Notes on Certainty Equivalent, Risk Aversion, and Arrow-Pratt Approximation

Bruno M. Parigi

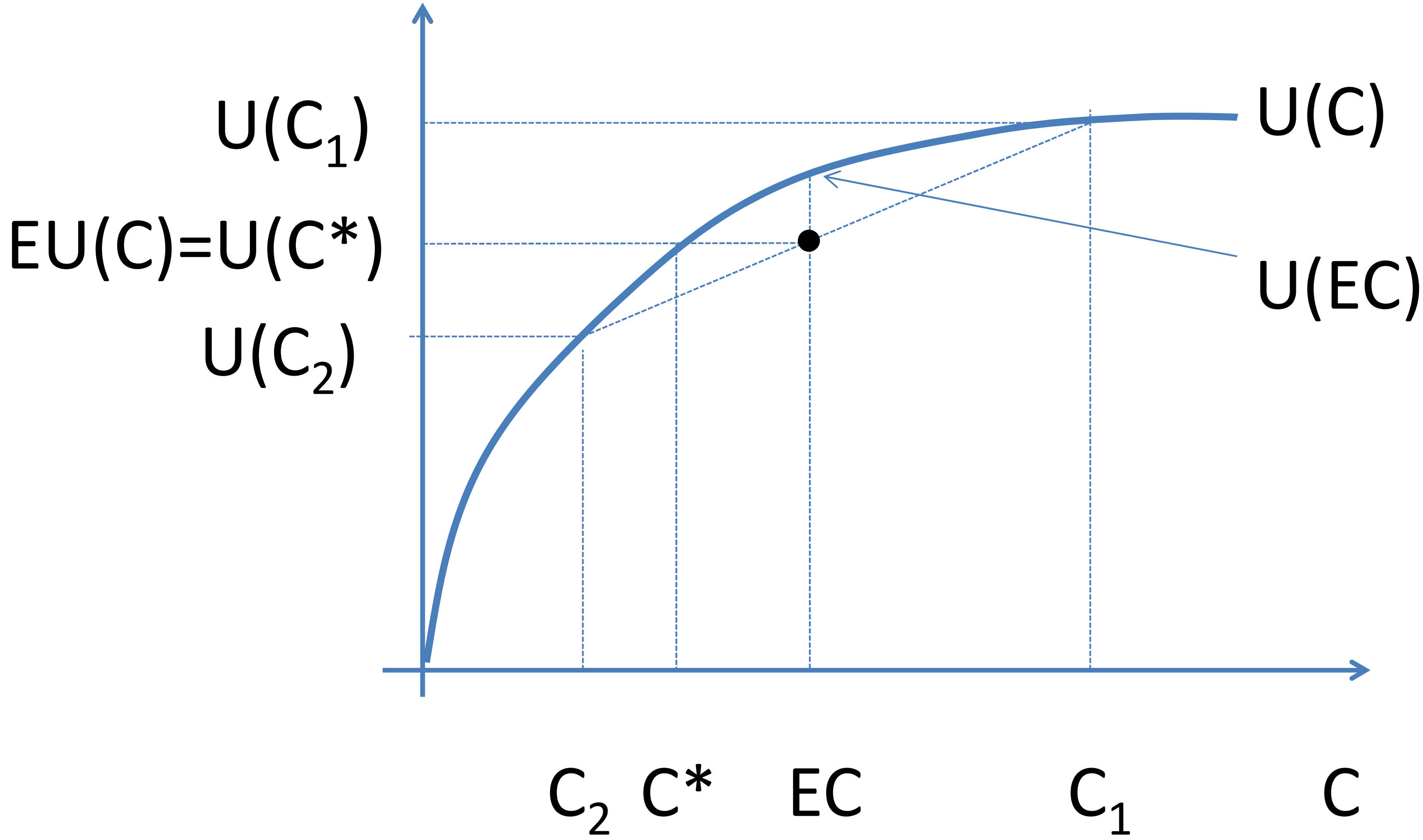
Certainty equivalent

Let C be random consumption that can take on two values C_1 and C_2 with probability p_1 and p_2 respectively and average consumption $EC = C_1 p_1 + C_2 p_2$ $\tilde{c}$

Expected utility of random consumption C is $EU(C) = U(C_1)p_1 + U(C_2)p_2 \equiv V(C) = U(C^*)$

where C^* is certainty equivalent consumption

Certainty equivalent consumption C*



Certainty equivalent and Risk premium

- Certainty equivalent consumption is that level of consumption that gives consumer same satisfation of random consumption C that can take on two values C_1 and C_2 with probability p_1 and p_2
- $\rho \equiv EC - C^*$ = risk premim
- Risk premium > 0 iff $U'' < 0$ (risk averse)
- Risk averse consumer would be willing to sacrifice something with respect to average consumption to avoid risk of random C

Certain equivalent and … (cont.ed)

- Risk premium = 0 iff $U'' = 0$ (risk neutral)
- Risk premium < 0 iff $U'' > 0$ (risk lover)
- Hence risk premium linked to shape of utility function

