation formula for the derivatives of u_i^N :

Proposition 14. The following equations for the derivative of u_i^N hold for all $j \neq i$:

$$D_{x_j} u_i^N(t, \mathbf{x}) = \frac{1}{N - 1} D_m U(t, x_i, m_{\mathbf{x}}^{N, i}, x_j),$$
(55)

$$D_{x_i,x_j}^2 u_i^N(t,\mathbf{x}) = \frac{1}{N-1} D_x D_m U(t,x_i, m_{\mathbf{x}}^{N,i}, x_j),$$
(56)

$$\left| D_{x_j, x_j}^2 u_i^N(t, \mathbf{x}) - \frac{1}{N - 1} D_y D_m U(t, x_i, m_{\mathbf{x}}^{N, i}, x_j) \right| \le \frac{C}{N^2} \,, \tag{57}$$

where the last inequality holds a.e. $x \in \Omega^N$.

Proof. We can limit ourselves to prove (55) and (57), since the second formula is a direct consequence of the first one. We consider $\mathbf{x} = (x_j)_{j \in \{1, ..., N\}}$, $\mathbf{v} = (v_j)$ s.t. $\mathbf{x}, \mathbf{v} \in \Omega^N$, with $x_j \neq x_k$ for any $j \neq k$, $v_i = 0$, i being fixed. By density of \mathbf{x} with $x_j \neq x_k$ in Ω^N , it is sufficient to prove the first formula just in this case.

For $\varepsilon = \min_{j \neq k} |x_j - x_k|$, we consider a smooth vector field $\varphi : \Omega \to \mathbb{R}^d$ such that $\varphi(x) = v_j$ for $x \in B(x_j, \varepsilon/3)$ and $x + \varphi(x) \in \Omega$ for all $x \in \Omega$. Therefore, since $u_i^N(t, \mathbf{x} + \mathbf{v}) = U(t, x_i, (id + \varphi) \sharp m_{\mathbf{x}}^{N,i})$, Lemma 54 implies:

$$u_i^N(t, \mathbf{x} + \mathbf{v}) = u_i^N(t, \mathbf{x}) + \int_{\Omega} D_m U(t, x_i, m_{\mathbf{x}}^{N,i}, y) \cdot \varphi(t) dm_{\mathbf{x}}^{N,i}(y) + o\left(||v||_{L^2(m_{\mathbf{x}}^{N,i}))}\right),$$

exploiting the definition of $m_x^{N,i}$, we compute the integral obtaining

$$u_i^N(t, \mathbf{x} + \mathbf{v}) = u_i^N(t, \mathbf{x}) + \frac{1}{N - 1} \sum_{i \neq i} D_m U(t, x_i, m_x^{N, i}, x_j) \cdot v_j + o(|v|),$$

concluding the proof of (55). Concerning the last inequality, we first show that $D_{x_i,x_i}^2 u_i^N$ exists almost everywhere. Indeed

$$\begin{split} |D_{x_{j}}u_{i}^{N}(t,\mathbf{x})-D_{x_{j}}u_{i}^{N}(t,\mathbf{y})| &\leq \frac{C}{N}|D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j})-D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},y_{j})| \\ &+\frac{C}{N}\left(|D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},y_{j})-D_{m}U(t,x_{i},m_{\mathbf{y}}^{N,i},y_{j})|\right) &\leq \frac{C}{N}|\mathbf{x}-\mathbf{y}|\,, \end{split}$$

where the last inequality comes from (50) and (51). This implies that $D^2_{x_j,x_j}u^N_i$ exists a.e., and that it is also globally bounded by $\frac{C}{N}$. Considering $\mathbf{e}^{\mathbf{k}}_{\mathbf{j}} = (e^1_{jk},...,e^N_{jk})$ with $e^l_{jk} = \delta_{lj}e_k$ as a basis over $(\mathbb{R}^d)^N$, we can write

$$\begin{split} & \left| \frac{D_{x_{j}}u^{N,i}(t,\mathbf{x} + h\mathbf{e_{j}^{k}}) - D_{x_{j}}u^{N,i}(t,\mathbf{x})}{h} - \frac{1}{N-1}\partial_{y_{k}}D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j}) \right| \\ & \leq \frac{C}{N} \left| \frac{D_{m}U(t,x_{i},m_{\mathbf{x}+h\mathbf{e_{j}^{k}}}^{N,i},x_{j} + he_{k}) - D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j} + he_{k})}{h} \right| + \frac{C}{N} \\ & \left| \frac{D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j} + he_{k}) - D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j})}{h} - \partial_{y_{k}}D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j}) \right| \\ & \leq \frac{C}{N} \left| \partial_{y_{k}}D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{h}) - \partial_{y_{k}}D_{m}U(t,x_{i},m_{\mathbf{x}}^{N,i},x_{j}) \right| \\ & + \frac{C}{Nh} \mathbf{d}_{1} \left(m_{\mathbf{x}+h\mathbf{e_{i}^{k}}}^{N,i},m_{\mathbf{x}}^{N,i} \right) \leq \frac{C}{N^{2}} \end{split}$$

for h sufficiently small, x_h being in the line segment between x_j and $x_j + he_k$, thanks to (51). Passing to the limit, as $h \to 0$, we conclude.

We can prove now that each finite projection $(u_i^N)_{1 \le i \le N}$ of U is an approximated solution of the Nash system (5).

Theorem 15. Assume Assumptions 1 hold, we have that $u_i^N \in C^1([0,T] \times \Omega^N)$, $u_i^N(t,\cdot) \in W^{2,\infty}(\Omega^N)$ and u_i^N solves almost everywhere the following equation

$$\begin{cases}
-\partial_{t}u_{i}^{N} - \sum_{j} \operatorname{tr}(a(x_{j})D_{x_{j}x_{j}}^{2}u_{i}^{N}) + H(t, x_{i}, D_{x_{i}}u_{i}^{N}) \\
+ \sum_{j \neq i} H_{p}(t, x_{j}, D_{x_{j}}u_{i}^{N}) \cdot D_{x_{j}}u_{i}^{N} = F(t, x_{i}, m_{x}^{N,i}) + r_{i}^{N}(t, x), \\
u_{i}^{N}(T, x) = G(x_{i}, m_{x}^{N,i}), \\
u_{i}^{N}(t, x)|_{x_{i} \in \partial \Omega} = 0, \qquad i = 1, ..., N.
\end{cases} (58)$$

where $||r_i^N||_{\infty} \leq \frac{C}{N}$.

