POLITECNICO DI MILANO

Spacecraft attitude dynamics and control

Project

Author:

Matteo Grechi 10621015

Professor:
Franco Bernelli

11th Jul, 2021

Contents

1 Introduction **2**

2 Reference frames **2**

3 Spacecraft description **3**
3.1 Structure 3
3.2 Sensors 3
3.2.1 Sun sensor and star tracker model 4
3.2.2 Gyroscope model 4
3.3 Actuators 5

4 Spacecraft kinematic and dynamic **6**
4.1 Environment disturbances 6

5 Attitude estimation **8**
5.1 State propagation 8
5.2 State update 9

6 Attitude control **9**
6.1 Detumbling law 9
6.2 General law 9
6.3 Control allocator 10

7 Results **10**

A Kalman filter **11**

B “Asyncronous” Kalman filter **13**

C Lyapunov control function **14**

1 Introduction

The goal of this project is to develop a simple ADCS system for a satellite. As imposed by requirements, quaternions are used as attitude parameters; Sun sensor is used as attitude sensor and constant thrust thruster are used as actuator.

2 Reference frames

The reference frame used in this project are the following:

- Earth centered inertial frame (called also NF or $\mathcal{N}$): its x and z axis correspond to the vernal equinox and to the north pole direction. The y direction is defined such that $\{x, y, z\}$ is right-handed.
- LVLH Frame (called also LF or $\mathcal{L}$): z axis is pointing at the center of Earth, y axis is aligned to minus the specific angular momentum of the orbit, and x is completing the right-handed triad.
- Body Frame (called also BF or $\mathcal{B}$): its x, y and z axis are the red, green and blue axis shown in figure 1.
- Reference attitude frame (called also RF or $\mathcal{R}$): when $\mathcal{B}$ is coincident to $\mathcal{R}$ then the spacecraft has the corrected attitude.
- Principal axis of Inertia frame (called also IF or $\mathcal{I}$): ideally this frame coincide with $\mathcal{B}$. In reality, because the mass is not distributed evenly, the principal axis of inertia are rotated.
- Sensor Frame (called also SF or $\mathcal{S}$): as specified by the sensor manufacturer.

The following conventions are used:

- $\{^{\mathcal{N}}v\}$ is a column array containing the components of the vector $\vec{v}$ in the $\mathcal{N}$ frame (lower-case v).
- $\{^{\mathcal{N}}J\}$ is a matrix containing the components of the rank 2 tensor J in the $\mathcal{N}$ frame (upper-case J).
- $A_{\mathcal{B}/\mathcal{N}}$ represent the rotation matrix such that $\{^{\mathcal{B}}v\} = A_{\mathcal{B}/\mathcal{N}}\{^{\mathcal{N}}v\}$.
- $q_{\mathcal{B}/\mathcal{N}}$ is the quaternion representation of $A_{\mathcal{B}/\mathcal{N}}$. Let's consider two vectors $\{^{\mathcal{B}}v\}$ and $\{^{\mathcal{N}}v\}$. $q_{\mathcal{B}/\mathcal{N}}$ is a unitary quaternion such that $\{0, \{^{\mathcal{B}}v\}\} = q_{\mathcal{B}/\mathcal{N}}^{-1} \cdot \{0, \{^{\mathcal{N}}v\}\} \cdot q_{\mathcal{B}/\mathcal{N}}$ where $\cdot$ is the standard quaternion product.
- $\vec{\omega}_{\mathcal{R}/\mathcal{B}}$ is the relative angular velocity of frame $\mathcal{R}$ respect to frame $\mathcal{B}$: $\vec{\omega}_{\mathcal{R}/\mathcal{B}} = \vec{\omega}_{\mathcal{R}} - \vec{\omega}_{\mathcal{B}}$.
  $\vec{\omega}_{\mathcal{R}}$ and $\vec{\omega}_{\mathcal{B}}$ are the angular velocity of $\mathcal{R}$ and $\mathcal{B}$ measured in an inertial frame.
  $\{^{\mathcal{B}}\omega_{\mathcal{R}/\mathcal{B}}\}$ is simply $\vec{\omega}_{\mathcal{R}/\mathcal{B}}$ expressed in the $\mathcal{B}$ frame.
- $q \otimes p$ is NOT the standard quaternion product between q and p. In particular $q \otimes p = p \cdot q$: in this way $A(q \otimes p) = A(q)A(p)$.

3 Spacecraft description

3.1 Structure

In this project a 3U satellite is considered and is shown in figure 1 and has the following dimensions: $a = 10$ cm, $b = 10$ cm and $c = 30$ cm. A typical mass value for a 3U spacecraft is 4kg so with an

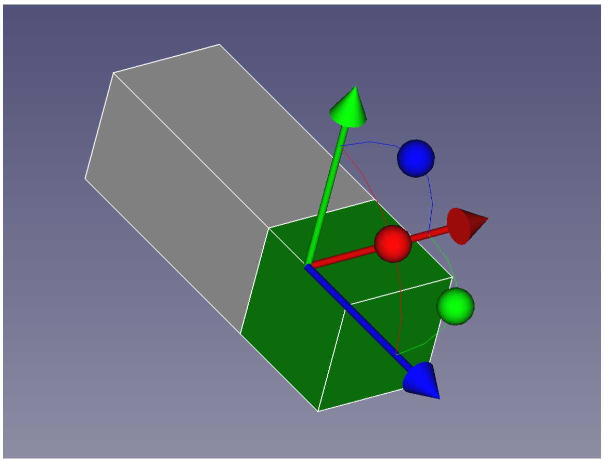


Figure 1: Spacecraft structure

uniform mass distribution the inertia matrix expressed in $\mathcal{I}$ is:

$$\{^{\mathcal{I}}J\} = \frac{1}{2}m \begin{bmatrix} b^2 + c^2 & & \\ & a^2 + c^2 & \\ & & a^2 + b^2 \end{bmatrix} = \begin{bmatrix} 0.2 & & \\ & 0.2 & \\ & & 0.04 \end{bmatrix} \text{kg m}^2$$