How to measure Risk aversion

- Absolute risk aversion coefficient
- $A(C) = - U''(C)/U'(C)$
- Problem with $A(C)$: it depends on the unit of measurement of the phenomen

How to measure … (cont.ed)

- Solution: use (negative of) elasticity with respect to consumption of marginal utility of consumption = RELATIVE RISK AVERSION COEFFICIENT = *R(C)*
- $R(C) = -[dU'(C)/U'(C)]/[dC/C] =$
  $-[dU'(C)/dC][C/U'(C)] = -[CU''(C)]/U'(C)$

Relative risk aversion coefficient

- *R(C)* is an elasticity; hence it a is ratio between two numbers; hence it is a pure number
- Example: if *R(C) = 1* it means that if *C* increases by 1%, then marginal utility of consumption declines by 1%
- Observe that *R(C) = CA(C)*

- coeff. of rel. risk aver. measures extent to which marginal utility of additional consumption declines; see power utility

Risk Aver. Coeff. and Risk Premium

- Consider a random consumption C such that C= $\overline{c}$ + h, where h can take on two values + δ and – δ with same probability and EC = $\overline{c}$
- Consider a second degree Taylor expansion of U(C) around $\overline{c}$.

$$U(c) = U(\overline{c}) + hU'(\overline{c}) + \frac{h^2}{2} U''(\overline{c}) + rest$$

Risk Aver. Coeff. … (Cont.ed)

- Applying expected value operator it becomes

$$EU(c) = U(\overline{c}) + EhU'(\overline{c}) + E\frac{h^2}{2}U''(\overline{c}) + E(rest)$$

$$EU(c) = U(\overline{c}) + E\frac{h^2}{2}U''(\overline{c}) + E(rest)$$

$$EU(c) = U(\overline{c}) + \frac{\delta^2}{2}U''(\overline{c}) + E(rest)$$

Risk Aver. Coeff. … (Cont.ed)

- Recall that certainty equivalent C* is such that

  $\rho = \overline{c} - c*$ where $EC = \overline{c}$

- Consider now a Taylor expansion of $U(c*)$

$$U(c*) = U(\overline{c} - \rho) = U(\overline{c}) - \rho U'(\overline{c}) + rest$$

Risk Aver. Coeff. … (Cont.ed)

- If δ small then rests can be ignored
- Observe that by definition EU(C) = U(C*)
- Hence

$$EU(c) \simeq U(\overline{c}) + \frac{\delta^2}{2} U''(\overline{c}) \simeq U(\overline{c}) - \rho U'(\overline{c}) \simeq U(c*)$$

$$\rho \simeq -\frac{\delta^2}{2} \frac{U''(\overline{c})}{U'(\overline{c})} = \frac{1}{2} A(\overline{c}) Var(c)$$

Arrow-Pratt Approximation

$$\rho \simeq -\frac{\delta^2}{2}\frac{U''(\overline{c})}{U'(\overline{c})} = \frac{1}{2}A(\overline{c})Var(c)$$

Meaning of Arrow-Pratt Approximation

- It allows to separate impact on risk premium of two components
  - Risk of phenomenon: Var(C)
  - Preferences: Absolute risk aversione coefficient evaluated at the mean consumption EC
- A(EC) measuress the max amount that an individual with mean consumption EC and utility function U(C) is willing to pay to eliminate a small risk with variance 2
- Observe that if A(EC) $\uparrow$ then ρ $\uparrow$

Question 2.
Intermediate Exam, Nov. 19, 2013

An individual has a utility function $U(C) = \ln(C)$ where C denotes consumption. Suppose C is risky and can take values 1,2,3, and that 1 and 3 happen each with the same probability of 1/100.

A) Find the amount the individual would be happy to pay to eliminate completely the risk.

B) Find the certainty equivalent consumption.

