# COMPUTATIONAL PHYSICS - EXERCISE SHEET 01
# Planetary evolution in the Solar System using different integrators (Euler, Verlet, Velocity-Verlet)

Matteo Garbellini*
*Department of Physics, University of Freiburg*

(Dated: May 3, 2021)

The following is the report for the Exercise Sheet 01 - Planetary Evolution. Along this report, a python script was also handed in. Additional code and snippets (e.g. code for the plots) can be found at the following github repository https://github.com/matteogarbellini/Computational-Physics-Material-Science/tree/main/Exercise-Sheet-01. Please note that I had some issues with the figures captions, thus they were generated outside of this document.

## I. VERLET AND VELOCITY-VERLET INTEGRATION SCHEMES

### A. Verlet and Velocity-Verlet equivalence

It is straight forward to show the equivalence between the Verlet and Velocity-Verlet integration schemes. Recalling the latter (note that the indices were dropped for convenience):

$$r(t+dt) = r(t) + v(t)dt + \frac{1}{2}a(t)dt^2 \tag{1}$$

$$v(t+dt) = v(t) + \frac{1}{2}[a(t+dt) + a(t)]dt \tag{2}$$

we can shift the time variable from $t$ to $t+dt$ (change of variable) and we get for Eq(1)

$$r(t+2dt) = r(t+dt) + v(t+dt)dt + \frac{1}{2}a(t+dt)dt^2 \tag{3}$$

By addition of Eq(3) and Eq(1) we get

$$r(t+2dt) + r(t) = 2r(t+dt) + [v(t+dt) - v(t)]dt + \frac{1}{2}[d(t+dt) - a(t)]dt \tag{4}$$

from which we get the coordinate version of the Verlet integration scheme by substitution of Eq(2) and by shifting back the time coordinate $t$ to $t-dt$

$$r(t+dt) = 2r(t) - r(t-dt) + a(t)dt^2 \tag{5}$$

### B. Initial conditions

The following are the initial conditions necessary for the Verlet and Velocity-Verlet integration schemes:

- **Verlet**: for computing the position at time $t+dt$ the algoritm requires as initial conditions the position at time $t-dt$ and $t$. This means that the algorithm starting point is at a *midpoint* between $r(t-dt)$ and $r(t+dt)$. We will later discuss how this is implemented for the first iteration
- **Velocity-Verlet**: this integrations scheme requires the positions at time $t$ for the position propagation, while for requires positions at time $t$ and $t+dt$ for the velocity (this implies an additional force evaluation).

* matteo.garbellini@studenti.unimi.it

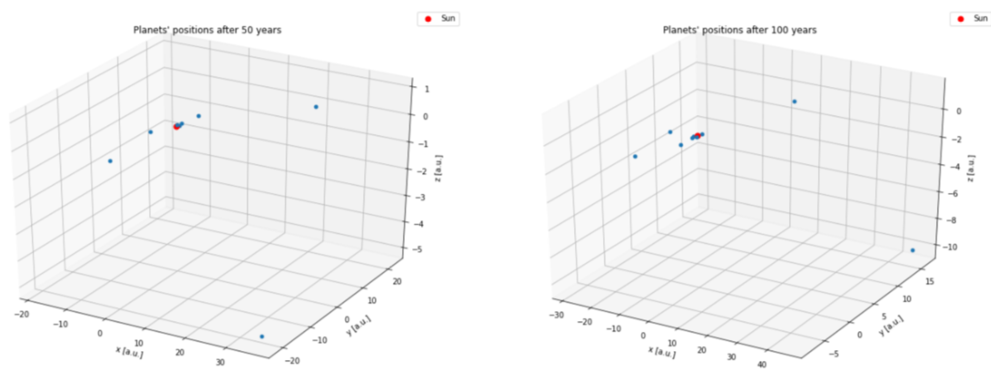
### C. First position propagation for the Verlet integration

As mentioned the Verlet integration scheme requires the initial conditions at time $t - dt$ and $t$. Therefore it is necessary to propagate the position to $t + dt$ in order to have the required initial conditions. To do so, the position are propagated with the Euler integration scheme. Although affected by a larger error, a single Euler propagation over the course of (possibly) thousands of iterations does not affect the precision of the Verlet algorithm. The Euler integration scheme can be found in the appendix.

## II. PLUTO'S TRAJECTORY USING THE VELOCITY-VERLET INTEGRATOR

### A. Planets location after 50 and 100 years

The following plots show the planet location after 50 years (left) and 100 years (right). The position were computed using the Velocity-Verlet integrator. A projection on the $x$-$y$ coordinates is also provided.