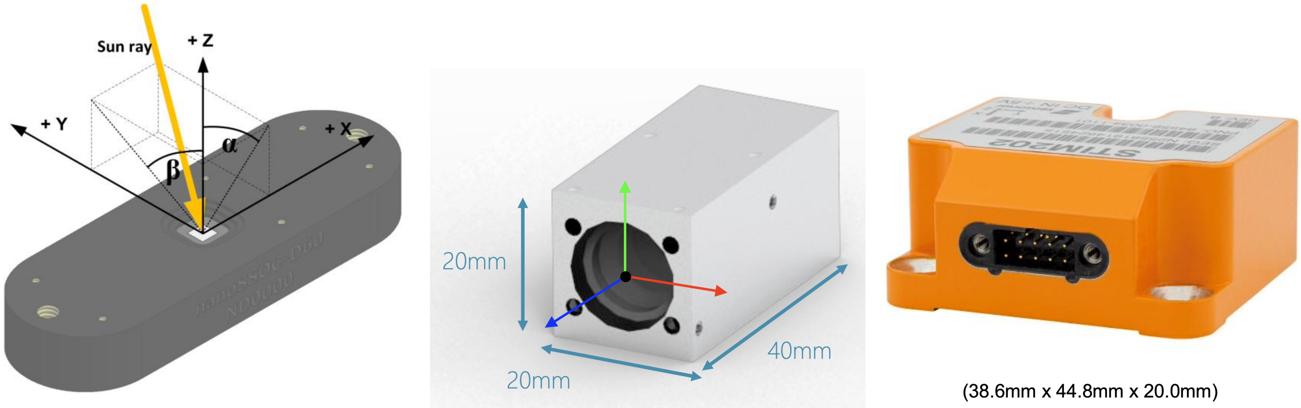
Considering the proposed geometry, in ideal conditions, $\mathcal{I}$ and $\mathcal{B}$ are coincident. In reality this is hardly achievable so a random rotation between the two frames is considered.
The rotation, represented by the quaternion $q_{\mathcal{I}/\mathcal{B}}$, has a random rotation axis with a random rotation angle distributed as $\mathcal{N}(0, (3\,\text{deg})^2)$.
Same considerations can be done on the center of mass of the s/c: ideally, given the proposed geometry, its position coincides with the geometric center of the solid. Because in reality the mass is not evenly distributed the center of mass position had been offsetted from the geometric center by a random direction multiplied by a random "length" distributed as $\mathcal{N}(0, (3\text{cm})^2)$.

3.2 Sensors

For this project only the Sun sensor is assigned. In order to increase the estimated attitude accuracy, a gyroscope and star trackers are used. The components shown in figure 2 are:

- Sun sensor: `nanoSSOC-D60` by `SOLARMEMS`
- Star tracker: `Mini Star Tracker` by `KU Leuven`
- Gyroscope: `STIM202` by `Sensonor`

The properties of the sensors are listed in table 1.



(a) Sun sensor (b) Star tracker (c) Gyroscope

Figure 2: Sensors

Sensor	Sampling freq [Hz]	FoW [deg]	Accuracy [arcsec]	ARW [deg/$\sqrt{\text{h}}$]	RRW [deg/h]
Sun sensor	50	60	20	-	-
Star tracker	2	60	240	-	-
Gyroscope	250	-	-	0.17	0.4

Table 1: Sensors properties

3.2.1 Sun sensor and star tracker model

Sun sensor and star tracker are very similar: their output consists of two angles, α and β, that gives the position of the target in the sensor celestial sphere.
In the case of the Sun sensor the Sun is the only target while with the star tracker the targets are multiple (the recognized stars).
Consider the sensor frame shown in figure 2 and a target direction $\{^{\mathcal{S}}n\}$. The target direction measured by the sensor is:

$$\{^{\mathcal{S}}n_{\text{meas}}\} = A(\delta q_{\text{cross boresight}} \otimes \delta q_{\text{about boresight}})\{^{\mathcal{S}}n\}$$

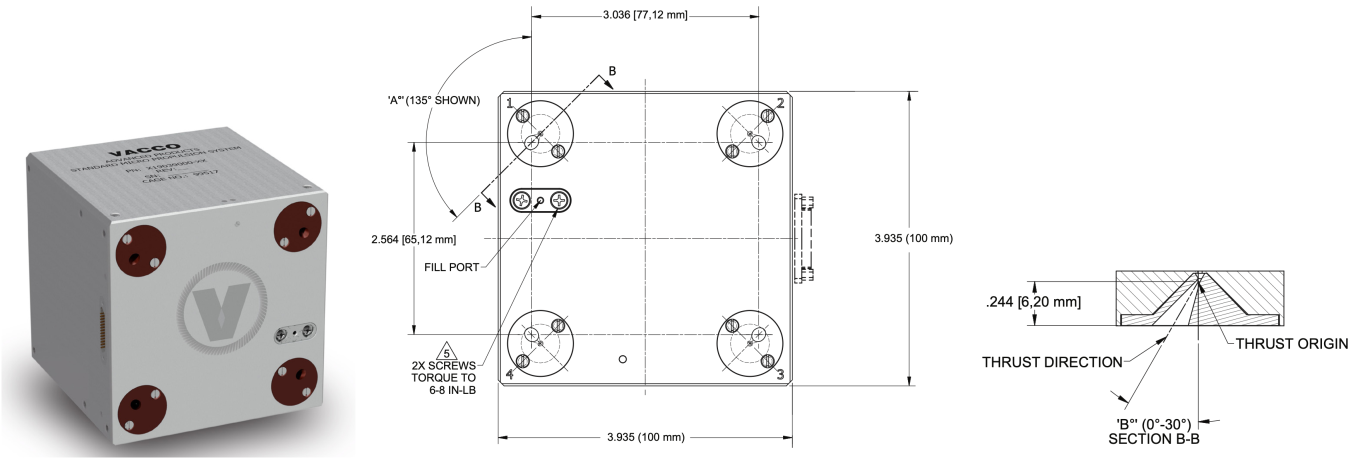
where $\delta q_{\text{about boresight}} = \cos(\delta\theta/2) + \mathbf{k}\sin(\delta\theta/2)$ and $\delta q_{\text{cross boresight}} = \cos(\delta\psi/2) + \mathbf{e}_{xy}\sin(\delta\psi/2)$. $\mathbf{e}_{xy}$ is a random direction in the xy plane while $\delta\theta$ and $\delta\psi$ are related to the about and cross boresight uncertainty: $\delta\theta \sim \mathcal{N}(0, \sigma^2_{\text{ab}})$, $\delta\psi \sim \mathcal{N}(0, \sigma^2_{\text{xb}})$.
The two angles α and β are then computed as:

$$\begin{cases} \alpha = \text{atan}\Big(\{^{\mathcal{S}}n\}_x / \{^{\mathcal{S}}n\}_z\Big) \\ \beta = \text{atan}\Big(\{^{\mathcal{S}}n\}_y / \{^{\mathcal{S}}n\}_z\Big) \end{cases}$$

where $\{^{\mathcal{S}}n\}_x$, $\{^{\mathcal{S}}n\}_y$ and $\{^{\mathcal{S}}n\}_z$ are the x, y and z components of $\{^{\mathcal{S}}n\}$.
This procedures is applied to each target.
For simplicity no visibility of the stars is checked.
For the sun sensor, instead, it has been checked that the chosen orbit is always exposed to the Sun.