Equation (58) models an approximated version of the Nash system (5). The homogeneous boundary condition for U implies that $u_i^N(t, \mathbf{x})\big|_{x_i \in \partial \Omega} = 0$ for each i = 1, ..., N. The rest of the proof follows analogously to the one of Prop. *Proposition 6.3* of [18], therefore we omit it.

6.2. Convergence

The main convergence result is realized by comparing (u_i^N) , the projections of the solution of the Master Equation defined in Equation (53), with (v_i^N) , namely with the solution of the Nash system defined in (5).

Since both of these solutions are symmetrical, there exist two functions V^N and $U^N: \Omega \times \Omega^{N-1} \to \mathbb{R}$, such that, for all $x \in \Omega$, the functions $(y_1, \dots, y_{N-1}) \to V^N(x, (y_1, \dots, y_{N-1}))$ are invariant under permutations. Therefore, we can write

$$v_i^N(t, \mathbf{x}) = V^N(t, x_i, (x_1, \dots x_{i-1}, x_{i+1}, \dots, x_N)),$$

$$u_i^N(t, \mathbf{x}) = U^N(t, x_i, (x_1, \dots x_{i-1}, x_{i+1}, \dots, x_N)).$$

We fix the initial condition and the corresponding initial spatial distribution $(t_0, m_0) \in [0, T) \times \mathcal{P}^{sub}(\Omega)$, also defining $\mathbf{M} = (M^i)_{i \in \{1,...,N\}}$ as a family of independent and identically distributed (i.i.d) random variables of law $m_0 \in P^{sub}$. Let also $((B_t^i)_{t \in [0,T]})_{i \in [1,...,N]}$ be a family of N independent d-dimensional Brownian motion associated with each player i, being also independent of $(Z_i)_{i \in \{1,...,N\}}$.

We consider the processes $(\mathbf{Y}_t)_t$ and their associated hitting times $\boldsymbol{\tau}_Y = \left[\tau_Y^1, \dots, \tau_Y^N\right]$, solution of the following coupled system of SDEs

$$\begin{cases}
dY_{t\wedge\tau_{Y}^{i}}^{i} = -H_{p}(t, Y_{t\wedge\tau_{Y}^{i}}^{i}, D_{x_{i}}v_{i}^{N}(t, \mathbf{Y}_{t\wedge\tau_{Y}}))dt + \sqrt{2}\sigma(Y_{t\wedge\tau_{Y}^{i}}^{i})dB_{t}^{i}, \\
Y_{t_{0}}^{i} = M^{i},
\end{cases}$$
(59)

with $\mathbf{Y}_{t \wedge \tau_Y} := \mathbf{Y}_{t \wedge \tau_Y} = \left[Y_{t \wedge \tau_Y^i}^1, \dots Y_{t \wedge \tau_Y^i}^i \dots, Y_{t \wedge \tau_Y^N}^N \right]$ while the process $dY_{t \wedge \tau_Y^i}^i$ evolves according to Eq. (59), but only before the corresponding hitting time for $t \in [t_0, \tau_Y^i]$. Accordingly, the correspondent processes $(X_t)_t$ and $\tau_X = [\tau_X^1, ... \tau_X^N]$, being related to the projections u_i^N solve the following system

$$\begin{cases} dX_{t \wedge \tau_X^i}^i = -H_p(t, X_{t \wedge \tau_X^i}^i, D_{X_i} u_i^N(t, \mathbf{X}_{t \wedge \tau_X})) dt + \sqrt{2}\sigma(X_{t \wedge \tau_X^i}^i) dB_t^i, \\ X_{t_0}^i = M^i. \end{cases}$$

$$(60)$$

with
$$\mathbf{X}_{t \wedge \tau_X} := \mathbf{X}_{t \wedge \tau_X} = \left[X_{t \wedge \tau_X^i}^1, \dots X_{t \wedge \tau_X^i}^i \dots, X_{t \wedge \tau_X^N}^N \right]$$
.

Moreover, we define the minimum between τ_X^i and τ_Y^i as follows $\tau^i := \tau_X^i \wedge \tau_Y^i$.

Remark 2. Let us underline that, evaluating projections over processes Y, u_i^N may show some frozen components corresponding to those players that have reached the boundary. For example, if we assume an arbitrate time $t = \tau_Y^i \ge \tau_Y^1$, u_i^N for which a player has already reached $\partial \Omega$, then u_i^N reads

$$u_i^N \left(t, Y_{\tau_Y^1}^1, \dots \{Y_t^i\}_{t \wedge \tau_Y^i} \dots, \{Y_t^N\}_{t \leq \tau_Y^N} \right)$$

with a constant, fixed component corresponding to $Y_{\tau_v^1}$.

The following result holds:

Proposition 16. Assume Hypotheses 1 hold, then, for any $1 \le i \le N$, we have

$$\mathbb{E}\left[\sup_{t\in[t_0,\tau^i]} |Y^i_{t\wedge\tau^i} - X^i_{t\wedge\tau^i}|\right] \le \frac{C}{N},\tag{61}$$

$$\mathbb{E}\left[\int_{t_0}^{t\wedge\tau_Y^i} |D_{x_i}v_i^N(t, \boldsymbol{Y}_{t\wedge\tau_Y^i}) - D_{x_i}u_i^N(t, \boldsymbol{Y}_{t\wedge\tau_Y^i})|^2 dt\right] \le \frac{C}{N^2}.$$
 (62)

Moreover, \mathbb{P} -almost surely and for all i = 1, ..., N, it holds

$$|u_i^N(t_0, M) - v_i(t_0, M)| \le \frac{C}{N},$$
 (63)

where C is a deterministic constant not depending on t_0 , m_0 and N.

Proof. First step. The proof proceeds as in Theorem 6.2.1 in [18], but we decided to include it both for the reader's convenience and since it is not restricted to the case a(x) = Id.

