L1 Stochastic Methods 30 sett 2024

marco.ferrante@unipd.it

A probabilistic model $\rightarrow$ define the set of all possible outcomes

<u>Ex1</u>: Coin toss $<^{T}_{H}$ $\Rightarrow$ $\Omega = \{H, T\}$

$\boxed{1.}$ $\Omega :=$ a set that contains all the possible outcomes of a random experiment

Ω is called SAMPLE SPACE : a way to describe all the possible outcomes

Ω è un insieme e posso fare operazioni tra insiemi : $\cup, \cap, \setminus, {}^c, \Delta$

<u>Ex2</u>. flip 2 coins $\Rightarrow$ $\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

<u>Ex3</u>. toss a dice $\Rightarrow$ $\Omega = \{1,2,3,4,5,6\}$ 

<u>Ex4</u>. toss 2 dice $\Rightarrow$ $\Omega = \{(1,1), (1,2), \dots, (6,6)\}$

$$= \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$$

$\boxed{2.}$ Events

if Ω is finite, the set of the events will be $2^{\Omega} = \mathcal{P}(\Omega) = \left\{ \begin{array}{l} \text{set of all the} \\ \text{subsets of } \Omega \end{array} \right\}$

ex: $\Omega = \{H, T\}$

$$\mathcal{P}(\Omega) = \{\{H\}, \{T\}, \{H,T\}, \{\}\}$$

Ex. $\Omega = \{1,2,3,4,5,6\}$

$2^{\Omega} \overset{?}{=} \{ \{\},$
$\{1\}, \{2\}, \{3\}, \ldots$
$\{1,1\}, \{1,2\}, \{1,3\}, \ldots$
$\{1,2,3\}, \ldots$
$\{1,2,3,4\}, \ldots$
$\{1,2,3,4,5\}, \ldots$
$\{1,2,3,4,5,6\} \}$

$|\Omega| = 6 \longrightarrow$ Cardinality of Ω
number of elements in Ω

$|2^{\Omega}| = 2^6 = 2^{|\Omega|}$, $\Omega \supseteq A = \{1,2,3,4,5,6\}$

$2 \times 2 \times 2 \times 2 \times 2 \times 2 \ldots$

there are 2 choices:
→ is present
→ is not present

Ex: toss 2 dice

$|\Omega| = |\{(i,j) : i,j \in \{1,2,\ldots,6\}\}|$
$= 6 \cdot 6 = 36$

$2^{|\Omega|} = 2^{36}$

Ω = SAMPLE SPACE
SET OF EVENTS $\longrightarrow 2^{\Omega}$

$\boxed{\mathcal{A}} \subseteq 2^{\Omega} = \{A : A \subseteq \Omega\}$

σ - FIELD

$\mathcal{A}$ = σ-Field

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) if $(A_n)_{n\in\mathbb{N}}$ is a infinite sequence of events $(A_n \in \mathcal{A}, \forall n)$

then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, $n \in \mathbb{N}$

$\boxed{\text{a σ-Field must contains } \Omega \text{ plus every complement of every element present}}$

$\mathcal{A} \subsetneq 2^{\Omega}$

Ex

$\mathcal{A}_1 = \{\phi, \Omega\}$ is a σ-FIELD?

(i) $\Omega \in \mathcal{A}_1$? yes

(ii) $A \in \mathcal{A}_1$? $\Rightarrow A^c \in \mathcal{A}_1$ yes $\Rightarrow A = \begin{cases} \phi \\ \Omega \end{cases}$

$A_i \in \mathcal{A}_1 \Rightarrow \bigcup_n A_n \in \mathcal{A}_1$, $\begin{matrix} \phi^c = \Omega \\ \Omega^c = \phi \end{matrix}$ yes

