European Journal of Combinatorics 31 (2010) 120–128

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The Fine numbers refined ☆

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ARTICLE INFO

Article history:
Received 13 November 2008
Accepted 10 April 2009
Available online 16 June 2009

ABSTRACT

We give a short combinatorial proof of a Fine number generating function identity and then explore some of the ramifications in terms of random walks, friendly walkers, and ordered trees. The results are also generalized to obtain similar results including those in Motzkin and Schröder settings.

1. Introduction

The principal objects of study in this paper are 2-*Motzkin paths* and *amicable path pairs*. The 2-Motzkin paths are paths from $(0, 0)$ to $(n, 0)$ with the possible steps being $U = (1, 1)$, an up step, $D = (1, -1)$, a down step, and $L = (1, 0)$, a level step which can be either red or green. We also require that the path not go below the x-axis. If the path only touches the axis at $(0, 0)$ and $(n, 0)$ we say that the path is an *elevated* 2-*Motzkin path*. The number of 2-Motzkin paths from $(0, 0)$ to $(n, 0)$ is C_{n+1} where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number [5].

A 2-Motzkin path with no level steps at height 0 is called a *Fine path*, which means a Dyck path without peaks at level 1. The set of all Fine paths is denoted $\mathcal{F}$ and those ending at $(n, 0)$ are denoted by $\mathcal{F}_n$. $F_n = |\mathcal{F}_n|$ and these are the *Fine numbers* [4,5]. The generating function for the Fine numbers is denoted $F(z)$ or more briefly F and

$$F(z) = F = \sum_{n=0}^{\infty} F_n z^n = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})} = 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + \cdots.$$

One fact that surprised us was that the main identity: $(z^2 + 2z)F^2 - (2z + 1)F + 1 = 0$ could be shown without recourse to the Catalan numbers and their generating function.

☆ This paper was supported by Faculty Research Fund, Sungkyunkwan University, 2008.

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doi:10.1016/j.ejc.2009.04.003

Amicable path pairs will be defined and discussed in the last section and are a geometric version of 2-Motzkin paths.

The following results will be needed as we proceed.

(i) $[z^m]\,C^s = \frac{s}{2m+s}\binom{2m+s}{m}$

(ii) $[z^m]\,BC^s = \binom{2m+s}{m}$

(iii) $F_n \sim \frac{4}{9}C_n$

where C and B denote the Catalan generating function and the central binomial generating function, respectively.

Proofs of (i) and (ii) can be found in many places including Graham, Knuth, and Patashnik's Concrete Mathematics [6] and Wilf's Generatingfunctionology [10]. A proof of (iii) can be found in [4].

For asymptotic estimates the following lemma of Bender [1] can be easily applied and is very useful.

Theorem 1.1 (*Bender's Lemma*). *Suppose that* $A(z) = \sum_{n\geq 0} a_n z^n$ *and* $B(z) = \sum_{n\geq 0} b_n z^n$ *are two generating functions, and the radius of convergence of* $A(z)$ *is larger than that of* $B(z)$. *Let* $C(z) = \sum_{n\geq 0} c_n z^n$ *be the product* $A(z)B(z)$. *Suppose further that* b_{n-1}/b_n *approaches a limit* b *as* $n \to \infty$. *If* $A(b) \neq 0$, *then* $c_n \sim A(b)\,b_n$.

We also need some basic facts about the Catalan generating function and some close relatives:

- $$C = 1 + zC^2 = \frac{1}{1-zC} = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}z^n$$
$$= 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots$$
- $$B = 1 + 2zBC = \frac{1}{1-2zC} = \frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0}\binom{2n}{n}z^n$$
$$= 1 + 2z + 6z^2 + 20z^3 + 70z^4 + 252z^5 + \cdots$$
- $$F = 1 + z^2C^2F = \frac{1}{1-z^2C^2} = \frac{C}{1+zC} = \frac{1}{z}\cdot\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$
$$= 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + 57z^6 + \cdots$$
- $$C^2 = \frac{F}{1-2zF}$$
- $$\frac{B}{C} = \frac{B+1}{2}$$
- $$C = \frac{F}{1-zF}.$$

The generating function C^2 counts 2-Motzkin paths.