For the sake of simplicity, but without loss of generality, we can assume $t_0 = 0$. In what follows we use the following notations:

$$\begin{split} U_{t}^{N,i} &= u_{i}^{N}\left(t, \ Y_{t \wedge \tau_{Y}^{1}}^{1}, \ \ldots, Y_{t \wedge \tau_{Y}^{N}}^{N}\right), \\ DU_{t}^{N,i,j} &= D_{x_{j}}u_{i}^{N}\left(t, \ Y_{t \wedge \tau_{Y}^{1}}^{1}, \ \ldots, Y_{t \wedge \tau_{Y}^{N}}^{N}\right), \\ V_{t}^{N,i} &= v_{i}^{N}\left(t, \ Y_{t \wedge \tau_{Y}^{1}}^{1}, \ \ldots, Y_{t \wedge \tau_{Y}^{N}}^{N}\right), \\ DV_{t}^{N,i,j} &= D_{x_{j}}v_{i}^{N}\left(t, \ Y_{t \wedge \tau_{Y}^{1}}^{1}, \ \ldots, Y_{t \wedge \tau_{Y}^{N}}^{N}\right). \end{split}$$

Since v_i^N solves the Nash system (5), applying the Itô formula, see [3], to $\left(U_t^{N,i} - V_t^{N,i}\right)^2$, we can deduce the following expansion for $d\left(U_t^{N,i} - V_t^{N,i}\right)^2$:

$$(A_t + B_t)dt + 2\sqrt{2} \left(U_t^{N,i} - V_t^{N,i} \right) \sum_{i=1}^{N} \left[\sigma(Y_{t \wedge \tau_y^i}^i) (DV_t^{N,i,j} - DU_t^{N,i,j}) \right] dB_t^i,$$

where

$$\begin{split} A_t &= 2 \left(U_t^{N,i} - V_t^{N,i} \right) \left(H(t, Y_{t \wedge \tau_Y^i}^i, DU_t^{N,i,i}) - H(t, Y_{t \wedge \tau_Y^i}^i, DV_t^{N,i,i}) \right) \\ &- 2 \left(U_t^{N,i} - V_t^{N,i} \right) \left(DU_t^{N,i,i} (H_p(t, Y_{t \wedge \tau_Y^i}^i, DU_t^{N,i,i}) - H_p(t, Y_t, DV_t^{N,i,i})) \right) \\ &- 2 \left(U_t^{N,i} - V_t^{N,i} \right) \left((DU_t^{N,i,i} - DV_t^{N,i,i}) H_p(t, Y_{t \wedge \tau_Y^i}, DV_t^{N,i,i}) - r_i^N(t, \mathbf{Y}_{t \wedge \tau_Y})) \right). \end{split}$$

and

$$\begin{split} B_t &= 2 \sum_{j=1}^{N} \langle a(Y_{t \wedge \tau_Y^i}^i) (DU_t^{N,i,j} - DV_t^{N,i,j}) \,, \, DU_t^{N,i,j} - DV_t^{N,i,j} \rangle \\ &- 2 (U_t^{N,i} - V_t^{N,i}) \sum_{i=1}^{N} DU_t^{N,i,j} \left(H_p(t, Y_{t \wedge \tau_Y^i}^j, DV_t^{N,j,j}) - H_p(t, Y_{t \wedge \tau_Y^i}^j, DU_t^{N,j,j}) \right). \end{split}$$

Let us underline that $|DU_t^{N,i,j}| \leq M$, $|DV_t^{N,i,j}| \leq M$ for a certain M > 0, and that H and D_pH are both locally Lipschitz continuous. Moreover, $DU_t^{N,i,i} = D_x U(t, Y_{t \wedge \tau_y^i}^i, m_{\mathbf{Y}_{t \wedge \tau_y}}^{N,i})$ is bounded, independently of $i, N, DU_t^{N,i,j} = (1/(N-1))D_m U(t, Y_{t \wedge \tau_y^i}^i, m_{\mathbf{Y}_{t \wedge \tau_y}}^{N,i})$ is bounded from C/N and, by Theorem 15, we also have that $t^{N,i} \leq C/N$

Integrating from t to τ_T^i the above-squared difference and taking the conditional expectation given **Z**, we deduce

$$\begin{split} \mathbb{E}^{\mathbf{Z}} \Big[|U_{t}^{N,i} - V_{t}^{N,i}|^{2} \Big] + 2 \sum_{j=1}^{N} \mathbb{E}^{\mathbf{Z}} \Big[\int_{t}^{T} |DU_{s}^{N,i,j} - DV_{s}^{N,i,j}|^{2} ds \Big] \\ & \leq \frac{C}{N} \int_{t}^{T} \mathbb{E}^{\mathbf{Z}} \Big[|U_{s}^{N,i} - V_{s}^{N,i}| \Big] \\ & + C \int_{t}^{T} \mathbb{E}^{\mathbf{Z}} \Big[|U_{s}^{N,i} - V_{s}^{N,i}| \cdot |DU_{s}^{N,i,i} - DV_{s}^{N,i,i}| \Big] ds \\ & + \frac{C}{N} \sum_{j=1, i \neq i}^{N} \int_{t}^{T} \mathbb{E}^{\mathbf{Z}} \Big[|U_{s}^{N,i} - V_{s}^{N,i}| \cdot |DU_{s}^{N,j,j} - DV_{s}^{N,j,j}| \Big] ds \,. \end{split}$$

Following a standard convexity argument, as, e.g., in [18], and by the generalized Young's inequality we get

$$\mathbb{E}^{\mathbf{M}}[|U_{t}^{N,i} - V_{t}^{N,i}|^{2}] + \mathbb{E}^{\mathbf{M}}\left[\int_{t}^{T} |DU_{s}^{N,i,i} - DV_{s}^{N,i,i}|^{2} ds\right]$$

$$\leq \frac{C}{N^{2}} + C \int_{t}^{T} \mathbb{E}^{\mathbf{Z}}[|U_{s}^{N,i} - V_{s}^{N,i}|]^{2} ds$$

$$+ \frac{1}{2N} \sum_{j=1}^{N} \mathbb{E}^{\mathbf{M}}\left[\int_{t}^{T} |DU_{s}^{N,i,j} - DV_{s}^{N,j,j}|^{2} ds\right].$$
(64)

By taking the mean over $i \in \{1, ..., N\}$, we can get rid of the second term in the left-hand side.