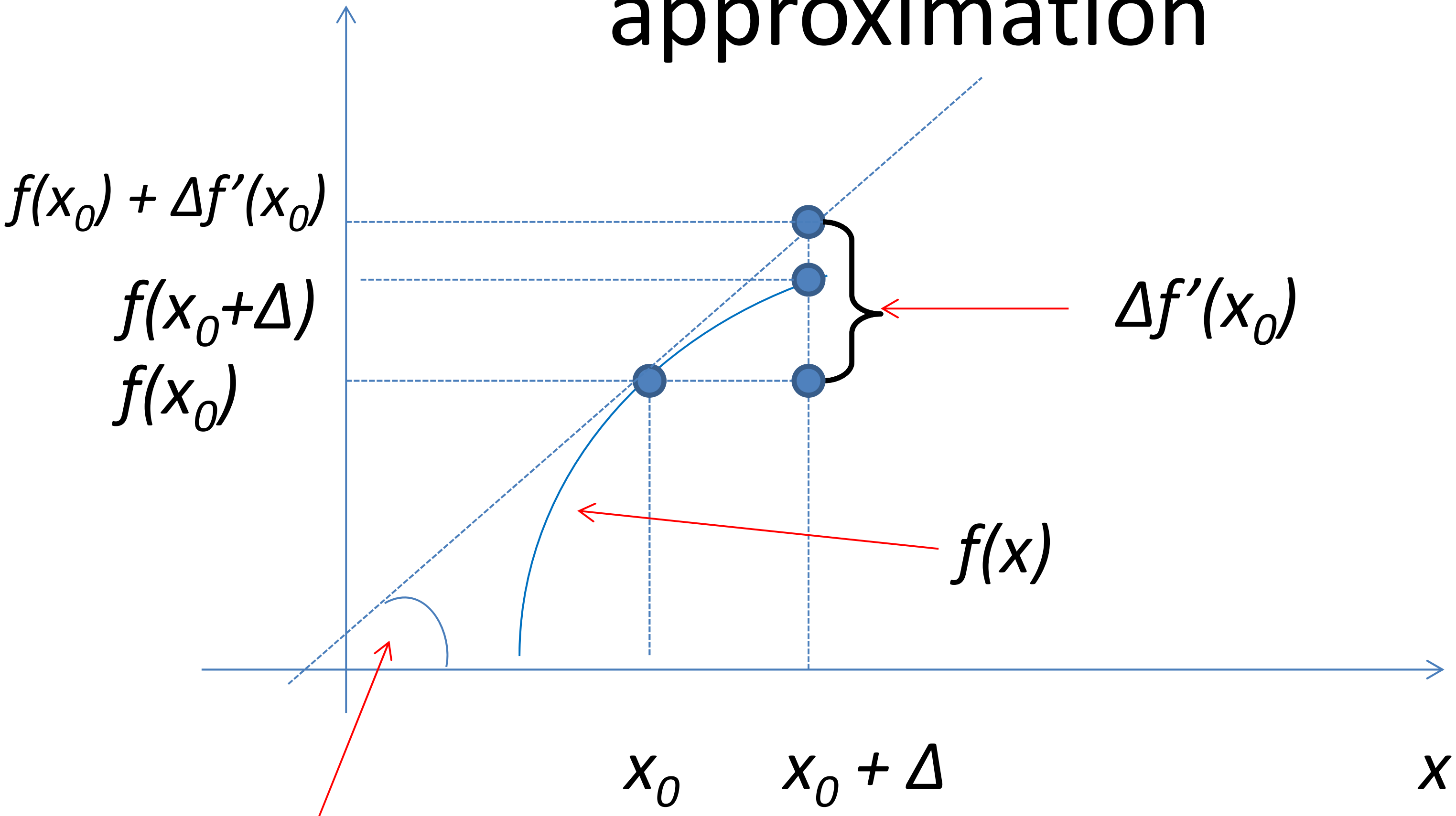
Question 2. SOLUTION _

A) $E(C) = 2$, $Var(C) = 2/100$. Relative risk aversion=1. Absolute risk av. =1/2. Hence since the risk is small, since the absolute risk aversion evaluated at E(C) is $1/E(C) = 1/2$, then from the Arrow-Pratt approximation it follows that the risk premium is

$\approx -(1/2)Var(C)U''(E(C))/U'(E(C)) =$
$(1/2)(2/100)(1/2) = 1/200 =$ risk pemium

B) Certainty equivalent consumption = E(C) - Risk premium = $2 - 1/200 = 1.995$

Note on first degree Taylor approximation



$f(x_0) + \Delta f'(x_0) \approx f(x_0+\Delta)$

$f(x_0) + \Delta f'(x_0) = f(x_0+\Delta) + rest$

Where rest $\rightarrow 0$ *if* $\Delta \rightarrow 0$.

Exercise July 10, 2017

Suppose $u(w) = \ln(w)$. The individual has wealth w and faces the gamble of winning or losing the same amount of wealth, h, with $0 < h < w$, with 50:50 odds.

Find the closed form solution (not the Arrow-Pratt approximation) of risk premium and determine whether it is positive, negative or zero.

SOLUTION

The certainty equivalent utility is

$\frac{1}{2}\ln(w+h) + \frac{1}{2}\ln(w-h) = \ln(w+h)^{1/2} + \ln(w-h)^{1/2} = \ln[(w+h)^{1/2}(w-h)^{1/2}] = \ln(w^2-h^2)^{1/2}$ =>

Certain equivalent wealth is $(w^2-h^2)^{1/2}$.

Expected wealth = w.

Hence risk premium is $\mathbf{w - (w^2-h^2)^{1/2}}$. To sign it consider that $w^2 > (w^2-h^2)$ => $\mathbf{w - (w^2-h^2)^{1/2} > 0}$ since $w>0$.

Exercise Nov 2014

Suppose you have an initial level of wealth W>100, and you have a well behaved utility function U over wealth and you are just indifferent between rejecting and accepting a bet that gives you the possibility of winning 110 euro or losing 100 euro with 50:50 chance. (To clarify: If you win you end up with W+110; if you lose you end up with W-100). What is your risk premium?

SOLUTION

Your risk premium is 5.

In fact $U(W) = ½U(W-100) + ½U(W+110) \Rightarrow$ certainty equivalent wealth is W.

Expected value of wealth is

$½(W-100) + ½(W+110) = W+5.$

Hence $(W + 5) – W = 5 =$ risk premium.