(a) 50 years

(b) 100 years

**Figure 1:** On the left, positions of planets after 50 years obtained with the Velocity Verlet algorithm in steps of half a day. Same on the right, after 100 years.

### B. Pluto's trajectory over the first and second 100 years

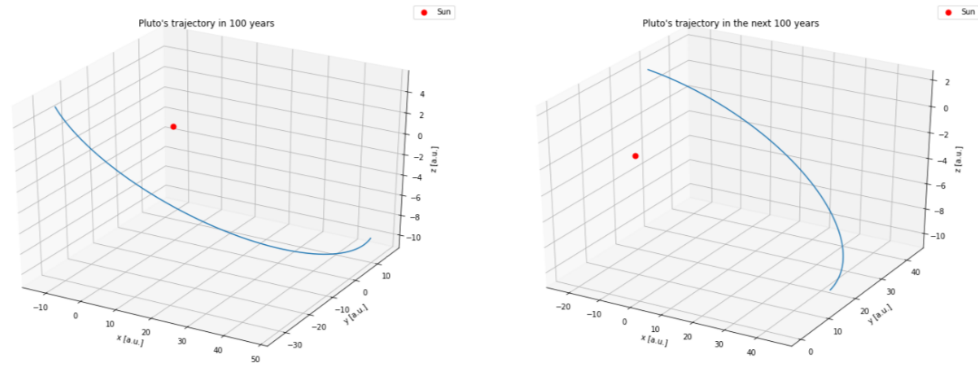
The following plots show the trajectory of Pluto in the first 100 years and the second 100 years (i.e. 101 to 200 years). The simulation was run using the Velocity-Verlet integrator.

### C. Pluto's orbital period around the sun

The planet's orbital period was estimated using Kepler's third law, namely:

$$\frac{a^3}{T^2} = \frac{GM}{4 * \pi^2} \tag{6}$$

where $M$ is the mass of the planet, $a$ the major semi-axis (estimated from the computed data) and $T$ the orbital period. The computed orbital period for Pluto is $T = 275$ years, while the correct value is $T = 249$ years.



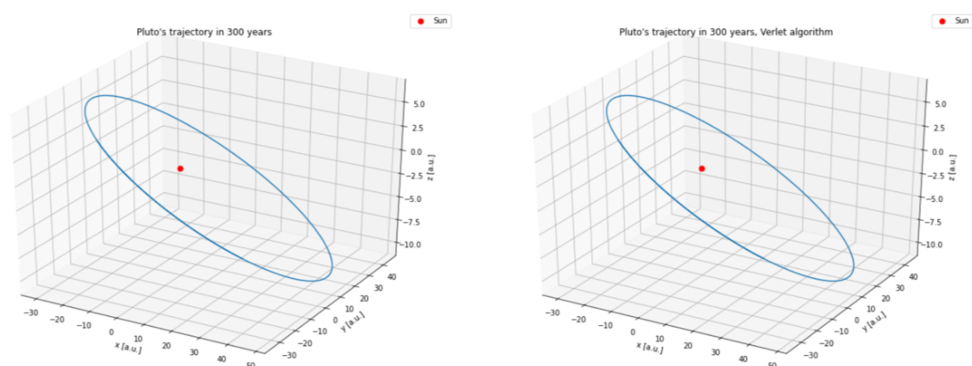

(a) First 100 years (b) Next 100 years

**Figure 2:** On the left, trajectory of Pluto over the first 100 years obtained with the Velocity Verlet algorithm in steps of half a day. On the right, trajectory of Pluto over the next 100 years obtained with the Velocity Verlet algorithm in steps of half a day.

## III. PLUTO'S AND MARS' TRAJECTORY USING THE VERLET INTEGRATOR

### A. Pluto's trajectory: Verlet vs Velocity-Verlet

The following plots compare Pluto's trajectory computed with the Verlet (left), Velocity-Verlet (right).




(a) Velocity Verlet (b) Verlet

**Figure 3:** Trajectory of Pluto over 300 years obtained with the Velocity Verlet algorithm in steps of half a day (left) and with the Verlet algorithm in steps of half a day (right).

### B. Mars' trajectory and orbital period

The Mars's orbital period was computed using Eq(6). The calculated value is $T \approx 1.79$ (correct value $T \approx 1.88$). The following plot shows Mars' trajectory over a period of 2 years, approximately its orbital period.