By the Gronwall's Lemma (eventually increasing the constant C), we deduce

$$\sup_{0 \le t < \tau_t^i} \left[\frac{1}{N} \sum_{i=1}^N \mathbb{E}^{\mathbf{M}} [U_t^{N,i} - V_t^{N,i}]^2 \right] \le \frac{C}{N^2} \,. \tag{65}$$

Plugging Eq. (65) into Eq. (64), we deduce that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}^{\mathbf{M}}\left[\int_{t}^{T}|DU_{s}^{N,j,j}-DV_{s}^{N,j,j}|^{2}ds\right]\leq\frac{C}{N^{2}}.$$

Inserting this inequality into Eq. (64) and applying again Gronwall's Lemma, we finally obtain

$$\sup_{t \in [0, \tau_{V}^{i}]} \mathbb{E}^{\mathbf{M}}[|U_{t}^{N, i} - V_{t}^{N, i}|^{2}] + \mathbb{E}^{\mathbf{M}}\left[\int_{0}^{T} |DU_{s}^{N, i, i} - DV_{s}^{N, i, i}|^{2} ds\right] \leq \frac{C}{N^{2}},$$
(66)

proving the bound in Eq. (62).

Second step. We now derive (63).

Evaluating Eq. (66) at t = 0, we obtain \mathbb{P} -almost surely for $i \in \{1, ..., N\}$

$$|U_0^{N,i} - V_0^{N,i}| = |u^{N,i}(0, \mathbf{M}) - v^{N,i}(0, \mathbf{M})| \le \frac{C}{N}$$

hence proving Eq. (63).

Third step. We are now in the position to prove bound for optimal trajectories, namely Eq. (61).

In view of the dynamics, encoded in (59) and (60) and respectively satisfied by processes $(Y_{i,t})_{t \in [0,\tau^i]}$ and $(X_{i,t})_{t \in [0,\tau^i]}$, we know that

$$\begin{split} \left(X_{t \wedge \tau_X^i}^i - Y_{t \wedge \tau_Y^i}^i\right)^2 & \leq \left(\int_0^{\tau^i} \left(D_p H\left(s, X_{s \wedge \tau_X^i}^i, D_{x_i} u^{N,i}(s, \mathbf{X}_{s \wedge \tau_X})\right)\right. \\ & \left. - D_p H\left(s, Y_{s \wedge \tau_Y^i}^i, D_{x_i} v^{N,i}(s, \mathbf{Y}_{s \wedge \tau_Y})\right)\right) ds\right)^2 \\ & \left. + \left(\int_0^{\tau^i} \sqrt{2} \left(\sigma(X_{s \wedge \tau_X^i}^i) - \sigma(Y_{s \wedge \tau_Y^i}^i)\right) dB_s^i\right)^2 \,. \end{split}$$

Since D_pH is Lipschitz continuous and U is regular, bounds in Prop. 14 implies

$$\begin{split} \left(X_{t \wedge \tau_X^i}^i - Y_{t \wedge \tau_Y^i}^i\right)^2 &\leq C \bigg[\int_0^{\tau^i} \bigg(|X_{s \wedge \tau_X^i}^i - Y_{s \wedge \tau_Y^i}^i| + \frac{1}{N} \sum_{j \neq i} |X_{s \wedge \tau_X^i}^j - Y_{s \wedge \tau_Y^i}^j| \bigg) ds \\ &+ \int_0^{\tau^i} D_p H \bigg(s, Y_{s \wedge \tau_Y^i}^i, D_{x_i} u^{N,i}(s, \mathbf{Y}_{s \wedge \tau_Y}) \bigg) \\ &- D_p H \bigg(s, Y_{s \wedge \tau_Y^i}^i, D_{x_i} v^{N,i}(s, \mathbf{Y}_{s \wedge \tau_Y}) \bigg) ds \bigg]^2 \\ &+ \bigg(\int_0^{\tau^i} \sqrt{2} \left(\sigma(X_s^i) - \sigma(Y_s^i) \right) dB_s^i \bigg)^2 \,. \end{split}$$

Taking the sup over $t \in [0, \chi]$ with $\chi \in [0, \tau^i]$ and the conditional expectation over **M**, we obtain

$$\mathbb{E}^{\mathbf{M}} \left[\sup_{t \in [0,\chi]} |X_{t \wedge \tau_{\chi}^{i}}^{i} - Y_{t \wedge \tau_{\chi}^{i}}^{i}|^{2} \right] \leq \mathbb{E}^{\mathbf{M}} \left[\int_{0}^{T} \left| DU_{s}^{N,i,i} - DV_{s}^{N,i,i} \right|^{2} ds \right] \\
+ C \int_{0}^{\chi} \left(\mathbb{E}^{\mathbf{M}} \left[\sup_{t \in [0,s]} |X_{t \wedge \tau_{\chi}^{i}}^{i} - Y_{t \wedge \tau_{\chi}^{i}}^{i}|^{2} \right] \right) \\
+ \frac{1}{N} \sum_{j \neq i} \mathbb{E}^{\mathbf{M}} \left[\sup_{t \in [0,s]} |X_{t \wedge \tau_{\chi}^{i}}^{j} - Y_{t \wedge \tau_{\chi}^{i}}^{j}|^{2} \right] ds \\
+ \sup_{t \in [0,s]} \mathbb{E}^{\mathbf{M}} \left[\left(\int_{0}^{\chi} \sqrt{2} \left(\sigma(X_{t \wedge \tau_{\chi}^{i}}^{i}) - \sigma(Y_{t \wedge \tau_{\chi}^{i}}^{i}) \right) dB_{t}^{i} \right)^{2} \right].$$
(67)

By Itô isometry and exploiting the Lipschitz assumption on σ , we know that

$$\begin{split} \sup_{t \in [0,s]} \mathbb{E}^{\mathbf{M}} \left[\left(\int_{0}^{\chi} \sqrt{2} \left(\sigma(X_{t \wedge \tau_{X}^{i}}^{i}) - \sigma(Y_{t \wedge \tau_{Y}^{i}}^{i}) \right) dB_{t}^{i} \right)^{2} \right] \\ &= \sup_{t \in [0,s]} \mathbb{E}^{\mathbf{M}} \left[\int_{0}^{\chi} 2 \left(\sigma(X_{t \wedge \tau_{X}^{i}}^{i}) - \sigma(Y_{t \wedge \tau_{Y}^{i}}^{i}) \right)^{2} dt \right] \\ &\leq \int_{0}^{\chi} \left(\mathbb{E}^{\mathbf{M}} \sup_{t \in [0,s]} \left| X_{t \wedge \tau_{X}^{i}}^{i} - Y_{t \wedge \tau_{Y}^{i}}^{i} \right|^{2} ds \right). \end{split}$$

We now take the sum over players $i \in \{1, ..., N\}$ and enforce Eq. (66). Then, by Gronwall's Lemma, we know that

$$\sum_{i=1}^{N} \mathbb{E}^{\mathbf{M}} \left[\sup_{t \in [0, \tau^{i}]} |X_{t \wedge \tau_{X}^{i}}^{i} - Y_{t \wedge \tau_{Y}^{i}}^{i}| \right] \le C.$$
 (68)

Inserting inequality (68) in (67), and applying again Gronwall's inequality, we obtain (61).