2. The main result

The first aim of this paper is to give a combinatorial proof of a Fine number generating function identity.

Theorem 2.1. *Let* F *be the generating function for the Fine numbers. Then we have the identity:*

$$(z^2 + 2z)F^2 - (2z+1)F + 1 = 0. \tag{1}$$

Proof. We start by rewriting the identity (1) as

$$F - 2zF^2 = 1 - 2zF + z^2F^2$$

and then as

$$F = 1 + \frac{z^2F^2}{1 - 2zF}.$$

We are now going to decompose $\mathcal{F}$ according to the number of level steps at height 1 in the first elevated subpath. If we let $\times$ be the symbol for the trivial path starting and ending at $(0, 0)$ we have

$$\mathcal{F} = \times \cup U\mathcal{F}D\mathcal{F} \cup U\mathcal{F}L\mathcal{F}D\mathcal{F} \cup U\mathcal{F}L\mathcal{F}L\mathcal{F}D\mathcal{F} \cup \cdots.$$

For instance in this decomposition, the term $U\mathcal{F}L\mathcal{F}D\mathcal{F}$ gives the paths with exactly one level step at height 1 in the first elevated subpath. The U and the D start and end the first elevated subpath and L is the unique level step at height 1. The level step can be of any of the two colors. If we convert this decomposition to generating functions, then we have;

$$\begin{aligned} F &= 1 + zFzF + zF(2z)FzF + zF(2z)F(2z)FzF + \cdots \\ &= 1 + z^2F^2 + 2z^3F^3 + 4z^4F^4 + \cdots \\ &= 1 + \frac{z^2F^2}{1 - 2zF}, \end{aligned}$$

which proves the identity (1). ■

Remark. The decomposition of $\mathcal{F}$ allows us to obtain very easily the bivariate generating function $G(t, z)$, where t marks the level steps at height 1. Indeed, one obtains

$$G = 1 + zFzG + zF(2tz)FzG + zF(2tz)F(2tz)FzG + \cdots$$

leading to $G = 1 + \frac{z^2F}{1-z^2F-2tzF}$. The alternative approach is more complicated: use the standard decomposition of 2-Motzkin paths and consider first the trivariate generating function $H(t, s, z)$, where t marks level steps at level 1 and s marks level steps at level 0.

Solving for F as given by (1) gives us that

$$F = \frac{1}{2z(z + 2)}(2z + 1 - \sqrt{1 - 4z}) = \frac{1}{z + 2}(1 + C).$$

Thus we have that

$$F_{n-1} + 2F_n = C_n, \quad n \geq 1$$

and thus $F_n/C_n \longrightarrow 4/9$. In the context of 2-Motzkin numbers the relevant ratio would be $F_n/C_{n+1} \longrightarrow 1/9$.

By writing the identity as

$$zF^2 = \frac{1 + 2z}{2 + z}F - \frac{1}{2 + z}$$

we can apply Bender's lemma to obtain an asymptotic result. First we note that $\frac{1}{2+z} = \frac{1}{2} \cdot \frac{1}{1+z/2}$ so for large n this term is making a negligible contribution. If we let $A(z) = F$ with radius of convergence $\alpha = 1/4$ and let $B(z) = \frac{1+2z}{2+z}$ with radius of convergence 2 then since $1/4 < 2$, Bender's lemma asserts that

$$\left[z^n\right] zF^2 \sim \left(\left[z^n\right] A(z)\right) \cdot B(\alpha) = F_n \cdot \frac{1 + 2(1/4)}{2 + (1/4)} = \frac{2}{3}F_n.$$

Since F has radius of convergence $1/4$ we have that $F_{n+1}/F_n \to 4$ and thus $[z^n] F^2 \sim \frac{8}{3}F_n$. This result will be used several times in Section 6.

