Quantum walks with time-dependent Hamiltonians and their application to the search problem on graphs

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Grover's quantum search algorithm has become one of the most celebrated algorithms in quantum computing and quantum information. It enables to outperform classical algorithms by achieving a quadratic speed up in the unstructured database search. Moreover, the search algorithm is *general* in the sense that it can be applied to a broad range of scenarios beyond the database search, from the *collision problem* to solving NP-complete problems.

The original Grover's algorithm was formulated at a time when the quantum circuit model was the mainstream tool in quantum computation, consisting in a series of quantum gates applied in discrete time. In 1998 Farhi and Gutmann introduced the idea of analog quantum computing, where the system evolves in continous-time following the Schroedinger equation. Later on, when quantum walks – that describe the free quantum evolution of a particle on a graph – were popularized as algorithmic tools, the search problem was soon investigated. Similarly is the case with the adiabatic evolution with time-dependent Hamiltonians emerged, the search problem was soon investigated. Therefore, when new models for quantum computing are introduced, new formulations of Grover's algorithm soon follow. It comes as a natural consequence the idea of comparing its different formulations, highlighting their differences and similarities.

In this thesis we studied the application of quantum walks with time-dependent Hamiltonian to the search problem on graphs. We compared the standard time-independent quantum walks search with the time-dependent one, with the goal of understanding if this implementation can lead to an improvement in performance for selected graphs. In particular, we studied its application to the *cycle graph*, where the search problem is not solved with the standard quantum walk approach of Farhi and Gutmann, and for completeness gave some results for the *complete graph*, in which the search may be accomplished exactly. In order to do so, we compared the two approaches in terms of *search*, *localization* – which represents a search without the needs to optimize the time – and a measure of *robustness*.

Next is a summary of what has been accomplished, some thoughts on the topic and future perspectives.

First we presented the concept of Laplacian, the matrix that determines the quantum evolution over the graph, followed by a brief description of the graphs considered (cycle and complete). We then introduced the search problem as originally posed by Grover, with particular emphasis on the action of the *oracle* – a black box able to recognize and mark the solutions. We then looked at the quantum walk approach of Farhi and Gutmann on the complete graph, where to the Laplacian of the quantum walk evolution is added an oracle term that marks the solution. We then discussed the adiabatic theorem, that states that if a system starts in the ground

state of an initial Hamiltonian, if it is made to evolve slow enough, it will end in the ground state of the final Hamiltonian. We studied its application to the unstructured search problem, focusing on the difference between the global adiabatic evolution of Childs and Goldstone and the local adiabatic evolution of Roland and Cerf. This is particularly interesting, since applying the adiabatic theorem *locally* enables to achieve the same quadratic speed up as the original Grover's search. To set the basis for our work we then presented the study by Wong *et. al*, where they showed that an adiabatic-quantum walk implementation of the search algorithm is not possible with the structure of the standard Grover's oracle, leaving however space to a merely time-dependent approach.

The main purpose of our work was thus to study a quantum walk search algorithm with time-dependent Hamiltonians, *inspired* by the adiabatic implementation, but free from the constraints of the adiabatic theorem. We considered an Hamiltonian based on the adiabatic evolution implementation of the form:

$$H(s) = (1-s)L - \gamma s \, |w\rangle \, \langle w|$$

where L is the Laplacian of the graph, $|w\rangle \, \langle w|$ is the the oracle Hamiltonian and s is the interpolating schedule that regulates the time-evolution of the Hamiltonian. In particular, we tried to improve the standard linear interpolating schedule of Farhi and Gutmann by taking the key aspects of the one derived by Roland and Cerf for the local adiabatic evolution and considered the following non-linear interpolating schedule:

$$s_{NL}(t) = \frac{1}{2}\Big[\Big(2\frac{t}{T} - 1\Big)^3 + 1\Big].$$

where T is the runtime of the search. We also considered the possibility of repeating the search multiple times if the search is not *perfect* – that is if the solution is not found with probability close to one. As a consequence, it was necessary to take into account an *initialization* time needed to prepare the system in the correct state and a physical *measure* time, that might affect the overall performance.

In order to compare the time-independent and time-dependent approach we differentiated between two types of results, namely the search and localization. The first describes the finding of the solution with high probability for the smallest time possible, while we call *localization* a search without the need to optimize the time. Additionally, we defined the *robustness*, a measure representing the variation on the outcome probability of the search due to noise on the γ parameter and runtime T.

We then turn our attention to the cycle graph, the main graph topology considered. We were able to show that the time-dependent approach has localization properties – it is therefore able to find the solution with high probability, although for a large runtime T – while the time-independent approach does not, since the outcome probability does not increase with time. This is indeed a consequence of the adiabatic nature of the time-dependent approach.

We then studied the search performance of the two approaches in terms of multiple runs search, since the search is not perfect for the time-independent Hamiltonian and the time is not optimized for the time-dependent one. To do so we introduce a new quantity which represents the minimum time necessary to get to unitary probability:

$$\tau = \min\left(\frac{T}{p}\right)_{T,\gamma}$$

and also consider the number of run iterations $I = \min(p^{-1})_{T,\gamma}$.

Additionally, since the probability p does not increase linearly with the time T, the quantity τ requires to consider a minimum time $T_{\min}$ to be effective at comparing the two approaches. To be consistent with the standar Grover's and the quantum walks search we set $T_{\min} = \pi/2\sqrt{N}$.

For completeness we finally turned our attention to the complete graph, qualitatively comparing probability and robustness. As expected the time-dependent approach was not able to achieve comparable performance with the standard time-independent algorithm. However, we discovered that the time-dependent approach was more robust than the time-independent one, although the improvement in robustness does not justify the much worse time scaling.

The complete graph case however illustrates the importance of the interpolating schedule. In fact, although it was not able to achieve the same time scaling as the time-independent approach, it suggests that the improvement on performance comes from the choice of the optimal interpolating schedule. Future investigation on the interpolating schedule can lead to improvements on performance, in particular for the cycle graph. Additionally one could also apply the time-dependent Hamiltonian to other graph topologies.