3.2.2 Gyroscope model

The output of the gyroscope is simply the body angular velocity expressed in the $\mathcal{B}$ frame. This velocity is only perturbed by a normally distributed term (n) and by a random walk term (b). In



(a) 3D view (b) Front view (c) Nozzle section

Figure 3: Propulsion unit

particular the measured angular velocity is:

$$\{^{\mathcal{B}}\omega_{\text{meas}}\} = \{^{\mathcal{B}}\omega\} + b + n$$

where $n \sim \mathcal{N}(0_3, (ARW\sqrt{f})^2 I_{3\times3})$ and $\frac{db}{dt} \sim \mathcal{N}(0_3, (RRW\sqrt{f})^2 I_{3\times3})$.
Because the control logic is expressed in discrete time the above relation shall be rewritten in:

$$\begin{cases} \{^{\mathcal{B}}\omega_{\text{meas}}\}_k = \{^{\mathcal{B}}\omega\}_k + b_k + n_k \\ b_k = b_{k-1} + \delta b_k \end{cases} \quad \text{where:} \quad \begin{cases} n_k \sim \mathcal{N}(0_3, (ARW\sqrt{f})^2 I_{3\times3}) \\ \delta b_k \sim \mathcal{N}(0_3, (RRW\sqrt{f})^2/f \; I_{3\times3}) \end{cases}$$

f (frequency), ARW and RRW are listed in table 1.

3.3 Actuators

For this project constant thrust jets are assigned. Given the spacecraft size, the `VACCO X19039000` propulsion unit has been selected. The propulsion unit is the green cube of figure 1. A detailed view of the propulsion unit is shown in figure 3.

The selected values for the angles A and B (see front view for A and nozzle section for B - figure 3) are respectively 135° and 30°.
Because it is required is to control only the attitude of the spacecraft, only the torque generated by each jet is computed as:

$$\{^{\mathcal{B}}T_i\} = (\{^{\mathcal{B}}P_i\} - \{^{\mathcal{B}}G\}) \times (-T\{^{\mathcal{B}}n_i\})$$

where $\{^{\mathcal{B}}P_i\} - \{^{\mathcal{B}}G\}$ and $\{^{\mathcal{B}}n_i\}$ are the nozzle position respect to the center of mass and the thrust direction of the jet, while T is the thrust magnitude.
The control torque is simply:

$$\{^{\mathcal{B}}T_{\text{control}}\} = \sum_i \{^{\mathcal{B}}T_i\} u_i$$

where $u_i \in \{0, 1\}$ is the on/off control signal of each jet.
Because no details are present, a minimum impulse bit of 0.1s is assumed.

4 Spacecraft kinematic and dynamic

The assigned attitude parameters are quaternions.
The attitude evolution is described by:

$$\frac{\mathrm{d}q_{\mathcal{B}/\mathcal{N}}}{\mathrm{d}t} = \frac{1}{2}\Omega(\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\})q_{\mathcal{B}/\mathcal{N}} \quad \text{with: } \Omega(u) = \begin{bmatrix} 0 & -u_1 & -u_2 & -u_3 \\ u_1 & 0 & u_3 & -u_2 \\ u_2 & -u_3 & 0 & u_1 \\ u_3 & u_2 & -u_1 & 0 \end{bmatrix} \tag{1}$$

In the above equation $q_{\mathcal{B}/\mathcal{N}}$ is a 4×1 array containing the real part of $q_{\mathcal{B}/\mathcal{N}}$ as first component (contrary to what seen in class).
Note: equation 1 describe the time evolution of the $\mathcal{B}$ frame. The counterpart describing the time evolution of frame $\mathcal{I}$ will be $\frac{\mathrm{d}q_{\mathcal{I}/\mathcal{N}}}{\mathrm{d}t} = \frac{1}{2}\Omega(\{^{\mathcal{I}}\omega_{\mathcal{I}/\mathcal{N}}\})q_{\mathcal{I}/\mathcal{N}}$.

The rotational dynamic of the body is easily described by Euler equations. These equations are written in the $\mathcal{I}$ frame.
The torque acting on the rigid body is the sum of the control torque and the environment disturbances.
Euler equations are:

$$\begin{cases} \dot{p} = \frac{B-C}{A}qr + \frac{M_x}{A} \\ \dot{q} = \frac{C-A}{B}pr + \frac{M_y}{B} \\ \dot{r} = \frac{A-B}{C}pq + \frac{M_z}{C} \end{cases} \quad \text{where: } \begin{cases} \{^{\mathcal{I}}\omega_{\mathcal{I}/\mathcal{N}}\} = \{p, q, r\}^T \\ \{^{\mathcal{I}}T_{\mathrm{TOT}}\} = \{M_x, M_y, M_z\}^T \\ \{^{\mathcal{I}}J\} = \mathrm{diag}(\{A, B, C\}^T) \end{cases} \tag{2}$$

with

$$\{^{\mathcal{I}}T_{\mathrm{TOT}}\} = \{^{\mathcal{I}}T_{\mathrm{control}}\} + \{^{\mathcal{I}}T_{\mathrm{GG}}\} + \{^{\mathcal{I}}T_{\mathrm{SRP}}\} + \{^{\mathcal{I}}T_{\mathrm{MD}}\} \tag{3}$$

4.1 Environment disturbances

The only missing disturbance torque is the one due by the aerodynamic drag. Since the height is greater than 1000km, its contribution had been discarded.
The considered disturbances are:

