Problem Set 0 Solutions

Problem 1

(a) Describe the probability space for rolling two dice.

Let the sample space be

$$\Omega = \{(i, j) : 1 \leq i, j \leq 6\}$$

where i and j represent the numbers on the first and second die, respectively. The total number of outcomes is $6 \times 6 = 36$.

Since each outcome is equally likely, the probability of each outcome is

$$P((i, j)) = \frac{1}{36}, \quad \text{for all } (i, j) \in \Omega.$$

(b) Compute the probability of obtaining an even number.

The possible sums of two dice are $2, 3, 4, \ldots, 12$. The even sums are $2, 4, 6, 8, 10, 12$. We count the number of outcomes for each even sum:

$$\begin{aligned}
\text{Sum} = 2 &: (1,1) \quad \text{(1 outcome)} \\
\text{Sum} = 4 &: (1,3), (2,2), (3,1) \quad \text{(3 outcomes)} \\
\text{Sum} = 6 &: (1,5), (2,4), (3,3), (4,2), (5,1) \quad \text{(5 outcomes)} \\
\text{Sum} = 8 &: (2,6), (3,5), (4,4), (5,3), (6,2) \quad \text{(5 outcomes)} \\
\text{Sum} = 10 &: (4,6), (5,5), (6,4) \quad \text{(3 outcomes)} \\
\text{Sum} = 12 &: (6,6) \quad \text{(1 outcome)}
\end{aligned}$$

The total number of outcomes with even sums is $1 + 3 + 5 + 5 + 3 + 1 = 18$.

Thus, the probability of getting an even sum is

$$P(\text{even sum}) = \frac{18}{36} = \frac{1}{2}.$$

Problem 2

(a) Describe the probability space for selecting two balls without replacement.

There are 6 balls labeled 1 to 6. The sample space consists of all possible pairs of balls selected without replacement:

$$\Omega = \{(i, j) : 1 \leq i < j \leq 6\}.$$

The number of possible outcomes is $\binom{6}{2} = 15$. Since each outcome is equally likely, the probability of each outcome is

$$P((i, j)) = \frac{1}{15}.$$

(b) Compute the probability of obtaining two balls with consecutive numbers.

The pairs that correspond to consecutive numbers are: $(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)$. There are 5 such pairs, so the probability of selecting two consecutive numbers is

$$P(\text{consecutive numbers}) = \frac{5}{15} = \frac{1}{3}.$$

Problem 3

Let $\Omega = \mathbb{R}$ and define

$$\mathcal{A} = \{A \subset \mathbb{R} : A \text{ is countable}\} \cup \{A \subset \mathbb{R} : A^c \text{ is countable}\}.$$

(a) Prove that $\mathcal{A}$ is a σ-field.

We need to verify the three properties of a σ-field:

1. $\mathbb{R} \in \mathcal{A}$: The set $\mathbb{R}$ is uncountable, but $\mathbb{R}^c = \emptyset$, which is countable. Thus, $\mathbb{R} \in \mathcal{A}$.

2. Closed under complements: If $A \in \mathcal{A}$, either A is countable or A^c is countable. In both cases, $A^c \in \mathcal{A}$.

3. Closed under countable unions: Let $A_1, A_2, \dots \in \mathcal{A}$. There are two cases:
   - If each A_i is countable, then $\bigcup A_i$ is countable, so $\bigcup A_i \in \mathcal{A}$.
   - If A_i^c is countable for each i, then $\bigcap A_i^c$ is countable, so $\bigcup A_i \in \mathcal{A}$.

Thus, $\mathcal{A}$ is a σ-field.

(b) Prove that $(-\infty, 0] \notin \mathcal{A}$.

The set $(-\infty, 0]$ is uncountable, and its complement $(0, \infty)$ is also uncountable. Hence, $(-\infty, 0] \notin \mathcal{A}$.

Problem 4

Let $\Omega = \mathbb{N}$ and define

$$\mathcal{A} = \{A \subset \mathbb{N} : A \text{ or } A^c \text{ is finite}\}.$$

Show that $\mathcal{A}$ is a field but not a σ-field.

- $\mathcal{A}$ is a field because it is closed under finite unions, intersections, and complements.
- $\mathcal{A}$ is not a σ-field because the infinite union $\bigcup_{n=1}^{\infty} \{n\} = \mathbb{N} \notin \mathcal{A}$, since $\mathbb{N}^c = \emptyset$ is finite.

Problem 5

(a) Prove that the intersection of σ-fields is a σ-field.

Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of σ-fields. The intersection $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-field because:

- It contains Ω,
- It is closed under complements, and
- It is closed under countable unions.

(b) Define the minimal σ-field containing $\{1\}$ and $\{2, 4\}$.

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and define the sets $A = \{1\}$ and $B = \{2, 4\}$. The minimal σ-field containing A and B must contain:

$$\emptyset, A, B, A^c, B^c, A \cup B, A \cap B, \Omega.$$

The minimal σ-field is the collection of all subsets of Ω formed by unions and complements of these sets.