Coming back to Theorem 2, it states the fundamental result about the convergence of $(v^{N,i})_{i \in \{1,\dots,N\}}$, the solution of the Nash system (58) towards U, namely the classical solution to the Master Equation (7) stated with Dirichlet boundary conditions.

Proof of Theorem 2. For the sake of simplicity, we can choose $m_0 = 1$, then apply the estimate provided by Equation (63). Thus,

$$\left| U(t_0, M_i, m_{\mathbf{M}}^{N,i}) - v^{N,i}(t_0, \mathbf{M}) \right| \le \frac{C}{N}$$
 a.e. $i \in \{1, ..., N\}$,

where $\mathbf{M} = (M_1, ..., M_N)$ are i.i.d. random variables, uniformly distributed in Ω . The support of \mathbf{M} is Ω^N , thus from the continuity of U we have

$$|U(t_0, x_i, m_{\mathbf{x}}^{N,i}) - v^{N,i}(t_0, \mathbf{x})| \le \frac{C}{N} \quad \forall \mathbf{x} \in \Omega^N, \quad i \in \{1, ..., N\}.$$

Let us underline that latter estimates hold for any $\mathbf{x} \in \Omega^N$. Moreover, since U is Lipschitz continuous with respect to m, we can replace $U(t_0, x_i, m_{\mathbf{x}}^{N,i})$ with $U(t_0, x_i, m_{\mathbf{x}}^N)$, since

$$\left| U(t_0, x_i, m_{\mathbf{x}}^{N,i}) - U(t_0, x_i, m_{\mathbf{x}}^N) \right| \le C \mathbf{d}_1(m_{\mathbf{x}}^{N,i}, m_{\mathbf{x}}^N) \le \frac{C}{N}$$

To prove (ii) we need the convergence results of empirical measure stated in [2, 32, 38].

First, we move from the sub-probability measure m_0 , defined over Ω to m_0^* , which corresponds to a possible probability measure defined over $\mathcal{P}(\mathbb{R}^n)$, and such that its restriction on Ω coincides with $m_0 \in \mathcal{P}^{sub}(\Omega)$, namely $m_0^*(x)|_{\Omega} = m_0(x)$, $\forall x \in \Omega$. We remark that m_0^* is one of the possible probability measures since the extension is not uniquely determined. Then, the following inequality holds

$$\begin{split} \int_{\Omega^{N-1}} |u_i^N(t_0,\mathbf{x}) - U(t_0,x_i,m_0)| \prod_{j \neq i} m_0(dx_j) \\ & \leq \int_{\mathbb{R}^{N-1}} |u_i^N(t_0,\mathbf{x}) - U(t_0,x_i,m_0^*)| \prod_{j \neq i} m_0^*(dx_j) \,. \end{split}$$

Since U is Lipschitz continuous, and we are now working with probability measures, we can apply results from [2, 32, 38], obtaining

$$\int_{\mathbb{R}^{N-1}} |u_i^N(t_0, \mathbf{x}) - U(t_0, x_i, m_0^*)| \prod_{j \neq i} m_0^*(dx_j)
\leq C \int_{\mathbb{R}^{N-1}} \mathbf{d}_1(m_x^{N,i}, m_0^*) \prod_{j \neq i} m_0^*(dx_j) \leq C\omega_N,$$

where ω_N is defined as the right side in Eq. (13).

The latter implies that

$$\begin{split} &\|w_{i}^{N}(t_{0},\cdot,m_{0})-U(t_{0},\cdot,m_{0})\|_{L^{1}(m_{0})} = \\ &= \int_{\Omega} \left| \int_{\Omega^{N-1}} \left(v_{i}^{N}(t_{0},\mathbf{x})-U(t_{0},x_{i},m_{0}) \right) \prod_{j\neq i} m_{0}(dx_{j}) \right| m_{0}(dx_{i}) \\ &\leq \mathbb{E} \left[|v_{i}^{N}(t,\mathbf{M})-u_{i}^{N}(t,\mathbf{M})| \right] + \int_{\Omega^{N}} |u_{i}^{N}(t_{0},\mathbf{x})-U(t_{0},x_{i},m_{0}^{*})| \prod_{j} m_{0}^{*}(dx_{j}) \\ &\leq \frac{C}{N} + C\omega_{N} \leq C\omega_{N} \end{split}$$

holds, hence proving both Eq. (13) and the Theorem.

7. Application to a Toy Model for Optimal Liquidations

We will study a toy model of mean field type which can be seen as the limiting problem of an N player game. In particular, we adapt the setting described in [20, 46, 54], with multiple investors, letting players stop their investments when a given, fixed, condition is met. We consider a variable interest rate - constant for certain periods of time and then variable at the discretion of the issuer of the loan. The investment ends when the price reaches a certain level, e.g. one prescribed by the contractor, which could also be (from the *low level* perspective) the default one. Thus, we introduce a set in which the players want to operate within a fixed lower bound $l \in \mathbb{R}$ and a fixed upper one $u \in \mathbb{R}$.