3. Generalization

Some of these results are known but this approach is a bit different. Since the Fine numbers have been surveyed thoroughly in [4], we look at other ways this refined approach might be used. The first generalization is to allow b kinds of level steps instead of 2. Then if the generating function is now called $\bar{F}$ the result is

$$\bar{F} = 1 + \frac{z^2\bar{F}^2}{1 - bz\bar{F}} = \frac{1 + bz - \sqrt{1 - 2bz + b^2z^2 - 4z^2}}{2(bz + z^2)}.$$

If we allow level steps at height 0, we have the generating function for b-Motzkin numbers denoted by $\bar{M}$ and we then have

$$\bar{M} = 1 + bz\bar{M} + z^2\bar{M}^2 = \frac{1 - bz - \sqrt{1 - 2bz + b^2z^2 - 4z^2}}{2z^2}.$$

From this it follows that

$$\bar{F} = \frac{z\bar{M} + b}{z + b}.$$

Since $\bar{M}_n/\bar{M}_{n-1} \longrightarrow b + 2$ and $\bar{F}_{n-1} + b\bar{F}_n = \bar{M}_{n-1}$ for $n \geq 1$ we get the limit

$$\frac{\bar{F}_n}{\bar{M}_n} \longrightarrow \frac{1}{(b+1)^2}.$$

When $b = 1$ we have the Motzkin numbers and the Fine analog (i.e. no level steps at height 0) is the *gamma* (also called *Riordan*) *numbers* so if we let Γ be the generating function we have

$$\Gamma = 1 + \frac{z^2\Gamma^2}{1 - z\Gamma} = \frac{1}{2(z + z^2)}(1 + z - \sqrt{1 - 2z - 3z^2}).$$

If m denotes the Motzkin number generating function we obtain

$$\Gamma = \frac{zm + 1}{1 + z}.$$

Thus $\Gamma_n + \Gamma_{n-1} = m_{n-1}$, and Γ_n/Γ_{n-1} approaches 3 while Γ_n/m_n approaches 1/4. As numerical reassurance we note that

$$\frac{\Gamma_{48}}{m_{48}} = \frac{8554387054045559778}{33449647319445900942} \doteq .25574.$$

If $b = 3$ the sequence counts tree-like polyhexes, a famous result of Harary and Read [7] concerning ways to attach benzene rings together. The generating function for these 3-Motzkin paths is given by

$$\begin{aligned}\bar{M} &= 1 + 3z\bar{M} + z^2\bar{M}^2 = \frac{1}{2z^2}(1 - 3z - \sqrt{1 - 6z + 5z^2}) \\ &= 1 + 3z + 10z^2 + 36z^3 + 137z^4 + 543z^5 + 2219z^6 + \cdots.\end{aligned}$$

If we have no level steps at height 0 then

$$\begin{aligned}\bar{F} &= 1 + \frac{z^2\bar{F}^2}{1 - 3z\bar{F}} = \frac{1 + 3z - \sqrt{1 - 6z + 5z^2}}{2(3z + z^2)} \\ &= \frac{z\bar{M} + 3}{3 + z} = 1 + z^2 + 3z^3 + 11z^4 + 42z^5 + 167z^6 + 684z^7 + O\left(z^8\right),\end{aligned}$$

which is the sequence A117641 of Sloane's Encyclopedia of Integer Sequences [9]. Hence we get $\bar{F}_n/\bar{M}_n \longrightarrow 1/16$.

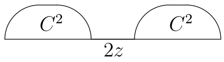

Fig. 1. 2-Motzkin path with a marked level step at height 0.

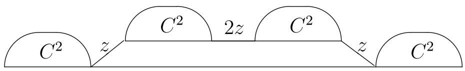

Fig. 2. 2-Motzkin path with a marked level step at height 1.

To generalize in another direction we consider Schröder paths using steps $U = (1, 1), D = (1, -1)$ and $L = (2, 0)$ with the usual condition of not going below the x-axis. The generating function for such paths is

$$\begin{aligned} R(z) &= \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z} \\ &= 1 + 2z + 6z^2 + 22z^3 + 90z^4 + 394z^5 + 1806z^6 + 8558z^7 + \cdots. \end{aligned}$$

The Fine analog here are the little Schröder paths with no level steps at height 0 and with the generating function $S(z) = \frac{1+R(z)}{2}$ so that $S_n/R_n = 1/2$ for $n \geq 1$.

4. The number of level steps

The total number of level steps at height 0 for all 2-Motzkin paths has the generating function

$$C^2(2z)C^2 = 2zC^4 = 2z + 8z^2 + 28z^3 + 96z^4 + \cdots,$$

and the guiding picture is Fig. 1.