- gravity gradient:

$$\{^{\mathcal{I}}T_{\mathrm{GG}}\} = \frac{3\mu}{R^3}\{(C-B)c_2c_3,\ (A-C)c_1c_3,\ (B-A)c_1c_2\}^T$$

  where

  - μ is Earth's planetary constant
  - R is the distance between the spacecraft center of mass and the center of the Earth
  - $\{A, B, C\}$ are the principal moments of inertia: $\{^{\mathcal{I}}J\} = \mathrm{diag}(\{A, B, C\}^T)$
  - $\{c_1, c_2, c_3\}$ are the component of the normalized spacecraft Center of Mass position in the $\mathcal{I}$ frame:

$$\{c_1, c_2, c_3\}^T = \frac{\{^{\mathcal{I}}R_{\mathrm{s/c\ CoM}}\} - \{^{\mathcal{I}}R_{\mathrm{CoE}}\}}{R}$$

    where $\vec{R}_{\mathrm{CoE}}$ is the Center of Earth position

- solar radiation pressure:

$$\{^{\mathcal{B}}T_{\text{SRP}}\} = \sum_i (\{^{\mathcal{B}}R_{\text{surf cent; }i}\} - \{^{\mathcal{B}}R_{\text{CoM}}\}) \times \{^{\mathcal{B}}F_i\}$$

with

$$\{^{\mathcal{B}}F_i\} = -PA_i(\{^{\mathcal{B}}u_{\text{Sun}}\} \cdot \{^{\mathcal{B}}n_i\}) \left[(1-\rho_{S;\,i})\{^{\mathcal{B}}u_{\text{Sun}}\} + \left(2\rho_{S;\,i}\,\{^{\mathcal{B}}u_{\text{Sun}}\} \cdot \{^{\mathcal{B}}n_i\} + 2/3\,\rho_{D;\,i}\right)\{^{\mathcal{B}}n_i\}\right]$$

where

  - $(\{^{\mathcal{B}}R_{\text{surf cent; }i}\} - \{^{\mathcal{B}}R_{\text{CoM}}\})$ is the position of the geometric center of surface i respect to the s/c center of mass.
  - P is the radiation pressure: $P = F_e/c_0$
  - A_i is the surface area of surface i
  - $\{^{\mathcal{B}}u_{\text{Sun}}\}$ is the Sun direction
  - $\{^{\mathcal{B}}n_i\}$ is the facing direction of surface i
  - $\rho_{S;\,i}$ and $\rho_{D;\,i}$ are the specular reflection and diffuse reflection factors of surface i

  Note: solar radiation pressure is acting on a surface only when light is present.
  Given the simple geometry of the spacecraft this condition is achieved when:

$$\begin{cases} \{^{\mathcal{B}}u_{\text{Sun}}\} \cdot \{^{\mathcal{B}}n_i\} > 0 \\ \frac{\{^{\mathcal{B}}R_{\text{s/c CoM}}\} - \{^{\mathcal{B}}R_{\text{CoE}}\}}{\left\|\{^{\mathcal{B}}R_{\text{s/c CoM}}\} - \{^{\mathcal{B}}R_{\text{CoE}}\}\right\|} \cdot (-\{^{\mathcal{B}}u_{\text{Sun}}\}) = \cos\theta < \cos\arcsin\frac{R_E}{\left\|\{^{\mathcal{B}}R_{\text{s/c CoM}}\} - \{^{\mathcal{B}}R_{\text{CoE}}\}\right\|} \end{cases}$$

  The first condition states that the surface shall have a direct view of the Sun (albedo has not been considered) while the second states that the s/c shall not be in the umbra region of Earth.

- magnetic dipole:

$$\{^{\mathcal{B}}T_{\text{MD}}\} = \{^{\mathcal{B}}m_{\text{s/c}}\} \times \{^{\mathcal{B}}B\} = \{^{\mathcal{B}}m_{\text{s/c}}\} \times \{^{\mathcal{B}} - \nabla V\}$$

with

$$V(\rho,\theta,\phi) = R_E \sum_{i=1}^{n} \left\{ (R_E/\rho)^{i+1} \sum_{j=0}^{i} \left[\left(g_i^j \cos(j\phi) + h_i^j \sin(i\phi) \right) P_i^j(\cos\theta) \right] \right\}$$

where

$$P_i^j(x) = \frac{1}{2^i\, i!} \sqrt{\frac{2(i-j)!}{(i+j)!}(1-x^2)^j}\; \frac{\mathrm{d}^{i+j}}{\mathrm{d}x^{i+j}}\left((x^2-1)^i\right)$$

  g_i^j and h_i^j are called Gaussian coefficients. `IGRF 1995` coefficients are used and their value in nT is reported on the course slides. In the project the maximum order n is 4.
  ρ, ϕ, and θ are respectively the distance from the center of Earth, the longitude, and $90° -$ latitude.
  Because V is expressed in spherical coordinates

$$-\nabla V = -\frac{\partial V}{\partial \rho}\vec{e}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \theta}\vec{e}_\theta - \frac{1}{\rho\sin\theta}\frac{\partial V}{\partial \phi}\vec{e}_\phi = B_\rho\vec{e}_\rho + B_\theta\vec{e}_\theta + B_\phi\vec{e}_\phi$$

  By looking figure 4 is possible to define a temporary $\mathcal{T}$ "spherical frame" by considering the triad $\{\vec{e}_\theta, \vec{e}_\phi, \vec{e}_\rho\}$.
  The attitude quaternion of $\mathcal{T}$ is:

$$q_{\mathcal{T}/\mathcal{N}} = (\cos(RA/2) + \mathbf{j}\sin(RA/2)) \otimes (\cos((90° - \delta)/2) + \mathbf{k}\sin((90° - \delta)/2))$$

  where RA is the right ascension and δ the declination of the spacecraft.
  The magnetic field array is $\{^{\mathcal{T}}B\} = \{B_\theta,\ B_\phi,\ B_\rho\}^T$ with $\{^{\mathcal{B}}B\} = A(q_{\mathcal{B}/\mathcal{N}} \otimes q_{\mathcal{T}/\mathcal{N}}^{-1})\{^{\mathcal{T}}B\}$.