We focus on an N-player game with an Itô-diffusion setting for a model with a controlled drift and state-independent controls. The N players, indexed by $i \in I = \{1, 2, ..., N\}$, have a common investment horizon [0, T]. Each player i uses a self-financing strategy $(\pi_t^i)_{t \in [0,T]}$ and can control its own drift. The corresponding wealth, denoted by $(X_t^i)_{t \in [0,T]}$, represents the amount of the investment, and it evolves accordingly to the following SDE:

$$dX_t^i = \pi_t^i dt + \sqrt{2}\sigma dB_t^i, \tag{69}$$

assuming a control $\pi_t^i \in \mathcal{A}$ and a constant volatility σ . The self-financing strategies π^i belong to the set of admissible policies \mathcal{A} defined as

$$\mathcal{A} = \left\{ \pi^i_\cdot : [0, T] \to L^2(\tilde{\Omega}) \ \mathcal{F}\text{-progr. measurable with } \mathbb{E}\left[\int_0^{\tau^i} \sigma^2 \pi_s^2 ds \right] < \infty \right\}$$

with τ^i corresponding to the hitting time of the i^{th} wealth, which corresponds to the time when X_t^i reaches one of the critical values defined by the bounds, l or u:

$$\tau^{i} = \inf \left\{ t \in [0, T] : X_{t}^{i} = l \vee X_{t}^{i} = u \right\}$$

Following the setting stated in [54], players optimize their expected terminal utility being also concerned with the performance of their competitors. Accordingly, we fix an arbitrary policy $(\pi_1, ..., \pi_{i-1}, \pi_{i+1}, ..., \pi_N)$ for the other players, whereas the i^{th} player aims to maximize

$$J^{i}(t, \boldsymbol{x}, \boldsymbol{\pi}_{s}^{i}) = \mathbb{E}\left[\int_{t}^{\tau^{i}} \frac{1}{2} \left(\boldsymbol{\pi}_{s}^{i}\right)^{2} ds + G\left(X_{\tau^{i}}^{i}, m_{X_{\tau^{i}}^{i}}^{N, i}\right) \middle| \boldsymbol{X}_{t} = \boldsymbol{x}\right],$$

with the terminal payoff G defined as follows:

$$G(x,m) = x(1-x) \int_0^1 z(1-z)dm(z) .$$

We underline that functions $H(p) = \frac{1}{2}|p|^2$ and G both satisfy hypotheses 1.

For our 1-dimensional trading example, we enforce a simplified choice of the domain by prescribing both lower and upper limits, i.e. l = 0 and u = 1. Moreover, we fix the volatility $\sigma = 1$.

Accordingly, the Nash system takes the following form.

$$\begin{cases} -\partial_{t}v_{i}^{N} - \sum_{j=1}^{N} \partial_{x_{j}x_{j}}^{2}v_{i}^{N}(t, \boldsymbol{x}) + \frac{1}{2} \left(\partial_{x_{i}}v_{i}^{N}(t, \boldsymbol{x})\right)^{2} \\ + \sum_{j\neq i}^{N} \partial_{x_{j}}v_{j}^{N}(t, \boldsymbol{x})\partial_{x_{j}}v_{i}^{N}(t, \boldsymbol{x}) = 0, \\ v_{i}^{N}(T, \boldsymbol{x}) = \frac{1}{N-1}x_{i}(1-x_{i})\sum_{j\neq i}^{N}x_{j}(1-x_{j}), \\ v_{i}^{N}(t, \boldsymbol{x})_{|x_{i}\in\{0,1\}} = 0, \quad i = 1, ..., N. \end{cases}$$

We will study models of mean field type which can be seen as the limiting problem of the above-described one. When $N \to \infty$, the dynamic for a representative player is described by

$$dX_t = \pi_t dt + \sqrt{2} dB_t.$$

The corresponding functional reads

$$J(x,\pi_t) = \mathbb{E}\left[\int_0^\tau \frac{1}{2} \pi_s^2 ds + X_\tau (1-X_\tau) \int_0^1 z (1-z) dm_\tau(z) \left| X_t = x \right| \right],$$

where m_{τ} is the density of the process (X_{τ}) .

The Master Equation for this simplified setting reads

$$\begin{cases} -\partial_{t}U(t,x,m) - \partial_{xx}U(t,x,m) + \frac{1}{2} (\partial_{x}U(t,x,m))^{2} + \\ - \int_{0}^{1} \partial_{y} (D_{m}U(t,x,m,y)) dm(y) + \\ + \int_{0}^{1} D_{m}U(t,x,m,y) \cdot \partial_{x}U(t,y,m) dm(y) = 0 \end{cases}$$

$$U(T,x,m) = x(1-x) \int_{0}^{1} z(1-z) dm(z)$$

$$U(t,0,m) = U(t,1,m) = 0$$

$$\frac{\partial U}{\partial m}(t,x,m,0) = \frac{\partial U}{\partial m}(t,x,m,1) = 0.$$

Although the solution of the Nash system is not easy to compute, we can directly approach the Master Equation, using the definition of U accordingly to (9). In particular, the Hamilton-Jacobi equation for our formulation reads as follows:

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2} (u_x)^2 = 0 \\ u(T, x) = x(1 - x) \int_0^1 z(1 - z) dm_T(z), & \forall x \in [0, 1], , \\ u(t, 0) = u(t, 1) = 0, & \forall t \in [t_0, T]. \end{cases}$$

By the change of variable $w = e^{-\frac{1}{2}u}$, see also [19], the equation for the function w becomes

$$\begin{cases} -w_t - w_{xx} = 0, \\ w(T, x) = \exp\left(-\frac{1}{2}x(1 - x)\int_0^1 z(1 - z)dm_T(z)\right), \\ w(t, 0) = w(t, 1) = 1. \end{cases}$$

hence a classical heat equation whose solution can be computed with standard methods. Coming back to the change of variable $u = -2 \log(w)$ and (9), the function $U(t_0, \cdot, m_0) = u(t_0, \cdot)$ admits the following representation:

$$U(t_0, x, m_0) = -2 \ln \left(1 + \sum_{\substack{k=1 \ 2Q}}^{+\infty} b_k e^{-k^2 \pi^2 (T - t_0)} \sin(k\pi x) \right),$$

being $b_k = 2 \int_0^1 e^{-\frac{1}{2}y(1-y)\int_0^1 z(1-z)dm_T(z)} \sin(k\pi y)dy$ and $m_T = m(T, \cdot)$, where m is the solution of the Fokker-Planck equation

$$\begin{cases} m_t - m_{xx} - (mu_x)_x = 0, \\ m(t_0, x) = m_0, & \forall x \in [0, 1], \\ m(t, 0) = m(t, 1) = 0, & \forall t \in [t_0, T]. \end{cases}$$

