Introduction

Grover's quantum search algorithm has become one of the most celebrated algorithms in quantum computing and quantum information. It enables to outperform classical algorithms by achieving a quadratic speed up in the unstructured database search. Moreover, the search algorithm is *general* in the sense that it can be applied to a broad range of scenarios beyond the database search [6], from the *collision problem* [1] to solving NP-complete problems [5].

The original Grover's algorithm was formulated in a time when the quantum circuit model was the mainstream tool in quantum computation, consisting in a series of quantum gates applied in discrete time [7]. In 1998 Farhi and Gutmann introduced the idea of analog quantum computing, where the system evolves in continous-time following the Schroedinger equation [4]. Later on, when quantum walks were popularized as algorithmic tools and the adiabatic evolution with time-dependent Hamiltonians emerged, the search problem was soon investigated [2, 3]. Therefore, when new models for quantum computing are introduced, new formulations of Grover's algorithm soon follow. It came as a natural consequence the idea of comparing its different formulations, highlighting their differences and similarities [7].

In this thesis we study the application of quantum walks with time-dependent Hamiltonian to the search problem on graph. We compare the standard time-independent quantum walks search with the time-dependent one, with the goal of understanding if this implementation can lead to an improvement in performance for selected graphs. In particular we study its application to the cycle graph, where the search problem is not solved with the standard quantum walk of Farhi and Gutmann, and for completeness we give some results for the complete graph, for which the quantum walk search is a perfect search. In order to do so we compare the two approaches in terms of search, localization and a measure of robustness.

Our work can be summarized as follows:

- In Chapter 1 we recall the minimum theoretical knowledge necessary to follow the work done in this thesis. We review some basic notion of graph theory, quantum walks and the main characteristics of the graph considered. We then introduce the quantum search problem as firstly posed by Grover, followed by its quantum walks implementation. Then we discuss the adiabatic theorem and its application to the search problem, focusing in particular on the difference between global and local adiabatic evolution. Lastly we show that an adiabatic-quantum walks search algorithm is not possible.

- In Chapter 2 we study quantum walks with time-dependent Hamiltonians, focusing on the application to the search problem on graph. In particular we focus on the cycle graph and the complete graph. The goal of this section is to determine if a time-dependent Hamiltonian, inspired by the adiabatic implementation, can bring any advantages to the search problem on selected graphs.

Finally, conclusions and future perspective complete this thesis.

Conclusions

We conclude our work with a summary of what has been accomplished in this thesis, some thoughts on the results and future perspective.

- CHAPTER1 - PRELIMINARIES

- In the preliminaries we reviewd graph theory and quantum walks

- Introduced the search problem as originally posed by grover

- We gave an overview of the various implementation of the search problem, from the quantum walks search on complete graph to the global and local adiabatic search

- Lastly we showed the difference between adiabatic computation and quantum walks based search in order to prove that an adiabatic-quantum-walk search is not possible.

- CHAPTER2 - TIME DEPENDENT HAMILTONIAN

- In Chapter 1 we introduced the main topic of our work which is a quantum walks search algorithm with a time-dependent Hamiltonian. It is inspired by the adiabatic implementation but free of the constrains of the adiabatic theorem

- We introduced a few classes of interpolating schedules with the goal of improving the standard linear one of Farhi and Gutmann, drawing from the Roland-Cerf one

- We also considered the possibility of repeating the search multiple times if the search is not perfect, which made us take into account an initialization and measure time.

- CHAPTER2 - SELECTED TOPOLOGIES

- We then gave some reasoning on the choice of topologies, namely the cycle graph since it does not work and any improvement is a great results, and for completeness the complete graph since it is known to work without the time-dependent approach.

- CHAPTER2 - CHARACTERIZATION OF THE RESULTS

- We then introduced the parameters used for the comparison. We explained the difference between search and localization, and gave a quantitative measure of robustness.

- The search represents the finding of the solution with high probability - as close as unitary as possible - with the smallest time possible

- We called localization the finding of the solution with high probability without the need for the time optimization

- The probability depends on the combination of γ and T, paramteres that can be affected by noise/perturbation. We call T-robustness and γ-robustness the variation of probability due to noise in the T and γ parameteres, respectively

- CHAPTER2 - RESULTS FOR THE CYCLE GRAPH

- We then turned our attention to the cycle graph, for which we compared the time-dependent and time-independent approaches using the parameters previously introduced

- In terms of localization we discovered that the probability distribution of the time-independent approach does not increase with time. Therefore we can safely say that it does not show any localization properties.

- On the other hand the time-dependent Hamiltonian, being based on the adiabatic implementation, for large T allows us to reach unitary probability. More interestingly we can acheive high enough probability, in the order of $p = 0.8 - 0.9$ for much less time.

- We then studied the search performance of the two approaches. Since the time-independent approach does not get to unitary probability, and the time-dependent one is not optimized on the time we studied the search in terms of multiple run search. Therefore we introduced a new quantity τ which represents the minimum time necessary to get to unitary probability.

- We notice that τ requires to consider a minimum time $T_{\min}$ to be effective at comparing the two approaches. To be consistent with the standard grover search and quantum walks search on complete graph we constrain the time at $T_{\min} = \pi/2\sqrt{N}$

- We discover that for the time-dependent approach the Roland-Cerf interpolating schedules performes the best, followed by the standard linear one. sqrt and cbrt worsen significantly the performance.

- Comparing the time-dependent approch with Roland-Cerf and linear, and the time-independent we see that for small $N \approx 25$ the performance is similar, while for large N it get significantly different.

- If we look at the iterations distribution we can see that both approaches perform slightly better than the classical search up to $N \approx 57$. After that the time-independent approach performs worse than classical, while the time-dependent approach still has some advantage, although we're not able to make predictions for N larger than 71 which is the maximum graph dimension studied.

- We then studied the robusteness for both time and gamma. The time-independent approach has a very discontinous probability distribution, made of peaks and valley of high and low probability, leading to having the worse γ-robustness of all the approaches considered. The time-dependent approach in general has a smooth probability distribution regardless of the interpolating schedule considered.

- We discover that the linear interpolating schedule is more γ-robust compared to the Roland-Cerf one. Overall the time-dependent approach is way more robust then the time-independent one.

- In terms of T-robustness surprisingly the time-independent approach is more robust than the other two. However looking at the numberical values, which we remember don't have an absolute physical meaning, we see that

the difference is minimal - order of 10^{-2}, while for the γ robustness it is in the order of $10^{0.5-1}$.

- CHAPTER2 - RESULTS FOR THE COMPLETE GRAPH

- For completeness we also consider the complete graph. As discussed in the preliminaries, we're able to solve the search problem with both the adiabatic implementation - though in that scenario it is unstructured search - and the standard quantum walk.

- As mentioned by wong and in the preliminaries an adiabatic-quantum walk search is not possible

- in particular we compared the standard time-independent probability distribution with the time-dependent one. As expected the time-independent performs much better and the time-dependent approach is comparable in terms of performance, even considering the multiple run search.

- In terms of qualitative robustness we find that the time-dependent approach has as smoother probability distribution and therefore better robustness

- This particular case illustrates the importance of the interpolating schedule. In fact, the probability distribution evaluated with the linear interpolating schedule is barely able to acheive probability $p = 0.5$ for $T = N$

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