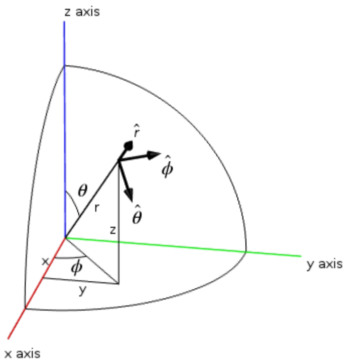


Figure 4: Spherical coordinates

5 Attitude estimation

The state estimator consists of an Unscented Kalman Filter.
Even if in this case there is no advantage in using an UKF over an Extended Kalman Filter (EKF), it is the choice for this project because it is derivative-less: the EKF needs the gradient of the state transition function and the gradient of the measurement function. Even if is possible to compute the gradient numerically when no analytic expression is available, by using the UKF the obstacle has been avoided.
To this project there attached a UKF implementation prototype: for simplicity reasons in the simulink file the preimplemented Matlab module has been used.

5.1 State propagation

The state vector of the filter has 7 components. The first 4 are the attitude quaternion $q_{\mathcal{B}/\mathcal{N}}$ while the last 3 are the gyro bias in $\mathcal{B}$:

$$\hat{x}_k = \{q_{\mathcal{B}/\mathcal{N}_{\text{estimated}}}|_{t=t_k},\ \{^{\mathcal{B}}b\}_{\text{estimated}}|_{t=t_k}\}^T = \{\hat{q}_k,\ \hat{b}_k\}^T$$

The propagation of the estimated attitude quaternion from t_k to t_{k+1} is obtained by a forward Euler integration step of equation 1:

$$\hat{q}_{k+1} = \hat{q}_k + \frac{1}{2}\Omega(\hat{\omega}_k)\hat{q}_k\Delta t \quad \text{with } \hat{\omega}_k = \{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\}_{\text{measured}}|_{t=t_k} - \hat{b}_k + \epsilon_{\omega,\,k}$$

where Δt is the time step. A quaternion normalization of $\hat{q}_{k+1}$ is highly reccomended at each integration step. The estimated bias update equation is:

$$\hat{b}_k = \hat{b}_k + \epsilon_{b,\,k}$$

Note: due to the nature of the equations the process noise is non additive. For this reason the two noise terms are present in the equations. Matlab, when the non additive noise option is selected, will automatically augment the state in order to treat the non additive noise properly.
The two noise terms are like:

$$\begin{cases} \epsilon_{\omega,\,k} \sim \mathcal{N}(0_3,\ (ARW\sqrt{f})^2 I_{3\times3}) \\ \epsilon_{b,\,k} \sim \mathcal{N}(0_3,\ (RRW\sqrt{f})^2 \Delta t I_{3\times3}) \end{cases}$$

5.2 State update

The state update is performed whenever a Sun sensor or a star tracker measurement is available. Since the two sensors has a different sampling rate, the state update is divided in 2 different steps:

- Sun sensor update
- Star tracker update

The two functions are in essence almost identical, the only difference is in the number of objects they are tracking, in the uncertainties and in the update frequency.
For the sake of simplicity is enough to see the Sun sensor update function.

6 Attitude control

The ideal control law is based on Lyapunov functions. For more details see the appendix.
Two different control laws are implemented:

- Detumbling law: used when only the angular velocity measure is available
- General law: whenever it is possible to estimate the attitude.

6.1 Detumbling law

This law is designed to work only with the angular velocity. This design choice is justified by the fact that when the angular velocity is high enough the star tracker loses its ability to detect stars. By using a specific control law that avoid the attitude information this problem is no more a problem.
The used Lyapunov function for this law is:

$$V = \frac{1}{2}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\}^T\{^{\mathcal{I}}J\}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\}$$

Note: the two frames $\mathcal{I}$ and $\mathcal{B}$ are considered coincident because no estimator able to determine the rotation between the two frames has been implemented.
The closed loop stability is reached when:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\}^T\left(\{^{\mathcal{I}}J\}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} \times \{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} + \{^{\mathcal{B}}u\}\right) < 0 \quad \forall t$$

So the following control law is derived:

$$\{^{\mathcal{B}}u\} = -k_1\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} - \{^{\mathcal{I}}J\}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} \times \{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\}$$

where $\{^{\mathcal{B}}u\}$ is the ideal control torque that shall be applied to the spacecraft.

6.2 General law

For a case in which the attitude quaternion can be estimated the control law described in this section can be used.
Consider the following Lyapunov function:

$$V = \frac{1}{2}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{R}}\}^T\{^{\mathcal{I}}J\}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{R}}\} + 2k_2\left(1 - \mathrm{Re}\left\{q_{\mathcal{B}/\mathcal{R}}\right\}^2\right)$$

As shown in the appendix, enforcing $\frac{\mathrm{d}V}{\mathrm{d}t} < 0\ \forall t$ leads to the following control expression:

$$\{^{\mathcal{B}}u\} = -k_1\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{R}}\} - \{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} \times (\{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{N}}\} - \{^{\mathcal{B}}\omega_{\mathcal{B}/\mathcal{R}}\}) - 2k_2\,\mathrm{Re}\left\{q_{\mathcal{B}/\mathcal{R}}\right\}\{\mathrm{Im}\left\{q_{\mathcal{B}/\mathcal{R}}\right\}\}$$

6.3 Control allocator

The ideal control torque is not directly applicable so a control allocator is needed. The control allocator goal is to find the activation sequence for each jet that gets closer to the ideal control.
Since constant thrust jets are assigned is not possible to modulate the single jet thrust to obtain the desired torque vector. However, by considering the activation time, is possible to mimic this behaviour.
By integrating the control torque over a period of Δt a delta in the angular momentum $\Delta\Gamma$ is obtained. The following optimization problem is then solved:

$$\begin{cases} \min_{\mathbf{x}} \left\|\Delta\Gamma - T\mathbf{x}\right\|^2 \\ 0 < x_i < \Delta t \quad \forall i \end{cases}$$

where $\Delta\Gamma = \int_t^{t+\Delta t} \{^{\mathcal{B}}u\} \mathrm{d}t$, T is a matrix which columns are the torque generated by the jets written in the $\mathcal{B}$ frame (the i-th column contain the torque generated by the i-th thruster).
Once the optimization has concluded and $\mathbf{x}_{\mathrm{opt}}$ is retrieved, the following activation law is considered:

$$u_i(t < \tau < t + \Delta t) = \begin{cases} 0 & \text{if } x_{\mathrm{opt}_i}/\Delta t \leq 0.5 \\ 1 & \text{otherwise} \end{cases} \qquad \forall i$$

where $u_i(t < \tau < t + \Delta t)$ is the on/off status of the i-th thruster for the time window between t and $t + \Delta t$.
The time Δt correspond to the thrusters minimum impulse bit (0.1s assumed).