Declarations

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References

- [1] Aïd, R., Dumitrescu R., Tankov P.(2021). The entry and exit game in the electricity market: a mean-field approach. Journal of Economic Dynamics & Control, 8(4):331-358.
- [2] Ajtai, M., Komlos, J., Tusnàdy, G. (1984). On optimal matchings. Combinatorica, Vol. 4(4), 259-264.
- [3] Alsmeyer, G., Jaeger, M. (2005). A Useful Extension of Ito's Formula with Applications to Optimal Stopping. Acta Math Sinica 21, 779–786 (). https://doi.org/10.1007/s10114-004-0524-y
- [4] Bayraktar, E., Cecchin, A., Cohen, A., Delarue, F. (2021). Finite state mean field games with wright-fisher common noise. Journal de Mathématiques Pures et Appliquées, Volume 147, Pages 98-162, ISSN 0021-7824, https://doi.org/10.1016/j.matpur.2021.01.003.
- [5] Bayraktar, E., Cohen, A. (2018). Analysis of a finite state many player game using its master equation. SIAM Journal on Control and Optimization, 56(5), 3538-3568.
- [6] Benazzoli, C., Campi, L., Di Persio, L. (2020). Mean field games with controlled jump-diffusion dynamics: Existence results and an illiquid interbank market model. In: Stochastic Processes and their Applications 130.11, pp. 6927–6964.
- [7] Bensoussan, A., Frehse, J. (2002). Smooth solutions of systems of quasilinear parabolic equations. ESAIM: Control, Optimisation and Calculus of Variations, 8, 169-193.
- [8] Bertucci, C. (2018). Optimal stopping in mean field games, an obstacle problem approach. J. Math. Pures Appl. 120, 165-194.
- [9] Bertucci, C. (2021). Monotone solutions for mean field games master equations: finite state space and optimal stopping. Journal de l'Ecole polytechnique Mathematiques, Tome 8 pp. 1099-1132. doi: 10.5802/jep.167.
- [10] Bertucci, C., Lasry, J.-M., Lions, P.-L. (2021). Master Equation for the Finite State Space Planning Problem. Archive for Rational Mechanics and Analysis 242 no. 1, p. 327, DOI:10.1007/s00205-021-01687-8.
- [11] Bouveret, G., Dumitrescu, R., Tankov, P. (2020). Mean-field games of optimal stopping: a relaxed solution approach. Siam J. Control Optim. 58, 1795-1821, 2020.
- [12] Buckdahn, R., Li, J., Peng, S., Rainer, C. (2017). *Mean-Field Stochastic Differential Equations and Associated PDEs.* The Annals of Probability, 45(2), 824-878.
- [13] Burger, M., Di Francesco, M., Markowich, P. A., Wolfram, M.T. (2014). *Mean field games with nonlinear mobilities in pedestrian dynamics*. Discrete and Continuous Dynamical Systems B, 19(5): 1311-1333.
- [14] Burzoni, M., Campi, L. (2021). Mean field games with absorption and common noise with a model of bank run. arXiv:2107.00603.
- [15] Campi, L., Fischer, M. (2016). N-player games and mean field games with absorption. The Annals of Applied Probability, 28(4), 2188-2242.
- [16] Cardaliaguet, P. (2012). Notes on Mean Field Games, Technical report.
- [17] Cardaliaguet, P., Cirant, M., Porretta, A. (2020). Remarks on Nash equilibria in mean field game models with a major player. Proceedings of the American Math. Society 148, 4241-4255.
- [18] Cardaliaguet, P., Delarue, F., Lasry, J.-M., Lions, P.-L. (2019). The Master Equation and the Convergence Problem in Mean Field Games. Annals of Mathematics Studies, Vol. 2.
- [19] Cardaliaguet, P., Lasry, J.-M., Lions, P.-L, and Porretta, A. (2012). Long time average of mean field games. Netw. Heterog. Media, 7(2):279–301.
- [20] Cardaliaguet, P., Lehalle, C.A.(2018). *Mean field game of controls and an application to trade crowding*. Mathematics and Financial Economics, 12, 335–363.
- [21] Carmona, R. (2016). Lectures on BSDEs, stochastic control, and stochastic differential games with financial applications. SIAM, DOi: 10.1137/1.9781611974249.
- [22] Carmona, R., Delarue, F. (2013). Probabilist analysis of Mean Field Games. SIAM Journal on Control and Optimization, 51(4), 2705-2734.
- [23] Carmona, R., Delarue, F. (2017). Probabilistic theory of mean field games with applications, Springer Verlag.
- [24] Carmona, R., Delarue, F. (2018). The Master Field and the Master Equation. Probabilistic Theory of Mean Field Games with Applications II. Springer, Cham, 239-321.
- [25] Cecchin, A., Delarue, F. (2020). Selection by vanishing common noise for potential finite state mean field games. Communications in Partial Differential Equations, 47(1), 89-168.
- [26] Cecchin, A., Pelino, G. (2019). Convergence, fluctuations and large deviations for finite state mean field games via the master equation. Stochastic Processes and their Applications, 129(11), 4510-4555.
- [27] Cecchin, A., Pra, P.D., Fischer, M., Pelino, G. (2019). On the convergence problem in mean field games: a two state model without uniqueness. SIAM Journal on Control and Optimization, 57(4), 2443-2466.