Note that C^2 is the generating function for the sequence counting the 2-Motzkin paths. Recall that $[z^m]C^s = \frac{s}{2m+s}\binom{2m+s}{m}$. Thus we have

$$\begin{aligned} [z^n]2zC^4 = 2[z^{n-1}]C^4 &= 2 \cdot \frac{4}{2(n-1)+4}\binom{2n+2}{n-1} = \frac{4}{n+1}\binom{2n+2}{n-1} \\ &= \frac{4n}{(n+1)(n+3)}\binom{2n+2}{n}. \end{aligned}$$

The number of 2-Motzkin paths is $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \frac{1}{n+1}\binom{2n+2}{n}$. Therefore the average number of level steps at height 0 is

$$\frac{[z^n]2zC^4}{C_{n+1}} = \frac{4n}{n+3} \to 4 \quad \text{as } n \to \infty.$$

A similar, but more complicated, computation shows that the variance is

$$\frac{12n^3 + 12n^2 - 24n}{(n+3)^2(n+4)} \to 12 \quad \text{as } n \to \infty.$$

For level steps at height 1 we similarly have Fig. 2.

The generating function for all level steps at height 1 is

$$C^2 \cdot z \cdot C^2 \cdot 2z \cdot C^2 \cdot z \cdot C^2 = 2z^3C^8$$

and

$$[z^n]2z^3C^8 = \frac{8}{n+1}\binom{2n+2}{n-3} = \frac{8}{n+1}\frac{n(n-1)(n-2)}{(n+5)(n+4)(n+3)}\binom{2n+2}{n}.$$

Hence the average number of level steps at height 1 is

$$\frac{[z^n]2z^3C^8}{C_{n+1}} = \frac{8n(n-1)(n-2)}{(n+5)(n+4)(n+3)} \to 8 \quad \text{as } n \to \infty.$$

For the total number of level steps at height k we have the generating function $2z^{2k+1}C^{4k+4}$. This gives us a limiting value of $4(k+1)$ steps at height k where k is fixed and n goes to infinity. The convergence slows considerably as k increases.

What is the total number of level steps? The generating function is

$$\begin{aligned}
&2zC^4(1 + z^2C^4 + (z^2C^4)^2 + (z^2C^4))^3 + \cdots = 2zC^4 \cdot \frac{1}{1 - z^2C^4} \\
&= 2zC^3 \cdot \frac{C}{1 - zC^2} \cdot \frac{1}{1 + zC^2} = 2zC^3 \cdot B \cdot \frac{1}{C} = 2zBC^2 \\
&= \sum_{n\geq 1} 2\binom{2n}{n-1} z^n = 2z + 8z^2 + 30z^3 + 112z^4 + 420z^5 + \cdots.
\end{aligned}$$

It seems intuitively very reasonable that the proportion of level steps should approach $\frac{1}{2}$ since 2 of the 4 possible steps at positive height are level steps. More precisely the proportion is

$$\frac{2\binom{2n}{n-1}}{nC_{n+1}} = \frac{n+2}{2n+1} \to \frac{1}{2} \quad \text{as } n \to \infty.$$

A second way to approach the number of level steps at height 0 is to let $m_{n,k}$ be the number of paths of length n with k level steps at height 0. From a simple decomposition picture, the generating function of the kth column is given $F(2zF)^k$ for $k = 0, 1, 2, \ldots$. Alternatively, from a different picture one can derive that the bivariate generating function $G(t, z)$, where t marks number of level steps at height 0, is given by $G = 1 + 2tzG + z^2C^2G$, leading easily to $[t^k]G = 2^kz^kF^{k+1}$. This yields the infinite lower triangular matrix

$$(m_{n,k})_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 0 & 2 & & & & \\ 1 & 0 & 4 & & & \\ 2 & 4 & 0 & 8 & & \\ 6 & 8 & 12 & 0 & 16 & \\ 18 & 26 & 24 & 32 & 0 & 32 \\ & & & \cdots & & \end{bmatrix}.$$