7 Results

As shown in figure 5 the detumbling mode is quite effective since it reduces the angular speed to acceptable levels. For the normal mode figure 6 show that the pointing accuracy achieved is acceptable for the type of actuator used.

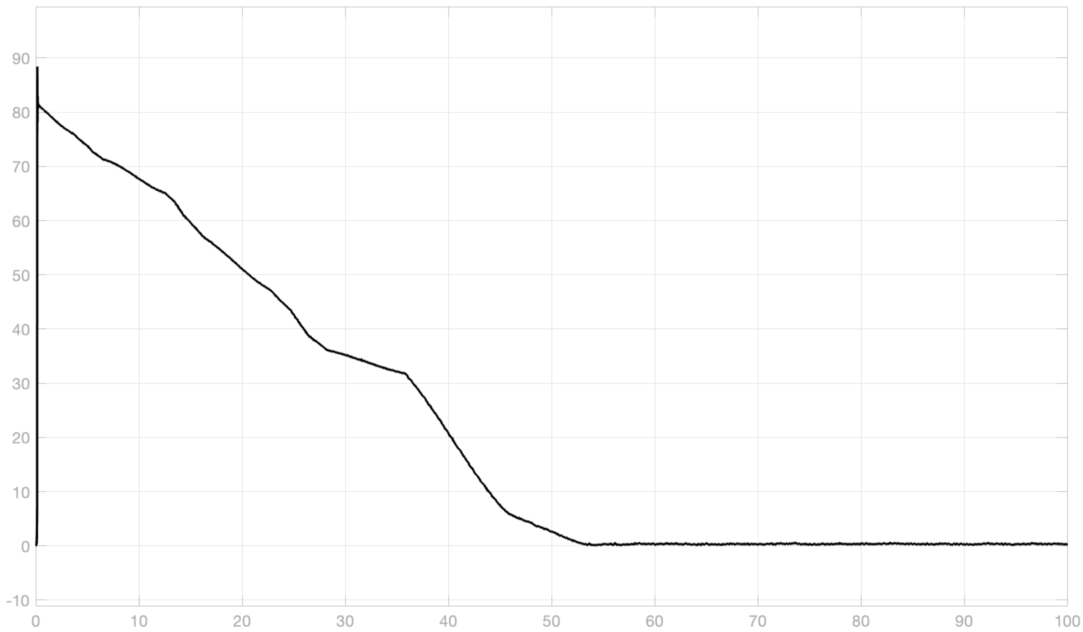


Figure 5: Angular rate [°/s] with the Detumbling control law vs time [s]

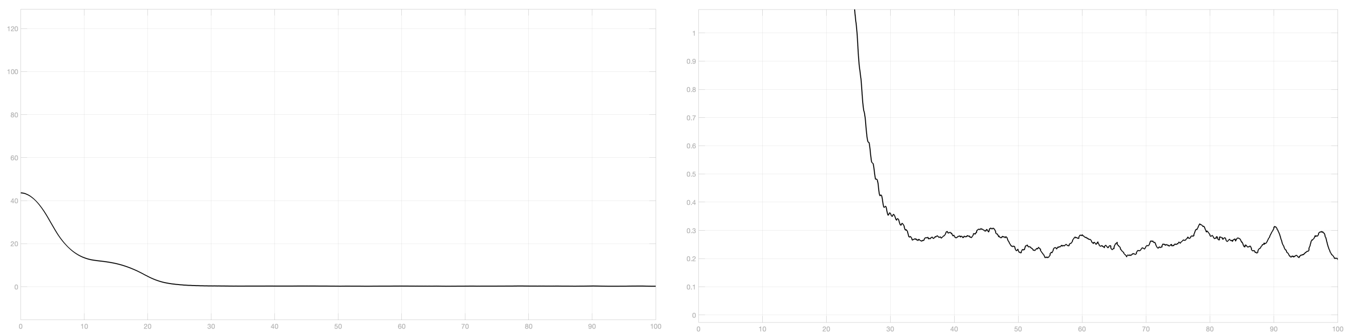


Figure 6: Attitude error [°] for the normal mode vs time [s]

A Kalman filter

A Kalman filter is a "statistically" optimal linear observer. Given a discrete time process described by the following equations:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + \epsilon(k) \\ y(k) = Cx(k) + \nu(k) \end{cases} \qquad \text{with} \begin{cases} \epsilon(k) \sim \mathcal{N}(0, R) \\ \nu(k) \sim \mathcal{N}(0, Q) \end{cases}$$

the state estimated by the Kalman filter $\hat{x}(k)$ is such that

$$\begin{cases} \mathbb{E}\{e(k)\} := \mathbb{E}\{x(k) - \hat{x}(k)\} = 0 \\ \mathbb{E}\{\|e(k)\|^2\} := \mathbb{E}\{\|x(k) - \hat{x}(k)\|^2\} \text{ is minimized} \end{cases} \tag{4}$$

The latter condition can be obtained by minimizing

$$\mathbb{E}\left\{\mathrm{Tr}\left\{e(k)e(k)^T\right\}\right\} = \mathrm{Tr}\{\mathrm{Cov}\{e(k)\}\} := \mathrm{Tr}\, P(k)$$

In the Kalman filter the evolution of the estimated state is described by the following equations:

$$\begin{cases} \hat{x}(k|k-1) = A\hat{x}(k-1) + Bu(k-1) \\ \hat{x}(k) = \hat{x}(k|k-1) + K[y(k) - C\hat{x}(k|k-1)] \end{cases}$$

where the matrix K is called Kalman gain.
The first condition of system 4, if $\mathbb{E}\{e(k-1)\} = 0$, is automatically satisfied:

$$\begin{aligned} e(k) = x(k) - \hat{x}(k) = & \underbrace{x(k)}_{Ax(k-1)+Bu(k-1)+\epsilon(k-1)} - \underbrace{\hat{x}(k|k-1)}_{A\hat{x}(k-1)+Bu(k-1)} - \\ & + K[\underbrace{y(k)}_{C[Ax(k-1)+Bu(k-1)+\epsilon(k-1)]+\nu(k)} - C\underbrace{\hat{x}(k|k-1)}_{A\hat{x}(k-1)+Bu(k-1)}] = \\ & = Ae(k-1) + \epsilon(k-1) - K[CAe(k-1) + C\epsilon(k-1) + \nu(k)] = \\ & = [1 - KC][Ae(k-1) + \epsilon(k-1)] - K\nu(k) \end{aligned}$$