- [28] Chassagneux, J.F., Crisan, D., Delarue, F. (2014). A Probabilistic approach to classical solutions of the master equation for large population equilibria. Accepted in Mem. Amer. Math. Soc., arXiv: 1411.3009.
- [29] Delarue, F., Lacker, D., Ramanan, K. (2018). From the master equation to mean field game limit theory: Large deviations and concentration of measure. Annals of Probability, 2020, 48 (1), pp.211-263.
- [30] Delarue, F., Lacker, D., Ramanan, K. (2019). From the master equation to mean field game limit theory: a central limit theorem. Electron. J. Probab. 24, no. 51, 1-54.
- [31] Delfour, M.C., Zolesio, J.-P. (1994). Shape analysis via oriented distance function. J. Funct. Anal., 123, 129-201.
- [32] Dereich, S., Scheutzow, M., Schottstedt, R. (2013). Constructive quantization: approximation by empirical measures. Annales de l'IHP, Probabilites et Statistiques, Vol. 49(4), 1183-1203.
- [33] Djete, M. F. (2020). Mean field games of controls: on the convergence of Nash equilibria. arXiv preprint arXiv:2006.12993.
- [34] Doncel, J., Gast, N., Gaujal, B. (2019). Discrete mean field games: Existence of equilibria and convergence. Journal of Dynamics & Games 6(3), 221-239.
- [35] Ferreira, R., Gomes, D. (2015). Existence of weak solutions to stationary mean-field games through variational inequalities, SIAM Journal on Mathematical Analysis, 50(6), 5969–6006.
- [36] Ferreira, R., Gomes, D., Tada, T. (2018). Existence of weak solutions to first-order stationary mean-field games with Dirichlet conditions. Proceedings of the American Mathematical Society, 147(11), 4713–4731.
- [37] Figalli, A., Gigli, N. (2010). A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions. J. Math. Pures et Appl., 94(2), 107-130.
- [38] Fournier, N., Guillin, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. Probability Theory and Related Fields, Vol. 162(3), 707-738.
- [39] Gangbo, W., Mészáros, A. R. (2020). Global well-posedness of Master Equations for deterministic displacement convex potential mean field games. arXiv preprint arXiv:2004.01660.
- [40] Gangbo, W., Mészáros, A. R., Mou, C., Zhang, J. (2021). Mean Field Games Master Equations with Non-separable Hamiltonians and Displacement Monotonicity. arXiv preprint arXiv:2101.12362.
- [41] Gangbo, W., Swiech, A. (2015)., Existence of a solution to an equation arising from the theory of mean field games, Journal of Differential Equations, 259, pp. 6573–6643.
- [42] Graber, P.J. (2016) Linear Quadratic Mean Field Type Control and Mean Field Games with Common Noise, with Application to Production of an Exhaustible Resource. Appl Math Optim 74, 459–486. https://doi.org/10.1007/s00245-016-9385-x
- [43] Graber, P.J., Mouzouni, C. (2020) On Mean Field Games models for exhaustible commodities trade ESAIM: COCV, 26 11 DOI: https://doi.org/10.1051/cocv/2019008
- [44] Graber, P.J., Sircar, R. (2023) Master equation for Cournot mean field games of control with absorption. Journal of Differential Equations, Volume 343, Pages 816-909, ISSN 0022-0396, https://doi.org/10.1016/j.jde.2022.10.031.
- [45] Grover, P., Bakshi, K., Theodorou, E.A. (2018). A mean-field game model for homogeneous flocking, Chaos 28, 061103.
- [46] Hu, R., Zariphopoulou, T. (2022). N-Player and Mean-Field Games in Itô-Diffusion Markets with Competitive or Homophilous Interaction. In: Yin, G., Zariphopoulou, T. (eds) Stochastic Analysis, Filtering, and Stochastic Optimization. Springer, Cham. https://doi.org/10.1007/978-3-030-98519-69
- [47] Huang, M., Caines, P.E., Malhamé, R.P. (2006). Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Comm. Inf. Syst. 6, 221–251.
- [48] Huang, M., Caines, P.E., Malhamé, R.P. (2007). Large population Cost-Coupled LQG Problems With Nonuniform Agents: Individual-Mass Behavior and Decentralized *ϵ*-Nash Equilibria. IEEE Transactions on Automatic Control, 52(9), 1560-1571.
- [49] Jakobsen, R. E., Rutkowski, A. (2023). The master equation for mean field game systems with fractional and nonlocal diffusions. Preprint arXiv:2305.18867
- [50] Kolokoltsov, V.N., Li, J., Yang, W. (2012). Mean Field Games and nonlinear Markov Processes. Preprint arXiv:1112.3744.
- [51] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N. (1967). Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence R.I.
- [52] Lacker, D. (2016). A general characterization of the mean field limit for stochastic differential games. Probability Theory and Related Fields, 165, 581-648.
- [53] Lacker, D., (2020). On the convergence of closed-loop Nash equilibria to the mean field game limit. Ann. Appl. Probab. 30(4): 1693-1761.
- [54] Lacker, D., Zariphopoulou, T. (2019). Mean field and n-agent games for optimal investment under relative performance criteria. Mathematical Finance, 29(4):1003–1038.
- [55] Lasry, J.-M., Lions, P.-L. (2006). Jeux à champ moyen. I. Le cas stationnaire. C. R. Math. Acad. Sci. Paris 343, 619-625.
- [56] Lasry, J.-M., Lions, P.-L. (2006). Jeux à champ moyen. II. Horizon fini et contrôle optimal. C. R. Math. Acad. Sci. Paris 343, 679-684.
- $[57] \ Lasry, J.-M., Lions, P.-L. \ (2007). \ \textit{Mean field games}. \ Jpn. \ J. \ Math. \ 2\ , \ no. \ 1, \ 229-260.$
- [58] Lasry, J.-M., Lions, P.-L., Guèant, O. (2011). Application of Mean Field Games to Growth Theory. In: Paris-Princeton lectures on mathematical finance; Lecture notes in Mathematics. Springer, Berlin.
- [59] Lunardi, A. (2012). Analytic Semigroups and Optimal Regularity in Parabolic Problems. Modern Birkhäuser Classics.
- [60] Mayorga, S. (2020). Short time solution to the master equation of a first order mean field game. Journal of Differential Equations, 268(10), 6251-6318.
- [61] Mazanti, G., Santambrogio, F. (2019). Minimal-time mean field games, Math. Models Methods Appl. Sci. 29, 1413-1464.
- [62] Nutz, M., Zhang, Y. (2019). A mean field competition. Mathematics of Operations Research, 44(4), 1245-1263.
- [63] Piccoli, B., Rossi, F. (2014). Generalized Wasserstein Distance and its Application to Transport Equations with Source. Archive for Rational Mechanics and Analysis, 211, 335-358. Springer Science and Business Media LLC.
- [64] Piccoli, B., Rossi, F. (2016). On Properties of the Generalized Wasserstein Distance. Arch Rational Mech Anal 222, 1339–1365 https://doi.org/10.1007/s00205-016-1026-7
- [65] Porretta, A. (2015). Weak Solutions to Fokker-Planck Equations and Mean Field Games. Arch Rational Mech Anal 216, 1-62.
- [66] Ricciardi, M. (2022). The Master Equation in a Bounded Domain with Neumann Conditions. Communications in Partial Differential Equations, 47(5), 912-947.
- [67] Ricciardi, M. (2023). The Convergence Problem in Mean Field Games with Neumann Conditions. SIAM Journal on Mathematical Analysis, 55(4), 3316-3343.
- [68] Santambrogio, F. (2017). {Euclidean, Metric, and Wasserstein} Gradient Flows: an overview. Bull. Math. Sci. 7, 87-154 (2017).