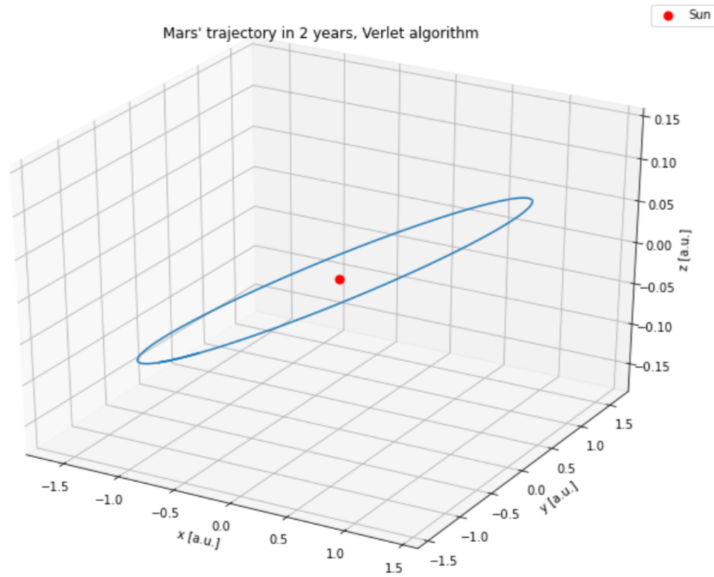



**Figure 4:** Mars' trajectory over 2 years, obtained with the Verlet algorithm in steps of half a day.

## IV. ENERGY CONSERVATION USING THE EULER INTEGRATOR

We then evaluated the energy of the system using the Euler integration scheme over a period of 300 years. As can be seen form the following plot, the energy oscillates throughout the simulation. Although the energy it is not constant, the trajectory of Pluto is quite similar to the Velocity-Verlet and Verlet integrators. Note however that the plot resolution does not allow for an accurate comparison of the trajectories.

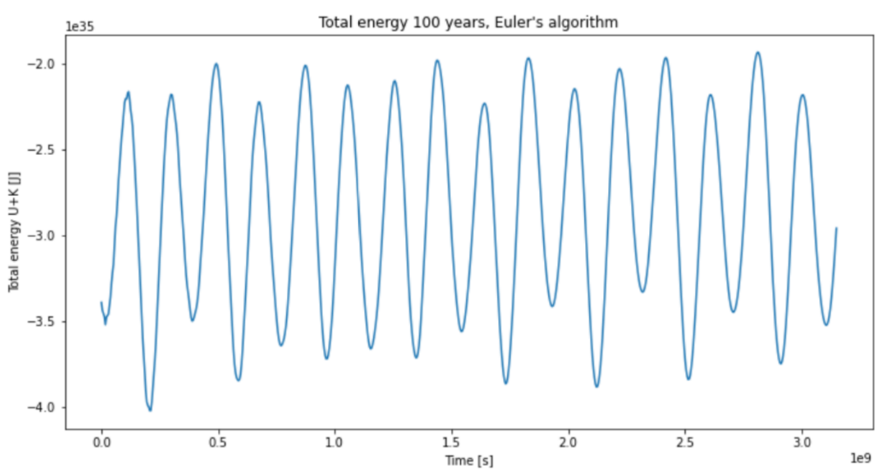



**Figure 5:** Total energy of the Solar System with Euler algorithm (1 measurement every 100 simulation steps).

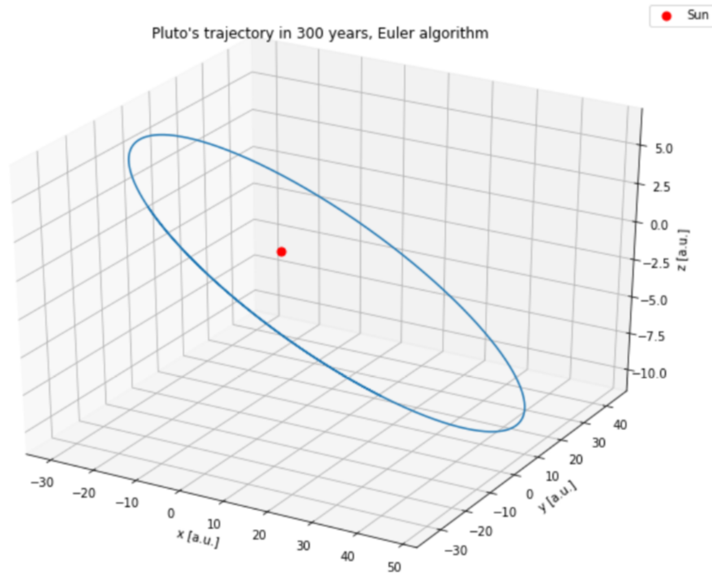



**Figure 6:** Pluto's trajectory over 300 years, obtained with the Euler algorithm in steps of half a day.