The latter condition of system 4 is guaranteed by choosing an optimum value for the Kalman gain. First of all compute how the covariance evolve:

$$\begin{aligned} P(k) = \mathbb{E}\{e(k)e(k)^T\} = & [1 - KC]\underbrace{[AP(k-1)A^T + R]}_{\bar{P}(k-1)}[1 - KC]^T + KQK^T = \\ & = \bar{P}(k-1) - KC\bar{P}(k-1) - P(k-1)C^TK^T + KC\bar{P}(k-1)C^TK^T + KQK^T = \\ & = K\underbrace{[C\bar{P}(k-1)C^T + Q]}_{\Phi}K^T - K\underbrace{[C\bar{P}(k-1)]}_{\Psi} - \underbrace{[C\bar{P}(k-1)]^T}_{\Psi^T}K^T + \underbrace{\bar{P}(k-1)}_{\Xi} = \\ & = K\Phi K^T - K\Psi - \Psi^TK^T + \Xi \end{aligned} \tag{5}$$

Then consider that it is sufficient to minimize the trace of the covariance matrix. Because $\mathrm{Tr}\{A\} = \mathrm{Tr}\left\{A^T\right\}$ then:

$$\mathrm{Tr}\{P(k)\} = \mathrm{Tr}\left\{K\Phi K^T - 2K\Psi + \Xi\right\}$$

Using the expression for matrix derivative present in [1] the optimal value for K is found by imposing

$$\frac{\partial\,\mathrm{Tr}\{P(k)\}}{\partial K} = K(\Phi + \Phi^T) - 2\Psi^T = 2K\Phi - 2\Psi^T = 0$$

where the matrix Φ is symmetric. This equation gives the expression of the Kalman gain:

$$K = \Psi^T\Phi^{-1} = \bar{P}(k-1)C^T[C\bar{P}(k-1)C^T + Q]^{-1}$$

Substituting back the expression for the Kalman gain in equation 5 the update equation for the covariance matrix becomes:

$$\begin{aligned} P(k) = & \underbrace{K\Phi K^T}_{K\Phi[\Psi^T\Phi^{-1}]^T = K\Phi[\Phi^T]^{-1}\Psi = K\Psi} - K\Psi - \Psi^TK^T + \Xi = -\Psi^TK^T + \Xi = \\ & = -\bar{P}(k-1)C^TK^T + \bar{P}(k-1) = \bar{P}(k-1)(1 - C^TK^T) \end{aligned}$$

Because $P(k)$ and $\bar{P}(k-1)$ are symmetric then:

$$P(k) = (1 - CK)\bar{P}(k-1)$$

B "Asyncronous" Kalman filter

In the reality usually the measurements are not available at the same time. In the case of this project the state is propagated using the gyroscope measurements while the correction step can be done only when the star tracker and the Sun sensor data are available. The operating frequency of the gyroscope is far greater than the operating frequency of the other two sensors.
An asyncronous Kalman filter is able to deal with asyncronous data.
In this case the state propagation and correction equations are the following:

$$\begin{cases} \text{State propagation from } k-i \text{ to } k: \\ \hat{x}(k-i+1|k-i) = A\hat{x}(k-i) + Bu(k-i) \\ ... \\ \hat{x}(k-1|k-i) = A\hat{x}(k-2|k-i) + Bu(k-2) \\ \hat{x}(k|k-i) = A\hat{x}(k-1|k-i) + Bu(k-1) \\ \text{State correction:} \\ \hat{x}(k) = \hat{x}(k|k-i) + K[y(k) - C\hat{x}(k|k-i)] \end{cases}$$

The real process is still characterized by:

$$\begin{cases} x(k) = Ax(k-1) + Bu(k-1) + \epsilon(k-1) \\ y(k) = Cx(k) + \nu(k) \end{cases} \quad \text{with} \quad \begin{cases} \epsilon(k) \sim \mathcal{N}(0, R) \\ \nu(k) \sim \mathcal{N}(0, Q) \end{cases}$$

Let's define and write the update equations of the quantities:

$$\begin{cases} e(k|k-i) := x(k) - \hat{x}(k|k-i) = Ax(k-1) + Bu(k-1) + \epsilon(k-1) - A\hat{x}(k-1|k-i) + Bu(k-1) = \\ \qquad = Ae(k-1|k-i) + \epsilon(k-1) \\ P(k|k-i) := \mathbb{E}\{e(k|k-i)e(k|k-i)^T\} = AP(k-1|k-i)A^T + R \end{cases}$$

Then the error becomes:

$$\begin{aligned} e(k) := x(k) - \hat{x}(k) &= x(k) - \hat{x}(k|k-i) - K[y(k) - C\hat{x}(k|k-i)] = \\ &= e(k|k-i) - K[Cx(k) + \nu(k) - C\hat{x}(k|k-i)] = \\ &= e(k|k-i) - KCe(k|k-i) - K\nu(k) = \\ &= [1 - KC]e(k|k-i) - K\nu(k) \end{aligned}$$

By doing the same calculation done in the simple Kalman filter appendix chapter is possible to compute $P(k)$ as follows:

$$P(k) := \mathbb{E}\{e(k)e(k)^T\} = K[CP(k|k-i)C^T + Q]K^T - K[CP(k|k-i)] - [CP(k|k-i)]^T K^T + P(k|k-i)$$