Multiplying by the column vector $(1, 1, 1, \ldots)^{\mathrm{T}}$ gives the row sums $1, 2, 5, 14, 42, \ldots$. Incidentally, the sequence determined by the triangular matrix is A096794 in [9]. To find the total number of level steps we multiply by $(0, 1, 2, 3, \ldots)^{\mathrm{T}}$ which has the generating function $\frac{z}{(1-z)^2}$ and which in turn gives us

$$F \cdot \frac{2zF}{(1 - 2zF)^2} = 2z(C^2)^2 = 2zC^4$$

as the generating function for the total number of level steps at height 0. Note that the matrix $(m_{n,k})_{n,k\geq 0}$ is a Riordan array $(F, 2zF)$ which is an element of the Riordan group. To derive the above identity, we used the fundamental theorem for Riordan matrix (see [8]): $(F, 2zF)\frac{z}{(1-z)^2}$. We will not pursue this connection, but we also note that this matrix is the unique 2-Bell matrix with row sums $(C_{n+1})_{n\geq 0}$ (see [8]).

5. Analogous identities

What happens when other families of paths are classified by the number of level steps at height 1 in the first elevated path? For 2-Motzkin paths themselves we obtain

$$C^2 = \frac{1}{1-2z} + \frac{1}{1-2z} z^2 (F + 2zF^2 + (2z)^2 F^3 + \cdots) C^2 \tag{2}$$

$$= \frac{1}{1-2z} + \frac{z^2 FC^2}{1-2z} \cdot \frac{1}{1-2zF}. \tag{3}$$

The term $\frac{1}{1-2z}$ counts paths which always stay on the x-axis while z^2 represents the up and down steps starting and ending the first elevated path. The identity (3) is simple to prove using the identity $C^2 = \frac{F}{1-2zF}$ which comes by partitioning 2-Motzkin paths by number of level steps at height 0.

If we modify our 2-Motzkin paths by only allowing red steps at height 0 the appropriate generating function is C and we now obtain

$$C = \frac{1}{1-z} + \frac{1}{1-z} z^2 (F + 2zF^2 + (2z)^2 F^3 + \cdots) C$$

$$= \frac{1}{1-z} + \frac{z^2 FC}{1-z} \cdot \frac{1}{1-2zF}.$$

6. Applications and amicable path pairs

A *path pair* is a pair of paths both starting at $(0, 0)$, each comprised of unit East and North steps and ending after n steps at a common end point $(k, n-k)$. If one path never goes below the other these are called *amicable* (called simply path pairs in [4]) and the number of possible pairs after n steps is C_{n+1} with the generating function C^2 (see [4], Proposition 6). We can subdivide amicable paths in rather natural ways.

An amicable pair is *reluctant* (called fat path pair in [4] but parallelogram polyomino in numerous other sources) if they only meet at the beginning and end points. A pair is *shy* (path pair with no joint steps in [4]) if they can meet occasionally in the middle of the walk but then immediately go separate ways. A pair is *sociable* (amicable but not shy) if they share at least one edge during the walk. A pair is *inseparable* if they always take the same edges.

The building blocks for amicable paths are the reluctant and inseparable pairs and these have generating functions z^2C^2 and $1/(1-2z)$ respectively. The name for a figure bounded by a reluctant pair is a *parallelogram polyomino*. If we join one or more of these parallelogram polyominos together with the end point of one being the starting point of the next we have the graph of a shy pair with a "bumper car" look.

There is a straightforward bijection from amicable pairs to 2-Motzkin paths. If the upper path goes north while the lower path goes east this maps to an Up step, if both paths go east we get a red level step, if both paths go north we have a green level step, and if the upper path goes east while the lower path goes north this maps to a Down step. Since the two paths meet at the end we have an equal number of Ups and Downs and in the middle we always have at least as many Ups as Downs. The map is easily translated back from 2-Motzkin paths to path pairs so we do have a bijection.

We summarize the equivalences and generating functions in a table.