Ex

$\Omega = \{H, T\}$

$2^{\Omega} = \{\phi, \{H\}, \{T\}, \{H,T\}\}$

$\mathcal{A} \subseteq 2^{\Omega}$, $\mathcal{A} = \{\phi, \{H\}\}$

$\Omega \notin \mathcal{A} \Rightarrow \mathcal{A}$ is not a σ-Field

Ex $\Omega = \{1,2,3,4,5,6\}$ $\quad \mathcal{A} = 2^{\Omega}$

$A = \{2,4,6\}$ $\rightarrow$ rappresenta l'unione di possibili outcome di un esperimento: dal lancio di 1 dado ottengo 2 o 4 o 6 (un pari)

$B = \{1,2\}$

$A, B \in \mathcal{A} \Rightarrow$
$$A \cup B = \{1,2,4,6\}$$
$$A \cap B = \{2\}$$
$$A^c = \{1,3,5\}$$

l'unione è sempre possibile, è una operazione chiusa in una σ-algebra

$(A \cup B)^c = A^c \cap B^c$

Come definisco un modello di probabilità:

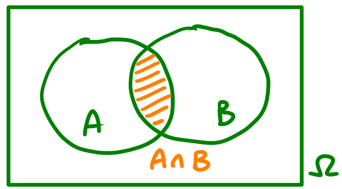
1. Ω = sample space (outcomes)
2. $\mathcal{A} \subseteq 2^{\Omega}$ σ-field (events)
3. A function $\mathbb{P}$ is called "probability" $\mathbb{P}: \mathcal{A} \longrightarrow [0,1] \subseteq \mathbb{R}$

   (i) $\mathbb{P}(\Omega) = 1$

   (ii) if $(A_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint event, i.e.

   $A_n \in \mathcal{A} \quad \forall n$, $A_n \cap A_m \neq \emptyset \quad \forall n \neq m$, then

$$\mathbb{P}\Big[\underbrace{\bigcup_{n \in \mathbb{N}} A_n}_{\in\, \mathcal{A}}\Big] = \sum_{n \in \mathbb{N}} \mathbb{P}[A_n]$$



Remark: (ii) $\Rightarrow A, B \in \mathcal{A}$, $A \cap B = \emptyset$

$$\mathbb{P}[A \cap B] = \mathbb{P}(A) + \mathbb{P}(B)$$

Definition : the triple $(\Omega, \mathcal{A}, \mathbb{P})$ is called probability space

Ex 1

$\Omega = \{H, T\}$ (flip of a coin)

$\mathcal{A} = 2^{\Omega} = \{\phi, \Omega, \{H\}, \{T\}\}$

$\mathbb{P} : \mathcal{A} \longrightarrow [0,1]$

$\mathbb{P}[\{H\}] = p \in [0,1]$ $\quad \mathbb{P}[A^c] = 1 - \mathbb{P}[A]$

$\mathbb{P}[\Omega] = 1$

$\mathbb{P}[\phi] = 0$

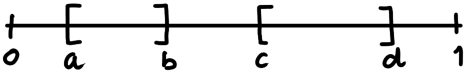
$\Omega = A \cup A^c \Rightarrow \boxed{\mathbb{P}[\Omega]} = \mathbb{P}[A \cup A^c] \overset{(ii)}{=} \mathbb{P}[A] + \mathbb{P}[A^c] \overset{(i)}{=} \boxed{1}$

$\mathbb{P}[\{H\}] = p \Rightarrow \mathbb{P}[\{T\}] = 1 - p$ perchè $\{T\} = \{H\}^c$

$\Omega = [0,1] \subseteq \mathbb{R}$, $[a,b] \subseteq [0,1]$

$\mathbb{P}\big[[a,b]\big] = b - a$

$\mathbb{P}[\Omega] = 1 - 0 = 1$



$$\mathbb{P}\Big[[a,b]\cup[c,d]\Big] = \mathbb{P}\Big[[a,b]\Big] + \mathbb{P}\Big[[c,d]\Big]$$

$$\mathbb{P}: \underbrace{\mathcal{A}}_{\substack{\| \\ 2^{\Omega} = 2^{[0,1]}}} \longrightarrow [0,1]$$

$$\mathbb{P}: \underbrace{\mathcal{B}([0,1])}_{\text{Borel } \sigma\text{-field}} \longrightarrow [0,1]$$