By minimizing the trace of $P(k)$ the following optimal Kalman gain is obtained:

$$K = P(k|k-i)C^T[CP(k|k-i)C^T + Q]^{-1}$$

Please note that $P(k) = P(k|k)$ (in general $\star(k) = \star(k|k)$). In this case the correction equation for $P(k)$ is:

$$P(k) = (1 - CK)P(k|k-i)$$

C Lyapunov control function

Lyapunov stability theorem assert that if exist a function (called Lyapunov function) such that:

$$\begin{cases} V(x^*) \le V(x) & \forall x \\ \frac{\mathrm{d}V}{\mathrm{d}t} < 0 & \forall t \end{cases}$$

than the point x^* is a stable equilibrium point.
The chosen Lyapunov function for this project is:

$$V = \frac{1}{2}\{^{\mathcal{B}}\omega_e\}^T\{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_e\} + 2k_2(1 - q_{0e}^2)$$

where the array $\{^{\mathcal{B}}\omega_e\}$ and the matrix $\{^{\mathcal{B}}J\}$ are the components of angular rate error and the components of the inertia tensor expressed in the body reference frame.
q_{0e} is the real part of the error quaternion defined as $q_e = q_{B/R} = q_{B/N} \otimes q_{R/N}^{-1}$ where $q_{R/N}$ is the quaternion representation of the attitude matrix $A_{R/N}$ while $q_{B/N}$ is the quaternion representation of the attitude matrix $A_{B/N}$.
This function automatically satisfy the first Lyapunov condition because the matrix J is a positive definite symmetric matrix and because k_2 is chosen to be > 0.
The expression for the control law is retrieved by imposing the second condition:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \{^{\mathcal{B}}\omega_e\}^T\{^{\mathcal{B}}J\}\frac{\mathrm{d}\{^{\mathcal{B}}\omega_e\}}{\mathrm{d}t} - 4k_2 q_{0e}\frac{\mathrm{d}q_{0e}}{\mathrm{d}t}$$

The angular velocity error is defined as $\vec{\omega}_e = \vec{\omega}_B - \vec{\omega}_R$ where $\vec{\omega}_R$ is the reference angular velocity. The reference angular velocity is defined in the $\mathcal{N}$ frame so $\{^{\mathcal{B}}\omega_e\} = \{^{\mathcal{B}}\omega_B\} - A_{B/N}\{^{\mathcal{N}}\omega_R\}$ where $\{^{\mathcal{B}}\omega_B\}$ is the body angular velocity expressed in the body frame while $\{^{\mathcal{N}}\omega_R\}$ is the reference angular velocity expressed in $\mathcal{N}$.
The equation above can be expressed as:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \{^{\mathcal{B}}\omega_e\}^T\left(\{^{\mathcal{B}}J\}\frac{\mathrm{d}\{^{\mathcal{B}}\omega_B\}}{\mathrm{d}t} - \{^{\mathcal{B}}J\}\frac{\mathrm{d}}{\mathrm{d}t}\Big(A_{B/N}\{^{\mathcal{N}}\omega_R\}\Big)\right) - 4k_2 q_{0e}\frac{\mathrm{d}q_{0e}}{\mathrm{d}t} \tag{6}$$

The rotational dynamics of a body with constant inertia is described by:

$$\{^{\mathcal{B}}J\}\frac{\mathrm{d}\{^{\mathcal{B}}\omega_B\}}{\mathrm{d}t} = \{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_B\} \times \{^{\mathcal{B}}\omega_B\} + \{^{\mathcal{B}}u\} \tag{7}$$

where $\{^{\mathcal{B}}u\}$ is the external torque.
The kinematic equations written in quaternion form states that

$$\frac{\mathrm{d}q_{0e}}{\mathrm{d}t} = -\frac{1}{2}\{^{\mathcal{B}}\omega_e\}^T\{q_{ve}\} \tag{8}$$

where $\{q_{ve}\}$ is the vector part of the quaternion q_e.
By substituting equations 7 and 8 in equation 6 the following equation is obtained:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \{^{\mathcal{B}}\omega_e\}^T\left(\{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_B\} \times \{^{\mathcal{B}}\omega_B\} + \{^{\mathcal{B}}u\} - \{^{\mathcal{B}}J\}\frac{\mathrm{d}}{\mathrm{d}t}\Big(A_{B/N}\{^{\mathcal{N}}\omega_R\}\Big) + 2k_2 q_{0e}\{q_{ve}\}\right) \tag{9}$$

From equation 9 it follows that a possible value for the control law is:

$$\{^{\mathcal{B}}u\} = -k_1\{^{\mathcal{B}}\omega_e\} - \{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_B\} \times \{^{\mathcal{B}}\omega_B\} + \{^{\mathcal{B}}J\}\frac{\mathrm{d}}{\mathrm{d}t}\Big(A_{B/N}\{^{\mathcal{N}}\omega_R\}\Big) - 2k_2 q_{0e}\{q_{ve}\} \tag{10}$$

By substituting equation 10 in equation 9 it can be checked that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$. The term $\frac{\mathrm{d}}{\mathrm{d}t}\Big(A_{B/N}\{^{\mathcal{N}}\omega_R\}\Big)$ can be rewritten as:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(A_{B/N}\{^{\mathcal{N}}\omega_R\}\Big) &= -\{^{\mathcal{B}}\omega_B\} \times A_{B/N}\{^{\mathcal{N}}\omega_R\} + A_{B/N}\frac{\mathrm{d}\{^{\mathcal{N}}\omega_R\}}{\mathrm{d}t} = \\
&= \{^{\mathcal{B}}\omega_B\} \times (\{^{\mathcal{B}}\omega_B\} - \{^{\mathcal{B}}\omega_B\} - A_{B/N}\{^{\mathcal{N}}\omega_R\}) + A_{B/N}\frac{\mathrm{d}\{^{\mathcal{N}}\omega_R\}}{\mathrm{d}t} = \\
&= \{^{\mathcal{B}}\omega_B\} \times (\{^{\mathcal{B}}\omega_e\} - \{^{\mathcal{B}}\omega_B\}) + A_{B/N}\frac{\mathrm{d}\{^{\mathcal{N}}\omega_R\}}{\mathrm{d}t} = \\
&= \{^{\mathcal{B}}\omega_B\} \times \{^{\mathcal{B}}\omega_e\} + A_{B/N}\frac{\mathrm{d}\{^{\mathcal{N}}\omega_R\}}{\mathrm{d}t}
\end{aligned}$$

By considering a constant $\{^{\mathcal{N}}\omega_R\}$, equation 10 can be rewritten as:

$$\{^{\mathcal{B}}u\} = -k_1\{^{\mathcal{B}}\omega_e\} - \{^{\mathcal{B}}J\}\{^{\mathcal{B}}\omega_B\} \times (\{^{\mathcal{B}}\omega_B\} - \{^{\mathcal{B}}\omega_e\}) - 2k_2 q_{0e}\{q_{ve}\}$$

References

[1] K. B. Petersen and M. S. Pedersen. *The Matrix Cookbook*. Nov. 2012. URL: http://www2.imm.dtu.dk/pubdb/p.php?3274.