Path pair type	2-Motzkin path	G.F.	Area
Amicable	2-Motzkin	C^2	$\frac{z^2C^2}{1-4z}$
Shy	Fine	F	$\frac{z^2F^2}{1-4z}$
Reluctant	Elevated 2-Motzkin path	z^2C^2	$\frac{z^2}{1-4z}$
Sociable	at least one level step at height 0	$C^2 - F$	$\frac{z^2(C^2-F^2)}{1-4z}$
Bicolored path pair, crossing allowed	Grand 2-Motzkin path	B	$\frac{2z^2B^2}{1-4z}$

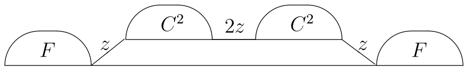

Fig. 3. Fine path with a marked level step at height 1.

There are now several natural questions to consider. What proportion of amicable pairs are reluctant, shy, or sociable? On average how sociable are amicable path pairs, that is how many edges or points do they have in common?

At this point we omit most of the proofs both to keep the paper short but also since they are much the same as earlier proofs. However to go through one such result we could ask about the number of points shared by amicable pairs. Translating to 2-Motzkin paths we are asking for the number of points at height 0. The generating function for 2-Motzkin paths with a distinguished point at height 0 is $C^2 \cdot 1 \cdot C^2 = C^4$ where the 1 is the generating function for a single distinguished point. Thus the total number of points at height 0 is

$$\left[z^n\right] C^4 = \frac{4}{2n+4}\binom{2n+4}{n}$$

and we have the sequence $1, 4, 14, 48, 165, 572, \ldots$. This is a classical sequence with a long history going back at least to Cayley [2].

Thus the average number of points at height 0 per 2-Motzkin path is

$$\frac{4}{2n+4}\binom{2n+4}{n} \Big/ \frac{1}{n+2}\binom{2n+2}{n+1} = 4\frac{(2n+3)\,(n+1)}{(n+3)\,(n+4)} \to 8.$$

For shy path pairs there are fewer points in common. The generating function for the total number of common vertices is $F \cdot 1 \cdot F = F^2$. Since $[z^n]\, F^2 \sim \frac{8}{3} F_n$, the average number of points in common approaches $8/3$. Since there are common points at the start and finish this leaves an average number of common points in the middle approaching $2/3$.

Using similar techniques, the probability that there are no common points other than the end points i.e. a shy path is actually reluctant is

$$\frac{[z^n]\, z^2 C^2}{[z^n]\, F} = \frac{C_{n-1}}{F_n} \sim \frac{\frac{1}{4} C_n}{\frac{4}{9} C_n} = \frac{9}{16}.$$

Equivalently we are counting the number of elevated 2-Motzkin paths. If the two paths meet once in the middle we have $\frac{2}{n-2}\binom{2n-4}{n}$ such paths and the proportion of such paths is

$$\frac{(n+2)\,(n+1)\,(n-3)}{4\,(2n+1)\,(2n-1)\,(2n-3)} \to \frac{1}{32}.$$

How would the absence of level steps at height 0 influence the number of level steps at height 1. The generating function for the total number of level steps at height 1 for Fine paths is $F^2 z^2 C^4 (2z)$ where the picture is as given in Fig. 3.

By the identity $\frac{1}{1-z^2C^2} = F$, we have $z^2C^2 = \frac{F-1}{F}$ and

$$2z^3C^4F^2 = \frac{2}{z}(F-1)^2.$$

Thus the average number is

$$\frac{[z^n] 2z^3C^4F^2}{[z^n]F} = \frac{2[z^{n+1}](F-1)^2}{[z^n]F} \to \frac{16}{3} \quad \text{as } n \to \infty.$$

It is known that the area of a reluctant pair of paths of length n is 4^{n-1}. One place this is discussed is [11]. This is easily extended to shy and amicable pairs as the areas are additive.

We can also view these results in terms of ordered (planar) trees. If we let each leaf at distance 1 from the root be either red or green we again get C^2 as the generating function counting such ordered trees. In this context a Fine tree is one with no leaves at distance 1 from the root. Thus the total number of leaves at distance 1 from the root is again $\frac{4n}{(n+1)(n+3)}\binom{2n+2}{n}$ and the average number is $\frac{4n}{n+3}$.

Amicable pairs are closely related to the idea of vicious or friendly walkers and are essentially 2-watermelons in the terminology of Chen et al. [3].

Acknowledgment

The authors are grateful to an anonymous referee for the help in improving the exposition and for suggesting the Remark in Section 2.

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